Finiteness obstruction for G-spectra and K-theory

Introduction

In classical categories, there are various possible notions of finiteness.

A popular one is *compactness*, which is defined so as to enjoy similar properties to "finite presentation" in algebraic contexts; and a second one is being built out of basic objects using finite colimits.

They are usually related, and a lot of the time (if the basic objects are themselves compact), finiteness in the second sense implies compactness. One may then ask whether the converse holds, and one then faces an obstruction theory problem: given an intrinsic property of a certain object, can I build it as a finite colimit of basic objects ?

In the context of homotopy types, Wall's finiteness obstruction famously gives a purely algebraic criterion to decide when that obstruction vanishes; and it is also classical that this obstruction vanishes stably.

One may then ask a similar question in the equivariant setting - this note is inspired by a question to that effect on MathOverflow by Tim Campion and by an answer of Oscar Randall-Williams. The first section is devoted to a K-theory computation (of independent interest) which will be used to set up a finiteness obstruction in the second section.

We recall in the appendix briefly how Wall's finiteness obstruction works, and relate it to other classical finiteness obstructions.

1 Stratification of G-spectra and a K-theory computation

The main tool in describing a finiteness obstruction for $\mathbf{Sp}_G^\omega$ is going to be a computation of $K(\mathbf{Sp}_G^\omega)$ (in fact we only need K_0); and this is going to be obtained with a stratification of $\mathbf{Sp}_G^\omega$.

We will not go into the whole theory of stratifications of stable categories as it would take us too far afield (see [AMGR19]), but it is the idea underlying this computation.

Before stating the main result, we need a bit of notation.

Definition 1.1. A *family* of subgroups of G is a subset of the poset of subgroups of G which is closed under subgroups and conjugation.

Definition 1.2. Given a family of subgroups $\mathcal{F}$, we let $\langle\mathcal{F}\rangle$ denote the thick subcategory of $\mathbf{Sp}_G^\omega$ generated by $\Sigma^\infty_+ G/H, H \in \mathcal{F}$.

For ease of notation, we write $\mathbf{Sp}_G^\omega/\mathcal{F}$ for the verdier quotient of $\mathbf{Sp}_G^\omega$ by $\langle\mathcal{F}\rangle$. Note that this need *not* be idempotent complete a priori.

Here, "thick subcategory generated by" means "smallest stable subcategory closed under retracts containing".

Definition 1.3. Let $H \leq G$ be a subgroup. We let $N_G(H)$ denote the normalizer of H inside G, and $W_G(H) := N_G(H)/H$.

Remark 1.4. On $\mathbf{Sp}_G$, the functor of geometric H-fixed points Φ^H can naturally be viewed as a functor to $\mathrm{Fun}(BW_G(H), \mathbf{Sp}) \simeq \mathbf{Mod}_{\mathbb{S}[W_G(H)]}$

Construction 1.5. In the category Fin_G of finite G-sets, the orbit G/H has a natural $W_G(H)$-action (in fact, $W_G(H)$ is its automorphism group there), so that $\Sigma^\infty_+ G/H$ also has a natural $W_G(H)$-action in $\mathbf{Sp}_G$.

It follows that there is a morphism of associative rings $\mathbb{S}[W_G(H)] \to \mathrm{End}(\Sigma^\infty_+ G/H)$ and hence (by the Schwede-Shipley theorem, as it implies that $\mathbf{Perf}(\mathrm{End}(\Sigma^\infty_+ G/H))$ is equivalent to the thick subcategory generated by $\Sigma^\infty_+ G/H$ in $\mathbf{Sp}^\omega_G$) a functor $\mathbf{Perf}(\mathbb{S}[W_G(H)]) \to \mathbf{Sp}^\omega_G$ sending $\mathbb{S}[W_G(H)]$ to $\Sigma^\infty_+ G/H$.

The main result is the following: fix a subgroup $H \leq G$ per conjugacy class of subgroups.

