Endomorphism monads

Introduction

The goal of this note was initially to prove that Morita equivalent commutative monads are isomorphic.

To do so, we introduce the notion of "endomorphism monad" of an object in a category with arbitrary products, and specialize it to $\mathrm{id}_{\mathbf{C}} \in \mathrm{Fun}(\mathbf{C}, \mathbf{C})$. We call the endomorphism monad of this object the center monad of $\mathbf{C}$, and we show that if $\mathbf{C} = \mathbf{Alg}_T$ for some monad T, then there is an injective morphism of monads from this center monad to T, and we call a monad commutative if it is surjective.

We also show that this center monad is always commutative by relating this notion of commutativity to the notion defined on the nLab [nLa21], namely, we prove that the nLab's definition is equivalent to ours.

The question of a universal property of the endomorphism monad is also left open.

All of this was worked out on Twitter with @AyeGill, @sarah_zrf and @schala163

Conventions

We will not worry about size issues, they can all be relatively easily bypassed here.

Monads will typically be denoted by T, sets by capital letters such as X, Y, A, B, and objects of categories typically with lower-case letters such as c, d. Categories will typically be denoted in capital boldface.

Categories that appear frequently are : $\mathbf{Set}$, the category of sets; $\mathbf{Alg}_T$, the category of T-algebras for some monad T; $\mathbf{Mod}_R$, the category of R-modules for some ring R; $\mathbf{Set}_M$, the category of M-sets for some monoid M; the category of categories.

1 Endomorphism monads of objects

Construction 1.1. Let $\mathbf{C}$ be a category with arbitrary coproducts, and $c \in \mathbf{C}$.

There is a (unique up to unique isomorphism) coproduct-preserving functor $\mathbf{Set} \to \mathbf{C}$ sending the singleton to c - it sends a set X to $\coprod_X c$, an X-indexed coproduct of c's.

By design, this functor has right adjoint, given by $\hom_{\mathbf{C}}(c, -)$. Indeed we clearly have $\hom_{\mathbf{Set}}(X, \hom(c, d)) \cong \hom_{\mathbf{C}}(\coprod_X c, d)$.

In particular the composition $\mathbf{Set} \to \mathbf{C} \to \mathbf{Set}$ is a monad on $\mathbf{Set}$, given by $X \mapsto \hom_{\mathbf{C}}(c, \coprod_X c)$.

Definition 1.2. Let $\mathbf{C}$ be a category with arbitrary products, and $c \in \mathbf{C}$. Applying the above construction to $\mathbf{C}^{\mathrm{op}}$ and the same object c, we get a monad on $\mathbf{Set}$, which we call the *endomorphism monad* of c, which we denote by $\mathcal{E}\mathrm{nd}_{\mathbf{C}}(c)$.

Remark 1.3. We chose to do the construction with coproducts because that is the natural universal property of $\mathbf{Set}$; but the definition with products is important for the applications we have in mind - the reason it will come up is that for a monad T, the forgetful functor from T-algebras to sets preserves products, and not coproducts.

Question 1.4. Does the endomorphism monad have a universal property ? If $\mathbf{C}$ has coproducts, then it is tensored over $\mathbf{Set}$, and the endomorphism monoid of c has a universal property, as the terminal monoid acting on c. Is there something similar for the endomorphism monad ?

Observation 1.5. We can unravel the definition of the endomorphism monad and describe its structure more concretely.

The value on a set X is $\hom_{\mathbf{C}}(c^X, c)$, where $c^X := \prod_X c$. The unit

$$\eta_X : X \to \hom_{\mathbf{C}}(c^X, c)$$

is given by sending x to the projection on x, while the multiplication

$$\mu_X : \hom_{\mathbf{C}}(c^{\hom_{\mathbf{C}}(c^X,c)}, c) \to \hom_{\mathbf{C}}(c^X, c)$$

is given by precomposing with the map $c^X \to c^{\hom_{\mathbf{C}}(c^X,c)}$ given on the component $f \in \hom_{\mathbf{C}}(c^X, c)$ by f. This is left as an exercise to the reader.

Remark 1.6. The value of this monad on a singleton is $\hom_{\mathbf{C}}(c, c)$. If we see monads as analogous to monoids, this monad is analogous to the endomorphism monoid, but it contains more information, as we will see later.

