Filtered colimits are left exact

In this short note, I explain a model-independent proof of the following result:

Theorem 0.1. *In the* ∞*-category* An *of anima, filtered colimits commute with finite limits.*

From now on the usual convention applies, that category means ∞-category.

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1 The proof

So far, the only proof available seems to be reduce to the same statement about homotopy groups, where it essentially reduces to a statement about Kan complexes. The following (kind of far-fetched, but not so ridiculous) argument is supposed to eliminate the reduction to simplicial sets and be more "model-independent". It relies, surprisingly, on Stasheff's recognition principle and the following corollary thereof:

Corollary 1.1. *The loops functor* $\Omega : \mathrm{An}^{\geq 1}_{\mathrm{pt}/} \to \mathrm{An}$ *from pointed connected anima to anima preserves sifted colimits.*

This is a corollary of Stasheff recognition since by the latter, the above functor is identified with the forgetful functor from groups to anima, which clearly commutes with sifted colimits.

Remark 1.2. As far as I know, Stasheff recognition does *not* rely on Theorem 0.1, making this proof in principle not circular.

We prove Theorem 0.1 by examining a series of special cases, and eventually reducing to the above corollary of Stasheff recognition.

The first special case is the following:

Lemma 1.3. *Suppose* $A_\bullet, C_\bullet : I \to \mathrm{An}$ *are functors from a sifted category with maps* $A_\bullet \to B \leftarrow C_\bullet$ *to a space* B*. The canonical map* $\mathrm{colim}_I(A_i \times_B C_i) \to (\mathrm{colim}_I A_i) \times_B (\mathrm{colim}_I C_i)$ *is an equivalence.*

Proof. The forgetful functor $\mathrm{An}_{/B} \to \mathrm{An}$ preserves colimits, so we essentially need to prove that in An/B, products commute with sifted colimits. This follows from the equivalence $\mathrm{An}_{/B} \simeq \mathrm{An}^B$, the pointwise computation of colimits, and the corresponding statement in An. □

Corollary 1.4. *Suppose $X_\bullet : I \to \mathrm{An}_{/X}$ is a filtered diagram, where all maps $X_i \to X$ are inclusions of components/monomorphisms. Then the induced map $\mathrm{colim}_I X_i \to X$ is also an inclusion of components/a monomorphism.*

Proof. It suffices to note that this is a condition about the pullback $\mathrm{colim}_I X_i \times_X \mathrm{colim}_I X_i$ which therefore falls into the previous lemma. □

Example 1.5. If the X_i's are sets and all transition maps are injections, the we can compute the colimit X in Set and obtain a diagram $I \to \mathrm{An}_{/X}$ fitting into the previous corollary, so that we find that the colimit is the same as in An.

Next, the key statement, the point where "something is happening", is the following special case:

Lemma 1.6. *Let $B_\bullet : I \to \mathrm{An}_{\mathrm{pt}/}$ be a filtered diagram. The natural map $\mathrm{colim}_I \Omega(B_i) \to \Omega(\mathrm{colim}_I B_i)$ is an equivalence.*

Note that this does not directly follow from Corollary 1.1 since the diagram does not take values in *connected* pointed anima. In fact, while Corollary 1.1 holds for sifted diagrams, the above lemma does *not* and really requires filtered.

Example 1.7. Let $x : \mathrm{pt} \to X$ be an arbitrary pointed anima, and represent it as $\mathrm{colim}_{\Delta^{\mathrm{op}}} X_n$ for a pointed simplicial set $X_\bullet$. Then $\Omega(X_n, x) = \mathrm{pt}$ for every n, so that $\mathrm{colim}_{\Delta^{\mathrm{op}}} \Omega(X_n, x) \simeq \mathrm{pt}$ which is, in general, different from $\Omega(X, x)$.

The key idea of the proof is exactly to use filteredness to reduce to Corollary 1.1.

Proof. We introduce a notation. For an anima X and a set $S \subset \pi_0(X)$, we let (X, S) denote $X \times_{\pi_0(X)} S$, that is, the full subspace of X spanned by points whose component is in S. If X is pointed, we only allow S's such that the basepoint lives in S, and then consider (X, S) also as a pointed anima.

Warning 1.8. This conflicts with usual notation in that for a pointed anima $x : \mathrm{pt} \to X$, (X, x) here refers to *the component of x in X* and *not* "the pointed anima X equipped with x". In the course of this proof (and exclusively here), the new notation takes precedence.

In the statement of the lemma, we let $b_i : \mathrm{pt} \to B_i$ denote the chosen basepoint and $b_\infty : \mathrm{pt} \to B_\infty := \mathrm{colim}_I B_i$ the corresponding basepoint in the colimit.

By Corollary 1.1, we have that the canonical map

$$\mathrm{colim}_I \Omega(B_i, b_i) \to \Omega(\mathrm{colim}_I (B_i, b_i))$$

is an equivalence.

Note also that $\Omega(B_i, b_i) \to \Omega B_i$ is an equivalence, and $\Omega(B_\infty, b_\infty) \to \Omega B_\infty$ also is. Thus, it suffices to prove that the canonical map $\mathrm{colim}_I (B_i, b_i) \to (B_\infty, b_\infty)$ is an equivalence.

We introduce an auxiliary category J for this purpose. It consists of objects i of I together with a *finite* subset $S \subset \pi_0(B_i) \times_{\pi_0(B)} \{b_\infty\}$ containing (the component of) b_i, and a map $(i, S) \to (j, T)$ is the data of a map $i \to j$ in I together with the property that the induced map $B_i \to B_j$ sends S to (a subset of) T. We leave to the reader the details of setting this category up in detail, but note also that every filtered category admits a cofinal map from a filtered poset, so one may restrict to this case, in which case the above description is sufficient.

