Equivariant stable homotopy theory

Introduction

The goal of this note is to collect some basic results about equivariant stable homotopy theory using spectral Mackey functors as our foundation, that is, the point of view developed in [Bar17].

We skip quite the number of details when it comes to results found in that source, so this text is *not* meant to be self-contained (other sources will be used without proof as well !). We do try to provide at least some intuitive explanation for these results.

One can think of this note as an attempt to explain or specialize some of the ideas in [Bar17] (and [BGS19]) in the special case where $C = \mathrm{Fin}_G$. We try to go a bit further, and introduce some ideas in stable equivariant homotopy theory.

Conventions

$\mathbf{S}, \mathbf{Sp}, \mathrm{Fin}, \mathrm{Fin}_G, \mathrm{Fin}_G^{free}, \mathcal{O}_G$ are respectively the ∞-category of spaces, of spectra, the category of finite sets, of finite G-sets (where G will be a finite or profinite group, in the latter case the finite G-sets are assumed to have open stabilizers), of finite free G-sets (only when G is finite), of finite transitive G-sets.

We use ∞-categorical language throughout, in particular any categorical notion which is not preceded by the prefix 1 refers to ∞-categories (so "category" will mean ∞-category, and "1-category" will mean a usual category). 1-categories are viewed as special cases of categories via the nerve functor.

All set-theoretic issues are swept under the rug, unless fixing them requires some actual work, in which case we will mention them.

Overview

- Section 1 introduces the span category, as well as the notion of Mackey functors, which will be our main focus throughout the notes.
- Section 2 introduces the classical functors in equivariant matters : restriction, induction, categorical fixed points, as well as their basic properties and relations. The Wirthmüller isomorphism is also proved there.
- In section 3 we briefly introduce unstable equivariant homotopy theory and connect it to our story via the infinite suspension functor, and the infinite loops. We prove the Segal-tom Dieck splitting there .

- In section 4, we compare categorical fixed points and homotopy fixed points and introduce the class of Borel-equivariant spectra.
- In section 5, we introduce the smash product on $\mathbf{Sp}_G$ and its basic properties. We introduce some standard algebras, and prove Balmer, Dell'Ambrogio and Sanders's result, which identifies restriction as extension of scalars for some of these.
- In section 6, we introduce geometric fixed points and prove some of their basic properties. We use them to show that the representation spheres are invertible, and state (but don't prove) the monoidal universal property of G-spectra with respect to G-spaces.
- In section 7, we give a brief introduction to induction theory, and explain how to change it to obtain derived induction theory, *à la* Mathew-Naumann-Noel.
- Finally, in section 8, we mention, for completeness, a description of G-spectra when G is profinite.

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1 What are Mackey functors ?

One can find numerous definitions of Mackey functors in the literature, but we will only present one, the one that seems best suited for our purposes - it is the one present in [Bar17].

However, we don't need the definition in there with as much generality, as we focus on the case of groups. We can thus freely give the following definition:

Definition 1.1. Let C be a category with pullbacks. $\mathrm{Span}(C)$ is the category of spans in C: its objects are given by objects of C, its morphisms by spans, and composition is given (up to contractible ambiguity - which is what ∞-categories excel at doing) by taking pullbacks.

We won't define it more precisely; it is done in [Bar17], and it is technical. See [Bar17, 3.7] for a description of the n-simplices. Note that Barwick calls this category "the effective Burnside category" and denotes it by A^{eff}.

Remark 1.2. For $x, y \in C$, $\mathrm{map}_{\mathrm{Span}(C)}(x, y) \simeq (C_{/\{x,y\}})^{\simeq}$, the core groupoid of the category of spans.

Notation 1.3. In the special case $C = \mathrm{Fin}_G$, we will let $\mathrm{Span}(G) := \mathrm{Span}(\mathrm{Fin}_G)$, although this is an evident abuse of notation.

Remark 1.4. There is a more general, but also more complicated definition where one can even specify which morphisms are allowed to go "backwards" and which are allowed to go in the right direction. This is all done in great detail in [Bar17], but we will not need it : in our situation of interest, every morphism is ingressive and eggressive.

There are natural functors $C \to \mathrm{Span}(C), C^{\mathrm{op}} \to \mathrm{Span}(C)$, as well as an equivalence

$$\mathrm{Span}(C)^{\mathrm{op}} \simeq \mathrm{Span}(C) \tag{1}$$

(simply given by changing the spans' direction) which are all compatible. For instance, if we look at $C \to \mathrm{Span}(C) \to \mathrm{Span}(C)^{\mathrm{op}}$, then this is the opposite functor of $C^{\mathrm{op}} \to \mathrm{Span}(C)$.

There are now two basic definitions we need from [Bar17]:

Definition 1.5. A category C is said to have direct sums if the following hold:

1. C is pointed (that is, it has a zero object)
2. C has finite products and coproducts
3. For any finite tuple $(X_i)_{i\in I}$ of objects, the natural map $\coprod_{i\in I} X_i \to \prod_{i\in I} X_i$ is an equivalence.

We also say that C is semiadditive in this situation.

Definition 1.6. A category C is said to be disjunctive if it has pullbacks, finite coproducts, and finite coproducts are *disjoint* and *universal*.

This latter condition can in turn be reformulated by saying that the natural map $\prod_{i\in I} C_{/X_i} \to C_{/\coprod_{i\in I} X_i}$ is an equivalence. Informally, its inverse would then be $(U \to \coprod_{i\in I} X_i) \mapsto (U \times_{\coprod_{i\in I} X_i} X_j \to X_j)_{j\in J}$

Lemma 1.7. *([Bar17, 4.3]) Suppose C is a disjunctive category. Then* $\mathrm{Span}(C)$ *has direct sums.*

Proof. One proves that $C \to \mathrm{Span}(C)$ preserves finite coproducts, which is enough given the compatibility between the various functors $C \to \mathrm{Span}(C)$ and $C^{\mathrm{op}} \to \mathrm{Span}(C)$. So for $x, y \in C$, their coproduct $x \sqcup y$ in C is their biproduct in $\mathrm{Span}(C)$, with the same inclusion maps, and with projection maps the span $x \sqcup y \leftarrow x = x$ (and similaly for y).

For this, one proves that, letting $\emptyset \in C$ denote the initial element, $\mathrm{map}_{\mathrm{Span}(C)}(\emptyset, x)$ is contractible. But this mapping set is the core groupoid of $C_{/\{\emptyset,x\}}$, which is contractible, as any object with a map $y \to \emptyset$ in C is initial (and then the category of initial objects is a contractible groupoid) - this last fact comes from the universality and disjointness of coproducts: $C_{/\emptyset} \simeq C^{\emptyset} \simeq *$.

Then one proves that $x \sqcup y$ is a coproduct in $\mathrm{Span}(C)$ which also comes from disjointness and universality of coproducts in C. The details are in the cited reference. $\square$

In particular, the cartesian monoidal structure on $\mathrm{Span}(C)$ is also cocartesian, and so the forgetful functor induces an equivalence $\mathrm{CMon}(\mathrm{Span}(C)) \to \mathrm{Span}(C)$ (by [Lur17, 2.4.3.10.]). The idea is simply that one can see from the definition of a commutative monoid that the multiplication $A \times A \to A$ is induced by the identity $A \to A, A \to A$ (which makes sense, as $A \times A$ is also the coproduct of A and A). It's an easy exercise 1-categorically to check it.

More generally,

Proposition 1.8. *Let E be a semiadditive category; then the forgetful functor* $\mathrm{CMon}(E) \to E$ *is an equivalence.*

This is a very interesting property, which we will use a couple of times.

Definition 1.9. Let C be a disjunctive category, and E a semiadditive category. We defined E-valued Mackey functors on C to be (co)product preserving functors $\mathrm{Span}(C) \to E$.

We put $\mathbf{Mack}(C; E) := \mathrm{Fun}^{\oplus}(\mathrm{Span}(C), E)$. When $C = \mathrm{Fin}_G$, we let $\mathbf{Mack}_G(E) := \mathbf{Mack}(\mathrm{Fin}_G; E)$.

When $C = \mathrm{Fin}_G$, $E = \mathbf{Sp}$, $\mathbf{Mack}_G(\mathbf{Sp})$ (the category of *spectral Mackey functors*) is equivalent to the category of genuine G-spectra, which is modelled by the model 1-category of orthogonal G-spectra (this is a theorem of Guillou-May). We will also denote it by $\mathbf{Sp}_G$.

Note that Span is functorial in categories with pullbacks and pullback preserving functors.

Lemma 1.10. *For E a semiadditive (resp. additive, resp. stable) category,* $\mathbf{Mack}(C; E)$ *is semiadditive (resp. additive, resp. stable).*

Remark 1.11. In fact, we will see in section 2 that $\mathrm{Fun}^{\times}(\mathrm{Span}(C), E)$ is semiadditive even if E only has products and is not semiadditive.

Proof. $\mathrm{Fun}(\mathrm{Span}(C), E)$ clearly satisfies this, and in all cases $\mathrm{Fun}^{\oplus}(\mathrm{Span}(C), E)$ is closed under the necessary operations for it to be true. ☐

2 Restriction, Induction, Fixed points

We can now define various functors usually introduced in equivariant homotopy theory: restriction, induction, (genuine) fixed points; and we can relate them. Their basic properties reflect properties of $\mathrm{Span}(G)$.

Definition 2.1. Let $K \leq G$ be an open subgroup, and $X \in \mathbf{Mack}_G(E)$. Then $X^K := X(G/K)$ is called the categorical fixed points of X.

Remark 2.2. So a Mackey functor, or a genuine G-spectrum, can be thought of as a spectrum with G-action where we somehow remember "strict" fixed points, as opposed to homotopy fixed points.

To make more sense of this last remark, note that G/e in Fin_G has a canonical G-action (by right translation), so it has one in $\mathrm{Span}(G)$, and so the evaluation of a Mackey functor X at G/e has a G-action: a Mackey functor is more data than just this E-object with G-action.

Now let $H \leq G$ be a subgroup. We have a functor $forget^G_H : \mathrm{Fin}_G \to \mathrm{Fin}_H$ given by forgetting the G-action and only remembering the H-action; but also a functor $\mathrm{Fin}_H \to \mathrm{Fin}_G$ given by induction, $X \mapsto G \times_H X := (G \times X)/((gh, x) \sim (g, hx), h \in H)$. We have $G \times_H - \dashv forget^G_H$

Both of these preserve pullbacks, and so they induce functors $\mathrm{Span}(G) \to \mathrm{Span}(H), \mathrm{Span}(H) \to \mathrm{Span}(G)$ which we denote in the same way. Since Span is functorial, we expect it to send adjunctions to adjunctions, and so we still have $G \times_H - \dashv forget^G_H$ at this level.

However, that is not quite true. Span is functorial on $\mathbf{Cat}^{lex}_\infty$ whose mapping spaces are the core groupoids of functor categories, in particular Span is not defined a priori on non invertible natural transformations (such as the ones appearing in an adjunction).

However, one can check that Span is defined (and functorial) on *cartesian* natural transformations, that is, the natural transformations in which each naturality square is cartesian, i.e. for any $p : Z \to X$, the canonical map $F(Z) \to F(X) \times_{G(X)} G(Z)$ must be an equivalence.

Accepting that, it's easy to show that in the case of $G \times_H - \dashv forget^G_H$, both the unit and the co-unit do satisfy this cartesianness condition, and therefore do induce natural transformations on Span, and thus an adjunction, as the triangle identities follow from the original ones.

Remark 2.3. This functoriality is explained in [BH17, Proposition C.20] and the discussion before that statement. Note, however, that it has a more concrete explanation: a natural transformation from F to G can be seen as a functor $\Delta^1 \times \mathbf{C} \to \mathbf{D}$. We'd want it to induce a functor $\Delta^1 \times \mathrm{Span}(\mathbf{C}) \to \mathrm{Span}(\mathbf{D})$, and for that we'd usually want to look at the composite $\Delta^1 \times \mathrm{Span}(\mathbf{C}) \to \mathrm{Span}(\Delta^1 \times \mathbf{C}) \to \mathrm{Span}(\mathbf{D})$.

But for that we need $\Delta^1 \times \mathbf{C} \to \mathbf{D}$ to be a functor to which Span can be applied. Using the more general setup from [Bar17], we see that if we take Δ^1 to only have equivalences as egressive morphisms ("wrongway morphisms"), then the condition that $\Delta^1 \times \mathbf{C} \to \mathbf{D}$ is a morphism to which Span can be applied is exactly the condition that the original natural transformation be cartesian.

In fact, the proof of the functoriality in *loc. cit.* follows the same idea. We don't need all this functoriality so let us stick to that remark.

But using the self-duality of $\mathrm{Span}(G)$ and the compatibility with the functors $\mathrm{Fin}_G \to \mathrm{Span}(G)$, we also get $forget^G_H \dashv G \times_H -$. This is the origin of the so-called Wirthmüller isomorphism, as we will later see.

Definition 2.4. We define $\mathrm{res}^G_H : \mathbf{Mack}_G(E) \to \mathbf{Mack}_H(E)$ by precomposition with $G \times_H -$; and Ind^G_H by precomposition with $forget^G_H$.

Remark 2.5. To see why this makes sense, let $X \in \mathbf{Mack}_G(E)$, and let $K \leq H \leq G$ be subgroups. Then $(\mathrm{res}^G_H X)^K = (\mathrm{res}^G_H X)(H/K) = X(G \times_H H/K) \simeq X(G/K) = X^K$. So the restriction of X has the same fixed points as X: that's why we defined it that way.

We defined Ind that way to get:

Lemma 2.6. Ind^G_H *is left adjoint to* res^G_H*.*

Proof. This follows at once from the definition of Mackey functors and the fact that $G \times_H - \dashv forget^G_H$ (precomposition with each one preserves additive functors so induces an adjunction in the other direction between the full subcategories of additive functors, i.e. categories of Mackey functors - note also that precomposition changes the handedness of adjunctions). $\square$

Theorem 2.7 (Wirthmüller isomorphism)**.** Ind^G_H *is also right adjoint to* res^G_H

Proof. Same as above, using this time $forget^G_H \dashv G \times_H -$. $\square$

Remark 2.8. This is often stated as saying that induction and coinduction agree. Indeed for other flavours of equivariant homotopy theory where some of the things we assumed to be finite here aren't, it might be the case that induction and coinduction are both defined but not equivalent.

However in this setup, this statement seems more natural as it is the one that follows immediately from what we observed earlier.

Let's look at some basic properties of these: both of these functors are left and right adjoints, so they commute with arbitrary limits and colimits; this will be rather useful.

Moreover, if we evaluate $\mathrm{Ind}^G_H X$ at G/e (so we take the "underlying spectrum"), it's like evalutating X at $\coprod_{x \in G/H} xH$ that is, something isomorphic to $\coprod_{x \in G/H} H$; and so, because X preserves coproducts, this is $\bigoplus_{x \in G/H} X(H/e)$. In other words:

Lemma 2.9. *The underlying spectrum of* $\mathrm{Ind}_H^G X$ *is a* G/H*-fold coproduct of the underlying spectrum of* X*.*

Remark 2.10. This is of course what we'd expect from any reasonable candidate to be called "induction".

We now want to introduce homotopy fixed points, and compare them with categorical fixed points. A good way of doing that is to relate E-objects with G-action and E-valued Mackey functors.

We feel that the clearest way of doing so is using the following result of Glasman [Gla15, A.1]:

Theorem 2.11. *Let* $\mathbf{E}$ *be a semiadditive category. Then the restriction functor*

$$\mathbf{Mack}(\mathrm{Fin}_G^{free}, \mathbf{E}) \to \mathrm{Fun}(BG, \mathbf{E})$$

is an equivalence.

In other words, $\mathrm{Span}(Fr^G)$ *is the free semiadditive* ∞*-category on* BG*.*

We will not prove it here, but the author has written another note about this theorem, where a somewhat simpler proof is given than in Glasman's paper (perhaps should we say "less technical" - using Lawvere technology).

In particular, right Kan extension along $\mathrm{Span}(\mathrm{Fin}_G^{free}) \to \mathrm{Span}(G)$ (when it is defined) provides a right adjoint to the forgetful $\mathbf{Mack}_G(E) \to \mathbf{Mack}(\mathrm{Fin}_G^{free}, E) \simeq \mathrm{Fun}(BG, E)$. We now assume E is presentable to have the existence of such adjoints guaranteed, although we could probably get away with much weaker hypotheses (but our examples of interest have $E = \mathbf{Sp}, \mathbf{Sp}_{\geq 0}, \mathrm{CMon}$, or perhaps $\mathbf{S}$ in the nonadditive case - all of which are presentable).

