p-adic stuff

Introduction

The first encounter with p-adic phenomena is often with $\mathbb{Z}_p$, the p-adic integers, or maybe $\mathbb{Q}_p$, the fraction field of the latter.

There are various ways of introducing them - perhaps the most popular one being through infinite series of the form $\sum_{i=0}^{\infty} a_i p^i, a_i \in \{0, ..., p-1\}$.

This point of view shows that p-adic phenomena can sometimes be viewed topologically in a sense - this tension between this topology and other important "topological dimensions" is at the heart of some very exciting new research.

We will however be content with studying the algebraic side of p-adic phenomena, only seeing this topological point of view as a tool at various points.

Studying p-adic stuff can lead one to a better understanding of usual stuff, via some decompositions (to study an object X, study its p-adic counterparts X_p, and its rational counterpart $X_{\mathbb{Q}}$, and you'll already know a whole lot) such as Sullivan's arithmetic square in topology (which reduces a lot of homotopy theory to p-adic homotopy theory and rational homotopy theory).

This document is an attempt by the author to understand the basics of p-adic phenomena, in various (but related) contexts: homological algebra, unstable homotopy theory, stable homotopy theory (or spectral algebra).

Contents

1 (part of) The underived story

We start by reviewing some of the classical story of p-adic phenomena in the classical (or "underived") setting. Fix a prime p throughout.

Let A be an abelian group. We can endow it with the so-called p-adic topology, where a subbasis of opens have the form $x + p^n A$ for some $n \in \mathbb{N}$ and $x \in A$.

In $\mathbb{Z}$ this is a topology that is closely related to the p-adic valuation : recall that $v_p(m)$ is by definition the highest k such that $p^k \mid m$ (with $v_p(0) := +\infty$). We can then define the p-adic norm $|m|_p := p^{-v_p(m)}$ (with $p^{-\infty} = 0$), and the p-adic distance as $d_p(m,n) = |m-n|_p$. This distance induces the p-adic topology (exercise).

If we Cauchy-complete $\mathbb{Z}$ with respect to this distance, we get a wild beast called $\mathbb{Z}_p$. Let's try to see what that looks like. Let $x \in \mathbb{Z}_p$.

Then it's the limit of a Cauchy sequence $(x_n)_n \in \mathbb{Z}^{\mathbb{N}}$ which means that for any k, for n, m large enough, $p^k \mid x_n - x_m$. So if we expand x_n in the p-decimal basis, we get $x_n = \sum_{i=0}^d a_i p^i, a_i \in \{0, ..., p-1\}$ and for n large enough, the $a_i, i \leq k$ don't depend on the specific n that we chose.

In particular, if we call a_i these "stable" values, then we can define a series $\sum_{i=0}^{\infty} a_i p^i$. From the definition, it's then easy to check that its partial sums (u_n) form a sequence such that $u_n - x_n \to 0$ in the p-adic norm, so that (u_n) is also a Cauchy sequence converging to x.

So we can represent any p-adic integer (this is what we call elements of $\mathbb{Z}_p$) as an infinite series of that form. It's then an easy exercise to show that any two such series define the same p-adic integer if and only if they have the same coefficients, *and* that any such list of coefficients defines a convergent series.

This gives us a topological understanding of $\mathbb{Z}_p$, which will be interesting later. Let's also give an algebraic interpretation of that construction.

Earlier when we had our Cauchy sequence (x_n), we saw that for n, m large enough, x_n and x_m were the same "up to a mod p^k ambiguity". This means that $x_n = x_m$ in $\mathbb{Z}/p^k$. If we moreover take a larger k' we can still find an appropriate rank from which x_j defines a fixed element of $\mathbb{Z}/p^{k'}$.

It's easy to check that this defines a map $\mathbb{Z}_p \to \varprojlim_k \mathbb{Z}/p^k$, and that this map is an isomorphism. This is a second, more algebraic description of $\mathbb{Z}_p$, this is the one we will use.

Definition 1. *Let A be an abelian group. Its p-completion is $A_p := \varprojlim_k A/p^k A$*

Remark 1. *Note that we have compatible canonical maps $A \to A/p^k A$ and therefore a map $A \to A_p$*

Definition 2. *An abelian group A is p-complete if the canonical map $A \to A_p$ is an isomorphism*

In particular, $\mathbb{Z}$ is not p-complete. As en exercise, prove that $\mathbb{Z}/p^k$ is p-complete, and that any product of p-complete abelian groups is p-complete.

Here's where the topological point of view comes in handy :

Proposition 1. *An abelian group A is p-complete if and only if it is complete for the p-adic topology, that is :*

1. *The p-adic topology is Hausdorff*
2. *Any sequence (a_n) such that for any k, for n, m large enough $a_n - a_m \in p^k A$ (a Cauchy sequence) converges*

Proof. The canonical map $A \to A_p$ is injective if and only if $\bigcap_n p^n A = 0$, if and only if the p-adic topology is Hausdorff.

As before, any Cauchy sequence in A defines a term in $\varprojlim_n A/p^n A = A_p$, and so surjectivity of the canonical map is equivalent to any Cauchy sequence converging.

Since a map of abelian groups is an isomorphism if and only if it's a bijection, we are done.

(Exercise : check the details !) $\square$

This immediately implies :

Corollary 1. *Let A be a p-complete abelian group. Then any closed subgroup (for the p-adic topology) is also p-complete.*

It will follow from this that any projective limit of p-complete abelian groups is p-complete (which is far from obvious from the description using the canonical map $A \to A_p$, as far as the author can tell).

Corollary 2. *Let (A_i) be an inverse system of p-complete abelian groups. Then $\varprojlim_i A_i$ is p-complete.*

Corollary 3. *For any abelian group A, A_p is p-complete.*

Proof. For any k, any p^k-torsion abelian group is p-complete (the canonical map $A \to A/p^n A$ is an isomorphism for $n \geq k$), hence $A/p^k A$ is p-complete for any k, hence by corollary 2, A_p is p-complete. $\square$

Proof of corollary 2. A product of p-complete abelian groups is p-complete, as projective limits commute with direct products, and $(\prod_i M_i)/p^k(\prod_i M_i) \cong \prod_i (M_i/p^k M_i)$.

Therefore it suffices to show that $\varprojlim_i A_i \subset \prod_i A_i$ is closed for the p-adic topology.

Assume (a_i) is not in $\varprojlim_i A_i$, this means there exist $i \leq j$ with $\phi_{i,j}(a_i) \neq a_j$ (where $\phi_{i,j}$ denotes the transition map of the inverse system).

As A_j is p-complete, hence Hausdorff for the p-adic topology, it follows that there is some k such that $\phi_{i,j}(a_i) + p^k A_j$ and $a_j + p^k A_j$ are disjoint. Now $(a_i) + p^k \prod_i A_i$ doesn't meet $\varprojlim_i A_i$, and is a neighbourhood of (a_i). $\square$

With either description of $\mathbb{Z}_p$, we easily get that it acquires a natural ring structure such that each of $\mathbb{Z} \to \mathbb{Z}_p$ and $\mathbb{Z}_p \to \mathbb{Z}/p^k$ are ring morphisms. If you don't know this stuff it's good to check that it's unique (in fact, the second requirement alone imposes uniqueness, and the first one follows).

A good thing about the algebraic description of p-completion is that one automatically gets a natural $\mathbb{Z}_p$-module structure on each p-complete abelian group.

Indeed, A/p^kA has a unique $\mathbb{Z}/p^k$-module structure, hence by the canonical ring morphism $\mathbb{Z}_p \to \mathbb{Z}/p^k$ it acquires a canonical $\mathbb{Z}_p$-module structure. These structures are obviously compatible with one another when k varies (if it's not obvious : exercise !), and hence provide the inverse limit $A_p = \varprojlim_k \mathbb{Z}/p^k$ with a canonical $\mathbb{Z}_p$-structure.

We have repeatedly used the word "canonical" thus far for this structure without giving it a proper meaning. In fact p-complete modules are especially nice because their $\mathbb{Z}_p$-module structure is unique (this is *not* true for general $\mathbb{Z}_p$-modules as we shall see later). Our goal is now to prove this statement. In fact it will follow from a stronger statement, which we now state.

Theorem 1. *Let $\mathbb{Z}_p - \widehat{\mathbf{Mod}}$ denote the full subcategory of $\mathbb{Z}_p$-modules on those modules that are p-complete as abelian groups. Then the forgetful functor $\mathbb{Z}_p - \widehat{\mathbf{Mod}} \to \mathbf{Ab}$ is fully faithful.*

In particular, if you have a p-complete abelian group A, it has at most one way of being a $\mathbb{Z}_p$-module.

The first step in this endeavour is to prove the intuitive fact that $\mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}/p^k$. This is easy to prove:

Lemma 1. *The canonical map $\mathbb{Z}_p \to \mathbb{Z}/p^k$ induces an isomorphism $\mathbb{Z}_p/p^k\mathbb{Z}_p \to \mathbb{Z}/p^k$*

Proof. The induced map is clearly surjective, so it suffices to prove injectivity. Assume $x = \sum_{i=0}^{\infty} a_i p^i$ is in the kernel.

Then by definition $(a_0, ..., a_{k-1}) = (0, ..., 0)$ (recall how we defined the map $\mathbb{Z}_p \to \mathbb{Z}/p^k$ in terms of infinite series), and so $x = p^k \sum_{i=k}^{\infty} a_i p^{i-k}$ so $x \in p^k\mathbb{Z}_p$, which proves the claim. □

With this at our disposal, we can get the following intermediate result:

Lemma 2. *Let A be ab arbitrary $\mathbb{Z}_p$-module. Then the canonical map $A \otimes_{\mathbb{Z}} \mathbb{Z}/p^k \to A \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^k$ is an isomorphism.*

Proof. Because $\mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}/p^k$, both tensor products are the cokernel of $A \xrightarrow{p^k} A$.

(Exercise : write this down carefully) □

We can now prove theorem 1:

Proof. Let A be a $\mathbb{Z}_p$-module, B a $\mathbb{Z}_p$-module which is p-complete as an abelian group.

Each $B \to B \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^k$ is a map of $\mathbb{Z}_p$-modules.

Let's now consider the map $B \otimes_{\mathbb{Z}} \mathbb{Z}/p^k \to B \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^k$. The abelian group $B \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$ has two natural $\mathbb{Z}_p$-module structures, but because the above map is a $\mathbb{Z}_p$-linear map for both, and an isomorphism for both, it follows that both $\mathbb{Z}_p$-module structures on $B \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$ are the same (one can also prove this by hand, somehow "less conceptually" – it's instructive to try and do so)

Therefore the map $B \to B \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$ is $\mathbb{Z}_p$-linear, and therefore so is $B \to \varprojlim_k B \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$. But this map is an isomorphism of abelian groups, therefore it's an isomorphism of $\mathbb{Z}_p$-modules.

Now

$$\begin{aligned}
\hom_{\mathbb{Z}_p}(A, B) &\cong \varprojlim_k \hom_{\mathbb{Z}_p}(A, B \otimes_{\mathbb{Z}} \mathbb{Z}/p^k) \\
&\cong \varprojlim_k \hom_{\mathbb{Z}/p^k}(A \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^k, B \otimes_{\mathbb{Z}} \mathbb{Z}/p^k) \\
&\cong \varprojlim_k \hom_{\mathbb{Z}/p^k}(A \otimes_{\mathbb{Z}} \mathbb{Z}/p^k, B \otimes_{\mathbb{Z}} \mathbb{Z}/p^k) \\
&\cong \varprojlim_k \hom_{\mathbb{Z}}(A, B \otimes_{\mathbb{Z}} \mathbb{Z}/p^k) \\
&\cong \hom_{\mathbb{Z}}(A, B)
\end{aligned}$$

where the first isomorphism follows from the above description of B, the second from extension of scalars along $\mathbb{Z}_p \to \mathbb{Z}/p^k$, the third from the previous lemma, the fourth from extension of scalars along $\mathbb{Z} \to \mathbb{Z}/p^k$, and the last one again from p-completeness of B.

One easily checks that the composite morphism is the forgetful morphism $\hom_{\mathbb{Z}_p} \to \hom_{\mathbb{Z}}$, which then proves the claim. □

One interesting class of examples consists of all finitely generated $\mathbb{Z}_p$-modules. Indeed one can prove that $\mathbb{Z}_p$ is a principal ideal domain (I leave it as an exercise – the important point is to notice is that $\mathbb{Z}_p$ is local, with maximal ideal $p\mathbb{Z}_p$), with all ideals of the form (p^k), so its finitely generated modules are of the form $\mathbb{Z}_p^r \oplus \bigoplus_i \mathbb{Z}/p^{n_i}$, and these are all p-complete as (finite) direct products of p-complete abelian groups.

It follows that for an abelian group, "is a finitely generated $\mathbb{Z}_p$-module" is a *property*, and not extra structure (analogously to how "being a rational vector space" is a property of an abelian group, and not extra structure).

Before we move on to the derived picture, I want to introduce another category of algebraic structures with a "p-adic feel" to them. I'm not exactly sure how they fit into the whole p-adic story, but they are certainly interesting and related to p-adic phenomena.

If you have an inverse system of finite abelian p-groups, you can take its inverse limit, and endow it with a topology: the limit topology with respect to the discrete topology on each term of your inverse system. This is what's usually called a pro-p-abelian group.

This definition is not very satisfying, because it's not clear where the topology comes from : why would we add the topology ? In fact there's another definition of pro-p-abelian groups, which works more generally: one may define pro-objects in an arbitrary (small) category. I will write it out as a defintion, but in fact one needs to prove that this exists (we will not do so here, and we won't prove the basic facts about it):

Definition 3. *Let C be a small category.* $\mathrm{Pro}(C)$ *is the category freely generated by C under inverse limits.*

More precisely, this means it's a category with all inverse limits, together with a functor $j : C \to \mathrm{Pro}(C)$ such that for any category D with all inverse limits, restriction along j induces an equivalence $Fun'(\mathrm{Proc}(C), D) \to Fun(C, D)$ where Fun' denotes the category of functors that preserve inverse limits.

Let's regroup the basic properties of this category in a proposition, which we won't prove:

Proposition 2.

