The effective Burnside category of finite free G-sets as the free semi-additive category on BG

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Introduction

The goal of this note is to provide a simpler proof of the following claim made in [BGS19, 8.1] and [Gla15, A.1]:

Theorem 0.1. *Let G be a finite group, and Fr^G the category of finite free G-sets. Let $\mathbf{E}$ be a semiadditive ∞-category. Then the restriction functor*

$$\mathbf{Mack}(Fr^G, \mathbf{E}) \to \mathrm{Fun}(BG, \mathbf{E})$$

is an equivalence.

In other words, $A^{eff}(Fr^G)$ is the free semiadditive ∞-category on BG.

The claim is not proved in the first named reference, and a long, technical proof is given in the second one, although most of the ideas of our proof are already found in the latter.

Conventions

We will not recall here the definition of $A^{eff}(Fr^G)$ and refer the reader to the aforementioned papers for a definition. This ∞-category is essentially the ∞-category of spans of Fr^G, except that since everything is done homotopically, we can afford not to take isomorphism classes of spans to define it; and let composition (defined by pullbacks) be defined only up to homotopy.

$\mathbf{Mack}(Fr^G, \mathbf{E})$ is simply defined as $\mathrm{Fun}^{\oplus}(A^{eff}(Fr^G), \mathbf{E})$, the category of (co)product preserving functors $A^{eff}(Fr^G) \to \mathbf{E}$.

In the rest of the document, we will only use the word "category" to refer to ∞-categories.

$\mathbf{S}$ refers to the category of spaces, CMon to the category of commutative monoids in $\mathbf{S}$ (and more generally, $\mathrm{CMon}(\mathbf{C})$ refers to the category of commutative monoids in $\mathbf{C}$, whenever $\mathbf{C}$ has all finite products), and Fin to the category of finite sets (viewed as a full subcategory of $\mathbf{S}$).

1 Outline of the proof

Our proof differs from that of [Gla15] in two ways:

First, the way we prove that $A^{eff}(Fr^G)$ is the Lawvere theory of commutative monoids with G-action is different, as we use the universal property of that Lawvere theory to achieve that goal. It makes for a less technical proof, which relies on well-known ideas in the theory of algebraic theories.

Second, the way we prove the theorem from this intermediate result is also different, and somewhat less technical. Indeed, once we know that $A^{eff}(Fr^G)$ is the Lawvere theory of commutative monoids with G-action, the result follows almost immediately, again from the universal property of that Lawvere theory.

2 Recollections on Lawvere theories

For an ∞-categorical account of Lawvere theories, we refer the reader to [Cra10]. The result we are mainly interested in is proposition 3.24 from that source, which states that a Lawvere theory is equivalent to the opposite of the category of its finitely generated free models.

For the convenience of the reader, we produce proofs of the results we are interested in, in our context, although they are by no means original, and the rest of the document can be safely read without these proofs.

Lemma 2.1. *Let $\mathcal{L}(G)$ denote the full subcategory of $\mathrm{Fun}(BG, \mathrm{CMon})$ on the essential image of $\mathrm{Fin} \subset \mathbf{S}$ under the left adjoint to the forgetful functor $\mathrm{Fun}(BG, \mathrm{CMon}) \to \mathbf{S}$.*

Then for any category $\mathbf{C}$ with all finite products, there is an equivalence of categories

$$\mathrm{Fun}^\times(\mathcal{L}(G)^{op}, \mathbf{C}) \simeq \mathrm{Fun}(BG, \mathrm{CMon}(\mathbf{C}))$$

More precisely, there is an objet $\mathcal{U} \in \mathrm{Fun}(BG, \mathrm{CMon}(\mathcal{L}(G)^{op}))$ such that $F \mapsto F(\mathcal{U})$ induces the above equivalence.

Proof. Start with $\mathbf{C} = \mathbf{S}$: the left hand side is $\mathrm{Fun}^\times(\mathcal{L}(G)^{op}, \mathbf{S})$, also known as $P_\Sigma(\mathcal{L}(G))$, the nonabelian derived category of $\mathcal{L}(G)$. It is the free cocompletion of $\mathcal{L}(G)$ under sifted colimits.