Assumption 2.12. From now on, E will be a presentable additive category.

We can thus define the homotopy fixed points of a Mackey functor, by first restricting it to BG, and then taking the homotopy fixed points, i.e. the limit over BG. One can also restrict from BG to BH for some subgroup H, and then taking the homotopy fixed points, or restrict from $\mathrm{Span}(G)$ to $\mathrm{Span}(H)$, then restrict to BH, and take the homotopy fixed points. The following is left as an exercise:

Lemma 2.13. *Let* $X \in \mathbf{Mack}_G(E)$*. Then* $(\mathrm{res}_H^G X)^{hH} \simeq X^{hH}$*, where* $(-)^{hH}$ *denotes the homotopy fixed points.*

Let X be an E-object with G-action; and let $\tilde{X}$ be the right Kan extension to $\mathrm{Span}(G)$, additivized to be a Mackey functor - that is, after right Kan extending, apply the right adjoint to the inclusion $\mathrm{Fun}^\times(\mathrm{Span}(G), E) \to \mathrm{Fun}(\mathrm{Span}(G), E)$. We would like to compare $\tilde{X}^H$ and X^{hH}, for $H \leq G$; more generally, we would like to compare X^{hH} and X^H for $X \in \mathbf{Mack}_G(E)$, but in the latter case we don't expect an equivalence between the two (cf. for instance the Segal-tom Dieck splitting, which we will discuss later).

We will need an important fact:

Proposition 2.14. *Assume* C *is disjunctive, so that* $\mathrm{Span}(C)$ *is semiadditive.*

Let E *be a category with finite products. Then the forgetful functor* $\mathbf{Mack}(C; \mathrm{CMon}(E)) \to \mathrm{Fun}^\times(\mathrm{Span}(C), E)$ *is an equivalence.*

This is a special case of:

Theorem 2.15. *Suppose C, D are categories with finite products, and suppose C is semi-additive. Then the forgetful functor* $\mathrm{Fun}^\times(C, \mathrm{CMon}(D)) \to \mathrm{Fun}^\times(C, D)$ *is an equivalence.*

This theorem is, in turn, equivalent to :

Proposition 2.16. *Suppose C is a semi-additive category. Then the forgetful functor* $\mathrm{CMon}(C) \to C$ *is an equivalence.*

Proof of the theorem from the proposition. Because products commute with products, $\mathrm{Fun}^\times(C, \mathrm{CMon}(D)) \simeq \mathrm{CMon}(\mathrm{Fun}^\times(C, D))$.

Now, $\mathrm{Fun}^\times(C, D)$ is equivalent, via the Yoneda embedding of D, to a certain full subcategory of $\mathrm{Fun}(C \times D^{\mathrm{op}}, \mathbf{S})$, namely, that of functors which preserve finite products in the first variable and are pointwise (in the first variable) representable (in the second variable).

It is in particular equivalent to the full subcategory of $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Fun}^\times(C, \mathbf{S}))$ on those functors that are representable after every evaluation at $c \in C$.

Suppose we can prove the case where $D = \mathbf{S}$. Then it follows from the proposition that this last category is equivalent, along the forgetful functor, to the full subcategory of $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Fun}^\times(C, \mathrm{CMon}))$ on those functors that are pointwise (in C) representable (after forgetting the monoid structure), i.e. to the full subcategory of $\mathrm{Fun}^\times(C, \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{CMon}))$ on pointwise representable functors. We conclude using the fact that $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{CMon}) \simeq \mathrm{CMon}(\mathrm{Fun}(D^{\mathrm{op}}, \mathbf{S}))$ and that the representable functors on the left are equivalent to CMon(the representable functors), i.e. to $\mathrm{CMon}(D)$, and that it is now easy to prove that $\mathrm{Fun}^\times(C, \mathrm{CMon}(D))$ is semi-additive.

For $D = \mathbf{S}$, we observe that $\mathrm{Fun}^\times(C, \mathbf{S})$ is the accessible localization of $\mathrm{Fun}(C, \mathbf{S})$ at the maps $よ(c) \coprod よ(d) \to よ(c \oplus d)$, where $よ : C^{\mathrm{op}} \to \mathrm{Fun}(C, \mathbf{S})$ is the contravariant Yoneda embedding.

In particular, because the "semi-additivity" comparison map is given by $よ(c) \coprod よ(d) \to よ(c \oplus d) \simeq よ(c \times d) \xrightarrow{\simeq} よ(c) \times よ(d)$, it becomes an equivalence in the localization.

Now, any $F \in \mathrm{Fun}^\times(C, \mathbf{S})$ is a canonical colimit of representables, and in fact this colimit is sifted when F preserves products (this is an instructive exercise), and so for F, G preserving finite products, we find that their "semi-additivity comparison map" is also an equivalence after localization (note if $I \to C$ is a sifted diagram, then $\mathrm{colim}_{i \in I}(よ(c_i) \coprod よ(d)) \simeq \mathrm{colim}_{i \in I} よ(c_i) \coprod よ(d)$, using that I being sifted implies that it is weakly contractible, and thus that $\mathrm{colim}_I よ(d) \simeq よ(d)$). This means precisely that $\mathrm{Fun}^\times(C, \mathbf{S})$ is semi-additive. $\square$

Proof of the proposition. This follows from [Lur17, 2.4.3.9.] and the fact that if C is semi-additive, then its cartesian symmetric monoidal structure is also its cocartesian symmetric monoidal structure.

$\square$

3 Relation to genuine G-spaces

It would be quite weird to write a whole note about equivariant stable homotopy theory without mentioning equivariant homotopy theory, and thus genuine G-spaces. As with spectra, spaces with G-action, in the ∞-categorical setting, aren't always sufficient for our purposes, and so we look at objects containing more data. This is the notion of a genuine G-space:

Definition 3.1. $\mathbf{S}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{S}) \simeq \mathrm{Fun}^\times(\mathrm{Fin}_G^{\mathrm{op}}, \mathbf{S})$[1] is the category of G-spaces; and $\mathbf{S}_{G,*} := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{S}_*)$ is the category of pointed G-spaces.

Remark 3.2. By Elmendorf's theorem, this is presented by the 1-category of topological spaces with G-action, the weak equivalences being G-equivariant maps $X \to Y$ inducing weak equivalences on each fixed point set $X^H \to Y^H$.

So 1-categorically, we may think of them as spaces with G-action, but ∞-categorically, or homotopically, one must think of them as objects containing more data.

Heuristically, G-spectra should be what happens when you take G-spaces and formally invert all representation spheres - as opposed to the nonequivariant situation, where we only invert the spheres S^n. A representation sphere is the one-point compactification of a (real) linear representation V, we usually denote it by S^V.

You want to make the operation of smashing with S^V invertible. Since $S^{V\oplus W} \cong S^V \wedge S^W$, and since any representation is a summand of the regular representation ρ, it suffices to invert that one. But ρ is induced from a G-action on a finite set, so we can already see some form of connection with our finite G-sets.

Our presentation will differ from that heuristic, although we hope to recover the heuristic from that presentation (to recover it, we would first need a smash product on G-spectra, which we can define but it will come later). What we will do is first define an infinite loop space functor $\Omega^\infty : \mathbf{Sp}_G \to \mathbf{S}_{G,*}$.

Definition 3.3. $\Omega^\infty : \mathbf{Sp}_G \to \mathbf{S}_{G,*}$ is defined as the following composite:

$$\mathbf{Sp}_G = \mathrm{Fun}^\oplus(\mathrm{Span}(G), \mathbf{Sp}) \overset{\Omega^\infty}{\to} \mathrm{Fun}^\times(\mathrm{Span}(G), \mathbf{S}_*) \to \mathrm{Fun}^\times(\mathrm{Fin}_G^{\mathrm{op}}, \mathbf{S}_*) \to \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{S}_*) = \mathbf{S}_{G,*}$$

(which we can continue by postcomposing with $\mathbf{S}_* \to \mathbf{S}$, to get a functor we would still denote by Ω^∞ - note that we can replace $\mathbf{S}_*$ by $\mathbf{S}$ at any stage of this composition without altering the end result)

Each of these functors obviously preserves limits, and is accessible, therefore so is the composite. In particular, Ω^∞ has a left adjoint, denoted by Σ^∞ (or Σ^∞_+ in the unpointed case).

Any 1-categorical G-space defines a genuine G-space by Elmendorf's theorem, which has $X(G/H) = X^H$ (the literal fixed points, nothing derived here). In particular, G-sets yield genuine G-spaces, which we will abusively denote in the same way (e.g. G/H).

Now note that $\mathbf{Sp}_G$ is stable by lemma 1.10, so it is canonically $\mathbf{Sp}$-enriched; in fact it is presentable, so it's a $\mathbf{Sp}$-module in $\mathbf{Pr}^L_{st}$ [Lur17, 4.8.2.18]. We will let Map_G denote the mapping spectra in $\mathbf{Sp}_G$. We have:

Proposition 3.4. *The functors* $\mathrm{Map}_G(\Sigma^\infty_+ G/H, -)$ *and* $(-)^H$ *are equivalent.*

[1]This equivalence is given by restriction/right Kan extension, it can be restated as "Fin_G is obtained from $\mathcal{O}_G$ by freely adding finite coproducts"

Proof.

$$\mathrm{map}_{\mathbf{Sp}}(X, \mathrm{Map}_G(\Sigma^\infty_+ G/H, Y)) \simeq \mathrm{map}_{\mathbf{Sp}_G}(X \otimes \Sigma^\infty_+ G/H, Y) \simeq \mathrm{map}_{\mathbf{Sp}_G}(\Sigma^\infty_+ G/H, \mathrm{Map}(X, Y))$$

$$\simeq \mathrm{map}_{\mathbf{S}_G}(G/H, \Omega^\infty \mathrm{Map}(X, Y)) \simeq \mathrm{map}_{\mathbf{S}_G}(G/H, \mathrm{map}_{\mathbf{Sp}}(X, Y))$$

where $\otimes$ here comes from the tensoring $\mathbf{Sp} \otimes \mathbf{Sp}_G \to \mathbf{Sp}_G$ we mentioned earlier (and so the first equivalence essentially comes from the definition of Map_G), $\mathrm{Map}(X, Y)$ is viewed as a functor, $T \mapsto \mathrm{Map}(X, Y(T))$, which is of course still a spectral Mackey functor.

Now, to conclude this computation, note that as a functor on Fin_G, G/H is defined by $(G/H)(G/K) := (G/H)^K = \mathrm{map}_{\mathrm{Fin}_G}(G/K, G/H)$, so G/H is more literally the representable functor on G/H. Therefore, by the Yoneda lemma,

$$\mathrm{map}_{\mathbf{S}_G}(G/H, \mathrm{map}_{\mathbf{Sp}_G}(X, Y)) \simeq \mathrm{map}_{\mathbf{Sp}}(X, Y)(G/H) \simeq \mathrm{map}_{\mathbf{Sp}}(X, Y^H)$$

and so, in the end,

$$\mathrm{map}_{\mathbf{Sp}}(X, \mathrm{Map}_G(\Sigma^\infty_+ G/H, Y)) \simeq \mathrm{map}_{\mathbf{Sp}}(X, Y^H)$$

All these equivalences are natural in X, Y (even H for that matter), and so, again by the Yoneda lemma, we get the desired result. $\square$

Corollary 3.5. *The $\Sigma^\infty_+ G/H, H \leq G$ form a set of compact generators of $\mathbf{Sp}_G$.*

Proof. $\mathbf{Mack}_G(\mathbf{Sp}) \subset \mathrm{Fun}(\mathrm{Span}(G), \mathbf{Sp})$ is closed under sifted colimits, therefore they are computed pointwise in the smaller category.

It follows that the $\Sigma^\infty_+ G/H$, which represent $\Omega^\infty(-)^H$, preserve filtered colimits, and so they are indeed compact.

Moreover, since this is the case in the bigger one, the evaluation functors $ev_{G/H}, H \leq G$ jointly detect zero objects, therefore by the previous propositions, the $\Sigma^\infty_+ G/H$ do form a system of generators. $\square$

Remark 3.6. It is clear from the definition of $(-)^H$ in both cases and the definition of Ω^∞ that there is a natural equivalence $\Omega^\infty(X^H) \simeq (\Omega^\infty X)^H$

One can also relate restriction in the case of G-spectra and G-spaces:

Definition 3.7. For G-spaces, we define res^G_H by precomposition along $\mathrm{Fin}^{\mathrm{op}}_H \to \mathrm{Fin}^{\mathrm{op}}_G, X \mapsto G \times_H X$, and CoInd^G_H by precomposition along the forgetful $\mathrm{Fin}^{\mathrm{op}}_G \to \mathrm{Fin}^{\mathrm{op}}_H$.

Remark 3.8. This definition is still the right one, as it sill gives $(\mathrm{res}^G_H X)^K = X^K, K \leq H$. However, as a functor defined on $\mathrm{Fin}^{\mathrm{op}}_H$ (not Fin_H), $G \times_H -$ does *not* have a right adjoint (for instance it does not preserve coproducts).

So we cannot define induction simply by precomposition along a certain functor. However, this is reasonable: induction of a space X looks "something like" a G/H-indexed *co*product of X, and precomposition along a functor could only relate this induction with a product.

A better argument for why this can't be is that if we could define it by precomposition, it would be compatible with Ω^∞, which it is not supposed to be.

Since, on $\mathrm{Fin}_H^{op}, \mathrm{Fin}_G^{op}$, the forgetful functor is left adjoint to $G \times_H -$, it follows that $\mathrm{res}_H^G \dashv \mathrm{CoInd}_H^G$.

Moreover, the functors we defined here are compatible with the relevant functors $\mathrm{Span}(G) \to \mathrm{Span}(H)$ (and backwards) along the inclusion $\mathrm{Fin}_G^{op} \to \mathrm{Span}(G)$, from which it follows immediately:

Proposition 3.9. $\mathrm{res}_H^G \Omega^\infty X \simeq \Omega^\infty(\mathrm{res}_H^G X)$*, and similarly for coinduction.*

On the other hand, res_H^G clearly commutes with all limits (they are computed in $\mathbf{S}$ or $\mathbf{S}_*$), and it is clearly accessible, so by the adjoint functor theorem it has a left adjoint: we define Ind_H^G to be that left adjoint.

Corollary 3.10. $\Sigma_+^\infty \mathrm{Ind}_H^G \simeq \mathrm{Ind}_H^G \Sigma_+^\infty$
Moreover, $\Sigma_+^\infty \mathrm{res}_H^G \simeq \mathrm{res}_H^G \Sigma_+^\infty$
(and similarly without the $+$*)*

Proof. The right adjoints commute, therefore so do the left adjoints. ☐

Definition 3.11. We let S_G^0 denote the G-equivariant sphere spectrum, that is, it is $\Sigma_+^\infty *$, where $*$ is the trivial G-space.

Corollary 3.12. $\mathrm{res}_H^G S_G^0 \simeq S_H^0$
$\mathrm{Ind}_H^G S_H^0 \simeq \Sigma_+^\infty G/H$

Proof. The first part follows at once from $\Sigma_+^\infty \circ \mathrm{res}_H^G \simeq \mathrm{res}_H^G \circ \Sigma_+^\infty$.

For the second part, note that $\mathrm{Ind}_H^G \Sigma_+^\infty * \simeq \Sigma_+^\infty \mathrm{Ind}_H^G *$.

Now note that on G-spaces, $\mathrm{map}(\mathrm{Ind}_H^G *, X) \simeq \mathrm{map}(*, \mathrm{res}_H^G X)$, but $*$ is also just $\mathrm{map}(-,*)$, so by the Yoneda lemma, $\mathrm{map}(*, \mathrm{res}_H^G X) \simeq X(G \times_H *) \simeq X(G/H) \simeq \mathrm{map}(G/H, X)$: $G/H \simeq \mathrm{Ind}_H^G *$. ☐

Remark 3.13. In particular, $\mathrm{Ind}_H^G S_H^0$ is *not* S_G^0. For instance, when $H = 1$, it is $\Sigma_+^\infty G$.

