$f^2 = 0$

Introduction

In stable homotopy theory, one encounters a seemingly new phenomenon : the self map of multiplication by $n \in \mathbb{Z}$ on the cofiber X/n might not be 0.

This is surprising because in classical algebra, if you have a group A and mod out an integer n, then multiplication by n on A/n is zero.

The reason that there's a difference is essentially that in homotopy theory, saying that a map is zero is additional structure and not just a property; so when you quotient out by n, what you're really doing is adding a homotopy between multiplication by n and 0 - but that needs to be taken into account if you want to define a homotopy between multiplication by n on this new structure and 0.

What remains however true is that multiplication by n^2 is 0 on X/n. In a first section, we will explain why this new phenomenon actually has a shadow in classical algebra, and in a second section we will prove a general version of this n^2 result.

Finally, in a third section we will talk about when n is 0 on X/n.

1 The shadow in classical algebra

Recall the following extremely useful statement:

Lemma 1.1 (5-lemma). *Suppose $0 \to A \to B \to C \to 0$ and $0 \to M \to N \to K \to 0$ are exact sequences of abelian groups, and suppose we have a commutative diagram :*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & K \arrow[r] & 0
\end{tikzcd}$$

Suppose $A \to M$ and $C \to K$ are isomorphisms, then so is $B \to N$.

We will not prove this here and leave it as an instructive exercise (if you're reading this you probably already know this), and we will also leave the reader with the task of finding a counterexample to the following plausible-sounding analog of this statement:

Lemma 1.2 (WRONG lemma). *In the same situation as above, suppose $A \to M$ is the 0 morphism and $C \to K$ is also the 0 morphism. Then $B \to N$ is also the 0 morphism.*

Remark 1.3. The wrongness of this lemma is somehow the reason why 0 morphisms are harder than isomorphisms. This is for instance a theme uniting the notion of phantom maps in stable homotopy theory, and the so-called "generating hypothesis", both of which are related to 0-ish analogs of statements that are true about isomorphisms or equivalences.

In fact one can find a counterexample with the same short exact sequence appearing twice, and this leads us to the following correction of this false lemma (and actually the correction can help you find counterexamples) :

Lemma 1.4 (This one is correct, don't worry)**.** *Let $0 \to A \to B \to C \to 0$ be an exact sequence as above, and let $f : B \to B$ be a morphism such that its restriction to A is 0, and such that the induced $C \to C$ is also 0.*

Then $f^2 = 0$. (even if f might be nonzero).

Proof. The proof of this will be the inspiration for the proof in the next section.

Consider the following large commutative diagram, which we simply get by pasting the diagram involving f with itself :

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] \arrow[dl, dotted] & C \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Let us explain the dotted arrow: $B \to C \to C$ is 0 because by assumption $C \to C$ is. But by commutativity of the diagram, this is equal to $B \to B \to C$. By exactness of the middle line, this implies that $B \to B$ factors as $B \to A \to B$, and this is what the dotted arrow represents.

But now note that $B \to B \to B$ can be written as $B \to A \to B \to B$ which, by commutativity, is nothing but $B \to A \to A \to B$; but now $A \to A$ is 0, so that $B \to B \to B$ must be 0 too, which is what was claimed. ☐

2 $f^2 = 0$ on X/f

Let us explain why the phenomenon in the previous section of f not being 0 on B is a shadow of the homotopical phenomenon I will mention here.

Suppose we are working in spectra, and $f : X \to X$ is an endomorphism. We can form its cofiber, X/f. Then the long exact sequence of homotopy groups of the cofiber sequence $X \to X \to X/f$ tells us that the homotopy groups of X/f fit into a short exact sequence :

$$0 \to \pi_n(X)/\pi_n(f) \to \pi_n(X/f) \to \ker(\pi_{n-1}(f)) \to 0$$

Now if we look at the induced $\overline{f} : X/f \to X/f$ (defined in the naive way), on homotopy groups this induces a commutative diagram :

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \pi_n(X)/\pi_n(f) \arrow[r] \arrow[d] & \pi_n(X/f) \arrow[r] \arrow[d] & \ker(\pi_{n-1}(f)) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \pi_n(X)/\pi_n(f) \arrow[r] & \pi_n(X/f) \arrow[r] & \ker(\pi_{n-1}(f)) \arrow[r] & 0 \end{tikzcd}$$

where both vertical maps are induced by $\pi_k(f)$, $k = n, n-1$, and in particular are obviously 0. Now this does not in general imply that $\pi_n(X/f) \to \pi_n(X/f)$ is 0, and the reason for that is exactly the phenomenon described before.

