Universität Münster

Chromatic redshift for rigid categories

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Mathematics Münster
Cluster of Excellence

Chromatic homotopy theory

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$$\mathrm{Sp}_{(p)} \to \cdots \to L_n^{(f)}\mathrm{Sp} \to \cdots \to L_2^{(f)}\mathrm{Sp} \to L_1\mathrm{Sp} \to \mathrm{Sp}_{\mathbb{Q}} \simeq D(\mathbb{Q})$$

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Remark

We now know (Burklund–Hahn–Levy–Schlank) that for all $n \geqslant 2$, $\mathrm{Sp}_{T(n)} \neq \mathrm{Sp}_{K(n)}$. For the purposes of this talk, this distinction is irrelevant.

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If R is a commutative ring spectrum and $L_{T(n)}R = 0$, then for all $m \geqslant n$, $L_{T(m)}R = 0$.

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For commutative ring spectra, qualitative form of redshift:

Theorems (Clausen–Mathew–Naumann–Noel, Land–Mathew–Meier–Tamme)

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Chromatic redshift for commutative ring spectra (cont'd)

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Together : K-theory raises the height of commutative ring spectra by exactly 1.

Redshift beyond commutative ring spectra

These results settle the "redshift *philosophy*" for commutative ring spectra, but leave many doors open:

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Arbitrary symmetric monoidal categories are too general.

Rigid categories

Much closer to commutative ring spectra: *rigid* stably symmetric monoidal categories (every object has a dual).

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Common slogan:

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"Whatever works for commutative ring spectra works for rigid categories"

Call them "rigid categories" for this talk.

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Let C be a rigid category. There is a commutative ring map

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Noshift Theorem

Let C be a $T(n)$-local rigid category. There exists a fully faithful symmetric monoidal embedding $C \to D$ such that D is $T(n)$-local rigid, is countably generated as a C-module and

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Take away

There is no *categorical* phenomenon underlying the lower bound for redshift. The proof of Y+BSY *had* to be ring-theoretic.

A stronger result

Main Theorem

Let C be a rigid category. There exists a fully faithful symmetric monoidal embedding $\mathsf{C} \to \mathsf{D}$ such that D is rigid and

$$\mathrm{Dim} : \mathrm{K}(\mathsf{D}) \xrightarrow{\simeq} \mathrm{End}(\mathbf{1})^{BS^1}$$

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Corollary

Let R be an ordinary commutative ring. There exists an idempotent complete rigid stable category D with $K_0(D) \cong R$. It can be chosen to be R-linear.

Dimensions

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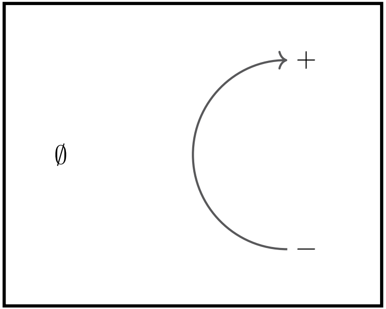
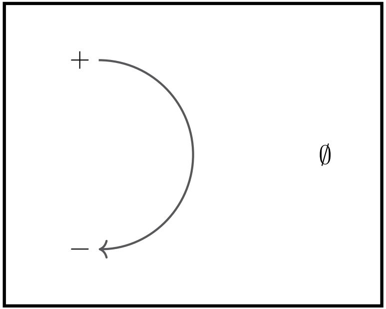
is, in this case, the circle S^1 viewed as an oriented cobordism from the empty manifold to itself



co-evaluation on $X = +$



evaluation pairing for $X = +$

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Theorem (Barkan–Steinebrunner, R.)

Let $\mathcal{C}$ be a rigid category (+...) and M a $\mathcal{C}$-module. The free rigid commutative $\mathcal{C}$-algebra on M, $\mathrm{Free}^{\mathrm{rig}}_{\mathcal{C}}(M)$ has endomorphism ring given by the free commutative algebra in $\mathrm{Ind}(\mathcal{C})$ on

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"Labelled" cobordisms

Relation with Dimensions

In the above context, the following square commutes :

$$\begin{tikzcd} \mathsf{K}(\mathcal{M})[BS^1] \arrow[r] \arrow[d, "\mathsf{tr}"'] & \mathsf{K}(\mathsf{Free}_{\mathcal{C}}^{\mathsf{rig}}(\mathcal{M}))[BS^1] \arrow[d, "\widetilde{\mathsf{Dim}}"] \\ \mathsf{HH}(\mathcal{M}/\mathcal{C})_{hS^1} \arrow[r, "\mathsf{Thm.}"'] & \mathsf{End}(\mathbf{1}_{\mathsf{Free}_{\mathcal{C}}^{\mathsf{rig}}(\mathcal{M})}) \end{tikzcd}$$

Proof ingredient 2

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Corollary (R–Sosnilo–Winges)

Let $\mathcal{C}$ be a rigid category.
Let $x \in \pi_n K(\mathcal{C})$. There exists a countably generated $\mathcal{C}$-module $\mathcal{E}$ with $K(\mathcal{E}) = \Sigma^n K(\mathcal{C})$, a $\mathcal{C}$-module $\mathcal{C}'$ with a K-theory equivalence $\mathcal{C} \to \mathcal{C}'$ and a zigzag $\mathcal{E} \to \mathcal{C}' \leftarrow \mathcal{C}$ of $\mathcal{C}$-modules representing x under K.

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Both K-theory descriptions are actually "motivic" and also hold for $\mathrm{HH}(-/\mathcal{C})$.

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Let C be a rigid category and $x \in \pi_n(\mathrm{End}(\mathbf{1}_C)^{BS^1})$. There exists a rigid D and a fully faithful symmetric monoidal embedding $C \to D$ such that x is in the image of $\pi_n K(D)$ under Dim.

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Proposition

Let C be a rigid category and $x \in \pi_n K(C)$ be in the kernel of $\pi_n \mathrm{Dim}$. There exists a rigid D and a fully faithful symmetric monoidal embedding $C \to D$ such that x is 0 in $\pi_n K(D)$.

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(Really uses RSW and BS/R)

Remark

For $n = 0$, both propositions are relatively easy exercises using only (a precise version of) BS/R.

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Conclude with a small object argument.

Thank you for your attention !