Theorem 1.6. *The above construction induces an equivalence of connective K-theory spectra:*

$$\bigoplus_{[H]\leq G} K(\mathbb{S}[W_G(H)]) \to K(\mathbf{Sp}^\omega_G)$$

The main technical tool will be the following result (compare [Kra20, Theorem 3.10]) :

Theorem 1.7. *Let $\mathcal{F}$ be a family of subgroups of G, and suppose $K \notin \mathcal{F}$ but $K' \in \mathcal{F}$ for all $K' < K$. Then the geometric K-fixed points spectra induces a cartesian square of stable categories:*

$$\begin{tikzcd}
\mathbf{Sp}^\omega_G/\mathcal{F} \arrow[r] \arrow[d, "\Phi^K"] & \mathbf{Sp}^\omega_G/(\mathcal{F} \vee \{K\}) \arrow[d, "\Phi^K"] \\
\mathrm{Fun}(BW_G(K), \mathbf{Sp}^\omega) \arrow[r] & \mathrm{Fun}(BW_G(K), \mathbf{Sp}^\omega)/\langle \mathbb{S}[W_G(K)] \rangle
\end{tikzcd}$$

Here, $\mathcal{F} \vee \{K\}$ is the smallest family of subgroups containing K and $\mathcal{F}$, namely it is the (set-theoretic) union of $\mathcal{F}$ and all conjugates of K (because everything below is already in $\mathcal{F}$ by assumption).

Let us now take this result for granted and explain how to prove the K-theory formula from this result. We will work by induction on $\mathcal{F}$ and prove the following more precise result:

Theorem 1.8. *Suppose $\mathcal{F}$ is a family of subgroups of G, and fix a representative H for each conjugacy class of subgroups of G not in $\mathcal{F}$.*

Then the previous construction induces an equivalence on connective K-theory spectra:

$$\bigoplus_{[H], H\notin\mathcal{F}} K(\mathbb{S}[W_G(H)]) \to K(\mathbf{Sp}^\omega_G/\mathcal{F})$$

Taking $\mathcal{F}$ to be the empty family reduces to the previous theorem. We now show how this theorem follows from the above "isotropy separation square".

Proof. We prove this by descending induction on the size of $\mathcal{F}$ (or by containment).

Let us start with the full family $\mathcal{A}ll$. Then $\langle \mathcal{A}ll \rangle = \mathbf{Sp}_G^\omega$, so that both sides are 0, and the claim follows.

Suppose now the claim has been proved for every family strictly bigger than $\mathcal{F}$, and assume $\mathcal{F}$ does not contain every subgroup. Let K be minimal among subgroups not in $\mathcal{F}$, and one of the chosen representatives.

Then theorem 1.7 applies, and it follows that we have a Verdier localization sequence

$$\langle \mathbb{S}[W_G(H)] \rangle \to \mathbf{Sp}_G^\omega/\mathcal{F} \to \mathbf{Sp}_G^\omega/(\mathcal{F} \vee \{K\})$$

Let us note that $\Phi^K(\Sigma^\infty_+ G/K) \simeq \mathbb{S}[W_G(K)]$ so that the composite

$$\mathbf{Perf}(\mathbb{S}[W_G(K)]) \to \mathbf{Sp}_G^\omega \to \mathbf{Sp}_G^\omega/\mathcal{F} \to \mathrm{Fun}(BW_G(K), \mathbf{Sp}^\omega)$$

is identified with the inclusion $\langle \mathbb{S}[W_G(K)] \rangle \to \mathrm{Fun}(BW_G(K), \mathbf{Sp}^\omega)$.

In other words, in the above Verdier localization sequence, the functor $\langle \mathbb{S}[W_G(K)] \rangle \to \mathbf{Sp}_G^\omega/\mathcal{F}$ is the one we described in construction 1.5.

Furthermore, since all functors $\mathbf{Perf}(\mathbb{S}[W_G(H)]) \to \mathbf{Sp}_G/(\mathcal{F} \vee \{K\}), H \notin \mathcal{F} \vee \{K\}$ appearing in this construction factor through $\mathbf{Sp}_G^\omega/\mathcal{F}$ (in fact, through $\mathbf{Sp}_G^\omega$), it follows (by the induction hypothesis) that the Verdier projection splits on K-theory.