We want to make this definition functorial, so we recall two definitions for convenience :

Definition 1.7. $\mathbf{Mnd}$ is the category of monads on $\mathbf{Set}$: its objects are monads, and its morphisms are morphisms of monads, that is, natural transformations which are compatible with the monad structure.

Remark 1.8. Monads are in general not accessible functors $\mathbf{Set} \to \mathbf{Set}$, so it is not clear that $\mathbf{Mnd}$ is locally small. I do not know whether it is, but my guess would be that it isn't.

Definition 1.9. $\mathbf{Cat}^{\times}$ is the category of categories with arbitrary products and product preserving functors, and $\mathbf{Cat}^{\times}_{*} := \mathbf{Cat}^{\times}_{\mathbf{Set}^{\mathrm{op}}/}$ is the category of these together with a distinguished object - we represent the distinguished object by a coproduct-preserving functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{C}$.

Remark 1.10. There is an obvious alternative definition of $\mathbf{Cat}^{\times}_{*}$ as $\mathbf{Cat}^{\times} \times_{\mathbf{Cat}} \mathbf{Cat}_{*}$ where we literally just pick out a distinguished object. The forgetful functor from our definition to this one (which evaluates at a specific singleton) is an equivalence of 2-categories, but our definition is more practical if we wish to stay in the land of 1-categories.

Fixing an inverse to this equivalence requires some form of the axiom of choice, and so would make the construction of a functor either less precise or less natural.

Notation 1.11. We abuse notation by writing $(\mathbf{C}, c)$ for an object of $\mathbf{Cat}_*^\times$, but it is understood that c stands not only for $c \in \mathbf{C}$, but for the choice of a product-preserving functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{C}$.

We will also write c^X for the image of X under this functor.

In the next construction, we show that the assignment $(\mathbf{C}, c) \mapsto \mathcal{E}\mathrm{nd}_{\mathbf{C}}(c)$ can be made into a functor $\mathbf{Cat}_*^\times \to \mathbf{Mnd}$.

Construction 1.12. Let $F : (\mathbf{C}, c) \to (\mathbf{D}, d)$ be a morphism in $\mathbf{Cat}_*^\times$, that is, a commutative diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] & \mathbf{D} \\ \mathbf{Set}^{\mathrm{op}} \arrow[u, "c"] \arrow[ur, "d"'] & \end{tikzcd}$$

Passing to op's, we get a commutative diagram

$$\begin{tikzcd} \mathbf{C}^{\mathrm{op}} \arrow[r, "F"] & \mathbf{D}^{\mathrm{op}} \\ \mathbf{Set} \arrow[u, "c"] \arrow[ur, "d"'] & \end{tikzcd}$$

and then a morphism, natural in X, $\hom_{\mathbf{C}}(c^X, c) \to \hom_{\mathbf{D}}(F(c^X), F(c)) = \hom_{\mathbf{D}}(d^X, d)$. Given our description of the unit and the multiplication, it is clear that this gives a morphism of monads $\mathcal{E}\mathrm{nd}_{\mathbf{C}}(c) \to \mathcal{E}\mathrm{nd}_{\mathbf{D}}(d)$.

Corollary 1.13. *There is a functor* $\mathbf{Cat}_*^\times \to \mathbf{Mnd}$ *which on objects acts as* $(\mathbf{C}, c) \to \mathcal{E}\mathrm{nd}_{\mathbf{C}}(c)$*, and acts on morphisms by postcomposition.*

Corollary 1.14. *There is a (unique up to unique isomorphism) functor* $\mathbf{Cat}^\times \times_{\mathbf{Cat}} \mathbf{Cat}_* \to \mathbf{Mnd}$ *which sends each* $(\mathbf{C}, c)$ *to an endomorphism monad, and acts by postcomposition (up to isomorphism) on morphisms.*

2 Center monads, commutative monads

Definition 2.1. If $\mathbf{C}$ is a category with arbitrary products, then $\mathrm{Fun}(\mathbf{C}, \mathbf{C})$ is the same, and it has a distinguished element, id.

We can therefore define $\mathcal{E}\mathrm{nd}_{\mathrm{Fun}(\mathbf{C},\mathbf{C})}(\mathrm{id})$ and call it the center monad of $\mathbf{C}$, and denote it by $\mathcal{Z}(\mathbf{C})$.