1. *The functor $j : C \to \mathrm{Pro}(C)$ is fully faithful, and preserves the finite (co)limits that exist in C*
2. *For all $c \in C, j(c)$ is cocompact in* $\mathrm{Pro}(C)$*; explicitly this means that the canonical map $\varinjlim_i \hom_{\mathrm{Pro}(C)}(d_i, j(c)) \to \hom_{\mathrm{Pro}(C)}(\varprojlim_i d_i, j(c))$ is an isomorphism for any inverse system (d_i) in* $\mathrm{Pro}(C)$.
3. *As a consequence of 1., 2., hom-sets in* $\mathrm{Pro}(C)$ *are computed as follows :* $\hom_{\mathrm{Pro}(C)}(\varprojlim_i j(c_i), \varprojlim_k j(d_k)) \cong \varprojlim_k \varinjlim_i \hom_C(c_i, d_k)$
4. *A possible construction of* $\mathrm{Pro}(C)$ *is as the opposite category of the smallest subcategory of $Fun(C, \mathrm{Set})$ containing the essential image of the Yoneda embedding $C^{op} \to Fun(C, \mathrm{Set})$ and stable under filtered colimits.*

We will use these to prove that, in the case where C is the category of finite abelian p-groups, this construction coincides with the topological one. This is quite surprising, as it means that a formal, algebraic construction, is completely encoded in some topology. In fact, one can use them to prove the following criterion to recognize $\mathrm{Pro}(C)$:

Proposition 3. *Let C be a small category, D a category with inverse limits and $f : C \to D$ a functor. Then it induces an inverse limit-preserving functor* $\mathrm{Pro}(C) \to D$.

1. *This functor is fully faithful if and only if f is, and $f(c)$ is cocompact for any $c \in C$*
2. *It's an equivalence if and only if it's fully faithful and D is generated under inverse limits by the image of f*

Proof. Statement 2. follows from the fact that $\mathrm{Pro}(C)$ has all inverse limits and that a functor is an equivalence if and only if it's fully faithful and essentially surjective.

For statement 1., this follows from the formula we gave above for the homsets in $\mathrm{Pro}(C)$.

(Exercise: fill in the details) □

We can now use this to prove the desired statement about pro-p-abelian groups. Let Ab_p^f denote the category of finite abelian p-groups and morphisms between them, and let $\mathrm{pro}-p$ denote the category of topological abelian groups which are inverse limits of discrete finite abelian p-groups and continuous maps between them.

We have an obvious fully faithful embedding $\mathrm{Ab}_p^f \to \mathrm{pro}-p$.

Moreover, $\mathrm{pro}-p$ clearly has all inverse limits, so we get an inverse-limit preserving functor $\mathrm{Pro}(\mathrm{Ab}_p^f) \to \mathrm{pro}-p$. Given the above proposition,the definition of $\mathrm{pro}-p$, and the fact that our initial functor was fully faithful, it suffices to check that a finite abelian p-group is cocompact in $\mathrm{pro}-p$ to prove that our functor establishes an equivalence $\mathrm{Pro}(\mathrm{Ab}_p^f) \simeq \mathrm{pro}-p$.

Proposition 4. *Any finite abelian p-group is cocompact in* $\mathrm{pro}-p$.

Lemma 3. *A pro-p-abelian group G (defined topologically) is the inverse limit of G/N, where N runs over all finite index normal open subgroups of G.*

Proof. Write $G \cong \varprojlim_i G_i$ where each G_i is a finite abelian p-group. Let $N_i := \ker(G \to G_i)$, and consider the corresponding inverse system (G/N_i). We clearly have a map $(G/N_i \to G_i)$ of inverse systems which induces continuous group morphisms $G \to \varprojlim_i G/N_i \to \varprojlim_i G_i$.

The composite map $G \to \varprojlim_i G_i$ is an isomorphism, therefore $\varprojlim_i G/N_i \to \varprojlim_i G_i$ is surjective.

It's also injective because each $G/N_i \to G_i$ is, therefore it's a continuous bijection. But $\varprojlim_i G_i$ is Hausdorff, and $\varprojlim_i G/N_i$ is compact, hence the map is a homeomorphism. It follows that $G \cong \varprojlim_i G/N_i$ as topological groups.

In particular, $\bigcap_i N_i = \{0\}$. Now let N be any open normal subgroup of G (which automatically has finite index anyway, as G is compact). Since N is open, it's also closed, and hence it's compact.

In particular, all other cosets $gN \subset G$ are compact as well. Since $\bigcap_i N_i = \{0\}$, if $g \notin N$, $\bigcap_i (N_i \cap gN) = \emptyset$; hence by compactness there is i_g such that $N_i \cap gN = \emptyset$.

Since N has finite index, there are only finitely many cosets, and so there is some fixed i such that $N_i \cap gN = \emptyset$ for all $g \notin N$. In particular, $N_i \subset N$.

It follows that the map $I \to \{N \triangleleft G \mid N \text{ is open }\}, i \mapsto N_i$ is coinitial, and so the inverse limits $\varprojlim_i G/N_i$ and $\varprojlim_{N \triangleleft G, \mathrm{open}} G/N$ are canonically isomorphic, which proves the desired result. $\square$

Lemma 4. *Let (X_i) be an inverse system of finite nonempty sets. Then $\varprojlim_i X_i$ is nonempty.*

Proof. Because the X_i's are finite, $\prod_i X_i$ is compact if you endow it with the product topology (where all the X_i's are discrete), by Tychonoff's theorem.

For each $i \leq j$, let $F_{i,j} \subset \prod_k X_k$ denote $\{(x_i) \mid \phi_{i,j}(x_i) = x_j\}$ where, again, $\phi_{i,j}$ denotes the transition map.

Then $F_{i,j}$ is clearly closed in the product topology; and $\bigcap_{i \leq j} F_{i,j} \cong \varprojlim_i X_i$.

By compactness, it then suffices to check that any finite intersection of $F_{i,j}$'s is nonempty. But if we have a finite subsets of $F_{i,j}$'s, then we can find i_0 which is below every i,j appearing in this subset. Since $X_{i_0} \neq \emptyset$, we can take any $x \in X_{i_0}$ and apply the transition maps to it, then complete the other coordinates arbitrarily to get an element of this finite intersection. $\square$

Proof of proposition 4. We only prove the following (a priori weaker) statement, which is enough for our purposes (and in fact, the stronger statement then follows from it): let (G_i) be an inverse system of finite abelian p-groups, and A a finite abelian p-group. Then the canonical map $\varinjlim_i \hom(G_i, A) \to \hom_{\text{pro}-p}(G, A)$ is an isomorphism.

This is a priori weaker because we should be dealing with arbitrary inverse systems of pro-p-abelian groups, whereas we're assuming we have an inverse system of *finite* abelian p-groups. But in fact this is enough to recover the equivalence $\text{Pro}(\text{Ab}_p^f)$, as is clear from the proof.

• We begin by proving surjectivity of the canonical map. Let $f : G \to A$ be a continuous morphism. Since A is discrete, $\ker(f)$ is open and has finite index in G. Therefore, by lemma 3, it contains the kernel N_i of $G \to G_i$ for some i, so that f factors through $G/N_i \to A$.

Now of course G/N_i is only a subgroup of G_i, so we don't quite get that it factors through $G_i \to A$ yet. Let $X := G_i \setminus (G/N_i)$ (warning : this is $\setminus$, we're taking the set-theoretic complement), and for $j \leq i, X_j := \phi_{j,i}^{-1}(X)$.

The X_j clearly form an inverse system of sets, and its inverse limit is empty (an element of the inverse limit would in particular yield an element of G, which lands in X, which is of course impossible), so by lemma 4, there is j such that $X_j = \emptyset$. In particular, the transition map $G_j \to G_i$ factors through G/N_i, and so we get a map $G_j \to A$ which clearly factors $G \to A$, which proves the claim.

• We now prove injectivity. Since both sides are naturally groups, and the map is a group morphism, it suffices to examine the kernel. So let $f : G_k \to A$ such that the composite $G \to G_k \to A$ is zero. Let $X := G_k \setminus \ker(f)$ (same warning as above), and for $j \leq k$, let $X_j := \phi_{j,k}^{-1}(X)$.

As above, the X_j form an inverse system of finite sets, whose inverse limit is empty; therefore, again by lemma 4, one of the X_j must be empty. It follows that $G_j \to G_k$ factors through $\ker(f)$, i.e. $G_j \to G_k \to A$ is zero. This proves the injectivity claim, and thus completes the proof. $\square$

Corollary 4. *The canonical functor* $\text{Pro}(\text{Ab}_p^f) \to \text{pro} - p$ *is an equivalence of categories.*

We will now relate pro-p-abelian groups to $\mathbb{Z}_p$-modules and p-complete abelian groups.

Indeed, there is a fully-faithful inclusion $\text{Ab}_p^f \to \mathbb{Z}_p - \mathbf{Mod}$ which therefore extends uniquely to an inverse-limit-preserving functor $\text{Pro}(\text{Ab}_p^f) \to \mathbb{Z}_p - \mathbf{Mod}$. In fact, it lands in $\mathbb{Z}_p - \widehat{\mathbf{Mod}}$, because each finite abelian p-group is p-complete, and by corollary 2, the result then follows.

Corollary 5. *There is a unique inverse-limit-preserving functor* $\mathrm{Pro}(\mathrm{Ab}_p^f) \to \mathbb{Z}_p - \widehat{\mathbf{Mod}}$ *extending the fully faithful inclusion* $\mathrm{Ab}_p^f \to \mathbb{Z}_p - \widehat{\mathbf{Mod}}$

We can summarize this section by the following diagram:

$$\begin{array}{ccccccc} & & \mathrm{Ab}_p^f & \subset & \mathbb{Z}_p - \mathbf{Mod} & & \\ & \supset & \cap & \subset & & \downarrow & \\ \mathrm{Pro}(\mathrm{Ab}_p^f) & \longrightarrow & \mathbb{Z}_p - \widehat{\mathbf{Mod}} & \subset & & \mathrm{Ab} & \end{array}$$

where arrows just mean that we have a "forgetful" functor, and inclusions mean fully-faithful embeddings.

In my opinion, it's important to have a clear picture of this before embarking upon the derived story.

2 Derived p-completion of abelian groups and chain complexes

For this part of our story, it seems to me that the best way to express it is via ∞-categories, so that's what we'll use.

The previous section was mostly elementary (except at the end where we start using categorical language), this one won't be so much. A lot of what's written here can be re-expressed in more elementary terms (e.g. using the usual language of homological algebra), and in fact the statements often take a form analogous to usual category theory so the reader who doesn't know about ∞-categories shouldn't be too afraid.

We let $\mathbf{D}(\mathbb{Z})$ denote the derived ∞-category of abelian groups. This is an "∞-categorical enhancement" of the usual derived category, $D(\mathbb{Z})$. We recall that it is a symmetric monoidal ∞-category (its tensor product is essentially the derived tensor product of chain complexes of abelian groups, and we will now denote it by $\otimes_{\mathbb{Z}}$), and that it is stable and presentable.
In particular, its tensor product has a left adjoint : we get a bifunctor $Hom : \mathbf{D}(\mathbb{Z})^{op} \times \mathbf{D}(\mathbb{Z}) \to \mathbf{D}(\mathbb{Z})$ such that for any $C \in \mathbf{D}(\mathbb{Z}), - \otimes_{\mathbb{Z}} C \dashv Hom(C, -)$ (this corresponds essentially to the derived hom).
Moreover, the natural analog of p-completion in this setting is defined as $C_p^\wedge := \varprojlim_n C \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$.

Remark 2. *Here,* $\varprojlim$ *is to be taken in the sense of* ∞*-categories, so it is already derived.*
Similarly, $\otimes_{\mathbb{Z}}$ *denotes the* ∞*-categorical tensor product, so it is already derived. Anything else wouldn't be homotopically relevant (for instance it wouldn't be invariant under equivalence), so wouldn't make sense.*

Remark 3. *It is interesting to note that in the* ∞*-categorical setting, this construction seems natural; whereas if you're thinking about it from a model categorical point of view, it's not clear what this is supposed to be doing : you're* *first deriving $\otimes_{\mathbb{Z}}$ on the left, then $\varprojlim$ on the right - it's not clear why such a construction (deriving two functors in two different directions) is meaningful. This remark is one of the reasons that prompted me to write this.*

Example 1. *If you take $C = \mathbb{Z}$ (concentrated in degree 0), since it is flat, and since the system of abelian groups $(\mathbb{Z}/p^n)$ has all transition maps surjective, this limit can be computed as a 1-categorical limit : $\mathbb{Z}_p^\wedge = \mathbb{Z}_p$, concentrated in degree 0.*

There is natural map $C \to C_p^\wedge$, just as in the 1-categorical case, and it is then natural to define :

Definition 4. *$C \in \mathbf{D}(\mathbb{Z})$ is derived p-complete if the natural map $C \to C_p^\wedge$ is an equivalence.*

Example 2. *$\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \simeq (\mathbb{Z}/p^n \xrightarrow{p} \mathbb{Z}/p^n)$, with the obvious transition maps (because $\mathbb{Z}/p \simeq (\mathbb{Z} \xrightarrow{p} \mathbb{Z})$ and the latter is a complex of flat modules); and therefore the limit can be computed as a 1-categorical limit : $(\mathbb{F}_p)_p^\wedge \simeq (\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p)$. Identifying $\mathbb{F}_p \simeq (\mathbb{Z} \xrightarrow{p} \mathbb{Z})$, one may then check that the natural map $\mathbb{F}_p \to (\mathbb{F}_p)_p^\wedge$ is the commutative square*

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "p"] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z}_p \arrow[r, "p"] & \mathbb{Z}_p \end{tikzcd}$$

which is clearly an equivalence. So $\mathbb{F}_p$ is derived p-complete. One may check in the same way that $\mathbb{Z}/p^n$ is derived p-complete for all n, and that $\mathbb{Z}_p$ is as well. It follows (from the classification of finitely generated $\mathbb{Z}_p$-modules) that any finitely generated $\mathbb{Z}_p$-module is derived p-complete.

Example 3. *Assume n is coprime to p. What is the derived p-completion of $\mathbb{Z}/n\mathbb{Z}$?*

Conclude that for finitely generated abelian groups, derived p-completion coincides with p-completion.

Now we would like to check that "derived p-completion" is actually a completion, so at the very least that $C_p^\wedge$ is derived p-complete.

Our current description of the derived p-completion is not good enough for that (tensor product doesn't commute with limits, so it's not clear how to compute $(C_p^\wedge)_p^\wedge$).

For that, we note that $\mathbb{Z}/p^n$ is the cofiber of $\mathbb{Z} \xrightarrow{p^n} \mathbb{Z}$, so $C \otimes \mathbb{Z}/p^n$ is the cofiber of $C \xrightarrow{p^n} C$.

In particular, up to a shift, it is the fiber of that same map. More precisely : $\Omega(C \otimes \mathbb{Z}/p^n) = \mathrm{fib}(C \xrightarrow{p^n} C)$; and this fiber is now the fiber of $Hom(\mathbb{Z}, C) \to Hom(\mathbb{Z}, C)$ induced by $\mathbb{Z} \xrightarrow{p^n} \mathbb{Z}$, so it's $Hom(\mathbb{Z}/p^n, C)$.

Therefore, if we take the limit: $\Omega(C_p^\wedge) \simeq \varprojlim Hom(\mathbb{Z}/p^n, C) \simeq Hom(\varinjlim_n \mathbb{Z}/p^n, C)$

Now, colimits along a filtered poset are already derived, so this colimit is just the 1-categorical colimit, i.e. $\mathbb{Z}/p^\infty$. Therefore, $C_p^\wedge \simeq \Sigma Hom(\mathbb{Z}/p^\infty, C)$. This is our new description.

