Ordinals in homotopy theory

The goal of this short note is to justify a method of constructing functors out of ordinals in ∞-categories. Specifically, we prove:

Proposition 0.1. *Let C be an ∞-category admitting κ-small ordinal indexed colimits, and let α be an ordinal such that for all $\beta < \alpha, \mathrm{cof}(\beta) < \kappa$*[1]*.*

Let T be an endofunctor of C equipped with a transformation $\eta : \mathrm{id}_C \to T$. Then there exists a functor $F : \alpha \to \mathrm{Fun}(C, C)$ with

- $F(0) = \mathrm{id}_C$*,*
- *For each $\beta + 1 < \alpha$, there is an identification*
$$(F(\beta) \to F(\beta + 1)) = (\eta_{F(\beta)} : F(\beta) \to T \circ F(\beta));$$
- *For each limit $\beta < \alpha$, $\mathrm{colim}_{\gamma<\beta} F(\gamma) \simeq F(\beta)$.*

Example 0.2. This is used implicitly in the proof of [Lur09, Proposition 6.2.2.7] with seemingly no justification.

Warning 0.3. We make no claim about unicity. It seems quite likely to the author that there is *no* unicity.

We make a remark before proving the claim: the claim looks essentially trivial (the three bullet points seem to give a well-defined functor in and of themselves!), but there is a subtlety: to define $F(\beta)$, for β a limit ordinal, one needs not only to have defined the $F(\gamma)$'s and maps between them for $\gamma < \beta$, but instead, one needs to define a whole β-indexed diagram $(\gamma \mapsto F(\gamma))$. Since β is *complicated* as an ∞-category, i.e. functors out of it truly involve higher coherence data, this is not an easy task, and seems to require precisely the kind of proposition which we are proving here.

Let us make precise the sense in which β can be complicated as an ∞-category: for each $n \in \mathbb{N}$, the ordinal ω_n has cohomological dimension exactly $n+1$ - e.g. ω, the first countable ordinal, has $\lim^1$'s but no $\lim^2$'s. Similarly, ω_n has (up to) $\lim^{n+1}$'s but no $\lim^{n+2}$'s. In particular, $\lim^{n+1}$'s *are* achievable, showing indeed some higher homotopical data. Above $\omega_{\omega+1}$, there is *no* bound[2] on homotopical dimension, so there is no *hope* of replacing a random β with even a finite dimensional object.

Example 0.4. There exists an ω_1-indexed diagram of anima X_α, all of which are connected, such that $\lim_{\alpha\in\omega_1} X_\alpha$ is empty.

[1] For example if C admits all filtered colimits.

[2] Of course ω_ω has cofinality ω and so while diagrams out of it are complicated, limits of these diagrams aren't so much.

Question 0.5. Can the fact that ω_n has homotopy-dimension $n+1$ be explained categorically ? e.g. is there some presentation of ω_n "up to cofinality" in terms of some $\leq n+1$-dimensional ∞-categories ? Is ω_n "absolutely" of homotopy dimension $n+1$, i.e. is this true for functors with values in an arbitrary ∞-topos ?

Remark 0.6. The proof that ω_n has no $\lim^{n+2}$ is originally due to Osofsky [Oso74] using homological algebraic methods, and is given a modern interpretation by Lurie [Lur18, Lemma D.3.3.6]. Both proofs directly use *models*, and indeed we will also use models in our proof of Proposition 0.1. It seems difficult to imagine a proof concerning a property of ordinals not phrased in terms of uniqueness that could go throuh model-independently and indeed it seems like any method to produce a functor out of ω_1, say, runs into circularities until one produces an honest-to-god functor-up-to-equality, precisely because ω_1 is a regular cardinal and so cannot be "accessed from below" in an order-theoretic way. I truly hope to be proven wrong.

Once one has tried enough times and failed to prove Proposition 0.1 model-independently, and accepted the need to use models, many approaches and strategies come to mind. They do not all converge, but the proof is after the fact not so difficult (thus maybe undermining the point that I am trying to make, namely that the question is subtle[3]!).

Before giving a proof, let us point out the following easy facts:

Lemma 0.7. *Let β be an ordinal.*

- *If β is limit, then $\beta \simeq \mathrm{colim}_{\gamma<\beta}\gamma$;*
- *If $\beta = \beta_0 + 1$, $\beta = \beta_0 * \Delta^0$.*

However, the first point is the source of the apparent circularity in model-independent attempts: as explained earlier, any attempt at creating functors out of ordinals in ∞-land kind of runs into circularities since to use things like "$\alpha \simeq \mathrm{colim}_{\beta<\alpha}\beta$" to build functors out of α, we already need to understand how to build points in α-indexed limits, which are essentially special kinds of α-indexed functors. Here is what may, however, lead us somewhere: since "$\alpha \simeq \mathrm{colim}_{\beta<\alpha}\beta$" is true both in simplicial sets and in ∞-categories, we have the following "surprise": given a quasicategory C, $\mathrm{Fun}(\alpha, C)$ is both a limit of the $\mathrm{Fun}(\beta, C)$'s in simplicial sets, and a (homotopy) limit of them in ∞-categories.

