Basics of adic spaces

The goal of these notes is to provide a brief introduction to adic spaces, in the context of the Spring 2024 BabyTop seminar at Harvard, on the recent preprint [BSSW24].

The author of the notes is not very familiar with adic spaces, or anything surrounding them - proper motivation for the concepts and results introduced might be lacking at times.

My sources were [Wei17, SW20, CS, Hub93, Hub94].

All errors are mine.

Conventions

All rings are commutative and unital. The word "space" here means "topological space", unless it is preceded by "adic".

1 A crash course on valuations

Valuations will play an important role in the discussion of adic spaces, so it is important to get a handle on them. We begin by discussing valuation fields.

1.1 Valuation fields and rings

Definition 1.1. A valuation field is a field K equipped with a *valuation* $v : K \to \Gamma \cup \{0\}$ where Γ is a totally ordered abelian group, denoted multiplicatively, 0 is declared to be $< \Gamma$ and $0 \cdot \gamma = 0$ for all $\gamma \in \Gamma \cup \{0\}$, that is, a function v satisfying:

1. $v(0) = 0, v(1) = 1$;
2. $v(ab) = v(a)v(b)$;
3. $v(a + b) \leq \max(v(a), v(b))$.

Example 1.2. For $K = \mathbb{Q}$, the p-adic valuations $v_p : \mathbb{Q} \to p^{\mathbb{Z}}$ are examples of such valuations (where $p^{\mathbb{Z}}$ is ordered in the opposite of the usual order, so that $p < 1$).

Example 1.3. Let R be a discrete valuation ring, such as $\mathbb{F}_p[t]_{(t)}$ or $\mathbb{F}_p[[t]]$ or $\mathbb{Z}_p$ with uniformizer ϖ, and consider the field $R[\varpi^{-1}]$. It has a ϖ-adic valuation, also with values in $\varpi^{\mathbb{Z}}$. These are "rank 1" valuations.

Example 1.4. There is always the trivial valuation on a field K, namely $v(K^\times) = 1, v(0) = 0$. If K is a finite field, this is the only valuation. Indeed, if $x \in K$, then x is a root of unity so that $v(x)^n = 1$ for some n. But Γ is a totally ordered abelian group, so it is torsion-free (exercise!) and so $v(x) = 1$.

This might sound like some kind of analytic notion, but it turns out to be completely algebraic. Up to replacing Γ with $v(K^\times)$, we may assume v is surjective, and:

Observation 1.5. Let v be a valuation on K, and let $R = \{x \in K \mid v(x) \leq 1\}$. In this case, R is a subring of K whose fraction field equals K and such that for all $x \in K$, either $x \in R$ or $x^{-1} \in R$.

Furthermore, if v is surjective then it gets identified with $p : K \to K^\times/R^\times \cup \{0\}$ defined by $p(k) = k \mod R^\times$ if $k \neq 0$, 0 if $k = 0$, where $K^\times/R^\times$ is equipped with the[1] order $k \leq k'$ if and only if $k(k')^{-1} \in R$.

Conversely, any subring R with fraction field K satisfying the condition that for all $x \in K$, either $x \in R$ or $x^{-1} \in R$, the above description defines a valuation v_R on K with $v_R(x) \leq 1$ if and only if $x \in R$.

Definition 1.6. Such a domain R is called a valuation ring.

In particular, the above suggests that valuations are completely algebraically encoded by the inclusion $R \subset K$. Nonetheless, it will be convenient to have the flexibility of a general $v : K \to \Gamma \cup \{0\}$.

Observation 1.7. Let $R \subset K$ be the inclusion of a valuation ring R into its fraction field. R has two key properties: firstly, it is local, and second, it is integrally closed.

Proof. Let $\mathfrak{m} \subset R$ be $\{x \in R, v(x) < 1\}$, where $v : K \to \Gamma \cup \{0\}$ is a valuation defining R. The axioms of valuations guarantee that $\mathfrak{m}$ is an ideal in R. Furthermore, it is clear that $R^\times = \{x \in R, v(x) = 1\}$, so $\mathfrak{m}$ is an ideal whose complement is the set of units - it follows that R is local with maximal ideal $\mathfrak{m}$.

