Finite colimits and excisiveness

Convention

"category" and all related notions are to be interpreted as their correspond ∞-categorical notions by default.

1 A finiteness result

Definition 1.1. Let $\mathbf{C}$ be a category with finite colimits, and $X : \Delta^{\mathrm{op}} \to \mathbf{C}$ is said to be n-skeletal if it is Kan extended from its restriction to $\Delta^{\mathrm{op}}_{\leq n}$.

A consequence of the definition is that $\mathbf{C}$ has colimits of n-skeletal simplicial objects, for all $n < \infty$.

Remark 1.2. This consequence is not immediate, as one would need to prove that $\Delta^{\mathrm{op}}_{\leq n}$ is a finite ∞-category, which is not clear as far as I can tell.

However, lemma 1.2.4.17. from Lurie's *Higher algebra* shows that any category with finite colimits has $\Delta^{\mathrm{op}}_{\leq n}$-indexed colimits. We will use this implicitly in this note.

We call those finite geometric realizations.

Example 1.3. If $\mathbf{C}$ is the category of sets, a simplicial set is n-skeletal if and only if all its m-simplices are degenerate for $m > n$.

Theorem 1.4. *Suppose K is a 1-category whose nerve is n-skeletal and finite, and let $f : K \to \mathbf{C}$ be a functor.*

Then the functor $F : \Delta^{\mathrm{op}} \to \mathbf{C}$ associated to f following Bousfield-Kan is n-skeletal.

Remark 1.5. F is informally given by $[n] \mapsto \coprod_{i_0 \to \ldots \to i_n} f(i_0)$. See my note about the Bousfield-Kan formula for an ∞-categorical approach to this functor. Note that the requirement that K be a 1-category is not essential, it just makes for a slightly easier proof.

Remark 1.6. As in my note about Bousfield-Kan, we could easily replace K by an arbitrary ∞-category, as long as it is represented by an n-skeletal simplicial set.

Proof. Consider the following diagram, where the Bousfield-Kan formula comes from:

$$\begin{tikzcd} \int K \arrow[r] \arrow[d] & K \arrow[r, "f"] & \mathbf{C} \\ \Delta^{\mathrm{op}} & & \end{tikzcd}$$

where $\int K$ denotes the Grothendieck construction of the nerve of K, viewed as a functor $\Delta^{\text{op}} \to \text{Set}$.

Recall that $\int K \to K$ is cofinal, and that F is the left Kan extension of $\int K \to K \to \mathbf{C}$ along $\int K \to \Delta^{\text{op}}$.

Consider furhter the restriction of the nerve $N(K)$ to $\Delta^{\text{op}}_{\leq n}$ and the associated Grothendieck construction $\int_n K$. These all fit into a bigger diagram :

$$\begin{tikzcd} \int_n K \arrow[r] \arrow[d] & \int K \arrow[r] \arrow[d] & K \arrow[r, "f"] & \mathbf{C} \\ \Delta^{\text{op}}_{\leq n} \arrow[r] & \Delta^{\text{op}} & & \end{tikzcd}$$

What we're going to prove is that the $\int K \to K \to \mathbf{C}$ is the left Kan extension of its restriction to $\int_n K$.

It will follow at once (by composability of left Kan extensions) that F is the left Kan extension along $\Delta^{\text{op}}_{\leq n} \to \Delta^{\text{op}}$ of a certain functor, and because the latter is fully faithful, it will be the left Kan extension of its restriction.

For this, let $([m], \sigma)$ denote an object of $\int K$, namely some $[m] \in \Delta^{\text{op}}$ together with a simplex $\sigma \in N(K)_m$, we want to consider the comma category $(\Delta^{\text{op}}_{\leq n})_{/[m]}$. We may of course assume $m > n$.

An object in this comma category is a pair $([d], \tau)$ together with a morphism $g : [m] \to [d]$ in Δ such that $g^*\tau = \sigma$, and a morphism $([d], \tau, g) \to ([d'], \tau', g')$ is a morphism $[d'] \to [d]$ in Δ such that the obvious triangle commutes, and compatible with τ, τ'.

The functor mapping $([d], \tau, g)$ to the image $i : [d'] \to [d]$ of $g : [m] \to [d]$ with $i^*\tau$ is a right adjoint to the inclusion of the $([d], \tau, g)$'s such that g is surjective. It follows that the inclusion is cofinal, so we may replace the category $(\Delta^{\text{op}}_{\leq n})_{/[m]}$ by its full subcategory $(\Delta^{\text{op}}_{\leq n})^{surj}_{/[m]}$ on those objects.

Letting π denote the functor $\int K \to K$, it follows that the left Kan extension of the restriction of $f \circ \pi$ to $\int_n K$ is given on $[m]$ by

$$\text{colim}_{([d],\tau,g) \in (\Delta^{\text{op}}_{\leq n})^{surj}_{/[m]}} f(\tau_0)$$

But now note that if $[m] \to [d]$ is surjective, then $0 \mapsto 0$, so that $\tau_0 = \sigma_0$. In other words, the diagram restricted to $(\Delta^{\text{op}}_{\leq n})^{surj}_{/[m]}$ is constant with value $f(\sigma_0) = f \circ \pi([m], \sigma)$. It therefore suffices to show that $(\Delta^{\text{op}}_{\leq n})^{surj}_{/[m]}$ is weakly contractible.

