Deducing the Bousfield-Kan formula for homotopy (co)limits from first principles

Introduction

Some silly notes about understanding the Bousfield-Kan formula for homotopy colimits (specifically the appearance of the simplicial object $[n] \mapsto \coprod_{i_0 \to \ldots \to i_n} F(i_0)$) from first principles of ∞-category theory. Somehow related to "synthetic" higher catgeory theory.

I will only be dealing with the case where the indexing category I is a 1-category, although the same thing should work for an ∞-category.

1 The main proof

Consider the nerve of I, $N(I) : \Delta^{op} \to \mathbf{Set}$. It is defined by $N(I)_n = \hom([n], I)$, where $[n] = \{0 < ... < n\}$ is the partially ordered set viewed as a category.

Then the Grothendieck construction yields a category $\int I$ described as follows: its objects are pairs $([n], x)$ with $x \in N(I)_n$, and a morphism $([n], x) \to ([m], y)$ is an arrow $f : [n] \to [m]$ in Δ^{op} (that is, a nondecreasing map $[m] \to [n]$) such that $f^*x = y$ (here, f^* is to be taken in the sense of $N(I)$).

This of course comes equipped with a functor $\pi : \int I \to \Delta^{op}$ sending $([n], x) \to [n]$. Less evidently, it also comes with a functor to I, which we now describe.

Define $\eta : \int I \to I$ as follows: on objects, $\eta([n], x) = x_0$. Note that $x \in N(I)_n = \hom([n], I)$, so we can identify x with a sequence $x_0 \to ... \to x_n$ of morphisms in I. This is what we mean by x_0.

On arrows, if $g : ([n], x) \to ([m], y)$ is defined by $f : [n] \to [m]$, then note that the condition $f^*x = y$ means that $y_0 \to ... \to y_m$ is exactly $x_{f(0)} \to ...x_{f(m)}$, and in particular, because $f(0) \geq 0$, we get an arrow $x_0 \to x_{f(0)} = y_0$ for free : this is $\eta(g)$.

Remark 1.1. This construction is the one that is less simple if we take I to be an ∞-category. It is, however, a good exercise to try and work it out in the most general case. We leave that to the interested reader.

The proof of the Bousfield-Kan formula now relies on two facts: the functor $\int I \to I$ is (homotopy) cofinal, and left Kan extension along $\int I \to \Delta^{op}$ produces the correct formula. Let's look at that situation diagrammatically. We start with a functor $I \to \mathbf{C}$, where $\mathbf{C}$ is an ∞-category with arbitrary colimits. This produces the following diagram:

$$\begin{tikzcd}
\int I \arrow[r] \arrow[d] & I \arrow[r, "F"] & \mathbf{C} \\
\Delta^{op} & &
\end{tikzcd}$$

Now, I claim that $\int I \to I$ is cofinal, which implies that I can compute $\mathrm{colim}_I F$ as $\mathrm{colim}_{\int I}(F \circ \eta)$.

Moreover, note that a colimit is just a left Kan extension along $\int I \to \bullet$, where $\bullet$ is the terminal category. But of course $\int I \to \bullet$ factors through Δ^{op}, and left Kan extensions compose. In particular, we can first left Kan extend along $\int I \to \Delta^{op}$, and then take the colimit:

$$\begin{tikzcd}
\int I \arrow[r] \arrow[d] & I \arrow[r] & \mathbf{C} \\
\Delta^{op} \arrow[urr] \arrow[d] & & \\
\bullet \arrow[uurr] & &
\end{tikzcd}$$

$\mathrm{colim}_I F$ can therefore be computed as $\mathrm{colim}_{\Delta^{op}}(\mathrm{Lan}_\pi(F \circ \eta))$. The second claim is then that the functor $\mathrm{Lan}_\pi(F \circ \eta)$ can be described as $[n] \mapsto \coprod_{i_0 \to \ldots \to i_n} F(i_0)$.

We now simply have to prove these two claims. They both follow from general categorical considerations, and we want to explain them to some extent.

2 A first detour: the formula for Kan extensions

We want to prove the following claim:

Proposition 2.1. *Let $i : \mathbf{C} \to \mathbf{D}$ be a functor of ∞-categories, and $F : \mathbf{C} \to \mathbf{E}$ another functor. Assume $\mathbf{E}$ has all colimits.*

Then $\mathrm{Lan}_i F$ exists, and there is a canonical equivalence

$$\mathrm{Lan}_i F(d) \simeq \mathrm{colim}_{(i(c) \to d) \in C_{/d}} F(c)$$

where $C_{/d}$ is defined as the pullback $C \times_D D_{/d}$

Remark 2.2. We could be more precise and require only that $\mathbf{E}$ have the necessary colimits for this to make sense. We are not interested in such subtleties for this document, an argument for cocomplete $\mathbf{E}$ is largely sufficient for our purposes.