As for integral closure, let $x \in K$ satisfy $x^n = \sum_{k<n} a_k x^k$ for some $a_k \in R$. Taking v of both sides, we find $v(x)^n \leq \max_k v(a_k)v(x)^k \leq \max_k v(x)^k$ as $v(a_k) \leq 1$. As Γ is a totally ordered group, this forces $v(x) \leq 1$. □

Example 1.8. Let v be a valuation on $\mathbb{Q}$, with valuation ring R. R has to be $\mathbb{Z}[s^{-1}, s \in S]$ for some set of primes S. Thus, for it to be local, S needs to be $\mathbb{P} \setminus \{p\}$ for some prime p, in which case $R = \mathbb{Z}_{(p)}$ for some prime p; or $S = \mathbb{P}$, in which case $R = \mathbb{Q}$. The first ones correspond to the p-adic valuations, and the latter corresponds to the trivial valuation. Thus the examples of Example 1.2 are the only examples of nontrivial valuations on $\mathbb{Q}$.

Example 1.9. Let K be a valuation field, with valuation v and valuation ring O_K, and L/K an algebraic extension of K. Let $B \subset L$ denote the integral closure of O_K in L. B need not be local, and so it will not, in general, be the valuation ring of a valuation on L. However one can prove that this is the only obstruction: for any maximal ideal $\mathfrak{n}$ of B, $B_\mathfrak{n}$ is a valuation ring such that $(\mathfrak{n}B_\mathfrak{n}) \cap O_K = \mathfrak{m}_K$, thus providing a valuation that extends v. If L is a *normal* extension of K, then in fact all of these maximal ideals and extended valuations are Galois conjugate.

1.2 Valuations on rings

One can replace the word "field" in the definition of a valuation field. Of course, the word "valuation ring" is already taken, so we can't call them that, but we can still say:

[1] Well-defined, and total

Definition 1.10. A valuation v on a ring A is a map $v : A \to \Gamma \cup \{0\}$ for some totally ordered abelian group Γ satisfying the same axioms as before, that is:

1. $v(0) = 0, v(1) = 1$;
2. $v(ab) = v(a)v(b)$;
3. $v(a + b) \leq \max(v(a), v(b))$.

Unlike for fields, now it can happen that $v(a) = 0$ without $a = 0$. However, the situation is not so bad:

Observation 1.11. Let v be a valuation on the ring A, and let $\mathfrak{p} = \{a \in A \mid v(a) = 0\}$. As the notation suggests and as is easy to prove, $\mathfrak{p}$ is a prime ideal, and using $v(a + x) \leq v(a)$ for all $a \in A, x \in \mathfrak{p}$, we easily find that v factors canonically through $A/\mathfrak{p}$, in fact through $\kappa(\mathfrak{p}) := \mathrm{Frac}(A/\mathfrak{p})$.

Thus, any valuation v on A factors through a valuation on a field, where it is completely understood in terms of the corresponding valuation ring $R_v \subset \kappa(v) := \kappa(\mathfrak{p})$.

Definition 1.12. The prime $\mathfrak{p}$ is called the *support* of v.

Definition 1.13. The set $\mathrm{Spv}(A)$ is defined to be the set of pairs $v = (\mathfrak{p}_v, R_v)$ where $\mathfrak{p}_v$ is a prime ideal of A and R_v a valuation ring inside $\kappa(v) := \kappa(\mathfrak{p}_v)$.

Equivalently, it is a set of equivalence classes of valuations on A, under a suitable and "evident" notion of equivalence.

Something which will be crucial later on, is the map of spaces $\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A)$ which only remembers the prime ideal. By design, its fiber over a prime ideal $\mathfrak{p}$ is $\mathrm{Spv}(\kappa(\mathfrak{p}))$, the set of valuations on the field $\kappa(\mathfrak{p})$. We can use this to give examples:

Example 1.14. Let us describe $\mathrm{Spv}(\mathbb{Z})$ using the map $\mathrm{supp} : \mathrm{Spv}(\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z})$. The fiber over a closed point p is $\mathrm{Spv}(\mathbb{F}_p)$. As $\mathbb{F}_p$ is a finite field, any valuation on it is trivial (cf. Example 1.4). We call the corresponding valuation $\bar{v}_p$.

The fiber over the generic point (0) is $\mathrm{Spv}(\mathbb{Q})$, which we identified in Example 1.8. We advise the reader to draw a picture of the map supp in this case, although we should point out that, in the coming topology, v_p and $\bar{v}_p$ are not as unrelated as they seem.

