Some examples of finiteness obstructions

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Introduction

In classical categories, there are various possible notions of finiteness.

A popular one is *compactness*, which is defined so as to enjoy similar properties to "finite presentation" in algebraic contexts; and a second one is being built out of basic objects using finite colimits.

They are usually related, and a lot of the time (if the basic objects are themselves compact), finiteness in the second sense implies compactness. One may then ask whether the converse holds, and one then faces a kind of obstruction theoretic problem: given an intrinsic property of a certain object, can I build it as a finite colimit of basic objects ?

In the context of homotopy types, Wall's finiteness obstruction famously gives a purely algebraic criterion to decide when that obstruction vanishes, living in a K_0-group. In a lecture course on algebraic K-theory and manifolds, Lurie proved a generalized version of Wall's obstruction, which essentially always provides such a K-theoretic criterion. The goal of this note is to make this obstruction explicit in a number of examples.

This note is structured as follows: in Section 1, we recall Lurie's generalized Wall obstruction and indicate the strategy to make the obstruction explicit, in Section 2, we look at Wall's original example, in Section 3, we look at the category of stable categories, where we observe that Lurie's result recovers a result that was independently rediscovered by Sasha Efimov and finally in Section 4, we review the example of separable commutative algebras, where we show an analogue of the classical étale presentation lemma, showing that compact étale algebras are not only compact, they are actually finitely presented.

In Appendix A, we shamelessly copy ("review") Lurie's proof of his generalized obstruction.

Conventions

All categories are ∞-categories unless stated otherwise (e.g. "1-category").

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1 Lurie's generalized Wall obstruction

1.1 Basics

Definition 1.1. A category $\mathcal{C}$ is called pointed if it has a zero object, that is, an object which is both initial and terminal.

Definition 1.2. If $\mathcal{C}$ is a pointed category, a cofiber sequence $X \to X' \to X''$ is a pushout square of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ 0 \arrow[r] & X'' \end{tikzcd}$$

where 0 is a 0-object.

Definition 1.3. Let $\mathcal{C}$ be a pointed, finitely cocomplete category. We let $K_0(\mathcal{C})$ be the abelian group generated by $\pi_0(\mathcal{C}^{\simeq})$ modulo the relations $[X'] = [X] + [X'']$ for any cofiber sequence $X \to X' \to X''$, where $[X]$ denotes the generator corresponding to the object $X \in \mathcal{C}$.

Remark 1.4. In fact, we could have defined it to be the abelian monoid with these generators and relations. Indeed, the cofiber sequence $0 \to X \to X$ shows that $[0] = 0$, and the cofiber sequence $X \to 0 \to \Sigma X$ shows that $[\Sigma X] + [X] = [0] = 0$.

Remark 1.5. In particular, every element of $K_0(\mathcal{C})$ can be represented as $[X]$ for some $X \in \mathcal{C}$.

Corollary 1.6. *$\Sigma : \mathcal{C} \to \mathcal{C}$, defined as $0 \coprod_X 0$, induces an isomorphism on K_0 (specifically, it induces -1).*

In particular, the canonical map from $\mathcal{C}$ to $SW(\mathcal{C}) := \operatorname{colim}(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \dots)$ induces an isomorphism on K_0.

Remark 1.7. $SW(\mathcal{C})$ is the "Spanier-Whitehead" category of $\mathcal{C}$. It is the universal stable category receiving a finite-colimit preserving map from $\mathcal{C}$.

In particular, note that our definition of K_0, being somewhat low-tech, gives 0 for ordinary 1-categories. To "fix" this one has to introduce more structure on $\mathcal{C}$/not consider all cofiber sequences.

1.2 Thomason's obstruction

The key observation that makes K_0 suitable for finiteness obstructions is the following observation, due to Thomason:

Proposition 1.8. *Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full stable subcategory of the stable category $\mathcal{C}$. Suppose that every object of $\mathcal{C}$ is a summand of some object in $\mathcal{C}_0$. In this case:*

1. *The morphism $K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ is injective.*
2. *If $X \in \mathcal{C}$ then $[X]$ is in the image of $K_0(\mathcal{C}_0)$ if and only if $X \in \mathcal{C}_0$.*

This a special case of Lurie's obstruction, which we are about to state, though we note that Lurie's proof uses this special case.

Remark 1.9. Let $\mathcal{C}$ be a small stable category, and consider the large category $\mathrm{Ind}(\mathcal{C})$. There is a canonical inclusion $\mathcal{C} \subset \mathrm{Ind}(\mathcal{C})^\omega$, but it is generally not an equivalence: trying to prove that every object on the right lies in $\mathcal{C}$ will lead one to observe that every object in $\mathrm{Ind}(\mathcal{C})^\omega$ is a *retract* of an object of $\mathcal{C}$. This is true generally, so this is why the above proposition is helpful to distinguish compactness from finiteness (where we fix $\mathcal{C}$ to be our "building blocks" of "finite" things).

In 1-category land, retracts are also finite colimits, and so the subtlety between finite and compact is not very visible. Here is what happens in pointed ∞-categories when one tries to implement the usual method for splitting off an idempotent:

Example 1.10. Let $\mathcal{C}$ be a pointed category and $x, y \in \mathcal{C}$. Let $z = x \vee y$ and let $e : z \to z$ be the projection onto the y-summand followed by inclusion of y into z.

It is not difficult to see that $\mathrm{cofib}(e : z \to z) \simeq x \vee \Sigma x$, whose K-theory class is 0. Thus one sees that in the situation of Thomason's theorem, if $x \in \mathcal{C}$ is a retract of some $z \in \mathcal{C}_0$, then x is a retract of $x \oplus \Sigma x$, and the latter is *also* in $\mathcal{C}_0$, so it is some kind of canonical choice of z.

1.3 Lurie's obstruction

Thomason's obstruction is a stable obstruction, and we saw in Corollary 1.6 that K_0 does not distinguish a category from its stabilization. Yet, in examples such as spaces, stabilizing *does* change whether an object is finite or not. In particular, for the unstable Wall obstruction it is crucial to have at hand some extra data to maintain *some* unstability.