However, it does imply that $\pi_n(\overline{f})^2 = 0$. Unfortunately this is not enough to imply that $\overline{f}^2 = 0$ (even for finite X, this would rely on the generating hypothesis which is still open). Moreover it wouldn't really be satisfying as it would be using the fact that we have homotopy groups. We want to prove the following more general claim :

Proposition 2.1. *Let* $\mathbf{C}$ *be a stable* ∞*-category, and* $f : \Sigma^n X \to X$ *a map in* $\mathbf{C}$*. Let* X/f *denote its cofiber, and let* $\overline{f} : \Sigma^n X/f \to X/f$ *denote the induced map.*

Then $\overline{f} \circ \Sigma^n \overline{f} \simeq 0$.

Remark 2.2. We add a sparkle of generality by allowing f to change degrees. This changes nothing to the essence of the argument, and we will only do the proof in the case $n = 0$ - the reader can generalize the proof, or find a way to deduce the general statement from the $n = 0$ case (one can do that using coproducts of suspensions of X).

Before going into the proof, we must make a warning about the term "induced map" here. In classical algebra, there is not much room for doubt here, but here we must say something. Namely, cofib is a functor from $\mathrm{Fun}(\Delta^1, \mathbf{C})$ to $\mathbf{C}$, and so the induced morphism on cofibers depends on the whole commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

including the homotopy witnessing it as commutative, and not only on the 4 morphisms that are involved.

This is of course related to the fact that in a triangulated category, cones aren't functorial, and induced morphisms on cones are not unique (they depend on the specific homotopy !).

Here, the square that we want to use has boundary given by

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

There is a very obvious homotopy filling it (because of course $f \circ f$ is homotopic to $f \circ f$!), but this is *not* the one coming into play for the statement that 2 is not zero on $S^0/2$, as we will see later.

In fact, what we mean for this statement is that *any* induced morphism has a trivial square. More precisely, what we need will be the following: a morphism $\overline{f} : X/f \to X/f$ such that *there exists* a morphism of fiber sequences of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "f"] & X/f \arrow[d, "\overline{f}"] \\ X \arrow[r, "f"'] & X \arrow[r] & X/f \end{tikzcd}$$

Of course, the obvious homotopy provides an induced morphism satisfying this, but any homotopy does. We will see later that this is relevant. Our updated statement (in the case $n = 0$) reads:

Proposition 2.3. *Let* $\mathbf{C}$ *be a stable* ∞*-category,* $f : X \to X$ *an endomorphism of some object* X*, with cofiber* X/f*.*

Let $\overline{f}$ *be* an *induced morphism in the sense just described. Then* $\overline{f} \circ \overline{f} \simeq 0$.

Proof. By assumption, we have the following diagram of fiber sequences :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & X \arrow[r] \arrow[d, "f"] & X/f \arrow[d, "\overline{f}"] \\ X \arrow[r, "f"] & X \arrow[r] & X/f \end{tikzcd}$$

Our goal is to adapt the proof of the lemma in the previous section, by using the short exact sequence of π_n as above for inspiration. For that, we see that the leftmost X contributes as the rightmost kernel in those sequences, so we shift the sequences :

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X/f \arrow[r] \arrow[d, "\overline{f}"] & \Sigma X \arrow[d, "\Sigma f"] \\ X \arrow[r] & X/f \arrow[r] & \Sigma X \end{tikzcd}$$