Remark 2.2. The endomorphism monoid $\hom(\mathrm{id}, \mathrm{id})$ is often called the "center" of $\mathbf{C}$, and denoted $Z(\mathbf{C})$.

An observation is that the endomorphism monoid of the identity functor of $\mathbf{C}$ is always commutative, we would like to find an analog of this statement in our situation. The appropriate notion seems to be that of a commutative monad.

The idea behind the notion of a commutative monad T is that the various "operations" defined by T should commute with each other. If f is an "X-ary operation" (i.e. an element of TX) , f defines a set-map $A^X \to A$ for any T-algebra A, but both A^X and A are T-algebras so we can ask whether this map is a morphism of T-algebras : this is essentially equivalent to the operations commuting with each other, and this is what we call a commutative monad:

Definition 2.3. Let T be a monad. T is said to be commutative if for each T-algebra A, every set X, and every $f \in TX$, the following map is a morphism of T-algebras:

$$A^X = \hom_{\mathbf{Set}}(X, A) \cong \hom_{\mathbf{Alg}_T}(TX, A) \to \hom_{\mathbf{Set}}(TX, A) \to A$$

where the first $\cong$ comes from the free-forgetful adjunction, the first $\to$ is simply the forgetful functor, and the last $\to$ is evaluation at f.

Proposition 2.4. *Let T be a monad, and $U : \mathbf{Alg}_T \to \mathbf{Set}$ be the forgetful functor. The above construction yields a morphism $TX \to \hom_{\mathrm{Fun}(\mathbf{Alg}_T,\mathbf{Set})}(U^X, U)$: this is an isomorphism.*

This is further a monad morphism, if we view the left hand side as $\mathcal{E}\mathrm{nd}_{\mathrm{Fun}(\mathbf{Alg}_T,\mathbf{Set})}(U)$; in particular it is a monad isomorphism.

Remark 2.5. This is analogous to the fact that the endomorphism ring of the forgetful functor $\mathbf{Mod}_R \to \mathbf{Ab}$ is isomorphic to R; or the endomorphism monoid of the forgetful functor $\mathbf{Set}_M \to \mathbf{Set}$ is isomorphic to M, for a monoid M.

Note that the endomorphism monoid of the forgetful functor $\mathbf{Mod}_R \to \mathbf{Set}$ is isomorphic to R as a multiplicative monoid - in particular it forgets all about the additive structure of R, whereas the endomorphism monad, by the proposition, is the free R-module monad, so it remembers all about it. In other words, while the endomorphism monoid cannot tell the difference between R and its underlying multiplicative monoid, the endomorphism monad can.

Proof. Observe that U^X is represented by TX, and so this follows at once from unwinding the Yoneda lemma in this specific case.

The claim that this is a morphism of monads is not too complicated to check by hand. One way to simplify the calculations is to use the (product-preserving) Yoneda embedding $\mathbf{Alg}_T^{\mathrm{op}} \to \mathrm{Fun}(\mathbf{Alg}_T, \mathbf{Set})$, and observe that our morphism factors through $T \to \mathcal{E}\mathrm{nd}_{\mathbf{Alg}_T^{\mathrm{op}}}(T1) \to \mathcal{E}\mathrm{nd}_{\mathrm{Fun}(\mathbf{Alg}_T,\mathbf{Set})}(U)$.

The second map in this factorization is an isomorphism of monads by functoriality of $\mathcal{E}\mathrm{nd}$, so it suffices to check the claim for the first one. But now, the monad associated to the adjunction $\mathbf{Set} \rightleftarrows (\mathbf{Alg}_T^{\mathrm{op}})^{\mathrm{op}} = \mathbf{Alg}_T$ is known to be T. □

Proposition 2.6. *Let T be a monad, and U as above. U preserves products, so post-composition with it induces a morphism $\hom_{\mathrm{Fun}(\mathbf{Alg}_T,\mathbf{Alg}_T)}(\mathrm{id}^X, \mathrm{id}) \to \hom_{\mathrm{Fun}(\mathbf{Alg}_T,\mathbf{Set})}(U^X, U)$. This morphism is injective.*

In particular, T is commutative if and only if this morphism is an isomorphism.

