Inverting elements in commutative algebras

Let $\mathcal{V}$ denote a presentably symmetric monoidal category with unit $\mathbf{1}$, $A \in \mathrm{CAlg}(\mathcal{V})$ a commutative algebra and $a : \mathbf{1} \to A$. One may want to make a invertible, and find the universal recipient of a morphism from A where a is invertible.

By formal nonsense, because $\mathcal{V}$ is presentable, we can always find such a universal commutative algebra, and such a universal associative algebra. The question to compare them then makes sense : is the latter the underying algebra of the former ?

We claim that they always agree under the given hypotheses.

Consider the category of A-bimoduels in $\mathcal{V}$, $\mathbf{BiMod}_A$, as well as its full subcategory of bimodules on which a acts invertibly on both sides, $\mathbf{BiMod}_A^a$.

$\mathbf{BiMod}_A$ is a (non-symmetric) monoidal category, with tensor product given by the relative tensor product of bimodules.

Because A is commutative, the category of left A-modules, $\mathbf{Mod}_A$, also has a relative tensor product. We also let $\mathbf{Mod}_A^a$ denote the full subcategory of modules on which a acts invertibly. We note the following fact, which is formal by the adjoint functor theorem :

Lemma 0.1. *Both inclusions $\mathbf{BiMod}_A^a \subset \mathbf{BiMod}_A, \mathbf{Mod}_A^a \subset \mathbf{Mod}_A$ have left adjoints.*

We want to prove that these adjunctions are smashing, with idempotent object given by some (commutative) A-algebra, that the underlying A-algebra of the one in $\mathbf{Mod}_A$ is equivalent to the one in $\mathbf{BiMod}_A$, and that these idempotent objects (which match up) are the universal (commutative) algebra under A where a is inverted.

Let us first prove the smashing behaviour.

As both $\mathbf{BiMod}_A, \mathbf{Mod}_A$ are monoidal, we can consider the following statement, which covers both cases (even though the second one is symmetric monoidal, and so easier) :

Theorem 0.2. *Let $\mathcal{W}$ be a presentably monoidal category, and $a : \mathbf{1} \to \mathbf{1}$ an endomorphism of the unit.*

Consider the full subcategory $\mathcal{W}^a$ on those objects where a acts invertibly on both sides, and $L : \mathcal{W} \to \mathcal{W}^a$ the left adjoint to the inclusion.

Then the canonical map $\mathbf{1} \to L(\mathbf{1})$ identifies $L(\mathbf{1})$ as an idempotent algebra, and $L \simeq L(\mathbf{1}) \otimes - \otimes L(\mathbf{1})$, and the forgetful functor $\mathcal{W}^a \to \mathcal{W}$ is identified with the forgetful functor from $L(\mathbf{1})$-bimodules.

Proof. We abuse notation by omitting the name of the inclusion $\mathcal{W}^a \subset \mathcal{W}$ from our notations.

Observe that $\mathcal{W}^a$ is stable under tensor product, so that the natural map $x \otimes y \to Lx \otimes Ly$ induces a natural map $L(x \otimes y) \to Lx \otimes Ly$.

We want to prove that it is an equivalence if either x or y is L-local. Of course, by symmetry, we may assume y is local.

For now, let us have no assumptions on y. Observe that $-\otimes y$ has a right adjoint, which we call $\hom^r(y,-)$. Then, naturally in x we have $\mathrm{map}(Lx\otimes Ly, z)\simeq \mathrm{map}(Lx, \hom^r(Ly,z))$.

Suppose z is L-local. We want to prove that $\hom^r(Ly,z)$ is then also L-local. For this, observe that a acts as follows on the right: $\hom^r(Ly,z)\simeq \hom^r(Ly,z)\otimes \mathbf{1} \xrightarrow{\mathrm{id}\otimes a} \hom^r(Ly,z)\otimes\mathbf{1}\simeq \hom^r(Ly,z)$. Call this map $\tilde{a}$

The corresponding $\hom^r(Ly,z)\otimes Ly\to z$ is then given by $\hom^r(Ly,z)\otimes Ly\xrightarrow{\tilde{a}} \hom^r(Ly,z)\otimes Ly\xrightarrow{ev} z$ where ev is the co-unit of the $-\otimes Ly \dashv \hom^r(Ly,-)$ adjunction.

In particular, by associativity of the tensor product, this adjoint of $\tilde{a}$ is given by the *left* action of a on Ly. In other words, letting a^r, a^l denote the natural transformations corresponding to the right or left action of a, we have $a^r_{\hom^r(y,z)} = \hom^r(a^l, z)$ for an arbitrary y. In particular, for Ly which is in $\mathcal{W}^a$, it is an equivalence.