The setting for Lurie's obstruction is as follows: Let $\mathcal{C}$ be a small, pointed, finitely cocomplete category. We can embed $\mathcal{C}$ in its Ind-completion, $\mathrm{Ind}(\mathcal{C})$, and in the compacts thereof, i.e. the idempotent completion of $\mathcal{C}$. We will view these as inclusions: $\mathcal{C} \subset \overline{\mathcal{C}} \subset \mathrm{Ind}(\mathcal{C})$, where $\overline{\mathcal{C}} := \mathrm{Ind}(\mathcal{C})^\omega \simeq \mathcal{C}^{\mathrm{idem}}$.

Remark 1.11. Recall that any object in $\overline{\mathcal{C}}$ is a retract of some object of $\mathcal{C}$.

Also recall that both inclusions $\mathcal{C} \subset \overline{\mathcal{C}} \subset \mathrm{Ind}(\mathcal{C})$ preserve finite colimits.

An observation is that while compactness is intrinsic, finiteness depends on our choice of $\mathcal{C}$. To generalize finiteness, we define the following notion of "cell-attachments" in $\mathcal{C}$:

Definition 1.12. We define a class of arrows in $\mathrm{Ind}(\mathcal{C})$: the class of $\mathcal{C}$-finite arrows is the smallest class of arrows containing arrows in $\mathcal{C}$, closed under pushouts along arbitrary maps, and under composition.

Example 1.13. If $\mathcal{C}$ = finite spaces, then $\mathcal{C}$-finite maps are maps $X \to Y$ where Y can be built from X by attaching finitely many cells.

Observation 1.14. The class of arrows $f : X \to Y$ that can be written as finite composites of arrows $Y_i \to Y_{i+1}$ which are pushouts $Y_i \to Y_i \coprod_C D$ where $C, D \in \mathcal{C}$ is closed under composition (obvious) and under pushouts (not hard), and thus coincides with the class of $\mathcal{C}$-finite arrows.

We now explain where the finiteness obstruction lives. The idea is that if $X \in \overline{\mathcal{C}}$, the finiteness of X can be phrased as the fact that $\emptyset \to X$ is $\mathcal{C}$-finite. Taking K_0 directly kills too much information because we essentially stabilize, but if we look at this map as living over X, then we retain *some* unstable information.

Because the transition from unstable to stable behaves slightly better in the pointed setting, we introduce the following definition:

Definition 1.15. Let $X \in \mathrm{Ind}(\mathcal{C})$. We let $\mathrm{Ind}(\mathcal{C})_{X//X} := (\mathrm{Ind}(\mathcal{C})_{/X})_{X/} \simeq (\mathrm{Ind}(\mathcal{C})_{X/})_{/X}$, and $\mathcal{C}[X] \subset \mathrm{Ind}(\mathcal{C})_{X//X}$ be the full subcategory spanned by the triangles $X \to Y \to X$ such that $X \to Y$ is $\mathcal{C}$-finite, and we let $\overline{\mathcal{C}[X]}$ denote the compacts of this over-under slice category, equivalently the idempotent completion of $\mathcal{C}[X]$.

Informally, this says that adding the reference map to X guarantees that the question of the "finiteness" of X has the same answer after stabilization, where it becomes a K_0-question by Thomason's result.

Notation 1.16. Let Y be a compact object of $\mathrm{Ind}(\mathcal{C})$ equipped with a map to X. Then $X \to X \coprod Y \to X$ is in $\overline{\mathcal{C}[X]}$ - we (abusively) denote its class in $K_0(\overline{\mathcal{C}[X]})$ by $[Y/X]$. For instance, if X is compact, and we take $\mathrm{id}_X : X \to X$, we get $[X/X]$.

The generalized Wall finiteness obstruction is the following theorem :

Theorem 1.17 (Lurie)**.** *Let $\mathcal{C}$ be a small finitely cocomplete category, and $X \in \mathrm{Ind}(\mathcal{C})$. Then:*

1. *The canonical map $\alpha : K_0(\mathcal{C}[X]) \to K_0(\overline{\mathcal{C}[X]})$ is injective;*
2. *Suppose X is compact. Then $[X/X] \in K_0(\overline{\mathcal{C}[X]})$ is in the image of α if and only if X is in $\mathcal{C}$.*

We note, in addition, that for any $\mathcal{C}$ $\mathrm{Ind}(SW(\mathcal{C})) \simeq \mathbf{Sp}(\mathrm{Ind}(\mathcal{C}))$, by comparing their universal properties. Thus, really $K_0(\overline{\mathcal{C}[X]}) \simeq K_0(SW(\overline{\mathcal{C}[X]})) \simeq K_0(\mathbf{Sp}(\mathrm{Ind}(\mathcal{C})_{X//X})^{\omega})$.

In practice, what this means is that to answer the question "Is $X \in \mathcal{C}$?" when we know that X is in $\mathrm{Ind}(\mathcal{C})^{\omega}$, we need to do two things:

a) Try to understand the stabilization $\mathbf{Sp}(\mathrm{Ind}(\mathcal{C})_{X//X})$ and its K_0;

b) Try to understand the image of α.

For the latter, we note the following helpful lemma (which is what the theorem above "wants to say", but the above version is more natural to prove):

Lemma 1.18. *Let $\mathcal{C}, X$ be as above. In this case, $K_0(\mathcal{C}[X])$ is generated as an abelian group by the $[Y/X]$'s, $Y \in \mathcal{C}$*

Proof. Fix $Y \in \mathcal{C}[X]$, and write $f : X \to Y$ as a composite $X \to Y_1 \to ... \to Y_n = Y$ where each $Y_i \to Y_{i+1}$ (including $i = 0$ where $Y_0 := X$) is a pushout along a map in $\mathcal{C}$. By defining the relevant maps $Y_i \to X$ as composites $Y_i \to Y_n = Y \to X$, we have a diagram $Y_1 \to ... \to Y_n$ in $\mathcal{C}[X]$, which induces an equality of K_0-classes: $[Y_n] = [Y_n/Y_{n-1}] + [Y_{n-1}/Y_{n-2}] + ... + [Y_1]$. Now each $Y_i \to Y_{i+1}$ is of the form $Y_i \to Y_i \coprod_C D$, so that the cofiber (in $\mathcal{C}[X]!$) is of the form $X \coprod_C D$ for some $C, D \in \mathcal{C}$. But this is clearly the cofiber of $X \coprod C \to X \coprod D$, which proves that its K-theory class is $[D/X] - [C/X]$, as was to be shown. □

This lemma shows that once we understand a) in the program above, that is, $\mathbf{Sp}(\mathrm{Ind}(\mathcal{C})_{X//X})$ and the K_0 of its compacts, we need only understand very specific classes $[Y/X]$ in its K_0.