Proof. U is faithful, and so post-composition with U is also faithful as a functor $\mathrm{Fun}(\mathbf{Alg}_T, \mathbf{Alg}_T) \to \mathrm{Fun}(\mathbf{Alg}_T, \mathbf{Set})$. This proves injectivity.

By definition, T is commutative if and only if the map $\eta_f : U^X \to U$ is a morphism of T-algebras for each $f \in TX$, i.e. if and only if it can be lifted to $\mathrm{id}^X \to id$; so T is commutative if and only if this morphism is surjective. Because it is injective, this is if and only if it is an isomorphism. □

Corollary 2.7. *If T is a commutative monad, then $T \cong \mathcal{Z}(\mathbf{Alg}_T)$. In particular, if T, S are commutative monads such that $\mathbf{Alg}_T \simeq \mathbf{Alg}_S$ (as abstract categories), then $T \cong S$.*

Remark 2.8. This is the analog of the situation for rings - note that this does not hold if we don't assume that both T, S are commutative, as the classical example of rings shows.

Two monads T, S such that $\mathbf{Alg}_T \simeq \mathbf{Alg}_S$ are said to be Morita equivalent, so the above result can be reformulated as "Two Morita equivalent commutative monads are isomorphic".

Proof. Let T be a monad, and $U : \mathbf{Alg}_T \to \mathbf{Set}$ the forgetful functor. Fix a coproduct-preserving functor $\mathbf{Set} \to \mathrm{Fun}(\mathbf{Alg}_T, \mathbf{Alg}_T)^{\mathrm{op}}$ which sends singletons to id, and postcompose it with U to get a morphism $(\mathrm{Fun}(\mathbf{Alg}_T, \mathbf{Alg}_T), \mathrm{id}) \to (\mathrm{Fun}(\mathbf{Alg}_T, \mathbf{Set}), U)$ in $\mathbf{Cat}_*^\times$.

This induces a morphism of monads $\mathcal{Z}(\mathbf{Alg}_T) = \mathcal{E}\mathrm{nd}_{\mathrm{Fun}(\mathbf{Alg}_T, \mathbf{Alg}_T)}(\mathrm{id}) \to \mathcal{E}\mathrm{nd}_{\mathrm{Fun}(\mathbf{Alg}_T, \mathbf{Set})}(U) \cong T$.

But the previous proposition states exactly that T is commutative if and only if this is an isomorphism. ☐

Examples 2.9. We leave it as an exercise to prove the following :

- The "free group" monad is not commutative
- The "free abelian group" monad is commutative
- The "free R-module" monad is commutative if and only if R is a commutative (semi-)ring.
- The ultrafilter monad is not commutative.

We will now relate this definition to the definition that can be found on the nLab [nLa21]-this is a much more checkable criterion, more "computational". We do it for two reasons: first, so that the results here relate to the ordinary definition of commutative monad, and second, because this criterion will help us show that $\mathcal{Z}(\mathbf{C})$ is a commutative monad for any $\mathbf{C}$ with products.

Construction 2.10. Let T be a monad on sets. We define two natural transformations, $\sigma_{A,B} : TA \times B \to T(A \times B)$ and $\tau_{A,B} : A \times TB \to T(A \times B)$.

$\sigma_{A,B}$ is defined as follows : fixing $b \in B$, the map $TA \to T(A \times B)$ is given by applying T to $a \mapsto (a, b)$.

$\tau_{A,B}$ is defined symmetrically.

Proposition 2.11. *A monad T is commutative if and only if for any sets A, B, the following two morphisms are equal:*

$$TA \times TB \xrightarrow{\sigma_{A,TB}} T(A \times TB) \xrightarrow{T\tau_{A,B}} TT(A \times B) \xrightarrow{\mu_{A \times B}} T(A \times B)$$

and

$$TA \times TB \xrightarrow{\tau_{TA,B}} T(TA \times B) \xrightarrow{T(\sigma_{A,B})} TT(A \times B) \xrightarrow{\mu_{A \times B}} T(A \times B)$$

Remark 2.12. For our applications, we only really need"if", but it is nice to know that our condition is equivalent to the classical definition.