Write $\sigma = (i_0 \to ... \to i_m)$. Because $N(K)$ is n-skeletal, there are at most n nonidentity morphisms in this simplex, and therefore there is unique $d \leq n$ and d-simplex τ with a surjective map $g : [m] \to [d]$ such that $g^*\tau = \sigma$ and τ has no identity morphism. It is easy to check that $([d], \tau, g)$ is initial in $(\Delta^{\text{op}}_{\leq n})^{surj}_{/[m]}$, so the latter must be weakly contractible, which proves the claim. $\square$

2 Application to n-excisive functors

We now assume $\mathbf{C}$ is stable; we want to prove the following result:

Proposition 2.1. *Suppose $\mathbf{C}_0 \subset \mathbf{C}$ is a full subcategory closed under finite coproducts; and suppose $\mathbf{C}$ is generated under finite colimits and desuspensions by $\mathbf{C}_0$.*

Let $\mathbf{D}_0 \subset \mathbf{D}$ be an exact inclusion of stable categories, and let $f : \mathbf{C} \to \mathbf{D}$ be an n-excisive reduced functor such that $f_{|\mathbf{C}_0}$ has values in $\mathbf{D}_0$. Then f does too.

This proof, due to Barwick, Glasman, Mathew and Nikolaus is in *K-theory and polynomial functors*, and part of it in Dold and Puppe's *Homologie nicht additiver Funktoren.*

We start with a lemma:

Lemma 2.2. *Suppose $f : \mathbf{C} \to \mathbf{D}$ is an n-excisive functor, and suppose further that $X : \Delta^{\mathrm{op}} \to \mathbf{C}$ is an k-skeletal simplicial object. Then $f(X_\bullet)$ is nk-skeletal.*

Proof. The proof uses the Dold-Kan correspondance. We first note that $\mathrm{Ho}(f)$ is n-polynomial in the sense of additive categories.

Indeed, it is not hard to see that the cross-effect in the sense of Goodwillie calculus agrees with the ordinary cross effect (see the proof of 6.1.3.22 in *Higher Algebra*), and so if f is n-excisive, its $(n+1)$st cross-effect vanishing implies that it is n-polynomial.

We then use the following observation: the Dold-Kan correspondance works for additive categories that have cokernels for split monomorphisms - and certainly the homotopy category of a stable category satisfies this condition.

In particular, because the chain complex in question has terms given by $|\mathrm{sk}_{i+1}(X)|/|\mathrm{sk}_i X|$ so that if X is n-skeletal, then the terms of the chain complex vanish above degree n.

Conversely, if those terms vanish above degree n, then the map $\mathrm{sk}_n X \to X$ induces an equivalence upon realizing finite skeleta, which, by the ∞-categorical Dold-Kan correspondance, implies that it is an equivalence.

Therefore X is n-skeletal if and only if the chain complex associated to it by the Dold-Kan correspondance (in $\mathrm{Ho}(\mathbf{C})$) vanishes above degree n. We are therefore reduced to a claim about 1-categories, namely the following lemma ☐

Lemma 2.3. *Suppose C, D are additive 1-categories with cokernels for split monomorphisms, and let $f : C \to D$ be an n-polynomial reduced functor.*

Let X be a nonnegatively graded chain complex in C, and $DK(X)$ its associated simplicial object. Suppose X vanishes above degree k, then the chain complex associated to $f(DK(X))$ vanishes above degree nk.

Proof. Let $\mathcal{X} := DK(X)$ and $\mathcal{Y} := f(\mathcal{X})$.

It suffices to show that the morphism $\bigoplus_i \mathcal{Y}_{nk} \to \mathcal{Y}_m$ induced by the degeneracies is surjective for $m > nk$.

Now $\mathcal{Y}_m = f(\mathcal{X}_m) = f(\bigoplus_{[m]\to[k]} X_k)$ where the direct sum ranges over surjections $[m] \to [k]$.

Now it's easy to see, by induction, that $f(\bigoplus_i A_i)$ is a direct sum of cross-effects $cr_s(f)(A_{i_1}, ..., A_{i_s})$, where $s \leq n$ because f is n-polynomial.

It follows that $\mathcal{Y}_m$ is a direct sums of $cr_s(f)(X_{k_1}, ..., X_{k_s})$ where $s \leq n$ and we have surjections $[m] \to [k_i]$, and $k_i \leq k$ (recall that $X_r = 0$ for $r > k$).