If one thinks of elements of a ring A as functions on $\mathrm{Spec}(A)$, then for a given $f \in A$ and a given valuation v, we can think of $v(f)$ as how "big" this function can get, with respect to a certain measure of big-ness, given precisely by the valuation. The following notation is supposed to encompass this idea, that valuations are given by hypothetical points or "domains" of the desired geometric object of interest.

Notation 1.15. Let v be a valuation and f an element of A. We let $|f(v)| := v(f)$. If v is only defined up to equivalence, this is not so well-defined. The reader may find solace in the following points:

- By the previous discussions, any equivalence class of valuations v has a canonical representative in $\Gamma = \kappa(v)^\times / R_v^\times$, so we could decide to mean that;

- We will only ever use this notation to indicate equalities or inequalities between the same kind of objects, in which case the notation is completely unambiguous.

Thus, the following subsets of $\mathrm{Spv}(A)$ describe inequalities imposed by these functions:

Definition 1.16. Let $f_1, ..., f_n, s \in A$. We let $U(\frac{f_1,...,f_n}{s}) \subset \mathrm{Spv}(A)$ denote $\{v \in \mathrm{Spv}(A) \mid v(f_i) \leq v(s) \neq 0\}$. Finite intersections of these kinds of subsets are called rational subsets.

With the fancy notation, this looks like $\{v \in \mathrm{Spv}(A) \mid |f(v_i)| \leq |s(v)| \neq 0\}$.

Notation 1.17. Suppose $T_1, ..., T_k$ are finite subsets of A and $s_1, ..., s_k \in A$. We let $U(\{\frac{T_i}{s_i}\}_i)$ denote the intersection of the $U(\frac{T_i}{s_i})$'s.

Definition 1.18. The topological space $\mathrm{Spv}(A)$ is defined as the set $\mathrm{Spv}(A)$ together with the topology where the rational subsets form a basis of opens.

What is going to happen now is as follows: Spv is a bit too algebraic, and does not take into account many topological properties that a ring A may have. We will thus introduce a class of topological rings of interest (Huber rings, and Huber pairs) for which we will partially restrict $\mathrm{Spv}(A)$ to get a new space, $\mathrm{Spa}(A, A^+)$. These adic spectra are supposed to be the local models of our adic spaces, and so they will be equipped with a structure sheaf. We will discuss the subtlety of whether this is actually a sheaf.

We will then define adic spaces to be locally ringed spaces locally modelled on these "affinoid[2] adic spaces".

Taking various topologies into account will recover classical scheme theory (with discrete topologies), formal scheme theory (with adic topologies on rings), and rigid-analytic spaces (with some fancier topology). Finally, there will also simply be new examples.

2 Huber rings, Huber pairs

A Huber ring (called "f-adic ring" in Huber's original work) is a topological ring which is not adic, i.e. its topology is not given by an ideal, but is close enough to such a thing. More precisely:

Definition 2.1. A topological ring A is a Huber ring if there exists an open subring A_0 such that the subspace topology on A_0 is adic, i.e. given by a finitely generated ideal I (which is in general not an ideal of A).

Remark 2.2. The pair (A_0, I) is not part of the structure of the Huber ring, only its existence is required.

Definition 2.3. A ring A_0 as above is called a ring of definition, and the pair (A_0, I), a couple of definition.

Remark 2.4. The pair (A_0, I) completely determines the topology on A, as a basis of open neighbourhoods of any $a \in A$ is given by $\{a + I^n\}_n$ (where $I^0 = A_0$).

Example 2.5. Any adic ring is Huber, with itself as a ring of definition. Thus, formal schemes will be part of the theory.

[2] We say "affinoid" to avoid confusion with affine schemes, I suppose.

Remark 2.6. Just as adic topological rings need not be complete, we do not require Huber rings to be complete.

Definition 2.7. A Huber ring is called Tate if there exists a topologically nilpotent unit. Such a unit is called a pseudo-uniformizer.

Example 2.8. Let K be a complete discrete valuation field with uniformizer ϖ. With the couple of definition $(O_K, (\varpi))$, it is a Tate Huber ring.

Observation 2.9. More generally, if A is a Tate Huber ring with couple of definition (A_0, I) and with pseudo-uniformizer u, then for some n, $u^n \in A_0$ (as A_0 is open and $u^n \to 0$). Thus $A \simeq A_0[(u^n)^{-1}]$.

