Hochschild homology as an obstruction to algebraic structures

This is just a short note to record an argument using HH that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p)$ is not an $\mathbb{F}_p$-algebra, even though it receives an algebra map from $\mathbb{F}_p$.

Note that it is a $\mathbb{Z}$-algebra, in fact:

Proposition 0.1. *Let $\mathbf{E} \in \mathrm{CAlg}(\mathbf{Pr}^L_{st})$ be a stable presentably symmetric monoidal category, and $x \in \mathbf{E}$ an object.*

Then $\underline{\mathrm{End}}_{\mathbf{E}}(x)$ has a canonical algebra structure in $\mathbf{E}$ given by composition, where $\underline{\mathrm{End}}_{\mathbf{E}}$ is the enriched endomorphism object.

Proof. See [Lur12] somewhere about endomorphism objects. □

One can prove that it is not an $\mathbb{F}_p$-algebra in the following way: by a $\mathbb{Z}$-linear Schwede-Shipley recognition theorem, one can easily prove that $\mathbf{Perf}(\mathbb{Z})^{p-nil} \simeq \mathbf{Perf}(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p))$, $\mathbb{Z}$-linearly, where $\mathbf{Perf}(\mathbb{Z})^{p-nil}$ denotes the full subcategory of $\mathbf{Perf}(\mathbb{Z})$ spanned by modules on which p acts nilpotently.

In particular, there is a fiber sequence

$$\mathrm{HH}_{\mathbb{Z}}(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p)) \to \mathrm{HH}_{\mathbb{Z}}(\mathbb{Z}) \to \mathrm{HH}_{\mathbb{Z}}(\mathbb{Z}[1/p])$$

from which it follows that $\mathrm{HH}_{\mathbb{Z}}(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p))$ is concentrated in degree -1 and has π_{-1} isomorphic to $\mathbb{Z}[1/p]/\mathbb{Z}$, in particular it is not p-torsion (it is, of course, p-power torsion).

But if it were an $\mathbb{F}_p$-algebra, its $\mathrm{HH}_{\mathbb{Z}}$ would be a module over $\mathrm{HH}_{\mathbb{Z}}(\mathbb{F}_p)$, and so over $\mathbb{F}_p$, hence it would have p-torsion homotopy groups.

Corollary 0.2. $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p)$ *is not an $\mathbb{F}_p$-algebra.*

Remark 0.3. We used $\mathrm{HH}_{\mathbb{Z}}$ and a $\mathbb{Z}$-linear Schwede-Shipley recognition theorem, but if one would prefer to stick with the ordinary Schwede-Shipley theorem, it is completely fine to use THH: indeed the other THH's appearing are still connective, and their THH_0 agrees with their HH_0, so $\pi_{-1}(\mathrm{THH}(\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p)))$ is still not p-torsion, while $\mathrm{THH}(\mathbb{F}_p)$ is p-torsion.

Remark 0.4. I don't know if there is an easier proof of that corollary...

Remark 0.5. Let us justify the claim that it receives an algebra map from $\mathbb{F}_p$: this is just the canonical map $\tau_{\geq 0}\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p) \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p)$, noting that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p)$ is coconnective and its π_0 is $\mathbb{F}_p$.

References

[Lur12] Jacob Lurie. Higher algebra, 2012.