Here's a possible approach to a proof from "first principles" - please note that this will not be a complete proof; it should, however give some idea of what is happening.

We first prove it in the case where $\mathbf{E} = \mathrm{Fun}(A, \mathbf{S})$ is a presheaf category on some small category A; where $\mathbf{S}$ is the ∞-category of spaces.

Since in such presheaf categories colimits are computed pointwise and so are Kan extensions, it suffices to deal with the case $\mathbf{E} = \mathbf{S}$.

Remark 2.3. The statement "colimits are computed pointwise" follows from the following fact which should hold for any reasonable theory of ∞-categories: if $A \leftrightarrows B$ is an adjunction, then for any V, so is $\mathrm{Fun}(V, A) \leftrightarrows \mathrm{Fun}(V, B)$; as well as $\mathrm{Fun}(A \times B, V) \simeq \mathrm{Fun}(A, \mathrm{Fun}(B, V))$ for arbitrary ∞-categories A, B, V - the latter can actually be proved from even "more primordial" principles, depending on how you define Fun.

The same can be said for Kan extensions.

In the case where $\mathbf{E} = \mathbf{S}$, we first deal with the case of representable functors $F = \mathrm{map}(c, -)$ for some $c \in \mathbf{C}$.

Then by adjunction, and the Yoneda lemma, we find

$$\mathrm{map}(\mathrm{Lan}_i \, \mathrm{map}(c, -), G) \simeq \mathrm{map}(\mathrm{map}(c, -), G \circ i) \simeq G(i(c)) \simeq \mathrm{map}(\mathrm{map}(i(c), G))$$

naturally in G and c, from which the Yoneda lemma implies $\mathrm{Lan}_i \, \mathrm{map}(c, -) \simeq \mathrm{map}(i(c), -)$, naturally in $c \in C$.

We take the Yoneda lemma as granted as well, for obvious reasons.

Now, note that by another version of the Yoneda lemma, which we also take for granted, for any copresheaf $F : \mathbf{C} \to \mathbf{S}$, we have a natural equivalence $\mathrm{colim}_{(y(c)\to F)\in \mathbf{C}_{/F}} y(c) \simeq F$, where $y : \mathbf{C} \to \mathrm{Fun}(\mathbf{C}, \mathbf{S})$ denotes the Yoneda embedding.

From this, and because Lan_i is a left adjoint, we find

$$\mathrm{Lan}_i F(d) \simeq \mathrm{colim}_{(y(c)\to F)\in \mathbf{C}_{/F}} \mathrm{Lan}_i \, \mathrm{map}(c, -)(d) \simeq \mathrm{colim}_{(y(c)\to F)\in \mathbf{C}_{/F}} \mathrm{map}(i(c), d)$$

But now, note that $\mathrm{map}(i(c), d) \simeq \mathrm{colim}_{\mathrm{map}(i(c),d)} *$ and so $\mathrm{Lan}_i F(d)$ is the colimit of the constant functor equal to a point on the following category: ${}_{d\backslash}\mathbf{C}_{/F}$. We can now "commute" the colimits, and we get that it is $\mathrm{colim}_{(i(c)\to d)\in \mathbf{C}_{/d}} \mathrm{colim}_{y(c)\to F} *$

But by the Yoneda lemma, $\mathrm{map}(y(c), F) \simeq F(c)$, so this is exactly $\mathrm{colim}_{(i(c)\to d)\in \mathbf{C}_{/d}} F(c)$, i.e. the formula that was claimed.

We get the claim for a general $\mathbf{E}$ from the following principle, which is again expected for a reasonable theory of ∞-categories:

Theorem 2.4. *Let A be a small category and $\mathbf{E}$ a cocomplete category. Then restriction along the Yoneda embedding $A \to \mathrm{Fun}(A^{op}, \mathbf{S})$ induces an equivalence*

$$\mathrm{Fun}^L(\mathrm{Fun}(A^{op}, \mathbf{S}), \mathbf{E}) \to \mathrm{Fun}(A, \mathbf{E})$$

where Fun^L *is the subcategory of* Fun *on those functors that preserve colimits, which is equivalently the subcategory of* Fun *on those functors that are left adjoints.*

From this, observe that our initial $A \to \mathbf{E}$ gives rise to a left adjoint $\mathrm{Fun}(A^{op}, \mathbf{S}) \to \mathbf{E}$.