Example 4. *This description allows us to compute $H_*(A_p^\wedge)$ when A is an ordinary abelian group (viewed as a complex concentrated in degree 0): H_0 is $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A)$ and H_1 is $\hom(\mathbb{Z}/p^\infty, A)$.*

We also need a new description of the canonical map:

Proposition 5. *Under the equivalence $C_p^\wedge \simeq \Sigma Hom(\mathbb{Z}/p^\infty, C)$, the canonical map $C \to C_p^\wedge$ corresponds to the map $Hom(\mathbb{Z}, C) \to \Sigma Hom(\mathbb{Z}/p^\infty, C)$ induced by the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}/p^\infty \to 0$*

Proof. The canonical map $C \to C \otimes \mathbb{Z}/p^n$ is the one in the cofiber sequence $C \xrightarrow{p^n} C \to C \otimes \mathbb{Z}/p^n$. So if we identify the cofiber with Σ of the fiber, the map $C \to \Sigma Hom(\mathbb{Z}/p^n, C)$ is naturally induced by the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^n \to 0$.
Taking the colimit of that sequene readily yields the result. □

Remark 4. *This proposition allows one to identifiy the canonical map $A \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty)$ for A an ordinary abelian group as coming from the* Ext*-long exact sequence of the same short exact sequence.*
This gives a criterion for detecting when an ordinary abelian group A is derived p-complete; it suffices that $A \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A)$ is an isomorphism, and that $\hom(\mathbb{Z}/p^\infty, A) = 0$*. In fact, it's an easy exercise to prove that the latter is a consequence of the former, so : an ordinary abelian group A is derived p-complete if and only if the canonical map $A \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A)$ is an isomorphism.*
Using the same interpretation, we can see that it is equivalent to $\hom(\mathbb{Z}[\frac{1}{p}], A)$ *and* $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{p}], A)$ *vanishing, or, said differently, to $Hom(\mathbb{Z}[\frac{1}{p}], A)$ vanishing.*

We use the above remark to give more examples of derived p-complete abelian groups :

Proposition 6. *If M is an abelian group which is p-complete, then it is derived p-complete.*

Proof. For this proof, we let $\otimes^u_{\mathbb{Z}}$ denote the underived tensor product.

Let M be a p-complete abelian group. Since $M \otimes^u_{\mathbb{Z}} \mathbb{Z}/p^{n+1} \to M \otimes^u_{\mathbb{Z}} \mathbb{Z}/p^n$ is surjective for all n, the 1-categorical limit of this system is the derived limit, i.e. M, because M is p-complete by assumption.

It follows that $Hom(\mathbb{Z}[\frac{1}{p}], M) = \varprojlim Hom(\mathbb{Z}[\frac{1}{p}], M \otimes^u_{\mathbb{Z}} \mathbb{Z}/p^n)$. Now on $Hom(\mathbb{Z}[\frac{1}{p}], M \otimes^u_{\mathbb{Z}} \mathbb{Z}/p^n)$, multiplication by p^n is 0 (because it is so on $M \otimes^u_{\mathbb{Z}} \mathbb{Z}/p^n$) and is an equivalence (because it is so on $\mathbb{Z}[\frac{1}{p}]$), so $Hom(\mathbb{Z}[\frac{1}{p}], M \otimes^u_{\mathbb{Z}} \mathbb{Z}/p^n) = 0$.
It follows that $Hom(\mathbb{Z}[\frac{1}{p}], M) = 0$ (note : this *Hom is* derived !). By the

criterion explained in the previous remark, it follows that M is derived p-complete. $\square$

So derived p-complete abelian groups are more general than p-complete ones. We will need this extra generality if we want to understand p-completion of (simple) spaces with no additional hypothesis on their homotopy groups.

Next, we can prove that the two natural maps $C_p^\wedge \to (C_p^\wedge)_p^\wedge$ are equivalences:

Lemma 5. *For any C, $C_p^\wedge$ is derived p-complete*

Proof. By the previous proposition, we have an exact triangle $Hom(\mathbb{Z}[\frac{1}{p}], C_p^\wedge) \to C_p^\wedge \to (C_p^\wedge)_p^\wedge$, where we want to prove that the second map is an equicalence. So it boils down to proving that $Hom(\mathbb{Z}[\frac{1}{p}], C_p^\wedge) = 0$. Since $Hom(\mathbb{Z}[\frac{1}{p}], -)$ is a right adjoint, it preserves limits, hence fiber sequences, hence (since we are in a stable ∞-category) cofiber sequences, and hence it preserves Σ. Therefore, since $C_p^\wedge \simeq \Sigma Hom(\mathbb{Z}/p^\infty, C)$, it suffices to show that $Hom(\mathbb{Z}[\frac{1}{p}], Hom(\mathbb{Z}/p^\infty, C)) = 0$.

It's not hard to notice that the tensor-hom adjunction also works in the "enriched" setting, i.e. $Hom(A \otimes_{\mathbb{Z}} B, C) \simeq Hom(A, Hom(B, C))$, this is left as an exercise (hint : use the Yoneda lemma), so that we are reduced to showing $Hom(\mathbb{Z}[\frac{1}{p}] \otimes_{\mathbb{Z}} \mathbb{Z}/p^\infty, C) = 0$, and so to showing $\mathbb{Z}[\frac{1}{p}] \otimes_{\mathbb{Z}} \mathbb{Z}/p^\infty = 0$.
However, $\mathbb{Z}[\frac{1}{p}]$ is flat, so this is computed as a usual (underived) tensor product : in this case, this equality is trivial. So we are done. $\square$

Lemma 6. *For any C, the map $(C \to C_p^\wedge)_p^\wedge$ is an equivalence.*

Remark 5. *Note that this is a priori a different map from the previous map : in the previous lemma, we were considering the canonical map $id \to (-)_p^\wedge$ evaluated at $C_p^\wedge$, whereas here we're evaluating it at C, and then applying the functor $(-)_p^\wedge$ to it.*

Proof. Recall that the canonical map $C \to C_p^\wedge$ sits in an exact triangle $Hom(\mathbb{Z}[\frac{1}{p}], C) \to C \to C_p^\wedge$, so if we apply $\Sigma Hom(\mathbb{Z}/p^\infty, -)$ (which is equivalent to $(-)_p^\wedge)$) to it, we get that we want to prove $Hom(\mathbb{Z}[\frac{1}{p}], C)_p^\wedge = 0$.
One way of proving this is using the tensor-hom adjunction once again, and reducing to the same computation as above ($\mathbb{Z}/p^\infty \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] = 0$).
Another way of proving this is to note that p^n is an equivalence on $\mathbb{Z}[\frac{1}{p}]$, so it is one on $Hom(\mathbb{Z}[\frac{1}{p}], C)$, so all the cofibers $Hom(\mathbb{Z}[\frac{1}{p}], C) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$ are 0, and so the limit is as well ($\mathbb{N}$ is weakly-contractible as an ordered set).
In any case, the result holds and so we are done. $\square$

From the two previous lemma, we conclude :

Corollary 6. *The full subcategory of derived p-complete complexes in $\mathbf{D}(\mathbb{Z})$ is a localization of $\mathbf{D}(\mathbb{Z})$, and $(-)_p^\wedge$ is the localization functor.*

Remark 6. *This means that $(-)^\wedge_p$ is the left adjoint to the inclusion of the full subcategory of derived p-complete chain complexes.*

Remark 7. *It's quite amusing to note that, although the property of being derived p-complete should initially be checked on the specific canonical map $C \to C^\wedge_p$, the previous results actually imply that it is sufficient to produce any equivalence $C \simeq C^\wedge_p$: indeed $C^\wedge_p$ is derived p-complete, so if $C \simeq C^\wedge_p$, then C is as-well. In particular the existence of an equivalence implies that our specific map is an equivalence.*

We can actually go a bit further:

Corollary 7. *The full subcategory of derived p-complete complexes in $\mathbf{D}(\mathbb{Z})$ is an accessible localization of $\mathbf{D}(\mathbb{Z})$.*

Proof. Note that $\varprojlim : Fun(\mathbb{N}^{op}, \mathbf{D}(\mathbb{Z})) \to \mathbf{D}(\mathbb{Z})$ is a right adjoint from a presentable category to another, therefore it is accessible.

In particular, $C \mapsto C^\wedge_p$ is accessible too, as a functor $\mathbf{D}(\mathbb{Z}) \to \mathbf{D}(\mathbb{Z})$. This is enough to conclude (see e.g. 5.5.1.2. in *Higher Topos Theory*) $\square$

In particular, the full subcategory of derived p-complete complexes is presentable.

We can now try to examine the class of maps at which this is a localization. That is, what are the maps $C \to D$ that become equivalences after applying the functor $(-)^\wedge_p$?

General theory tells us that they are precisely the maps that induce an equivalence $\mathrm{map}(D, E^\wedge_p) \to \mathrm{map}(C, E^\wedge_p)$ for all E.

Proposition 7. *This is equivalent to $Hom(D, E^\wedge_p) \to Hom(C, E^\wedge_p)$ being an equivalence for all E*

Proof. Clearly, if it's an equivalence on Hom, then it's one on map too.

For the converse, using the Yoneda lemma, we are reduced to proving that if $C \to D$ becomes an equivalence under $(-)^\wedge_p$, then so does $K \otimes_\mathbb{Z} C \to K \otimes_\mathbb{Z} D$ for all K. This is clearly true for $K = \mathbb{Z}$, and since $\mathcal{D}(\mathbb{Z})$ is generated by $\mathbb{Z}$ under colimits and desuspensions (and that all of $- \otimes_\mathbb{Z} C, - \otimes_\mathbb{Z} D$ and $(-)^\wedge_p$ preserve these), this is true for any K. $\square$

Remark 8. *We used the fact that $(-)^\wedge_p$ preserves desuspensions. This is a priori not clear since it's a left adjoint. But it's easy enough (with our second description of derived p-completion) to see the full subcategory of derived p-complete chain complexes is a stable subcategory; therefore preserving suspensions implies preserving desuspensions as well.*

Now since $Hom(-, \Sigma -) = \Sigma Hom(-,-)$ for the same reasons as usual, it follows that our condition is equivalent to $Hom(D, Hom(\mathbb{Z}/p^\infty, E)) \to Hom(C, Hom(\mathbb{Z}/p^\infty, E))$ being an equivalence for all E, which is equivalent (by the tensor-hom adjunction and the Yoneda lemma) to $C \otimes_\mathbb{Z} \mathbb{Z}/p^\infty \to D \otimes_\mathbb{Z} \mathbb{Z}/p^\infty$ being an equivalence.

Corollary 8. *Derived p-completion is the localization at the class of maps $C \to D$ such that $C \otimes_{\mathbb{Z}} \mathbb{F}_p \to D \otimes_{\mathbb{Z}} \mathbb{F}_p$ is an equivalence.*

Proof. By the work done just above, it suffices to show that a map $C \to D$ becomes an equivalence after tensoring with $\mathbb{F}_p$ if and only if it becomes one after tensoring with $\mathbb{Z}/p^\infty$.

Suppose $C \otimes \mathbb{F}_p \to D \otimes \mathbb{F}_p$ is an equivalence. Then using the short exact sequences $0 \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n \to 0$ and induction on n, one proves that $C \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \to D \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$ is an equivalence for all n; and then taking the colimit produces the desired result.

Conversely, assume $C \otimes_{\mathbb{Z}} \mathbb{Z}/p^\infty \to D \otimes_{\mathbb{Z}} \mathbb{Z}/p^\infty$ is an equivalence. Note that the cofiber of $\mathbb{Z}/p^\infty \xrightarrow{p} \mathbb{Z}/p^\infty$ is $\Sigma\mathbb{F}_p$, so we get a cofiber sequence $C \otimes_{\mathbb{Z}} \mathbb{Z}/p^\infty \to C \otimes_{\mathbb{Z}} \mathbb{Z}/p^\infty \to C \otimes_{\mathbb{Z}} \Sigma\mathbb{F}_p$ and simiarly with D. Comparing them yields the result. □

Before moving on to the topological side of things (at least, a first approximation to it), let's make a comment about finiteness. Derived p-complete complexes are very well controlled by their reduction mod p (this is in a sense what the definition tells you), and we want to prove a finiteness criterion that exemplifies that. First, we have to relate derived p-complete complexes to complexes of $\mathbb{Z}_p$-modules.

Indeed, we can play the same game over $\mathbb{Z}_p$, rather than over $\mathbb{Z}$. But since $\mathbb{Z}/p^k$ is the cofiber in $\mathbf{D}(\mathbb{Z}_p)$ of $\mathbb{Z}_p \xrightarrow{p^k} \mathbb{Z}_p$, we get similarly that $A \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^k \simeq A \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$, even with the derived tensor product; and in particular, the "derived p-complete objects" of $\mathbf{D}(\mathbb{Z}_p)$ will be derived p-complete as complexes of abelian groups.

Notation 1. *We let $\mathbf{D}(\mathbb{Z})^\wedge_p$ (resp. $\mathbf{D}(\mathbb{Z}_p)^\wedge_p$) denote the full subcategory of $\mathbf{D}(\mathbb{Z})$ on derived p-complete complexes (resp. derived p-complete complexes of $\mathbb{Z}_p$-modules)*

We therefore get a forgetful functor $\mathbf{D}(\mathbb{Z}_p)^\wedge_p \to \mathbf{D}(\mathbb{Z})^\wedge_p$.

Theorem 2. *The forgetful functor $\mathbf{D}(\mathbb{Z}_p)^\wedge_p \to \mathbf{D}(\mathbb{Z})^\wedge_p$ is an equivalence.*

Proof. The essential image of this functor is clearly closed under inverse limits, so we only need to show that it is fully faithful and that it contains $A \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$ for any $A \in \mathbf{D}(\mathbb{Z})$ to conclude that it is essentially surjective. It will then follow that it is an equivalence.

Since $A \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$ is canonically a $\mathbb{Z}/p^k$-module hence a $\mathbb{Z}_p$-module, it is clearly in the essential image of this functor, so we are reduced to showing fully-faithfulness.

Just as in the underived case, the main point is that the canonical map $A \to \varprojlim_n A \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$ can be promoted to a map of $\mathbb{Z}_p$-modules, where the right hand side has the $\mathbb{Z}_p$-module structure coming from $\mathbb{Z}/p^n$. But this again follows from the fact that $A \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \simeq A \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n$.

The details are left as an exercise. $\square$

So from now on we'll mainly focus on p-complete complexes viewed as $\mathbb{Z}_p$-modules.