Since $\mathcal{C}$ is typically itself defined in terms of certain generators under finite colimits, this limits the search heavily. This will be our strategy in all examples below.

Example 1.19. Let $\mathcal{C}$ be a stable category. In this case, taking fibers (or cofibers) induces an equivalence $\mathcal{C}_{X//X} \simeq \mathcal{C}$. Unwinding the above theorem in this special case recovers exactly Thomason's theorem.

2 Example I: the Wall obstruction

In this section, we study the case of spaces, treated by Wall originally.

Let $\mathbf{S}$ be the category of spaces, and $\mathbf{S}^{\mathrm{fin}} \subset \mathbf{S}$ the smallest subcategory containing $\emptyset, \mathrm{pt}$ and closed under pushouts, called finite spaces. It is well known that $\mathrm{Ind}(\mathbf{S}^{\mathrm{fin}}) \simeq \mathbf{S}$, so that $\mathbf{S}^{\mathrm{fin}} \subset \mathbf{S}^{\omega}$, and objects of the latter are all retracts of objects of the former (hence their classical name: *finitely dominated* spaces).

Fix $X \in \mathbf{S}$. By un/straightening, we have an equivalence $\mathbf{S}_{/X} \simeq \mathbf{S}^X$ under which $\mathrm{id}_X : X \to X$ goes to the constant local system pt (the terminal object). Thus, $\mathbf{S}_{X//X} \simeq \mathbf{S}_*^X$, local systems of pointed spaces and so $\mathbf{Sp}(\mathbf{S}_{X//X}) \simeq \mathbf{Sp}^X$, local systems of spectra.

Unwinding the identifications, we find that for any $x : \mathrm{pt} \to X$, the suspension of the object $X \coprod \mathrm{pt}$ in $\mathbf{S}_{X//X}$ in $\mathbf{Sp}^X$ is none other than $x_!\mathbb{S} \in \mathbf{Sp}^X$, where for a map $f : Y \to Z$ of spaces we let $f_!$ denote left Kan extension along f, while that of the object $X \coprod X$ is $\mathbb{S}_X$, the constant local system with value $\mathbb{S}$.

Concretely, if X is connected and we use x to identify $\mathbf{Sp}^X \simeq \mathbf{Mod}_{\mathbb{S}[\Omega(X,x)]}$, $x_!\mathbb{S}$ corresponds to $\mathbb{S}[\Omega(X,x)]$. Since finite spaces are generated under pushouts by the point, Lemma 1.18 tells us that in this case, Lurie's Wall obstruction is as follows:

Corollary 2.1. *Let X be a compact space. X is finite if and only if the K-theory class $[\mathbb{S}_X] \in K_0((\mathbf{Sp}^X)^{\omega})$ is in the subgroup generated by those of the form $[x_!\mathbb{S}], x \in X$ (which only depends on the image of x in $\pi_0(X)$).*

If X is connected, one has $\mathbf{Sp}^X \simeq \mathbf{Mod}_{\mathbb{S}[\Omega(X,x)]}$ for any choice of a basepoint x, and this induces a sequence of isomorphisms

$$K_0((\mathbf{Sp}^X)^\omega) \cong K_0(\mathbb{S}[\Omega(X,x)]) \cong K_0(\mathbb{Z}[\pi_1(X,x)]).$$

Under these isomorphisms (which are given by K_0 applied to certain concrete functors), $[x_!\mathbb{S}]$ goes to the class of $\mathbb{Z}[\pi_1(X,x)]$ itself, while $[\mathbb{S}_X]$ goes to the class of $C_*(\tilde{X},\mathbb{Z})$, the singular chain complex of the universal cover of X, equipped with its $\pi_1(X,x)$-action. This happens to be perfect as a $\mathbb{Z}[\pi_1(X,x)]$-chain complex when X is compact, and recovers exactly Wall's obstruction, so we obtain the following:

Theorem 2.2 (Wall)**.** *Let (X,x) be a pointed connected finitely dominated space. X is finite if and only if the class of $C_*(\tilde{X},\mathbb{Z})$ in $K_0(\mathbb{Z}[\pi_1(X,x)])$ is in the image of the basechange map $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\pi_1(X,x)])$.*

Remark 2.3. One of the results in Wall's paper is that any class in $\tilde{K}_0(\mathbb{Z}[G])$ can be represented as w_X for some X with $\pi_1(X) \cong G$. It seems like this can not be stated in the level of generality of Lurie's Wall obstruction.

It would be interesting to see what can be said about which classes correspond to finiteness obstructions in more general contexts, e.g. in the equivariant case.

3 Example II: stable categories

In this section, we consider finiteness obstructions in the setting of $\mathbf{Cat}^{\mathrm{ex}}$, the category of stable categories.

3.1 Finiteness and Efimov's obstruction

A reasonable notion of finiteness in this context (as in the context of $\mathbf{Cat}$ itself) is the following: $(\mathbf{Cat}^{\mathrm{ex}})^{\mathrm{fin}}$ is the smallest subcategory of $\mathbf{Cat}^{\mathrm{ex}}$ containing $0, \mathbf{Sp}^\omega$ and $(\mathbf{Sp}^\omega)^{\Delta^1} = (\mathbf{Sp}^{\Delta^1})^\omega$ and closed under pushouts.

It is clear that $\mathrm{Ind}((\mathbf{Cat}^{\mathrm{ex}})^{\mathrm{fin}}) = \mathbf{Cat}^{\mathrm{ex}}$, so we are in position to apply our general strategy. For this, we first need to understand $\mathbf{Sp}(\mathbf{Cat}^{\mathrm{ex}}/C)$ for $C \in \mathbf{Cat}^{\mathrm{ex}}$.

To state the result, we need a bit of notation.

Notation 3.1. Let $f : C \to \mathrm{Ind}(C)$ be an exact functor. We let $\mathrm{Lace}(C,f)$ be the stable category informally described as pairs $(x, \alpha : x \to f(x))$ of "f-twisted endomorphisms" in C.

