THE EILENBERG-MOORE SPECTRAL SEQUENCE

Introduction

This is essentially just a rewriting of Toën's argument for the Eilenberg-Moore spectral sequence in [Toë19, Appendix A] - the two main changes are that it's in english, and that instead of using a topological model for the homotopy types involved, we do the same proof but using only homotopy types (the heart of the matter is the same though, and there is no claim to originality in these notes, at all).

There will also be more detail than in Toën's paper, at some points perhaps too much.

1. The setting

We work over a base ring k - we will see what additional hypotheses we need to have on it for this to work.

We have a pullback square of spaces

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

with two additional assumptions: for each $b \in B$, the fiber $X_b := f^{-1}(b)$ is of finite cohomological type over k (this will be used in two places), and the action of $\pi_1(B, b)$ on $H^i(X_b; k)$ is nilpotent for every i.

For simplicity, we will also assume that B is connected.

The claim we want to prove is that under suitable hypotheses on k (which are satisfied e.g. when k is a PID, in particular $\mathbb{Z}$ or a field), this pullback induces a pushout square of commutative k-algebras by taking k-valued cochains, i.e. the following natural map is an equivalence:

$$C^*(X; k) \otimes_{C^*(B;k)} C^*(Y; k) \xrightarrow{\sim} C^*(W; k)$$

From this, using the Tor-spectral sequence associated to any such tensor product (see e.g. [Lur12, 7.2.1.19.]), we get the Eilenberg-Moore spectral sequence.

2. The proof

Let $\mu := f \circ p$ which is canonically identified with $g \circ q$; and consider the functors $f_* : \mathrm{Fun}(X, \mathbf{Mod}_k) \to \mathrm{Fun}(B, \mathbf{Mod}_k)$, and g_*, μ_*, q_*, p_* defined similarly via right Kan extensions.

As the right adjoints of the restriction functors which are symmetric monoidal for the pointwise symmetric monoidal structure, they all have a canonical lax symmetric monoidal structure.

In particular, we have canonical maps $k \to p_*k, k \to q_*k$, where k denotes the constant k-valued functor on the appropriate categories, which induce a canonical map

$$f_*k \otimes_k g_*k \to \mu_*k \otimes_k \mu_*k \to \mu_*k$$

At a point $b \in B$, the explicit formula for right Kan extension tells us that this map is $C^*(X_b; k) \otimes_k C^*(Y_b; k) \to C^*(W_b; k)$. But of course, $W_b \simeq X_b \times Y_b$ canonically, and so this map is a "Künneth map". We want it to be an equivalence, so because of our assumption on the k-cohomology of X_b this a "cohomological Künneth"-type statement, which holds for instance if k is noetherian and has finite global dimension - see proposition A.1 in the appendix. We therefore assume it to be the case from now on.

Let us now use $\Gamma : \mathrm{Fun}(B, \mathbf{Mod}_k) \to \mathbf{Mod}_k$ to denote the limit functor (I use this notation because it'll be lighter than having lim's everywhere, and to be absolutely sure not to confuse it with the colim's that will appear).

Γ is also lax symmetric monoidal as the right adjoint to the trivial functor, and so it lifts to a lax symmetric monoidal functor, which we still denote by Γ, $\Gamma : \mathrm{Fun}(B, \mathbf{Mod}_k) \to \mathbf{Mod}_{C^*(B;k)}$ (as $\Gamma(k) \simeq C^*(B;k)$)

This in turn induces a canonical map

$$\Gamma(f_*k) \otimes_{C^*(B;k)} \Gamma(g_*k) \to \Gamma(f_*k \otimes_k g_*k) \tilde{\to} \Gamma(\mu_*k)$$

where the equivalence is because of the previous argument. Now, because right adjoints compose, we see that this map is exactly $C^*(X;k) \otimes_{C^*(B;k)} C^*(Y;k) \to C^*(W;k)$. So it suffices to prove that the first portion, $\Gamma(f_*k) \otimes_{C^*(B;k)} \Gamma(g_*k) \to \Gamma(f_*k \otimes_k g_*k)$, is an equivalence.

Consider the following property of $E, F \in \mathrm{Fun}(B, \mathbf{Mod}_k)$: "$\Gamma(E) \otimes_{C^*(B;k)} \Gamma(F) \to \Gamma(E \otimes_k F)$ is an equivalence".

It is clearly satisfied when $E = k$, and so, it is satisfied whenever E is in the (thick) stable subcategory generated by k in $\mathrm{Fun}(B, \mathbf{Mod}_k)$.

Note that if E is concentrated in one degree, then up to a shift, it belongs to $\mathrm{Fun}(B, \mathbf{mod}_k) \simeq \mathrm{Fun}(\tau_{\leq 1} B, \mathbf{mod}_k) \simeq \mathrm{Fun}(B\pi_1(B,b), \mathbf{mod}_k)$, where $\mathbf{mod}_k$ denotes the 1-category of discrete k-modules.