Furthermore, the topology on A_0 is u^n-adic: for some k, $u^{nk} \in I$, and conversely, $u^n A_0 \subset A_0$ is open (as u is invertible in A) so it contains I^k for some power k.

Example 2.10. Let $\mathbb{Q}_p\langle T\rangle$ denote the subring of $\mathbb{Q}_p[[T]]$ given by power series whose coefficients tend to 0, and let $\mathbb{Z}_p\langle T\rangle$ be the obvious subring. The latter has its p-adic topology, making $\mathbb{Q}_p\langle T\rangle$ a Tate Huber ring[3].

We think of this last example as a closed unit disk D: functions on it are exactly power series that converge on the whole unit disk.

Similarly, one can define the open unit disk.

Non-example 2.11. $\mathbb{Q}_p[[T]]$ is not a Huber ring with couple of definition $(\mathbb{Z}_p[[T]], (p, T))$. Indeed, in such a topology $T^n \to 0$ and hence $p^{-1}T^n \to 0$, but this never enters the supposedly open subring $\mathbb{Z}_p[[T]]$.

Example 2.12. On the other hand, with itself as a ring of definition, $\mathbb{Z}_p[[T_1, ..., T_n]]$ is a Huber ring, with ideal of definition $(p, T_1, ..., T_n)$ (we leave the obvious generalization to DVR's to the reader).

Now we can adapt the definition of valuations to the present setting. Of course we could give the definition more generally, but we restrict to Huber rings:

Definition 2.13. Let A be a Huber ring. A continuous valuation on A is a valuation v such that for any $\gamma \in \Gamma$, $v^{-1}(\{x, x < \gamma\})$ is open in A.

Notation 2.14. For a Huber ring A, we let $\mathrm{Cont}(A)$ denote the subspace of $\mathrm{Spv}(A)$ spanned by continuous valuations, although we restrict the rational subsets that we allow: we only allow $U(\{\frac{T_i}{s_i}\}_i)$ if for all i, $T_i \cdot A$[4] is open in A.

The desired local model $\mathrm{Spa}(A)$ will come from a further restriction. The motivation for this is well-explained in [Wei17, Page 7], [SW20, §3.3], but we will take for granted that this is a reasonable definition. To give it, we need to introduce a few notions.

Definition 2.15. Let A be a topological ring and $S \subset A$ a subset. S is called bounded if for any open $U \ni 0$, there exists an open $V \ni 0$ with $SV \subset U$.

[3] The topology on $\mathbb{Q}_p\langle T\rangle$ is *not* the subspace topology of that on $\mathbb{Q}_p[[T]] \cong \mathbb{Q}_p^{\mathbb{N}}$.

[4] In this notation, we take the additive subgroup generated by the products.

Example 2.16. Since multiplication is bounded, any singleton is bounded. Since bounded subsets are closed under finite unions, any finite set is bounded.

Remark 2.17. If A is Huber, given a couple of definition (A_0, I), because the I^n's form a basis of open neighbourhoods of 0, the above is equivalent to "for all n, there exists $N >> 0$ such that $SI^N \subset I^n$". In particular the latter definition is independent of (A_0, I).

Example 2.18. Let A be a Huber ring and A_0 a ring of definition. By the previous remark, *since I is an ideal in A_0*, A_0 is bounded.

Definition 2.19. An element f of A is called power-bounded if the set $\{f^n, n \in \mathbb{N}\}$ is bounded.

Example 2.20. Elements of A_0 are power-bounded.

Example 2.21. The subset $A^\circ \subset A$ is the set of power-bounded elements is a subring. It contains all rings of definition of A.

In particular it is an open subring.

Proof. Any subgroup of a topological group containing an open subgroup is open, so the last statement is clear.

By the previous example, we simply need to prove that A° is a ring. Closure under products is clear, and closure under sums is an instructive exercise. □

Warning 2.22. In general, $A^\circ \subset A$ is not a ring of definition. For example it need not even be bounded.

Example 2.23. Let $\mathbb{Q}_p[T]/T^2$ be the Huber ring with the obvious topology, with a couple of definition $(\mathbb{Z}_p[T]/T^2, p)$. For any $q \in \mathbb{Q}_p$, qT is power-bounded for obvious reasons, and hence $A^\circ = \mathbb{Z}_p \oplus \mathbb{Q}_p T$, which is not itself bounded.