We will now analyze Σ_+^∞ a tad closer and see that it is actually related to K-theory. This will be the "equivariant Barratt-Priddy-Quillen theorem".

Recall the definition of Ω^∞ as a long composite:

$$\mathbf{Sp}_G = \mathrm{Fun}^\oplus(\mathrm{Span}(G), \mathbf{Sp}) \xrightarrow{\Omega^\infty} \mathrm{Fun}^\times(\mathrm{Span}(G), \mathbf{S}_*) \to \mathrm{Fun}^\times(\mathrm{Fin}_G^{op}, \mathbf{S}_*) \to \mathrm{Fun}(\mathcal{O}_G^{op}, \mathbf{S}_*) = \mathbf{S}_{G,*}$$

Let us now add some steps inbetween. Notice, indeed, that $\Omega^\infty : \mathbf{Sp} \to \mathbf{S}$ factors through an equivalence of connective spectra with grouplike commutative monoids in spaces, so that this functor can be written as:

$$\mathbf{Sp}_G = \mathrm{Fun}^\oplus(\mathrm{Span}(G), \mathbf{Sp}) \xrightarrow{\Omega^\infty} \mathrm{Fun}^\times(\mathrm{Span}(G), \mathrm{CMon}) \to \mathrm{Fun}^\times(\mathrm{Span}(G), \mathbf{S}) \to \mathrm{Fun}^\times(\mathrm{Fin}_G^{op}, \mathbf{S}) = \mathbf{S}_G$$

Now the left adjoint, Σ_+^∞, can thus be described as a composite of left adjoints. In the usual story, there are two points of complexity : going from $\mathbf{S}$ to CMon, which takes a space to the free commutative monoid on it ($* \mapsto \coprod_n B\Sigma_n$ for instance), and then going from

CMon $\to$ **Sp**, which takes a commutative monoid to its group completion. Noticing that gives a quick proof of the usual Barratt-Priddy-Quillen theorem.

Here we have the same complexity, but we also have a problem, as we need to go from $\mathrm{Fun}(\mathrm{Span}(G), \mathrm{CMon})$ to $\mathrm{Fun}^\times(\mathrm{Span}(G), \mathrm{CMon})$ as well : indeed the operation of taking the free commutative monoid on a space is not product preserving - this additional complexity could be problematic.

However, we are in luck, because of the work we did in section 2 ! Indeed, the functor $\mathrm{Fun}^\times(\mathrm{Span}(G), \mathrm{CMon}) \to \mathrm{Fun}^\times(\mathrm{Span}(G), \mathbf{S})$ is an equivalence, by proposition 2.14; so its left adjoint essentially does nothing: it takes an "**S**-valued Mackey functor" and endows it with its natural commutative monoid structure.

It follows that the left adjoint, Σ^∞_+, is given by Kan extending along $\mathrm{Fin}_G^{\mathrm{op}} \to \mathrm{Span}(G)$, remembering that we were a commutative monoid, and then group completing (it turns out that group completing preserves products: indeed those are the same as coproducts in $\mathbf{Sp}, \mathrm{CMon}$, and since group completion is a left adjoint, it preserves that - hence, no need to "mackeyfy", we just need to group complete !)[2]. This yields the equivariant Barratt-Priddy-Quillen theorem:

Theorem 3.14 (Equivariant BPQ)**.** *For any $X \in \mathrm{Span}(G)$, $S^0_G(X) \simeq K((\mathrm{Fin}_G)_{/X})$, where $(\mathrm{Fin}_G)_{/X}$ is the symmetric monoidal category with tensor product given by $(Y \to X) \otimes (Z \to X) = (Y \sqcup Z \to X)$*

Remark 3.15. Taking $X = G/e$, we see that $(\mathrm{Fin}_G)_{/G} \simeq \mathrm{Fin}$ symmetric monoidally, and with trivial G-action on the right hand side, so that the underlying spectrum with G-action of S^0_G is the sphere spectrum with trivial G-action. In particular, $(S^0_G)^{hG} \simeq \mathrm{Map}(\Sigma^\infty_+ BG, S^0)$.

On the other hand, taking $X = G/G$, we see that $(\mathrm{Fin}_G)_{/*} \simeq \mathrm{Fin}_G$ symmetric monoidally, so $(S^0_G)^G \simeq K(\mathrm{Fin}_G)$. We will later describe this K-theory, and get the Segal-tom Dieck splitting out of this.

The comparison of these two objects is the object of the Segal conjecture (now a theorem).

Proof. Recall that S^0_G is $\Sigma^\infty_+ *$, so we may start with $* \in \mathbf{S}_G$. As a functor $\mathrm{Fin}_G^{\mathrm{op}} \to \mathbf{S}$, this can be described as $\mathrm{map}(*, -)$ ($*$ is initial in $\mathrm{Fin}_G^{\mathrm{op}}$).

Therefore (this is a general fact, left as an exercise) its left Kan extension along $\mathrm{Fin}_G^{\mathrm{op}} \to \mathrm{Span}(G)$ is just $\mathrm{map}_{\mathrm{Span}(G)}(*, -)$. Its commutative monoid structure comes from the fact that $\mathrm{Span}(G)$ is semiadditive.

Therefore $S^0_G(X)$ is just the group completion of $\mathrm{map}_{\mathrm{Span}(G)}(*, X)$. But this is a symmetric monoidal 1-groupoid, given by $((\mathrm{Fin}_G)_{/\{*,X\}})^{\simeq}$ under disjoint union, by remark 1.2. This is clearly equivalent symmetric monoidally to $(\mathrm{Fin}_G)^{\simeq}_{/X}$; therefore its group completion is exactly the K-theory $K((\mathrm{Fin}_G)_{/X})$, as claimed. $\square$

Remark 3.16. In a very similar way, this can be used to describe $\Sigma^\infty_+ X, X \in \mathrm{Fin}_G$.

We will now use this description to get a version of the Segal-tom Dieck splitting theorem.

[2] We are using implicitly that left Kan extending preserves product-preserving presheaves. Indeed, left Kan extension sends representables to representables, and product-preserving presheaves are precisely sifted colimits of representables.

Corollary 3.17 (Segal-tom Dieck splitting)**.** *The categorical fixed points of the equivariant sphere are described as follows:*

$$(S^0_G)^G \simeq \bigoplus_{[H]} \Sigma^\infty_+ BW_G(H)$$

where $[H]$ runs over conjugacy classes of subgroups of G, and for a subgroup H, $W_G(H) := N_G(H)/H$, where $N_G(H)$ is the normalizer of H, that is, the subgroup of g's such that $gHg^{-1} = H$.

Remark 3.18. For a subgroup H, $(S^0_G)^H = (\mathrm{res}^G_H S^0_G)^H = (S^0_H)^H$, so this description actually tells us about all the categorical fixed points.

Remark 3.19. In [Bar17], Barwick gives a much more general version of the Segal-tom Dieck splitting, one for each topos with some additional hypotheses. This is the case where the topos is the topos of G-sets.

Proof. We already know that the left hand side is $K(\mathrm{Fin}_G)$, so we are reduced to describing that.

The point is that $\mathrm{Fin}_G^\simeq$ is the free commutative monoid on the G/H, for some fixed representatives of conjugacy classes of subgroups of G. Now each G/H has automorphism group exactly $W_G(H)$, so we get this description.

More precisely, consider $\coprod_{[H]} BW_G(H)$. Clearly, its Σ^∞_+ is the right hand side of the above equation.

But we can decompose it, because Ω^∞ factors as $\mathbf{Sp} \to \mathrm{CMon} \to \mathbf{S}$, and so Σ^∞_+ factors as $\mathbf{S} \overset{free}{\to} \mathrm{CMon} \overset{(-)^{gp}}{\to} \mathbf{Sp}$, where $(-)^{gp}$ is the group completion. So we are left with proving that $\mathrm{Fin}_G^\simeq \simeq free(\coprod_{[H]} BW_G(H))$.

This is done in [Bar17], but hopefully the idea is clear. Let us give a sketch, in the case where G is finite.

If G is finite, there is an easier proof which consists in noting that both sides are just $\prod_{[H]}$ of something, and we just have to identify these somethings, specifically, $free(BW_G(H))$ and the core-groupoid of the category of finite G-sets all of whose stabilizers belong to $[H]$.

But now both are fibered over Fin, and there is a natural map from the first to the second; it's then easy to check the equivalence on fibers over Fin (or better yet, on pullbacks over $B\Sigma_n$).

This proof actually works without the finiteness assumption. $\square$

Remark 3.20. We will describe later a map $X^H \to X^{hH}$. Here this will give a comparison map $\bigoplus_{[H]} \Sigma^\infty_+ BW_G(H) \to \mathrm{Map}(\Sigma^\infty_+ BG, S^0)$.

Note that on π_0, the left hand side is the *Burnside ring* $A(G)$ of G, which, as an abelian group, is freely generated by isomorphism classes of finite transitive G-sets, and on the right hand side, we have stable cohomotopy of $BG : \pi^0_s(BG)$. Segal's conjecture states that, up to completion, this map $(S^0_G)^G \to (S^0_G)^{hG}$ is an equivalence. In this sense, it is related to the Atiyah-Segal completion theorem, and to the Sullivan conjecture. It is one big point where the whole genuine equivariant structure is helpful, rather than simply G-actions.

Let us now again use proposition 2.14 to construct this map $X^G \to X^{hG}$. The idea is as follows: we saw earlier that we had an adjunction between $\mathbf{Sp}_G$ and $\mathrm{Fun}(BG, \mathbf{Sp})$ which sent a G-spectrum to its underlying spectrum with G-action. The point is that this is a colocalization and that $(-)^{hG}$ only depends on the spectrum with G-action (by definition).

So the plan is as follows: prove that for X in the image of $\mathrm{Fun}(BG, \mathbf{Sp})$, $X^G \simeq X^{hG}$ (more precisely, we will define a natural equivalence between the two), and then we will use the unit of the adjunction to define a map $X^G \to X^{hG}$ more generally (which will be $X^G \to (RLX)^G \simeq (RLX)^{hG} \simeq X^{hG}$).

4 Comparison between different types of fixed points

We first want to prove that on G-spectra coming from spectra with G-action, homotopy and categorical fixed points coincide.

We first want to prove that we can restrict to looking at G-fixed points throughout.

Definition 4.1. Let $u_G : \mathbf{Sp}_G \to \mathrm{Fun}(BG, \mathbf{Sp})$ be given by restriction along $BG \to \mathrm{Span}(G)$ and $b_G : \mathrm{Fun}(BG, \mathbf{Sp}) \to \mathbf{Sp}_G(E)$ be its right adjoint, given by right Kan extension and mackeyfication (b for Borel)

We also let res and Ind denote the corresponding functors on $\mathrm{Fun}(BG, \mathbf{Sp})$.

We want to prove two things:

1. That $u_H \mathrm{res}^G_H X \simeq \mathrm{res}^G_H u_G X$
2. That $b_H \mathrm{res}^G_H X \simeq \mathrm{res}^G_H b_G X$

With those two statements, we will indeed be reduced to the $H = G$ case, as if we can prove that for any G, then it is true for H and so

$$(b_G X)^{hH} = (\mathrm{res}^G_H u_G b_G X)^{hH} \simeq (u_H \mathrm{res}^G_H b_G X)^{hH} \simeq (u_H b_H \mathrm{res}^G_H X)^{hH} \simeq (\mathrm{res}^G_H X)^{hH}$$

$\simeq (b_H \mathrm{res}^G_H X)^H \simeq (\mathrm{res}^G_H b_G X)^H \simeq (b_G X)^H$

The first statement is fairly straightforward, it simply follows from the commutativity of the following diagram:

$$\begin{tikzcd} BH \arrow[r] \arrow[d] & \mathrm{Span}(H) \arrow[d] \\ BG \arrow[r] & \mathrm{Span}(G) \end{tikzcd}$$

because all of the functors appearing are just precompositions by some arrows in this diagram.

For the second statement, there is probably a more abstract way to do it, using right adjointable squares (see [Lur17, 6.1.6.3] for instance) together with the Wirthmüller isomorphism, but we will do it by hand. The statement is a claim of equivalence of right adjoints, which we will prove by looking at the left adjoints instead: we want to prove that $\mathrm{Ind}^G_H \circ u_H \simeq u_G \circ \mathrm{Ind}^G_H$.

For this we construct $u_H \to u_H \circ \mathrm{res}^G_H \mathrm{Ind}^G_H \simeq \mathrm{res}^G_H u_G \mathrm{Ind}^G_H$ to be u_H applied to the unit of the induction-restriction adjunction, and then take the adjoint map $\mathrm{Ind}^G_H u_H \to u_G \mathrm{Ind}^G_H$. Since we have a map of spectra with G-action, we can just prove that it's an equivalence on underlying spectra.

The left hand side is a Kan extension along $BH \to BG$, the fiber being G/H as a discrete category, so the underlying spectrum of the left hand side applied to X is just $\bigoplus_{G/H} X^e$, and the right hand side is $X(G)$, where G is viewed as an H-set, so it's just $\bigoplus_{G/H} X^e$ as well. To check that the map we obtain between the two is the correct one, one can just look in the homotopy category, and get that it is indeed an equivalence.

So we are now reduced to the claim about G-fixed points.

Theorem 4.2. *Let E be a presentable semiadditive category. Then (abusing notations) for $X \in \mathrm{Fun}(BG, E)$, $X^{hG} \simeq (b_G X)^G$.*

Proof. We want to compare $(-)^{hG}$ and $ev_{G/G} \circ b_G$. We will compare their left adjoints: for $(-)^{hG}$, this is nothing but the diagonal $\Delta : E \to \mathrm{Fun}(BG, E)$, and we will decompose the left adjoint of $ev_{G/G} \circ b_G$.

Indeed, b_G is defined as the right adjoint of $\mathrm{Fun}^\times(\mathrm{Span}(G), E) \to \mathrm{Fun}(\mathrm{Span}(G), E) \to \mathrm{Fun}(BG, E)$, and $ev_{G/G}$ can be seen as right adjoint to $E \overset{\mathrm{Lan}}{\to} \mathrm{Fun}(\mathrm{Span}(G), E) \to \mathrm{Fun}^\times(\mathrm{Span}(G), E)$.

Note that that the functor $\mathrm{Fun}^\times(\mathrm{Span}(G), E) \to \mathrm{Fun}(\mathrm{Span}(G), E)$ has both a left and a right adjoint, and both appear in this proof ! The right adjoint appears in the definition of b_G, while the left adjoint appears in the decomposition of the left adjoint of $ev_{G/G}$. Both are some form of additivization, so we will denote them *Ladd* and *Radd* (L for left, R for right).

So we want to compare Δ and $i^* \circ forget \circ Ladd \circ \mathrm{Lan}_{G/G}^{\mathrm{Span}(G)}$ where $i : BG \to \mathrm{Span}(G)$ denotes the canonical map.

We first deal with the case $E = \mathrm{CMon}$. The point will be that we can compare things to $\mathbf{S}$ (which is not semiadditive), and use proposition 2.14.

Indeed, we have the following commutative diagram of left adjoints, where $\mathbf{S} \to \mathrm{CMon}$ is left adjoint to the forgetful functor:

$$\begin{tikzcd}
\mathbf{S} \arrow[r, "\sim"] \arrow[d] & \mathrm{Fun}(*, \mathbf{S}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathrm{Span}(G), \mathbf{S}) \arrow[r] \arrow[d] & \mathrm{Fun}^\times(\mathrm{Span}(G), \mathbf{S}) \arrow[d, "\sim"] \\
\mathrm{CMon} \arrow[r, "\sim"] & \mathrm{Fun}(*, \mathrm{CMon}) \arrow[r] & \mathrm{Fun}(\mathrm{Span}(G), \mathrm{CMon}) \arrow[r] & \mathrm{Fun}^\times(\mathrm{Span}(G), \mathrm{CMon})
\end{tikzcd}$$

Note that by proposition 2.14, the rightmost vertical arrow is indeed an equivalence, as it's the left adjoint to the forgetful functor, which is an equivalence.