Proof. We observe that the claim that the morphism $A^X \to A$ associated to $f \in TX$ is a morphism of T-algebras for all $f \in TX$ is equivalent to the claim that there exists a dotted lift in the following diagram :

$$\begin{tikzcd} & \hom_{\mathbf{Alg}_T}(A^X, A) \arrow[d] \\ TX \arrow[r] \arrow[ur, dotted] & \hom_{\mathbf{Set}}(A^X, A) \end{tikzcd}$$

i.e. that the map $\coprod_{f\in TX} A^X \to A$ factors through the coproduct *in T-algebras*.

Now, there is a natural coequalizer diagram of T-algebras $TTA \rightrightarrows TA \to A$, so the Z-fold coproduct of A in T-algebras is the coequalizer of $T(Z \times TA) \rightrightarrows T(Z \times A)$.

It is worth expliciting the two maps in this diagram: one of them is simply given by the action map $TA \to A$, while the other one is given by $\tau_{Z,A} : Z \times TA \to T(Z \times A)$, which is then canonically extended to an algebra map $T(Z \times TA) \to T(Z \times A)$.

So the TX-fold coproduct of A^X is the coequalizer of $T(TX \times T(A^X)) \to T(TX \times A^X)$, and we're asking whether the map $TX \times A^X \to A$ factors through this. Certainly, it extends canonically to an algebra map from $T(TX \times A^X)$, so the question is really whether the two composite $T(TX \times T(A^X)) \rightrightarrows T(TX \times A) \to A$ are equal.

Of course, because $T(TX \times T(A^X))$ is a free T-algebra, this is equivalent to asking whether the two maps $TX \times T(A^X) \rightrightarrows T(TX \times A^X) \to A$ are equal, where the first map is τ_{TX,A^X} while the second map is $TX \times T(A^X) \to TX \times A^X \to T(TX \times A^X)$.

The next observation is that the map $TX \times A^X \to A$ has an interpretation in terms of σ. Indeed, it is given by $TX \times A^X \overset{\sigma_{X,A^X}}{\to} T(X \times A^X) \overset{T(ev)}{\to} TA \to A$. Indeed, fix $\alpha : X \to A$. Then this composite is $TX \overset{T(x\mapsto(x,\alpha))}{\to} T(X \times A^X) \overset{T(ev)}{\to} TA \to A$, i.e. $T(ev \circ (x \mapsto (x, \alpha)))$ followed by $TA \to A$, i.e. $T\alpha$ followed by this, which is exactly the definition of $TX \times A^X \to A$.

In particular, the first map (the one involving τ) is the following composite:

$$TX \times T(A^X) \overset{\tau}{\to} T(TX \times A^X) \overset{T\sigma}{\to} TT(X \times A^X) \overset{TT(ev)}{\to} TTA \overset{T\rho_A}{\to} TA \to A$$

Because A is a T-algebra, the last bit can be changed into $TTA \overset{\mu_A}{\to} TA \to A$, and because $\mu : TT \to T$ is natural, the composite $\mu_A \circ TT(ev)$ is the same as $T(ev) \circ \mu_{X\times A^X}$, so overall this composite can be rewritten as

$$TX \times T(A^X) \overset{\tau}{\to} T(TX \times A^X) \overset{T\sigma}{\to} TT(X \times A^X) \overset{\mu}{\to} T(X \times A^X) \overset{T(ev)}{\to} TA \to A$$

Now, the first chunk of this composite (until μ) is exactly one of the two maps appearing in the statement of the proposition.

Let us move on to the second composite, given by

$$TX \times T(A^X) \to TX \times A^X \overset{\sigma}{\to} T(X \times A^X) \to TA \to A$$

Because σ is natural, we can replace the first chunk with

$$TX \times T(A^X) \overset{\sigma}{\to} T(X \times T(A^X)) \to T(X \times A^X) \to TA \to A$$

Next we want to compare the last chunk with something involving τ, namely with $T(X \times T(A^X)) \stackrel{T\tau}{\to} TT(X \times A^X) \stackrel{\mu}{\to} T(X \times A^X) \to TA \to A$.

Because both are T-algebra morphisms, and the source is free, it suffices to compare them after restricting to $X \times T(A^X)$, and then to $T(A^X)$ after fixing an $x \in X$.