In particular, using the induction hypothesis, and the fiber sequence of spectra induced on K-theory by our Verdier sequence, we find that the functors from construction 1.5 induce an equivalence on K-theory spectra

$$K(\mathbb{S}[W_G(K)]) \oplus \bigoplus_{[H], H \notin \mathcal{F} \vee \{K\}} K(\mathbb{S}[W_G(H)]) \xrightarrow{\simeq} K(\mathbf{Sp}_G^\omega/\mathcal{F})$$

.

Any conjugacy class of subgroups not in $\mathcal{F}$ is either $[K]$ or something not in $\mathcal{F} \vee \{K\}$, by our choice of K, so this proves the claim. □

2 The finiteness obstruction

With theorem 1.6 in hand, we can now describe a finiteness obstruction. We first specialize to K_0, and obtain the following:

Corollary 2.1. *There is an isomorphism of abelian groups*

$$K_0(\mathbf{Sp}_G^\omega) \to \bigoplus_{[H] \leq G} K_0(\mathbb{Z}[W_G(H)])$$

sending the class of $\Sigma^\infty_+ G/H$ to $\mathbb{Z}[W_G(H)]$ in the $[H]$-component.

Proof. We apply π_0 to theorem 1.6.

Noting that for any connective ring spectrum R, $K_0(R) \cong K_0(\pi_0(R))$, this proves that there exists an isomorphism.

To prove that it does what we claim, note that $\mathbb{S}[W_G(H)]$ corresponds to $\mathbb{Z}[W_G(H)]$, and gets mapped to $\Sigma^\infty_+ G/H$. Because it is an isomorphism, our description is correct. □

Definition 2.2. We let $\mathbf{Sp}_G^f$ denote the full stable subcategory of *finite* G-spectra, namely the stable subcategory generated by the $\Sigma^\infty_+ G/H$.

Remark 2.3. This subcategory is *dense* in $\mathbf{Sp}_G^\omega$, in the sense that any object of $\mathbf{Sp}_G^\omega$ is a retract of some object of $\mathbf{Sp}_G^f$.

Luckily, there is a classification of dense subcategories based on K_0. We recall it below.

Recall the following theorem due essentially to Thomason:

Theorem 2.4 ([Tho97])**.** *Let $\mathbf{C}$ be a stable category. Call a stable subcategory $\mathbf{D}$ dense if any $x \in \mathbf{C}$ is a retract of some $y \in \mathbf{D}$. Then there is a bijection between dense subcategories of $\mathbf{C}$ and subgroups of $K_0(\mathbf{C})$, given by the following explicit maps:*

- *To a dense subcategory $\mathbf{D} \subset \mathbf{C}$, associate the image of the induced morphism $K_0(\mathbf{D}) \to K_0(\mathbf{D})$.*
- *To a subgroup $A \subset K_0(C)$, associate $\{x \in \mathbf{C} \mid [x] \in A\}$, where $[x]$ is the K-theory class of x.*

Remark 2.5. In the above statement, if $\mathbf{D}$ is a dense subcategory, $K_0(\mathbf{D}) \to K_0(\mathbf{C})$ is injective.

Remark 2.6. To convince oneself that $\{x \in \mathbf{C} \mid [x] \in A\}$ is dense, note that for every x, $[x \oplus \Sigma x] = 0 \in A \subset K_0(\mathbf{C})$.

Because any object in $\mathbf{Sp}_G^f$ is obtained by taking coproducts, cofibers suspensions, etc. of objects of the form $\Sigma^\infty_+ G/H$, we find that the image of $K_0(\mathbf{Sp}_G^f) \to K_0(\mathbf{Sp}_G^\omega) \to \bigoplus_{[H]\leq G} K_0(\mathbb{Z}[W_G(H)])$ is exactly the image of $\bigoplus_{[H]\leq G} K_0(\mathbb{Z}) \to \bigoplus_{[H]\leq G} K_0(\mathbb{Z}[W_G(H)])$.