Moreover, by [GGN16, 4.9], the bottom composite, followed by $\mathrm{Fun}^\times(\mathrm{Span}(G), \mathrm{CMon}) \to \mathrm{Fun}(BG, \mathrm{CMon})$ is completely determined by where it sends the image of $* \in \mathbf{S}$. But for this image we can just trace through the above diagram: $\mapsto \mathrm{map}(*, -) \mapsto \mathrm{map}(*, -)$, as the left Kan extension of $\mathrm{map}(x, -)$ along F is always $\mathrm{map}(F(x), -)$; but now $\mathrm{map}_{\mathrm{Span}(G)}(*, -)$ is already product preserving ! Therefore its image in $\mathrm{Fun}^\times(\mathrm{Span}(G), \mathrm{CMon})$ is nothing but $\mathrm{map}(*, -)$ as well, which we then see is clearly sent in $\mathrm{Fun}(BG, \mathrm{CMon})$ to $\mathrm{Fin}^\simeq$ with the trivial G-action (because that's what $\mathrm{map}_{\mathrm{Span}(G)}(G/G, G/e)$ is), i.e. $\mathrm{CMon} \to \mathrm{Fun}(BG, \mathrm{CMon})$ agrees, on $\mathrm{Fin}^\simeq$, with the diagonal Δ.

It follows that they agree on all of CMon. It follows, again essentially by [GGN16, 4.9] that this holds for all presentable semiadditive categories (for instance one can tensor the bottom composite with E in $\mathbf{Pr}^L$, and get the corresponding composite for E)

In particular, this identifies the left adjoints, hence the right adjoints, which is exactly what we wanted to prove. The claim follows. □

Remark 4.3. Let us explain why we went through $\mathbf{S}$. Suppose we only looked at the bottom line in the above diagram, which is essentially the universal case of the comparison we want to make.

Start with $X \in$ CMon (for instance $\mathrm{Fin}^{\simeq}$, which is all that really matters), then Kan extend this to $\mathrm{Span}(G)$. The thing you obtain might look very complicated (specifically, as $\mathrm{Fin}^{\simeq} = free(*)$, the Kan extension you obtain is $free(\mathrm{map}_{\mathrm{Span}(G)}(*, -))$, which is a complicated object, and most importantly, far from additive !). Therefore the operation of additivization, which is benign in the case of $\mathbf{S}$, is actually doing something here. And indeed, look at the proof: the additivization is nothing but $\mathrm{map}_{\mathrm{Span}(G)}(*, -)$ itself ! This is why proposition 2.14 came in really handy here.

Construction 4.4. We define the comparison map $X^H \to X^{hH}$ as follows: the unit $X \to b_G u_G X$ induces $X^H \to (b_G u_G X)^H$, and we constructed earlier an equivalence $(b_G u_G X)^H \simeq (u_G X)^{hH} = X^{hH}$, so we get $X^H \to X^{hH}$

Remark 4.5. Note that the two sides of this transformation are quite different: $X \mapsto X^H$ preserves arbitrary colimits, while in general $X \mapsto X^{hH}$ can only be said to preserve finite colimits. Both sides preserve limits, however.

As we already mentioned, seeing how far this arrow is from being an equivalence is the object of a lot of work in equivariant homotopy theory: the Sullivan and Segal conjectures are questions of that type; and the Atiyah-Segal completion theorem as well.

Remark 4.6. One can prove analogous results for the left adjoint to the restriction functor $\mathbf{Sp}_G \to \mathbf{Sp}^{BG}$, which sends a spectrum with G-action to the corresponding "coBorel" genuine G-spectrum. We leave this as an exercise to the reader.

5 The monoidal structure

In this section, we introduce a monoidal structure on $\mathbf{Sp}_G$, which is needed to even make sense of the requirement that representation spheres be invertible. It's also needed to start talking about commutative algebra in G-spectra, and for instance work the machinery of [MNN17] and [MNN19], to study descent problems and induction-restriction type questions (which relate to some classical theorems such as Quillen's F-isomorphism theorem, or the Cartan-Eilenberg stable elements formula).

It is also, of course, interesting to develop this monoidal structure in order to define things such as multiplicative equivariant cohomology theories (that aren't Borel equivariant cohomology theories).

This work relies on Day convolution, a well-known structure in 1-category theory, investigated for ∞-categories in [Gla16] and [Lur17].

First we start by noting the following fact:

Theorem 5.1. *The functor* Span : $\mathbf{Cat}_\infty^{lex} \to \mathbf{Cat}_\infty$ *preserves products. In particular it sends finitely complete symmetric monoidal categories C where the tensor product $C \times C \to C$ preserves finite limits to symmetric monoidal categories; furthermore the natural transformation $i \to$* Span*, where i is the forgetful functor, naturally lifts to a symmetric monoidal* *natural transformation for these categories together with left exact symmetric monoidal functors.*

Remark 5.2. One has to be careful about the hypothesis about the tensor product $C\times C \to C$ here. We are not asking that it preserve finite limits in each variable as is often the case in this kind of statement, we are really asking that it preserve finite limits as a one-variable functor.

This is related to the fact that we are considering $\mathbf{Cat}_\infty^{lex}$ as a cartesian symmetric monoidal category, and *not* with (a variation on) the Lurie tensor product.

For instance, if C is finitely complete, then the cartesian monoidal structure on C satisfies this hypothesis: this is exactly the claim that products commute with (finite) limits.

Proof. The product-preservation follows from the fact that for any categories C, D and objects $c_0, c_1 \in C, d_0, d_1 \in D$, the canonical functor is an equivalence $(C \times D)_{\{(c_0,d_0),(c_1,d_1)\}} \to C_{\{c_0,c_1\}} \times D_{\{d_0,d_1\}}$ and the fact that the "core-groupoid" functor preserves products.

The claim then follows at once, as commutative monoids in $\mathbf{Cat}_\infty$ are exactly symmetric monoidal categories, and in $\mathbf{Cat}_\infty^{lex}$ (*with the cartesian monoidal structure !*) are exactly the symmetric monoidal categories we described, and so both the symmetric monoidal structure on $\mathrm{Span}(C)$ and on the functor $C \to \mathrm{Span}(C)$ are given by post-composition. ☐

Corollary 5.3. *Suppose C is a disjunctive category, which we view as a symmetric monoidal category with the cartesian monoidal structure. Then the symmetric monoidal structure on* $\mathrm{Span}(C)$ *we described above is compatible with direct sums: the tensor product commutes with direct sums in each variable.*

Proof. C satisfies the hypothesis of the previous statement because products commute with limits.

Furthermore, $C \to \mathrm{Span}(C)$ is essentially surjective, symmetric monoidal, and sends coproducts to direct sums. The claim therefore follows from the fact that in C, products distribute over coproducts, which is part of the definition of a disjunctive category. ☐

Example 5.4. This applies in our main example of interest, i.e. Fin_G. It follows that $\mathrm{Span}(G)$ has the above structure and properties.

Remark 5.5. In [BGS19], the authors take a more complicated route to the monoidal structure on $\mathrm{Span}(C)$. This is because they achieve that result in much greater generality than what we are interested in here: Fin_G is particularly simple.

Also, they are not interested so much in the monoidal structure on $\mathrm{Span}(C)$ as they are in the one on $\mathbf{Mack}(C; E)$, and so they can relax their hypotheses on C even further because they need a slightly weaker structure on $\mathrm{Span}(C)$. We refer to their paper for more detail.

Corollary 5.6. *Suppose C is has finite limits. Then every object in* $\mathrm{Span}(C)$ *is self-dual.*

Proof. In fact, we can exhibit the duality data: let $X \in C$ be any object. Then the span $* \leftarrow X \xrightarrow{\Delta} X \times X$ and its "dual" $X \times X \leftarrow X \to *$ correspond to a duality data $1 \to X \otimes X, X \otimes X \to 1$ in $\mathrm{Span}(C)$.

We leave the details to the reader. ☐

This will be a very important property in what follows.

We now want to define a symmetric monoidal structure on $\mathbf{Mack}(C;E)$ when E has one as well. The point is that we can localize the Day convolution structure on $\mathrm{Fun}(\mathrm{Span}(C),E)$. The following theorem is folklore, it is e.g. [BGS19, Lemma 3.7] in this specific case, see [CDH+20, Lemma 5.3.4] or [HR20, Theorem 4.5] for a more general account.

Theorem 5.7. *Let C be as above, and suppose E is a presentably symmetric monoidal semiadditive category. Then the localization $\mathrm{Fun}(\mathrm{Span}(C),E) \to \mathbf{Mack}(C;E)$ is compatible with the Day convolution symmetric monoidal structure.*

In particular, the space of symmetric monoidal structures on $\mathbf{Mack}(C;E)$ together with symmetric monoidal structures on the localization functor is contractible.

Furthermore, for this symmetric monoidal structure on $\mathbf{Mack}(C;E)$, the Yoneda embedding $\mathrm{Span}(C)^{\mathrm{op}} \to \mathbf{Mack}(C;E)$ has a symmetric monoidal structure.

Remark 5.8. The last point implies that the image of any $X \in \mathrm{Span}(C)$ in $\mathbf{Mack}(C;E)$ is dualizable.

In particular, by taking $E = \mathbf{Sp}, \mathrm{CMon}$ and $C = \mathrm{Fin}_G$, we get symmetric monoidal structures on $\mathbf{Sp}_G$ and $\mathbf{Mack}(G;\mathrm{CMon})$.

Let us note an interesting relation between the monoidal structure and the restriction structure :

Construction 5.9. Let C be as above, and $S \in C$. Then $S \times -$ preserves coproducts and pullbacks, so it induces an additive functor $\mathrm{Span}(C) \to \mathrm{Span}(C)$, and therefore, by precomposition, a functor $(-)_S : \mathbf{Mack}(C;E) \to \mathbf{Mack}(C;E)$. Concretely, $M_S = M(S \times -)$. This construction is functorial in S.

Now consider the following construction: every object in $\mathrm{Span}(C)$ has a dual so we get a duality functor $\mathbb{D} : \mathrm{Span}(C) \to \mathrm{Span}(C)^{\mathrm{op}}$, which we can then compose with the Yoneda embedding to get a functor $\mathrm{Span}(C) \to \mathbf{Mack}(C;\mathrm{CMon})$, and, using the unique symmetric monoidal left adjoint $\mathrm{CMon} \to E$ [GGN16] we get a symmetric monoidal functor $\mathrm{Span}(C) \to \mathbf{Mack}(C;E)$. Let us call it Σ^∞_+. The second construction sends $S \mapsto \Sigma^\infty_+ S \otimes -$, where $\otimes$ is the tensor product on $\mathbf{Mack}(C;E)$.

Proposition 5.10. *The two constructions above are equivalent.*

Proof. It is clear from their definition that they are both base-changed from the universal case with $E = \mathrm{CMon}$, so we may assume $E = \mathrm{CMon}$. In particular, $\mathbf{Mack}(C;\mathrm{CMon}) \simeq \mathrm{Fun}^\times(\mathrm{Span}(C), \mathbf{S})$ and it is therefore the nonabelian derived category of $\mathrm{Span}(C)^{\mathrm{op}}$, i.e. it is freely generated by $\mathrm{Span}(C)$ under sifted colimits.

It is quite clear that both functors $(-)_S$ and $\Sigma^\infty_+ S \otimes -$ commute with sifted colimits, so it suffices to check the claim that they are equivalent on $\mathrm{Span}(C)$. In other words, one only needs to check that $(S,T) \mapsto \mathrm{map}(T, S \times -)$ and $(S,T) \mapsto \Sigma^\infty_+ S \otimes \mathrm{map}(T,-)$ are naturally equivalent. The latter is, by definition, $\mathrm{map}(\mathbb{D}S,-) \otimes \mathrm{map}(T,-)$ and so, because the Yoneda embedding is symmetric monoidal, this is equivalent to $\mathrm{map}(\mathbb{D}S \times T, -)$.

By definition of the duality functor, this is now equivalent to $\mathrm{map}(T, S \times -)$, and we are done. ☐

Remark 5.11. As an object, $\mathbb{D}S = S$, but it is important to keep the $\mathbb{D}$ to note the appropriate functoriality of the objects involved.

Remark 5.12. On objects and arrows, one can check that $\mathbb{D}$ is the identity on objects, and switches the direction of spans. With a bit more work (and more precise definitions), one can show that $\mathbb{D}$ is equivalent to the canonical equivalence $\mathrm{Span}(C) \simeq \mathrm{Span}(C)^{\mathrm{op}}$ from the beginning.

Example 5.13. Consider $C = \mathrm{Fin}_G, E = \mathbf{Sp}$. Then this says in particular that for any G-spectrum M,

$$M^H = M(G/H) \simeq M(G/H \times G/G) = M_{G/H}(G/G) = (M_{G/H})^G \simeq (\Sigma^\infty_+ G/H \otimes M)^G$$

This allows one to have a "restriction functor" internal to $\mathbf{Sp}_G$. In fact we will later see a stronger version of that statement, due to Balmer-Dell'Ambrogio-Sanders (and refined ∞-categorically in [MNN17])

This last construction can actually be interpreted slightly more generally. Indeed, any semiadditive presentable category E is canonically a CMon-module in $\mathbf{Pr}^L$, from which we deduce a $\mathbf{Mack}(C; \mathrm{CMon})$-module structure on $\mathbf{Mack}(C; E)$, so even if E is not symmetric monoidal, we can make sense of $\Sigma^\infty_+ S \otimes -$ as a functor on $\mathbf{Mack}(C; E)$. It is not hard to see that the above proof works in that setting as well. We leave the details to the interested reader.

We now move on to the Balmer-Dell'Ambrogio-Sanders interpretation of restriction. Note that for a disjunctive category C and an object $c \in C$, $C_{/c}$ is still disjunctive : indeed the forgetful functor $C_{/c} \to C$ preserves coproducts and pullbacks, and for an arrow $f : y \to c$, $(C_{/c})_{/f} \simeq C_{/y}$.

In fact we can organise $c \mapsto C_{/c}$ as a functor on C^{op} by noting that the functor $\mathrm{Fun}(\Delta^1, C) \to C$ given by evaluation at 1 is a cartesian fibration because C has pullbacks. This cartesian fibration classifies a functor $C^{\mathrm{op}} \to \mathbf{Cat}_\infty$ given exactly by these comma categories.

Furthermore, each pullback functor $C_{/c} \to C_{/d}$ along an edge $d \to c$ preserves pullbacks and the terminal object, hence it preserves finite limits. It follows that the functor $C^{\mathrm{op}} \to \mathbf{Cat}_\infty$ factors through $\mathbf{Cat}_\infty^{lex}$. We can therefore composite it with Span and get a functor $C^{\mathrm{op}} \to \mathbf{Cat}_\infty$ given by $c \mapsto \mathrm{Span}(C_{/c})$, which sends coproducts in C to products in $\mathbf{Cat}_\infty$.

This functor also factors through the category of preadditive categories and additive functors.

Composing this with $D \mapsto \mathrm{Fun}^\oplus(D, E)$ for some fixed preadditive presentable category, we get a functor $C^{\mathrm{op}} \to \mathbf{Pr}^L_{preadd}$ given by $c \mapsto \mathbf{Mack}(C_{/c}, E)$.

Example 5.14. Suppose $C = \mathrm{Fin}_G$ as usual, and $c = G/H$. Then $C_{/c} = (\mathrm{Fin}_G)_{/(G/H)} \simeq \mathrm{Fin}_H$ as functors on C^{op}, an equivalence being given by $X \mapsto (G \times_H X \to G \times_H * = G/H)$ (pullback then corresponds to restriction).

Taking e.g. $E = \mathbf{Sp}$, the above therefore constructs a functor $\mathrm{Fin}_G^{\mathrm{op}} \to \mathbf{Pr}^L_{st}$ given on orbits by $G/H \mapsto \mathbf{Sp}_H$ and for $H \leq K$, given by restriction $\mathbf{Sp}_K \to \mathbf{Sp}_H$.

Remark 5.15. Using the $(\infty, 2)$-categorical universal property of the $(\infty, 2)$-category of spans, and then restricting to Span, we could actually construct this as a functor from $\mathrm{Span}(C)$ itself.

We haven't mentioned this, and won't really need this so we leave it at that for now.

The point of defining this is that we wanted to state the Balmer-Dell'Ambrogio-Sanders result not only for orbits but for more general G-sets.

Consider now the following : suppose $E = \mathrm{CMon}$, so that $\mathbf{Mack}(C; E)$ is freely generated under sifted colimits by $\mathrm{Span}(C)$. Then we can look at the map induced by $c \to *$: it induces a morphism $\mathbf{Mack}(C; \mathrm{CMon}) \to \mathbf{Mack}(C_{/c}; \mathrm{CMon})$ given on the Yoneda image of $\mathrm{Span}(C)$ by $d \mapsto d \otimes c$ (where we use $\otimes$ to denote the cartesian product in C, so the monoidal structure in $\mathrm{Span}(C)$ or equivalently $\mathbf{Mack}(C; \mathrm{CMon})$).