Given $f : D \to C$, we let $\Sigma^\infty_+ f$ denote the composite exact functor $f_! f^R : C \to \mathrm{Ind}(D) \to \mathrm{Ind}(C)$.

The following is proved e.g. in [HNS24]:

Theorem 3.2 ([HNS24, Theorem 1.1])**.** *There is an adjunction $\Sigma^\infty_+ \dashv \mathrm{Lace}$ between*

$$\mathbf{Cat}^{\mathrm{ex}}_{/C} \rightleftarrows \mathrm{Fun}^{\mathrm{ex}}(C, \mathrm{Ind}(C))$$

which witnesses $\mathrm{Fun}^{\mathrm{ex}}(C,\mathrm{Ind}(C))$ as $\mathbf{Sp}(\mathbf{Cat}^{\mathrm{ex}}/C)$.

Note that $\mathrm{Fun}^{\mathrm{ex}}(C, \mathrm{Ind}(C)) \simeq \mathrm{Ind}(C^{\mathrm{op}} \otimes C)$, so that if C is compact in $\mathbf{Cat}^{\mathrm{ex}}$, its Wall obstruction lives in $K_0(C^{\mathrm{op}} \otimes C)$, where here $\otimes$ is the idempotent-completed tensor product of stable categories.

In fact, the above description tells us that it is is the class of $[\mathrm{id}_C] \in K_0(C^{\mathrm{op}} \otimes C)$, so we are left with identifying the classes $[Y/C]$ for $Y \in (\mathbf{Cat}^{\mathrm{ex}})^{\mathrm{fin}}$. Since those Y's are generated by $\mathbf{Sp}^\omega$ and $(\mathbf{Sp}^\omega)^{\Delta^1}$ under pushouts, it suffices to calculate the relevant K-theory classes of those.

An exact functor $F : (\mathbf{Sp}^{\Delta^1})^\omega \to C$ is the data of an arrow $f : x \to y$ in C, and for such an arrow one can compute $\Sigma^\infty_+ F \in \mathrm{Fun}^{\mathrm{ex}}(C, \mathrm{Ind}(C))$: it is the cofiber of a certain map $\mathrm{map}(y/x, -) \otimes x \to \mathrm{map}(y, -) \otimes y$.

In particular, it lies in the stable subcategory generated by objects of the form $\mathrm{map}(z, -) \otimes x, z, x \in C$. Similarly, $\mathrm{map}(y, -) \otimes y$ is $\Sigma^\infty_+ F$ for the functor $F : \mathbf{Sp}^\omega \to C$ picking out y, so that these also lie in this stable subcategory.

Conversely, since $\mathrm{map}(y, -) \otimes y$ is obtainable, it follows that $\mathrm{map}(y/x, -) \otimes x$ also is by taking suitable cofibers, and since $x \to y$ is arbitrary, this means that the stable subcategory generated by the $\Sigma^\infty_+ F$'s for $F : \mathbf{Sp}^\omega \to C$ or $F : (\mathbf{Sp}^\omega)^{\Delta^1} \to C$ is exactly that spanned by the functors of the form $\mathrm{map}(z, -) \otimes x, z, x \in C$.

In particular, their K-theory classes are exactly the image of the canonical map

$$K_0(C^{\mathrm{op}}) \otimes K_0(C) \to K_0(C^{\mathrm{op}} \otimes C)$$

. We thus obtain Efimov's obstruction:

Corollary 3.3 (Efimov)**.** *Let C be a compact stable category. C is finite if and only if $[\mathrm{id}_C] \in K_0(C^{\mathrm{op}} \otimes C)$ is in the image of the canonical map $K_0(C^{\mathrm{op}}) \otimes K_0(C) \to K_0(C^{\mathrm{op}} \otimes C)$.*

Remark 3.4. C being compact implies that C is *smooth*, that is, that the coevaluation copairing

$$\mathbf{Sp} \xrightarrow{\mathrm{id}_{\mathrm{Ind}(C)}} \mathrm{Fun}^L(\mathrm{Ind}(C), \mathrm{Ind}(C)) \simeq \mathrm{Ind}(C^{\mathrm{op}}) \otimes \mathrm{Ind}(C)$$

preserves compact objects.

It is a classical fact that being compact implies being smooth. If the evaluation pairing preserves compact objects, equivalently if $\mathrm{map}_C(x, y)$ is compact for all $x, y \in C$, C is called proper. Being smooth and proper is equivalent to being dualizable in $\mathbf{Cat}^{\mathrm{perf}}$ ($\mathbf{Cat}^{\mathrm{ex}}$ up to idempotent-completion), and it is an instructive exercise, using the above obstruction to finiteness, to see that if C is smooth and proper *and* finite, then we have a Künneth isomorphism $K(C^{\mathrm{op}}) \otimes_{K(\mathbb{S})} K(C) \simeq K(C^{\mathrm{op}} \otimes C)$, and in fact, a general Künneth formula $K(D) \otimes_{K(\mathbb{S})} K(C) \simeq K(D \otimes C)$. In fact, more is true: it follows that if C is smooth and proper and finite, its additive motive $\mathcal{U}_{\mathrm{add}}(C)$ is a direct summand of some finite sum $\bigoplus_i \mathcal{U}_{\mathrm{add}}(\mathbb{S})$ (and in fact, is itself equivalent to such a finite sum).

So while smooth and proper categories abound in geometry (think of smooth and proper schemes), and finite categories abound in combinatorics (think of finite categories), the intersection is quite small.

Thus one can think of algebraic geometry as providing a big source of examples of compact non-finite categories[1].

[1] Some may object to this perspective on algebraic geometry.

Remark 3.5. It is easy to see that in all the previous constructions, considering the question of finiteness in $\mathbf{Cat}^{\mathrm{ex}}$ or in $\mathbf{Cat}^{\mathrm{perf}}$ yields more or less the same answer. Only for the notion of smoothness is there a stark difference.

3.2 A fun example

This subsection discusses a fun application of Lurie's obstruction based around the following easy observation:

Observation 3.6. The functor $C \mapsto C^{\simeq}, \mathbf{Cat}^{\mathrm{ex}} \to \mathbf{S}$ is conservative.

Remark 3.7. This is also true for additive, not-necessarily stable categories.

