Dualizability of presentable categories

The goal of this note is to understand a bit about dualizability of presentable categories. We being by proving the well-known fact that a presentable (stable) category is dualizable if and only if it is a retract of a (stably) compactly generated category, and we then prove that $\mathbf{Sh}(X;\mathbf{Sp})$, although dualizable in many cases, is rarely fully dualizable.

In particular, it is never the case for, say, a manifold of positive dimension.

Lemma 0.1. *Let $\mathbf{C}, \mathbf{D}$ be presentable categories, and $x \in \mathbf{C} \otimes \mathbf{D}$. There exists a (small) diagram $f : I \to \mathbf{C} \times \mathbf{D}$ such that $x \simeq \operatorname{colim}_I p \circ f$, where $p : \mathbf{C} \times \mathbf{D} \to \mathbf{C} \otimes \mathbf{D}$ is the canonical map.*

For this lemma, we begin with two sub-lemmas:

Lemma 0.2. *If $\mathbf{D} \to \mathbf{D}'$ is an accessible localization, at the set of arrows W, then $\mathbf{C} \otimes \mathbf{D} \to \mathbf{C} \otimes \mathbf{D}'$ is also an accessible localization at the set of arrows $\mathbf{C}_0 \otimes W$ where $\mathbf{C}_0 \subset \mathbf{C}$ is any generating set of objects.*

Proof. Let $\mathbf{E}$ be a presentable category. We let $\operatorname{Fun}^L_W(\mathbf{D}, \mathbf{E})$ denote the full subcategory of $\operatorname{Fun}^L(\mathbf{D}, \mathbf{E})$ on functors that invert W. This is equivalent to $\operatorname{Fun}^L(\mathbf{D}', \mathbf{E})$ under restriction along $\mathbf{D} \to \mathbf{D}'$.

We have

$$\operatorname{Fun}^L(\mathbf{C} \otimes \mathbf{D}', \mathbf{E}) \simeq \operatorname{Fun}^L(\mathbf{C}, \operatorname{Fun}^L(\mathbf{D}', \mathbf{E})) \simeq \operatorname{Fun}^L(\mathbf{C}, \operatorname{Fun}^L_W(\mathbf{D}, \mathbf{E})) \simeq \operatorname{Fun}^L_{\mathbf{C}_0 \otimes W}(\mathbf{C} \otimes \mathbf{D}, \mathbf{E})$$

which proves the claim. $\square$

Tim Campion made me realize that the following was easy:

Lemma 0.3. *Lemma 0.1 holds if $\mathbf{C}, \mathbf{D}$ are presheaf categories.*

Proof. Say $\mathbf{C} = \mathbf{Psh}(\mathbf{C}_0), \mathbf{D} = \mathbf{Psh}(\mathbf{D}_0)$. Then $\mathbf{C}_0 \times \mathbf{D}_0 \to \mathbf{C} \times \mathbf{D} \to \mathbf{C} \otimes \mathbf{D}$ witnesses the target as the free cocompletion of the domain, so it is identified with $\mathbf{C}_0 \times \mathbf{D}_0 \to \mathbf{Psh}(\mathbf{C}_0 \times \mathbf{D}_0)$.

In particular, any object in $\mathbf{C} \otimes \mathbf{D}$ is the *canonical* colimit of a (small) diagram with values in $\mathbf{C}_0 \times \mathbf{D}_0$ so a fortiori in $\mathbf{C} \times \mathbf{D}$. $\square$

Proof of lemma 0.1. Write $\mathbf{C}$ (resp. $\mathbf{D}$) as a localization of some presheaf category $\mathbf{Psh}(\mathbf{C}_0)$ (resp. $\mathbf{Psh}(\mathbf{D}_0)$).

Then by iterating lemma 0.2, $\mathbf{Psh}(\mathbf{C}_0) \otimes \mathbf{Psh}(\mathbf{D}_0) \to \mathbf{C} \times \mathbf{D}$ is a localization as well, so if $x \in \mathbf{C} \otimes \mathbf{D}$ we can view it as the image of some element in $\mathbf{Psh}(\mathbf{C}_0) \otimes \mathbf{Psh}(\mathbf{D}_0)$ which, by lemma 0.3, is the colimit of some diagram with values in $\mathbf{Psh}(\mathbf{C}_0) \times \mathbf{Psh}(\mathbf{D}_0)$.