Now note that if we shifted the initial cofiber sequence twice we would have $X/f \to \Sigma X \overset{\Sigma f}{\to} \Sigma X$. In particular this composite is nullhomotopic. But this is exactly the rightmost composite in the above diagram “right-down”. So in the following diagram, because our cofiber sequences are also fiber sequences, there exists a dotted arrow making it commute:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X/f \arrow[r] \arrow[d, "\overline{f}"] \arrow[dl, dotted] & \Sigma X \arrow[d, "\Sigma f"] \\ X \arrow[r] \arrow[d, "f"] & X/f \arrow[r] \arrow[d, "\overline{f}"] & \Sigma X \arrow[d, "\Sigma f"] \\ X \arrow[r] & X/f \arrow[r] & \Sigma X \end{tikzcd}$$

(indeed note that $X/f \to X/f \to \Sigma X$ is homotopic to $X/f \to \Sigma X \to \Sigma X$ which we just explained was nullhomotopic)

But now $\overline{f} \circ \overline{f}$ factors as $X/f \to X \to X/f \to X/f$ which is homotopic to $X/f \to X \to X \to X/f$, but $X \to X \to X/f$ is nullhomotopic, therefore so is the composite, therefore so is $\overline{f} \circ \overline{f}$, which is what was claimed. $\square$

Remark 2.4. In fact, this shows even better - as the proof of section 1 seems to indicate -, namely that the composite of any two induced morphisms $\overline{f}, \tilde{f}$ in the sense described is automatically nullhomotopic.

This is of course much better than any proof involving homotopy groups, and in fact it *implies* the result on homotopy groups in a "second" way.

As an application, for any p, p^2 is nullhomotopic on S^0/p; indeed p is an induced morphism in the sense described above (because multiplication by p is a natural morphism). It can be proved that p is not in general (specifically for $p = 2$)

As another, perhaps more surprising application, note the following: given a spectrum X, I can consider a fiber sequence $X \to X \to X/p^n$, and in fact I can arrange them in an inverse system $\{X\}_n \to \{X\}_n \to \{X/p^n\}_n$, where the middle one is the constant inverse system, and the left hand one is $\ldots \xrightarrow{p} X \xrightarrow{p} X$.

In particular, If I tensor this fiber sequence of inverse systems, the left hand one becomes $\ldots \xrightarrow{p} X/p \xrightarrow{p} X/p$, which, as a pro-object, is equivalent to the one where we compose consecutive maps, i.e. $\ldots \xrightarrow{p^2} X/p \xrightarrow{p^2} X/p$. But all the maps in this inverse system are nullhomotopic, which means that the system is pro-zero. In particular, the map $\{X \otimes S^0/p\} \to \{X \otimes S^0/p^n \otimes S^0/p\}$ is a pro-equivalence.

3 Sometimes, $\overline{f} = 0$

In this section we want to expand on warning 2.

Indeed, there is a universal way to make the following square commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

and if we take this specific nullhomotopy, the induced morphism $\overline{f}$ *will* in fact be nullhomotopic. Denis Nardin made me realize that this wasn't wrong, but didn't contradict what we know about Moore spectra.

The first subsection is here to explain that result.

In a second subsection, we will explain why that does not contradict the fact that 2 is not 0 on the Moore spectrum $S^0/2$; in a third section we will study the $\mathbb{Z}$-linear case; and finally we will give a general criterion in monoidal categories.

3.1 A general nullhomotopy

Let us look at the universal case to understand what is happening.

We want to talk about nullhomotopies, so the most general scenario should involve pointed categories. Furthermore, it should involve the universal endomorphism, namely $B\mathbb{N}$.

Finally, we want to compute cofibers, so our category better have them - and for good measure, we'll add in all finite colimits. In other words, the universal example should be the pointed category freely generated by $B\mathbb{N}$ under finite colimits.

So as not to worry about the word "finite", let us simplify things (there is no real loss of generality) by considering the pointed category freely generated by $B\mathbb{N}$ under colimits, namely $\mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*) = \mathrm{Fun}(B\mathbb{N}, \mathbf{S})_*$.

Another point of view on this category is that it's simply $\mathbb{N}_+$-modules in pointed spaces - we'll use both points of view interchangeably. Now, the universal endomorphim is simply $n \mapsto n+1$ as an endomorphism of the $\mathbb{N}_+$-module $\mathbb{N}_+$; call it s.