The proof is simple: one notices that for $x, y \in C$, the constructions $f \mapsto \begin{pmatrix} \mathrm{id}_x & f \\ 0 & \mathrm{id}_y \end{pmatrix}$ and $g \mapsto \mathrm{pr}_y \circ g \circ \mathrm{in}_x$ provide a natural retraction of $\mathrm{Aut}(x \oplus y)$ onto $\mathrm{Map}(x, y)$. Since retracts of equivalences are equivalences, if $f^{\simeq} : C^{\simeq} \to D^{\simeq}$ is fully faithful, then so is $f : C \to D$; and essential surjectivity is obvious.

The following is a rather surprising, if obvious, corollary:

Corollary 3.8. $\mathbf{Sp}^{\omega}$ *generates* $\mathbf{Cat}^{\mathrm{ex}}$ *under colimits, and hence* $(\mathbf{Cat}^{\mathrm{ex}})^{\omega}$ *under finite colimits and retracts.*

It raises the question: does it generate it $(\mathbf{Cat}^{\mathrm{ex}})^{\mathrm{fin}}$ under finite colimits ? We now have the tools to answer this: we already know that the obstruction is $[\mathrm{id}_{\mathbf{Sp}^{\Delta^1}}] \in K_0(\mathrm{Fun}^{\mathrm{ex}}(\mathbf{Sp}^{\Delta^1}, \mathbf{Sp}^{\Delta^1})^{\omega})$ and we also already know that we need to understand the subgroup of this generated by objects of the form $[\mathrm{map}(x, -) \otimes x], x \in \mathbf{Sp}^{\Delta^1}$.

It is also elementary to show that $K_0(\mathrm{Fun}^{\mathrm{ex}}(\mathbf{Sp}^{\Delta^1}, \mathbf{Sp}^{\Delta^1})^{\omega}) \to \hom(K_0(\mathbf{Sp}^{\Delta^1}), K_0(\mathbf{Sp}^{\Delta^1})) \cong \hom(\mathbb{Z}^2, \mathbb{Z}^2)$ is an isomorphism, sending id to id.

Thus it remains to analyze the effect on K-theory of $\mathrm{map}(x, -) \otimes x : \mathbf{Sp}^{\Delta^1} \to \mathbf{Sp}^{\Delta^1}$, for all $x \in \mathbf{Sp}^{\Delta^1}$. To make sense of the question, we need to pick an isomorphism $K_0(\mathbf{Sp}^{\Delta^1}) \cong \mathbb{Z}^2$. Let us pick the one that sends $[x \to y]$ to $\chi(x), \chi(y)$. This means that we have canonical generators given by the classes of $[0 \to \mathbb{S}]$ and $[\mathbb{S} \to 0]$, and we see that under our chosen isomorphisms, the matrix induced by a chosen $x = (x_0 \to x_1)$ is

$$\begin{pmatrix} \chi(x_0^{\vee})\chi(x_0) & \chi((x_1/x_0)^{\vee})\chi(x_0) \\ \chi(x_0^{\vee})\chi(x_1) & \chi((x_1/x_0)^{\vee})\chi(x_1) \end{pmatrix}$$

We use three facts to simplify this: first, χ is invariant under duality, $\chi(x_1/x_0) = \chi(x_1) - \chi(x_0)$, and finally, $\chi(x_0), \chi(x_1)$ can be arbitrary integers. Thus, we find that the general form of such a matrix is

$$\begin{pmatrix} n^2 & (m-n)n \\ nm & (m-n)m \end{pmatrix}$$

with $m, n \in \mathbb{Z}$. Adding the first column to the second yields the following matrix:

$$\begin{pmatrix} n^2 & mn \\ nm & m^2 \end{pmatrix}$$

where the off-diagonal entries are equal to one another.

On the other hand, the identity matrix is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and after the same column-operation, it becomes $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ where the off-diagonal entries are *not* equal. Thus, the latter cannot be in the subgroup generated by the former and a fortiori, the class of id cannot be in the subgroup generated by the $[\mathrm{map}(x,-)\otimes x]$, which proves, ultimately, that

Corollary 3.9. $(\mathbf{Sp}^{\Delta^1})^\omega$ *is not generated under finite colimits by* $\mathbf{Sp}^\omega$ *in* $\mathbf{Cat}^{\mathrm{ex}}$.

4 Example III: separable commutative algebras

In this section we discuss separable commutative algebras, which are a generalization of étale algebras.

We refer to [Ram23] for a general theory of separable algebras, for now let us simply give the simplest version of the definition:

Definition 4.1. Let C be a symmetric monoidal stable category, and $A \in \mathrm{CAlg}(C)$ a commutative algebra. A is called separable if there is an idempotent e in $A\otimes A$ such that the multiplication map witnesses $(A\otimes A)[e^{-1}] \simeq A$.

A practical consequence of this is that A is 0-cotruncated in $\mathrm{CAlg}(C)$, and hence its cotangent complex L_A vanishes.

We also recall the following calculation:

Theorem 4.2 ([Lur12, Theorem 7.3.4.13])**.** *Let C be a presentably symmetric monoidal stable category and $A \in \mathrm{CAlg}(C)$. The construction of trivial square zero extensions is implemented by an equivalence $\mathbf{Sp}(\mathrm{CAlg}(C)_{/A}) \simeq \mathbf{Mod}_A(C)$. The left adjoint $\Sigma^\infty_+ : \mathrm{CAlg}(C)_{/A} \to \mathbf{Mod}_A(C)$ is given by the cotangent complex.*

In particular, we get:

Corollary 4.3. *Let (C, A) be as above, and assume now that C is compactly generated and that $A \in \mathrm{CAlg}(C)^\omega$. The Lurie–Wall obstruction class $[A/A] \in K_0((\mathrm{CAlg}(C)_{A//A})^\omega)$ vanishes. In particular, it lies in any subgroup.*

Remark 4.4. In fact, we do not need such precise information as the calculation of $\mathbf{Sp}(\mathrm{CAlg}(C)_{/A})$. The 0-cotruncatedness of A itself guarantees that its image under any left adjoint to a stable category is 0.

As a further corollary, we see that for any reasonable notion of "finite" commutative algebra, A is finite. For example, suppose that C is generated by a collection of compact objects $C_0 \subset C$, and define "finite" to mean "in the smallest subcategory containing the free algebras on objects of C_0 and closed under pushouts and finite coproducts"; then A is finite in that sense.