In particular, if we left Kan extend the Yoneda embedding $\mathbf{C} \to \mathrm{Fun}(\mathbf{C}^{op}, \mathbf{S})$ along $i : \mathbf{C} \to \mathbf{D}$, we get the above formula. But both sides of the formula are preserved by our left adjoint $\mathrm{Fun}(\mathbf{C}^{op}, \mathbf{S}) \to \mathbf{E}$ (for the RHS, this comes from preservation of colimits by left adjoints, and for the LHS, this follows from the "pointwise adjunction" property of functor categories that we mentioned earlier), and so the formula is correct there too.

Note that by dualizaing the main result we get a similar formula for right Kan extensions when $\mathbf{E}$ is complete. This will be useful in the next section (what's remarkable is that this formula will be useful three times in this note, for two different reasons).

3 A second detour: Quillen's theorem A

This section is devoted to proving the following result, which is a weakening (along the same lines as before) of Quillen's theorem A:

Theorem 3.1. *Let $f : \mathbf{C} \to \mathbf{D}$ be a functor of ∞-categories. If for all $d \in \mathbf{D}$, $\mathbf{C}_{d/} = \mathbf{C} \times_{\mathbf{D}} \mathbf{D}_{d/}$ is weakly contractible, then f is cofinal; i.e. precomposing by f doesn't change the value or existence of a colimit.*

Remark 3.2. We will not prove such a strong statement. We will assume the target is cocomplete and prove the result in that situation.

Remark 3.3. In the 1-categorical situation, we require $\mathbf{C} \times_{\mathbf{D}} \mathbf{D}_{d/}$ to be *weakly connected* rather than weakly contractible. The reason why this is the case will be clear from the proof.

Remark 3.4. Note that $\mathbf{C}_{d/}$ appears in this statement, and not the $\mathbf{C}_{/d}$ from left Kan extensions.

Why we have $d/$ is clear from the intuition of what cofinality means and how this condition works; but this suggests that we might be dealing with right Kan extensions rather than left ones. This is perhaps surprising as cofinality is a statement about colimits, but again this will be clear from the proof.

Let us rephrase our cofinality statement: for any category $\mathbf{E}$, we have a restriction functor $\mathrm{Fun}(\mathbf{D}, \mathbf{E}) \to \mathrm{Fun}(\mathbf{C}, \mathbf{E})$. If $\mathbf{E}$ is cocomplete, we can compose both of them with the colimit functor and ask whether this diagram commutes:

$$\begin{tikzcd} \mathrm{Fun}(\mathbf{D},\mathbf{E}) \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\mathbf{C},\mathbf{E}) \arrow[dl] \\ & \mathbf{E} & \end{tikzcd}$$

By the same trick as in the previous section, we can reduce to the situation where $\mathbf{E} = \mathrm{Fun}(A, \mathbf{S})$ for some small A, and so in particular, we may assume $\mathbf{E}$ is bicomplete. This is actually all we will need.

Since E is complete, all the arrows in that diagram have right adjoints, and so to check whether the diagram commutes, one may check whether the diagram of right adjoints does:

$$\begin{tikzcd} \mathrm{Fun}(\mathbf{D},\mathbf{E}) & & \mathrm{Fun}(\mathbf{C},\mathbf{E}) \arrow[ll] \\ & \mathbf{E} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where the horizontal arrow is now *right* Kan extension along $\mathbf{C} \to \mathbf{D}$ and the vertical arrows are both the functor sending an object e to the constant functor at e, which we will denote by $\Delta(e)$.

But now note the following : for $e \in \mathbf{E}$ and $d \in \mathbf{D}$,

$$\mathrm{Ran}_f \Delta(e)(d) \simeq \lim_{(d \to f(c)) \in \mathbf{C}_{d/}} \Delta(e)$$

by the (dual to the) previous section.

Now we have:

Lemma 3.5. *Let $\mathbf{E}$ be a complete category, and X a weakly contractible category. For any $e \in \mathbf{E}$, the constant functor $X \to \mathbf{E}$ with value e has limit e.*

Remark 3.6. In 1-category land, this works whenever X is weakly connected: one can indeed easily write down a proof "by hands" that it does.

There is in fact a more general statement: for any category X and any constant functor $X \to \mathbf{S}$, with value Y, the limit of that functor is $\mathrm{map}(|X|, Y)$ (where $|X|$ is the homotopy type of X); and so in particular, by the Yoneda lemma, if $\mathbf{E}$ is complete, the limit of a constant functor at e is $e^{|X|}$ (where exponentiation here denotes the "cotensorisation" of e by $|X|$).