But the following diagram commutes, where the left hand vertical functor preserves colimits:

$$\begin{tikzcd} \mathbf{Psh}(\mathbf{C}_0) \times \mathbf{Psh}(\mathbf{D}_0) \arrow[r] \arrow[d] & \mathbf{Psh}(\mathbf{C}_0) \otimes \mathbf{Psh}(\mathbf{D}_0) \arrow[d] \\ \mathbf{C} \times \mathbf{D} \arrow[r] & \mathbf{C} \otimes \mathbf{D} \end{tikzcd}$$

which proves the claim. □

Corollary 0.4. *Let $\mathcal{M}$ be a mode (e.g. $\mathbf{S}$ or $\mathbf{Sp}$), and $\mathbf{C} \in \mathbf{Mod}_{\mathcal{M}}(\mathbf{Pr}^L)$, dualizable as an $\mathcal{M}$-module. Then $\mathbf{C}$ is a retract in $\mathbf{Pr}^L$ of some $\mathcal{M}$-valued presheaf category.*

If $\mathcal{M}$ is compactly generated, then $\mathbf{C}$ is therefore a retract of a compactly generated $\mathcal{M}$-module.

Proof. Let $\mathbf{C}^\vee$ be the dual of $\mathbf{C}$. Then $\mathbf{C} \otimes \mathbf{C}^\vee = \mathrm{Fun}^L(\mathbf{C}, \mathbf{C})$. Using lemma 0.1, let $f : I \to \mathbf{C} \times \mathrm{Fun}^L(\mathbf{C}, \mathcal{M})$ be a diagram whose colimit is $\mathrm{id}_{\mathbf{C}} \in \mathrm{Fun}^L(\mathbf{C}, \mathbf{C})$.

We consider the two projections of f, call them $p : I \to \mathbf{C}, q : I \to \mathrm{Fun}^L(\mathbf{C}, \mathcal{M})$. We then define $\mathbf{C} \to \mathrm{Fun}(I, \mathcal{M}), c \mapsto (i \mapsto q_i(c))$, and $\mathrm{Fun}(I, \mathcal{M}) \to \mathbf{C}, H \mapsto \mathrm{colim}_I(H(i) \otimes p(i))$.

Note that in this second expression, $\otimes$ denotes the action of $\mathcal{M}$ on $\mathbf{C}$. It is clear that both preserve colimit, and it is clear by design that the composite is equivalent to $\mathrm{id}_{\mathbf{C}}$, so $\mathbf{C}$ is a retract in $\mathbf{Pr}^L$ (equivalently in $\mathbf{Mod}_{\mathcal{M}}(\mathbf{Pr}^L)$) of $\mathrm{Fun}(I, \mathcal{M})$, as claimed. □

Corollary 0.5. *Let $\mathcal{M}$ be a mode as above, in which we further assume that the unit is compact; and let $\mathbf{C} \in \mathbf{Mod}_{\mathcal{M}}(\mathbf{Pr}^L)$ be dualizable as above. We further assume that the right adjoint of the coevaluation map $\mathcal{M} \to \mathrm{Fun}^L(\mathbf{C}, \mathbf{C})$ preserves filtered colimits.*

In this situation, there is a finite set of objects in $\mathbf{C}$ such that the smallest full subcategory of $\mathbf{C}$ containing it, closed under colimits and $\mathcal{M}$-tensors is the whole of $\mathbf{C}$.

Proof. This is essentially obvious: the assumptions impose that the image of the unit (i.e. $\mathrm{id}_{\mathbf{C}} \in \mathrm{Fun}^L(\mathbf{C}, \mathbf{C})$) is compact.

In particular, filtering I by finite simplicial sets, $I = \mathrm{colim}_{J \subset I, \text{ finite}} J$, so that in $\mathrm{id}_{\mathbf{C}} = \mathrm{colim}_I p(i) \boxtimes q(i)$ we find that $\mathrm{id}_{\mathbf{C}}$ is a retract of some $\mathrm{colim}_J p(j) \boxtimes q(j)$ for J a finite simplicial set.