Remark 4.5. For simplicity, we worked with commutative algebras, but all the arguments have an obvious analogue for $\mathbb{E}_k$-algebras, $k \geq 2$. We do warn the reader that for $k = 1$, the situation is more delicate because A is not necessarily 0-cotruncated, cf. [Ram23, Example 3.36].

A A proof of Lurie's Wall obstruction

We begin with a proof of Thomason's obstruction which we recall below:

Proposition A.1. *Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full stable subcategory of the stable category $\mathcal{C}$. Suppose that every object of $\mathcal{C}$ is a summand of some object in $\mathcal{C}_0$. In this case:*

1. *The morphism $K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ is injective.*
2. *If $X \in \mathcal{C}$ then $[X]$ is in the image of $K_0(\mathcal{C}_0)$ if and only if $X \in \mathcal{C}_0$.*

Lurie's proof uses the following variant:

Definition A.2. Let $\mathcal{C}$ be a semi-additive category. In this case, $\pi_0(\mathcal{C}^{\simeq})$ acquires a canonical commutative monoid structure via $\oplus$. We let $K_0^{\oplus}(\mathcal{C})$ denote the group completion of this monoid.

In particular, for stable $\mathcal{C}$ there is a canonical surjection $K_0^{\oplus}(\mathcal{C}) \to K_0(\mathcal{C})$. We let $I(\mathcal{C})$ denote its kernel.

Lemma A.3. *In the situation of proposition 1.8, the morphism $I(\mathcal{C}_0) \to I(\mathcal{C})$ is surjective.*

Proof. Clearly, $I(\mathcal{C})$ is generated by $[X'] - [X] - [X'']$ for cofiber sequences $X \to X' \to X''$.

If we have such a cofiber sequence, we can find Z, Z'' such that $X \oplus Z, X'' \oplus Z'' \in \mathcal{C}_0$, and then we have a fiber sequence in $\mathcal{C}$ of the form $X \oplus Z \to X' \oplus Z \oplus Z'' \to X'' \oplus Z''$ which shows that $[X' \oplus Z \oplus Z''] - [X \oplus Z] - [X'' \oplus Z'']$ is in the image of $I(\mathcal{C}_0)$.

But in $K_0^{\oplus}(\mathcal{C})$, this is equal to $[X'] + [Z] + [Z''] - [X] - [Z] - [X''] - [Z''] = [X'] - [X] - [X'']$. $\square$

Proof of proposition 1.8. The monoid $\pi_0(\mathcal{C}^{\simeq})$ is abelian, so its group completion is explicit, and in particular if $[X] - [Y] \in K_0^{\oplus}(\mathcal{C}_0)$ goes to 0 in $K_0^{\oplus}(\mathcal{C})$, there exists $Z \in \mathcal{C}$ such that $X \oplus Z \simeq Y \oplus Z$. But Z is a summand of an object Z' in $\mathcal{C}_0$, so the same happens in $\mathcal{C}_0$, and $[X] - [Y] = 0$.

In particular, the surjectivity of $I(\mathcal{C}_0) \to I(\mathcal{C})$ shows (by some version of the 5-lemma) that $K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ is injective.

For the image, we look at the following diagram in more detail:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & I(\mathcal{C}_0) \arrow[r] \arrow[d] & K_0^{\oplus}(\mathcal{C}_0) \arrow[r] \arrow[d] & K_0(\mathcal{C}_0) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & I(\mathcal{C}) \arrow[r] & K_0^{\oplus}(\mathcal{C}) \arrow[r] & K_0(\mathcal{C}) \arrow[r] & 0
\end{tikzcd}$$

Suppose $X \in \mathcal{C}$ and $[X]$ is in the image of $K_0(\mathcal{C}_0)$, say the image of α. Lift α to $\tilde{\alpha} \in K_0^{\oplus}(\mathcal{C}_0)$, then in $K_0^{\oplus}(\mathcal{C})$, $[X]$ and the image of $\tilde{\alpha}$ differ by something in $I(\mathcal{C})$, which is in the image of $I(\mathcal{C}_0)$, so up to changing $\tilde{\alpha}$, we have that $[X] \in K_0^{\oplus}(\mathcal{C})$ is in the image of $K_0^{\oplus}(\mathcal{C}_0)$. This means that there exists $Y \in \mathcal{C}_0$ and $Z \in \mathcal{C}$ such that $X \oplus Z \simeq Y \oplus Z$. Up to changing Z, we may assume $Z \in \mathcal{C}_0$, i.e. there exists $Z \in \mathcal{C}_0$ such that $X \oplus Z \in \mathcal{C}_0$.

But then we have a cofiber sequence $Z \to X \oplus Z \to X$ which shows that $X \in \mathcal{C}_0$ as well. $\square$

Let us now recall Lurie's obstruction theorem. We begin with the set-up: Let $\mathcal{C}$ be a small, pointed, finitely cocomplete category. We embed $\mathcal{C}$ in its Ind-completion, $\mathrm{Ind}(\mathcal{C})$, and in the compacts thereof, i.e. the idempotent completion of $\mathcal{C}$. We will view these as inclusions: $\mathcal{C} \subset \overline{\mathcal{C}} \subset \mathrm{Ind}(\mathcal{C})$, where $\overline{\mathcal{C}} := \mathrm{Ind}(\mathcal{C})^\omega$ is the idempotent-completion of $\mathcal{C}$.

Let $SW(-)$ be the left adjoint to the forgetful functor from stable categories to finitely cocomplete pointed categories, i.e. the Spanier–Whitehead stabilization.

We will need at some point a technical lemma:

Lemma A.4. *The canonical morphism $SW(\overline{\mathcal{C}}) \to \overline{SW(\mathcal{C})}$ is an equivalence.*

Proof. This canonical map is determined by $\mathcal{C} \to SW(\mathcal{C}) \to \overline{SW(\mathcal{C})}$: this induces $\overline{C} \to \overline{SW(\mathcal{C})}$ because the target is idempotent complete, and then $SW(\overline{\mathcal{C}}) \to \overline{SW(\mathcal{C})}$ because the target is stable, as the idempotent-completion of a stable category.