Now looking at the adjunction $C \leftrightarrows C_{/c}$, we see that both the unit and the co-unit are cartesian, so that they induce an adjunction on span categories. Actually, because span categories are self-dual (1) they induce a chain of adjunctions, where each adjoint is both left and right adjoint to the other one.

Therefore the right adjoint of $\mathbf{Mack}(C; \mathrm{CMon}) \to \mathbf{Mack}(C_{/c}; \mathrm{CMon})$ is also given on $\mathrm{Span}(C_{/c})$ by $(y \to c) \mapsto y$. So the monad induced by the adjunction is $d \mapsto d \otimes c$, which looks a lot like a monad induced by an induction-restriction adjunction. This is no coincidence (note that we use CMon here, but because both adjoints preserve all colimits, a similar story could be told for any presentable preadditive category E, simply by tensoring with E)

To state this more precisely, note that both C and $C_{/c}$ fall into the hypotheses of theorem 5.7, and the functor $C \to C_{/c}$ is symmetric monoidal for the cartesian symmetric monoidal structures, so that the functor $\mathbf{Mack}(C; \mathrm{CMon}) \to \mathbf{Mack}(C_{/c}; \mathrm{CMon})$ also acquires a canonical monoidal structure.

Its right adjoint therefore also acquires a lax symmetric monoidal structure, and therefore the adjunction factors as follows, where $\mathbf{1}_c$ denotes the symmetric monoidal unit of $\mathbf{Mack}(C_{/c}; \mathrm{CMon})$ and U the right adjoint:

$$\mathbf{Mack}(C; \mathrm{CMon}) \rightleftarrows \mathbf{Mod}_{U(\mathbf{1}_c)}(\mathbf{Mack}(C; \mathrm{CMon})) \rightleftarrows \mathbf{Mack}(C_{/c}; \mathrm{CMon})$$

We can almost state the Balmer-Dell'Ambrogio-Sanders theorem now, but we need one last notion, namely that of a separable algebra.

Definition 5.16. Let D be a symmetric monoidal category, and $A \in \mathrm{Alg}(D)$. Then there is an A-bimodule structure on A, as well as a bimodule map $A \otimes A \to A$ given by multiplication.

A is said to be *separable* if this has a bimodule section s.

Remark 5.17. This in particular implies that for any A-module M, the co-unit map $A \otimes M \to M$ has a natural A-module section (it always has a section in the base category). Indeed, consider the following composite : $M \overset{\eta \otimes M}{\to} A \otimes M \overset{s}{\to} A \otimes A \otimes M \overset{A \otimes \rho}{\to} A \otimes M$, where ρ is the action of A on M.

This is clearly a section, and I claim that it is A-linear. I'm not sure of an easy way to see this, but here's at least a convoluted way: up to embedding D in its presheaf category with the symmetric monoidal Yoneda embedding (where the latter has the Day convolution monoidal structure), we may assume D is presentable and the tensor product respects colimits in each variable. In this case we can define a relative tensor product $\otimes_A$, and the above sequence gets identified with the following : $M \simeq A \otimes_A M \to (A \otimes A) \otimes_A M \simeq A \otimes (A \otimes_A M) \simeq A \otimes M$, where this time each map is A-linear.

Example 5.18. Suppose $C = \mathrm{Fin}_G$. Then every $X \in \mathrm{Fin}_G$ yields a separable commutative algebra in $\mathrm{Span}(\mathrm{Fin}_G)$. Indeed, in this case, because separable algebras are stable under finite products if they commute with the tensor product (exercise !), it suffices to prove it for $X = G/H$. But then we have a decomposition $\coprod_{r\in H\backslash G/H} G/(H\cap H^r) \cong G/H \times G/H$ in Fin_G, which implies that the following (commutative) algebra morphism in $\mathrm{Fin}_G^{\mathrm{op}} : G/H \times G/H \to G/H$ gets identified with a projection $\prod_{i\in I} A_i \to A_i$, which always has bimodule sections.

Remark 5.19. More generally, if C is disjunctive and $x \in C$, then the corresponding commutative algebra x comes with a morphism $x \otimes x \to x$. The diagonal map $x \to x \otimes x$ is always an x-bimodule map, but the proof I know is a bit involved. Then, the composition $x \to x \otimes x \to x$ is the span $x \times_{x\times x} x$, i.e. x^{S^1}, the "free loop space" on x - in Fin_G, this is equivalent to x, and so this is indeed a bimodule section.

More generally, this is equivalent to x via the canonical map precisely if x is 0-truncated in C, and this suffices for the corresponding commutative algebra to be separable.

Finally, we can state the Balmer-Dell'Ambrogio-Sanders theorem :

Theorem 5.20 (Restriction as extension of scalars)**.** *Suppose that the algebra given by c in* $\mathrm{Span}(C)$ *is separable. Then, in the above sequence of adjunctions, the rightmost adjunction is an adjoint equivalence, namely* $\mathbf{Mod}_{U(\mathbf{1}_c)}(\mathbf{Mack}(C;\mathrm{CMon})) \simeq \mathbf{Mack}(C_{/c};\mathrm{CMon})$

Remark 5.21. This factorization is natural in $c \in C^{\mathrm{op}}$, therefore so is the equivalence.

Remark 5.22. Here $U(\mathbf{1}_c)$ has the algebra structure coming from the lax symmetric monoidality of U and the fact that $\mathbf{1}_c$ is the unit in $\mathbf{Mack}(C_{/c};\mathrm{CMon})$. Because all of this structure comes from the adjunction $C \leftrightarrows C_{/c}$, more precisely its dual $C^{\mathrm{op}} \rightleftarrows (C_{/c})^{\mathrm{op}}$, we can identify explicitly this algebra structure: it is the natural co-algebra structure on $c \in C$ with respect to the cartesian symmetric monoidal structure (note that C^{op} has the cocartesian monoidal structure, and there any object is canonically and uniquely an algebra).

This identification is natural in $c \in C^{\mathrm{op}}$.

Example 5.23. Following remark 5.22, if $C = \mathrm{Fin}_G$, the algebra structure on $U(\mathbf{1}_c)$ for $c = G/H$ is the dual of the co-algebra structure on $G/H \in \mathrm{Fin}_G$.

Forgetting the equivariant structure, this algebra structure is informally given by $(gH, g'H) \mapsto 1$ if $gH = g'H$, 0 else.

Before the proof, we need a few lemmas.

Lemma 5.24. *The adjunction $C \leftrightarrows C_{/c}$ is comonadic and identifies $C_{/c}$ with the category of c-comodules in C.*

Proof. The dual situation is clearer: suppose D has finite coproducts and pushouts, then we're claiming that $D_{d/}$ is the same thing as d-modules (where d has its canonical commutative algebra structure coming from the fact that the forgetful functor $\mathrm{CAlg}(D) \to D$ is an equivalence - here, D is equipped with its cocartesian symmetric monoidal structure). Intuitively this should be clear - and it is so in the 1-categorical context.

For a more precise proof, observe that $\mathrm{CAlg}(D)_{d/}$ is on the one hand equivalent to $D_{d/}$ via the forgetful functor (because $\mathrm{CAlg}(D) \to D$ is an equivalence), and on the other hand it is equivalent to commutative algebras in d-modules, i.e. to d-modules (because the monoidal structure there is still the coproduct). $\square$

Remark 5.25. We are only interested in this statement for $C = \mathrm{Fin}_G$, so if you are satisfied with the 1-categorical proof, then it is good enough for the rest of the document.

Lemma 5.26. *The dual adjunction* $\mathrm{Span}(C) \rightleftarrows \mathrm{Span}(C_{/c})$ *is monadic, and identifies* $\mathrm{Span}(C_{/c})$ *with* c*-modules in* $\mathrm{Span}(C)$*, where* c *has its natural commutative algebra structure coming from the symmetric monoidal functor* $C^{\mathrm{op}} \to \mathrm{Span}(C)$

Sketch of proof. The monadicity from the previous statement can be expressed as a certain diagram of categories with finite limits along pullback-preserving functors being an equivalence.

Span preserves limits along pullback-preserving functors, so that this diagram remains a limit diagram after passing to span-categories.

This also shows monadicity with respect to the appropriate monad. □

Remark 5.27. See remark 5.29 below to see why the sketchiness of this proof is not an issue for our (very restricted) purposes.

Proof of theorem 5.20. Let D be a small symmetric monoidal category that has finite coproducts, and where the tensor product commutes with coproducts in each variable separately, and let $d \in \mathrm{CAlg}(D)$. We furthermore assume that d has the following property : any d-module in D is a retract of a free module (i.e. of the form $d \otimes x$) - this is the case e.g. if d is separable.

We want to compare d-modules in $\mathrm{Fun}^{\sqcup}(D^{\mathrm{op}}, \mathbf{S}) = P_\Sigma(D)$ and $P_\Sigma(\mathbf{Mod}_d(D))$. If we can prove that they are equivalent via the natural functor, then by lemma 5.26, we will be done (note that we assumed that c was separable in $\mathrm{Span}(C)$ so the hypothesis about retracts is satisfied).

Consider the natural fully faithful inclusion $\mathbf{Mod}_d(D) \to \mathbf{Mod}_d(P_\Sigma(D))$. To show that it induces an equivalence $P_\Sigma(\mathbf{Mod}_d(D)) \to \mathbf{Mod}_d(P_\Sigma(D))$, we need to show two things: that its image lands in projective objects, and that its image generates $\mathbf{Mod}_d(P_\Sigma(D))$ under sifted colimits.

For the first claim, note that it suffices to show (by the freeness assumption) that free d-modules, i.e. modules of the form $d \otimes x$, are sent to projective objects of $\mathbf{Mod}_d(P_\Sigma(D))$. But this is the case because $d \otimes x$ is also free in the latter, and x is projective in $P_\Sigma(D)$ (by definition), and finally the forgetful functor $\mathbf{Mod}_d(P_\Sigma(D)) \to P_\Sigma(D)$ preserves sifted colimits.

For the second claim, writing a d-module in $P_\Sigma(D)$ as the colimit of the monadic bar construction reduces the claim again to free d-modules, so of the form $d \otimes x$ for some $x \in P_\Sigma(D)$, and then we are reduced to the corresponding claim for $P_\Sigma(D)$, which follows by definition of P_Σ.

The claim thus follows. □

Now that theorem 5.20 will imply similar statements for categories that are tensored over $\mathbf{Mack}(C; \mathrm{CMon})$. Specializing to the case $C = \mathrm{Fin}_G$ and taking $\mathbf{Sp}$ instead of CMon, we get:

Corollary 5.28. *For the following examples, "restriction to the subgroup* H*" is naturally identified with "extension of scalars along* A^G_H*", where* $A^G_H = \mathbb{D}(\Sigma^\infty_+ G/H)$ *with algebra structure given informally by the dual of the diagonal,* $(gH, g'H) \mapsto 1$ *if* $gH = g'H$*,* 0 *else; interpreted suitably.*

- $\mathbf{Sp}_G \to \mathbf{Sp}_H$*; same thing with modules over* R *for some* $R \in \mathrm{CAlg}(\mathbf{Sp}_G)$*.*

- $\mathrm{Fun}(BG, \mathbf{Sp}) \to \mathrm{Fun}(BH, \mathbf{Sp})$*; same thing with modules over R for some $R \in \mathrm{Fun}(BG, \mathrm{CAlg}(\mathbf{Sp}))$*
- *For e.g. a finite field k, $\mathbf{stmod}_{kG} \to \mathbf{stmod}_{kH}$, where* **stmod** *is the stable module category.*

Proof. This follows, because these categories ($\mathrm{Fun}(BG, \mathbf{Sp})$ or $\mathbf{StMod}_{kG}$, etc.) are tensored over $\mathbf{Mack}_G(\mathrm{CMon})$. □

Remark 5.29. The last example was the original statement in Balmer-Dell-Ambrogio-Sanders. It of course suffices to prove the first about $\mathbf{Sp}_G$, where, if the previous proofs of monadicity feel too sketchy, we can actually use the Barr-Beck-Lurie monadicity theorem. This is, e.g. [MNN17, Theorem 5.32]

Remark 5.30. We stated things for subgroups $H \leq G$, but we could actually use finite G-sets instead. Note that for a finite G-set $X = \coprod_i G/H_i$, the algebra $\mathbb{D}(\Sigma^\infty_+ X)$ is $\prod_i \mathbb{D}(\Sigma^\infty_+ G/H_i)$, and this is compatible with $\mathbf{Sp}_X \simeq \prod_i \mathbf{Sp}_{H_i}$.

The above corollary is natural in $X \in \mathrm{Fin}_G^{\mathrm{op}}$, not only in H.

We conclude this section by describing a strong symmetric monoidal structure on $\Sigma^\infty : \mathbf{S}_{G,*} \to \mathbf{Sp}_G$ (and therefore a lax symmetric monoidal structure on Ω^∞).

Observe that $\mathbf{S}_{G,*} = \mathbf{S}_G \otimes \mathbf{S}_*$, so we may as well describe a monoidal structure on $\Sigma^\infty_+ : \mathbf{S}_G \to \mathbf{Sp}_G$. But now $\mathbf{S}_G$ is cartesian symmetric monoidal, and this corresponds exactly to the day convolution monoidal structure on $\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{S})$ as $\mathcal{O}_G^{\mathrm{op}}$ is cocartesian symmetric monoidal.

Now $\mathcal{O}_G \to \mathrm{Span}(G)^{\mathrm{op}}$ is (almost) by definition symmetric monoidal, and so it induces a strong symmetric monoidal colimit preserving functor on categories of presheaves with Day convolutional monoidal structure.

But given how we defined Σ^∞_+, if we compose the above with pointwise Σ^∞_+ and later, Mackeyfication, we get exactly a symmetric monoidal structure on Σ^∞_+.

Our goal in the next section will be to introduce a new fixed points functor which captures some compatibility with Σ^∞_+, and allows us to prove, e.g. that S^V is invertible in $\mathbf{Sp}_G$ for any representation V of G.

We will also briefly sketch the (monoidal) universal property of $\mathbf{Sp}_G$ compared to $\mathbf{S}_G$.

6 Geometric fixed points and isotropy separation

A possible intuition for geometric fixed points comes from classical algebra: if X is a G-set, and we linearize it by taking $\mathbb{Z}[X]$, then the fixed points of that abelian groups will contain too much information, compared to just $\mathbb{Z}[X^G]$. For instance, we get $\sum_g gx$ for any $x \in X$, and more generally for each orbit of the form G/H in X we get a term $\sum_{gH} gHx$.

In other words, $\mathbb{Z}[X]^G$ contains information about all orbits of X instead of just the trivial ones (which correspond to $\mathbb{Z}[X^G]$). To "correct" this and get access to the "geometric fixed points" of $\mathbb{Z}[X]$ (namely those coming from the geometry, from X), we kill all information related to proper isotropy. This is the point of view we'll take on geometric fixed points.

Definition 6.1. Let $\mathbf{Sp}_G^{\mathcal{P}}$ denote the localizing subcategory of $\mathbf{Sp}_G$ generated by the orbits $\Sigma^\infty_+ G/H, H < G$.

Momentarily denote by φ^G the localization functor $\mathbf{Sp}_G \to \mathbf{Sp}_G/\mathbf{Sp}_G^{\mathcal{P}}$

The claim is that $\mathbf{Sp}_G/\mathbf{Sp}_G^{\mathcal{P}} \simeq \mathbf{Sp}$, so that φ^G can be identified with a functor $\Phi^G : \mathbf{Sp}_G \to \mathbf{Sp}$, which will be the geometric fixed points. This corresponds to the intuition we alluded to above: we get access to geometric fixed points by killing all proper isotropy.

We will later see that geometric fixed points carry more structure, and in particular a (strict) symmetric monoidal structure. This will allow us to relate this Mackey story to the more classical story of inverting representation spheres.

Theorem 6.2. *$\mathbf{Sp}_G/\mathbf{Sp}_G^{\mathcal{P}}$ is equivalent to $\mathbf{Sp}$ as a symmetric monoidal category.*

Before proving this, we need some notation.

