Associative square zero extensions

Let $R \to S$ be a square zero extension of associative k-algebras, for a field k. That is, a surjective ring morphism with kernel I such that $I^2 = 0$.

What we want to prove here is that there are S-bimodules M, N together with a surjective bimodule morphism $M \to N$ and a pullback square of rings of the form

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \oplus M \arrow[d] \\ S \arrow[r] & S \oplus N \end{tikzcd}$$

Note that the bottm horizontal map is *not* the inclusion $s \mapsto (s, 0)$.

We will see a more precise version of that claim.

In fact, this follows easily from the derived theory of the cotangent complex : indeed, in this case, the E_1-cotangent complex of S over k is the fiber of $S \otimes_k S \to S$, but because k is a field, this tensor product is "underived", and because this morphism is always surjective, the fiber is just the kernel. In particular, all the derived notions coincide in this case with underived notions, so that the square zero extension is classified by a derivation, and putting things together shows the claim.

The goal of this note is to specialize this theory to this very concrete case, where no derived machinery should be needed (given the look of the claim).

1 Associative Kähler differentials

Definition 1.1. Let A be an associative ring, and M an A-bimodule. A k-linear derivation of A with values in M is a k-linear morphism $f : A \to M$ such that for all $a, b \in A$, $f(ab) = f(a)b + af(b)$, where we use the bimodule structure of M.

Clearly, if $M \to N$ is a morphism of A-bimodules, post-composition by it preserves derivations. In particular we can define a functor $\mathrm{Der}(A, -)$ of derivations.

Observation 1.2. If M is an A-bimodule, and $A \oplus M$ has the trivial square zero extension structure, then a k-linear ring section of $A \oplus M \to A$ is exactly the same data as that of a k-linear derivation $A \to M$.

Definition 1.3. We define $\Omega^1_{A/k} := \ker(A \otimes_k A \to A)$ as an A-bimodule.

Warning 1.4. This notation is non-standard : $\Omega^1_{A/k}$ is typically reserved for the commutative case, where derivations take values in modules and not bimodules (and A is commutative). I simply did not know the standard notation.

We wish to prove that it represents the functor $\mathrm{Der}(A, -)$.

For this, we first define a derivation $d : A \to \Omega^1_{A/k}$ by $a \mapsto a \otimes 1 - 1 \otimes a$. We leave the verification that it is a derivation as an exercise.

Proposition 1.5. *$(\Omega^1_{A/k}, d)$ is the universal derivation.*

Proof. We first prove that the image of d generates $\Omega^1_{A/k}$. Suppose $\sum_i a_i b_i = 0$. Then $\sum_i a_i \otimes b_i = \sum_i a_i \otimes b_i - (a_i b_i) \otimes 1 = \sum_i a_i \cdot (1 \otimes b_i - b_i \otimes 1) = -\sum_i a_i \cdot d(b_i)$, which proves the claim.

So this proves the uniqueness part of the definition of universal derivation, i.e. if $k \circ d = k' \circ d$, then $k = k'$, for k, k' morphisms of A-bimodules.

Now if (M, f) is an A-bimodule with a derivation, we can define $g : A \otimes A \to M$ via $a \otimes b \mapsto f(a)b$ (this is well-defined), and we observe then that $g(d(a)) = f(a)1 - f(1)a = f(a) - f(1)a$, but $f(1) = f(1 \cdot 1) = f(1) + f(1)$, so $f(1) = 0$.

In particular, the restriction of g to the image of d agrees with f. So if we can prove that it is an A-bimodule map, we are done. Suppose $x \in \Omega^1_{A/k}$, $x = \sum_i a_i \otimes b_i$. Then $g(ax) = \sum_i f(aa_i)b_i = f(a)\sum_i a_i b_i + a\sum_i f(a_i)b_i = a\sum_i f(a_i)b_i = ag(x)$, and on the right it's even easier to show $g(xa) = g(x)a$, as this is true on the whole of $A \otimes A$.

This proves existence and so we are done. $\square$

Remark 1.6. By observation 1.2, we get a section $A \to A \oplus \Omega^1_{A/k}$, which is universal among sections of square zero extensions.

Observe that $\Omega^1_{-/k}$ is functorial in the sense that for any ring map $f : A \to A'$, we get a canonical A-bimodule morphism $\Omega^1_{A/k} \to f^*\Omega^1_{A'/k}$, where for an A'-bimodule M, f^*M is the A-bimodule obtained by restriction of scalars along f. This morphism is compatible with the derivations, in the sense that it maps $d(a)$ to $d(f(a))$, where we abuse notation by using d for both derivations.

