Small idempotent algebras

The goal of this short note is to prove the following :

Proposition 0.1. *Suppose $\mathbf{C}$ is at least E_3-monoidal, and $\mathbb{1} \to E$ is an idempotent object, where E is dualizable. In this case, the unit has a splitting $E \to \mathbb{1}$, i.e. E is a retract of the unit.*

In particular, if $\mathbf{C}$ is, say, additive and idempotent complete, we have a decomposition $\mathbb{1} \simeq E \times F$ where F is also idempotent, and $\mathbf{C}$ splits as $\mathbf{Mod}_E(\mathbf{C}) \times \mathbf{Mod}_F(\mathbf{C})$.

Remark 0.2. E_2 is probably sufficient, but I did not want to have to think about what "dualizable" means in this context. It is also very likely that E_1 is enough.

Recall the following definition:

Definition 0.3. Let $\mathbf{C}$ be a monoidal category with unit $\mathbb{1}$. An idempotent object is an object $x \in \mathbf{C}$ together with a morphism $\mathbb{1} \to x$ such that the morphisms $x \otimes (\mathbb{1} \to x)$ and $(\mathbb{1} \to x) \otimes x$ are equivalences.

Remark 0.4. In an E_2-monoidal category, you only need to check one of these morphisms.

Recall the following important principle, of which we sketch a proof for convenience:

Proposition 0.5. *Suppose $\mathbf{C}$ is at least E_2-monoidal, and let $\mathbb{1} \to E$ be an idempotent object. There exists an essentially unique algebra structure on E with this map as the unit.*

If $\mathbf{C}$ is E_n-monoidal, $n \in [2, \infty]$, for all $k \leq n$ there exists a unique E_k-algebra structure on E with this map as the unit.

Sketch of proof. Let $\eta : \mathrm{id} \to E \otimes -$ be the natural transformation given by $(\mathbb{1} \to E) \otimes x$, and call $L = E \otimes -$ the target functor.

The assumption implies that ηL and $L\eta$ are equivalences $L \to L^2$, so that η is the unit of an adjunction $L \dashv i$ between $\mathbf{C}$ and the essential image of L.

Furthermore, the class W of maps that are L-equivalences is closed under tensor product : indeed, if $x \to y$ is a map such that $E \otimes x \to E \otimes y$ is an equivalence, then so is $x \otimes z \to y \otimes z$ for any z. (Note : this is the part that fails if $\mathbf{C}$ is only E_1, even if we adjust L to be $E \otimes - \otimes E$)

It follows that the essential image $L\mathbf{C}$ which is also the localization $\mathbf{C}[W^{-1}]$ admits an essentially unique E_n-monoidal structure for which the adjunction $\mathbf{C} \rightleftarrows L\mathbf{C}$ is E_n-monoidal.

In particular, this induces an adjunction with the same underlying co-unit $\mathrm{Alg}_{E_n}(\mathbf{C}) \rightleftarrows \mathrm{Alg}_{E_n}(L\mathbf{C})$, in particular the right adjoint is fully faithful, so that $L\mathbb{1} = E$ has a unique E_n-structure compatible with the unit map $\mathbb{1} \to E$, i.e. with our given unit. $\square$

Remark 0.6. For $n = 1$, we still get an essentially unique algebra structure, but this sketch does not work. In particular, there is no monoidal localization of this form. One can take as a counterexample the category of accessible functors with composition $\mathrm{Fun}^{acc}(\mathbf{Sp}, \mathbf{Sp})$. Consider for instance the idempotent monad given by inverting a prime p $L = (-)[\frac{1}{p}]$, the functors $F = p$-completion and $G = \Sigma^\infty_+ \Omega^\infty$.

Then $F \to 0$ is an L-local equivalence: $FL = 0$ because p-completion exactly kills spectra on which p is invertible. However, FG is not an L-local equivalence : take any p-complete coconnective spectrum E with a nonzero π_0, then $LFGL(E) = LFG(E) = ((\Sigma^\infty_+ \pi_0(E))^\wedge_p)[\frac{1}{p}]$. This is a direct sum of $\mathbb{Q}_p$'s if for instance $\pi_0(E)$ is finite (more generally it will be a bit more complicated because one p-completes a direct sum; but in any case it has $\mathbb{Q}_p$ as a direct summand and so is nonzero).

Recall that our goal is to prove:

Proposition 0.7. *Suppose* $\mathbf{C}$ *is at least* E_3*-monoidal, and* $\mathbb{1} \to E$ *is an idempotent object, where* E *is dualizable. In this case, the unit has a splitting* $E \to \mathbb{1}$*, i.e.* E *is a retract of the unit.*

In particular, if $\mathbf{C}$ *is, say, additive and idempotent complete, we have a decomposition* $\mathbb{1} \simeq E \times F$ *where* F *is also idempotent, and* $\mathbf{C}$ *splits as* $\mathbf{Mod}_E(\mathbf{C}) \times \mathbf{Mod}_F(\mathbf{C})$*.*

Remark 0.8. In a stable/triangulated setting, this has a neat proof based on Balmer spectra. The point of me writing this is that this statement is much more elementary and doesn't need that machinery.

Proof. This is a statement about the homotopy category, so up to going there we may assume $\mathbf{C}$ is symmetric monoidal.

Let DE be a dual of E. In particular DE is a retract of $DE \otimes E \otimes DE$, and so it is E-local - it follows that $DE = DE \otimes E$.

Furthermore, the evaluation $\mathbb{1} \to E \otimes DE \simeq DE$ is easily checked to witness DE as an idempotent object as well (essentially because $DE \otimes DE = D(E \otimes E) = DE$), so that we dually have $E = E \otimes DE$.

But then the map $E \simeq DE \otimes E \to \mathbb{1} \to E \otimes DE \simeq E$ is homotopic to the identity, as can be checked after tensoring with E (because the domain and the target are E-local), and the map $\mathbb{1} \to E \otimes DE \simeq E$ is identified with the unit of E. $\square$

References