# The nil- and $\mathbb{A}^1$-invariance of rational periodic cyclic homology

**Abstract**

We give a modern account of two fundamental properties of periodic cyclic homology over a rational base, namely its truncating character in the sense of [LT19] and its $\mathbb{A}^1$-invariance.

# Introduction

Periodic cyclic homology is a classical variant of Hochschild homology which has numerous nice features, such as (as the name suggests) periodicity. Rationally, it is closely related to (rational) algebraic $K$-theory via Goodwillie's theorem about nilpotent extensions.

My goal in this note is to give a modern account of two fundamental properties of (rational) periodic cyclic homology, namely its nil-invariance and its $\mathbb{A}^1$-invariance. For $\mathbb{A}^1$-invariance, I mainly resort to [LT23, Theorem D]. In short, the main thing I prove is the following:

**Theorem 0.1.** *Let $A^\eta \to A$ be a square zero extension of a rational ring spectrum $A$ by an $A$-bimodule $M$, and let* HP *denote periodic cyclic homology.*

- *If $A^\eta \to A$ is a* trivial *square zero extension, then it induces an equivalence $\mathrm{HP}(A^\eta) \to \mathrm{HP}(A)$;*
- *If $A^\eta, M$, and therefore $A$ are connective, then $\mathrm{HP}(A^\eta) \to \mathrm{HP}(A)$ is an equivalence.*

Building on the second bullet point with further connectivity estimates, we also obtain:

**Corollary 0.2.** *Let $A$ be a rational connective ring spectrum. The map $A \to \pi_0(A)$ induces an equivalence $\mathrm{HP}(A) \to \mathrm{HP}(\pi_0(A))$.*

The key to proving all of this is connectivity estimates, together with the first bullet point. Essentially, we deduce the second bullet point from the first by suitably filtering the algebra $A^\eta$ and reducing to a trivial square zero extension by taking associated gradeds. The filtered object in question is complete, subject to connectivity assumptions, so that knowledge of the associated graded is sufficient.

In the trivial square zero extension case, we deduce the result from expressing the $S^1$-Tate construction as the inversion of an Euler class $e \in H^2(BS^1;\mathbb{Q})$, and observing that this Euler class vanishes in $H^2(BC_n;\mathbb{Q}) = 0$.

## Outline

In the first section, I remind the reader of the formula for the Hochschild homology of a trivial square zero extension, which involves cyclic groups and will be the crucial point where rationality is used - this is where I prove the first bullet point, as well as its consequence for $\mathbb{A}^1$-invariance.

In the second section, I do some recollections on filtered objects, mostly to set up the basics and conventions I'll use later on.

In the third section, I finally treat the case of nontrivial square zero extensions in the connective case by using filtered methods and reducing to the trivial case.

In the appendix, I talk a bit about complex orientations and explain how to connect Tate constructions and Euler classes.

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# 1 Hochschild homology of trivial square zero extensions

The goal of this section is to describe the Hochschild homology of trivial square zero extensions, and prove already in the rational case that their periodic cyclic homology trivializes. This special case of the main result will be the key step. Throughout, we will work in more generality.

**Assumption 1.1.** We assume throughout that $\mathcal{C} \in \mathrm{CAlg}(\mathbf{Pr}^{\mathrm{L}}_*)$, that is, $\mathcal{C}$ is a pointed, presentably symmetric monoidal category.

The precise result is now as follows - we take it for granted but (a refined version of it) can be found in forthcoming work of Keenan:

**Proposition 1.2.** *Let $(A, M) \in \mathbf{LMod}(\mathcal{C})$ be a pair of an algebra in $\mathcal{C}$ and a module thereon.*

*There is a canonical equivalence of objects with $S^1$-action of the form*

$$\mathrm{HH}_{\mathcal{C}}(A \coprod M) \simeq \coprod_n \mathrm{Ind}_{C_n}^{S^1} \Sigma^n \mathrm{HH}_{\mathcal{C}}(A; M^{\otimes_A n})$$

*where $A \coprod M$ is the trivial square zero extension of $A$ by $M$, which is well defined as $\mathcal{C}$ is pointed.*

**Remark 1.3.** This also has a more general version for trivial square zero extensions of categories, cf. [Ras18], but the proof is more involved.