Proposition 3.1. *The cofiber of s is S^0, with $\mathbb{N}_+$-action given by the trivial morphism $S^0 \to S^0$.*

Proof. Note that the evaluation at a point $\mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*) \to \mathbf{S}_*$ preserves all colimits, so let us start by computing the colimit in pointed spaces.

There, up to equivalence, our map is of the form $X_+ \vee * \to X_+ \vee S^0$, and so its cofiber is the coproduct (in pointed spaces) of the cofiber of $X_+ \to X_+$, and the cofiber of $* \to S^0$.

Therefore the cofiber of our map is $* \vee S^0 = S^0$.

This computation also works in pointed sets, in other words this specific colimit is preserved by the inclusion functor $\mathrm{Fun}(B\mathbb{N}, \mathrm{Set}_*) \to \mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*)$, so that it actually suffices to compute the action in pointed sets.

Now it's just an easy 1-categorical verification that the $\mathbb{N}_+$-action is as described. □

Let us now explain how that implies the general claim, namely that if we choose for the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

the universal homotopy, then the induced morphism on X/f *is* nullhomotopic.

Let $X \xrightarrow{f} X$ be an endomorphism in a cocomplete pointed category $\mathbf{C}$. This induces a unique colimit-preserving (in particular pointed) functor $\mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*) \to \mathbf{C}$ sending $\mathbb{N}_+ \mapsto X$ and s to f.

In particular, the cofiber of s is mapped to the cofiber of f. This cofiber is S^0 with the trivial action given above.

Furthermore, the filling of the square

$$\begin{tikzcd} \mathbb{N}_+ \arrow[r, "s"] \arrow[d, "s"] & \mathbb{N}_+ \arrow[d, "s"] \\ \mathbb{N}_+ \arrow[r, "s"] & \mathbb{N}_+ \end{tikzcd}$$

is the universal one.

Therefore the induced endomorphism on S^0 gets mapped to the induced endomorphism on X/f. It is now a 1-categorical matter to check that the induced endomorphism on S^0 is nullhomotopic, which implies the claim. So we have proved (up to adding enough colimits):

Proposition 3.2. *Let $X \xrightarrow{f} X$ be an endomorphism of a pointed category with cofibers, and let $\overline{f} : X/f \to X/f$ be induced by the universal way to fill the square $f \circ f \simeq f \circ f$. Then $\overline{f}$ is nullhomotopic.*

3.2 The case of Moore spectra

You have guessed it, the point is now that in the case of the Moore spectrum S^0/p, the induced morphism $\overline{p}$ is simply not multiplication by p.

This can be confusing, because the following diagram *does* commute:

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d, "p"] & S^0/p \arrow[d, "p"] \\ S^0 \arrow[r] & S^0/p \end{tikzcd}$$

but of course, the map $S^0/p \to S^0/p$ does not only depend on the map $S^0 \to S^0$, but also on a chosen homotopy filling the square

$$\begin{tikzcd} S^0 \arrow[r, "p"] \arrow[d, "p"] & S^0 \arrow[d, "p"] \\ S^0 \arrow[r, "p"] & S^0 \end{tikzcd}$$

The claim is that the nullhomotopy that induces p is not the universal one (which induces $\overline{p} \simeq 0$) - or perhaps more precisely, the universal homotopy does not induce multiplication by p (at least in the case $p = 2$).

In particular, this is saying that there is more than one self-homotopy of p^2 in S^0, equivalently, that S^0 has a nontrivial π_1. But this is the case: $\pi_1(S^0) \cong \mathbb{Z}/2$.

But in fact, this must come from much earlier, as this p here is simply induced by the semi-additivity of **Sp**, so the particular self homotopy of p^2 (or indeed, n^2) is induced by the universal one, in the universal semi-additive category on a point.

This is known as Span(Fin), the $(2,1)$-category of spans of finite sets, or the Lawvere theory of commutative monoids (in the homotopy-theoretic sense), or the effective Burnside category of the trivial group.

