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Goodwillie calculus of localizing invariants

(an ahistorical approach to the Dundas-Goodwillie-McCarthy theorem)

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MM Mathematics Münster Cluster of Excellence

Motivation

Theorem (Dundas-Goodwillie-McCarthy)

Let $R \to S$ be a map of connective ring spectra with $\pi_0(R) \to \pi_0(S)$ a surjection with nilpotent kernel. The following naturality square is cartesian:

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & K(S) \arrow[d] \\ TC(R) \arrow[r] & TC(S) \end{tikzcd}$$

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Goal: Understand this and more generally the related Goodwillie calculus of localizing invariants

Outline of the proof

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A word about Step 2

Square zero extensions fit into pullback squares:

$$\begin{tikzcd} \tilde{R} \arrow[r] \arrow[d] & R \arrow[d, "i_0"] \\ R \arrow[r, "d_\eta"] & R \oplus \Sigma M \end{tikzcd}$$

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Land-Tamme's map from the "circle-dot ring" $\odot \to R \oplus \Sigma M$ is highly connective.

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Follows from similar statement for $M \mapsto \mathrm{BGL}_k(A \oplus M)$.

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Theorem (Harpaz-Nikolaus-Saunier, R.)

"Naturally" in C, $\mathrm{Sp}(\mathrm{Cat}^{\mathrm{perf}}_{/C}) \simeq \mathrm{Fun}^L(\mathrm{Ind}(C), \mathrm{Ind}(C))$.

These assemble in $\mathrm{TCat}^{\mathrm{perf}}$, the tangent category, which can be informally described as the category of pairs (C, S), $S : \mathrm{Ind}(C) \to \mathrm{Ind}(C)$, and morphisms are oplax transformations:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "S"'] & D \arrow[d, "R"] \\ C \arrow[r] \arrow[ur, Rightarrow] & D \end{tikzcd}$$

Example: rings

Above,

$$\Omega^\infty_C : \mathsf{Fun}^L(\mathsf{Ind}(C), \mathsf{Ind}(C)) \to \mathsf{Cat}^{\mathsf{perf}}_{/C},$$

$$S \mapsto \mathsf{End}(C; S) := \{(x \to Sx) \mid x \in C\}$$

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A ring, M bimodule:

$$\mathrm{Perf}(A \oplus M) \hookrightarrow \mathrm{End}(\mathrm{Perf}(A), \Sigma M \otimes_A -)$$

Equivalence if (A, M) connective.

Deriving localizing invariants

$\mathrm{E} : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ localizing invariant (e.g. K or TC).

$$\mathrm{E}(C; S) := \mathrm{fib}(\mathrm{E}(\mathrm{End}(C; S)) \to \mathrm{E}(C))$$

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Analogous to $\mathrm{fib}(\mathrm{K}(A \oplus M) \to \mathrm{K}(A))$.
This is the functor we will consider derivatives of.

Trace theories I

Beautiful insight of Nikolaus (following Kaledin): $\mathrm{TCat}^{\mathrm{perf}} \subset \Lambda^{\mathrm{st}}$ and $\mathrm{End}(-;-)$ extends to Λ^{st}.

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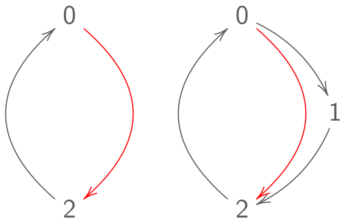
Morphisms: Functors inducing degree 1 maps on realizations.

Example



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Morphisms: coCartesian edges over Λ^{op} correspond to composition, and over the same object of Λ, correspond to oplax transformations as in TCat^{perf}:

$$\begin{tikzcd} C_i \arrow[r] \arrow[d, "S_i"'] & D_i \arrow[d, "R_i"] \\ C_{i+1} \arrow[r] \arrow[ur, Rightarrow] & D_{i+1} \end{tikzcd}$$

Example

$$
\begin{tikzcd}
C_0 \arrow[d, bend left, red, "F_1 \circ F_0"] \\
C_2 \arrow[u, bend left, "F_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C_0 \arrow[dr, "F_0"] \arrow[dd, bend left, red] & \\
& C_1 \arrow[dl, "F_1"] \\
C_2 \arrow[uu, bend left, "F_2"] &
\end{tikzcd}
$$

Cyclic theories

“Definition”

$\mathrm{End}(\vec{C};\vec{S}) := \{(x_1 \to S_0(x_0), x_2 \to S_1(x_1), \dots, x_0 \to S_n(x_n)) \mid x_i \in C_i\}$