We can deduce from this a first nil-invariance result:

**Corollary 1.4.** *Assume now that $\mathcal{C}$ is stable and rational, and let $(A, M) \in \mathbf{LMod}(\mathcal{C})$. The canonical map $\mathrm{HH}_{\mathcal{C}}(A \oplus M) \to \mathrm{HH}_{\mathcal{C}}(A)$ induces an equivalence*

$$\mathrm{HH}_{\mathcal{C}}(A \oplus M)^{tS^1} \to \mathrm{HH}_{\mathcal{C}}(A)^{tS^1}$$

*i.e. an equivalence $\mathrm{HP}_{\mathcal{C}}(A \oplus M) \to \mathrm{HP}_{\mathcal{C}}(A)$.*

*Proof.* Let $X \in \mathcal{C}^{BC_n}$ for some $n \geq 1$. We claim that the map $e : \mathrm{Ind}_{C_n}^{S^1} X \to \Sigma^2 \mathrm{Ind}_{C_n}^{S^1} X$ given by the degree $-2$ element in $\mathbb{Q}^{hS^1}$ is 0 in $\mathcal{C}^{BS^1}$.

This is because $\pi_{-2}(\mathbb{Q}^{BC_n}) = 0$, so that the $C_n$-equivariant map $X \to \Sigma^2 X$ given by $e_{|BC_n}$ is 0, and thus so is $X \to \Sigma^2 \mathrm{Ind}_{C_n}^{S^1} X$ and thus so is the corresponding map in $\mathcal{C}^{BS^1}$.

It follows that this map is also 0 as a self map of $\widetilde{\mathrm{HH}}_{\mathcal{C}}(A \oplus M) := \mathrm{fib}(\mathrm{HH}_{\mathcal{C}}(A \oplus M) \to \mathrm{HH}_{\mathcal{C}}(A))$ in $\mathcal{C}^{BS^1}$.

Now, by Proposition A.7[1], if $X \in \mathcal{C}^{BS^1}$ is such that $e : X \to \Sigma^2 X$ is $S^1$-equivariantly null, then $X^{tS^1} = 0$. The result follows.

(Essentially we are re-explaining the discussion at the end of Appendix A in the case $X = BC_n$.) ☐

As we do not need any connectivity results for this statement, we can use [LT23, Theorem D] to prove that HP is $\mathbb{A}^1$-invariant over rational rings:

**Corollary 1.5.** *Let $k$ be a commutative $\mathbb{Q}$-algebra.* HP *relative to $k$ is $\mathbb{A}^1$-invariant.*

*Proof.* It suffices to use invariance under trivial square zero extensions, which we just established, together with [LT23, Theorem D] ☐

**Remark 1.6.** Note that to obtain classical $\mathbb{A}^1$-invariance, it is crucial to allow nonconnective bimodules: in the notation of [LT23, Theorem D], we want $M = C$ and so we need an extension by $\Omega M = \Omega C$.

# 2 Filtered and graded objects

Let $\mathbb{Z}$ denote the usual ordered abelian group, and $\mathbb{Z}^\delta$ the same abelian group viewed as a discrete set.

**Definition 2.1.** For $\mathcal{C}$ a category, we let $\mathrm{Fil}\mathcal{C} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$ and $\mathrm{Gr}\mathcal{C} := \mathrm{Fun}((\mathbb{Z}^\delta)^{\mathrm{op}}, \mathcal{C})$ be the categories of filtered and graded objects in $\mathcal{C}$ respectively.

If $\mathcal{C}$ is symmetric monoidal and has the relevant colimits, we can equip them with the Day convolution symmetric monoidal structure, and this is the one which will always be used implicitly.

Objects of $\mathrm{Fil}\mathcal{C}$ can be described as sequences $\cdots \to X_2 \to X_1 \to X_0 \to X_{-1} \to \ldots$. The unit of Day convolution is $\ldots 0 \to 0 \to \mathbf{1}_{\mathcal{C}} \to \mathbf{1}_{\mathcal{C}} \to \ldots$. More generally:

**Construction 2.2.** The functor $\nu : \mathcal{C} \to \mathrm{Fil}\mathcal{C}$ sending $X$ to $\cdots \to 0 \to 0 \to X \to X \to \ldots$ is strong symmetric monoidal

**Construction 2.3.** For any integer $n$, we let $(n) : \mathrm{Fil}\mathcal{C} \to \mathrm{Fil}\mathcal{C}$ denote the functor that shifts by $n$ to the left. For example if $X$ is $\cdots \to X_2 \to X_1 \to X_0 \to X_{-1} \to \ldots$, $X(1)$ is $\ldots X_1 \to X_0 \to X_{-1} \to X_{-2} \to \ldots$.