In other words, an object $x \in \mathbf{Sp}_G^\omega$ is in $\mathbf{Sp}_G^f$ if and only if for all H, its image in $K_0(\mathbb{Z}[W_G(H)])$ is in the image of $K_0(\mathbb{Z})$.

In particular we get the following:

Corollary 2.7. $\mathbf{Sp}_G^f = \mathbf{Sp}_G^\omega$ *if and only if* $\bigoplus_{[H]\leq G} \widetilde{K}_0(\mathbb{Z}[W_G(H)]) = 0$, *if and only if for all* $H \leq G$, $\widetilde{K}_0(\mathbb{Z}[W_G(H)]) = 0$

Remark 2.8. For each prime p, there are easy p-local variants of these results.

Example 2.9. Suppose $G = C_2$. Then there are two subgroups, 1 and C_2. For C_2, we get $\widetilde{K}_0(\mathbb{Z}) = 0$ and for 1 we can prove that $\widetilde{K}_0(\mathbb{Z}[C_2])$ is also 0. So $\mathbf{Sp}_{C_2}^f = \mathbf{Sp}_{C_2}^\omega$.

Example 2.10. Examples where $\widetilde{K}_0(\mathbb{Z}[G]) \neq 0$ are not extremely easy to come by, and proofs would take us a bit too far afield. Let us mention that the quaternion group Q_8 is a relatively small group for which this happens; the cyclic group C_{23} is an abelian example (where the nonvanishing is related to the projective class group of a cyclotomic extension). For these groups, $\mathbf{Sp}_G^f \neq \mathbf{Sp}_G^\omega$.

A Other finiteness obstructions

A.1 For spaces

The category of finite spaces is the smallest full subcategory of $\mathbf{S}$ containing pt and stable under finite colimits.

From a model-categorical point of view, it is exactly those spaces that can be represented by finite simplicial sets.

Compact spaces, on the other hand, are spaces that are compact in $\mathbf{S}$. Because pt is a compact generator of $\mathbf{S}$, it formally follows that compact spaces are exactly retracts of finite spaces.

Classically, these are called finitely dominated spaces, because they are homotopy retracts of finite CW-complexes. A natural question is then whether these are different notions.

This question was answered by Wall in the following form :

Theorem A.1 (Wall finiteness obstruction)**.** *Given a connected compact space X, with basepoint x there exists a class $\eta_X \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X,x))]$ such that $\eta_X = 0$ if and only if X is finite.*

Furthermore, given a finitely presented group G, every class $\eta \in \widetilde{K}_0(\mathbb{Z}[G])$ is represented as η_X for some finitely dominated connected space X with $\pi_1(X,x) \cong G$.

Remark A.2. If X is a retract of Y, then $\pi_1(X,x)$ is a retract of $\pi_1(Y,y)$ for some y. In particular, if Y is finite, because the latter is then finitely presented, so is the former.

Remark A.3. It is remarkable that this theorem gives a purely algebraic criterion for a topological question.

We will explain in the subsection about chain complexes one way to understand this finiteness obstruction (although we will not prove this theorem), but in the meantime, here is a slightly more topological way to understand it.

Recall that for a connective ring spectrum R, the morphism $R \to \pi_0(R)$ induces an isomorphism $\tau_{\leq 1}K(R) \to \tau_{\leq 1}K(\pi_0(R))$. In particular, we may interpret $K_0(\mathbb{Z}[\pi_1(X,x)])$ as $K_0(\mathbb{S}[\Omega X])$. Now $\mathbf{Mod}_{\mathbb{S}[\Omega X]}$ can be seen to be equivalent to $\mathrm{Fun}(X, \mathbf{Sp})$, so that this K-group is in fact $K_0(\mathrm{Fun}(X,\mathbf{Sp})^\omega)$. Now if X is a compact space, $\lim_X : \mathrm{Fun}(X,\mathbf{Sp}) \to \mathbf{Sp}$ is a finite limit, so it preserves all colimits (these are stable categories), and in particular its left adjoint (the diagonal) preserves compact objects, so the trivial sphere functor $X \to \mathbf{Sp}, x \mapsto \mathbb{S}$ is in $\mathrm{Fun}(X,\mathbf{Sp})^\omega$.