The first easy observation is that J is filtered: given a diagram $K \to J$ from a finite K, we can first find a cocone diagram extending it in I, and then adjust the finite subset accordingly.

The same details will show that there is a functor $J \to \mathrm{An}_{\mathrm{pt}/}$ sending (i, S) to (B_i, S). Now, there is a functor $I \to J$ sending i to (i, b_i). We claim that this functor is *cofinal*. Indeed, I, J are both filtered, so the slices $I \times_J J_{j/}$ are either empty or filtered, and it therefore suffices to prove that they are not empty to obtain their weak contractibility. But now we note that if $S \subset \pi_0(B_i) \times_{\pi_0(B_\infty)} \{b_\infty\}$ is finite, then since π_0 commutes with colimits, there is some j with a map $i \to j$ so that all of S maps to b_j in $\pi_0(B_j)$. Here we use the description of filtered colimits in Set[1].

Remark 1.9. It is in this last paragraph that things break if I is only sifted.

Thus, to prove $\mathrm{colim}_I(B_i, b_i) \to (B_\infty, b_\infty)$ is an equivalence, it suffices to prove the same for $\mathrm{colim}_{(i,S)\in J}(B_i, S) \to (B_\infty, b_\infty)$. Now, there is a fibration $J \to I$ so we may compute the source instead as $\mathrm{colim}_{i\in I}\mathrm{colim}_{S\in\mathcal{P}_f(\pi_0(B_i)\times_{\pi_0(B_\infty)}\{b_\infty\})}(B_i, S)$ where $\mathcal{P}_f$ of a set is the poset of its finite subsets.

So let $T \subset \pi_0(X)$ be a set - we claim that $\mathrm{colim}_{S\in\mathcal{P}_f(T)} X \times_{\pi_0(X)} S \simeq X \times_{\pi_0(X)} T$, which is now a special case of Lemma 1.3 (here we are using also Example 1.5 for $\mathrm{colim}_{S\in\mathcal{P}_f(T)} S \simeq T$).

Thus, $\mathrm{colim}_{i\in I}\mathrm{colim}_{S\in\mathcal{P}_f(\pi_0(B_i)\times_{\pi_0(B_\infty)}\{b_\infty\})}(B_i, S) \simeq \mathrm{colim}_{i\in I}(B_i, \pi_0(B_i)\times_{\pi_0(B_\infty)}\{b_\infty\}) \simeq \mathrm{colim}_I B_i \times_{\pi_0(B_\infty)} \{b_\infty\}$. Finally, we can use Lemma 1.3 again to obtain that the latter is simply (B_∞, b_∞), as claimed. $\square$

Corollary 1.10. *Let $B_\bullet : I \to \mathrm{An}_{A,C/}$ be a filtered diagram. The map $\mathrm{colim}_I(A \times_{B_i} C) \to A \times_{\mathrm{colim}_I B_i} C$ is an equivalence.*

Proof. Without loss of generality, we may assume I has an initial object, $i_0 \in I$.We may then view the canonical map as a natural transformation between functors $(\mathrm{An}_{/B_{i_0}})^2 \to \mathrm{An}$ of the variable (A, C). Both functors commute with colimits in each variable, $\mathrm{An}_{/B_{i_0}}$ is generated under colimits by points, so we are reduced to the case $A = C = \mathrm{pt}$.

In that situation either $\mathrm{pt} = A \to \mathrm{colim}_I B_i$ and $\mathrm{pt} = C \to \mathrm{colim}_I B_i$ are not homotopic, in which case both sides are simply empty, or they are homotopic. In the latter case, since π_0 commutes with filtered colimits, we find that they are homotopic already in some B_j, and so up to considering $I_{j/}$, we are in the situation of Lemma 1.6, and may thus conclude. $\square$

We are now in a position where we can conclude:

Proof of Theorem 0.1. Let I be a filtered category. As I is in particular sifted, $|I| \simeq \mathrm{pt}$ and so colim_I preserves the terminal object.

Thus, to prove that it preserves finite limits it suffices to prove that it preserves pullbacks. So let $A_\bullet \to B_\bullet \leftarrow C_\bullet$ be a cospan of I-indexed diagrams. We note that the diagonal map $I \to I^{\Lambda^2_2}$ is cofinal, so we may rewrite $\mathrm{colim}_I(A_i \times_{B_i} C_i) \simeq \mathrm{colim}_{i\to j\leftarrow k}(A_i \times_{B_j} C_k) \simeq \mathrm{colim}_{i,k\in I^2}\mathrm{colim}_{j\in I_{i,k/}}(A_i \times_{B_j} C_k)$.

[1]But crucially, *not* the fact that they are the same as in An.

By Corollary 1.10, this is $\mathrm{colim}_{i,k\in I^2}(A_i \times_{\mathrm{colim}_j B_j} C_k)$, and so by Lemma 1.3, this is now $(\mathrm{colim}_i A_i) \times_{\mathrm{colim}_j B_j} (\mathrm{colim}_k C_k)$. We leave to the reader the details of checking that the map is the correct one. $\square$

In the "model-independent proofs" enterprise, it can be worth recording explicitly the "big results" one has used to ultimately measure exactly where and for what results one has used models. Here, I believe we have essentially used: Stasheff's recognition principle, Quillen's theorem A to discuss cofinality, and of course the un/straightening equivalence