Alternatively, letting $(-) : \mathbb{Z} \to \mathrm{Fil}\mathbf{S} \to \mathrm{Fil}\mathcal{C}$ be the composition of the Yoneda lemma with the symmetric monoidal colimit-preserving functor[2] $\mathbf{S} \to \mathcal{C}$, we have $X(n) = X \otimes \mathbf{1}_{\mathcal{C}}(n)$.

[1] Which applies, by Example A.6

[2] Assuming $\mathcal{C}$ has enough colimits.

We use the same notation, $(n)$, for the graded case, hoping no confusion arises.

**Observation 2.4.** Limits and colimits in $\mathrm{Fil}\mathcal{C}$ are computed pointwise. If we let $\mathrm{Fil}\mathcal{C}^{neg.cst.} \subset \mathrm{Fil}\mathcal{C}$ denote the full subcategory spanned by the $X$'s such that $X_n \to X_{n-1}$ is an equivalence for all $n \leq 0$, we thus find that it is closed under limits and colimits. If $\mathcal{C}$ has enough co/limits, the inclusion therefore has both a left and a right adjoint.

We now construct an associated graded functor - we were told about this construction by Robert Burklund.

**Construction 2.5.** Let $D \subset \mathbb{Z} \times \mathrm{Gr}\mathbf{S}_*$ be the subcategory spanned by objects of the form $(n, S^0(n)), n \in \mathbb{Z}$, and by maps that are either lying over a strict inequality $n < m$ in $\mathbb{Z}$, or invertible. We claim that this subcategory is a symmetric monoidal subcategory.

Further, the forgetful functor $D \to \mathbb{Z}$ is an equivalence, so that we may pick a symmetric monoidal functor $\mathbb{Z} \to D \to \mathrm{Gr}\mathbf{S}_*$ sending $n$ to $S^0(n)$ and the transition maps to 0.

By the universal property of Day convolution this extends uniquely to a symmetric monoidal colimit-preserving functor $\mathrm{Fil}(\mathbf{S}_*) \to \mathrm{Gr}\mathbf{S}_*$. By examning the underlying functor, we find that this is the associated graded functor, $X \mapsto (X_n/X_{n+1})_n$.

Tensoring this with $\mathcal{C}$ provides a symmetric monoidal, colimit preserving associated graded functor $\mathrm{gr} : \mathrm{Fil}\mathcal{C} \to \mathrm{Gr}\mathcal{C}$.

As a corollary, we observe:

**Observation 2.6.** If $\mathcal{C}$ is stable, then the full subcategory $\mathrm{Fil}\mathcal{C}^{neg.cst} \subset \mathrm{Fil}\mathcal{C}$ is closed under tensor products.

*Proof.* Note that $X \in \mathrm{Fil}\mathcal{C}^{neg.cst}$ if and only if $\mathrm{gr}(X)^i = 0$ for $i \leq -1$. Now suppose $X, Y$ satisfy this property, and let us compute $\mathrm{gr}(X \otimes Y)^i = \bigoplus_{p+q=i} \mathrm{gr}(X)^p \otimes gr(X)^q$. If $i \leq -1$, then either $p$ or $q$ is $\leq -1$, so that $\mathrm{gr}(X)^p = 0$ or $\mathrm{gr}(X)^q = 0$, and therefore $\mathrm{gr}(X \otimes Y)^i = 0$ too. $\square$

Similarly, we find that $\mathrm{Fil}_{\leq 0}\mathcal{C} \subset \mathrm{Fil}\mathcal{C}$, the full subcategory spanned by those filtered objects $X$ where $X_i = 0$ for $i > 0$ is closed under tensor products, and so we have a symmetric monoidal adjunction

$$\mathrm{Fil}_{\leq 0}\mathcal{C} \rightleftarrows \mathrm{Fil}\mathcal{C} \tag{1}$$

One of our key tools will be the observation that HH admits a filtered variant:

**Definition 2.7.** We define $\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}$ to be Hochschild homology relative to $\mathrm{Fil}_{\mathcal{C}}$.

**Remark 2.8.** By the previous observations, $\mathrm{Fil}\mathcal{C}^{neg.cst}$ is closed under $\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}$.