Definition 5. *Let R be a ring, $\mathbf{D}(R)$ its derived ∞-category. The full subcategory* $\mathrm{Perf}(R)$ *of perfect complexes is the smallest stable full subcategory of* $\mathbf{D}(R)$ *containing R and stable under retracts.*

Remark 9. *This definition can be given for any ring spectrum, more generally; but we want the necessary background to increase with time, and for now we don't need the extra generality.*

A very important property is the following:

Proposition 8. $\mathrm{Perf}(R)$ *is exactly the full subcategory of* $\mathbf{D}(R)$ *on compact objects.*

Proof. We won't prove the whole result, as the converse (compact $\implies$ perfect) is mainly category-theoretic.

R is compact, and the subcategory of compact objects is stable and closed under retracts, therefore every perfect complex is compact.

For the converse, the main idea is that $\mathrm{Perf}(R)$ is idempotent complete, so it's the full subcategory of $\mathrm{Ind}(\mathrm{Perf}(R))$ (the ind-completion) on compact objects (this is a category-theoretic statement). It then suffices to prove that $\mathrm{Ind}(\mathrm{Perf}(R)) \simeq \mathbf{D}(R)$. To prove this, you extend the inclusion $\mathrm{Perf}(R) \to \mathbf{D}(R)$ to a filtered-colimit-preserving functor $\mathrm{Ind}(\mathrm{Perf}(R)) \to \mathbf{D}(R)$, which is actually colimit preserving; and fully faithful.

It then suffices to check that it is essentially surjective. There are various ways of proving this, and we won't do it here. One way to prove this is written up in *Higher Algebra*, proposition 7.2.4.2. in the current version. $\square$

Under nice conditions, we can test perfectness (and so, compactness) on homology. *HA* gives some very general conditions; but we're only interested in $\mathbb{Z}/p, \mathbb{Z}_p$ and $\mathbb{Z}$, which are all PIDs and thus make the property easier to prove.

Proposition 9. *Let R be a PID. Then $M \in \mathbf{D}(R)$ is perfect if and only if $H_*(M)$ is finitely generated, that is, each $H_k(M)$ is finitely generated over R and only finitely many of them are nonzero.*

Proof. Perfectness always implies the condition on homology if R is noetherian: indeed this condition is satisfied by R and is stable under shifts, retracts, finite colimits.

Remark 10. *If R is not noetherian, there might be a perfect R-module with infinitely generated homology.*

For the converse, we need the hypothesis on R. Note that $M \simeq \bigoplus_{n\in\mathbb{Z}} H_n(M)[n]$, where for an R-module A, $A[n]$ denotes the same R-module viewed as a complex concentrated in degree n.

This holds because R is a PID. Then, since only finitely many $H_n(M)$'s are nonzero, this is a finite sum and it follows that we only need to prove that if A is finitely generated, then $A[0]$ is perfect.

But if $0 \to R^n \to R^m \to A \to 0$ is a short exact sequence (which exists, as R is a PID and A is finitely generated), $A \simeq \mathrm{cofib}(R^n \to R^m)$, and clearly $R^n, R^m \in \mathrm{Perf}(R)$; which proves the result. □

All these results and the definition as well show that perfectness is a good analogue of finiteness. The "finiteness criterion" we want to prove is then the following:

Proposition 10. *Let $M \in \mathbf{D}(\mathbb{Z}_p)^\wedge_p$. Then M is perfect if and only if $M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$ is perfect as a $\mathbb{Z}_p$-module.*

Remark 11. *With the above interpretation, this tells us that the finiteness of a derived p-complete module can be tested modulo p.*

Remark 12. *As an exercise, you may try to prove the following underived analogue: let M be a p-complete abelian group. Then M is finitely generated as a $\mathbb{Z}_p$-module if and only if $M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$ is finitely generated $\mathbb{Z}/p$-vector space.*

Remark 13. *The forward direction is very easy based on the definition of perfect modules. We leave it as an exercise to the reader, and will only be concerned with proving the converse.*

Lemma 7. *M is derived p-complete if and only if for each k, $H_k(M)$ is derived p-complete.*

Proof. Again, $\mathbb{Z}_p$ is a PID so $M \simeq \prod_n H_n(M)[n]$. Since derived p-complete complexes are closed under limits, it follows that if each $H_n(M)$ is derived p-complete, so is M.

Conversely, this description implies that each $H_n(M)$ is a retract of M (up to a shift). Now you may prove as an exercise that derived p-complete complexes are closed under retracts. □

Lemma 8. *In proving proposition 10, we may assume M is concentrated in degree* 0

Proof. Assume we know the result for complexes concentrated in degree 0 (and hence in an arbitrary degree), and assume $M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$ is perfect.

Write $M \simeq \bigoplus_n H_n(M)[n]$. Then $M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p \simeq \bigoplus_n H_n(M)[n] \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$.

By lemma 7, each $H_n(M)[n]$ is derived p-complete, so that $H_n(M)[n] \xrightarrow{p} H_n(M)[n]$ is an equivalence if and only if $H_n(M)[n] = 0$, and so $H_n(M)[n] \otimes_{\mathbb{Z}_p} \mathbb{Z}/p = 0$ if and only if $H_n(M)[n] = 0$.

In particular $\bigoplus_n H_n(M)[n] \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$ can be perfect only if all but finitely many $H_n(M)$'s are 0.

Moreover, each $H_n(M) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$ is a retract of this big sum (up to a shift), and so if the big sum is perfect, this must be perfect too. Therefore, if we know the result for complexes concentrated in degree 0, we know that each $H_n(M)$ is perfect as a $\mathbb{Z}_p$-complex.

Hence, M is perfect, as a finite direct sum of perfect complexes.

$\square$

I owe the following proof to Achim Krause who answered a question on MathOverflow.

Lemma 9. *Let K be a derived p-complete complex. Then K is concentrated in nonnegative degrees if and only if $K \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$ is.*

Proof. If K is concentrated in nonnegative degrees, since $\mathbb{Z}/p$ is too, the result immediately follows. We now focus on the converse.

Consider the exact triangle $K \xrightarrow{p} K \to K \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$, and the associated long exact sequence: $\dots \to H_{-1}(K) \xrightarrow{p} H_{-1}(K) \to 0 \to H_{-2}(K) \to H_{-2}(K) \to 0$.

It follows that for $n < 0, p : H_n(K) \to H_n(K)$ is surjective. By lemma 7, $H_n(K)$ is derived p-complete, and so by remark 4, $\hom(\mathbb{Z}[\frac{1}{p}], H_n(K)) = 0$.

Therefore, $H_n(K) = 0, n < 0$, and the result follows. $\square$

Lemma 10. *Let M, N be derived p-complete $\mathbb{Z}_p$-modules, and $f : M \to N$ a map of $\mathbb{Z}_p$-modules.*

Then f is surjective if and only if $f \otimes^u_{\mathbb{Z}_p} \mathbb{Z}/p$ is surjective

Remark 14. *Here we let $\otimes^u$ denote the underived tensor product.*

Proof. The forward direction is obvious, so we only look at the converse.

We look at f as a map in $\mathbf{D}(\mathbb{Z}_p)$. Let $K = \mathrm{fib}(f)$, so we have an exact triangle $K \to M \to N$.

Derived p-complete complexes are closed under finite limits, so K is derived p-complete.

Now $K \otimes_{\mathbb{Z}_p} \mathbb{Z}/p \to M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p \to N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$ is an exact triangle, and the associated long exact sequence looks like $\dots \to M \otimes^u_{\mathbb{Z}_p} \mathbb{Z}/p \to N \otimes^u_{\mathbb{Z}_p} \mathbb{Z}/p \to H_{-1}(K \otimes_{\mathbb{Z}_p} \mathbb{Z}/p) \to 0 \to \dots$

By hypothesis, it follows that $H_{-1}(K \otimes_{\mathbb{Z}_p} \mathbb{Z}/p) = 0$, and similarly for $H_k, k < 0$. Therefore $K \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$ is concentrated in nonnegative degrees, and by lemma 9, so is K.

The statement then follows from the long exact sequence of $K \to M \to N$. $\square$

Proof of proposition 10. Let M be a derived p-complete complex concentrated in degree 0, such that $M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p$ is perfect. In particular, $H_0(M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p) \cong M/pM$ is finitely generated.

Let $x_1, ..., x_n$ be a lift in M of a finite system of generators of M/pM, and $\mathbb{Z}_p^n \to M$ the associated $\mathbb{Z}_p$-linear map.

Then by construction, this becomes surjective after (underived) tensoring with $\mathbb{Z}/p$, so by lemma 10, it is surjective (both M and $\mathbb{Z}_p^n$ are derived p-complete). Therefore M is finitely generated, hence perfect. $\square$

As an application, we propose the following exercise (which should also have an elementary proof, but the author doesn't know it; and actually thought about a lot of this stuff in an effort to understand that result) : let (X, Y) be a pair of spaces such that $H^*(X, Y; \mathbb{F}_p)$ is finitely generated in each degree and nonzero only in finitely many degrees. Then so is $H^*(X, Y; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

This (rather long) comment now having been made, we can move on to the topological story.

3 Topological p-completion: first attempt

In this section we will define p-completion of spaces, study its basic properties. This will be a first, somewhat "naive" attempt.

There is another version of that story, instead looking at some pro-objects, which at this point in time the author doesn't know yet, but will try to understand.

We let $\mathcal{S}$ denote the ∞-category of spaces.

Definition 6. *Let $f : X \to Y$ be a map in $\mathcal{S}$. It's an $\mathbb{F}_p$-equivalence if $H^*(f; \mathbb{F}_p)$ is an isomorphism.*

Definition 7. *A space Z is p-complete if for any $\mathbb{F}_p$-equivalence $f : X \to Y$, $\mathrm{map}(Y, Z) \to \mathrm{map}(X, Z)$ is an equivalence.*

Example 5. *For any n, $K(\mathbb{F}_p, n)$ is p-complete. Indeed, for a p-local equivalence $f : X \to Y$, $\pi_k \mathrm{map}(Y, K(\mathbb{F}_p, n)) \to \pi_k \mathrm{map}(X, K(\mathbb{F}_p, n))$ is identified with $H^{n-k}(Y; \mathbb{F}_p) \to H^{n-k}(X; \mathbb{F}_p)$, which is assumed to be an isomorphism.*

The same argument and an easy induction show that $K(\mathbb{Z}/p^n, k)$ is p-complete for any n and k.

Like in the "homological" case, we have the following easy lemma :

Lemma 11. *p-complete spaces are stable under limits in $\mathcal{S}$.*

Lemma 12. *For Z to be p-complete, it suffices that for any $\mathbb{F}_p$-equivalence $f : X \to Y$, $\pi_0 \mathrm{map}(Y, Z) \to \pi_0 \mathrm{map}(X, Z)$ is an isomorphism*

Proof. Clearly if $X \to Y$ is an $\mathbb{F}_p$-equivalence, so is $X \times T \to Y \times T$ for any T. Thus, assuming the condition, we get that $\pi_0 \mathrm{map}(T \times Y, Z) \to \pi_0 \mathrm{map}(T \times X, Z)$ is an isomorphism, so $\pi_0 \mathrm{map}(T, \mathrm{map}(Y, Z)) \to \pi_0 \mathrm{map}(T, \mathrm{map}(X, Z))$ is an isomorphism, so by the Yoneda lemma, $\mathrm{map}(Y, Z) \to \mathrm{map}(X, Z)$ is an isomorphism in the homotopy category, so it's an equivalence. □

To make an interesting connection with $\mathbf{D}(\mathbb{Z})$ and use our previous work, let's now define more general Eilenberg-MacLane spaces, which will serve as our building blocks for p-complete spaces.

Definition 8. *For each $n \in \mathbb{Z}$, let $K(-, n)$ denote the functor $\mathbf{D}(\mathbb{Z}) \to \mathcal{S}$ defined by* $\mathrm{map}(\mathbb{Z}, \Sigma^n -)$

Remark 15. *Here $\Sigma : \mathbf{D}(\mathbb{Z}) \to \mathbf{D}(\mathbb{Z})$ is the suspension functor. In usual homological algebraic terms, it can be defined as the shift functor (so that $H_{*+1}(\Sigma C) \cong H_*(C)$)*

Proposition 11. *For any $A \in \mathbf{D}(\mathbb{Z}), n \geq 0, k \in \mathbb{Z}$, we have $\pi_n K(A,k) \cong H_{n-k}(A)$*

Proof. $\pi_n K(A,k) \cong \pi_0(\Omega^n \mathrm{map}(\mathbb{Z}, \Sigma^k A)) \cong \pi_0(\mathrm{map}(\mathbb{Z}, \Omega^n \Sigma^k A))$

But $\mathbf{D}(\mathbb{Z})$ is stable, so Ω and Σ are inverse to one another, so that this is $\pi_0 \mathrm{map}(\mathbb{Z}, \Omega^{n-k}(A)) \cong H_{n-k}(A)$ $\square$

Remark 16. *When A is concentrated in degree 0, we thus recover the usual Eilenberg-MacLane spaces, as $\pi_n K(A,k) \cong A$ when $n = k$, and 0 elsewhere.*

It is clear that this functor preserves arbitrary limits, and so $K(\mathbb{Z}_p, n) \simeq \varprojlim_k K(\mathbb{Z}/p^k\mathbb{Z}, n)$ is also p-complete. More generally, we have the following claim :

Proposition 12. *Let $A \in \mathbf{D}(\mathbb{Z})$ be derived p-complete. Then for all n, $K(A,n)$ is p-complete.*

Proof. A being derived p-complete means $A \simeq \varprojlim_k A \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$, so by preservation of limits, we are reduced to showing $K(A \otimes_{\mathbb{Z}} \mathbb{Z}/p^k, n)$ is p-complete.

Moreover, we have fiber sequences $A \otimes_{\mathbb{Z}} \mathbb{Z}/p^{k+1} \to A \otimes_{\mathbb{Z}} \mathbb{Z}/p \to \Sigma A \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$, which are preserved by $K(-,n)$; and clearly $K(\Sigma M, n) \simeq K(M, n+1)$.

Therefore, since p-complete spaces are stable under limits, we are reduced to showing that $K(A \otimes_{\mathbb{Z}} \mathbb{Z}/p, n)$ is p-complete.

Now note that $\mathrm{map}_{\mathcal{S}}(X, K(A,n)) \simeq map_{\mathbf{D}(\mathbb{Z})}(\mathrm{colim}_X \mathbb{Z}, \Sigma^n A) \simeq \mathrm{map}_{\mathbf{D}(\mathbb{Z})}(C_*(X), \Sigma^n A)$

Note that here we took a colimit indexed over a space X: this is possible thanks to the formalism of ∞-categories, where spaces can be seen as ∞-groupoids, hence special cases of ∞-categories. The fact that $\mathrm{colim}_X \mathbb{Z} \simeq C_*(X)$, the singular chain complex of X, is well-known, and comes from the fact that $C_*(-) : \mathcal{S} \to \mathbf{D}(\mathbb{Z})$ preserves colimits and is equivalent to $\mathbb{Z}$ when evaluated on $*$ (this is enough because $\mathcal{S}$ is freely generated under colimits by $*$)

Now if $X \to Y$ is an $\mathbb{F}_p$-equivalence, $\hom(C_*(Y), \mathbb{Z}/p) \to \hom(C_*(X), \mathbb{Z}/p)$ is an equivalence, so is $C_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p \to C_*(Y) \otimes_{\mathbb{Z}} \mathbb{Z}/p$.