The composite we have, after this precomposition, is given by $T(A^X) \stackrel{T\mathrm{pr}_x}{\to} TA \to A$, and an easy calculation shows that the same holds for the composite that we want, involving τ.

So we can replace the second composite with

$$TX \times T(A^X) \stackrel{\sigma}{\to} T(X \times T(A^X)) \stackrel{T\tau}{\to} TT(X \times A^X) \stackrel{\mu}{\to} T(X \times A^X) \to TA \to A$$

.

It is now clear that the condition in the proposition implies that these two composites are equal (it is the special case $A = X, B = A^X$ - with a different A), so the map factors uniquely through the coequalizer, i.e. the map $TX \times A^X \to A$ factors through the coproduct in $\mathbf{Alg}_T$, i.e. we have a dotted lift as desired, i.e. T is commutative.

So we have proved that the condition in the proposition implies the commutativity of T.

In fact, we have proved that T being commutative is equivalent to the two following maps being equal for any T-algebra A and any set X:

$$TX \times T(A^X) \stackrel{\sigma}{\to} T(X \times T(A^X)) \stackrel{T\tau}{\to} TT(X \times A^X) \stackrel{\mu}{\to} T(X \times A^X) \stackrel{T(ev)}{\to} TA \to A$$

and

$$TX \times T(A^X) \stackrel{\tau}{\to} T(TX \times A^X) \stackrel{T\sigma}{\to} TT(X \times A^X) \stackrel{\mu}{\to} T(X \times A^X) \stackrel{T(ev)}{\to} TA \to A$$

We use this for the converse. Indeed, fix X, B two sets. Then the unit $\eta : X \times B \to T(X \times B)$ induces a map $B \to T(X \times B)^X$. Letting $A := T(X \times B)$, we have a map $X \to X$ as a well as a map $B \to A^X$.

The two maps that we want to show commute are natural, and so we can compare them for the pair (X, B) and the pair (X, A^X), we get a commutative diagram of the form :

$$\begin{tikzcd}
TX \times TB \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & TX \times T(A^X) \arrow[d, shift left] \arrow[d, shift right] \\
T(X \times B) \arrow[r] & T(X \times A^X)
\end{tikzcd}$$

Now the two arrows on the right, we do not know a priori whether they are equal, but we know that they are coequalized by the map $T(X \times A^X) \to TA \to A$, and therefore by commutativity, the two maps $TX \times TB \to T(X \times B)$ are coequalized by $T(X \times B) \to T(X \times A^X) \to TA \to A$. But $T(X \times B) \to T(X \times A^X)$ is T applied to $X \times B \to X \times A^X$, and the second one is T applied to evaluation, so the composite is T applied to $X \times B \to X \times A^X \to A$, i.e. T applied to the unit $X \times B \to T(X \times B)$. Applying $\rho_A = \mu_{X \times B}$ to this yields the identity $T(X \times B) \to T(X \times B)$: our two maps are coequalized by the identity, so they are equal. This proves the claim. $\square$

Question 2.13. Is there a shorter proof ? A lot of the proof is computing the two maps in the coequalizer diagram that define the coproduct $\coprod_{TX} A^X$ in $\mathbf{Alg}_T$, maybe one can do this in a slicker way. Or maybe one can prove the equivalence without this coequalizer trick.

Using this characterization, we can prove that $\mathcal{Z}(\mathbf{C})$ is commutative:

Proposition 2.14. *Let* $\mathbf{C}$ *be a category with arbitrary products. Then* $\mathcal{Z}(\mathbf{C})$ *is a commutative monad.*

Remark 2.15. This is analogous to the fact that if R is a ring, or M a monoid, then $\mathrm{End}(\mathrm{id}_{\mathbf{Mod}_R})$ is the center of R as a ring, or $\mathrm{End}(\mathrm{id}_{\mathbf{Set}_M})$ is the center of M as a monoid.

In fact, in both cases, the interested reader should compute as an exemple that $\mathcal{Z}(\mathbf{Mod}_R)$ is the free $Z(R)$-module monad, and $\mathcal{Z}(\mathbf{Set}_M)$ is the free $Z(M)$-set monad.

Proof. Let $\mathbf{D}$ be a category with arbitrary products, and $d \in \mathbf{D}$. We compute the two morphisms of proposition 2.11 for $T = \mathcal{E}\mathrm{nd}_{\mathbf{D}}(d)$.

