On Bass' trace conjecture

Introduction

Bass' trace conjecture is a conjecture about the behaviour of the trace map from algebraic K-theory to Hochschild homology (HH) for integral group rings.

It is a generalization of a well-known result of Swan's :

Theorem 0.1. *Let G be a finite group, and P a finitely generated projective $\mathbb{Z}[G]$-module. In this case, $\mathbb{Q} \otimes_{\mathbb{Z}} P$ is a free $\mathbb{Q}[G]$-module.*

This is not true for more general groups, but Bass came up with a character theoretic analogue that is more likely to hold - this is stated in terms of the trace map from algebraic K-theory to Hochschild homology.

For any group G, one has a description of $\mathrm{HH}_0(\mathbb{Z}[G])$: it is a free abelian group spanned by conjugacy classes of G (more generally, $\mathrm{HH}(\mathbb{Z}[G]) = \mathbb{Z} \otimes \mathrm{THH}(\mathbb{S}[G]) \simeq \mathbb{Z} \otimes \Sigma^\infty_+ LBG$, where L denotes the free loop space).

The trace map works in the following, concrete way:

Construction 0.2. Let P be a finitely generated projective left module over a ring R. By definition, $P \cong R^n e$ for some idempotent matrix $e \in M_n(R)$. The Hattori-Stallings trace of P is the image of $\sum_i e_{ii}$ in the quotient $R/[R,R] = \mathrm{HH}_0(R)$.

It can be shown that this does not depend on the choice of e, and only on P. Furthermore, the Hattori-Stallings trace of $P \oplus Q$ is the sum of that of P and of Q, and so it factors as a morphism $K_0(R) \to \mathrm{HH}_0(R)$. We call this the trace map for R.

Corollary 0.3. *If P is a free $\mathbb{Z}[G]$-module, then e can be chosen to be* id *and so the Hattori-Stallings trace of P is the image of n in* HH0(R) *for some integer $n \in \mathbb{N}$.*

Conjecture 0.4 (Bass' trace conjecture)**.** For any group G, the trace map

$$K_0(\mathbb{Z}[G]) \to \mathrm{HH}_0(\mathbb{Z}[G]) = \mathbb{Z}[G/\mathrm{conj}]$$

lands entirely in the summand of the trivial conjugacy class.

Swan's result, the previous corollary and the fact that $\mathbb{Z} \to \mathbb{Q}$ is injective imply altogether that the conjecture holds for finite groups, and the conjecture can be seen as an analogue of Swan's result for infinite groups.

The goal of this note is to use trace methods to recover a result of Linnell (and later Berrick-Hesselholt, using slightly different trace methods), namely the following:

Theorem 0.5 (Linnell)**.** *Let G be a group and g be an element of G such that the summand corresponding to g's conjugacy class is hit by the trace map $K_0(\mathbb{Z}G) \to \mathrm{HH}_0(\mathbb{Z}G)$.*

There exists an integer $m \geq 1$ such that for any integer $s \geq 1$, g is conjugate to g^{s^m}.

Remark 0.6. This of course recovers the claim that the conjecture holds for finite groups.

It also constrains a lot the groups that can be counter-examples to the conjecture. Berrick-Hesselholt go a bit further in their analysis and show in particular (although they still show more) that if G does not contain a subgroup isomorphic to $\mathbb{Q}$, then it satisfies Bass' trace conjecture.

1 TC of spherical group rings

Berrick-Hesselholt use trace methods to derive Linnell's constraint, but they use TR, topological restriction homology.

We will use TC, topological cyclic homology instead, and one of the first computations relating to it, namely the computation for spherical group rings. To explain why this helps, let us first point out:

Lemma 1.1. *The maps $K(\mathbb{S}[G]) \to K(\mathbb{Z}[G])$ and $\mathrm{THH}(\mathbb{S}[G]) \to \mathrm{HH}(\mathbb{Z}[G])$ are isomorphisms on π_0, and they are compatible with the trace map.*

In particular we can use the spherical group ring $\mathbb{S}[G]$ instead of the group ring $\mathbb{Z}[G]$. This is what we will do, because we have a reasonable description of TC for the former.