So let $\mathcal{L}(G) \to \mathrm{Fun}(BG, \mathrm{CMon})$ be the canonical inclusion. By the universal property of P_Σ, this extends uniquely to a sifted colimit preserving functors $P_\Sigma(\mathcal{L}(G)) \to \mathrm{Fun}(BG, \mathrm{CMon})$.

$\mathcal{L}(G) \to \mathrm{Fun}(BG, \mathrm{CMon})$ is fully faithful, and for each object $x \in \mathcal{L}(G)$, $\mathrm{map}_{\mathrm{Fun}(BG,\mathrm{CMon})}(x, -)$ preserves sifted colimits, therefore $P_\Sigma(\mathcal{L}(G)) \to \mathrm{Fun}(BG, \mathrm{CMon})$ is also fully faithful.

To prove that it is an equivalence it therefore suffices to check that it is essentially surjective, i.e. that $\mathrm{Fun}(BG, \mathrm{CMon})$ is generated under sifted colimits by $\mathcal{L}(G)$. By the monadic bar construction, we see that it is generated under Δ^{op}-indexed (hence sifted) colimits by the essential image of the left adjoint to $\mathrm{Fun}(BG, \mathrm{CMon}) \to \mathbf{S}$; but $\mathbf{S}$ is clearly generated under sifted colimits by Fin, so the result follows.

Now, for $\mathbf{C}$ a presentable category, we get $\mathrm{Fun}^\times(\mathcal{L}(G)^{op}, \mathbf{S}) \otimes \mathbf{C} \simeq \mathrm{Fun}(BG, \mathrm{CMon}) \otimes \mathbf{C}$ and so (see for instance [GGN16, B.3]) we get an natural equivalence $\mathrm{Fun}^\times(\mathcal{L}(G)^{op}, \mathbf{C}) \to \mathrm{Fun}(BG, \mathrm{CMon}(\mathbf{C}))$.

Specializing to $\mathbf{C} = \mathrm{Psh}(\mathbf{D})$ for a category $\mathbf{D}$ shows that this works for an arbitrary $\mathbf{D}$. Since the equivalence is natural in $\mathbf{D}$, the result about $\mathcal{U}$ follows. $\square$

This result is the reason why we can actually conclude pretty quickly from [Gla15, A.4]:

Corollary 2.2. *Assume $A^{eff}(Fr^G) \simeq \mathcal{L}(G)$. Then theorem 0.1 follows.*

Proof. Indeed, $A^{eff}(Fr^G)^{op} \simeq A^{eff}(Fr^G)$, so if we have this assumption, we get $\mathrm{Fun}^{\oplus}(A^{eff}(Fr^G), \mathbf{E}) \simeq \mathrm{Fun}^{\times}(A^{eff}(Fr^G)^{op}, \mathbf{E}) \simeq \mathrm{Fun}^{\times}(\mathcal{L}(G)^{op}, \mathbf{E}) \simeq \mathrm{Fun}(BG, \mathrm{CMon}(\mathbf{E}))$.

But now, $\mathbf{E}$ is semiadditive, so by [GGN16, 2.3], the forgetful functor $\mathrm{CMon}(\mathbf{E}) \to \mathbf{E}$ is an equivalence, so we get

$$\mathrm{Fun}^{\oplus}(A^{eff}(Fr^G), \mathbf{E}) \simeq \mathrm{Fun}(BG, \mathbf{E})$$

Tracing through the equivalences, one may check that this is actually coming from restriction along $BG \to A^{eff}(Fr^G)$. □

But actually, that is not all: using this result about Lawvere theories, one can actually improve the proof of [Gla15, A.4]. Indeed, the main difficulty in that proof is constructing the functor $A^{eff}(Fr^G) \to \mathcal{L}(G)$; then it is actually easy (modulo some details) to prove that it is an equivalence.

But now, we have an easy way of constructing a functor $\mathcal{L}(G)^{op} \to A^{eff}(Fr^G)$

3 Relating the effective Burnside category and a Lawvere theory

This section is devoted to proving [Gla15, A.4], that is:

Theorem 3.1. *There is an equivalence $A^{eff}(Fr^G) \simeq \mathcal{L}(G)$.*

Proof. As mentioned earlier, the main difficulty of the proof in the cited paper is the construction of the functor $A^{eff}(Fr^G) \to \mathcal{L}(G)$.