A similar argument shows that $a^l_{\hom^r(y,z)} = \hom^r(y,a^l)$, and in particular is an equivalence for $z\in\mathcal{W}^a$.

This shows that $\hom^r(y,z)$ is in $\mathcal{W}^a$ as soon as y and z are.

In particular, we can continue our chain of equivalences from above: $\mathrm{map}(Lx\otimes Ly,z)\simeq \mathrm{map}(Lx,\hom^r(Ly,z))\simeq \mathrm{map}(x,\hom^r(Ly,z))\simeq \mathrm{map}(x\otimes Ly, z)$. We can't really go further, as x is not local and so it is not clear how to prove that $\hom^l(x,z)$ is L-local (with a hopefully clear notation).

But in any case, this proves that the canonical map $L(x\otimes Ly)\to Lx\otimes Ly$ is an equivalence, i.e. that if y is L-local, the canonical map $L(x\otimes y)\to Lx\otimes Ly$ is an equivalence.

The symmetric statement follows. It follows that $L\mathbf{1}$ is an idempotent: indeed the canonical map $L\mathbf{1}\simeq L(\mathbf{1}\otimes L\mathbf{1})\to L\mathbf{1}\otimes L\mathbf{1}$ is identified with $\eta\otimes\mathrm{id}_{L\mathbf{1}}$ (where $\eta:\mathbf{1}\to L\mathbf{1}$ is the unit of the adjunction), and symmetry shows that $\mathrm{id}_{L\mathbf{1}}\otimes\eta$ is also an equivalence. This shows that $L\mathbf{1}$ is an idempotent by definition [Lur12, 4.8.2.1].

In particular, we observe that $\mathcal{W}^a$ defines a sub-operad of $\mathcal{W}$, and because it is stable under non-nullary tensor products and has a unit ($L\mathbf{1}$ by the previous equivalence), it is a monoidal category, and the inclusion into $\mathcal{W}$ is lax monoidal, while L is strong monoidal.

It follows that $L\mathbf{1}\otimes -$ is a localization functor, but in fact $\tilde{L} = L\mathbf{1}\otimes - \otimes L\mathbf{1}$ too. Furthermore, observe that its essential image is exactly $\mathcal{W}^a$. Indeed, $L\mathbf{1}\otimes x\otimes L\mathbf{1}$ is clearly in $\mathcal{W}^a$.

Conversely, if x is in $\mathcal{W}^a$, then it is an $L\mathbf{1}$-bimodule, and so is a retract of $L\mathbf{1}\otimes x\otimes L\mathbf{1}$, and so must be $\tilde{L}$-local as a retract of a $\tilde{L}$-local object. This proves the claim about essential image.

The claim about bimodules follows from the fact that $\tilde{L}:\mathcal{W}\to\mathcal{W}^a$ lifts through $L\mathbf{1}$-bimodules and has the same underlying (idempotent) monad. ☐

In particular, we obtain a commutative A-algebra $A[1/a]^{comm}\in \mathrm{CAlg}(\mathbf{Mod}_A^a)$, as well as an associative algebra with a map from A, $A[1/a]^{ass}\in\mathrm{Alg}(\mathbf{BiMod}_A^a)$ (recall that $\mathrm{Alg}(\mathbf{BiMod}_A)\simeq \mathrm{Alg}_{A/}$) as the respective LA in $\mathcal{W} = \mathbf{Mod}_A, \mathbf{BiMod}_A$ respectively.

In particular, they satisfy the expected universal properties:

Corollary 0.3. *$A[1/a]^{comm}$ is the universal commutative algebra with a commutative algebra map from A where a is invertible.*

$A[1/a]^{ass}$ is the universal algebra with an algebra map from A where a is invertible.

Proof. Let $\mathrm{CAlg}^a_{A/}$ denote the full subcategory of $\mathrm{CAlg}_{A/}$ on those maps $A\to B$ where a is mapped to an invertible element of B.

Then, along the equivalence $\mathrm{CAlg}_{A/} \simeq \mathrm{CAlg}(\mathbf{Mod}_A)$, it is easy to check that $\mathrm{CAlg}^a_{A/}$ corresponds to $\mathrm{CAlg}(\mathbf{Mod}^a_A)$, where indeed $A[1/a]^{comm}$ is initial (it is the unit of $\mathbf{Mod}^a_A$).