In particular if, as an element of this category, it has a nilpotent $\pi_1(B,b)$-action, and is finitely generated as a k-module, then it is in said stable subcategory, and in particular the property holds for E (and any F).

By Postnikov approximation, we automatically get :

Lemma 2.1. *Suppose E is bounded and its homotopy groups are finitely generated over k and nilpotent $\pi_1(B,b)$-modules. Then the property holds for E and any F.*

Now we want to use this for E looking like $C^*(X;k)$, which obviously isn't bounded in general, so we have to use some further property of F to be able to approximate $C^*(X;k)$ by its truncations $\tau_{\geq -n} C^*(X;k)$ (note that I'm sticking to homological notations, as opposed to Toën - he is probably wiser on that one).

In particular, note that $C^*(X;k) \simeq \mathrm{colim}_{\geq n} \tau_{\geq -n} C^*(X;k)$. We have the following lemma:

Lemma 2.2. *For any E, $\Gamma(E) \simeq \mathrm{colim}_{\geq n} \Gamma(\tau_{\geq -n} E)$.*

Proof. Note that we have a fiber sequence $\tau_{\geq -n} E \to E \to \tau_{< -n} E$. Taking limits over B, since $(\mathbf{Mod}_k)_{\leq -(n+1)}$ is stable under limits, we have a fiber sequence $\Gamma(\tau_{\geq -n} E) \to \Gamma(E) \to T$, where T is concentrated in degrees $< -n$.

Simlarly, we have a fiber sequence $\Gamma(\tau_{\geq -n} E) \to \Gamma(\tau_{\geq -(n+1)} E) \to Z$ with a similar property for Z.

In particular, the natural map $\mathrm{colim}_n \Gamma(\tau_{\geq -n} E) \to \Gamma(E)$ induces, on homotopy groups, an isomorphism : it suffices to take $i >> |*|$ to get that $\Gamma(\tau_{\geq -i} E) \to \mathrm{colim}_n \Gamma(\tau_{\geq -n} E)$ and $\Gamma(\tau_{\geq -i} E) \to \Gamma(E)$ induce isomorphisms on π_*. $\square$

In particular, we now only have to deal with $\Gamma(E \otimes_k F)$; this is another place where we'll need the hypotheses on k, and also where we'll use that F is the cochain complex of a space - more precisely, we only really need F to be coconnective.

Lemma 2.3. *If F is coconnective, then $\Gamma(E \otimes_k F) \simeq \mathrm{colim}_n \Gamma(\tau_{\geq -n} E \otimes_k F)$.*

Remark 2.4. For this lemma (and not the rest of the proof), it actually suffices to assume that every module over k has a flat resolution of length $\leq m$, for some fixed m. I do not know if this is enough for the whole theorem.

If k is noetherian, this is essentially the same as our hypothesis for finitely generated modules, see the answer to this MO question [Per] - therefore we will not linger on whether better hypotheses can be found.

Once we prove this lemma, we can piece everything together and get the main theorem:

Theorem 2.5. *Assume k is noetherian and has a finite global dimension, and let X, Y, B, W be as in the setting.*

Then the canonical map $C^*(X;k) \otimes_{C^*(B;k)} C^*(Y;k) \to C^*(W;k)$ is an equivalence.

Proof. By lemma 2.1, and our assumption on the fibers of X,

$$\Gamma(\tau_{\geq -n} f_* k) \otimes_{C^*(B;k)} \Gamma(g_* k) \to \Gamma(\tau_{\geq n} f_* k \otimes_k g_* k)$$

is an equivalence.

By lemmas 2.2 and 2.3, because tensor products are compatible with colimits and $g_* k$ is coconnective, it follows that $\Gamma(f_* k) \otimes_{C^*(B;k)} \Gamma(g_* k) \to \Gamma(f_* k \otimes_k g_* k)$ is an equivalence, which is what we wanted to prove. □

Corollary 2.6. *Let k, X, Y, B, W be as in theorem 2.5. Then there exists a spectral sequence with E^2-term $E^2_{p,q} \cong \mathrm{Tor}_p^{H^*(B;k)}(H^*(X;k), H^*(Y;k))_q$ converging to $H^{p+q}(W;k)$*

Proof. See [Lur12, 7.2.1.19.]. □

It now suffices to prove the lemma:

Proof of lemma 2.3. This is essentially the same proof as the previous lemma, we need the extra assumptions to control the coconnectivity of the cofibre:

We have a fiber sequence

$$\tau_{\geq -n} E \otimes_k F \to E \otimes_k F \to \tau_{< -n} E \otimes_k F$$

Because F is coconnective, and by assumption on the global dimension m of k (here flat dimension would be enough), $\tau_{<-n} E \otimes_k F$ is concentrated in degrees $< m - n$, for a fixed m. In particular, taking n large enough we can make it be concentrated in degrees as negative as we want; similarly for $\pi_{-(n+1)}(E)[-(n+1)] \otimes_k F$ and the fiber sequence

$$\tau_{\geq -n} E \otimes_k F \to \tau_{\geq -(n+1)} E \otimes_k F \to \pi_{-(n+1)}(E)[-(n+1)] \otimes_k F$$

.