The universal object there is just the point. There, multiplication by p is simply given by the span

$$\begin{tikzcd} & \{1, ..., p\} \arrow[dl] \arrow[dr] & \\ \mathrm{pt} & & \mathrm{pt} \end{tikzcd}$$

.

Therefore a filling homotopy is the same data as an automorphism of $\{1, ..., p\}^2$. Let's try to work out what self-homotopy is induced by the semi-additive structure and how it differs from the universal one.

Let us focus on the case $p = 2$.

Before writing out the details, which are tedious to some extent, let us explain the intuition here.

Morally, the morphism $2 : X \to X$ can be described as $x \mapsto x + x$. To keep track of things, let us for a moment distinguish those two x's as x_0 and x_1.

Now if we go to $X \oplus X$, x gets mapped to (x, x), which we will also distinguish notationally as (x_a, x_b) for now. This is then sent in X (via the codiagonal) to $x_a + x_b$, which is itself sent (via 2 again) to $(x_a + x_b)_0 + (x_a + x_b)_1 = x_{a0} + x_{b0} + x_{a1} + x_{b1}$ while $x_0 + x_1$ is sent (via the diagonal) to $((x_0 + x_1)_a, (x_0 + x_1)_b)$ which is itself sent (via the codiagonal) to $x_{0a} + x_{1a} + x_{0b} + x_{1b}$.

Now our homotopy identifies these in X. Of course the indices are lies, and both are just literally $x + x + x + x$, and the universal homotopy notices that and does nothing, but the homotopy coming from the additive structure does things "in the right order" and so it actually switches the two middle terms.

On $\{1, 2, 3, 4\}$ this automorphism has sign -1, and although not much else is retained by passing from $\mathrm{Fin}^{\simeq}$ to the sphere, the sign *is*. In particular, the two homotopies are different.

Let us now try to write this down a bit more precisely - we still leave many details to the reader.

We are interested in the semi-additive category $\mathrm{Fun}(\Delta^1, \mathrm{Span}(\mathrm{Fin}))$, and the morphism 2 from $(\mathrm{pt} \xrightarrow{2} \mathrm{pt})$ to itself.

Note that in a semi-additive category, "2" is always defined unambiguously as the composite $X \to X \oplus X \to X$.

Furthermore, co/limits in a functor category are computed pointwise, so that our homotopy is the composite of two homotopies, namely one of the form

$$\begin{tikzcd} X \arrow[r, "2"] \arrow[d, "\Delta"] & X \arrow[d, "\Delta"] \\ X \oplus X \arrow[r, "2\oplus 2"] & X \oplus X \end{tikzcd}$$

and one of the form

$$\begin{tikzcd} X \oplus X \arrow[r, "2\oplus 2"] \arrow[d, "\nabla"] & X \oplus X \arrow[d, "\nabla"] \\ X \arrow[r, "2"] & X \end{tikzcd}$$

Let us start by pointing out that biproducts are indexed by $\{0, 1\}$, so we can write these coproducts as $X_0 \oplus X_1$, which is what we will do.

So the first square for instance is now written as

$$\begin{tikzcd} X \arrow[r, "2"] \arrow[d, "\Delta"] & X \arrow[d, "\Delta"] \\ X_0 \oplus X_1 \arrow[r, "2\oplus 2"] & X_0 \oplus X_1 \end{tikzcd}$$

Let us furthermore note that $X \xrightarrow{2} X$ is given by a composite $X \to X_a \oplus X_b \to X$, where the indices a and b serve the same purpose as 0 and 1.

The bottom map is then $X_0 \oplus X_1 \to (X_a \oplus X_b)_0 \oplus (X_a \oplus X_b)_1 \to X_0 \oplus X_1$.

Let us specialize the first one to Span(Fin) and $X = \mathrm{pt}$. Then the composite right-down is given by the span $\{a,b\} \times \{0,1\}$, the leg pointing to pt is determined and the leg pointing to $\{0,1\}$ is the second projection.

On the other hand, the composite down-right is easily seen to be $\{a,b\} \times \{0,1\}$ with second projection to $\{0,1\}$. So they are of course isomorphic, but what is the isomorphism (i.e. our homotopy ?).