Construction 6.3. Let $\mathcal{O}_{\mathcal{P}}$ denote the full subcategory of $\mathcal{O}_G$ on orbits with proper isotropy, and let $E\mathcal{P}$ be the colimit of the restricted Yoneda embedding $\mathcal{O}_{\mathcal{P}} \to \mathbf{S}_G$.

Lemma 6.4. *$(E\mathcal{P})^H$ is contractible if $H < G$, and empty if $H = G$.*

Proof. Each of the fixed point functors $(-)^H$ is given by "evaluation" and so it preserves colimits.

Therefore it suffices to compute the colimit of $\mathcal{O}_{\mathcal{P}} \to \mathbf{S}, X \mapsto \mathrm{map}(G/H, X)$ for all H.

But this is the colimit of $\mathrm{map}(G/H, -)$, which is representable on $\mathcal{O}_{\mathcal{P}}$ if $H < G$, and so it is contractible in this case.

If $H = G$, then because all $X \in \mathcal{O}_G$ have $X^H = \emptyset$, the colimit is also empty. □

Proposition 6.5. *The previous lemma completely characterizes $E\mathcal{P}$ as a G-space.*

Proof. Indeed we can consider the full subcategory of $\mathbf{S}_G$ on those G-spaces that have no G-fixed points. Then $E\mathcal{P}$ is in there, and any G-space satisfying the previous lemma is, too. We claim that any space satisfying that must be terminal in that full subcategory.

The proof for this is to observe that the restriction functor $\mathbf{S}_G \to \mathrm{Fun}(\mathcal{O}_{\mathcal{P}}^{\mathrm{op}}, \mathbf{S})$ has a fully faithful left adjoint given by left Kan extension, and that its essential image is exactly our full subcategory.

If we prove that, then the result follows because any G-space satisfying the previous lemma must map to a pointwise terminal, hence terminal object.

But now, suppose $X \in \mathrm{Fun}(\mathcal{O}_{\mathcal{P}}^{\mathrm{op}}, \mathbf{S})$. Then its left Kan extension at G/G is $\mathrm{colim}_{(G/H \to G/G) \in \mathcal{O}_G^{\mathrm{op}}} X^H$, but there are no maps $G/H \to G/G$ in $\mathcal{O}_G^{\mathrm{op}}$, if $H < G$, so this colimit is empty.

In particular, $\mathrm{Lan}_i(X \circ i) \to X$ is an equivalence if and only if it is one at G/G (it is an equivalence on the rest), if and only if $X^G = \emptyset$. □

Remark 6.6. It is easy to generalize that space $E\mathcal{F}$ and the above results to any subset $\mathcal{F}$ of subgroups of G that is closed under subconjugation.

Construction 6.7. We then define the space $\widetilde{E\mathcal{P}}$ as the cofiber in pointed G-spaces of the map $E\mathcal{P}_+ \to S^0$ which sends $E\mathcal{P}$ to the non-basepoint.

We abuse notation by also letting $\widetilde{E\mathcal{P}}$ denote its (pointed) infinite suspension.

In particular, we have a cofiber sequence of G-spectra $\Sigma^\infty_+ E\mathcal{P} \to S^0_G \to \widetilde{E\mathcal{P}}$

(incomplete) Proof of the theorem. The localization is clearly generated by a single object, namely the image of the sphere S^0_G, which is compact in there, because the inclusion of $\mathbf{Sp}_G^{\mathcal{P}}$ preserves compacts.

It follows, by the Schwede-Shipley recognition theorem, that to prove an equivalence of categories, it suffices to check that the endomorphism ring spectrum of the image of S^0_G is equivalent to the sphere.

But the sphere has a unique ring structure in $\mathbf{Sp}$, because it is the unit of $\mathbf{Sp}$, so it suffices to prove that as spectra, forgetting about ring structures.

Note that the canonical map $S^0_G \to \widetilde{E\mathcal{P}}$ is an equivalence upon localization because $\Sigma^\infty_+ E\mathcal{P} \in \mathbf{Sp}^{\mathcal{P}}_G$, so it suffices to prove the same for $\widetilde{E\mathcal{P}}$.

But in fact, it should be clear that $\widetilde{E\mathcal{P}}$ is actually $\mathbf{Sp}^{\mathcal{P}}_G$-local, in the sense that $\mathrm{map}(X, \widetilde{E\mathcal{P}}) = 0$ as soon as $X \in \mathbf{Sp}^{\mathcal{P}}_G$. Indeed, it suffices to prove so for X of the form $\Sigma^\infty_+ G/H, H < G$, and then this is equivalent (by duality) to $(\mathbb{D}(G/H_+) \otimes \widetilde{E\mathcal{P}})^G$, and so it suffices to show that $\mathbb{D}(G/H_+) \otimes \widetilde{E\mathcal{P}} \simeq 0$.

In particular, it suffices to show that $\Sigma^\infty_+ E\mathcal{P} \to S^0_G$ becomes an equivalence after this tensoring. But by theorem 5.20, this amounts to proving the same thing after restricting to H. But $E\mathcal{P}$ is contractible after such a restriction, and so this map *is* an equivalence.

It follows from that that the endomorphism spectrum of $\widetilde{E\mathcal{P}}$ in the localization is the same as the mapping spectrum $\mathrm{Map}(S^0_G, \widetilde{E\mathcal{P}})$ in $\mathbf{Sp}_G$, i.e. $(\widetilde{E\mathcal{P}})^G$.

We now wish to show that this is S^0, and this is where we take something for granted.

Namely, we take for granted that we have a cofiber sequence of pointed spaces of the form $\mathrm{colim}_{G/H \in \mathcal{O}_\mathcal{P}} \mathcal{O}^{\simeq}_{H+} \to \mathcal{O}^{\simeq}_{G+} \to * \coprod +$. This is proved in [CMNN20], as part of the proof of lemma A.3.

Taking the free E_∞-monoid on those pointed spaces and then group completing yields exactly the cofiber sequence $\mathrm{colim}_{G/H \in \mathcal{O}^{op}_G} (\Sigma^\infty_+ G/H)^G \to (S^0_G)^G \to S^0$ which is compatible with the map $\Sigma^\infty_+ E\mathcal{P} \to S^0_G$ from before.

This proves the claim.

So $\mathbf{Sp}_G/\mathbf{Sp}^{\mathcal{P}}_G \simeq \mathbf{Sp}$, and the left hand side has a symmetric monoidal structure with unit $S^0_G \mapsto S^0$, in which the tensor product commutes with colimits in either variable.

It follows that it is symmetric-monoidally equivalent to $\mathbf{Sp}$, by uniqueness of the symmetric monoidal structure on the latter. □

Remark 6.8. The unique colimit preserving symmetric monoidal functor $\mathbf{Sp} \to \mathbf{Sp}_G$ maps S^0 to S^0_G, and so it maps S^0 to a generator of $\mathbf{Sp}/\mathbf{Sp}^{\mathcal{P}}_G$; in particular it induces an equivalence $\mathbf{Sp} \simeq \mathbf{Sp}_G/\mathbf{Sp}^{\mathcal{P}}_G$, or rather the unique equivalence sending S^0 to the image of S^0_G.

We call $\Phi^G : \mathbf{Sp}_G \to \mathbf{Sp}$ the composite of (the symmetric monoidal) φ^G with the symmetric monoidal identification $\mathbf{Sp}_G/\mathbf{Sp}^{\mathcal{P}}_G \simeq \mathbf{Sp}$, and call it the geometric fixed points functor - note that it has, by design, a canonical (strong) symmetric monoidal structure. We now investigate some of its key properties.

The first thing is that the above spectrum $\widetilde{E\mathcal{P}}$ represents Φ^G in the following sense:

Proposition 6.9. *There is an equivalence of functors:*

$$\Phi^G \simeq (\widetilde{E\mathcal{P}} \otimes -)^G$$

Proof. The latter is a colimit-preserving functor $\mathbf{Sp}_G \to \mathbf{Sp}$ which sends S^0_G to S^0.

Furthermore, for $H < G$, we saw that $G/H_+ \otimes \widetilde{E\mathcal{P}} = 0$, so that it vanishes on $\mathbf{Sp}^{\mathcal{P}}_G$. It follows that it is Φ^G, by definition. □

In fact, $\widetilde{E\mathcal{P}} \otimes -$ vanishes on $\mathbf{Sp}_G^{\mathcal{P}}$ and $\mathbf{Sp}_G^{\mathcal{P}}$ is generated by compacts, so that $\mathbf{Sp}_G^{\mathcal{P}}$-local objects are closed under colimits and they contain $\widetilde{E\mathcal{P}} \otimes \Sigma^\infty_+ G/H$ for all H (we observed this for $H < G$, and $\Sigma^\infty_+ G/G = S^0_G$ is the unit).

It follows that all $\widetilde{E\mathcal{P}} \otimes X$'s are $\mathbf{Sp}_G^{\mathcal{P}}$-local. Conversely, if X is $\mathbf{Sp}_G^{\mathcal{P}}$-local, then $G/H_+ \otimes X = 0$ for all $H < G$ (indeed its K-fixed points are $\mathrm{Map}(G/H_+ \otimes G/K_+, X) = 0$ as $G/H_+ \otimes G/K_+ \in \mathbf{Sp}_G^{\mathcal{P}}$), and so $X \otimes \Sigma^\infty_+ E\mathcal{P} = 0$. In particular, $X \to X \otimes \widetilde{E\mathcal{P}}$ is an equivalence.

We obtain :

Corollary 6.10. *The fully faithful right adjoint to $p : \mathbf{Sp}_G \to \mathbf{Sp}_G/\mathbf{Sp}_G^{\mathcal{P}}$ identifies the latter with the full subcategory of $\mathbf{Sp}_G$ spanned by $\widetilde{E\mathcal{P}} \otimes X, X \in \mathbf{Sp}_G$.*

In particular, this localization is smashing and $\widetilde{E\mathcal{P}}$ is the idempotent commutative algebra that classifies it.

Now, observe that we can define Φ^H on $\mathbf{Sp}_G$ as the composite $\mathbf{Sp}_G \overset{\mathrm{res}^G_H}{\to} \mathbf{Sp}_H \overset{\Phi^H}{\to} \mathbf{Sp}$. The first claim is that, just as categorical fixed points jointly detect equivalences, so do geometric fixed points:

Proposition 6.11. *Let $X \in \mathbf{Sp}_G$ satisfy $\Phi^H(X) = 0$ for all $H \leq G$. Then $X = 0$.*

In particular, if $f : X \to Y$ is such that $\Phi^H(f)$ is an equivalence for all $H \leq G$, then f is an equivalence too.

Proof. We prove this by induction on the order of G.

For $G = e$, Φ^e is the identity $\mathbf{Sp} \to \mathbf{Sp}$ so the claim is trivial.

Suppose now the claim holds for all groups of order $< |G|$, and suppose $\Phi^H(X) = 0$ for all H.

In particular, $\Phi^G(X) = 0$, so $X \otimes \widetilde{E\mathcal{P}} = 0$, and $X = \mathrm{colim}_{G/H \in \mathcal{O}_{\mathcal{P}}} X \otimes G/H_+$. So it suffices to now show that $X \otimes G/H_+ = 0$ for $H < G$ which, by theorem 5.20, is equivalent to $\mathrm{res}^G_H X = 0$.

But for $K \leq H, \Phi^K(\mathrm{res}^G_H X) = \Phi^K(X) = 0$ by definition and by assumption. So by the induction hypothesis, $\mathrm{res}^G_H X = 0$, and we are done. ☐

Our original motivation for geometric fixed points is also satisfied (and was one of the original motivations for their historical introduction):

Proposition 6.12. *The symmetric monoidal functor $\Phi^G \circ \Sigma^\infty$ is equivalent to $\Sigma^\infty \circ (-)^G$.*

Proof. Both functors are colimit-preserving symmetric monoidal functors $\mathbf{S}_{G,*} \to \mathbf{Sp}$. In particular it suffices to show that they agree (as symmetric monoidal functors) on $\mathcal{O}_G$.

However they both vanish on $\mathcal{O}_{\mathcal{P}}$ and the full subcategory of functors $\mathcal{O}_G \to \mathbf{Sp}$ that vanish on $\mathcal{O}_{\mathcal{P}}$ is equivalent to $\mathbf{Sp}$ by evaluation at G/G, and this equivalence is in fact symmetric monoidal (under Day convolution for the former).

Accepting this for now, the claim follows as both have value S^0 on G/G (and in a unique symmetric monoidal way, as S^0 is the unit of $\mathbf{Sp}$).

To prove this, we observe that the unit of $\mathrm{Fun}(\mathcal{O}_G, \mathbf{Sp})$ is $\Sigma^\infty_+ \mathrm{map}(G/G, -)$, which vanishes on $\mathcal{O}_{\mathcal{P}}$, and that vanishing on $\mathcal{O}_{\mathcal{P}}$ is preserved under Day convolution (if $X \times Y \to G/H$, then one of X or Y must be in $\mathcal{O}_{\mathcal{P}}$, so that the terms that appear in the colimit defining Day convolution must vanish). In particular, the subcategory we defined has a symmetric monoidal structure for which the inclusion is canonically symmetric monoidal. It therefore suffices to show that evaluation at G/G is symmetric monoidal and an equivalence. For the latter, it is purely a Kan extension property ($\{G/G\}$ is a full subcategory of $\mathcal{O}_G$), and for the former it is a general fact that evaluation at $\mathbf{1}$ is lax symmetric monoidal, and we simply have to observe that the restriction to our full subcategory is in fact strict symmetric monoidal, which is an easy thing to check. $\square$

Corollary 6.13. *Let V be a finite dimensional real representation of G. Then $\Sigma^\infty S^V$ is invertible in $\mathbf{Sp}_G$.*

Proof. We first observe that S^V has a finite G-CW-structure, which implies that it can be written as a finite colimit of G/H_+'s in $\mathbf{S}_{G,*}$, and therefore $\Sigma^\infty S^V$ is dualizable in $\mathbf{Sp}_G$.

In particular, its dual is preserved by any symmetric monoidal functor: $F(D(\Sigma^\infty S^V)) \simeq D(F(\Sigma^\infty S^V))$.

To check invertibility, it suffices to check that the coevaluation map $D(\Sigma^\infty S^V)\otimes\Sigma^\infty S^V \to S^0_G$ is an equivalence, and to do so it suffices to check it after applying Φ^H for all H. But by symmetric monoidality and the previous result, $\Phi^H(\Sigma^\infty S^V) \simeq \Sigma^\infty((S^V)^H) \simeq \Sigma^\infty S^{(V^H)}$ and so this coevaluation map is the coevaluation map of a finite dimensional sphere in $\mathbf{Sp}$: this is an equivalence. $\square$

Notation 6.14. We often abuse notation and write S^V instead of $\Sigma^\infty S^V$.

Remark 6.15. If $\mathbf{C}$ is a symmetric monoidal stable category, it makes sense to index homotopy groups by $\mathrm{Pic}(\mathbf{C})$ rather than $\mathbb{Z}$, and here we have just seen that we have a map $RO(G) \to \pi_0\mathrm{Pic}(\mathbf{Sp}_G)$, where$RO(G)$ is the real representation ring (as an additive group), which is why we often speak of "$RO(G)$-graded (co)homology groups".

This fact is part of the relation between $RO(G)$-graded (co)homology theories and Mackey-functor valued (co)homology theories on G-spaces.

The approach we took to $\mathbf{Sp}_G$ clearly insisted on the Mackey functor perspective, and on transfers along maps in $\mathcal{O}_G$, but this last part about representation spheres was one of the original motivations for G-spectra : a universal place where representation spheres become invertible (in the same way that spectra are a universal place where ordinary spheres are invertible). In fact, it turns out (we won't prove it) that this slogan/motivation can also be completely justified and clarified by the following theorem, which is the main result in the appendix of [CMNN20]:

Theorem 6.16. *The infinite suspension functor $\Sigma^\infty : \mathbf{S}_{G,*} \to \mathbf{Sp}_G$ is the initial colimit-preserving symmetric monoidal functor with domain $\mathbf{S}_{G,*}$ that sends all the representation spheres to invertible objects.*

More precisely, let $\mathbf{D}$ be an arbitrary presentably symmetric monoidal category. Then restriction along Σ^∞ induces a fully faithful embedding $\mathrm{Fun}^{L,\otimes}(\mathbf{Sp}_G, \mathbf{D}) \to \mathrm{Fun}^{L,\otimes}(\mathbf{S}_{G,*}, \mathbf{D})$ with essential image those functors that send representation spheres to invertible objects.