Definition 2.24. A Huber ring is called *uniform* if A° is bounded.

Lemma 2.25. *If A is a uniform Huber ring, then A° is a ring of definition.*

Proof. Let (A_0, I) be a couple of definition. As A_0 is bounded, we have $A_0 \subset A^\circ$. So let $J := A^\circ I$ - this is a finitely generated ideal of A°, and we claim it is in ideal of definition of the topology.

Since A° is bounded (by definition of uniform), for every n there is an N such that $A_0 I^N \subset I^n$, and $A_0 I^N = J^N$.

Conversely, $I \subset J$ so for every n, $I^n \subset J^n$. Thus the topologies given by $\{I^n\}$ and $\{J^n\}$ are the same, as claimed. □

More generally, we can still relate A° and rings of definitions. Recall that they are all included in A°. Furthermore:

Proposition 2.26. *For a Huber ring A, A° is the filtered colimit (union) of the rings of definition of A.*

Proof. It suffices to prove that for any ring of definition A_0 and power-bounded subset S, there exists a ring of definition A_1 containing A_0 and S.

Indeed, from this, by taking S to be another ring of definition, we deduce that the poset of rings of definition is filtered; and by taking S to be $\{f\}$ for some power-bounded f, we find that A° is containg in the filtered union of the rings of definition.

So let A_0 be a ring of definition, and S such that $S^\infty = \bigcup_n S^n$ is bounded. By the definition of bounded, find an integer n such that $I^n S^\infty \subset I$. Since A_0 is closed under sums, we may as well close S^∞ under sums.

We then claim that the subring A_1 generated by A_0 and S^∞ is a ring of definition. Let $J = IA_1$, which is a finitely generated ideal of A_1. A_1 contains A_0, so it is open. Furthermore, $J^n \subset I^n A_1 \subset I^n A_0 + I^n A_0 S^\infty + I^n S^\infty \subset I$, and $I^n \subset J^n$. So in total, the topology on A_1 is J-adic, which is what we needed to show. ☐

We now determine A° in the given examples.

Example 2.27. If A is adic, i.e. A is itself a ring of definition, then $A = A^\circ$.

Let K be a complete discrete valuation field with couple of definition (O_K, ϖ) with ϖ a uniformizer. If $x = \varpi^k y, y \in O_K$ then x is power bounded if and only if $k \geq 0$. So $K^\circ = O_K$.

More generally, if A is Tate with a pseudo-uniformizer u which we assume without loss of generality to be in a ring of definition A_0, then a set S is bounded if and only if $S \subset u^{-n} A_0$ for some n, and so an element f is power bounded if and only if there is a uniform n such that for all k, $u^n f^k \in A_0$.

If $A = \mathbb{Q}_p\langle T\rangle$, $A^\circ = \mathbb{Z}_p\langle T\rangle$: any $f \in \mathbb{Q}_p\langle T\rangle$ has coefficients eventually in $\mathbb{Z}_p$, so up to something in the ring of definition $\mathbb{Z}_p\langle T\rangle \subset A^\circ$, it is a polynomial. Removing all monomials with coefficients in $\mathbb{Z}_p$ in a similar fashion, we are reduced to a polynomial all of whose coefficients have negative p-adic valuation. Such a thing is clearly not power-bounded, unless it is 0.

Definition 2.28. Let A be a Huber ring. A ring of integral elements $A^+ \subset A$ is an open subring which is integrally closed and contained in A°.

A Huber pair[5] is a pair (A, A^+) where A is a Huber ring and A^+ is a ring of integral elements.

Example 2.29. A° is a ring of integral elements, so (A, A°) is a Huber pair. This is often what is considered, but the theory benefits from the added flexibility of A^+.

Definition 2.30. Let (A, A^+) be a Huber pair. $\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A)$ is the space of continuous valuations v such that $v(A^+) \leq 1$, with the subspace topology.

If $A^+ = A^\circ$, we denote it simply by $\mathrm{Spa}(A)$.

An important topological property of $\mathrm{Spa}(A, A^+)$, which was already true of $\mathrm{Spv}(A)$ and $\mathrm{Cont}(A)$ is:

Theorem 2.31 ([Hub93, Theorem 3.5])**.** *For any Huber pair (A, A^+), the space $\mathrm{Spa}(A, A^+)$ is spectral. In particular, it is quasi-compact.*

Remark 2.32. This is not an easy theorem, but it is quite convenient. See also [Hub93, Proposition 2.2, Corollary 3.2] for the cases of $\mathrm{Spv}(A), \mathrm{Cont}(A)$ which are preludes to the case of $\mathrm{Spa}(A, A^+)$.