Topological cyclic homology is an invariant built out of THH, which is somewhat closer to K-theory than THH is. In particular, the trace map $K \to \mathrm{THH}$ factors through the *cyclotomic trace* $K \to \mathrm{TC} \to \mathrm{THH}$.

The computation we will need is the following p-adic computation - in which we will take $X = BG$:

Theorem 1.2 (Bökstedt-Hsiang-Madsen)**.** *Let X be a space, and let LX denote its free loop space. Recall that $\mathrm{THH}(\mathbb{S}[\Omega X]) \simeq \Sigma^\infty_+ LX$; after p-completion, there is a pullback square :*

$$\begin{array}{ccc} \mathrm{TC}(\mathbb{S}[\Omega X]) & \longrightarrow & \Sigma(\Sigma^\infty_+ LX)_{hS^1} \\ \downarrow & & \downarrow \\ \Sigma^\infty_+ LX & \xrightarrow{1-\tilde{\varphi}_p} & \Sigma^\infty_+ LX \end{array}$$

The proof of this will not be relevant (although it is quite nice !) for our proof, and most of the details of this diagram are irrelevant too, what we need is :

1. The left vertical map is the canonical map $\mathrm{TC} \to \mathrm{THH}$ after the identification $\mathrm{THH}(\mathbb{S}[\Omega X]) \simeq \Sigma^\infty_+ LX$
2. $\tilde{\varphi}_p : LX \to LX$ is given by precomposing with the p-sheeted covering $S^1 \to S^1$.
3. $(-)_{hS^1}$ is taken with respect to some S^1-action on THH

The third point does not really matter, it was simply to indicate that there is an S^1-action and the diagram makes sense.

In fact, for our proof, it does not even matter that this is a pullback square, what really matters is that there is a commutative square of this form.

Corollary 1.3. *On π_0, the bottom map in the above square sends the conjugacy class of the loop $g : S^1 \to X$ to the conjugacy class of g^p .*

2 The proof

We make two reductions:

Lemma 2.1. *Let g be as in the main theorem. To conclude, it suffices to prove that there is an integer m such that for any prime p, g is conjugate to g^{p^m}.*

Proof. Suppose that this is the case. Because g is conjugate to g^{p^m}, then the same condition applies to g^{p^m}, and so for any prime q, g^{p^m} is conjugate to $g^{(pq)^m}$.

By induction on the prime factorization of s, it is now clear that for any integer s, g is conjugate to g^{s^m}. □

The second reduction is obvious from the factorization of the trace map as $K \to \mathrm{TC} \to \mathrm{THH}$:

Lemma 2.2. *It suffices to prove the statement for the image of* $\mathrm{TC}_0(\mathbb{S}[G]) \to \mathrm{THH}_0(\mathbb{S}[G])$.

Proof of the main theorem. Following the second reduction, fix $x \in \mathrm{TC}_0(\mathbb{S}[G])$ getting sent to $\sum_{i\in I} n_i[g_i]$ under the map $\mathrm{TC}_0(\mathbb{S}[G]) \to \mathrm{THH}_0(\mathbb{S}[G]) \cong \mathbb{Z}[G/\mathrm{conj}]$.

Note that this is independent of any prime p, and in particular $|I|$ is independent of the prime p, which we now fix, following the first reduction.

Because $\mathrm{HH}_0(\mathbb{Z}[G])$ is a free abelian group, the map to its p-completion is injective. We may therefore work after p-completion and apply the Bökstedt-Hsiang-Madsen computation.