Geometric fixed points are also related to other kinds of fixed points, namely we have:

Proposition 6.17. *There exists a unique (lax) symmetric monoidal transformation $(-)^G \to \Phi^G$.*

This proposition in fact follows from:

Theorem 6.18. $(-)^G$ *is initial among colimit preserving lax symmetric monoidal functors* $\mathbf{Sp}_G \to \mathbf{Sp}$.

Proof. The category of these functors is, by standard properties of the Day convolution monoidal structure, equivalent to the category of lax symmetric monoidal finite coproduct-preserving functors $\mathrm{Span}(G) \to \mathbf{Sp}$, i.e. (again by standard properties of Day convolution) to the category of commutative algebras in $\mathrm{Fun}^{\oplus}(\mathrm{Span}(G), \mathbf{Sp}) \simeq \mathbf{Sp}_G$ under Day convolution.

Therefore the initial object there is the unit, which, if we unravel all these equivalences, is just $(-)^G$. $\square$

Remark 6.19. We will not prove it here, but it is not too hard to see (and it is an important fact !) that over $G = C_p$, the geometric fixed points of a Borel-equivariant spectrum coincide with its Tate construction X^{tC_p}.

Let us conclude this section by saying a few words about isotropy separation: geometric fixed points are only a special case of this.

More generally, we might sometimes be interested in some phenomena that behave differently for a certain class of groups, and in this case want to separate the information which happens at this class, and the rest. A way to achieve this is with *isotropy separation.* Given a certain family of subgroups $\mathcal{F}$, by which we mean a set of subgroups of G closed under conjugation and subgroups, we can define a G-space $E\mathcal{F}$ such that $(E\mathcal{F})^H \simeq *$ if $H \in \mathcal{F}$, and $\emptyset$ if $H \notin \mathcal{F}$ (see remark 6.6).

We can then similarly define $\widetilde{E\mathcal{F}}$ as well as their infinite suspensions, lying in a cofiber sequence $E\mathcal{F}_+ \to S^0_G \to \widetilde{E\mathcal{F}}$. Tensoring a G-spectrum X with this cofiber sequence allows one to separate the information of X "at" (subgroups in) $\mathcal{F}$ from the information of X "away" from $\mathcal{F}$ (see the first paragraph of this section for the case $\mathcal{F} = \mathcal{P}$, the family of all proper subgroups of G).

7 Induction theory

In this section, we review the basics of induction theory, and explain how in the derived setting it leads to Mathew, Naumann and Noel's derived induction theory.

The idea behind induction theory is to have powerful tools to get results for larger groups from results for certain subgroups, e.g. p-subgroups, or elementary abelian subgroups, ...

Our presentation of induction theory is very much ahistorical, and uses proposition 5.10 as its main tool. Classically, induction theory is formulated using $M_S := M(S \times -)$ for a Mackey functor M, but we feel that the story is much clearer with $\mathbb{D}(\Sigma^\infty_+ S) \otimes M$, because we can interpret things in terms of modules over a commutative algebra instead of this funny functor $(-)_S$.

The point is the following : $\mathbb{D}(\Sigma^\infty_+ S) \otimes M$ is a free $\mathbb{D}(\Sigma^\infty_+ S)$-module, which, by 5.20, means we can view it as induced from S; so it is in some sense determined by S-phenomena. So if M is "close" to something of this form, then it will also be determined by S-phenomena. To make this more precise, we introduce the so-called Amitsur complex of an algebra A:

Construction 7.1. Let A be a commutative algebra in a symmetric monoidal category $\mathbf{C}$. There is a cosimplicial object $\Delta \to \mathbf{C}$ given informally (this is an accurate description 1-categorically) by $[n] \mapsto A^{\otimes n+1}$, and where the coface maps introduce 1's as appropriate and the codegeneracies multiply together the appropriate elements.

Example 7.2. In low degrees, for instance, we have $A \rightrightarrows A^{\otimes 2}$ given by $a \mapsto a \otimes 1$ and $1 \otimes a$ respectively, and the degeneracy given by $a \otimes b \mapsto ab$.

Remark 7.3. This is the cosimplicial object appearing in classical (and derived) descent theory.

Example 7.4. Suppose $A = \mathbb{D}(G/H_+)$. This complex is then dual to the simplicial object $\cdots \to G/H \times G/H \rightrightarrows G/H$, where the two arrows are the two projections, and the degeneracy is the diagonal.

If $A = \prod_{H \in \mathcal{F}} A_H^G$ for some family $\mathcal{F}$ of subgroups of G, this diagram is related to the orbit category $O_{\mathcal{F}}$ as we will see later.

Note that this cosimplicial diagram is actually augmented, with 1 the unit of $\mathbf{C}$ in degree -1, so for all $M \in \mathbf{C}$, we have a canonical morphism $M \to M \otimes A^{\otimes \bullet + 1}$, and a reasonable question is: for which M does this exhibit M as the limit of this cosimplicial diagram ?

The full subcategory of M's for which this is the case contains A, because the cosimplicial diagram is then what is called *split*, i.e. it has a simplicial homotopy which forces it to be a *universal limit diagram* (preserved by any functor).

Furthermore, if we call D_M this diagram, then for any M, N, we have $D_{M \otimes N} = M \otimes D_N$, so, because D_A is a universal limit diagram, $M \otimes D_A$ is a limit diagram for any M, and so $D_{M \otimes A} \simeq D_{A \otimes M}$ is a(n absolute) limit diagram for any M.

Corollary 7.5. *For any M such that the canonical map $M \to A \otimes M$ has a retraction, the diagram D_M is a(n absolute) limit diagram.*

Proof. $M \mapsto D_M$ is a functor, and (absolute) limit diagrams are stable under retracts. Therefore it suffices to prove that $D_{A \otimes M}$ is a(n absolute) limit diagram, but this is the content of the previous discussion. ☐

Remark 7.6. To go from induction theory to derived induction theory, one just needs to note that in a stable category, limit diagrams are also stable under co/fiber sequences. For now we'll stick to this version.

We now specialize to the case of Mackey functors.

Definition 7.7. Let $S \in \mathrm{Fin}_G$, $M \in \mathbf{Mack}(G; E)$ where E is preadditive and presentable. Then M is called S-injective if the canonical morphism $M \to \mathbb{D}(\Sigma^\infty_+ S) \otimes M$ has a retraction.

It is called S-projective if the canonical map $\Sigma^\infty_+ S \otimes M \to M$ has a section.

Remark 7.8. Here $\otimes$ is to be understood as the action of $\mathbf{Mack}(G; \mathrm{CMon})$ on $\mathbf{Mack}(G; E)$

Lemma 7.9. *M is S-projective if and only if it is S-injective.*

Proof. Note that $\mathbb{D}(\Sigma^\infty_+ S) \otimes M$ has the structure of a $\mathbb{D}(\Sigma^\infty_+ S)$-module.

We prove that the two conditions are equivalent to "M is a retract of a $\mathbb{D}(\Sigma^\infty_+ S)$-module".

Indeed, both clearly imply this (note that as objects, $\Sigma^\infty_+ S \simeq \mathbb{D}(\Sigma^\infty_+ S)$). Conversely, because these maps are natural, the existence of a retraction (resp. a section) is preserved under retracts, so it suffices to show that $\mathbb{D}(\Sigma^\infty_+ S)$-modules have both a retraction to the first and a section to the second to prove the converse.

For the first map, this is clear: for any algebra A and A-module M, the map $M \to A \otimes M$ has a retraction given by the structure map $A \otimes M \to M$ (aka the co-unit map of the free-forgetful adunction).

For the second map, note again that the underlying object of a $\mathbb{D}(\Sigma^\infty_+ S)$-module is a retract of an object of the form $\mathbb{D}(\Sigma^\infty_+ S) \otimes N$, so it really suffices to prove the claim for $M = \mathbb{D}(\Sigma^\infty_+ S) \simeq \Sigma^\infty_+ S$. But then the canonical map $\Sigma^\infty_+(S \times S) \to \Sigma^\infty_+ S$ is induced by the projection onto the second factor $S \times S \to S$, which clearly has a section given by the diagonal. $\square$

Remark 7.10. This is again one of those cases where we have a specific map that we ask has a retraction/section, but it suffices to check that M be a retract of an arbitrary $\mathbb{D}(\Sigma^\infty_+ S)$-module for that to exist.

So to *check* the condition it suffices to exhibit any such retraction, but when you want to *use* the condition you can use the natural map.

Corollary 7.11. *If M is S-projective or S-injective, the diagram D_M is an absolute limit diagram.*

For a family $\mathcal{F}$ of subgroups, let $S = \coprod_{H \in \mathcal{F}} G/H$. Define $\mathrm{Fin}_{\mathcal{F}}$ to be the full subcategory of Fin_G on objects whose isotropy lies in $\mathcal{F}$. We then have a functor $\mathrm{Fin}_{\mathcal{F}}^{\mathrm{op}} \to \mathbf{Mack}_G(\mathrm{CMon})$ given by $X \mapsto \mathbb{D}(\Sigma^\infty_+ X)$, it preserves products and is symmetric monoidal.

Consider now the functor $\Delta^{\mathrm{op}} \to \mathrm{Fin}_{\mathcal{F}}$ given by $[n] \mapsto S^{\times(n+1)}$ whose face maps are obtained by deleting the appropriate coordinate, and whose degeneracies are obtained by suitable diagonals (it is the "Čech nerve" of the morphism $S \to *$). In any case, the composition $\Delta \to \mathrm{Fin}_{\mathcal{F}}^{\mathrm{op}} \to \mathbf{Mack}_G(\mathrm{CMon})$ is the cosimplicial object from before, applied to $\mathbb{D}(\Sigma^\infty_+ S)$.

Now we will use this to relate the limit of this cosimplicial object to limits over $O_{\mathcal{F}}$. We first have:

Lemma 7.12. *The functor $S^{\bullet+1} : \Delta^{\mathrm{op}} \to \mathrm{Fin}_{\mathcal{F}}$ described above is cofinal.*

Proof. The homotopy type of $\Delta^{\mathrm{op}}_{x/}$ is the geometric realization of $\mathrm{map}(x, S^{\bullet+1})$: indeed, the fibration $\mathrm{Fin}_{\mathcal{F},x/} \to \mathrm{Fin}_{\mathcal{F}}$ classifies the functor $\mathrm{map}(x,-)$, so its pullback over $S^{\bullet+1} : \Delta^{\mathrm{op}} \to \mathrm{Fin}_{\mathcal{F}}$ classifies the precomposition by this functor, which is exactly the simplicial set we described. It follows that the homotopy type of $\Delta^{\mathrm{op}}_{x/}$ is the colimit of this functor, i.e. the geometric realization described above.

But now this simplicial space is equivalently given as $\mathrm{map}(x,S)^{\bullet+1}$, i.e. the Čech nerve of $\mathrm{map}(x,S) \to *$. But now because $x \in \mathrm{Fin}_{\mathcal{F}}$, $\mathrm{map}(x,S) \neq \emptyset$, and the Čech nerve of a nonempty space mapping to $*$ is contractible (this is because groupoids are effective in $\mathbf{S}$)

If one is not comfortable with Čech nerves, one can also notice that in this case, $\mathrm{Fin}_{\mathcal{F}}$ is a 1-category, so $\mathrm{map}(x,S) = \hom(x,S)$ is just a nonemptyset, and this simplicial set is nothing but the nerve of the contractible 1-groupoid whose object set is $\hom(x,S)$ - it is therefore contractible. $\square$

It follows that $\lim_\Delta F(S^{\bullet+1}) \simeq \lim_{\mathrm{Fin}_{\mathcal{F}}^{\mathrm{op}}} F$. Furthermore, suppose $F : \mathrm{Fin}_{\mathcal{F}}^{\mathrm{op}} \to \mathbf{C}$ preserves products, then it is right Kan extended from $O_{\mathcal{F}}^{\mathrm{op}}$ (this is the statement that $\mathrm{Fin}_{\mathcal{F}}$ is freely generated from $O_{\mathcal{F}}$ under coproducts, which we leave as an exercise to the reader), and so $\lim_{\mathrm{Fin}_{\mathcal{F}}^{\mathrm{op}}} F \simeq \lim_{O_{\mathcal{F}}^{\mathrm{op}}} F$.

We apply this to the functor F defined by $X \mapsto \mathbb{D}(\Sigma^\infty_+ X) \otimes M$ for some S-projective $M \in \mathbf{Mack}_G(E)$. Then, by S-projectivity, the corresponding diagram $\Delta \to \mathbf{Mack}_G(E)$ has an absolute limit given by M, and therefore so does the corresponding diagram $O_{\mathcal{F}}^{\mathrm{op}} \to \mathbf{Mack}_G(E)$ (note that this limit is absolute modulo the requirement of finite product preservation, because of the arguments that went into comparing the limit over Δ and the one over $O_{\mathcal{F}}^{\mathrm{op}}$).

So for S-projective M, we find

$$M \simeq \lim_{O_{\mathcal{F}}^{\mathrm{op}}} \mathbb{D}(\Sigma^\infty_+ G/H) \otimes M$$

One could tell a similar story about colimits, but let us stick to that one.

Note that this limit is absolute with respect to product-preserving functors, so that we find:

Corollary 7.13. *Suppose M is S-projective. Then;*

- $M \simeq \lim_{O_{\mathcal{F}}^{\mathrm{op}}} \mathbb{D}(\Sigma^\infty_+ G/H) \otimes M$
- $M^G \simeq \lim_{O_{\mathcal{F}}^{\mathrm{op}}} M^H$ *(apply the G-fixed points functor)*
- *if* $M \in \mathbf{Mack}_G(\mathbf{Sp})$, $\pi_*^G M \simeq \lim_{O_{\mathcal{F}}^{\mathrm{op}}} \pi_*^H M$

Warning 7.14. The converse of the first point does *not* hold.

Remark 7.15. $\pi_*(-)^G$ preserves finite products, both as a functor $\mathbf{Sp} \to \mathbf{Ab}$, and as a functor $\mathbf{Sp} \to \mathbf{Sp}$. Therefore the last limit is a limit in $\mathbf{Ab}$ and in $\mathbf{Sp}$, that is, it's a 1-limit and a derived limit. In other words, all the higher limit functors (the derived functors of the limit functor) vanish if M is S-projective. This is extremely strong, and that's why S-projectivity results are extremely sought after.

Remark 7.16. Following remark 7.6, to do this in the setting of derived induction theory requires a bit more work, because you need to phrase the absoluteness of the diagram in stable terms rather than just additive terms - for that, [MNN17] and [MNN19] use pro-objects and the ∞-categorical Dold-Kan correspondance.

Because the diagrams D_M are then only "absolute with respect to exact functors between stable categories", you can't have the exact same result for homotopy groups and derived limits. What you instead get is that the limit spectral sequences collapse at a finite stage, and have a horizontal vanishing line : this means that the higher limits don't necessarily vanish for "S-nilpotent" objects (which is the suitable replacement for "S-projective" in that setting), but they are nonetheless well-behaved and "bounded" in a sense.

So MNN's "derived induction theory" might actually be called "stable induction theory", while the one we presented here is an additive one. The additive version is very nice in that it connects the 1-categorical world and the homotopy theoretic world, but in cases where stability is available it is definitely less powerful.

Remark 7.17. Following up on the mention of Dold-Kan in the previous remark, note that these statements were historically phrased in terms of chain complexes being contractible or similar ideas.

We now give a few criteria as to how to check S-projectivity. For $S = \coprod_{H\in\mathcal{F}} G/H$, we define $\mathcal{F}$-projectivity to mean S-projectivity.

The first criterion is one that allows us to define "defect bases":

Lemma 7.18. *Let $\mathcal{F}_0, \mathcal{F}_1$ be families. Suppose M is $\mathcal{F}_i$-projective, $i \in \{0,1\}$. Then M is $\mathcal{F}_0 \cap \mathcal{F}_1$-projective.*

Proof. Let $S_i = \coprod_{H\in\mathcal{F}_i} G/H$. Then M is a retract of $\Sigma^\infty_+ S_0 \otimes M$, which is a retract of $\Sigma^\infty_+ S_0 \otimes \Sigma^\infty_+ S_1 \otimes M \simeq \Sigma^\infty_+(S_0 \times S_1) \otimes M$.