The same argument then applies upon taking limits over B, to show that for n large enough, both (compatible with the comparison map) maps $\Gamma(\tau_{\geq -n} E \otimes_k F) \to \Gamma(E \otimes_k F)$ and $\Gamma(\tau_{\geq -n} E \otimes_k F) \to \Gamma(\tau_{\geq -(n+1)} E \otimes_k F)$ induce an isomorphism on π_*. □

We prove something which we left implicit in the above proof and is not obvious at a first glance if we start thinking in terms of flat resolutions, but is nonetheless easy to prove:

Lemma 2.7. *Let R be an arbitrary commutative ring, and assume every R-module has a flat resolution of length $\leq m$, and let M, N be coconnective R-modules. Then $M \otimes_R N$ is concentrated in degrees $\leq m$.*

Proof. We prove the equivalent statement that if M is concentrated in degrees $\leq a$, N in degrees $\leq b$, then $M \otimes_R N$ is concentrated in degrees $\leq a+b+m$.

This is true by assumption on flat resolutions when M and N are concentrated in a single degree.

By using Postnikov-type cofiber sequences, we see that this is true whenever M, N are bounded; and finally we use that $M \simeq \mathrm{colim}_n \tau_{\geq -n} M$ and similarly for N, and that $(\mathbf{Mod}_R)_{\leq a+b+m}$ is stable under filtered colimits (a fact which is not true for all t-structures, but is true in the case of modules over a ring) □

Remark 2.8. Is it true that any coconnective complex over such a ring is quasi-isomorphic to a complex of flat modules concentrated in degrees $\leq m$?

Appendix A. The cohomological Künneth formula

In this appendix we give a similar type of proof for the cohomological Künneth formula over a noetherian ring with finite global dimension k.

Recall that we can state it as follows:

Proposition A.1. *Let X, Y be homotopy types, and suppose X is of finite cohomological type over k. Then the canonical map $C^*(X;k) \otimes_k C^*(Y;k) \to C^*(X \times Y;k)$ is an equivalence.*

Remark A.2. This is somehow a "baby" case of the result we just proved. Of course, we cannot deduce it from that, as the proof above uses that statement.

Lemma A.3. *Suppose X is of finite cohomological type over k; then for all $n \geq 0$, $\tau_{\geq n} C^*(X;k)$ is a perfect k-module.*

Proof. This requires k to be regular noetherian, which is weaker than having finite global dimension: indeed $\tau_{\geq n} C^*(X;k)$ can be built by extensions from a finite number of $H^i(X;k)$'s, and so it suffices to check that these are perfect: but they're finitely generated, so over a regular noetherian ring, they are represented by a finite complex of finitely generated projectives, so they are perfect, and from that the claim follows. □

Lemma A.4. *Let $F_\bullet : \mathbb{N} \to \mathrm{Fun}(\mathbf{S}^{\mathrm{op}}, \mathbf{Mod}_k)$ denote a family of functors, where $\mathbb{N}$ is viewed as a poset with the usual order.*

Suppose for each i, F_i sends colimits in $\mathbf{S}$ to limits in $\mathbf{Mod}_k$; and suppose there is a nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ that tends to infinity and such that $\mathrm{cofib}(F_n \to F_{n+1})$ has values in $f(n)$-coconnective k-modules.

Then $\mathrm{colim}_i F_i : \mathbf{S}^{\mathrm{op}} \to \mathbf{Mod}_k$ sends colimits in $\mathbf{S}$ to limits in $\mathbf{Mod}_k$.

Proof. In $\mathbf{Mod}_k$, filtered colimits commute with finite limits, hence the claim is clear for these.

We will moreover prove that the claim holds for sequential colimits, and arbitrary coproducts, from which the claim will follow (indeed from sequential colimits and finite colimits one deduces that the claim holds for Δ^{op}-indexed colimits, and thus for arbitrary filtered colimits one can use Δ^{op}-indexed colimits and arbitrary coproducts)

In fact, the proof for arbitrary coproducts resembles the one for sequential colimits, and it's even easier, so we will leave that one to the reader and focus on sequential colimits.