Both are morphisms $\hom(d^A, d) \times \hom(d^B, d) \to \hom(d^{A\times B}, d)$, so we start with $f : d^A \to d, g : d^B \to d$. The claim is that one of the morphisms we get is $g \circ f^B$ and the other is $f \circ g^A$. The reader is advised to work this out on their own, as it is the only way to make it clear. We will try our best to be clear.

Following $\sigma_{A,TB}$, we first get a morphism $d^{A\times \hom(d^B,d)} \to d$ which is given by the composite

$$d^{A\times \hom(d^B,d)} \xrightarrow{\mathrm{pr}_g} d^A \xrightarrow{f} d$$

where pr_g is projection onto the g-component.

Then, following $T\tau_{A,B}$, we get a morphism $d^{\hom(d^{A\times B},d)} \to d$ which is given by

$$d^{\hom(d^{A\times B},d)} \to d^{A\times \hom(d^B,d)} \xrightarrow{\mathrm{pr}_g} d^A \xrightarrow{f} d$$

where the first map, on a component (a, h) projects onto the component of $d^{A\times B} \xrightarrow{\mathrm{pr}_a} d^B \xrightarrow{h} d$. In other words the composite with pr_g is given by

$$d^{\hom(d^{A\times B},d)} \to d^A$$

which on the a-component simply projects onto the component of $d^{A\times B} \xrightarrow{\mathrm{pr}_a} d^B \xrightarrow{g} d$.

We finally apply the multiplication to get a map $d^{A\times B} \to d^{\hom(d^{A\times B},d)} \to d^A \to d$.

This map, on the component $j : d^{A\times B} \to d$ is given by j (cf. observation 1.5). In particular, $d^{A\times B} \to d^A$, on the component of a, is given by $d^{A\times B} \xrightarrow{\mathrm{pr}_a} d^B \xrightarrow{g} d$.

In other words, $d^{A\times B} \to d^A$ is simply $g^A : (d^B)^A \to d^A$, and so, the full composite $d^{A\times B} \to d$ that we get is $f \circ g^A$.

In a completely symmetric way, one computes that the other morphism from proposition 2.11 sends (f, g) to $g \circ f^B$.

So by proposition 2.11, $\mathcal{E}\mathrm{nd}_{\mathbf{D}}(d)$ is commutative if and only if for all f, g as above, $f \circ g^A = g \circ f^B$.

We prove this for $\mathbf{D} = \mathrm{Fun}(\mathbf{C}, \mathbf{C}), d = \mathrm{id}_C$. The first thing we need to prove this is to show that for an arbitrary $g : \mathrm{id}^B \to \mathrm{id}$, and an arbitrary set A, $(g_c)^A = g_{c^A} : c^{B\times A} \to c^A$. Because the target is c^A, it suffices to prove that they agree after projecting onto the a-component for each $a \in A$, but this projection is a morphism in $\mathbf{C}$, so by naturality of g we have a commutative diagram :

$$\begin{tikzcd}
(c^A)^B \arrow[r, "g_{c^A}"] \arrow[d, "\mathrm{pr}_a^B"'] & c^A \arrow[d, "\mathrm{pr}_a"] \\
c^B \arrow[r, "g_c"] & c
\end{tikzcd}$$

which shows that $\mathrm{pr}_a \circ g_{c^A} = \mathrm{pr}_a \circ (g_c)^A$.

In particular, if we now use naturality of f, we find that the following diagram commutes for all c :

$$\begin{tikzcd} c^{A\times B} \arrow[r, "(g_c)^A"] \arrow[d, "f_{c^B}"'] & c^A \arrow[d, "f_c"] \\ c^B \arrow[r, "g_c"] & c \end{tikzcd}$$

and so by the previous computation applied to f, the following diagram also commutes:

$$\begin{tikzcd} c^{A\times B} \arrow[r, "(g_c)^A"] \arrow[d, "(f_c)^B"'] & c^A \arrow[d, "f_c"] \\ c^B \arrow[r, "g_c"] & c \end{tikzcd}$$

which was exactly the claim that $f \circ g^A = g \circ f^B$, and so $\mathcal{Z}(\mathbf{C})$ is commutative. $\square$

Remark 2.16. In general, the endomorphism monoid of the unit of an arbitrary monoidal category is commutative. Here, by suitably embedding $\mathbf{C}$ in $\mathrm{Fun}(\mathbf{C}, \mathbf{C})$, we find that if tensor products in $\mathbf{C}$ commute with arbitrary powers of the unit (which is a *strong* condition), the endomorphism monad of the unit is commutative.