There at least two ways to answer that question: one is to argue by naturality, namely there is only one isomorphism which is natural in $\{a,b\}$ and a morphims of spans (the twist $\tau : A \times B \cong B \times A$).

Another one is to "desynchronize", by replacing the second X with a possibly different Y and an arbitrary morphism $X \to Y$ (represented by a span Z). One can then see that there is, in general, only one such isomorphism, and in our specific case ($Y = X, Z = X_0 \coprod X_1$) it must be the twist map.

In the end, the main point is that the isomorphism is the twist map.

A similar analysis for the second square shows that when we compose everything, the two homotopies are distinct, essentially because one has signature -1 with respect to the other: if one identifies a and 0 and b to 1, we get two identifications of $\{a,b\} \times \{0,1\}$ with $\{0,1\} \times \{a,b\}$, namely the identity and the twist map, and they differ by a transposition -the identity corresponds to the universal homotopy, and the twist to our additive one.

Note that the difference between these two homotopies from $4 : \mathrm{pt} \to \mathrm{pt}$ to itself is therefore detected by going to the group completion (we know the sign $\Sigma_n \to C_2$ is the same as the map $\pi_1(\mathrm{Fin}^{\simeq}) \to \pi_1(S^0)$), so it follows that the "additive" homotopy for S^0 is distinct from the universal one. That does not seem to imply at once that 2 is nonzero on $S^0/2$, even if it would be really nice to find a combinatorial explanation for this fact.

3.3 The $\mathbb{Z}$-linear case

In this section, we give an example where the induced morphism is zero, namely the case of multiplication by an element in a commutative discrete $\mathbb{Z}$-algebra.

Namely, let R be a discrete commutative ring, $r \in R$ and M an R-module. Then $M/r \simeq M \otimes_R R/r$, and the action of r on M/r is induced from the action of r on R/r. The reason is that $r : \mathrm{id}_{\mathbf{D}(R)} \to \mathrm{id}_{\mathbf{D}(R)}$ is a $\mathbf{D}(R)$-linear natural transformation ($\mathrm{Fun}^L_{\mathbf{D}(R)}(\mathbf{D}(R), \mathbf{D}(R)) \simeq \mathbf{D}(R)$).

It therefore suffices to show that $r : R/r \to R/r$ is 0, R-linearly. There are several ways to do so, we list some of them here.

The first one is an analysis at the level of chain complexes: R/r is represented by the 2-term chain complex $R \xrightarrow{r} R$, and it's not hard to find an R-linear nullhomotopy of the action of r on this complex (one can more generally describe such a nullhomotopy for the chain-level mapping cone of r on a chain complex C).

Another one is to reduce to the universal case again, which is $\mathbb{Z}[x]$. Note that here, $\mathbb{Z}[x]/x$ is literally $\mathbb{Z}$ as a discrete module, and so all calculations can be made in $\mathbf{Ab}$, and discrete $\mathbb{Z}[x]$-modules, where they are obvious. Then, one notices that $R/r \simeq R \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x]/x$, and the action of r on that is the action of x on the right, so we are done.

The last version we will present is more general, and so we deal with it in the last subsection.

Remark 3.3. Before passing on to the next subsection, the reader is invited to spot all the details swept under the rug in this subsection.

Specifically, at several places we use something about R without explicitly saying what. This is not so important because of what we do in the next section.

3.4 A multiplicative criterion

Let us come back to the universal case of earlier, namely $\mathbb{N}_+$ in $\mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*)$.

Note that (by "accident") $\mathbb{N}_+$ is a commutative algebra, and this category is its category of modules in $\mathbf{S}_*$. In particular, it has a (symmetric) monoidal structure given by relative tensor product, but also we have an algebra morphism $\mathbb{N} \to \mathrm{End}(\mathrm{id})$, informally given by having s act on a module M by the action of 1 on M.

In particular, note that in our previous example, the induced morphism $S^0 \to S^0$ is not only induced from the universal homotopy, it is also induced by the above construction (as can be checked in the 1-category $\mathrm{Fun}(B\mathbb{N}, \mathrm{Set}_*)$.