Since the colimit functor $\mathrm{Fil}\mathcal{C} \to \mathcal{C}$ is symmetric monoidal and colimit preserving, it preserves HH. In particular, if $A$ is a filtered ring with colimit $A_\infty$, $\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A)$ is a filtered object with colimit $\mathrm{HH}_{\mathcal{C}}(A_\infty)$.

The final tool we need concerns complete filtered objects. Note that we are picturing our filtered objects as being filtrations on $\mathrm{colim}_{n \to -\infty} X_n$, and so $\lim_n X_n$ is thought of as the "intersection of the filtered pieces". We thus have:

**Definition 2.9.** An object $X \in \mathrm{Fil}\mathcal{C}$ is called complete if $\lim_n X_n = 0$.

And then we have a detection result, namely:

**Lemma 2.10.** *Assume $X$ is a complete filtered object. If $\mathrm{gr}(X) = 0$, then $X = 0$.*

*Proof.* If $\mathrm{gr}(X) = 0$, then by stability, each $X_n \to X_{n-1}$ is an equivalence. Therefore, for any $n$, $\lim_k X_k \to X_n$ is an equivalence, as $\mathbb{Z}$ is weakly contractible. By completeness, the latter is 0, therefore so is $X_n$. $\square$

# 3 Hochschild homology of square zero extensions and periodic cyclic homology

We fix a background cocompletely symmetric monoidal category $\mathcal{C}$.

Let $A^\eta \to A$ be a square-zero extension of a ring $A$ by a bimodule $M$. It is given (non-canonically) by a pullback of the form:

$$\begin{tikzcd} A^\eta \arrow[r] \arrow[d] & A \arrow[d, "i"] \\ A \arrow[r, "d_\eta"'] & A \oplus \Sigma M \end{tikzcd}$$

where $i$ is the canonical inclusion and $d_\eta$ is some derivation.

The trivial square zero extension $A \oplus \Sigma M$ can be performed in $\mathrm{Fil}\mathcal{C}$ as the trivial extension of $\nu A$ by $\nu(\Sigma M)(1)$. By the adjunction (1), both maps $i, d_\eta : A \to A \oplus \Sigma M$ can be canonically realized as filtered maps $\nu A \to \nu A \oplus \nu(\Sigma M)(1)$, so that, in total $A^\eta$ admits a canonical filtration $\mathrm{fil}^\star A^\eta$, which looks like $\ldots 0 \to M \to A^\eta \to A^\eta \to \ldots$ with $M$ in weight 1.

Thus, by the discussion following Definition 2.7, we find that the map $\mathrm{HH}_{\mathcal{C}}(A^\eta) \to \mathrm{HH}_{\mathcal{C}}(A)$ can be canonically enhanced to a filtered map $\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta) \to \mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A) = \nu(\mathrm{HH}_{\mathcal{C}}(A))$. We let $\widetilde{\mathrm{HH}}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)$ denote its fiber.

We make the following observation:

**Lemma 3.1.** *$\mathrm{gr}(A^\eta)$ is a trivial square zero extension of $A$ by $M(1)$ in $\mathrm{Gr}\mathcal{C}$.*

*Proof.* This follows from the fact that $\mathrm{gr}(d_\eta) = \mathrm{gr}(i)$, which in turn follows from the monoidal adjunction between nonnegatively graded objects and graded objects. $\square$

Thus, we find:

**Corollary 3.2.** *There is a canonical equivalence $\mathrm{gr}(\widetilde{\mathrm{HH}}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)) \simeq \bigoplus_{n\geq 1} \mathrm{Ind}^{S^1}_{C_n} \Sigma^n \mathrm{HH}_{\mathcal{C}}(A; M^{\otimes_A n}(n))$.*

*In particular, if $\mathcal{C}$ is rational and admits limits, $\widetilde{\mathrm{HH}}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)^{tS^1}$ is a constant filtered object.*

Thus we would be able to conclude if we knew that it was also a complete filtered object. This is in general not true, even in the split square zero case. However, with enough connectivity assumptions, this is true.

**Question 3.3.** Can one prove nil-invariance without completeness, and so without connectivity assumptions?

The connectivity assumptions are going to involve a reasonable t-structure on $\mathcal{C}$, such as the Postnikov t-structure on $\mathbf{Sp}_{\mathbb{Q}}$. We make the following ad-hoc definition:

**Definition 3.4.** A t-structure on $\mathcal{C}$ is called good if it is left and right complete, and if $\mathcal{C}_{\leq 0}$ is closed under filtered colimits and $\mathcal{C}_{\geq 0}$ under arbitrary products.