So suppose we have a sequence $Y_0 \to Y_1 \to \dots$ of spaces, we then have a natural map $\mathrm{colim}_n \lim_p F_n(Y_p) \to \lim_p \mathrm{colim}_n F_n(Y_p)$, and the claim is that this is an equivalence. We prove that by proving it induces an isomorphism on homotopy groups. So fix an index $*$.

I claim that there is an m such that the canonical map from $\lim_p F_m(Y_p)$ to each side induces an isomorphism on π_*, from which the claim will follow.

Fix n, then consider the map $\lim_p F_n(Y_p) \to \lim_p F_{n+1}(Y_p)$. By the Milnor exact sequence (this is why we need the colimit to be sequential and not an arbitrary filtered colimit), on homotopy groups, this induces a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \lim^1_p \pi_{*+1} F_n(Y_p) \arrow[r] \arrow[d] & \pi_*(\lim_p F_n(Y_p)) \arrow[r] \arrow[d] & \lim_p \pi_* F_n(Y_p) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \lim^1_p \pi_{*+1} F_{n+1}(Y_p) \arrow[r] & \pi_*(\lim_p F_{n+1}(Y_p)) \arrow[r] & \lim_p \pi_* F_{n+1}(Y_p) \arrow[r] & 0 \end{tikzcd}$$

Now, if n is large enough, $f(n)$ will be large enough compared to $|*|$ for the left and right vertical maps to be isomorphisms; by the 5-lemma, for n large enough, the middle vertical map is also an isomorphism. Since f is nondecreasing, we get that from this n onwards, the map $\lim_p F_n(Y_p) \to \lim_p F_m(Y_p)$ is an isomorphism on π_* for $m > n$ (note that $*$ is fixed !).

In particular, for m large enough, $\lim_p F_m(Y_p) \to \mathrm{colim}_n \lim_p F_m(Y_p)$ is an isomorphism on π_*.

On the other hand, for m large enough, and for all p (m is independent of p !) $F_m(Y_p) \to \mathrm{colim}_n F_n(Y_p)$ is an equivalence on π_* and π_{*+1}, so that $\lim_p F_m(Y_p) \to \lim_p \mathrm{colim}_n F_n(Y_p)$ is also an isomorphism in π_*, again by the Milnor exact sequence.

Since the following triangle commutes by definition :

$$\begin{tikzcd} & \lim_p F_m(Y_p) \arrow[dl] \arrow[dr] & \\ \mathrm{colim}_n \lim_p F_n(Y_p) \arrow[rr] & & \lim_p \mathrm{colim}_n F_n(Y_p) \end{tikzcd}$$

we get that the bottom map is an isomorphism on π_*, for every $*$, in particular, it is an equivalence.

It works the same way for coproducts, the key point being again that on π_*, for a fixed $*$, the colimit is "attained" independently of Y_p; and products commute with π_*. □

Proof of proposition A.1. Fix X, and consider the natural map $C^*(X;k) \otimes_k C^*(Y;k) \to C^*(X \times Y;k)$ as a natural transformation of functors (of Y). The right hand side sends colimits of spaces to limits of k-modules, and the natural transformation is obviously an equivalence when $Y = *$, therefore it suffices to prove that the left hand side also sends colimits of spaces to limits of k-modules to prove that it is an equivalence.

But the left hand side is the colimit of the $\tau_{\geq n} C^*(X;k) \otimes_k C^*(-;k)$, and each $\tau_{\geq n} C^*(X;k) \otimes_k -$ preserves limits, as by lemma A.3, $\tau_{\geq n} C^*(X;k)$ is perfect, hence dualizable. So it suffices to check that the hypotheses of lemma A.4 apply to conclude that $C^*(X;k) \otimes_k C^*(-;k)$ also sends colimits of spaces to inverse limits.

This uses again the hypothesis on k: the cofiber of

$$\tau_{\geq n} C^*(X;k) \otimes_k C^*(-;k) \to \tau_{\geq n+1} C^*(X;k) \otimes_k C^*(-;k)$$

is $H^{n+1}(X;k)[-(n+1)] \otimes_k C^*(-;k)$, and because $C^*(Y;k)$ is coconnective, and the global dimension of k is finite, we see that $H^{n+1}(X;k)[-(n+1)] \otimes_k C^*(Y;k)$ is concentrated in degrees $\leq \mathrm{gld}(k) - (n+1)$, which is indeed nonincreasing and converges to $-\infty$ as $n \to \infty$. We can thus apply lemma A.4 and conclude from there. □

References

[Lur12] Jacob Lurie. Higher algebra, 2012.

[Per] PersonX. Tor and projective dimension. MathOverflow. URL:https://mathoverflow.net/q/33209 (version: 2010-07-24).

[Toë19] Bertrand Toën. Le problème de la schématisation de Grothendieck revisité. *arXiv preprint arXiv:1911.05509*, 2019.