To prove the lemma it suffices to do so when $\mathbf{E} = \mathbf{S}$ by the Yoneda lemma. Notice the following: viewing $|X|$ as a category (an ∞-groupoid), we have a functor $X \to |X|$ with the following property, by definition of $|X|$:

Lemma 3.7. *For any $\mathbf{E}$, precomposition along $X \to |X|$:*

$$\mathrm{Fun}(|X|, \mathbf{E}) \to \mathrm{Fun}(X, \mathbf{E})$$

is a fully faithful embedding, with essential image those functors $X \to \mathbf{E}$ which send every edge in X to an invertible one.

In particular, if $|X|$ is contractible, evaluation at any point induces an equivalence $\mathrm{Fun}(|X|, \mathbf{E}) \to \mathbf{E}$, which is of course inverse to the diagonal functor $\mathbf{E} \to \mathrm{Fun}(|X|, \mathbf{E})$.

It follows that the diagonal functor $\mathbf{E} \to \mathrm{Fun}(X, \mathbf{E})$, which factors as $\mathbf{E} \to \mathrm{Fun}(|X|, \mathbf{E}) \to \mathrm{Fun}(X, \mathbf{E})$, is a fully faithful functor. In particular, its unit transformation

$$e \to \lim_X e$$

is an equivalence, which is what was claimed.

From this it follows at once that the canonical map $\Delta(e) \to \mathrm{Ran}_f \Delta(e)$ is an equivalence, by the assumption that $\mathbf{C}_{d/}$ is contractible. It follows that the diagram of right adjoints commutes, and thus so does the diagram of left adjoints: colimits on $\mathbf{D}$ can be computed by first restricting to $\mathbf{C}$, this Quillen's theorem A.

We note an important corollary:

Corollary 3.8. *Let $\mathbf{C}$ be a category with a terminal object c. Then for any functor $F : \mathbf{C} \to \mathbf{E}$, the following canonical map is an equivalence:*

$$F(c) \xrightarrow{\simeq} \mathrm{colim}_{\mathbf{C}} F$$

Proof. It suffices to prove that the functor $\bullet \to \mathbf{C}$ with value c is cofinal; by Quillen's theorem A it suffices to prove that for each $d \in \mathbf{C}$, $\bullet_{d/}$ is weakly contractible.

But that is precisely $\{c\} \times_{\mathbf{C}} \mathbf{C}_{d/} \simeq \mathrm{map}(d, c)$, so it is contractible by definition of terminal object. $\square$

More generally, suppose we can write $\mathbf{C}$ as a disjoint union $\coprod_i \mathbf{C}_i$, with each $\mathbf{C}_i$ having a terminal object c_i. Then there is a cofinal functor $I \to \mathbf{C}$ (where I is a discrete set), so we can compute colimits over $\mathbf{C}$ as coproducts : $\mathrm{colim}_{\mathbf{C}} F \simeq \coprod_i F(c_i)$.

4 Conclusion

We now have at our disposal everything we need to get the desired formula.

Proposition 4.1. *The functor $\eta : \int I \to I$ is cofinal.*

Proof. By Quillen's theorem A, it suffices to consider $\int I_{i/}$ for each $i \in I$.

Let's have a look at what that category looks like: an object is a pair $([n], x), x \in N(I)_n$ together with an arrow $i \to x_0$.

But if we have an arrow $i \to x_0$, we can form an $n+1$-simplex $i \cup x := i \to x_0 \to ... \to x_n$ and then the inclusion as the last elements $[n] \to [n+1]$ induces a morphism in $\int I$: $([n+1], i \cup x) \to ([n], x)$.

It is easy to check that $([n], x) \mapsto ([n+1], i \cup x)$ defines a functor $\int I_{i/} \to \int I_{i/}$ and that $([n+1], i \cup x) \to ([n], x)$ then defines a natural transformation between this functor and the identity.

But now there is also a natural transformation $([n+1], i \cup x) \to ([0], i)$, where $([0], i)$ is just the constant functor (this arrow is just induced by the inclusion at 0: $[0] \to [n+1]$).

So we have created a zigzag of natural transformations $id \leftarrow F \to *$ where $*$ is a constant functor.