Finally, since $\mathrm{map}(C_*(Y), \Sigma^n A \otimes_{\mathbb{Z}} \mathbb{Z}/p) \to \mathrm{map}(C_*(X), \Sigma^n A \otimes_{\mathbb{Z}} \mathbb{Z}/p)$ is identified with

$$\mathrm{map}_{\mathbf{D}(\mathbb{Z}/p)}(C_*(Y) \otimes_{\mathbb{Z}} \mathbb{Z}/p, \Sigma^n A \otimes_{\mathbb{Z}} \mathbb{Z}/p) \to \mathrm{map}_{\mathbf{D}(\mathbb{Z}/p)}(C_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p, \Sigma^n A \otimes_{\mathbb{Z}} \mathbb{Z}/p)$$

this means that the latter also is one, which is what we wanted. $\square$

Remark 17. *This shows that derived p-completion is more analogous to topological p-completion than usual, underived p-completion – the latter is too restrictive.*

Corollary 9. *Let X be a connected simple space (this means that $\pi_1(X)$ acts trivially on all homotopy groups, including itself – this implies in particular that $\pi_1(X)$ is abelian) with derived p-complete homotopy groups.*

Then X is p-complete.

Proof. As X is simple, its Postnikov tower is made up of principal fibrations, so we may work our way up the Postnikov tower by induction, using the fact that p-complete spaces are stable under limits (hence pullbacks), and that $K(\pi_n X, k)$ is p-complete. □

Example 6. *If X is simply-connected and its homotopy groups are finitely generated $\mathbb{Z}_p$-modules, then it's p-complete.*

Remark 18. *The section about underived p-adic stuff justifies the use of "are" in the previous example.*

The converse also holds for simple spaces, this is what we try to prove next, by first introducing the notion of p-completion.

The ideas here are again very much categorical in nature: we have a collection of spaces (p-complete spaces) that is closed under limits in $\mathcal{S}$, a reasonable question is to ask whether the inclusion of the subcategory has a left adjoint. It turns out that this is very much related to the presentability of the subcategory of p-complete spaces.

Notation 2. *We let $\mathcal{S}_p^\wedge$ denote the full subcategory of $\mathcal{S}$ on p-complete spaces.*

Proposition 13. *$\mathcal{S}_p^\wedge$ is an accessible localization of $\mathcal{S}$.*

In particular it is presentable, and the inclusion $\mathcal{S}_p^\wedge \to \mathcal{S}$ has a left adjoint $(-)_p^\wedge : \mathcal{S} \to \mathcal{S}_p^\wedge$

Proof. The proof is mainly categorical. It hinges on the fact that $X \to Y$ is an $\mathbb{F}_p$-equivalence if and only if $C_*(X;\mathbb{Z}/p) \to C_*(Y;\mathbb{Z}/p)$ is an equivalence in $\mathbf{D}(\mathbb{Z}/p)$, and the functor $C_*(-;\mathbb{Z}/p) : \mathcal{S} \to \mathbf{D}(\mathbb{Z}/p)$ preserves colimits.

The result then immediately follows from 5.5.4.16. and the beginning of section 5.5.4. in *Higher Topos Theory*, and is related to the theory of presentable ∞-categories rather than to our specific situation. □

Remark 19. *This is one very important place where working with ∞-categories is better than working with the homotopy categories: $Ho(\mathcal{S})$ is far from presentable, and so the existence of a left adjoint is not formal.*

Remark 20. *Here we used singular (co)homology with $\mathbb{F}_p$-coefficients, but thanks to spectra, this theorem would hold for an arbitrary homology theory.*

In particular, if we suppress the inclusion functor from the notation, we get a natural map $X \to X_p^\wedge$. Again, for general reasons, this map is an $\mathbb{F}_p$-equivalence. We call this map (or by abuse, the space $X_p^\wedge$) the p-completion of X.

It is in fact uniquely determined, up to equivalence, by the fact that it is an $\mathbb{F}_p$-equivalence from X to a p-complete space. So if we want to compute

p-completions, one way to do so is to exhibit an $\mathbb{F}_p$-equivalence $X \to Y$ with Y p-complete. We leave this as an exercise.

In general this is a complicated endeavour, but under nice assumptions on X (regarding its π_1), we can get some information.

Example 7. *Let Σ_3 denote the symmetric group on 3 letters, and let $B\Sigma_3$ be its classifying space.*

We have a fiber sequence $B\mathbb{Z}/3 \to B\Sigma_3 \to B\mathbb{Z}/2$. The Serre spectral sequence coupled with some group homology (specifically: $H_*(B\mathbb{Z}/3; \mathbb{F}_2) = 0$) shows that $B\Sigma_3 \to B\mathbb{Z}/2$ is an $\mathbb{F}_2$-equivalence.

Moreover, $B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2, 1)$ is 2-complete. It follows that $(B\Sigma_3)^\wedge_2 \simeq B\mathbb{Z}/2$.

Warning: this is not so easy for 3-completion! Indeed, in the Serre spectral sequence, $\pi_1(B\mathbb{Z}/2) \cong \mathbb{Z}/2$ acts nontrivially on $H_*(B\mathbb{Z}/3; \mathbb{F}_3)$ and so $B\mathbb{Z}/3 \to B\Sigma_3$ is not an $\mathbb{F}_3$-equivalence.

One may compute the $\mathbb{F}_3$-homology of $B\Sigma_3$ by what's known as Cartan's stable elements formula, which in general involves the combinatorics of Σ_3's 3-Sylow subgroups.

Example 8. *A finite product of $\mathbb{F}_p$-equivalences is again an $\mathbb{F}_p$-equivalence by the Künneth formula, therefore $(-)^\wedge_p$ commutes with finite products (note that it is a left adjoint so a priori it commutes with arbitrary colimits).*

We thus get an expression of, e.g. $(\mathbb{T}^2)^\wedge_p$ in terms of $(S^1)^\wedge_p$.

We now embark on describing the p-completion of a simply-connected space (note that essentially the same story can be told for simple, or even nilpotent spaces, but it requires a bit more work). We start by noting the following easy criterion:

Lemma 13. *Let $F \to X \xrightarrow{f} Y$ be a fiber sequence of spaces, where X, Y are simply-connected.*

Then f is an $\mathbb{F}_p$-equivalence if and only if $\pi_*(F)$ is a $\mathbb{Z}[\frac{1}{p}]$-module.

Proof. We use the Serre spectral sequence: under the given hypotheses, using the Künneth theorem, we have a spectral sequence with

$$E^2_{m,n} = H_m(Y; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_n(F; \mathbb{F}_p) \implies H_{m+n}(X; \mathbb{F}_p)$$

We use f to compare it to the spectral sequence of the trivial fibration $* \to X \to X$.

Assume that $\pi_*(F)$ is a $\mathbb{Z}[\frac{1}{p}]$-module. By the mod-$\mathcal{C}$ Hurewicz theorem, this implies that $H_*(F; \mathbb{Z})$ is as well (because F is simple – it follows e.g. from exercise 4.3.10 in Hatcher's *Algebraic Topology*), and therefore, by the universal coefficient theorem, $H_*(F; \mathbb{F}_p) = 0, * > 0$. It follows from the above spectral sequence and the comparison that $H_*(X; \mathbb{F}_p) \to H_*(Y; \mathbb{F}_p)$ is an isomorphism, which is what we wanted.

Conversely, assume that f is an $\mathbb{F}_p$-equivalence. Then again, by comparing the two spectral sequences, one can prove by induction on q that $H_n(F; \mathbb{F}_p) = 0$

for $q > 0$. The universal coefficient theorem thus implies that $H_*(F;\mathbb{Z})$ is uniquely p-divisible, i.e. a $\mathbb{Z}[\frac{1}{p}]$-module. We use the mod-$\mathcal{C}$ Hurewicz theorem again to conclude.

The details for this second spectral sequence argument are left as an instructive exercise. $\square$

Remark 21. *For this proof, it suffices that $\pi_1(Y)$ acts trivially on $H_*(F;\mathbb{F}_p)$ and that F is a connected simple space.*

For instance this would work for any fibre sequence of the form $K(A,n) \to K(B,n) \to K(C,n)$ induced by a fiber sequence $A \to B \to C$ in $\mathbf{D}(\mathbb{Z})$ (indeed this fiber sequence is just the looping of another fiber sequence, namely $K(A,n+1) \to K(B,n+1) \to K(C,n+1)$)

The reader can use this to improve on the statements that follow, and reach a description for simple spaces. For nilpotent spaces, more work is needed.

We use this lemma to describe the p-completion of a (generalized) Eilenberg-MacLane space:

Proposition 14. *Let $A \in \mathbf{D}(\mathbb{Z})$ be a complex concentrated in nonnegative degrees, and let $n \geq 2$.*

Then $K(A,n) \to K(A^\wedge_p, n)$ is a p-completion.

Remark 22. *The condition that $n \geq 2$ and that A be concentrated in nonnegative degrees is here to ensure that $K(A,n)$ is simply-connected, but as pointed out in remark 21, this is not actually necessary.*

Proof. By proposition 12, $K(A^\wedge_p, n)$ is p-complete, so we only need to show that the map is an $\mathbb{F}_p$-equivalence.

We have the following fiber sequence in $\mathbf{D}(\mathbb{Z})$:

$$Hom(\mathbb{Z}[\frac{1}{p}], A) \to A \to A^\wedge_p$$

and therefore we have the following fiber sequence in $\mathcal{S}$:

$$K(Hom(\mathbb{Z}[\frac{1}{p}], A), n) \to K(A,n) \to K(A^\wedge_p, n)$$

Both $K(A,n)$ and $K(A^\wedge_p, n)$ are simply-connected, so we may apply the previous lemma. But $\pi_*(K(Hom(\mathbb{Z}[\frac{1}{p}], A), n)) \cong H_{*-n}(Hom(\mathbb{Z}[\frac{1}{p}], A))$ is uniquely p-divisible. $\square$

This can be rewritten symbolically as $K(A,n)^\wedge_p \simeq K(A^\wedge_p, n)$.

We can now use this and Postnikov towers to describe the p-completion of a simply-connected space. Write $X \simeq \varprojlim_n X_n$, where X_n is the nth Postnikov stage of X.

X being simply-connected, we have, for each $n \geq 1$, a fiber sequence

$$X_{n+1} \to X_n \to K(\pi_{n+1}(X), n+2)$$

(in particular, $X_2 = K(\pi_2(X), 2)$)

To simplify the notations, we momentarily let $\pi_n := \pi_n(X)$.

By the above proposition, $(X_2)^\wedge_p = K((\pi_2)^\wedge_p, 2)$. We also have a map $X_2 \to K(\pi_3, 4)$, which we can p-complete to get $(X_2)^\wedge_p \to K((\pi_3)^\wedge_p, 4)$.

Let $\tilde{X}_3$ be the fiber of that map. Since p-complete spaces are stable under limits, $\tilde{X}_3$ is p-complete.

Moreover, we get a comparison of fiber sequences:

$$\begin{tikzcd}
X_3 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & K(\pi_3, 4) \arrow[d] \\
\tilde{X}_3 \arrow[r] & (X_2)^\wedge_p \arrow[r] & K((\pi_3)^\wedge_p, 4)
\end{tikzcd}$$

and it follows that the fiber of $X_3 \to \tilde{X}_3$ is the same as the fiber of $\mathrm{fib}(X_2 \to (X_2)^\wedge_p) \to \mathrm{fib}(K(\pi_3, 4) \to K(\pi_3, 4)^\wedge_p)$.

But each of those two has homotopy groups that are $\mathbb{Z}[\frac{1}{p}]$-modules, so by the 5-lemma, $\mathrm{fib}(X_3 \to \tilde{X}_3)$ also has such homotopy groups.

In particular, by the previous lemma, $X_3 \to \tilde{X}_3$ is an $\mathbb{F}_p$-equivalence. In other words, $\tilde{X}_3 = (X_3)^\wedge_p$.

We can then iterate: at each stage, define $\tilde{X}_{n+1}$ as the fiber of $(X_n \to K(\pi_{n+1}, n+2))^\wedge_p$ and prove that it is actually $(X_{n+1})^\wedge_p$.

We can then define $\tilde{X} := \varprojlim_n (X_n)^\wedge_p$. Note that although the inclusion $\mathcal{S}^\wedge_p \to \mathcal{S}$ preserves limits, $(-)^\wedge_p$ does not a priori, and so a priori it's not clear that this is $X^\wedge_p$.

It is however clear that $\tilde{X}$ is p-complete, and that the maps $X_n \to (X_n)^\wedge_p$ induce a map $X \to \tilde{X}$.

Limits commute with limits, so its fiber is $\varprojlim_n \mathrm{fib}(X_n \to (X_n)^\wedge_p)$. Letting fib_n denote $\mathrm{fib}(X_n \to (X_n)^\wedge_p)$, the above proof shows that we have fiber sequences

$$\mathrm{fib}_{n+1} \to \mathrm{fib}_n \to K(Hom(\mathbb{Z}[\frac{1}{p}], \pi_{n+1}), n+2)$$

It follows that $\mathrm{fib}_{n+1} \to \mathrm{fib}_n$ is n-connected, and so by the Milnor exact sequence (see the appendix), the homotopy groups of $\varprojlim_n \mathrm{fib}_n$ are just the limit of the homotopy groups of fib_n. But those are all $\mathbb{Z}[\frac{1}{p}]$-module, therefore, the homotopy groups of $\mathrm{fib}(X \to \tilde{X})$ are also $\mathbb{Z}[\frac{1}{p}]$-modules.

Therefore $\tilde{X} \simeq X^\wedge_p$. We have the following corollaries:

Corollary 10. *Let X be a simply-connected space.*
If (X_n) denotes the Postnikov tower of X, $X^\wedge_p \simeq \varprojlim_n (X_n)^\wedge_p$

Corollary 11. *Let X be a simply-connected space. We have the following short exact sequences for the homotopy groups of $X^\wedge_p$:*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{n+1}) \to \pi_{n+1}(X^\wedge_p) \to \hom(\mathbb{Z}/p^\infty, \pi_n) \to 0$$

Proof. We have fiber sequences

$$(X_{n+1})^\wedge_p \to (X_n)^\wedge_p \to K(\pi_{n+1}, n+2)^\wedge_p$$

$K(\pi_{n+1}, n+2)^\wedge_p$ only has homotopy in degrees $n+2, n+3$, so $(X_{n+1})^\wedge_p \to (X_n)^\wedge_p$ is $(n+1)$-connected, therefore $\pi_n((X_n)^\wedge_p) \cong \pi_n(X^\wedge_p)$ by the Milnor exact sequence and because of the previous corollary.