**Proposition 3.5.** *Let $\mathcal{C}$ be equipped with a good t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, and assume that $A^\eta, M$ (and therefore $A$) are connective.*

*In this case, the filtered object $\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)$ is complete. Therefore, so is $\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)^{hS^1}$.*

*Proof.* We wish to prove that $\lim_n \mathrm{HH}^{\mathrm{fil}}(A^\eta)_{\geq n} = 0$. By the Milnor exact sequence (coming from closure of $\mathcal{C}_{\geq 0}$ under products) and by completeness of the t-structure, it suffices to prove that for every $k$, $\{\pi_k^{\heartsuit}\mathrm{HH}^{\mathrm{fil}}(A^\eta)_{\geq n}\}_n$ is 0 as a pro-object.

Now write $\mathrm{HH}^{\mathrm{fil}}(A^\eta)_{\geq n} = \mathrm{colim}_{[\ell]\in\Delta^{\mathrm{op}}}((\mathrm{fil}^\star A^\eta)^{\otimes \ell+1})_{\geq n}$. In $\pi_k^{\heartsuit}$, the only contributions are from $\ell \leq k+1$. Recall that $k$ is fixed, so that for $n$ large enough $((\mathrm{fil}^\star A^\eta)^{\otimes \ell+1})_{\geq n} = 0$ for all $\ell \leq k+1$ (indeed, $\mathrm{fil}^\star A^\eta$ is in weights $\leq 1$, so that $(\mathrm{fil}^\star A^\eta)^{\otimes \ell+1}$ is in weights $\leq \ell+1$), and so for $n > k+2$, $\pi_k^{\heartsuit}\mathrm{HH}^{\mathrm{fil}}(A^\eta)_{\geq n} = 0$, thus proving the claim.

See also [LL23, Lemma 2.18]. $\square$

Under the same connectivity hypotheses, one can prove that also $\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)_{hS^1}$ is complete.

**Proposition 3.6.** *Let $\mathcal{C}$ have a good t-structure as before, and again assume $A^\eta, M$ (and therefore $A$) are connective. In this case, $\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)_{hS^1}$ is complete.*

*Proof.* Note that each $\mathrm{colim}_{\mathbb{C}P^n}\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)$ is complete, as a finite colimit of complete filtered objects.

Furthermore, for any $X \in \mathcal{C}^{\mathbb{C}P^{n+1}}$, the cofiber of map $\mathrm{colim}_{\mathbb{C}P^n}X \to \mathrm{colim}_{\mathbb{C}P^{n+1}}X$ is $\Sigma^{2n+2}X_e$ (for some basepoint $e$ of $\mathbb{C}P^{n+1}$), so that if $X$ is connective, this cofiber is $2n+2$-connective, and thus the original map was an equivalence on $\pi^{\heartsuit}_{\leq 2n}$.

By completeness of the t-structure, it follows that for any connective $X \in \mathcal{C}^{BS^1}$, $\tau_{\leq n}(X_{hS^1}) \simeq \tau_{\leq n}(\mathrm{colim}_{\mathbb{C}P^n}X)$, and so, in total,

$$\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)_{hS^1} \simeq \lim_n \tau_{\leq n}(\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)_{hS^1}) \simeq \lim_n \tau_{\leq n}(\mathrm{colim}_{\mathbb{C}P^n}\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta))$$

is a limit of complete filtered objects, and hence is itself complete.

Note that the truncation of a complete filtered object is itself filtered because of the Milnor sequence. $\square$

We finally obtain Goodwillie's result, which is the case where $\mathcal{C} = \mathbf{Mod}_{\mathbb{Q}}$ of the statement below:

**Corollary 3.7.** *Let $\mathcal{C}$ have a good t-structure as before, and assume $\mathcal{C}$ is rational. In this case $\mathrm{HP}_{\mathcal{C}} := \mathrm{HH}^{tS^1}_{\mathcal{C}}$ is invariant under square-zero extensions of connective rings by connective bimodules.*

*Furthermore, for any connective $A$, the map $A \to \pi_0^{\heartsuit}(A)$ induces an equivalence in $\mathrm{HP}_{\mathcal{C}}$.*

*Proof.* Let $A^\eta \to A$ be a square zero extension as before. By Corollary 3.2, $\widetilde{\mathrm{HH}}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)^{tS^1}$ is a constant filtered object, and by Proposition 3.5 and Proposition 3.6 it is complete.