It therefore suffices to show that for any family of s surjections $[m] \to [k_i]$, $s \leq n, k_i \leq k$, there is $m' < m$ and a simultaneous factorization $[m] \to [m'] \to [k_i]$. This is now pure combinatorics: if the map $[m] \to [k_1] \times ... \times [k_s]$ were injective, its image would contain a chain of size m, but the maximal size of a chain in such a product is $\sum_i k_i \leq \sum_i k = ks \leq nk$, so if $m > nk$, this cannot happen: the map is not injective and so we have such a factorization. ☐

Proposition 2.4. *Let $f : \mathbf{C} \to \mathbf{D}$ be an n-excisive reduced functor. Then f preserves finite geometric realizations.*

Proof. Because $\mathbf{D} \subset \mathrm{Ind}(\mathbf{D})$ preserves finite geometric realizations (it preserves all finite colimits), we may as well assume that $\mathbf{C}, \mathbf{D}$ are presentable and f preserves filtered colimits.

In this case we may fully apply Goodwillie calculus and work by induction: indeed f is the fiber of a map from an $(n-1)$-excisive functor to an n-homogeneous functor, so that by induction we are reduced to n-homogeneous functors.

But an n-homogeneous functor is of the form $x \mapsto B(x, ..., x)_{h\Sigma_n}$ for some symmetric n-linear B, so that it suffices to prove it for $x \mapsto B(x, ..., x)$. This preserves finite geometric realizations because it preserves finite colimits in each variable, and the diagonal functor $\Delta^{\mathrm{op}} \to (\Delta^{\mathrm{op}})^n$ is cofinal. $\square$

We can now prove the main result:

Proof. We start by proving that the subcategory of $\mathbf{C}$ generated by $\mathbf{C}_0$ under colimits is mapped to $\mathbf{D}_0$.

We proceed by induction: Let $\mathbf{C}_{k+1} :=$ the full subcategory of $\mathbf{C}$ on objects that are finite colimits of diagrams in $\mathbf{C}_k$.

Note that by induction, $\mathbf{C}_{k+1}$ is stable under finite coproducts: if $X : I \to \mathbf{C}_k$ and $Y : J \to \mathbf{C}_k$ are finite diagrams, then by the main result of the first section, the colimit of X is the same as that of a diagram $\Delta^{\mathrm{op}}_{\leq p} \to \mathbf{C}_n$ for some sufficiently large p (this already uses that $\mathbf{C}_k$ is stable under finite coproducts, see the Bousfield-Kan formula), and we may choose the same p for Y. Therefore the coproduct of the colimit of X with the colimit of Y is the colimit of the coproducts of the (truncated) simplicial diagrams. This proves that $\mathbf{C}_{k+1}$ is closed under coproducts.

We now prove that $f(\mathbf{C}_k) \subset \mathbf{D}_0$. Again, we proceed by induction, so assume the result for k.

Let $g : I \to \mathbf{C}_k$ be a finite diagram. Its Bousfield-Kan diagram $G : \Delta^{\mathrm{op}} \to \mathbf{C}_k$ is m-skeletal for some m and has the same colimit.

By the previous lemma, $f \circ G$ is mn-skeletal, and all its values are of the form $f($ some coproduct of terms in $\mathbf{C}_k) = f($ some terms in $\mathbf{C}_k)$.

Because f preserves finite geometric realizations, it follows that $f(\mathrm{colim}_I g) = f(|G_\bullet|) = |f(G_\bullet)|$ is a finite geometric realization of terms of $\mathbf{D}_0$, i.e. a finite colimit of terms of $\mathbf{D}_0$. This concludes the claim: $f(\mathbf{C}_\infty) \subset \mathbf{D}_0$, where $\mathbf{C}_\infty := \bigcup_k \mathbf{C}_k$.

We now deal with desuspensions. First note that $\bigcup_p \Omega^p \mathbf{C}_\infty = \mathbf{C}$: indeed the left hand side is closed under desuspensions and finite colimits (any finite diagram factors through some $\Omega^p \mathbf{C}_\infty$, and then through some $\Omega^p \mathbf{C}_k$, so that its colimit belongs to $\Omega^p \mathbf{C}_{k+1}$) and it contains $\mathbf{C}_0$, so that our hypothesis says that it is $\mathbf{C}$.

It therefore suffices to prove that if $\mathbf{E} \subset \mathbf{C}$ is closed under finite colimits and $f(\mathbf{E}) \subset \mathbf{D}_0$, then $f(\Omega \mathbf{E}) \subset \mathbf{D}_0$.

For this, we note that if X is any object, then the strongly coCartesian n-cube which is extended from a diagram with an X at the tip and 0's elsewhere has all its terms of the form "a finite direct sum of ΣX's". In particular, if $X \in \Omega \mathbf{E}$, then this strongly coCartesian n-cube has all its terms in $\mathbf{E}$.

It follows (by n-excisiveness) that $f(X)$ is the limit of a cube where all the terms are of the form $f($ something in $\mathbf{E})$, and therefore is a finite limit of terms of $\mathbf{D}_0$: it follows that $f(X) \in \mathbf{D}_0$, and we are done. $\square$