It follows that $\int I_{i/}$ is weakly contractible, and by Quillen's theorem A, $\int I \to I$ is cofinal. □

Remark 4.2. Let X, Y be categories, and $F \leftarrow G \to H$ a zigzag of natural transformations of functors $X \to Y$. Then they can be seen as morphisms in $\mathrm{Fun}(X, Y)$ and so they induce morphisms in $\mathrm{Fun}(X, |Y|)$. But now all edges in X are sent to equivalences, so they induce morphisms in $\mathrm{Fun}(|X|, |Y|)$. Since every arrow in $|Y|$ is invertible, we can see the zigzag as just an equivalence $F \to H$.

Apply this to $X = Y$ and $F = id_X, H = *$ a constant functor, then if such a zigzag exists, $id_{|X|}$ is equivalent to a constant functor: in particular, $|X|$ is contractible, so X is weakly contractible. So in any category, if we can build a zigzag of natural transformations between id_X and a constant functor, then X is weakly contractible.

We are halfway done, it now suffices to compute the left Kan extension of $F\eta$ along $\pi : \int I \to \Delta^{op}$.

For this, let $[n] \in \Delta^{op}$ and consider $\int I_{/[n]}$. This is a category whose objects are pairs $([m], x)$ together with a morphism $[m] \to [n]$ (in Δ^{op}).

Moreover, note that a morphism between two such objects is a morphism $[m_0] \to [m_1]$ over $[n]$ which does the correct thing on the second components. But now, if such a morphism exists, then the image of x_0 under f^* (where $f : [m_0] \to [n]$) must be the same as the image of x_1 under g^* (where $g : [m_1] \to [n]$).

In particular a given object of this category $(([m], x), f : [m] \to [n])$ defines an element $f^*x \in N(I)_n$, and any other object which is related to it by a morphism defines the same. In particular, we get a decomposition of our category as a coproduct $\coprod_{x \in N(I)_n} C_x$ where C_x is the subcategory of those $(([m], y), f)$ such that $f^*y = x$.

But now it should be clear that $(([n], x), id_{[n]})$ is terminal in C_x (if it is not clear, it is left as an exercise).

It follows that $\mathrm{Lan}_\pi(F \circ \eta)([n]) \simeq \mathrm{colim}_{(([m],x),f) \in \int I_{/[n]}} F(x_0)$ can be computed as a coproduct $\coprod_{x \in N(I)_n} F(x_0)$.

But now by definition $N(I)_n$ is the set of sequences of composable morphisms $x_0 \to ... \to x_n$, and so we do get the following description: $\coprod_{x_0 \to ... \to x_n} F(x_0)$. If you follow carefully the proof, you can also see that the morphisms induced by $[n] \to [m]$ are indeed the correct ones too.

We can therefore conclude:

Theorem 4.3. *Let $F : I \to \mathbf{C}$ be a functor from I to a cocomplete category $\mathbf{C}$. Then* $\mathrm{colim}_I F$ *is naturally equivalent to the colimit along Δ^{op} of*

$$[n] \mapsto \coprod_{i_0 \to ... \to i_n} F(i_0)$$

Before finishing this note, let us see what we used to get here.

- we used the fact that any 1-category could be seen as an ∞-category, and that the constructions we did in 1-category land (comma categories, functors) carried over to ∞-categories.
- we used properties of presheaf categories (bicompleteness of $\mathrm{Fun}(A, \mathbf{S})$ when A is small, and the fact that it is freely generated by A^{op} under colimits)
- some reasonable properties of adjunctions (that they work pointwise in functor categories, that they give the usual equivalences on mapping spaces and come with units and co-units that do the correct thing)
- We extensively used the Yoneda lemma (the usual one as well as the so-called "ninja" Yoneda lemma which expresses every presheaf as a canonical colimit of representables - of course this is related to the free cocompletion theorem)
- Invariance under equivalence of *everything*

We probably used some other stuff as well, but all in all, only very reasonable things. In particular, what I'm trying to show with this list is that these results that we used don't really rely on Lurie/Joyal's theory of quasicategories.

Of course, this theory is a good place for these results (where they have actually been *proved*); but there are ways in which it is unsatisfactory. Specifically, it is just a model for a theory of $(\infty, 1)$-categories; but any good model should have the aforementioned results. In particular, the proofs in this note should carry over to any reasonable model with minimal change (once the basic theory has been set up in said model), or at least this is the hope I have in writing this.

The *best* model would of course be a model where I can write such proofs verbatim and not need to say "this can easily be turned into a proof".

The hope is that the more we move forward in higher category land, the more people write model-independent proofs such as this one, so that we get a proper model independent foundation for most of our results. That way, if (when ?) some day we decide to change models because a better one has been proposed, we don't have to reprove everything that was proved "simplex by simplex".

Mathematics is robust, as it relies on ideas rather than particular technical questions - in that sense, the above should be doable.