Thus it is 0 by Lemma 2.10 and so by definition $\mathrm{HH}^{\mathrm{fil}}_{\mathcal{C}}(A^\eta)^{tS^1} \to \mathrm{HH}_{\mathcal{C}}(A)^{tS^1}$ is an equivalence. Both are eventually constant filtered objects, so we can take their colimits and obtain the desired result.

For the statement about $A \to \pi_0^{\heartsuit}(A)$, we observe that each $\tau_{\leq n+1}A \to \tau_{\leq n}A$ is a square zero extension by [Lur12, Corollary 7.4.1.28.], so by the square-zero case it suffices to prove that $\mathrm{HP}_{\mathcal{C}}(A) \simeq \lim_n \mathrm{HP}_{\mathcal{C}}(\tau_{\leq n}A)$.

However, this property is true for $\mathrm{HH}_{\mathcal{C}}$, and hence for $\mathrm{HH}^{hS^1}_{\mathcal{C}}$, and the same argument as for $\mathrm{HH}_{\mathcal{C}}$ shows it is true for $(\mathrm{HH}_{\mathcal{C}})_{hS^1}$. Thus in total it is true for $\mathrm{HP}_{\mathcal{C}} := \mathrm{HH}^{tS^1}_{\mathcal{C}}$ $\square$

**Remark 3.8.** By [LT19, Corollary 3.5], one could in principle deduce the square-zero statement from the $\pi_0$-statement, but we do not know how to prove the $\pi_0$-statement without proving the square zero statement first.

# A Complex orientations and $S^1$-actions

In this appendix, we explain how to understand $S^1$-Tate constructions as inverting an Euler class in complex oriented situations.

**Definition A.1.** Let $(X, x)$ be a pointed space, and $\mathcal{C}$ a pointed category. A reduced morphism between functors $F, G : \mathcal{C}^X \to \mathcal{C}^X$ is a morphism $F \to G$ equipped with a a nullhomotopy of $F_x \to G_x$.

**Example A.2.** Let $(X, x)$ be a pointed space and $\mathcal{C}$ a pointed cocomplete category. There is a canonical reduced morphism $\mathrm{id} \to X \otimes \mathrm{id}$ where the tensor is the canonical tensoring of $\mathbf{S}_*$ on $\mathcal{C}$.

It suffices to construct this on $\mathbf{S}_*$ itself, and for this, it suffices to construct this on the unit of $\mathbf{S}_*^X$, i.e. $S^0$ as a constant functor. It is now clear what to do.

**Definition A.3.** Let $\mathcal{C}$ be a stable symmetric monoidal category with unit $\mathbf{1}_{\mathcal{C}}$. $\mathcal{C}$ is called *complex orientable* if there exists a reduced morphism between endofunctors of $\mathcal{C}^{BS^1}$ of the form $e : \mathrm{id} \to \Sigma^2\mathrm{id}$ which restricts, in $\mathcal{C}^{S^2}$, to the canonical reduced morphism.

A complex orientation of $\mathcal{C}$ is the choice of such an $e$.

**Remark A.4.** If $\mathcal{C}$ is symmetric monoidal with unit $\mathbf{1}_{\mathcal{C}}$, and if we require $e$ to be $\mathcal{C}$-linear, we can also phrase this as follows: we want a map $\mathbf{1}_{\mathcal{C}} \to \Sigma^2\widetilde{\mathbf{1}}_{\mathcal{C}}^{BS^1}$ which restricts to the unit in $\Sigma^2\widetilde{\mathbf{1}}_{\mathcal{C}}^{S^2} \simeq \mathbf{1}_{\mathcal{C}}$.

Note that if $\mathcal{C}$ is complex orientable and symmetric monoidal, it admits *a* complex orientation of the form described above, because one can pick $e$ evaluated at $\mathbf{1}_{\mathcal{C}}$ with a trivial action and tensor it with any other object to get a new $\tilde{e}$. Since the canonical reduced morphism $\mathrm{id} \to \Sigma^2\mathrm{id}$ is of this form, this does produce a complex orientation, regardless of whether $e \simeq \tilde{e}$.