We can now use the long exact sequence of this fiber sequence:

$$\pi_{n+2}((X_n)^\wedge_p) \to H_0((\pi_{n+1})^\wedge_p) \to \pi_{n+1}((X_{n+1})^\wedge_p) \to \pi_{n+1}((X_n)^\wedge_p) \to 0$$

Note that by induction one can easily prove (from the same exact sequence and the same analysis of the homotopy groups of $K(\pi^\wedge_p, n)$) that $(X_n)^\wedge_p$ has no homotopy groups in degrees $\geq n+2$.

Example 4 and what came before now allow us to simplify the above exact sequence to:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{n+1}) \to \pi_{n+1}(X^\wedge_p) \to \pi_{n+1}((X_n)^\wedge_p) \to 0$$

.

We now use another bit of the long exact sequence of our fibration:

$$\pi_{n+3}((X_n)^\wedge_p) \to \pi_{n+3}(K(\pi_{n+1}, n+2)^\wedge_p) \to \pi_{n+2}((X_{n+1})^\wedge_p) \to \pi_{n+2}((X_n)^\wedge_p)$$

For the same reasons (again using example 4), this sequence becomes

$$0 \to \hom(\mathbb{Z}/p^\infty, \pi_{n+1}) \to \pi_{n+2}((X_{n+1})^\wedge_p) \to 0$$

and so we get: $\hom(\mathbb{Z}/p^\infty, \pi_{n+1}) \cong \pi_{n+2}((X_{n+1})^\wedge_p)$.

Using this for $n-1$, we may plug it into the previous short exact sequence to finally get the desired result:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{n+1}) \to \pi_{n+1}(X^\wedge_p) \to \hom(\mathbb{Z}/p^\infty, \pi_n) \to 0$$

□

Corollary 12. *Let X be a simply-connected space with finitely generated homotopy groups. Then the map $X \to X^\wedge_p$ induces an isomorphism $\pi_*(X^\wedge_p) \cong \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.*

Remark 23. *The short exact sequence in the previous statement is analogous to the above statement, indeed that short exact sequence is the closest thing we can hope for in the direction of "the homotopy groups of the p-completion are the derived p-completion of the homotopy groups".*

Proof. Everything we did so far shows that it suffices to prove the statement for X_n, the nth stage of the Postnikov tower.

But this follows from the case of Eilenberg-MacLane spaces, which follows from the fact that if A is a finitely generated abelian group, $A^\wedge_p \cong A_p \cong A \otimes_{\mathbb{Z}} \mathbb{Z}_p$. □

Corollary 13. *Let X be a simply-connected space. If X is p-complete, then its homotopy groups are derived p-complete.*

Remark 24. *Note that the converse was proved earlier so this is really an "if and only if" statement.*

Proof. We propose two proofs, one closer to the space level, and one using the previous short exact sequence.

1. We have the following commutative square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_p^\wedge \arrow[d] \\ X_n \arrow[r] & (X_n)_p^\wedge \end{tikzcd}$$

We proved earlier that the right vertical map is an isomorphism on $\pi_k, k \leq n$; the left vertical map is one too by definition. Therefore, if $X \to X_p^\wedge$ is an equivalence, $X_n \to (X_n)_p^\wedge$ is an isomorphism on $\pi_k, k \leq n$, and this, for all n.

Now, let us compare the two following fiber sequences, and their associated long exact sequences:

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & K(\pi_{n+1}, n+2) \arrow[d] \\ (X_{n+1})_p^\wedge \arrow[r] & (X_n)_p^\wedge \arrow[r] & K((\pi_{n+1})_p^\wedge, n+2) \end{tikzcd}$$

On homotopy groups, this yields the following exact sequences:

$$\begin{tikzcd} \pi_{n+2}K(\pi_{n+1}, n+2) \arrow[r] \arrow[d] & \pi_{n+1}(X_{n+1}) \arrow[r] \arrow[d] & \pi_{n+1}(X_n) \arrow[r] \arrow[d] & \pi_{n+1}(K(\pi_{n+1}, n+2)) \arrow[d] \\ \pi_{n+2}K((\pi_{n+1})_p^\wedge, n+2) \arrow[r] & \pi_{n+1}((X_{n+1})_p^\wedge) \arrow[r] & \pi_{n+1}((X_n)_p^\wedge) \arrow[r] & \pi_{n+1}K((\pi_{n+1})_p^\wedge, n+2) \end{tikzcd}$$

What we already know allows us to simplify this to:

$$\begin{tikzcd} \pi_{n+1}(X) \arrow[r] \arrow[d] & \pi_{n+1}(X_{n+1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ H_0(\pi_{n+1}(X)_p^\wedge) \arrow[r] & \pi_{n+1}((X_{n+1})_p^\wedge) \arrow[r] & \pi_{n+1}((X_n)_p^\wedge) \arrow[r] & 0 \end{tikzcd}$$

The middle vertical map is an isomorphism, therefore the composite $\pi_{n+1}(X_{n+1}) \to 0 \to \pi_{n+1}((X_n)_p^\wedge)$ must be surjective, so that $\pi_{n+1}((X_n)_p^\wedge) = 0$.

Since $\pi_{n+1}(X) \to \pi_{n+1}(X_{n+1})$ is an isomorphism, and since we could have added 0's on the left hand side (can the reader see why ?), it follows that $\pi_{n+1}(X) \to H_0(\pi_{n+1}(X)_p^\wedge)$ is an isomorphism.

It follows from remark 4 that $\pi_{n+1}(X)$ is derived p-complete.

2. We use the short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{n+1}(X)) \to \pi_{n+1}(X^\wedge_p) \to \hom(\mathbb{Z}/p^\infty, \pi_n(X)) \to 0$$

and induction.

Indeed, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{n+1}) = H_0(\pi_{n+1}(X)^\wedge_p)$ and so, by lemma 7 it is derived p-complete. Therefore to prove the result in degree $n+1$ it suffices to show that $\hom(\mathbb{Z}/p^\infty, \pi_n(X)) = 0$.

But this is one degree lower, so by induction we may assume that $\pi_n(X)$ is derived p-complete, and then this equality is automatic (see remark 4), which means we are done. $\square$

Remark 25. *As we said before, we focused on simply-connected spaces because it makes some of the arguments easier, but essentially the same story works for simple spaces; and with more work, it can be adapted to nilpotent spaces (for those, one must first explain what "derived p-complete" could mean for a nonabelian π_1)*

We now move on to our first attempt at the stable picture. The reader who is interested in topological p-completion from the point of view of pro-objects but does not know about spectra (or doesn't wish to read about them) is invited to scroll further and look for the appropriate section.

We placed the sections in this order, because the first attempt at describing the stable story is extremely analogous to the unstable story, and we feel it would be a shame to move it too far away.

In some respects, the stable story is significantly easier than the unstable one (once we move past the apparent difficulty of spectra), in part because there is no longer a question of whether something is simple or simply-connected or anything of that sort; but also because fiber sequences yield long exact sequences, rather than spectral sequences, and of course those are much easier to understand.

The main difficulty lies in the presence of 2 possible definitions, even in this first "naive" attempt. Our main goal is to relate them in the case of connective spectra, and to relate that story to derived p-completion in $\mathbf{D}(\mathbb{Z})$ (via the homotopy groups, just as in the unstable picture)

4 Stable p-completion: first attempt

In the land of spectra, there are two natural notions that would deserve to be called "p-adic equivalence". Indeed, we still have $\mathbb{Z}/p$, seen as an Eilenberg-MacLane spectrum, but there is a new player too: $\mathbb{Z}/p$ is the cofiber of $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$, and this is reasonable in homological algebra, as $\mathbb{Z}$ is the ground ring.

In spectra however, $\mathbb{S}$, the sphere spectrum, is the ground ring. It, too, has a "multiplication by p" map, and so we may consider $\mathbb{S}/p := \mathrm{cofib}(\mathbb{S} \to \mathbb{S})$.

The two notions of p-equivalence we get, and the two notions of p-completion we get are related, but not equivalent.

Definition 9. *A map $X \to Y$ of spectra is called a p-adic equivalence if $X \otimes \mathbb{S}/p \to Y \otimes \mathbb{S}/p$ is an equivalence.*

It's called an $\mathbb{F}_p$-equivalence if $X \otimes \mathbb{F}_p \to Y \otimes \mathbb{F}_p$ is an equivalence.

Remark 26. *Here $\otimes$ with no index denotes the smash-product of spectra. It can be extended to a symmetric monoidal structure on* $\mathbf{Sp}$*, the ∞-category of spectra.*

One of the added values of ∞-category theory is that we can have a good notion of smash-product of spectra, whereas there are some impossibility theorems for 1*-categorical models.*

Note that $\mathbb{Z} \otimes \mathbb{S}/p \simeq \mathbb{F}_p$. It follows that a p-adic equivalence is in particular an $\mathbb{F}_p$-equivalence.

Definition 10. *A spectrum E is called p-complete if for any p-adic equivalence $X \to Y$, the map* $\mathrm{map}_{\mathbf{Sp}}(Y, E) \to \mathrm{map}_{\mathbf{Sp}}(X, E)$ *is an equivalence.*

It is called $\mathbb{F}_p$-complete if the analogous statement holds for $\mathbb{F}_p$-equivalences.

As a p-adic equivalence is an $\mathbb{F}_p$-equivalence, any $\mathbb{F}_p$-complete spectrum is p-complete.

Notation 3. *We let $\mathbf{Sp}_p^\wedge$ denote the full subcategory of* $\mathbf{Sp}$ *on p-complete spectra, and $\mathbf{Sp}_{\mathbb{F}_p}$ the one on $\mathbb{F}_p$-complete spectra.*

The same proof as that of proposition 13 establishes:

Proposition 15. *The inclusion $\mathbf{Sp}_p^\wedge \to \mathbf{Sp}$ (resp. $\mathbf{Sp}_{\mathbb{F}_p} \to \mathbf{Sp}$) has a left adjoint $(-)_p^\wedge$ (resp. $(-)_{\mathbb{F}_p}$) which exhibits $\mathbf{Sp}_p^\wedge$ (resp. $\mathbf{Sp}_{\mathbb{F}_p}$) as an accessible localization of* $\mathbf{Sp}$.

In particular it is presentable.

Remark 27. *We could have replace $\mathbb{S}/p$ and $\mathbb{F}_p$ by any spectrum E, as $- \otimes E$ always preserves colimits.*

As in the case of spaces, we have a natural map $X \to X_p^\wedge$ (resp. $X \to X_{\mathbb{F}_p}$) which is a p-adic equivalence (resp. $\mathbb{F}_p$-equivalence), and where $X_p^\wedge$ is p-complete (resp. $\mathbb{F}_p$-complete).

In particular, $X_{\mathbb{F}_p}$ is p-complete so we get a natural factorization $X \to X_p^\wedge \to X_{\mathbb{F}_p}$.

A reasonable question is:

Question 1. *When is $X_p^\wedge \to X_{\mathbb{F}_p}$ an equivalence ?*

This question is equivalent to: when is $X_p^\wedge$ an $\mathbb{F}_p$-complete spectrum ?

We will see that it's the case whenever X is bounded below.

Notation 4. *For spectra X, Y, we let* $\mathrm{Map}(X,Y)$ *denote the mapping spectrum.*

$\mathrm{Map}(X,-)$ *is right adjoint to* $- \otimes X$

Recall that $\mathrm{map}(X,Y) \simeq \Omega^\infty \mathrm{Map}(X,Y)$.

Lemma 14. *The following are equivalent:*

1. *The spectrum E is p-complete (resp. $\mathbb{F}_p$-complete)*

2. *For any p-adic equivalence (resp. $\mathbb{F}_p$-equivalence) $X \to Y$, $\pi_0\mathrm{map}(Y,E) \to \pi_0\mathrm{map}(X,E)$ is an isomorphism.*

3. *For any p-adic equivalence (resp. $\mathbb{F}_p$-equivalence) $X \to Y$, $\mathrm{Map}(Y,E) \to \mathrm{Map}(X,E)$ is an isomorphism.*

Remark 28. *A priori, the condition with $\pi_0\mathrm{map}$ is weaker than the definition, which is weaker than the condition with* Map*.*

The point is then that we can prove the weakest version and get the strongest version for free.

Proof. We do the proof for p-adic equivalences, but of course it works for $\mathbb{F}_p$-equivalences the same way.

1. obviously implies 2.

Assume 2. Let F be a spectrum, then for any p-adic equivalence $X \to Y$, $X \otimes F \to Y \otimes F$ is also a p-adic equivalence, and so $\pi_0\mathrm{map}(Y \otimes F, E) \to \pi_0\mathrm{map}(X \otimes F, E)$ is an isomorphism.

But this is isomorphic to $\pi_0\mathrm{map}(F, \mathrm{Map}(Y,E)) \to \pi_0\mathrm{map}(F, \mathrm{Map}(X,E))$. By the Yoneda lemma applied in $Ho(\mathbf{Sp})$, it follows that $\mathrm{Map}(Y,E) \to \mathrm{Map}(X,E)$ is an equivalence, this proves 3.

3. obviously implies 1. so we are done. $\square$

Remark 29. *It follows, by looking at the cofiber, that E is p-complete (resp. $\mathbb{F}_p$-complete) if and only if whenever $X \otimes \mathbb{S}/p \simeq 0$ (resp. $X \otimes \mathbb{F}_p \simeq 0$), $\mathrm{Map}(X,E) \simeq 0$, if and only if for such X, $\mathrm{map}_{\mathbf{Sp}}(X,E) \simeq *$.*

Let $A \in \mathbf{D}(\mathbb{Z})$ be a chain complex. We view it as an Eilenberg-MacLane spectrum. Note that for any spectrum E, $\mathrm{map}_{\mathbf{Sp}}(E,A) \simeq \mathrm{map}_{\mathbf{D}(\mathbb{Z})}(\mathbb{Z} \otimes E, A)$.

Moreover, $(E \otimes \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/p \simeq E \otimes \mathbb{Z}/p$, so $X \otimes \mathbb{Z} \to Y \otimes \mathbb{Z}$ is a p-equivalence in $\mathbf{D}(\mathbb{Z})$ if and only if $X \to Y$ is an $\mathbb{F}_p$-equivalence in $\mathbf{Sp}$.

It follows that if A is derived p-complete, then as an Eilenberg-MacLane spectrum, it is $\mathbb{F}_p$-complete.

Corollary 14. *Let E be a bounded below spectrum with all homotopy groups derived p-complete abelian groups. Then E is $\mathbb{F}_p$-complete (and hence p-complete).*

Proof. We may write $E \simeq \varprojlim_n \tau_{\leq n} E$, where $\tau_{\leq n}$ is the truncation functor.

Since $\mathbb{F}_p$-complete spectra are stable under inverse limits, we may therefore assume that E is bounded above.

We can therefore proceed by induction on the number of nonzero homotopy groups of E. Let n be the lowest index where E has a nonzero homotopy group.