With our recollections on Lawvere theories though, we can now construct a functor (in the opposite direction) much more easily.

Indeed, to define a functor $\mathcal{L}(G)^{op} \to A^{eff}(Fr^G)$, it now suffices to construct an object in $\mathrm{Fun}(BG, \mathrm{CMon}(A^{eff}(Fr^G))$. But $A^{eff}(Fr^G)$ is semiadditive (this is [Bar17, 4.3]), so this is equivalent to $\mathrm{Fun}(BG, A^{eff}(Fr^G))$.

Also recall from [Bar17] that we have a canonical functor $Fr^G \to A^{eff}(Fr^G)$, so we are now reduced to constructing an element in $\mathrm{Fun}(BG, Fr^G)$. But now these are nerves of 1-categories, so we can just do this "by hand": note that the free G-set G has an action of G given by $(g,h) \mapsto hg^{-1}$ (the $^{-1}$ being there only to ensure that we get a left action, but this is far from essential).

We thus get an element of $\mathrm{Fun}(BG, \mathrm{CMon}(A^{eff}(Fr^G)))$ informally given by G, with its commutative monoid structure coming from semiadditivity, and its G-action coming from the following spans:

$$\begin{tikzcd} & G \arrow[dl, "="'] \arrow[dr, "r_{g^{-1}}"] & \\ G & & G \end{tikzcd}$$

(where r_h denotes right multiplication by h)

This corresponds to a product-preserving functor $\mathcal{L}(G)^{op} \to A^{eff}(Fr^G)$, sending the free commutative monoid with G-action on 1 to G. By product preservation, it follows immediately that this functor is essentially surjective, so to prove that it is an equivalence, we are reduced to showing its fully-faithfulness.

However, since it is product preserving and both categories are semiadditive, we are reduced to showing that

$$\mathrm{map}_{\mathcal{L}(G)}(Free_G(1), Free_G(1)) \to \mathrm{map}_{A^{eff}(Fr^G)}(G, G)$$

is an equivalence; where we let $Free_G(X)$ denote the free commutative monoid with G-action on a set X; and $Free(X)$ will be the free commutative monoid on that set.

Note that since this is a product-preserving functor of semiadditive categories, this map automatically lifts to a map of commutative monoids. The left hand side is clearly equivalent to $Free(G)$ as a commutative monoid, so we are reduced to showing the same of the right hand side, and to showing that generators are sent to generators.

Remark 3.2. The proof that $\mathrm{map}_{A^{eff}(Fr^G)}(G, G)$ is free as a commutative monoids on the spans indicated earlier is explicitly stated, but not proved in [Gla15].

Though it is is easily checked: by [Bar17], $\mathrm{map}_{A^{eff}(Fr^G)}(G, G)$ is equivalent, as a space, to $(Fr^G_{/\{G,G\}})^{\simeq}$, the core-groupoid of $Fr^G_{/\{G,G\}}$. Now this is actually a 1-category, so its core-groupoid is a 1-groupoid, which we can check to be equivalent to $(\mathrm{Fin}^{\simeq})^{\times G}$, as a monoidal category under disjoint unions (this is left as an exercise to the reader), whose nerve is precisely the free commutative monoid on G-generators, with the generators being exactly the spans indicated above.

Unwinding the definitions, we see that the free generators of $\mathrm{map}_{\mathcal{L}(G)}(Free_G(1), Free_G(1))$ are indeed sent to those spans, which concludes the proof. □

Remark 3.3. We actually needed this proof to conclude properly in corollary 2.2, because we needed that specific equivalence to check that the equivalence $\mathbf{Mack}(Fr^G, \mathbf{E}) \to \mathrm{Fun}(BG, \mathbf{E})$ was indeed given by restriction along $BG \to A^{eff}(Fr^G)$. But now this is quite clear by construction.

Remark 3.4. Theorem 0.1 is actually an important theorem if one tries to understand the difference between spectra with G-action and genuine G-spectra. Indeed, spectra with G-action are Mackey functors on Fr^G, while genuine G-spectra are Mackey functors on $G-\mathbf{set}$, the category of finite G-sets, which indicates that the latter contains more information; specifically information about subgroups of G.

References

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