In view of Remark A.4, we find that the above definition includes the classical setting of complex orientations. As a special case, we find:

**Corollary A.5.** *Suppose $E$ is a complex orientable ring spectrum in the usual sense. In this case, $\mathbf{Mod}_E(\mathbf{Sp})$ is complex orientable - this is the case, e.g. if $E$ is even.*

*Furthermore, if $E$ is an $\mathbb{E}_2$-ring spectrum, then any $E$-linear category is complex orientable.*

**Example A.6.** $\mathbb{Z}$ is complex orientable, so any $\mathbb{Z}$-linear category is complex orientable. This is in particular true for rational categories.

The main result for our purposes[3] is:

**Proposition A.7.** *Let $\mathcal{C}$ be a complex orientable stable category with enough co/limits, and let $e : \mathrm{id} \to \Sigma^2\mathrm{id}$ be a choice of complex orientation.*

*This choices induces a map, which we still denote e by abuse of notation, of the form $(-)^{hS^1} \to \Sigma^2(-)^{hS^1}$. The functor $(-)^{hS^1}[e^{-1}] := \mathrm{colim}_{\mathbb{N}}((-)^{hS^1} \to \Sigma^2(-)^{hS^1} \to \Sigma^4(-)^{hS^1} \to \dots$ is equivalent to $(-)^{tS^1}$.*

*Proof.* Let $T$ denote this functor. Clearly there is a map $(-)^{hS^1} \to T$. Among functors with such a map, $(-)^{tS^1}$ is initial among exact functors that vanish on induced objects.

Thus, it suffices to prove two things:

1. $T$ vanishes on induced objects;
2. If $F$ vanishes on induced objects, then $e$ induces an equivalence $\mathrm{map}(\Sigma^2(-)^{hS^1}, F) \to \mathrm{map}((-)^{hS^1}, F)$.

We first prove point 2. By the Yoneda lemma, it suffices to prove that for every $x \in \mathcal{C}$, $F(x) \to F(\Sigma^2 x)$ is an equivalence, where $x$ is taken to have the trivial $S^1$-action.

As $F$ is exact, it suffices to prove that $F$ vanishes on the fiber of $x \to \Sigma^2 x$, and so it suffices to prove that this fiber is induced.

Since $x \to \Sigma^2 x$ is equipped with an underlying nullhomotopy, we find a map $x \to \mathrm{fib}$ in $\mathcal{C}$, and thus a map $x[S^1] \to \mathrm{fib}$ in $\mathcal{C}^{BS^1}$.

Comparing this whole thing to $S^2$, we also find a factorization $x[S^1] \to x[\Omega S^2] \to \mathrm{fib}$, where, because the restriction of $e$ to $S^2$ is the canonical reduced morphism, we find that the null-sequence[4] $x[\Omega S^2] \to x \to \Sigma^2 x$ is $x[-]$ applied to the null-sequence $(\Omega S^2)_+ \to S^0 \to S^2$, and so the nullsequence $x[S^1] \to x \to \Sigma^2 x$ is $x[-]$ applied to $(S^1)_+ \to S^0 \to S^2$, a null-sequence which participates in a pushout square, and thus is sent to a pushout square by $x[-]$.

Thus, by stability, it is also a fiber sequence. It follows that the map $x[S^1] \to \mathrm{fib}$ (which was constructed $S^1$-equivariantly !) is nonequivariantly an equivalence, and hence an equivalence, thus proving the claim.

Now let us prove that $T$ vanishes on induced objects. It suffices to prove that $e$ is null on induced objects. But if $x$ is induced, say $x \simeq x_0[S^1]$, to prove that $x \to \Sigma^2 x$ is $S^1$-equivariantly null, it suffices to prove that $x_0 \to x \to \Sigma^2 x$ is (underlying) null, which follows from the fact that $x \to \Sigma^2 x$ is itself underlying null.

This concludes the proof. $\square$

[3] Complex orientability has many more interesting consequences.

[4] Which is not a fiber sequence in general!

**Remark A.8.** While $e$ is a choice, namely, there are possibly different complex orientations of $\mathcal{C}$, the resulting functor is independent of this choice, because $(-)^{tS^1}$ is.

Finally, in view of Remark A.4, we point out that if $\mathcal{C}$ is symmetric monoidal and complex oriented via some $e \in \Sigma^2 \widetilde{\mathbf{1}}_{\mathcal{C}}^{BS^1}$, and if $f : X \to BS^1$ is a map such that $e_{|X} = 0$, then the same argument as the end of the previous proof shows that $(-)^{tS^1}$ vanishes on objects of the form $f_! c, c \in \mathcal{C}^X$.

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