In particular, if f is surjective, then so is the adjoint morphism $f_!\Omega^1_{A/k} \to \Omega^1_{A'/k}$, as it is an A'-bimodule map and hits a set of generators, where $f_!$ is the left adjoint to f^*, i.e. it is extension of scalars along $A \otimes A^{\mathrm{op}} \to A' \otimes A'^{\mathrm{op}}$.

We wish to compare the kernel K of $f_!\Omega^1_{A/k} \to \Omega^1_{A'/k}$ with I. Firstly, observe that there is a morphism $I \to K$ given by $i \mapsto d(i)$. It is easy to check that this is a morphism of A-bimodules.

Proposition 1.7. *This morphism is surjective.*

Proof. In a similar way as the proof of proposition 1.5, one shows that the kernel of $\Omega^1_{A/k} \to \Omega^1_{A'/k}$ is generated by $I \cdot \Omega^1_{A/k} + A \cdot d(I)$.

Furthermore, as f is surjective, so is $\Omega^1_{A/k} \to f_!\Omega^1_{A/k}$, and $I \cdot \Omega^1_{A/k}$ vanishes under this morphism, so $d(I)$ generates the kernel of our morphism.

This is exactly saying that $I \to K$ is surjective. $\square$

We can then also define a ring morphism $A \to A' \oplus f_!\Omega^1_{A/k}$ by sending a to $(f(a), d(a))$, where the latter has the trivial square zero extension structure.

We then get a commutative square as follows :

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \oplus f_! \Omega^1_{A/k} \arrow[d] \\ A' \arrow[r] & A' \oplus \Omega^1_{A'/k} \end{tikzcd}$$

Of course this square is in general not a pullback, because we started with an arbitrary surjection and the pullback of square zero extensions is always a square zero extension.

The induced morphism from A to the pullback is, however, surjective.

Indeed, if (a, m) is in the pullback, i.e. the image of m in $\Omega^1_{A'/k}$ is equal to $d(a)$, then we can first find a lift of a, say $\tilde{a}$ in A, and then it is mapped to $(a, d(\tilde{a}))$, but then $d(\tilde{a}) - m$ is in K, so it is in the image of I, and so we can change $\tilde{a}$ so that $d(\tilde{a}) = m$ without changing the fact that it's a lift of a.

2 The case of square zero extensions

So to prove our main claim, it now suffices to show that if $A \to A'$ is a square zero extension, the map to the pullback is also injective, i.e. the map $I \to K$ is injective.

It would be nice to have a better but still concrete proof of that than what I am going to write down.

We want to identify the kernel of $f_! \Omega^1_{A/k} \to \Omega^1_{A'/k}$.

For this, we observe that the kernel is the same as the one after we composte with the inclusion of the latter in $A' \otimes A'$, after what the morphism in question becomes identified with $f_!$ of the inclusion $\Omega^1_{A/k} \to A \otimes A$, as is easy to see from the definitions.

We can now contemplate the following snake-lemma-type diagram :

$$\begin{tikzcd} & L \otimes_{A \otimes A^{\mathrm{op}}} \Omega^1_{A/k} \arrow[r] \arrow[d] & \Omega^1_{A/k} \arrow[r] \arrow[d] & f_! \Omega^1_{A/k} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L \arrow[r] & A \otimes A^{\mathrm{op}} \arrow[r] & f_! A \otimes A^{\mathrm{op}} & \end{tikzcd}$$

where L is by definition the kernel of $A \otimes A^{\mathrm{op}} \to A' \otimes A'^{\mathrm{op}}$.

The horizontal sequences are exact, so we can apply the snake lemma. The middle vertical arrow is injective, so the snake lemma exact sequence looks like

$$0 \to K \to L \otimes_{A \otimes A^{\mathrm{op}}} A \to A \to f_! A \to 0$$

Observe that (following the proof of the snake lemma) if $x \in \Omega^1_{A/k}$ lifts $\tilde{x} \in K$, the image of $\tilde{x}$ under the leftmost morphism in this sequence is given by $x \in A \otimes A^{\mathrm{op}}$, which happens to be in L, so that we can then look at its image in $L \otimes_{A \otimes A^{\mathrm{op}}} A$.