In particular, because both have a universal property below $\mathcal{C}$, the claim boils down to checking that $SW(\overline{\mathcal{C}})$ is idempotent complete. But this is because an idempotent can be split after using only finitely many pieces of data, namely a homotopy $r^2 \simeq r$ and a homotopy witnessing that the two equivalences $r^3 \simeq r$ are the same. In particular, in the filtered colimit defining $SW(\overline{\mathcal{C}})$, this data appears after a finite amount of steps in $\overline{\mathcal{C}}$, and can therefore be split therein. $\square$

Recall now the definition of $\mathcal{C}$-finite maps:

Definition A.5. A map $f : X \to Y$ in $\mathrm{Ind}(\mathcal{C})$ is $\mathcal{C}$-finite if it can be written as a composition $X = X_0 \to X_1 \to ... \to X_n = Y$ where each $X_i \to X_{i+1}$ fits in a pushout square:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & D \arrow[d] \\ X_i \arrow[r] & X_{i+1} \end{tikzcd}$$

where $C, D \in \mathcal{C}$.

Here are two easy properties of $\mathcal{C}$-finite morphisms in the following lemma, whose proof we leave to the reader:

Lemma A.6. *1. Equivalences are $\mathcal{C}$-finite.*

2. $X \in \mathrm{Ind}(\mathcal{C})$ belongs to $\mathcal{C}$ if and only if $\emptyset \to X$ is $\mathcal{C}$-finite.

A key point in the proof will be the following less obvious lemma:

Lemma A.7. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two maps. Suppose f and $g \circ f$ are $\mathcal{C}$-finite. Then g is $\mathcal{C}$-finite.*

Proof. By induction, we may reduce to the case where f is of the form $X \to X \coprod_C D$ where $C \to D$ is a map in $\mathcal{C}$.

In this case, write g as $X \coprod_C D \to Z \coprod_C D \to Z$, where the first map is manifestly $\mathcal{C}$-finite as a pushout of $g \circ f$, so we are left with the latter.

But the latter can be viewed as a pushout of the fold-map $D \coprod_C D \to D$ along $D \coprod_C D \to Z \coprod_C D$. $\square$

We recall the following definition of where the obstruction lives:

Definition A.8. Let $X \in \mathrm{Ind}(\mathcal{C})$. We let $\mathrm{Ind}(\mathcal{C})_{X//X} := (\mathrm{Ind}(\mathcal{C})_{/X})_{X/} \simeq (\mathrm{Ind}(\mathcal{C})_{X/})_{/X}$, and $\mathcal{C}[X] \subset \mathrm{Ind}(\mathcal{C})_{X//X}$ be the full subcategory spanned by the triangles $X \to Y \to X$ such that $X \to Y$ is $\mathcal{C}$-finite.

The following is stated but not spelled out in Lurie's notes.

Lemma A.9. *$\mathcal{C}[X] \subset \mathrm{Ind}(\mathcal{C})_{X//X}$ forms a set of compact generators, i.e. its objects are compact and the induced $\mathrm{Ind}(\mathcal{C}[X]) \to \mathrm{Ind}(\mathcal{C})_{X//X}$ is an equivalence.*

The compact objects of $\mathrm{Ind}(\mathcal{C})_{X//X}$ are more generally those for which $X \to Y$ is compact in $\mathrm{Ind}(\mathcal{C})_{X/}$.

Proof. Let $\mathcal{D}$ be a presentable compactly generated category and $Z \in \mathcal{D}$.

Compact objects in $\mathcal{D}_{/Z}$ are precisely those whose underlying object is compact. One inclusion is obtained from the fact that colimits in $\mathcal{D}_{/Z}$ are computed underlying and that pullbacks commute with filtered colimits. The other is obtained by observing that the $d \to Z, d \in \mathcal{D}^{\omega}$ form compact generators of $\mathcal{D}_{/Z}$, closed under retracts, so any compact object is one of them.

This shows the second half of the statement. For the first half, we start by proving that if $X \to Y$ is $\mathcal{C}$-finite, then it is compact in $\mathrm{Ind}(\mathcal{C})_{X/}$.

For this, we make the following elementary observations, which we leave to the reader:

1. If $c, d \in \mathcal{D}$ are compact and $c \to d$ is an arrow, then it is compact in $\mathcal{D}_{c/}$.

2. If $x_0 \to x_1 \to x_2$ are arrows, $x_0 \to x_1$ is compact in $\mathcal{D}_{x_0/}$, $x_1 \to x_2$ is compact in $\mathcal{D}_{x_1/}$, then $x_0 \to x_2$ is compact in $\mathcal{D}_{x_0/}$.

3. If $x \to y$ is compact in $\mathcal{D}_{x/}$, then the pushout of this map along $x \to x'$ is compact in $\mathcal{D}_{x'/}$.

In particular, the class of arrows $X \to Y$ which are compact in $\mathrm{Ind}(\mathcal{C})_{X/}$ is closed under pushouts and composition, and contains $\mathcal{C}^{\Delta^1}$, so it contains all $\mathcal{C}$-finite arrows. This proves that the $X \to Y \to X$'s such that $X \to Y$ is $\mathcal{C}$-finite are compact in $\mathrm{Ind}(\mathcal{C})_{X//X}$.

Furthermore, we have seen also that the ones where we only assume $X \to Y$ to be compact in $\mathrm{Ind}(\mathcal{C})_{X/}$ generate the whole thing, so it suffices to show that any such is a retract of a $\mathcal{C}$-finite one.

For this, it suffices to show that the $\mathcal{C}$-finite maps $X \to Y$ are closed under finite colimits, and that any object in $\mathrm{Ind}(\mathcal{C})_{X/}$ is generated under colimits by $\mathcal{C}$-finite maps.

The latter is easy : $X \to Y$ is the pushout of $X \to X \coprod Y$ and $X \to X$ along $X \to X \coprod X$ (pushouts are computed underlying), so it suffices to prove that all of these are generated under colimits. $X \to X$ is $\mathcal{C}$-finite, $X \to X \coprod X$ is of the form $X \to X \coprod Y$, and, writing Y as a filtered colimit of objects in $\mathcal{C}$, we find that the latter is a filtered colimit of $X \to X \coprod c$'s, $c \in \mathcal{C}$, which are all $\mathcal{C}$-finite.