However, this is wrong in general. Indeed, consider for instance a finite field k, and a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then $f = \lim_{\mathcal{U}} : k^{\mathbb{N}} \to k$ (where we give k the discrete topology) is k-linear, however I claim that f does not "commute with itself" (note that if we use the notations from above, this looks like $f \circ f^{\mathbb{N}} \neq f \circ f^{\mathbb{N}}$, but this is because the notation is slightly abusive : they are "different n's").

Indeed, consider the sequence $x_{n,m} = 1$ if $n > m$, 0 else. Then if we take $\lim_{\mathcal{U}}$ with respect to m first, and then n, we find 0, whereas if we take it with respect to n first, and then m, we find 1.

So, to sum up, we proved:

Theorem 2.17. *Given an arbitrary category with products $\mathbf{C}$ and $c \in \mathbf{C}$, there is an endomorphism monad $\mathcal{E}\mathrm{nd}_{\mathbf{C}}(c)$ which enhances the endomorphism monoid of c.*

If we apply this to $\mathrm{id}_{\mathbf{C}} \in \mathrm{Fun}(\mathbf{C}, \mathbf{C})$*, we get a* commutative monad $\mathcal{Z}(\mathbf{C})$*, the center monad of* $\mathbf{C}$.

If $\mathbf{C} = \mathbf{Alg}_T$ is the category of algebras for a given monad T, there is a canonical injective morphism $\mathcal{Z}(\mathbf{Alg}_T) \to T$, which is an isomorphism if and only if T is commutative.

Something interesting is that this construction works for arbitrary monads T, so we always get a "submonad" $\mathcal{Z}(T) \to T$.

Question 2.18. Can we compute this monad in interesting examples ?

Does it have a universal property ?

Remark 2.19. In the case of the free R-module or the free M-set monads, we can compute it as the free $Z(R)$-module or the free $Z(M)$-set monads.

In the case of the ultrafilter monad, one can show that the center is just the trivial monad $X \mapsto X$.

Remark 2.20. If T comes from a Lawvere theory, we can restrict $\mathcal{Z}(T)$ to finite sets to also get a Lawvere theory which we could call the center of the Lawvere theory. It is not hard to check that $\mathcal{Z}(T)$ comes from this Lawvere theory: an X-ary operation always come from a Y-ary operation for some finite Y, and so the X-ary operation is a map of T-algebras if and only if the finitary one is, so $\mathcal{Z}(T)$ is finitary.

3 Closing remarks

We essentially answered the initial question, but some important questions remain.

For instance, while the endomorphism monoid of an object is a conceptually clear object, one can not say the same of the endomorphism monad. Is it the endomorphism object of c (or some related object) for some action of $\mathrm{Fun}(\mathbf{Set}, \mathbf{Set})$ on $\mathbf{C}$ (or some related category) ?

Can the proof that $\mathcal{Z}(\mathbf{C})$ is commutative be made more conceptual than the calculation that it currently is ? A lot of the statements that we have here cannot directly be adapted to $(\infty, 1)$-categories, it would be nice to understand what is going on.

For instance, it seems relatively reasonable to expect the construction of $\mathcal{E}\mathrm{nd}_{\mathbf{C}}(c)$ to carry over to $(\infty, 1)$-categories which have arbitrary powers by spaces, but what does the "commutativity" of $\mathcal{Z}(\mathbf{C})$ become in this setting ?

Certainly, finding a universal property should help in answering these questions; but also getting a clearer conceptual understanding of what "commutative monad" should mean.

Note that whatever it means in the $(\infty, 1)$-categorical setting, we should not expect it to imply that T is equivalent to its center : if R is an E_∞-ring spectrum, $\mathrm{End}_{\mathrm{Fun}^L(\mathbf{Mod}_R, \mathbf{Mod}_R)}(\mathrm{id})$ is not equivalent to R in general, it is what's known has (topological) Hochschild cohomology.

References

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