Symmetric monoidal adjunctions

This is just a short note to show that two reasonable definitions of "symmetric monoidal adjunctions" actually agree - there is no claim of originality here whatsoever, this is a well-known fact (e.g. documented on the corresponding nLab page), and this note is just here to record it.

We phrase everything in terms of symmetric monoidal things, but the same holds when we replace every occurrence of "symmetric monoidal" with "monoidal" or "braided monoidal" (or in the ∞-categorical context, "E_n-monoidal" - we will explain shortly how to make sense of this ∞-categorically)

Definition 0.1. Let C, D be two symmetric monoidal categories. A symmetric monoidal adjunction is a pair of adjoint functors (L, R) that are both equipped with a lax symmetric monoidal structure, and such that the unit and co-unit are symmetric monoidal.

Remark 0.2. In an ∞-categorical context, this last requirement must also be the requirement of a lax symmetric monoidal *structure* on the unit and co-unit.

The result we want to prove is the following:

Proposition 0.3. *Suppose (L, R) is a symmetric monoidal adjunction. Then L is strong symmetric monoidal.*

We will in fact state a more precise result. Before going there let us make a few remarks.

Remark 0.4. Given a lax symmetric monoidal functor $F : C \to D$, we have in particular a natural transformation $F(x) \otimes F(y) \to F(x \otimes y)$, as well as a morphism $1_C \to F(1_D)$.

F is strong symmetric monoidal if and only if these are isomorphisms - in other words, it is a property of F (as opposed to the lax symmetric monoidal *structure* on a plain functor), which is why the above makes sense.

Remark 0.5. Suppose C, D are symmetric monoidal ∞-categories, and (L, R) is as above. Then $(\mathrm{Ho}(L), \mathrm{Ho}(R))$ is a symmetric monoidal adjunction between the homotopy categories $\mathrm{Ho}(C)$ and $\mathrm{Ho}(D)$.

The above proposition therefore implies that $\mathrm{Ho}(L)$ is strong symmetric monoidal, which, in turn (because $C \to \mathrm{Ho}(C)$ is conservative) implies that L is strong symmetric monoidal. In other words, this is really a statement about 1-categories, the result for ∞-categories follows formally.

Remark 0.6. It is important to note the asymmetry of the situation. It is in general (and most often in practice) not the case that R is also strong symmetric monoidal.

This asymmetry is not a counterexample to the duality principle of course - it is probably instructive to try to dualize the statement for oneself and observe that it does not imply that R is strong symmetric monoidal.

The asymmetry is also probably more poignant from the ∞-categorical point of view.

Speaking of asymmetry, recall the following general statements: an oplax symmetric monoidal structure on L gives rise to a lax symmetric monoidal structure on R, and vice versa - the two kinds of data are equivalent.

In particular, in our situation, the lax symmetric monoidal structure on R provides an oplax symmetric monoidal structure on L. Our more precise claim is:

Proposition 0.7. *The natural transformation $L(x\otimes y) \to L(x)\otimes L(y)$ induced by the oplax structure on L induced by the lax structure on R is an inverse to the natural transformation $L(x)\otimes L(y) \to L(x\otimes y)$.*

The same holds for the morphism $L(1_C) \to 1_D$, which is an inverse to the morphism $1_D \to L(1_C)$.

In particular L is strong symmetric monoidal.

In fact, we will not need to know that this oplax structure is in fact an oplax structure, we will simply need the definition of the natural transformation in question.

The last thing we want to do before going into the proof is to make explicit the hypothesis that the unit and co-unit are monoidal. We only deal with the unit, the co-unit case being similar (and spelled out again in the proof regardless). Namely, the unit is a natural transformation $\mathrm{id}_C \to RL$, and id_C has an obvious monoidal structure.

The monoidal structure on RL is just given by "composing" the two monoidal structure, i.e. $1_C \to RL(1_C)$ is given by the composite $1_C \to R(1_D) \to RL(1_C)$, and $RL(x)\otimes RL(y) \to RL(x\otimes y)$ is given by the composite $RL(x)\otimes RL(y) \to R(L(x)\otimes L(y)) \to R(L(x\otimes y))$.

The hypothesis is then saying that the composite $x\otimes y \to RL(x)\otimes RL(y) \to RL(x\otimes y)$ is nothing but the unit of $x\otimes y$ (exercise: check that this agrees with the general definition of monoidal transformation !)

Proof. We begin with some notations. η, ϵ are respectively the unit and the co-unit of the adjunction.

We let $\alpha_{x,y} : L(x)\otimes L(y) \to L(x\otimes y)$ denote the natural transformation which is part of the lax structure of L, and β the one for R.