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$E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$; then

$$E^{\mathrm{cyc}}(\vec{C},\vec{S}) := \mathrm{fib}(E(\mathrm{End}(\vec{C},\vec{S})) \to E(\prod_i C_i))$$

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Theorem (Nikolaus)

$E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ localizing invariant. In this case, $E^{\mathrm{cyc}} : \Lambda^{\mathrm{st}} \to \mathcal{E}$ sends coCartesian edges to equivalences.

$$\begin{tikzcd} C \arrow[d, bend left, "F"] \\ D \arrow[u, bend left, "G"] \end{tikzcd}$$

has coCartesian maps to (C, GF) and (D, FG).

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Definition

A trace theory is a functor $T : \Lambda^{st} \to \mathcal{E}$ that inverts coCartesian edges.

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Definition

A trace theory is a functor $T : \Lambda^{st} \to \mathcal{E}$ that inverts coCartesian edges.
$TrThy(\mathcal{E})$ is the category of trace theories.

Features of trace theories

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4. $P_n^{\mathrm{fbw}} T$ is still a trace theory.

The Goodwillie calculus of localizing invariants redux

Try to understand $P_n E^{cyc}$ as a trace theory.

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Get $T(\mathrm{Sp}, \mathrm{map}(x, y)) \simeq T(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \otimes \mathrm{map}(x, y)$.

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More generally,

Corollary (R.)

Let E be a finitary localizing invariant. There is a canonical $X_E \in \mathcal{E}^{BS^1}$ (namely $P_1 E^{cyc}(Sp^\omega, id_{Sp})$) with $P_1 E^{cyc} \simeq X_E \otimes THH$.

Frobenii

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For E finitary localizing invariant, X_E has a natural structure of a cyclotomic spectrum, and $P_1 E^{cyc} \simeq X_E \otimes \mathrm{THH}$ as polygonic spectra.

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More generally, can do:

$$P_1E^{cyc}(C, S) \to P_1E^{cyc}(C, S^{\otimes n})^{\tau C_n}$$

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Theorem (R.)

Let T be an n-homogeneous, finitary, *splitting*[1] trace theory. There is an S^1-spectrum X_T and an equivalence

$$T \simeq (X_T \otimes \mathrm{THH}(-, -^{\circ n}))_{hC_n}$$

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To understand $P_n E^{cyc}$, we need to understand:

1. $X_{D_n E^{cyc}}$;
2. The glueing between different layers.

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Higher derivatives

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Theorem (Harpaz-Nikolaus-Saunier, R.)

E localizing invariant. There is a natural cartesian square:

$$\begin{tikzcd}
P_n E^{\mathrm{cyc}} \arrow[r] \arrow[d] & (P_1 E^{\mathrm{cyc}}(-, -^{\otimes n}))^{hC_n} \arrow[d] \\
P_{n-1} E^{\mathrm{cyc}} \arrow[r] & (P_1 E^{\mathrm{cyc}}(-, -^{\otimes n}))^{tC_n}
\end{tikzcd}$$

and so $P_n E^{\mathrm{cyc}} \simeq \mathrm{TR}^n(P_1 E^{\mathrm{cyc}})$ with the polygonic structure described earlier.

Back to DGM

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TC is a localizing invariant built out of P_1K^{cyc} and its extra structure to ensure that $P_1K \simeq P_1TC$. From this, the way to the DGM theorem is relatively formal and mostly relies on connectivity properties of K-theory.

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The Goodwillie theory of a finitary localizing invariant E is controlled by a single cyclotomic spectrum, namely $\mathrm{P}_1\mathrm{E}^{\mathrm{cyc}}(\mathrm{Sp}^\omega, \mathrm{id}_{\mathrm{Sp}})$.

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The Goodwillie theory of a finitary localizing invariant E is controlled by a single cyclotomic spectrum, namely $\mathrm{P}_1\mathrm{E}^{\mathrm{cyc}}(\mathrm{Sp}^{\omega}, \mathrm{id}_{\mathrm{Sp}})$. For K-theory, this happens to be $\mathbb{S}^{\mathrm{triv}}$ for universality reasons.

A conjecture

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Conjecture

There is a variant of the nth jet bundle $\mathcal{P}_n(\mathrm{Cat}^{\mathrm{perf}})$ which takes into account splitting-ness. This variant is equivalent to $\Lambda^{\mathrm{st}}_{\leqslant n}$