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Rigid commutative algebras and a generalized 1D cobordism hypothesis

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Examples

1. A vector space over k is dualizable if and only if it is finite dimensional;
2. A chain complex is dualizable in the derived category $D(\mathbb{Z})$ if and only if it has finitely many nonzero homology groups, all of which are finitely generated.

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Question

How do we describe free rigid categories on categories ?

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In higher dimensions, work by Grady–Pavlov to generalize it further.

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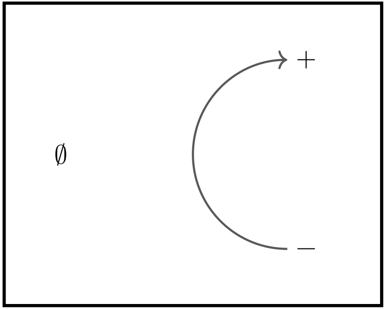
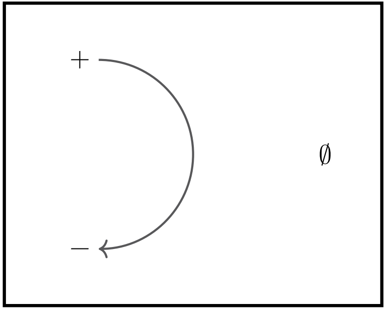


co-evaluation on $X = +$



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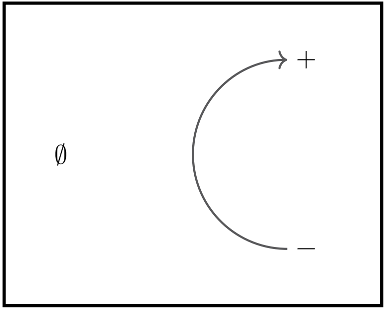

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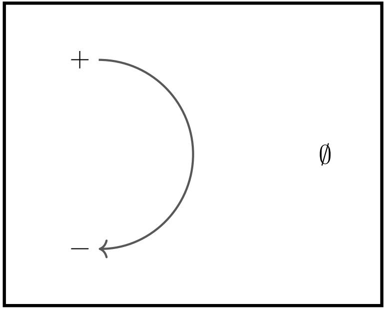

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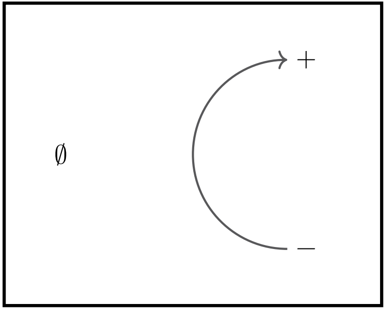

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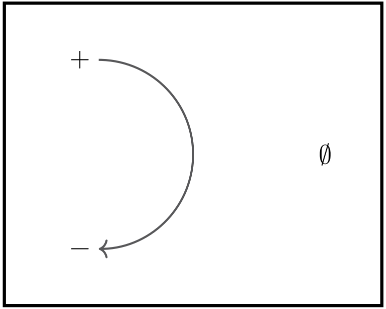

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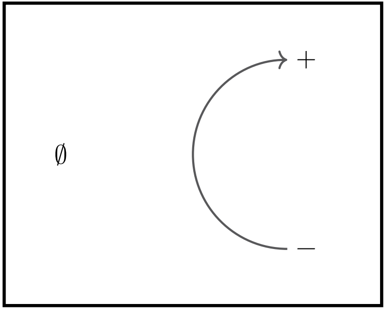

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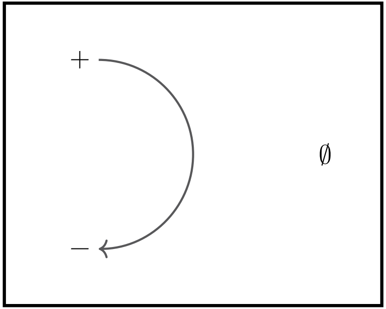

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Generally, in this case, a cobordism is a bunch of arcs like the one above, straight lines $+\to+$ or $-\to-$, and “floating” circles.

Dimensions

The circles correspond to the *dimension* of $+$, in general defined as the composite

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In a general rigid category, easy to see what the simply connected part of the cobordism maps to, the difficulty lies in the circles: $\mathrm{Diff}^+(S^1) \simeq S^1$ gives an S^1-action on dimensions.

Generalizing the Cobordism Hypothesis

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For every collection of objects $c_1, \ldots, c_n, d_1, \ldots, d_m$ in C, get $\bigotimes_i f(c_i) \otimes \bigotimes_j f(d_j)^\vee$

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For every $c \in C$, get $f(c)^\vee \otimes f(c) \to \mathbf{1}_D$ and hence

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Cyclic invariance of traces: $\mathrm{tr}(g \circ f) = \mathrm{tr}(f \circ g)$, i.e. composing in either direction produces the same map

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More generally, get a map

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This piece of data turns out to be the most fundamental one.

The main theorem

Theorem (Barkan–Steinebrunner, R.)

Let C be a category. $\mathsf{Free}^{\mathsf{rig}}(C)$ has objects given by $\bigotimes_i c_i \otimes \bigotimes_j d_j^\vee$, $c_i, d_j \in C$, and:

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Let C be a $\mathcal{V}$-enriched category. $\mathrm{Free}^{\mathrm{rig}}_{\mathcal{V}}(C)$ has objects given by $\bigotimes_i c_i \otimes \bigotimes_j d_j^\vee$, $c_i, d_j \in C$, and:

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Large categories

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Also generalizes to $\mathcal{V}$-enriched situation, with $\mathbf{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ and enriched presheaves.