Therefore, suppose $F : \mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*) \to \mathbf{C}$ is a colimit-preserving *monoidal functor* to a pointedly cocomplete monoidals category. Then in particular $\mathbb{N}_+$ is mapped to $\mathbf{1}$, the unit of $\mathbf{C}$, and s to an endomorphism f of $\mathbf{1}$.

More generally, the endomorphism induced by s on an $\mathbb{N}_+$-module M is mapped to the endomorphism induced by f on $F(M)$: this is an easy application of monoidality.

It follows that in this case, the endomorphism of $\mathbf{1}/f$ induced by f (in the sense of the monoidal structure on $\mathbf{C}$, but therefore also in the sense of the universal homotopy) is nullhomotopic, because it is so on $S^0 \in \mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*)$.

So the reasonable question is now: what does such a monoidal functor correspond to ? Without monoidality, just the requirement say that $\mathbb{N}_+$ be mapped to $\mathbf{1}$, this amounts to the data of an endomorphism of $\mathbf{1}$.

In other words, such a functor is an endomorphism on $\mathbf{1}$ with some extra structure. What exactly is that extra structure ?

Concretely, because $\mathbf{C}$ is pointed cocomplete and monoidal, it receives an essentially unique colimit-preserving symmetric monoidal functor. This functor has a right adjoint given by $\mathrm{map}(\mathbf{1}, -)$ and in particular it lands in $\mathrm{End}(\mathbf{1})$-modules in $\mathbf{S}_*$.

The classical Eckmann-Hilton argument upgrades to the statement that $\mathrm{End}(\mathbf{1})$ is an E_2-algebra in $\mathbf{S}_*$, and in fact we formally obtain a monoidal adjunction $\mathrm{End}(\mathbf{1}) - \mathbf{Mod} \rightleftarrows \mathbf{C}$.

Then fixing an endomorphism $f \in \mathrm{End}(\mathbf{1})$, we get an essentially unique colimit preserving functor $\mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*) \to \mathrm{End}(\mathbf{1}) - \mathbf{Mod}$ sending $\mathbb{N}_+$ to $\mathrm{End}(\mathbf{1})$ and s to f (and therefore, upon composition to $\mathbf{C}$, sending $\mathbb{N}_+$ to $\mathbf{1}$ and s to f).

It is now not hard to see that the functor $\mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*) \to \mathbf{C}$ will upgrade to a monoidal functor if and only if the functor $\mathrm{Fun}(B\mathbb{N}, \mathbf{S}_*) \to \mathrm{End}(\mathbf{1}) - \mathbf{Mod}$ does, if and only if the E_1-algebra morphism $\mathbb{N}_+ \to \mathrm{End}(\mathbf{1})$ determined by f upgrades to an E_2-algebra morphism.

In fact, more precisely, the spaces of monoidal enhancements (resp. E_2-enhancement of the algebra morphism) are equivalent.

In other words, the structure we alluded to before on f is exactly the structure of an E_2-enhancement of $\mathbb{N}_+ \to \mathrm{End}(\mathbf{1})$.

Corollary 3.4. *Let R be an E_2-ring spectrum and $r \in \pi_0(R)$.*

Suppose that the E_1-algebra morphism $S^0[t] := \Sigma^\infty_+ \mathbb{N} \to R$ determined by r enhances to an E_2-algebra morphism.

Then r is nullhomotopic on R/r.

Corollary 3.5. *Let R be a discrete commutative ring, and $r \in R$. Then r is nullhomotopic on R/r.*

Question 3.6. Are there examples of (derived) commutative $\mathbb{Z}$-algebras R and $r \in \pi_0(R)$ such that r does not act trivially on R/r ?

Corollary 3.7. *The morphism $S^0[t] \to S^0$ sending t to 2 does not enhance to an E_2-algebra morphism.*

Question 3.8. For odd p, p is apparently nullhomotopic on S^0/p. Does there exist an E_2-enhancement of the corresponding map $S^0[t] \to S^0$?

There are higher obstructions to there being an E_∞-map though, if the answer to the above is yes, are these obstructions enough to bound by some k (depending linearly on p ?) the highest n such that there is an E_n-enhancement ?