In particular, for any $x \in \mathbf{C}$, x is a retract of $\mathrm{colim}_J(p(j) \otimes q(j)(x))$, which shows that the smallest full subcategory of $\mathbf{C}$ closed under colimits and $\mathcal{M}$-tensors containing the $p(j)$'s is $\mathbf{C}$. □

Observation 0.6. The proof of the existence of the diagram from lemma 0.1 is very explicit. In particular, we can strengthen all the previous results by saying that if we know a generating set S of $\mathbf{C}$ under colimits, then we can pick the diagram to have values in the full subcategory spanned by S.

In particular, in the above corollary, we find that for any such generating set S, we can pick the $p(j)$'s to be in S. This will be important for the next result.

Corollary 0.7. *Let X be a topological space, and consider the category of sheaves of spectra on X (as a topological space, not presheaves on its homotopy type), $\mathbf{Sh}(X)$.*

Suppose that $\mathbf{Sh}(X)$ is dualizable over $\mathbf{Sp}$ and that the coevaluation map $\mathbf{Sp} \to \mathrm{Fun}^L(\mathbf{Sh}(X), \mathbf{Sh}(X))$ has a colimit-preserving right adjoint.

In this situation, there exists a non-empty open set $V \subset X$ such that the global sections functor $\mathbf{Sh}(V) \to \mathbf{Sp}$ is conservative.

Remark 0.8. I forget the precise hypotheses, but $\mathbf{Sh}(X)$ is often dualizable. What this corollary shows is that $\mathbf{Sh}(X)$ is rarely *fully dualizable*. In particular, it never is if X is a manifold.

Proof. We apply corollary 0.5 together with observation 0.6, for $\mathcal{M} = \mathbf{Sp}, \mathbf{C} = \mathbf{Sh}(X)$.

The generating set we choose for $\mathbf{Sh}(X)$ under colimits is the image under sheafification of $\mathbf{Psh}(X; \mathbf{Sp}) \to \mathbf{Sh}(X)$ of the $\Sigma^n \mathbb{S}[h_U], n \in \mathbb{Z}$, where h_U is the Yoneda image of the open $U \subset X$, $\mathbb{S}[-]$ is left adjoint to $\Omega^\infty : \mathbf{Sp} \to \mathbf{S}$. We use the same notation for their sheafification, hoping no confusion arises.

Corollary 0.5 and observation 0.6 together imply that there is a finite family of opens U_i and integers n_i such that the smallest stable subcategory of $\mathbf{Sh}(X)$ containg the $\Sigma^n \mathbb{S}[h_{U_i}]$ and closed under colimits is $\mathbf{Sh}(X)$.

In particular the smallest stable subcategory of $\mathbf{Sh}(X)$ containing the $\mathbb{S}[h_{U_i}]$ and closed under colimits is all of $\mathbf{Sh}(X)$.

In particular, the functor $\mathbf{Sh}(X) \to \prod_i \mathbf{Sp}, X \mapsto (\mathrm{Map}(\mathbb{S}[h_{U_i}], X))_i$ is conservative, where Map denotes the mapping spectrum in $\mathbf{Sh}(X)$.

Using the sheafification-forgetful, $\mathbb{S}[-] \dashv \Omega^\infty$ adjunctions and finally the Yoneda lemma, we see that this functor is just $X \mapsto (X(U_i))_i$.

Now let V be a minimal non-empty intersection of the U_i's. For each U_i, either $V \subset U_i$ or $V \cap U_i = \emptyset$. We claim that V has the desired property, i.e. it has no proper non-empty open subset (V is of course open and non-empty by design).

Let $\mathcal{F}$ be a sheaf on V, and let $i_V : V \to X$ denote the inclusion. Then $(i_V)_* \mathcal{F}(U_i) = \mathcal{F}(V \cap U_i) = \mathcal{F}(V)$ if $V \subset U_i$ or 0 if $V \cap U_i = \emptyset$.

So if $\mathcal{F}$ is any sheaf on V such that $\mathcal{F}(V) = 0$, then $\mathcal{F} = 0$ by conservativity: this proves the claim. □

Remark 0.9. Any open subset of a manifold is a manifold. We prove below that global sections are never conservative on a manifold.

Lemma 0.10. *The condition on a topological space Y "the global sections functor $\mathbf{Sh}(Y) \to \mathbf{Sp}$ is conservative" is hereditary, i.e. any subspace of Y also has that property.*

Proof. Let Z be a subspace of Y and let $\mathcal{F}$ be a sheaf on Z such that $\mathcal{F}(Z) = 0$.