This defines a class $[\mathbb{S}_X] \in \widetilde{K}_0(\mathbb{S}[\Omega X])$, and this is the wall finiteness obstruction.

A.2 For ordinary spectra

For spectra $\mathbf{Sp}$, we can make the same definitions as for spaces, replacing pt with $\mathbb{S}$, and ask the same question. Note that $\mathbf{Sp}$ is the colimit (in $\mathbf{Pr}^L$) of $\mathbf{S}_* \xrightarrow{\Sigma} \mathbf{S}_* \xrightarrow{\Sigma} \mathbf{S}_* \to \dots$ so it follows formally that compact spectra are the colimit of compact pointed spaces along the same suspension functor.

In particular, a compact spectrum X is of the form $\Sigma^{-n}\Sigma^\infty Y$ for some compact pointed space Y.

But now note that $\Sigma^{-n}\Sigma^\infty Y = \Sigma^{-n-2}\Sigma^\infty\Sigma^2 Y$.

For any pointed Y, ΣY is connected, and if Y is connected, ΣY is simply-connected. In particular, $\Sigma^2 Y$ is simply-connected, so that its Wall finiteness obstruction vanishes : in particular, by Wall's theorem, $\Sigma^2 Y$ is in fact finite, and so X is in fact finite as well.

We have been a bit sloppy about basepoints, but the moral of this story is that stably, Wall's finiteness obstruction vanishes. So, while for G-spectra there was a subtlety due to the representation theory of G, for spectra everything disappears. We thus find:

Corollary A.4. *Compact spectra are finite colimits of spectra of the form $\Sigma^{-n}\mathbb{S}, n \in \mathbb{N}$.*

Remark A.5. We haven't been very careful with pointed versus unpointed compactness/finiteness, but in fact they are equivalent.

Namely, for any pointed space X there is a cofiber sequence $S^0 \to X_+ \to X$, which implies that if X is unpointed finite, then it is pointed finite. Conversely, if $f : K \to \mathbf{S}_*$ is an arbitrary diagram, then its colimit is computed as the cofiber of $\mathrm{colim}_K \mathrm{pt} \to \mathrm{colim}_K f$, where the colimits are computed in $\mathbf{S}$. Therefore if X is pointed-finite, it is finite.

One can prove the same thing about compactness.

A.3 For parametrized spectra, module spectra, or chain complexes

Parametrized spectra over a space X are simply $\mathrm{Fun}(X, \mathbf{Sp})$. In there, we have the generator given by the infinite suspension of the Yoneda embedding $X \to \mathrm{Fun}(X, \mathbf{S})$, for any point $x \in X$ (if X is connected, say).

If we identifty $\mathrm{Fun}(X, \mathbf{Sp})$ with $\mathbf{Mod}_{\mathbb{S}[\Omega X]}$, this is simply $\mathbb{S}[\Omega X]$.

In particular, we can ask again what is the difference between $\mathbf{Perf}(\mathbb{S}[\Omega X])$ (compact modules over $\mathbb{S}[\Omega X]$) and finite modules, namely those that are finite colimits of shifts of $\mathbb{S}[\Omega X]$.

Phrased that way, this question is valid over any ring spectrum R (including $\mathbb{Z}$-algebras, which is essentially making us work over chain complexes up to quasi-isomorphism). The finiteness obstruction is again K-theoretic, because of the previously mentioned result of Thomason (2.4)

In particular, here the full subcategory generated under finite colimits by R in $\mathbf{Perf}(R)$ has as image the image of $\mathbb{Z} \cong K_0(\mathbb{S}) \to K_0(R)$, and so all compacts are finite if and only if this image is the whole of $K_0(R)$.

In other words, the obstruction to finiteness is exactly a class in $\widetilde{K}_0(R)$. This gives another explanation for why $\mathbf{Sp}^f = \mathbf{Sp}^\omega$: $K_0(\mathbb{S}) \cong K_0(\mathbb{Z})$, and a finitely generated projective abelian group is free.