The natural transformation $L(x\otimes y) \to L(x)\otimes L(y)$ that we then care about is defined as the following composite:

$$L(x\otimes y) \overset{L(\eta_x\otimes\eta_y)}{\longrightarrow} L(RL(x)\otimes RL(y)) \overset{L(\beta_{L(x),L(y)})}{\longrightarrow} LR(L(x)\otimes L(y)) \overset{\epsilon_{L(x)\otimes L(y)}}{\longrightarrow} L(x)\otimes L(y)$$

We will show that this is an inverse to α - in fact, as we mentioned, we don't need to know a priori that this defines an oplax structure on L (but it does, whenever β defines a lax structure on R).

The first part of the proof consists of the following diagram:

$$\begin{tikzcd}
L(x) \otimes L(y) \arrow[r, "L(\eta_x)\otimes L(\eta_y)"] \arrow[d, "\alpha_{x,y}"] & LRL(x) \otimes LRL(y) \arrow[d, "\alpha_{LR(x),LR(y)}"] \\
L(x \otimes y) \arrow[r, "L(\eta_x \otimes \eta_y)"] \arrow[ddr] & L(RL(x) \otimes RL(y)) \arrow[d, "L(\beta_{L(x),L(y)})"] \\
& LR(L(x) \otimes L(y)) \arrow[d, "\epsilon_{L(x)\otimes L(y)}"] \\
& L(x) \otimes L(y)
\end{tikzcd}$$

The upper square commutes because α is natural, the lower triangle commutes by definition of our natural transformation $L(x \otimes y) \to L(x) \otimes L(y)$.

The second thing to note is that the co-unit $LR \to \mathrm{id}_D$ is assumed to be symmetric monoidal, in particular the following diagrams commute:

$$\begin{tikzcd}
LR(x) \otimes LR(y) \arrow[r, "\epsilon_x \otimes \epsilon_y"] \arrow[d] & x \otimes y \\
LR(x \otimes y) \arrow[ur, "\epsilon_{x \otimes y}"'] &
\end{tikzcd}$$

where the vertical map is the lax monoidal structure on LR. But this lax monoidal structure on LR is nothing but the composite of the one on L and the one on R.

In other words, the right vertical composite in our earlier diagram,

$$LRL(x) \otimes LRL(y) \to L(RL(x) \otimes RL(y)) \to LR(L(x) \otimes L(y))$$

is nothing but this lax monoidal structure on LR. So the monoidality of the co-unit tells us that the total right-hand vertical composite of our big diagram is exactly $\epsilon_{L(x)} \otimes \epsilon_{L(y)}$.

Therefore, the composite $L(x) \otimes L(y) \to L(x \otimes y) \to L(x) \otimes L(y)$ is equal to $(\epsilon_{L(x)} \otimes \epsilon_{L(y)}) \circ (L(\eta_x) \otimes L(\eta_y)) = (\epsilon_{L(x)} \circ L(\eta_x)) \otimes (\epsilon_{L(y)} \otimes L(\eta_y))$, which is the identity by the triangular identities.

Let us now consider the composite in the other direction. This now uses that the unit is monoidal, so that, for the same reasons as above, we have a commutative diagram as follows:

$$\begin{tikzcd}
x \otimes y \arrow[r, "\eta_x \otimes \eta_y"] \arrow[d, "\eta_{x \otimes y}"] & RL(x) \otimes RL(y) \arrow[d, "\beta_{L(x),L(y)}"] \\
RL(x \otimes y) & R(L(x) \otimes L(y)) \arrow[l, "R(\alpha_{x,y})"]
\end{tikzcd}$$

In particular, it still commutes after applying L, and we get the following bigger diagram, which constitutes the second part of the proof:

$$
\begin{tikzcd}
L(x \otimes y) \arrow[r, "L(\eta_x \otimes \eta_y)"] \arrow[d, "L(\eta_{x \otimes y})"] & L(RL(x) \otimes RL(y)) \arrow[d, "L(\beta_{L(x), L(y)})"] \\
LRL(x \otimes y) \arrow[ddr, "\epsilon_{L(x \otimes y)}"'] & LR(L(x) \otimes L(y)) \arrow[l, "LR(\alpha_{x,y})"] \arrow[d, "\epsilon_{L(x) \otimes L(y)}"] \\
 & L(x) \otimes L(y) \arrow[d, "\alpha_{x,y}"] \\
 & L(x \otimes y)
\end{tikzcd}
$$

Here the outer composite is the map we want to show is the identity, the middle square commutes as we just explained, and the leftmost composite is the identity by the triangular identity. So we are left with showing that the lower triangle commutes.

But this is literally a naturality square for ϵ, so it commutes, and we are done.

The same proof shows that $1_D \to L(1_C) \to 1_D$ is the identity, and so is $L(1_C) \to 1_D \to L(1_C)$. $\square$

Remark 0.8. This proof, although tedious to some extent, is relatively elementary and straightforward: if you draw the correct diagrams, there is not much left to do.

It does require writing out the correct diagrams, and for that it is clear that the diagrammatic approach (as opposed to writing down equations $f \circ g = h$) is of immense help.

Remark 0.9. The corresponding nLab page does not provide such an explicit proof, it refers to a more general statement about doctrines. One hope for this note is that it would provide an explicit and completely elementary proof