Now $S_0 \times S_1$ has all its isotropy groups in $\mathcal{F}_0 \cap \mathcal{F}_1$, and any $H \in \mathcal{F}_0 \cap \mathcal{F}_1$ appears as the isotropy group of $(H,H) \in G/H \times G/H \subset S_0 \times S_1$, so letting $\mathcal{F} = \mathcal{F}_0 \cap \mathcal{F}_1$ and S the corresponding G-set, we find that a certain coproduct of S's retracts onto $S_0 \times S_1$. So M is a retract of $\bigoplus_i \Sigma^\infty_+ S \otimes M$, which is S-projective. The claim follows. $\square$

Furthermore, note that any M is G/G-projective, because $\Sigma^\infty_+ G/G$ is the unit in $\mathbf{Mack}_G(\mathrm{CMon})$, so that any M is $\mathcal{A}ll$-projective, where $\mathcal{A}ll$ is the family of all subgroups of G. It follows that any M has a minimal family $\mathcal{F}$ for which it is $\mathcal{F}$-projective.

Definition 7.19. We call that family $\mathcal{F}$ the defect base of M.

Remark 7.20. There is an analogous notion of "derived defect base" - they do not in general coincide.

Remark 7.21. This notion is related to the notion of a vertex of a G-module in modular representation theory, and the defect of a block.

Lemma 7.22. *If A is an algebra which is $\mathcal{F}$-projective, and M is an A-module, then M is $\mathcal{F}$-projective.*

Proof. M is a retract of $A \otimes M$ which is a retract of $\Sigma^\infty_+ S \otimes A \otimes M$, which is a $\mathbb{D}(\Sigma^\infty_+ S)$-module. $\square$

Remark 7.23. We only need A to be an algebra up to homotopy, and M to be a module up to homotopy.

So if we have nicer criteria for algebras, we can easily get nice projectivity statements. Here's one :

Proposition 7.24. *Suppose E is symmetric monoidal and $A \in \mathrm{Alg}(\mathbf{Mack}_G(E))$; and let $S = \coprod_{H\in\mathcal{F}} G/H$. Then A is $\mathcal{F}$-projective if and only if the unit $1 \to A$ lifts along the canonical projection $\mathbb{D}(\Sigma^\infty_+ S) \otimes A \to A$.*

Proof. Clearly if there is a retraction, then the unit lifts.

conversely, suppose the unit lifts and let $1 \to \mathbb{D}(\Sigma^\infty_+ S) \otimes A$ be such a lift. Then define $A \to \mathbb{D}(\Sigma^\infty_+ S) \otimes A$ as the composite $A \simeq 1 \otimes A \to \mathbb{D}(\Sigma^\infty_+ S) \otimes A \otimes A \to \mathbb{D}(\Sigma^\infty_+ S) \otimes A$, where the last map is just the multiplication of A. It is then straightforward to check that this is a section of the canonical projection. $\square$

Remark 7.25. Note that we only need A to be an algebra up to homotopy for this to work.

Corollary 7.26. *Suppose $A \in \mathrm{Alg}(\mathbf{Mack}_G(\mathbf{Sp}))$, and suppose the induction map $\bigoplus_{H\in\mathcal{F}} \pi_0^H(A) \to \pi_0^G(A)$ is surjective. Then A is $\mathcal{F}$-projective.*

In particular, if $\pi_0(A)$, viewed as $\mathbf{Ab}$-valued Mackey functor is $\mathcal{F}$-projective, so is A.

Proof. π_0^G is represented by the unit in $\mathbf{Mack}_G(\mathbf{Sp})$, so the hypothesis is that $\mathrm{map}(1, \mathbb{D}(\Sigma^\infty_+, S)\otimes A) \to \mathrm{map}(1, A)$ be surjective on π_0, which clearly implies that the unit lifts. $\square$

Note that the converse is clear, because split morphisms are universal epimorphisms.

This gives a clear computational way to prove that certain objects are $\mathcal{F}$-projective: find a nice ring over which they are a module, and compute the image of induction on π_0.

We now give an example of application of this theory. Because the proof of surjectivity is the meat of the game, we will not delve into it and instead explain how we can get an interesting result from a "mere" π_0-surjectivity result. These sections will be lacking some detail.

Remark 7.27. In stable induction theory, this criterion is no longer valid. In a chromatic context, there are analogous criteria involving rational surjectivity.

For the stable context, there are slightly more subtle criteria involving nilpotence, of similar type to Quillen's F-isomorphism theorem - otherwise one can try and relate derived defect bases and defect bases and use those relations.

7.1 Example: reduction of the Segal conjecture to p-groups

The Postnikov t-structure on $\mathbf{Sp}$ induces a t-structure on $\mathbf{Sp}_G$, whose heart is the 1-category of $\mathbf{Ab}$-valued Mackey functors. For a given G-spectrum X, we let $\pi_0(X)$ denote the relevant Mackey functor.

In particular, the sphere S^0_G has as its π_0 the Mackey Burnside ring. The G-fixed points of the latter are $A(G)$, the usual Burnside ring, with its augmentation ideal $I(G)$. We can consider $I(G)$-completion on $\mathbf{Sp}_G$ (this is a procedure analogous to completion at an ideal in classical algebra), and it's not hard to show that Borel G-spectra are $I(G)$-complete.

In particular, we have a canonical morphism $(S^0_G)^\wedge_{I(G)} \to (S^0)^{Bor}$ from the $I(G)$-adic completion of the sphere to the Borel sphere.

Theorem 7.28 (Segal conjecture, proved by Carlsson)**.** *This morphism is an equivalence.*

Remark 7.29. The conjecture was proved by Carlsson, but based on a lot of work of others, Lin, May.

In fact the initial conjecture was stated differently: it was that $A(G)^\wedge_{I(G)} \cong \pi^0(BG_+)$, but it was realized later that this should only be the π_0 part of the above statement - and for the statement we gave, equivariant homotopy theory is actually helpful, whereas the original one seemed to be an ordinary homotopy theoretic question.

In particular, that's where induction methods really shone: in the original statement, it's not clear how to apply induction, whereas in the statement we provided, we are in the equivariant world, and induction techniques are made for that world.

Now this morphism is also an algebra morphism, and so the Borel sphere is a module over the $I(G)$-complete equivariant sphere, and so whatever family the latter is projective over, so is the Borel sphere.

In particular, let $\mathcal{F}$ be such a family. Then to check that our morphism is an equivalence, it suffices to show that it's an equivalence after restriction to H for each $H \in \mathcal{F}$ (because that then implies that $G/H_+ \otimes f$ is an equivalence for all $H \in \mathcal{F}$, and f is equivalent to the limit of such morphisms, by $\mathcal{F}$-projectivity).

Now the restriction of the Borel sphere to H is also the Borel sphere, so we're left with understanding the restriction of the $I(G)$-complete sphere. But the restriction of the sphere is the sphere, and one can show that the $I(G)$-adic topology on $A(H)$ (induced by the restriction morphism $A(G) \to A(H)$) coincides with the $I(H)$-adic topology (this uses a knowledge of the prime ideals of $A(H)$).

Therefore we can state the "abstract and not-so-useful in this form" principle:

Proposition 7.30. *Suppose $\mathcal{F}$ is a family of subgroups of G such that the $I(G)$-completed Burnside ring Mackey functor $\widehat{A}$ is $\mathcal{F}$-projective; and suppose that the Segal conjecture holds for all $H \in \mathcal{F}$. Then it holds for G.*

The main point is then the following (this is where a big reduction happens):

Theorem 7.31. *The family of subgroups of prime power order of G is such a family.*

Remark 7.32. Because every p-subgroup of G is contained in a Sylow p-subgroup of G, all the induction morphisms factor through some Sylow. Therefore the statement is equivalent to the statement that $\bigoplus_p \widehat{A}(G_p) \to \widehat{A}(G)$ be surjective, where G_p is a chosen p-Sylow, for each prime $p \mid |G|$.

In particular, to prove the Segal conjecture, it suffices to do so for p-groups. This was a reduction step done by May and McClure, and then Carlsson proved that version of it.

7.2 Example: hyperelmentary induction in algebraic K-theory

Here we give an example coming from algebraic K-theory.

Remark 7.33. We will use the notation Π because there will be a functor called G-theory coming up soon, and though there is no possible confusion, this would lead to unfortunate notations.

Consider the Mackey functor given by $\Pi/H \mapsto D^b(RH)$ for some noetherian ring R - the derived category of bounded complexes of finitely generated RH-modules; the covariant map induced by an inclusion $H \leq K$ is given by induction, and the contravariant map induced by the same inclusion is given by the forgetful map. This defines a Green functor if R is furthermore regular, or more generally if (derived) tensor products of bounded complexes of finitely generated R-modules remain so. Recall the definition of a Green functor :

Definition 7.34. A Green functor is a commutative algebra in the category of Mackey functors.

Each $D^b(RH)$ contains $\mathbf{Perf}(RH)$, the category of perfect RH-modules, which is an ideal of $D^b(RH)$. One can then show that this provides $\Pi/H \mapsto \mathbf{Perf}(RH)$ the structure of a module over the previous functor. Algebraic K-theory is a lax monoidal functor, so we can apply it to these Mackey functors and obtain Π-spectra: the G-theory $G(RH)$ and the K-theory $K(RH)$, the latter being a module over the former.

In particular, if we want to get projectivity results for both Π-spectra, it suffices to do so for G-theory.

We then use the following general induction theorem, due to tom Dieck (see the section on induction theory in [Die06]) :

Theorem 7.35 (Hyperelementary induction)**.** *Suppose U is a Green functor such that $U(\Pi)$ is torsion-free and the resriction map $U(\Pi) \to \prod_{H\in\mathcal{C}yc} U(H)$ is injective, where $\mathcal{C}yc$ is the family of cyclic subgroups of Π.*

Then U is $\mathcal{H}y$-projective, where $\mathcal{H}y$ is the family of hyperelementary subgroups of Π, that is, those subgroups which are the extension of a p-group by a cyclic subgroup.

Suppose for instance that R is a field L. Then $G_0(L\Pi)$ is indeed torsion-free, as it is free on the isomorphism classes of simple $L\Pi$-modules. Furthermore, the injectivity condition is satisfied - this is not necessarily trivial, but it's essentially the idea that the class in G_0 is determined by its character. Let us outline a proof:

Proof. By Swan's book [Swa06, Corollary 2.12], we know that the $|\Pi|$-inverted G-theory of $L\Pi$ is $\mathcal{C}yc$-projective (regardless of the characteristic of L ! - in fact, stated as such, this is true for an arbitrary ring R) so that, after inverting $|\Pi|$, we find that $G_0(L\Pi) \to \lim_{H\in\mathcal{C}yc} G_0(LH) \to \prod_{H\in\mathcal{C}yc} G_0(LH)$ is injective. But everyone here is torsion-free, in fact free over $\mathbb{Z}$, therefore if it's injective after inverting $|\Pi|$, it is integrally injective. $\square$

It follows that $G(L-)$ is $\mathcal{H}y$-projective (in fact, as pointed out in the above proof, if we invert the order of Π, it is $\mathcal{C}yc$-projective ! for many purposes, this is good enough).

The same therefore holds for $K(L-)$, and therefore, if L was the residue field of a complete discrete valuation ring A, because $K_0(A-) \to K_0(L-)$ is an isomorphism, the same holds for $K_0(A-)$.

But we want to say more, and for this we need a more precise statement from [Die06] (the same section there):

Theorem 7.36 (p-local hyperelementary induction)**.** *Suppose again U is a Green functor, and suppose any torsion element of $U(\Pi)$ is nilpotent. Suppose further that the restriction map to the family of cyclic subgroups is rationally injective.*

Then $U_{(p)}$ is $\mathcal{H}y$-projective.

We will apply this to $U =$ the G-theory of $A[-]$. Because, as above, after inverting $|\Pi|$ this G-theory is $\mathcal{C}yc$-projective, the rational injectivity of the restriction map is clear. It therefore suffices to analyze the torsion in $G_0(A[\Pi])$. Note that we have a fiber sequence $G(k[\Pi]) \to G(A[\Pi]) \to G(K[\Pi])$, where k is the residue field of A, and K its fraction field.

This comes from the fact that the inclusion $D^b(k[\Pi]) \to D^b(A[\Pi])^{\varpi-tors}$ induces an equivalence on K-theory via the theorem of the heart and the dévissage theorem, where ϖ is a uniformizer for A.

But now we know from [Ser77, Theorem 33] that the "decomposition morphism" $G_0(K[\Pi]) \to G_0(k[\Pi])$ is surjection (this is the d in the "cde-triangle"), this sends the class of a $K\Pi$-module V to the class of V_0/ϖ, where $V_0 \subset V$ is an A-lattice for V.

But now note that the class of V_0/ϖ is 0 in $G_0(A[\Pi])$, because we have the following short exact sequence $0 \to V_0 \to V_0 \to V_0/\varpi \to 0$ in $A[\Pi]$-modules. It follows that $G_0(A[\Pi]) \to G_0(K[\Pi])$ is injective, so we get that the former is torsion-free, which is enough to apply the above statement.

We therefore get that $G(A[-])_{(p)}$ and therefore $K(A[-])_{(p)}$, and variant such as $K(A[-])^\wedge_p$ are all $\mathcal{H}y$-projective. In "Algebraic K-theory and traces", Madsen uses the latter to get a limit decomposition of $K(A[G])^\wedge_p$, which he can then relate to topological cyclic homology via trace methods, the latter being more computable.

8 The profinite case

In this section, we give another description of $\mathbf{Sp}_G$ when G is profinite, analogous to descriptions one might have in the 1-categorical case of continuous G-actions on sets. Note that in this case G-spectra are defined as Mackey functors on Fin_G, which is the category of finite sets with *continuous* G-action, that is, a G-action that factors through G/N for some open subgroup N of G. In particular, if $G \cong \lim_i G_i$ with each G_i a finite group, we can describe this category as $\mathrm{colim}_i \mathrm{Fin}_{G_i}$ - this is a filtered colimit of 1-categories, and so taking nerves gives a filtered colimits of simplicial sets.

In particular, given the description of $\mathrm{Span}(G_i)$, and the fact that $O(\Delta^n)$ (see [Bar17] for the notation) is a finite simplicial set, it follows that $\mathrm{Span}(\mathrm{Fin}_G)$ is the colimit, as simplicial sets but therefore also as categories, of the $\mathrm{Span}(G_i)$. One can also phrase this in terms of complete Segal spaces rather than simplicial sets.

We have the following corollary :

Corollary 8.1. *The restrictions* $\mathbf{Sp}_G \to \mathbf{Sp}_{G_i}$ *induced by the inclusions* $\mathrm{Span}(G_i) \to \mathrm{Span}(G)$ *induce an equivalence of categories* $\mathbf{Sp}_G \to \lim_i \mathbf{Sp}_{G_i}$.

Proof. It follows at once from the fact that each inclusion $\mathrm{Span}(G_i) \to \mathrm{Span}(G_j)$ preserves direct sums. □

Remark 8.2. Each $\mathbf{Sp}_{G_i} \to \mathbf{Sp}_{G_j}$ induced by $G_i \to G_j$ is a categorical fixed points functor, and is a right adjoint. In particular this limit of categories is also a limit in $\mathbf{Pr}^R$ – we can therefore view it as a colimit in $\mathbf{Pr}^L$ where each $G_i \to G_j$ induces $\mathbf{Sp}_{G_j} \to \mathbf{Sp}_{G_i}$ given informally by "the action that is trivial on $\ker(G_i \to G_j)$".

Note that the categorical fixed points, being given by a precomposition, preserve filtered colimits. In particular its left adjoint preserves all compact objects, and so we can phrase this colimit in $\mathbf{Pr}^L$ also as a colimit in $\mathbf{Cat}_\infty$ of the full subcategories of compact objects. Concretely, the functors appearing there map the compact generator $\Sigma^\infty_+ G_j/K$ to $\Sigma^\infty_+ G_i/\pi_{ij}^{-1}(K)$, where $\pi_{ij} : G_i \to G_j$ is the projection.

Remark 8.3. This is analogous to the fact that in 1-categories, continuous G-actions on sets is the limit of G_i-actions on sets, where a continuous G-set X corresponds to the sequence $(X^{H_i})_i$ of fixed points along $H_i = \ker(G \to G_i)$. The dual point of view, describing it as a colimit, is analogous to the description of the category of continuous finite G-sets we gave earlier.

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