So for instance for $x = d(i), i \in I$, this would be the image of $i \otimes 1 - 1 \otimes i$ in that quotient. To show that $I \to K$ is injective, it therefore suffices to show that this is nonzero if i is, for then it has to be nonzero in K too.

But now we observe that L is the quotient of $I \otimes A \oplus A \otimes I$ by their intersection in $A \otimes A$, i.e. by $I \otimes I$, *because we are working over a field.*

Now we can observe that $A \otimes_{A \otimes A^{\mathrm{op}}} (I \otimes A) \cong I$ via $i \otimes a \mapsto ai$, and similarly for $A \otimes I$. In particular, *because I is square zero*, the image of $I \otimes I$ vanishes in $A \otimes_{A \otimes A^{\mathrm{op}}} L$, so that it becomes in fact isomorphic to $I \oplus I$. In particular, the image of $i \otimes 1 - 1 \otimes i$ in that quotient is $(i, -i)$, which is nonzero as soon as i is.

This proves our claim, so that our square from before is in fact a pullback.

In conclusion, we get the following more precise claim, which was our goal from the start:

Theorem 2.1. *Let $f : R \to S$ be a square zero extension of associative k-algebras, for a field k.*

Then the canonical morphism $f_! \Omega^1_{R/k} \to \Omega^1_{S/k}$ is surjective, and the following is a pullback diagram of rings

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \oplus f_! \Omega^1_{R/k} \arrow[d] \\ S \arrow[r] & S \oplus \Omega^1_{S/k} \end{tikzcd}$$

where the left vertical map is f, the right one is the identity on the S-factor, and the canonical map on the bimodule factor, the bottom horizontal map is the universal derivation, and the top horizontal map is the composite of the universal derivation for R with the canonical maps.

3 Application to formal smoothness

We conclude this note by showing how it helps to prove a characterization of formally smooth (associative algebras), appearing in [KR00] as an "easy exercise in homological algebra":

Corollary 3.1. *Given a k-algebra A, the following are equivalent :*

1. $\Omega^1_{A/k}$ *is projective*
2. $\mathrm{Ext}^2_{A \otimes A^{\mathrm{op}}}(A, -)$ *vanishes*
3. *A is formally smooth, in the sense that if B is an arbitrary k-algebra with a nilpotent two-sided ideal I, then any k-algebra morphism $A \to B/I$ lifts to a k-algebra morphism $A \to B$*

Proof. The equivalence of (1) and (2) is really just an easy exercise in homological algebra, in the vein of "dimension-shifting", simply using the short exact sequence of A-bimodules $0 \to \Omega^1_{A/k} \to A \otimes A^{\mathrm{op}} \to A \to 0$ and the fact that a module is projective if and only if its Ext^1 vanishes.

Let us now show that (1) and (3) are equivalent.

Suppose (3) and let $M \to \Omega^1_{A/k}$ be a surjection of A-bimodules. Then $A \oplus M \to A \oplus \Omega^1_{A/k}$ is a square zero extension. It follows from (3) that the morphism $A \to A \oplus \Omega^1_{A/k}$ classified by the universal derivation lifts to a morphism $A \to A \oplus M$, but now this corresponds to a derivation with values in M lifting the universal one, i.e. by the universal property of the Kähler differentials, to a splitting of $M \to \Omega^1_{A/k}$. This shows projectivity.

Conversely, assume (1). By the usual trick of filtering B by B/I^n, we reduce to the case where $I^2 = 0$.

To get a lift $A \to B$, it suffices to find a splitting of the pullback $A \times_{B/I} B \to A$.

But this pullback is still a square zero extension, so by the main result of this note, it is a pullback of a square zero extension of the form $A \oplus M \to A \oplus \Omega^1_{A/k}$ - it therefore suffices to find a lift against that one.

But now this is clearly equivalent to finding a splitting of $M \to \Omega^1_{A/k}$, as before. But this morphism is surjective, and $\Omega^1_{A/k}$ is projective, so there is such a lift. $\square$

Remark 3.2. In the commutative case, there is no such nice formula for the full cotangent complex (whose π_0 is the usual Kähler differentials), and in particular even in such a nice situation (working over a field) it need not be concentrated in degree 0, so that the main result of this note fails.

References

[KR00] Maxim Kontsevich and Alexander L Rosenberg. Noncommutative smooth spaces. In *The Gelfand mathematical seminars, 1996–1999*, pages 85–108. Springer, 2000.