With $X = BG$, this in particular says that there is a commutative diagram of the form :

$$\begin{array}{ccc} \mathrm{TC}(\mathbb{S}[G]) & \longrightarrow & \Sigma(\Sigma^\infty_+ LBG)_{hS^1} \\ \downarrow & & \downarrow \\ \Sigma^\infty_+ LBG & \xrightarrow{1-\tilde{\varphi}_p} & \Sigma^\infty_+ LBG \end{array}$$

where on π_0, the bottom map sends the conjugacy class of g to that of g^p. The main observation is that the term in the top right hand corner is 1-connective, i.e. it has no π_0. In particular, on π_0 this diagram looks like

$$\begin{array}{ccc} \mathrm{TC}_0(\mathbb{S}[G]) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{Z}[G/\mathrm{conj}] & \xrightarrow{1-(-)^p} & \mathbb{Z}[G/\mathrm{conj}] \end{array}$$

So if $x \in \mathrm{TC}_0(\mathbb{S}[G])$ gets sent to $\sum_i n_i[g_i]$, we find that $\sum_i n_i[g_i] = \sum_i n_i[g_i^p]$.

We let $I_+ := \{i \in I \mid n_i > 0\}, I_- := \{i \in I \mid n_i < 0\}$ (and we assume the conjugacy classes $[g_i]$ are pairwise distinct). Our equation now reads

$$\sum_{i \in I_+} n_i [g_i] + \sum_{i \in I_-} |n_i| [g_i^p] = \sum_{i \in I_-} |n_i| [g_i] + \sum_{i \in I_+} n_i [g_i^p]$$

The $[g_i]$'s appearing on the left hand side must appear the same number of times on the right hand side, and so they must appear the same number of times in the $\sum_{i \in I_+} n_i [g_i^p]$ part, because for $i \in I_+, j \in I_-, [g_i] \neq [g_j]$. In particular for each i there must be some j with $[g_i] = [g_j^p]$. Because for $i \neq i', [g_i] \neq [g_{i'}]$, the j's in question must also be distinct.

In other words, $I_+ \to I_+, i \mapsto$ a j such that $[g_j^p] = [g_i]$ is injective, and therefore bijective.

We reason similarly with I_-, so that at the end of the day, for each $i \in I$, there is a unique j such that $[g_i] = [g_j^p]$, and therefore for each j there is a unique i such that $[g_j^p] = [g_i]$. In other words, $[g] \mapsto [g^p]$ restricts to a permutation of $\{[g_i], i \in I\}$. This set has size $|I|$ (independent of the prime p), so with $m = |I|!$ we find that for any $i \in I, [g_i^{p^m}] = [g_i]$. This is what we wanted to prove. □

Remark 2.3. Lars Hesselholt communicated to me a very nice proof that the conjecture holds for abelian groups (which of course also follows from the result mentioned here): the observation is that HH_0 is a Zariski sheaf (say when restricted to $\mathrm{Spec}(R)$ for some commutative ring R), and that the Zariski sheafification of K_0 is the constant sheaf with value $\mathbb{Z}$ (because projective modules are locally free).

In particular, on global sections, the trace map $K_0 \to \mathrm{HH}_0$ factors through $K_0(R) \to H^0(\mathrm{Spec}(R); \mathbb{Z}) \to \mathrm{HH}_0(R)$. If $\mathrm{Spec}(R)$ is connected, then $H^0(\mathrm{Spec}(R); \mathbb{Z}) \cong \mathbb{Z}$, spanned by the constant 1-valued function on $\mathrm{Spec}(R)$, which is of course the value of $[R] \in K_0(R) \to H^0(\mathrm{Spec}(R); \mathbb{Z})$, and therefore gets sent in $\mathrm{HH}_0(R) \cong R$ to 1.

But now, if G is abelian, it is apparently well-known that $\mathbb{Z}[G]$ has no idempotents, i.e. $\mathrm{Spec}(\mathbb{Z}[G])$ is connected.

It is apparently difficult to develop noncommutative geometry to a point where this proof could be made to work for arbitrary groups.

References