Then we have a map $E \to \tau_{\leq n} E \simeq \pi_n(E)$, its fiber satisfies the same hypotheses, and therefore we may assume by induction that it is $\mathbb{F}_p$-complete. By the above, $\pi_n(E)$ is also $\mathbb{F}_p$-complete, and so, writing $E \simeq \mathrm{fib}(\pi_n(E) \to \Sigma\mathrm{fib}(E \to \pi_n(E)))$, we get that E is also $\mathbb{F}_p$-complete. $\square$

Remark 30. *$\mathbb{F}_p$-complete (resp. p-complete) spectra are stable under Σ, one can see this for instance using the characterisation using* Map *in the previous lemma.*

Lemma 15. *Let $X \to Y$ be a map of spectra, and let F denote its fiber. It is a p-adic equivalence if and only if $\pi_*(F)$ is a $\mathbb{Z}[\frac{1}{p}]$-module.*

This holds for $\mathbb{F}_p$-equivalences if X, Y are bounded below.

Remark 31. *The reader will note the similarity with the lemmas proved earlier about topological spaces.*

The stable story so far is entirely parallel to the unstable one.

One can also note that this is a first important distinction between p-adic equivalences and $\mathbb{F}_p$-equivalences.

Proof. $F \to Y \to X$ is a cofiber sequence, therefore so is $\mathbb{S}/p \otimes F \to \mathbb{S}/p \otimes X \to \mathbb{S}/p \otimes Y$.

It follows that $\mathbb{S}/p \otimes X \to \mathbb{S}/p \otimes Y$ is an equivalence if and only if $\mathbb{S}/p \otimes F = 0$, and since we have a cofiber sequence $F \xrightarrow{p} F \to \mathbb{S}/p \otimes F$, it follows that this happens if and only if $F \xrightarrow{p} F$ is an equivalence.

The result now follows because π_* detects equivalences.

The case for $\mathbb{F}_p$-equivalences is slightly more subtle. Following the same reasoning, the map is an $\mathbb{F}_p$-equivalence if and only if $F \otimes \mathbb{Z}/p = 0$, that is, if and only if $(F \otimes \mathbb{S}/p) \otimes \mathbb{Z} = 0$.

But if X, Y are bounded below, so is F, and so is $F \otimes \mathbb{S}/p$.

The stable Hurewicz theorem says that if E is bounded below and $E \otimes \mathbb{Z} = 0$, then $E = 0$. The result thus follows. ☐

Corollary 15. *Let X, Y be bounded below spectra. A map $f : X \to Y$ is a p-adic equivalence if and only if it's an $\mathbb{F}_p$-equivalence.*

Proof. It follows from the previous lemma, and the fact that the condition is the same on F when X, Y are bounded below. ☐

Remark 32. *So p-adic equivalences and $\mathbb{F}_p$-equivalences are different only when we consider unbounded spectra.*

If one has a way of proving that $X_p^\wedge$ and $X_{\mathbb{F}_p}$ are bounded below whenever X is, then the above corollary would be enough to prove that when X is bounded below, the two completions agree (that's left as an exercise).

The author doesn't know at this point whether there's an easy proof of that boundedness property without an explicit construction. But the explicit construction is somewhat easier than in the unstable case, so this is the route we will follow.

Let X be a bounded below spectrum. Up to a shift, we may assume that X is connective. We write $X \simeq \varprojlim_n \tau_{\leq n} X$, and we'll work on the $X_n := \tau_{\leq n} X$ first.

Note that for $A \in \mathbf{D}(\mathbb{Z})$, $A \otimes \mathbb{S}/p \simeq A \otimes_{\mathbb{Z}} \mathbb{Z}/p$, so a map $A \to B \in \mathbf{D}(\mathbb{Z})$ which is a p-equivalence in $\mathbf{D}(\mathbb{Z})$ is a p-adic equivalence in $\mathbf{Sp}$. In particular:

Proposition 16. *Let $A \in \mathbf{D}(\mathbb{Z})$. Then, as an Eilenberg-MacLane spectrum, $A_p^\wedge \simeq A_{\mathbb{F}_p}$ and both are the same as the derived p-completion of A in $\mathbf{D}(\mathbb{Z})$.*

Proof. Let A^p momentarily denote the derived p-completion of A. Then we saw that A^p was $\mathbb{F}_p$-complete (and thus p-complete).

Moreover, $A \to A^p$ is a p-equivalence in $\mathbf{D}(\mathbb{Z})$, and therefore by the above a p-adic equivalence of spectra, and thus an $\mathbb{F}_p$-equivalence of spectra.

These are enough to conclude that $A \to A^p$ is both a p-completion and an $\mathbb{F}_p$-completion of A, which is what was claimed. □

Now both $(-)_p^\wedge$ and $(-)_{\mathbb{F}_p}$ are left adjoints from a stable category to another one, therefore they are exact, and thus preserve finite limits. Since X_n can be built from shifts of Eilenberg-MacLane spectra (concentrated in one degree) by taking successive fibers, it follows that we may build both its p-completions by replacing those Eilenberg-MacLane spectra by their p-completion (equivalently, their $\mathbb{F}_p$-completion). We get:

Corollary 16. *If X is bounded below, then for any n we have $(X_n)_p^\wedge \simeq (X_n)_{\mathbb{F}_p}$.*

Corollary 17. *If X is bounded below, then for any n, we get a fiber sequence*

$$(X_{n+1})_p^\wedge \to (X_n)_p^\wedge \to \Sigma^{n+2}\pi_{n+1}(X)_p^\wedge$$

or equivalently

$$(X_{n+1})_{\mathbb{F}_p} \to (X_n)_{\mathbb{F}_p} \to \Sigma^{n+2}\pi_{n+1}(X)_p^\wedge$$

Corollary 18. *If X is bounded below, the canonical map $(X_{n+1})_p^\wedge \to (X_n)_p^\wedge$ (or equivalently $(X_{n+1})_{\mathbb{F}_p} \to (X_n)_{\mathbb{F}_p}$) is n-connected.*

We now have a canonical map $X \simeq \varprojlim_n X_n \to \varprojlim_n (X_n)_p^\wedge \xrightarrow{\simeq} \varprojlim_n (X_n)_{\mathbb{F}_p}$. Its fiber is $\varprojlim_n \mathrm{fib}(X_n \to (X_n)_p^\wedge)$.

Just as in the unstable case, we get that this fiber has homotopy groups that are $\mathbb{Z}[\frac{1}{p}]$-modules, but now we have the advantage of knowing that $\varprojlim_n (X_n)_p^\wedge$ is bounded below (by construction and by the Milnor short exact sequence), from which it follows that this map is a p-adic equivalence (hence an $\mathbb{F}_p$-equivalence) to a spectrum that is both p-complete and $\mathbb{F}_p$-complete.

Corollary 19. *When X is a bounded below spectrum, the canonical map $X_p^\wedge \to X_{\mathbb{F}_p}$ is an equivalence.*

Moreover, the above construction is a construction of the p-completion (equivalently, the $\mathbb{F}_p$-completion); in particular $X_p^\wedge \simeq \varprojlim_n (\tau_{\leq n} X)_p^\wedge$

Just as in the unstable case, we can then get extra mileage from this construction, to get information about the homotopy groups of $X_p^\wedge$. We won't do the proofs, because they're essentially the same as before, using the same arguments – once the construction has been established.

Corollary 20. *Let X be a bounded below spectrum. We have the following short exact sequences for the homotopy groups of $X_p^\wedge$:*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{n+1}(X)) \to \pi_{n+1}(X_p^\wedge) \to \hom(\mathbb{Z}/p^\infty, \pi_n(X)) \to 0$$

for all n (this time in $\mathbb{Z}$)

Sketch of proof. This works exactly the same, studying the connectivity of $(X_{n+1})_p^\wedge \to (X_n)_p^\wedge$ via the fiber sequence and the knowledge of $\Sigma^{n+2}\pi_{n+1}(X)_p^\wedge$. □

Corollary 21. *Let X be a bounded below spectrum with finitely generated homotopy groups. Then the map $X \to X_p^\wedge$ induces an isomorphism $\pi_*(X_p^\wedge) \cong \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.*

Corollary 22. *Let X be a bounded below spectrum. If X is p-complete, then its homotopy groups are derived p-complete.*

Again, these statements are proved in exactly the same way as in the unstable case. Notice of course that the "simply-connected" hypothesis has disappeared; and that the "bounded below" hypothesis is one that is somehow "automatically satisfied" (or doesn't make sense) in the unstable case.

Let's relate the two stories via Ω^∞. Note that for a space X and a spectrum E, $\mathrm{map}_{\mathcal{S}}(X, \Omega^\infty E) \simeq \mathrm{map}_{\mathbf{Sp}}(\Sigma^\infty_+ X, E)$.

Moreover, $X \to Y$ is a p-equivalence of spaces if and only if $\Sigma^\infty_+ X \to \Sigma^\infty_+ Y$ is an $\mathbb{F}_p$-equivalence of spectra. Therefore, if E is $\mathbb{F}_p$-complete, $\Omega^\infty E$ is p-complete.

Let C denote the cofiber of $E \to E_{\mathbb{F}_p}$. Then $\pi_*(\Omega^\infty C) \cong \pi_*(C)$ (for $* \geq 0$) and we have a fiber sequence of simple spaces $\Omega^\infty E \to \Omega^\infty(E_{\mathbb{F}_p}) \to \Omega^\infty C$. If E is 1-connective, so is C, and thus by a spectral sequence argument similar to the one in lemma 13, $\Omega^\infty E \to \Omega^\infty(E_{\mathbb{F}_p})$ is a p-equivalence, and therefore a p-completion. Therefore:

Corollary 23. *Let E be a* 1*-connective spectrum. Then $\Omega^\infty(E_{\mathbb{F}_p}) \simeq (\Omega^\infty E)_p^\wedge$.*

For any spectrum E, if E is $\mathbb{F}_p$-complete, then $\Omega^\infty E$ is p-complete; if E is 1*-connective, the converse holds as well.*

Proof. The only thing we need to say something about is the converse. But this is because if E is connective, so is $E_{\mathbb{F}_p}$, and a map $X \to Y$ of connective spectra is an equivalence if and only if $\Omega^\infty X \to \Omega^\infty Y$ is. □

Remark 33. *This statement is wrong without connectivity assumptions.*

Indeed, consider an abelian group A which is not derived p-complete, viewed as a spectrum. Then $\Omega^\infty A$ is discrete and hence p-complete, so it is its own p-completion. However, it is not equivalent to $\Omega^\infty(A_{\mathbb{F}_p})$.

Similarly, $\Omega^\infty A$ is p-complete, but A is not $\mathbb{F}_p$-complete, for the same reason; so the converse fails.

Most of the interesting results so far have been about connective, or at least bounded below spectra. Let us give one sample computation in the case of a nonconnective spectrum where, luckily, everything goes well. Let $K = KU$ be the (complex) topological K-theory spectrum, and $k = ku$ its connective cover; and let $\beta \in \pi_2 k$ denote the Bott element. Recall that $k[\beta^{-1}] \simeq K$. We then have:

Proposition 17. *There is an equivalence $K_p^\wedge \simeq k_p^\wedge[\beta^{-1}]$.*

In particular, $\pi_*(K_p^\wedge)$ is "the same as if K was connective", i.e. $\mathbb{Z}_p$ if $*$ is even, 0 else.

Proof. Let $\{K \to K \to K/p^n\}_n$ be the inverse system of fiber sequences of spectra, where $K \to K$ is multiplication by p^n.

Taking the limit of this inverse system, one finds a fiber sequence $\varprojlim_n K \to K \to \varprojlim_n K/p^n$.

Now, the Milnor short exact sequence will allow us to compute the homotopy groups of the leftmost and rightmost terms.

Indeed, $\pi_*(K/p^n) \cong \mathbb{Z}/p^n$ if $*$ is odd, and 0 if it's even, in particular all $\varprojlim^1$-terms vanish in the Milnor short exact sequence, which implies that $\varprojlim_n K/p^n$ has the desired homotopy groups.

On the other hand, for the inverse system $\dots \xrightarrow{p^n} K \xrightarrow{p^n} K$, we see that its homotopy groups are $\varprojlim^1 \mathbb{Z}$ and 0, because the $\varprojlim$-terms vanish in the Milnor exact sequence. One may easily compute $\varprojlim^1 \mathbb{Z}$ to be $\mathbb{Z}_p/\mathbb{Z}$; on which p acts invertibly. In particular p acts invertibly on $\varprojlim_n K$, from which it follows that $K \to \varprojlim_n K/p^n$ is a p-adic equivalence. But now each K/p^n is clearly p-complete, from which it follows that $K \to \varprojlim_n K/p^n$ is a p-completion. This proves the claim about homotopy groups.

Now to get the claim about $k_p^\wedge[\beta^{-1}]$, one considers the composition $k \to K \to \varprojlim_n K/p^n$, and see that it factors as $k \to k_p^\wedge[\beta^{-1}] \to K_p^\wedge$. It is easy to conclude from there, and we leave it as an exercise. $\square$

In this example, we see that $K_p^\wedge \simeq \varprojlim_n K/p^n$. I initially thought that this was wrong in general, but in fact it is quite easy to prove that this formula is true more generally, and I owe this realization to Andrea Bianchi. Here's a formal statement and proof:

Proposition 18. *Let $X \in \mathbf{Sp}$. Then the canonical map $X \to \varprojlim_n X/p^n$ is a p-completion of X.*

Proof. Consider the following tower of fiber sequences $\{X \xrightarrow{p^n} X \to X/p^n\}_n$.

Since limits commute with limits, the fiber of $X \to \varprojlim_n X/p^n$ is $\varprojlim(\dots \to X \xrightarrow{p} X \xrightarrow{p} X)$. Now it's an easy exercise to check that p is invertible on this limit, from which it follows that $X \to \varprojlim_n X/p^n$ is a p-adic equivalence.

It thus suffices to prove that X/p^n is p-complete, for if this is the case, then the limit is p-complete as well, and so we have a p-adic equivalence with a p-complete target, which characterizes p-completions.

But now note that if $Y \otimes \mathbb{S}/p \simeq 0$, that is, $p : Y \to Y$ is an equivalence, then $\mathrm{Map}(Y, X) \xrightarrow{p} \mathrm{Map}(Y, X)$ is an equivalence, i.e. $\mathrm{Map}(Y, X/p^n) \simeq 0$. Therefore X/p^n is p-complete, and this concludes the proof. $\square$

This statement actually allows to remove "bounded below" in some statements from before. Indeed, an immediate corollary is:

Corollary 24. *Suppose $X \in \mathbf{Sp}$ is coconnective. Then $X_p^\wedge$ is 1-coconnective, i.e. its homotopy groups are concentrated in degrees ≤ 1.*

Proof. This follows from the above proposition and the Milnor exact sequence. $\square$

Corollary 25. *Corollaries 20 and 21 hold without the assumption that X is bounded below.*

Proof. There is a cofiber sequence $\tau_{\geq m}X \to X \to \tau_{<m}X$, which remains a cofiber sequence upon p-completion. The right hand side is concentrated in degrees $\leq m$ after p-completion, and the left hand side is bounded below so the two corollaries apply to it, and taking m to be small enough (large enough in absolute value), we find that this implies the result for X. $\square$

We will conclude this section with a "computation" of $\mathbb{S}_p^\wedge$. More precisely, we will show that it's a Moore spectrum, that is, it's a connective spectrum whose homology is concentrated in degree 0 (and is $\mathbb{Z}_p$).