Thus to create a point in $\mathrm{Fun}(\alpha, C)$, if we are willing to use models, in principle we only have to construct points in the $\mathrm{Fun}(\beta, C)$'s that strictly restrict to one another : it is now a *property* to check that $(F_\beta)_{|\gamma} = F_\gamma$ if one really writes $=$ and not "is equivalent to".

This is a sign that for a given quasicategory (i.e. a simplicial set with a certain horn-filling condition), the diagram $\alpha \mapsto \mathrm{Fun}(\alpha, C)$ is a particularly nice diagram (specifically, a Reedy fibrant one). We will exploit this in the proof of Proposition 0.1.

1 Proof

The key statement we actually prove is the following:

[3] I do not mind undermining my own points.

Theorem 1.1. *Let α be an arbitrary ordinal and let $X : \alpha^{\mathrm{op}} \to \mathbf{S}$ be a space-valued diagram. Suppose each "transition map"*

$$X_\beta \to \lim_{\gamma<\beta} X_\gamma$$

is π_0-surjective.

Then $\lim_\alpha X$ is non-empty.

Proof. Using unstraightening, view X as a right fibration $\mathcal{X} \to \alpha$ of simplicial sets. From this point of view, $\lim_\alpha X$ is represented by $\mathrm{Fun}_{/\alpha}(\alpha, \mathcal{X})$ and $\lim_{\gamma<\beta} \mathcal{X}_\gamma$ by $\mathrm{Fun}_{/\alpha}(\beta, \mathcal{X})$.

Now note that for $\gamma < \beta$, the inclusion $N(\gamma) \to N(\beta)$ is *left* anodyne[4] and so $\mathrm{Fun}_{/\alpha}(\beta, \mathcal{X}) \to \mathrm{Fun}_{/\alpha}(\gamma, \mathcal{X})$ is a left fibration.

Since by assumption, $\mathrm{Fun}_{/\alpha}(\beta+1, \mathcal{X}) \to \mathrm{Fun}_{/\alpha}(\beta, \mathcal{X})$ is surjective on π_0 and we just saw it is also a left fibration, it follows that it is surjective on $(-)_0$, the *set* of 0-simplices. Hence by looking at 0-simplices, we are reduced to the same statement but for set-valued diagrams, where it is a classical transfinite induction argument. □

We can use this to prove Proposition 0.1

Proof. Consider the two maps $\mathrm{Map}(\beta, \mathrm{Fun}(C, C)) \to \prod_{\gamma, \gamma+1<\beta} \mathrm{Fun}(C, C)^{\Delta^1}$ given on the one hand by evaluation at $\gamma \leq \gamma + 1$, and on the other hand by evaluation at γ followed by applying $\eta : \mathrm{id}_C \to T$.

Let X_β denote the full subspace of the equalizer of these two maps spanned by those points whose underlying map $\beta \to \mathrm{Fun}(C, C)$ also sends limit ordinals to colimits.

This is closed under precomposition and so it defines a contravariant functor on $\alpha + 1$ with values in $\mathbf{S}$. It satisfies the assumption from Theorem 1.1: for β limit, simply because the relevant map is actually an equivalence.

For $\beta = \beta_0 + 1$, it depends on whether β_0 is limit or successor: if β_0 is successor, this follows just by applying T and using Lemma 0.7 and if β_0 is limit, this follows by the existence of the relevant colimits in C.

Therefore, by Theorem 1.1, X_α is nonempty, which is exactly what we set out to prove. □

2 A proof of Osofsky's theorem

We follow Lurie except that we no longer need to invoke point-set models since this was taken care of "once and for all" in Theorem 1.1:

Lemma 2.1. *Suppose $\alpha = \omega_n$ and $X : \omega_n^{\mathrm{op}} \to \mathbf{S}$ is a diagram of m-connective spaces. In this case, $\lim_{\omega_n^{\mathrm{op}}} X$ is $(m-n)$-connective.*

Proof. We prove the result by induction on n.

Fixing n and assuming the result known for all smaller n's, we now induct on $m - n$: by considering loop spaces, we may reduce to the case where $m = n$ and in that case we simply wish to prove that the limit is non-empty. We apply Theorem 1.1: for $\beta < \omega_n$, note that by induction hypothesis, $\lim_{\gamma<\beta} X_\gamma$ is at least $(n - (n-1)) = 1$-connective, that is,

[4] This uses that left anodyne morphisms are closed under transfinite composition, which is an instance of the kind of thing we are doing here but in 1-category land.

connected. Hence $X_\beta \to \lim_{\gamma<\beta} X_\gamma$ is surjective (since the source is n-connective and hence nonempty). Thus the corollary applies and we are done. $\square$

Remark 2.2. This turns out to be optimal, but I will not prove this here.

References

[Lur09] Jacob Lurie. *Higher topos theory.* Princeton University Press, 2009.

[Lur18] Jacob Lurie. Spectral algebraic geometry. *preprint*, 2018.

[Oso74] Barbara L Osofsky. The subscript of $\aleph_n$, projective dimension, and the vanishing of $\varprojlim^{(n)}$. *Bulletin of the American Mathematical Society*, 80(1):8–26, 1974.