So we are left with the former. To get all finite colimits, it suffices to prove that the initial object is $\mathcal{C}$-finite, and that they are closed under pushouts. The initial object is $X \to X$ which is clearly $\mathcal{C}$-finite.

Now consider a diagram as follows, where the arrows from X are $\mathcal{C}$-finite:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & Y \\ W & Z \arrow[l] \arrow[u] \end{tikzcd}$$

By the 2-out-of-3 property from lemma A.7, we find that $Z \to Y$ is also $\mathcal{C}$-finite. In particular, $W \to W \coprod_Z Y$ is also $\mathcal{C}$-finite, and so $X \to W \to W \coprod_Z Y$ is $\mathcal{C}$-finite, as a composite of $\mathcal{C}$-finite maps (note that the pushout is computed underlying). $\square$

In particular, the compact objects in $\mathrm{Ind}(\mathcal{C})_{X//X}$ are the idempotent completion $\overline{\mathcal{C}[X]}$ of $\mathcal{C}[X]$.

Notation A.10. Let Y be a compact object of $\mathrm{Ind}(\mathcal{C})$ equipped with a map to X. Then $X \to X \coprod Y \to X$ is in $\overline{\mathcal{C}[X]}$ - we (abusively) denote its class in $K_0(\overline{\mathcal{C}[X]})$ by $[Y/X]$. For instance, if X is compact, and we take $\mathrm{id}_X : X \to X$, we get $[X/X]$.

We now recall Lurie's generalized Wall finiteness obstruction theorem :

Theorem A.11. *Let $\mathcal{C}$ be a small finitely cocomplete category, and $X \in \mathrm{Ind}(\mathcal{C})$. Then:*

1. *The canonical map $\alpha : K_0(\mathcal{C}[X]) \to K_0(\overline{\mathcal{C}[X]})$ is injective;*
2. *Suppose X is compact. Then $[X/X] \in K_0(\overline{\mathcal{C}[X]})$ is in the image of α if and only if X is in $\mathcal{C}$.*

Proof. Note that $K_0(\mathcal{C}[X]) \to K_0(\overline{\mathcal{C}[X]})$ gets identified with $K_0(SW(\mathcal{C}[X])) \to K_0(\overline{SW(\mathcal{C}[X])})$ so it is injective by Thomason's theorem.

Furthermore, the "if" part of 2. is obvious, so we now move on to proving "only if", and assume $[X/X]$ is in the image of α.

For simplicity of writing, we assume that X is terminal in $\mathrm{Ind}(\mathcal{C})$. Up to changing $\mathcal{C}$ with $\mathcal{C}_{/X}$ and thus $\mathrm{Ind}(\mathcal{C})$ with $\mathrm{Ind}(\mathcal{C})_{/X}$, this is without loss of generality.

In this case, $\mathrm{Ind}(\mathcal{C})_{*//*} = \mathrm{Ind}(\mathcal{C})_*$ and $\overline{\mathcal{C}[*]} = \overline{\mathcal{C}}_*$, pointed objects in $\overline{\mathcal{C}}$, and $\mathcal{C}[*] \subset \overline{\mathcal{C}}_*$ is the full subcategory spanned by $\mathcal{C}$-finite maps $* \to Y$.

By assumption, $*$ is a retract of some $C \in \mathcal{C}$, so that we get a map $* \to C$ for some C. In particular, we have a cofiber sequence $* \coprod * \to * \coprod C \to C$ in $\overline{\mathcal{C}}_*$. The first term is $[*/*]$, which we assumed is in the image of α, and the second term is $\mathcal{C}$-finite, so also in the image of α.

It follows that the class of C in $K_0(\overline{\mathcal{C}}_*)$ is in the image of α.

By the discussion at the beginning of the proof, and by the stable case, it shows that the image of C in $SW(\overline{\mathcal{C}}_*)$ is in the image of $SW(\mathcal{C}[*])$, i.e. for some $n \geq 0$, $\Sigma^n C$ in $\overline{\mathcal{C}}_*$ is in fact in $\mathcal{C}[*]$: the map $* \to \Sigma^n C$ is $\mathcal{C}$-finite. It follows by the 2-out-of-3 property (lemma A.7) that $\Sigma^n C \to *$ is $\mathcal{C}$-finite.

We want to lower n, because if $n = 0$, then $C \to *$ is $\mathcal{C}$-finite, and C is in $\mathcal{C}$ and so $* \in \mathcal{C}$. So we prove:

If $n \geq 0$ and there is a $C \in \mathcal{C}$ with a basepoint $* \to C$ for which $\Sigma^{n+1} C \to *$ is $\mathcal{C}$-finite, then there is a D with a basepoint $* \to D$ for which $\Sigma^n D \to *$ is $\mathcal{C}$-finite.

i.e. we can lower n until it reaches 0, in which case we are done. For this, we are going to use two descriptions of iterated suspensions.

Consider such a C, and let D be any object of $\mathcal{C}$ with a basepoint (e.g. C, but it is less confusing to keep it notationally distinct).

We then have a pushout diagram, where $\otimes$ denotes the tensoring of $\mathcal{C}$ with finite spaces:

$$\begin{tikzcd}
C \coprod_{S^n \otimes C} S^n \otimes \mathrm{pt} \arrow[r] \arrow[d] & C \coprod_{S^n \otimes C} S^n \otimes D \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathrm{pt} \coprod_{S^n \otimes \mathrm{pt}} S^n \otimes D
\end{tikzcd}$$

Note that $C \coprod_{S^n \otimes C} S^n \otimes \mathrm{pt} \simeq \Sigma^{n+1} C$, and so the left vertical map is $\mathcal{C}$-finite - therefore the right vertical map is so too.

But the upper right hand corner belongs to $\mathcal{C}$, so that the bottom right hand corner does too.

Now observe that we also have a pushout diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & \mathrm{pt} \coprod_{S^n \otimes \mathrm{pt}} S^n \otimes D \arrow[d] \\
\mathrm{pt} \arrow[r] & \Sigma^n D
\end{tikzcd}$$

and the top map is a map between objects of $\mathcal{C}$, so $\mathcal{C}$-finite, therefore the bottom map is $\mathcal{C}$-finite. Again by 2-out-of-3, $\Sigma^n D \to *$ is $\mathcal{C}$-finite, which is what we wanted to prove.

□

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