Ramified extensions never lift to the sphere

In [SVW99], the authors prove that "$\mathbb{S}[i]$" doesn't exist, or more precisely, that there is no ring spectrum R such that $R \otimes \mathbb{Z} \simeq \mathbb{Z}[i]$. Their method proves much more, and the goal of this short note is to make that explicit. The precise result we prove is:

Theorem 0.1. *Let R a complete DVR of mixed characteristic with perfect residue field k of characteristic p. If the canonical map $W(k) \to R$ is not an isomorphism, then there is no p-complete $\mathbb{E}_1$-ring $\mathbb{S}_R$ such that $(\mathbb{S}_R \otimes \mathbb{Z})_p \simeq R$.*

The content of the note was worked out in a "Homotopy hang out session" in Münster, with Konrad Bals, Nikolaus Betker, Felix Nass, Thomas Nikolaus, Phil Pützstück and Rin Ray.

The key input for this theorem is the calculation of $\mathrm{THH}(R)_p$ for such rings R by Lindenstrauss–Madsen [LM00], revisited by Krause–Nikolaus [KN22, Theorem 4.4].

Theorem 0.2. *Let R be as above. Fix a uniformizer π and let E be the minimal polynomial of π over $W(k)$, where k is the residue field of R. There is a noncanonical isomorphism*

$$\mathrm{THH}_{2n-1}(R;\mathbb{Z}_p) \cong R/nE'(\pi),$$

the even positive homotopy groups vanish, and $\mathrm{THH}_0(R;\mathbb{Z}_p) \cong R$.

Proof of Theorem 0.1. For the duration of this proof, all spectra and tensor products appearing are implicitly p-completed. Suppose such a ring $\mathbb{S}_R$ exists. We then have the following THH calculation:

$$\mathrm{THH}(R) \simeq \mathrm{THH}(\mathbb{S}_R) \otimes \mathrm{THH}(\mathbb{Z}_p)$$

Now, $\mathrm{THH}(\mathbb{Z}_p)$ is a $\mathbb{Z}_p$-module, so we may rewrite the right hand term as $(\mathrm{THH}(\mathbb{S}_R) \otimes \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{THH}(\mathbb{Z}_p)$.

By standard base-change properties of relative THH, we have $\mathrm{THH}(\mathbb{S}_R)\otimes\mathbb{Z}_p \simeq \mathrm{THH}(\mathbb{S}_R\otimes\mathbb{Z}_p/\mathbb{Z}_p) \simeq \mathrm{THH}(R/\mathbb{Z}_p)$. Note that this is now a $\mathbb{Z}_p$-module, and is in particular formal, so its π_0 is a summand thereof. Since R is commutative, this π_0 is simply R.

In particular, if $\mathbb{S}_R$ exists, then $\mathrm{THH}(R)$ admits $R \otimes_{\mathbb{Z}} \mathrm{THH}(\mathbb{Z}_p)$ as a summand, so π_{2p-1} as well. Since R is flat over $\mathbb{Z}_p$, we find that $R/pE'(\pi)$ admits $R \otimes_{\mathbb{Z}} \mathbb{Z}/p \cong R/p$ as a retract.

Now, $R/pE'(\pi)$ and R/p have the same (finite) rank, and they are both p-power torsion. Thus, one can only be the retract of the other if they are isomorphic, so in particular $R/pE'(\pi)$ is p-torsion. Considering the residue class of 1, we find that $E'(\pi)$ has to be a unit in R, which implies that R is unramified, or equivalently, the map $W(k) \to R$ is an isomorphism. □

We used:

Lemma 0.3. *Let A, B be two finite, abelian p-groups of the same (p-)rank. Suppose A is a retract of B. Then $A \cong B$.*

Proof. Let $A \oplus C \cong B$ be a direct sum decomposition. Rank is additive, so C must have rank 0, and therefore it must be 0. □

References

[KN22] Achim Krause and Thomas Nikolaus. Bökstedt periodicity and quotients of DVRs. *Compositio Mathematica*, 158(8):1683–1712, 2022.

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[SVW99] R. Schwänzl, R.M. Vogt, and F. Waldhausen. Adjoining roots of unity to E_∞ ring spectra in good cases - a remark. *Contemporary Mathematics*, 1999.