The same argument applies for the associative case, as $\mathrm{Alg}_{A/} \simeq \mathrm{Alg}(\mathbf{BiMod}_A)$, and an invertible element acts invertibly on both sides (and conversely). □

Corollary 0.4. *The underlying associative algebra map of $A \to A[1/a]^{comm}$ is $A \to A[1/a]^{ass}$.*

Proof. We wish to show that the following diagram commutes:

$$\begin{tikzcd} \mathbf{Mod}_A \arrow[r, "\{1/a\}"] \arrow[d, "\mu^*"] & \mathbf{Mod}^a_A \arrow[d, "\mu^*"] \\ \mathbf{BiMod}_A \arrow[r, "{[1/a]}"] & \mathbf{BiMod}^a_A \end{tikzcd}$$

where we let $X \mapsto X\{1/a\}$ denote the functor L from before for $\mathbf{Mod}_A$, and $X \mapsto X[1/a]$ the one for $\mathbf{BiMod}_A$, and where μ is the multiplication $A \otimes A \to A$, viewed as an associative algebra map $A \otimes A^{\mathrm{op}} \to A$.

Indeed, if it does, because A as an A-bimodule is $\mu^* A$, it will show that $\mu^* A\{1/a\} \simeq A[1/a]$ as modules, but the latter has an essentially unique algebra structure (also, μ^* is symmetric monoidal), so that the claim will follow at once. For this, we consider the corresponding square of right adjoints, where $\mu^* : \mathbf{Mod}_A \to \mathbf{BiMod}_A$ has a right adjoint given by $\mu_* : X \mapsto \mathrm{Map}_{A \otimes A^{\mathrm{op}}}(A, X)$. To prove that this square commutes, it suffices to show that if $M \in \mathbf{BiMod}^a_A$, then $\mu_* M \in \mathbf{Mod}^a_A$.

But μ_* is lax monoidal, as the right adjoint of a strong monoidal functor, so that the action of a on $\mu_* M$ can be factored as $A \otimes_A \mu_*(M) \to \mu_*(A \otimes_A M) \to \mu_*(A \otimes_A M) \to \mu_*(M)$, where all the maps are equivalences : for the first and last one because of general monoidal category facts, and the middle one because M is a-local in $\mathbf{BiMod}_A$.

This proves that the square of right adjoints commutes, hence the square of left adjoints commute.

Hence $\mu^*(A[1/a]^{comm})$ is an algebra in $\mathbf{BiMod}^a_A$ with the same underlying A-bimodule as $A[1/a]^{ass}$. By uniqueness, we have a unique equivalence of algebras in bimodules $A[1/a]^{ass} \to \mu^*(A[1/a]^{comm})$.

We now conclude by observing that the following diagram commutes:

$$\begin{tikzcd} \mathrm{CAlg}(\mathbf{Mod}_A) \arrow[r, "\simeq"] \arrow[d] & \mathrm{CAlg}_{A/} \arrow[d] \\ \mathrm{Alg}(\mathbf{Mod}_A) \arrow[r] \arrow[d, "\mu^*"] & \mathrm{Alg}_{A/} \arrow[d, "\simeq"] \\ \mathrm{Alg}(\mathbf{BiMod}_A) \arrow[r, "\simeq"] & \mathrm{Alg}_{A/} \end{tikzcd}$$

(note that the middle vertical map is *not* an equivalence) □

Remark 0.5. Despite the notation, even for a module M, it is not in general the case that $M[1/a]$ is the telescope of a, i.e. $\mathrm{colim}_n(M \xrightarrow{a} M \xrightarrow{a} M \to \dots)$. The following example was pointed out to me by Denis Nardin: for $\mathcal{V} = \mathbf{Cat}_\infty^{ex}$, there is no nontrivial stable category on which 2 acts invertibly (observe that $2 : \mathbf{Sp}^\omega \to \mathbf{Sp}^\omega$ is $S^0 \mapsto S^0 \oplus S^0$ so on a general C it's $X \mapsto X \oplus X$, so if it were an equivalence, then we would have an equivalence of mapping spectra $\mathrm{map}(X,X) \to \mathrm{map}(X \oplus X, X \oplus X)$ *given by* $f \mapsto f \oplus f$. Taking e.g. $\mathrm{pr}_1 : X \oplus X \to X \to X \oplus X$ shows that this implies $X = 0$), but the mapping telescope is in general nonzero (and in particular, 2 doesn't act invertibly on it) - one way to observe this is that K-theory preserves filtered colimits and is additive, so e.g. $K_0(T_2\mathbf{Perf}(\mathbb{Z})) = T_2 K_0(\mathbb{Z}) = \mathbb{Z}[1/2] \neq 0$, where T_2 denotes the telescope for 2.

References

[Lur12] Jacob Lurie. Higher algebra, 2012.