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Example : interpreting $\mathrm{THH}^{\mathrm{un}}(\mathrm{C})$

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Remark

Cat is cartesian, and in cartesian categories, the only dualizable object is the terminal object. The moral is that Pr^{L} is a bit more “linear algebraic”.

This whole story also works in the $\mathcal{V}$-enriched/linear world.

Dualizable objects in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$

Brief digression: in the case $\mathcal{V} = \mathrm{Sp}$, enriched presheaf categories correspond exactly to compactly generated stable categories

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Invariants like K-theory and THH were usually defined on the latter, but recently Efimov observed that all "K-theory-like" invariants[3] extend uniquely to dualizable Sp-modules, thus bringing a lot of attention to them.

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Let X be a compact Hausdorff topological space. The category $\mathrm{Sh}(X; \mathrm{Sp})$ of sheaves of spectra is dualizable in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

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These "dualizable categories" and examples had already been noticed before by Gaitsgory–Rozenblyum and Lurie, though not pursued as much.

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In their study of dualizable not-necessarily-compactly-generated presentable stable categories, Gaitsgory–Rozenblyum identified the following reformulation of rigidity:

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Remark

The right adjoint of a linear functor is automatically *lax* linear, so the second condition is a property of this lax linear structure, rather than extra structure.

Examples

- If C_0 is a small[4] symmetric monoidal stable category, then $C := \mathrm{Ind}(C_0)$ is large rigid if and only if C_0 is rigid;

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- When $\mathrm{B} = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, recovers the Gaitsgory–Rozenblyum definition

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New guess

The map $\mathrm{Map}_C(c, c) \to \mathrm{Map}(\mathbf{1}_D, f(c) \otimes f(c)^\vee)$ we used in the previous guess can be reinterpreted here as a(n op)lax commutative triangle:

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Here the top map is $f_\# \otimes (f_\#)^\vee$ followed by the self duality $\mathrm{Psh}(D)^\vee \simeq \mathrm{Psh}(D)$ and the multiplication map.

New guess (ct'd)

Similarly, the evaluation map $f(c) \otimes f(c)^\vee \to \mathbf{1}_D$ we combined it with to get $\mathrm{Map}_C(c,c) \to \mathrm{End}(\mathbf{1}_D)$ can be interpreted as a(n op)lax commutative triangle:

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More generally, the unique symmetric monoidal functor $\mathrm{Cob}_1^{\mathrm{fr}} \to \mathrm{Pr}^{\mathrm{L}}, + \mapsto \mathrm{Psh}(C)$ acquires a symmetric monoidal oplax natural transformation to the constant diagram at $\mathrm{Psh}(D)$.

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Theorem

In $\mathbf{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$, the free rigid commutative algebra on a dualizable $\mathcal{V}$-module $\mathcal{M}$ is the oplax colimit over $\mathrm{Cob}_1^{\mathrm{fr}}$ of the symmetric monoidal functor $\mathrm{Cob}_1^{\mathrm{fr}} \to \mathbf{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ classifying $\mathcal{M}$.

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Oplax colimits are nice enough to get the results from the previous theorem out of this one (in fact, to get more general results too).

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In this case, the universal B is

$$\mathsf{Fun}((\mathsf{Cob}_1^{\mathsf{fr}})^{\mathsf{op}}, \mathsf{Pr}^{\mathsf{L}})$$

(with Day convolution)

Proof sketch

By the explicit construction from a few slides ago, prove *a priori* that the free rigid commutative algebra on b in B exists and is a retract[6] of $\mathrm{colim}^{\mathrm{oplax}}_{\mathrm{Cob}_1^{\mathrm{fr}}}(\mathrm{b}^{\otimes \bullet})$

[6] This shows the a priori nontrivial fact that free rigid commutative algebras are preserved under reasonable functors between B's.

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In $\mathrm{Fun}(\mathrm{I}, \mathrm{Pr}^{\mathrm{L}})$, I small rigid, prove that $\mathrm{colim}^{\mathrm{oplax}}_{\mathrm{I}}\mathfrak{y}$ has no nontrivial retracts for $\mathfrak{y}$ the (Pr^{L}-linearized) Yoneda embedding.

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Apply this to $\mathrm{I} = \mathrm{Cob}_1^{\mathrm{fr}}$, where $\mathfrak{y} : \mathrm{Cob}_1^{\mathrm{fr}} \to \mathrm{Fun}((\mathrm{Cob}_1^{\mathrm{fr}})^{\mathrm{op}}, \mathrm{Pr}^{\mathrm{L}})$ classifies the universal dualizable object.

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Equality in the universal case guarantees equality in general.

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Extracting calculations

To get from the oplax colimit description to the mapping space descriptions, interpret "$\mathrm{Map}_{\mathrm{Free}^{\mathrm{rig}}(C)}(c, d)$" as the composite

$$C \xrightarrow{\iota} \mathrm{Free}^{\mathrm{rig}}(C) \xrightarrow{\iota^R} C$$

and similar for higher tensor powers.

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and similar for higher tensor powers.
These can again be computed in the universal case where everything is explicit and "degenerate".