By the stable Hurewicz theorem, this characterizes $\mathbb{S}_p^\wedge$ uniquely.

To make this computation, we use a slight generalization of Serre's mod $\mathcal{C}$ theory to be able to apply it to $\mathcal{C} =$ the class of derived p-complete abelian groups. This is a natural thing to attempt to do, and it was for instance explained in a paper by Barthel and Bousfield (in an attempt to relate the p-completion of X to that of $\Sigma^\infty_+ X$): *On the comparison of stable and unstable p-completion* (which we'll call Barthel-Bousfield in what follows).

Let us first see why we might need to use this generalized mod $\mathcal{C}$ theory. Let $\mathbb{S} \to \mathbb{S}_p^\wedge$ denote the completion map, and let F denote its fiber.

F has homotopy groups that are $\mathbb{Z}[\frac{1}{p}]$-modules, therefore by the usual mod $\mathcal{C}$-theory, so are its homology groups, and it follows that $\mathbb{Z} \otimes \mathbb{S} \to \mathbb{Z} \otimes \mathbb{S}_p^\wedge$ is a p-adic equivalence (equivalently an $\mathbb{F}_p$-equivalence, since everything in sight is connective). But of course $\mathbb{Z} \otimes \mathbb{S} \simeq \mathbb{Z}$, and of course $\mathbb{Z}_p^\wedge \simeq \mathbb{Z}_p$.

Therefore if we can prove that $\mathbb{Z} \otimes \mathbb{S}_p^\wedge$ is p-complete, it will follow that it's a p-completion of $\mathbb{Z}$, i.e. $\mathbb{Z}_p$; the description as a Moore spectrum then follows.

To show that it's p-complete, since it's connective, it would be enough to prove that its homotopy groups are derived p-complete; but its homotopy groups are exactly the homology groups of $\mathbb{S}_p^\wedge$. So if we had a theorem of the form "if the homotopy groups are derived p-complete, so are the homology groups", we could conclude.

This is what this generalization of Serre's mod $\mathcal{C}$ theory will do.

Definition 11. *Let $\mathcal{C}$ be a class of abelian groups. We say that $\mathcal{C}$ is a weak Serre class if for any exact sequence $A_1 \to A_2 \to A_3 \to A_4 \to A_5$ of abelian groups with $A_1, A_2, A_4, A_5 \in \mathcal{C}$, A_3 is also in $\mathcal{C}$.*

Proposition 19. *The class of derived p-complete abelian groups is a weak Serre class.*

Proof. It suffices to prove that it's closed under quotients, kernels and extensions (it's an easy exercise to show that this is in fact equivalent).

So let $f : A \to B$ be a map between derived p-complete abelian groups; see it as a map in $\mathbf{D}(\mathbb{Z})$ and let K denote its fiber. The long exact sequence of homology groups shows that $H_0(K) = \ker(f)$ and $H_{-1}(K) = (f)$.

Derived p-complete complexes are stable under limits, so K is derived p-complete, therefore so is its homology, therefore so are $\ker(f)$ and (f).

We are left to deal with extensions. But if $0 \to A \to B \to C \to 0$ is a short exact sequence, then $A \to B \to C$ is a fiber sequence in $\mathbf{D}(\mathbb{Z})$, and therefore so is $B \to C \to \Sigma A$. $B \simeq \mathrm{fib}(C \to \Sigma A)$ and both $C, \Sigma A$ are derived p-complete, so B must be too. $\square$

We now know that if we can prove a mod $\mathcal{C}$ Hurewicz theorem for weak Serre classes, it will hold for derived p-complete abelian groups, and we will be able to conclude. In fact, there is such a result:

Proposition 20. *Let $\mathcal{C}$ be a weak Serre class. Then for any bounded below spectrum X, the following are equivalent:*

1. $\pi_n(X) \in \mathcal{C}$ *for all* $n \in \mathbb{Z}$
2. $H_n(X) \in \mathcal{C}$ *for all* $n \in \mathbb{Z}$

Lemma 16. *Let $\mathcal{C}$ be a weak Serre class. Then $\mathcal{C}$ is stable under (underived) tensoring with finitely generated abelian groups, and similarly under taking* $\mathrm{Tor}_1^{\mathbb{Z}}$ *with a finitely generated abelian group.*

Proof. Left as an exercise. $\square$

Proof of the proposition. We will only prove the statement of interest to us, that is 1. $\implies$ 2. The converse is proved in Barthel-Bousfield.

The first step is to prove the result for Eilenberg-MacLane spectra. So let A be an abelian group in $\mathcal{C}$, we want to prove that $\pi_n(A \otimes \mathbb{Z}) \in \mathcal{C}$. But this can be seen as the homology of the spectrum $\mathbb{Z}$ with coefficients in A, and so the universal coefficient theorem yields a short exact sequence:

$$0 \to \pi_n(\mathbb{Z} \otimes \mathbb{Z}) \otimes^u_{\mathbb{Z}} A \to \pi_n(\mathbb{Z} \otimes A) \to \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{n-1}(\mathbb{Z} \otimes \mathbb{Z}), A) \to 0$$

Now it's an important result that $\pi_*(\mathbb{Z} \otimes \mathbb{Z})$ is finitely generated in each degree, and therefore the previous lemma and stability of $\mathcal{C}$ under extensions shows that $\pi_n(\mathbb{Z} \otimes A) \in \mathcal{C}$.

Now we proceed by induction on the number of nonzero homotopy groups to show the proposition for bounded spectra.

Suppose X is a bounded spectrum with all homotopy groups in $\mathcal{C}$, and with maximum nonzero homotopy group in degree $n + 1$. Then we have a fiber sequence $\Sigma^{n+1}\pi_{n+1}(X) \to X \to \tau_{\leq n} X$.

By induction, we may assume $H_*(\tau_{\leq n} X) \in \mathcal{C}$ for all $*$, and since $\pi_{n+1}(X) \in \mathcal{C}$, the case for Eilenberg-MacLane spectra shows that the same can be said about $\Sigma^{n+1}\pi_{n+1}(X)$.

Now the long exact sequence in homotopy for $\mathbb{Z}\otimes\Sigma^{n+1}\pi_{n+1}(X)\to\mathbb{Z}\otimes X\to\mathbb{Z}\otimes\tau_{\leq n}X$ and the definition of a weak Serre class shows that $H_*(X)\in\mathcal{C}$ for each $*$.

Finally, we use the Postnikov tower to go from there to arbitrary bounded below spectra: let X be a bounded below spectrum with $\pi_n(X)\in\mathcal{C}$ for all n. Then $X\to\tau_{\leq n}$ is n-connected, and so is an isomorphism on homology up to degree n.

Letting n go to ∞, because of what we proved earlier, it follows that the homology of X is in $\mathcal{C}$, and we are done. $\square$

Remark 34. *We didn't prove it, but the converse can be proved quite easily if one knows the Atiyah-Hirzebruch spectral sequence for stable homotopy, and that $\pi_*\mathbb{S}$ is a finitely generated abelian group for any $*$.*

We can then apply this to $\mathcal{C}=$ derived p-complete abelian groups.

It follows from this, corollaries 22 and 14 that if E is bounded below and p-complete, then so is $E\otimes\mathbb{Z}$. In particular we get the desired result:

Corollary 26. *$\mathbb{S}_p^\wedge$ is a Moore spectrum for $\mathbb{Z}_p$, that is, it is connective and $\mathbb{S}_p^\wedge\otimes\mathbb{Z}\simeq\mathbb{Z}_p$.*

Even for this first attempt, there is still much more to be said (should it only be the whole contents of Barthel-Bousfield), but we will stop there; and move on to the pro-homotopy theoretic story.

Indeed, just as in the underived setting, we have spaces and spectra that can play the role of "finite abelian groups", and it is reasonable to consider the associated pro-objects, as another attempt to understand "p-adic phenomena". This is the object of the following sections.

5 Topological p-completion: second attempt

6 Stable p-completion: second attempt

A Appendix: The Milnor exact sequence

We are interested in the following statement:

Theorem 3. *Let (X_k) be an inverse system of pointed spaces (resp. spectra), and $X:=\varprojlim_k X_k$. Then for all $n\in\mathbb{N}$ (resp. $\mathbb{Z}$) there is a short exact sequence:*

$$0\to\varprojlim_k{}^1\pi_{n+1}(X_k)\to\pi_n(X)\to\varprojlim_k\pi_n(X_k)\to 0$$

(in the case of spaces, if $n=0$, this is an exact sequence of pointed sets)

In particular, in nice situations where the system $(\pi_n(X_k))_k$ satisfies the Mittag-Leffler condition, the $\varprojlim^1$ term vanishes. This is the case in Postnikov systems for instance.

This theorem relies on the following lemma:

Lemma 17. *Consider an ∞-category $\mathcal{C}$ with all (small) limits, and let (X_k) denote an inverse system in $\mathcal{C}$. Then we have a pullback square:*

$$\begin{tikzcd}
\varprojlim_k X_k \arrow[r] \arrow[d] & \prod_k X_k \arrow[d] \\
\prod_k X_k \arrow[r] & \prod_k X_k \times X_k
\end{tikzcd}$$

where the vertical arrow $\prod_k X_k \to \prod_k X_k \times X_k$ is just $\prod_k \Delta_k$ ($\Delta_k : X_k \to X_k \times X_k$ is the diagonal arrow), and the horizontal arrow is, on the k-coordinate, $\prod_k X_k \to X_k \times X_{k+1} \xrightarrow{id_{X_k} \times p_{k+1}} X_k \times X_k$ ($p_{k+1} : X_{k+1} \to X_k$ being the transition map of the inverse system)

Proof. There is probably a more conceptual proof.

Thanks to the Yoneda embedding, we reduce the problem to proving it in $\mathcal{S}$, where we can use the fact that our ∞-categorical limit can be computed as a homotopy limit.

But for the homotopy limit, we may first of all assume that our inverse system is composed of fibrations, and then replace $\prod_k \Delta_k : \prod_k X_k \to \prod_k X_k \times X_k$ with $\prod_k \mathrm{Path}(X_k) \to \prod_k X_k \times X_k$ which is a fibration.

The claim is then that this 1-categorical pullback is a model for the homotopy limit. But this is done in Goerss-Jardine, *Simplicial Homotopy Theory*, as lemma VI.1.12. $\square$

From this lemma, we can describe the fiber of $\varprojlim_k X_k \to \prod_k X_k$, as the fiber of $\prod_k X_k \to \prod_k X_k \times X_k$, which is clearly $\prod_k \Omega X_k$.

We therefore have a long exact sequence

$$\dots \to \pi_{n+1} \prod_k X_k \to \pi_n \prod_k \Omega X_k \to \pi_n(\varprojlim_k X_k) \to \pi_n \prod_k X_k \to \pi_{n-1} \prod_k \Omega X_k \to \dots$$

Knowing that $\pi_m \Omega X_k \cong \pi_{m+1} X_k$, we may rewrite this as

$$\dots \to \pi_{n+1} \prod_k X_k \to \pi_{n+1} \prod_k X_k \to \pi_n(\varprojlim_k X_k) \to \pi_n \prod_k X_k \to \pi_n \prod_k X_k \to \dots$$

The last step to get the Milnor exact sequence is to identify the map $\pi_n \prod_k X_k \to \pi_n \prod_k X_k$. Just as in the proof of the lemma, via Yoneda we are reduced to the case of pointed spaces.

We first want to identify the boundary map in the fiber sequence $\prod_k \Omega X_k \to \prod_k X_k \to \prod_k X_k \times X_k$.

The latter is just a product of fiber sequences of the form $\Omega X \to X \to X \times X$ so by naturality it suffices to identify the boundary for those : what is $\pi_{n+1}(X \times X) \to \pi_n \Omega X \cong \pi_{n+1} X$?

But again, by naturality, because π_{n+1} is represented by S^{n+1}, it suffices to deal with the case $X = S^{n+1}$, and in this case our boundary is a map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

that fits into a short exact sequence $0 \to \mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z} \to 0$; so it must be $(a,b) \mapsto a-b$. Therefore that's what it is for $\Omega X \to X \to X \times X$ as well.

Now, to conclude, we use our pullback square to compare fiber sequences:

$$\begin{tikzcd}
\prod_k \Omega X_k \arrow[r, "\simeq"] \arrow[d] & \prod_k \Omega X_k \arrow[d] \\
\varprojlim_k X_k \arrow[r] \arrow[d] & \prod_k X_k \arrow[d] \\
\prod_k X_k \arrow[r] & \prod_k X_k \times X_k
\end{tikzcd}$$

and so by naturality we can finally compare the boundary maps:

$$\begin{tikzcd}
\pi_{n+1} \prod_k X_k \arrow[r] \arrow[d] & \pi_{n+1} \prod_k X_k \times X_k \arrow[d] \\
\pi_{n+1} \prod_k X_k \arrow[r, "\cong"] & \pi_{n+1} \prod_k X_k
\end{tikzcd}$$

The map we're interested in is the left vertical map, and we know all the rest: the top horizontal one is just $x \mapsto (x, p_* x)$ (here we slightly abuse notation, hopefully what is meant is clear), the bottom horizontal one is the identity, and the right vertical one, as we've just established, is $(a,b) \mapsto a-b$.

It follows that the left vertical map is $x \mapsto x - p_*(x)$ with the same abuse of notation. Using $\pi_{n+1} \prod_k X_k \simeq \prod_k \pi_{n+1} X_k$, a better notation is $(a_1, ..., a_k, ...) \mapsto (a_1 - (p_2)_*(a_2), a_2 - (p_3)_*(a_3), ...)$.

The kernel of that map is exactly $\varprojlim_k \pi_{n+1}(X_k)$, and its cokernel $\varprojlim_k{}^1 \pi_{n+1}(X_k)$.

Putting things together, we get the desired Milnor exact sequence:

$$0 \to \varprojlim_k{}^1 \pi_{n+1}(X_k) \to \pi_n(X) \to \varprojlim_k \pi_n(X_k) \to 0$$

Note that this works for pointed spaces, spectra (one could in fact deduce it for spectra from the pointed spaces-version), and therefore also chain complexes (the forgetful functor $\mathbf{D}(\mathbb{Z}) \to \mathbf{Sp}$ preserves limits, and homotopy groups of an Eilenberg-MacLane spectrum correspond to the homology groups of the corresponding chain complex)