We then have $i_* \mathcal{F}(Y) = \mathcal{F}(Z) = 0$ so $i_* \mathcal{F} = 0$, and so, because any open W of Z is of the form $U \cap Z$ for some open $U \subset Y$, we find $\mathcal{F}(W) = \mathcal{F}(U \cap Z) = i_* \mathcal{F}(U) = 0$, as claimed. □

Corollary 0.11. *Global sections is not a conservative functor $\mathbf{Sh}(M) \to \mathbf{Sp}$ for any manifold M of dimension ≥ 2.*

Remark 0.12. We give a separate argument for 1-dimensional manifolds below, as it is slightly more complicated.

Note that for 0-dimensional manifolds, global sections are conservative, as they are just a product.

Proof. If M is of dimension ≥ 2, then by picking a euclidean open inside of M, we see that there is a subspace of M homeomorphic to S^1. By the previous lemma, it suffices to deal with S^1.

But then the composition $\mathrm{Fun}(S^1, \mathbf{Sp}) \to \mathbf{Sh}(S^1) \to \mathbf{Sp}$ is equivalent to $\lim_{S^1}$, where the first functor is the inclusion of locally constant sheaves.

In particular, because the first functor is fully faithful, it is conservative: it suffices to prove that the composite is not conservative. But, for instance, $\mathbb{Q}$ equipped with the self-automorphism 2 (viewed as a functor $S^1 = B\mathbb{Z} \to \mathbf{Sp}$) has a trivial limit, because this limit over S^1 is the fiber of $1-$the automorphism, i.e. $1 - 2 = -1$, which is invertible. $\square$

Corollary 0.13. *Global sections is not a conservative functor* $\mathbf{Sh}(M) \to \mathbf{Sp}$ *for any manifold* M *of dimension* ≥ 1*.*

Proof. Just as above, it suffices to deal with the case of $\mathbb{R}$.

Now observe that global sections is right adjoint to the "constant sheaf functor", and in fact, because $\mathbb{R}$ is contractible, the global sections of a constant sheaf are just the value of that sheaf; more precisely the unit of this adjunction is an equivalence. Therefore it is a colocalization, and if it were conservative, it would be an equivalence $\mathbf{Sh}(\mathbb{R}) \simeq \mathbf{Sp}$.

But of course, this is not the case, as there are non-constant sheaves on $\mathbb{R}$. $\square$

Corollary 0.14. *For any manifold of dimension* ≥ 1*,* $\mathbf{Sh}(M)$ *is not fully dualizable over* $\mathbf{Sp}$.

We conclude with some examples of topological spaces for which global sections is a conservative functor.

Example 0.15. Let X be an extremally disconnected set, i.e. a topological space where every open subset is closed. The global sections functor is conservative. Indeed, if $\mathcal{F}(X) = 0$, then for any open U with open complement V, $\mathcal{F}(U)$ is a retract of $\mathcal{F}(U)\times\mathcal{F}(V) \simeq \mathcal{F}(X) = 0$.

Example 0.16. Let $X = \mathbb{N}\cup\{\infty\}$. In a similar way as before, global sections is conservative. This time it is not true that every open is closed (consider $\mathbb{N} = \bigcup_n \{n\}$). However, X has a basis of open sets, each of which is closed - namely, the $\{n\}$ and $[n, \infty], n \in \mathbb{N}$.

Remark 0.17. $\mathbf{Sh}(X)$ only depends of course on the poset of opens $\mathcal{O}(X)$.

Therefore, any map of topological spaces $Y \to Z$ which induces an isomorphism $\mathcal{O}(Z) \to \mathcal{O}(Y)$ preserves and reflects the property that global sections be conservative.

As an example, if $\pi : Y \to Z$ is an open quotient map for which $\pi^{-1}\pi(U) = U$ for any U, then $\pi^{-1} : \mathcal{O}(Z) \cong \mathcal{O}(Y)$. This happens for instance if we kill some indiscrete connected components of Y.

Question 0.18. Is there a topological characterization of topological spaces X for which global sections is conservative ? Of locales ?

Of Grothendieck sites ?

References