Let us make this obstruction extra explicit in the case of a discrete ring R.

Note that two projective modules P, Q, have equal class in $K_0(R)$ if and only if there exists some n such that $P \oplus R^n \cong Q \oplus R^n$. In particular, $[P] = 0$ in $\widetilde{K}_0(R)$ if and only if P is *stably free*, that is, if there exist n, m such that $P \oplus R^n \cong R^m$.

In terms of chain complexes of projectives, let $P_n \to \dots \to P_0$ represent a perfect R-module, where each P_i is projective. Then, letting Q_0 be such that $P_0 \oplus Q_0$ is free, and up to adding a complex of $Q_0 \to Q_0$ (which does not change the homotopy type of the original complex), we find that we may assume P_0 is free. Doing the same thing with $P_1 \oplus Q_0$, we may assume P_1 is free; then similarly with P_2 and so on.

Up until P_n. Now, if we want to add $Q_n \to Q_n$, we will lengthen the complex, and Q_n might still not be free.

So the point is that any complex bounded of finitely generated projectives is, up to quasi-isomorphism, a complex of finitely-generated free modules, except maybe for the last term. It can be made into a complex of finitely generated free modules if and only if the last term is stably free (assuming the previous ones are already free) : so we find again this obstruction in $\widetilde{K}_0(R)$. In fact, it is not hard to see that it is the same: a perfect R-module is representable by a chain complex of finitely generated free modules if and only if it is finite, in the sense that it is a finite colimit of shifts of R.

Let us now specialize to $R = \mathbb{Z}[\pi_1(X,x)]$, for some connected compact space X. We want to relate the description we gave of Wall's finiteness obstruction to a more classical one.

The point is that the following morphisms are isomorphisms on K_0 ((because they are so on π_0) : $\mathbb{S}[\Omega X] \to \mathbb{S}[\pi_1(X,x)] \to \mathbb{Z}[\pi_1(X,x)]$.

Writing the morphism $\mathbb{S}[\Omega X] \to \mathbb{Z}[\pi_1(X,x)]$ as this composite allows one to see that it is given by first left Kan extending along $X \to B\pi_1(X,x)$, and then tensoring with $\mathbb{Z}$ (or the other way around).

Now because the fiber of $X \to B\pi_1(X,x)$ is $\widetilde{X}$, the universal cover of X, one sees that this composite, on module categories, sends $M \in \mathrm{Fun}(X, \mathbf{Sp})$ to $\mathrm{colim}_{\widetilde{X}} \mathbb{Z} \otimes M \in \mathrm{Fun}(B\pi_1(X,x), \mathbf{Mod}_{\mathbb{Z}})$.

In particular, this isomorphism sends $\mathbb{S}_X$ to $\mathrm{colim}_{\widetilde{X}} \mathbb{Z} = C_*(\tilde{X};\mathbb{Z})$ with $\pi_1(X,x)$-action (equivalently, $B\pi_1(X,x)$-functoriality) given by deck transformations.

It follows that the finiteness obstruction for $\mathbb{S}_X$ in $\mathbf{Perf}(\mathbb{S}[\Omega X])$ is the same as the one for $C_*(\widetilde{X};\mathbb{Z})$ in $\mathbf{Perf}(\mathbb{Z}[\pi_1(X,x)])$, which can be described in terms of stable freeness as before.

This reformulation allows us to prove one of the directions of Wall's theorem. Indeed, if X is a finite CW-complex, $\widetilde{X}$ can be given a compactible $\pi_1(X,x)$-cell structure, so that the chain complex $C_*(\widetilde{X};\mathbb{Z})$ can be viewed (as a $\mathbb{Z}[\pi_1(X,x)]$-chain complex) as the chain complex of cells, which is a complex of finitely generated free $\mathbb{Z}[\pi_1(X,x)]$-modules.

Wall's theorem says that this algebraic obstruction is the only one. The converse can be proved by using the vanishing of the algebraic obstruction to construct explicit maps from spheres.

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