[5] "Affinoid algebra" in the work of Huber

We do make two remarks about the introduction of A^+. The first is that the choice of "open integrally closed subrings, consisting of power-bounded elements" can be justified almost purely topologically, by the following correspondence [Hub93, Lemma 3.3.(i)] (see also [SW20, Proposition 3.3.1] for context):

Theorem 2.33. *There is a bijection of the form*

$$A^+ \mapsto \{v \mid v(A^+) \leq 1\}, \qquad F \mapsto \{f \in A \mid \forall v \in F, v(f) \leq 1\}$$

between integrally closed open subrings of A consisting of power bounded elements on the one hand and subsets of $\mathrm{Cont}(A)$ of the form $\bigcap_{f\in I}\{|f(-)| \leq 1\}$ as I runs through subsets of A°.

The second remark is about the example of the closed unit disk D, with algebra of functions $\mathbb{Q}_p\langle T\rangle$. Certainly, the function "T" should never take a value > 1. However, there are valuations v in $\mathrm{Cont}(\mathbb{Q}_p\langle T\rangle)$ such that $v(T)$ is (infinitesimally) greater than 1.

Example 2.34. Let $|-| : \mathbb{Q}_p \to \mathbb{R}_{>0}$ denote the p-adic valuation. Let $\Gamma = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ be given the lexicographic order, and let $v : \mathbb{Q}_p\langle T\rangle \to \Gamma \cup \{0\}$ be $\sum_n a_n T^n \mapsto \sup_n(|a_n|, 2^n)$.

One can check that v is well-defined and is a valuation, and $v(T) = (1, 2)$ is $> (1, 1)$ but it is $< (1 + \epsilon, x)$ for any $\epsilon, x > 0$.

Of course, this valuation v does not lie in $\mathrm{Spa}(\mathbb{Q}_p\langle T\rangle) = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$, so this is one way to motivate the introduction of a second factor.

3 The structure presheaf, adic spaces and preadic spaces

We now introduce the structure presheaf on the adic spectrum of a Huber pair. We make ahead of time a warning:

Warning 3.1. The structure presheaf will in general not be a sheaf.

In particular, the local geometry of adic spaces will be a bit more complicated than for schemes. We will nonetheless be able to assign a "preadic space" to all Huber pairs, so while the main objects of interest will be adic spaces, these preadic spaces will add a convenient flexibility to the theory.

The structure presheaf will be defined in two steps: first, we define the Huber pairs of sections on rational subsets of $\mathrm{Spa}(A, A^+)$, and then right Kan extend to all open subsets. We will then provide a number of special cases in which this structure presheaf is a sheaf (called *sheafy* Huber pairs).

The following is a theorem of Huber - we make it concrete after stating it in terms of universal properties [Hub94, Proposition 1.3] (see also [SW20, Theorem 3.1.3]):

Theorem 3.2. *Let (A, A^+) be a Huber pair, and $U = U(\{\frac{T_i}{s_i}\}_i)$ a rational subset of $\mathrm{Spa}(A, A^+)$. There exists a complete*[6] *Huber pair $(\mathcal{O}(U), \mathcal{O}^+(U))$ and a morphism of Huber pairs $(A, A^+) \to (\mathcal{O}(U), \mathcal{O}^+(U))$ such that the induced morphism on $\mathrm{Spa}(-)$ factors through U, and is initial among such (complete) Huber pairs.*

Moreover, the induced map on $\mathrm{Spa}(-)$'s is a homeomorphism onto its image, $\mathrm{Spa}(\mathcal{O}(U), \mathcal{O}^+(U)) \cong U$. In particular, U is quasi-compact.

[6] Here and throughout, we include Hausdorffness in the definition of "complete".

Concretely, $(\mathcal{O}(U), \mathcal{O}^+(U))$ can be described in terms of the T_i's and s_i's. Let us describe the construction in the case where there is only one i, so $U = U(\frac{T}{s})$ for some $T = \{t_1, ..., t_n\}$. Let us fix a couple of definition (A_0, I) for A.

First, the definition of U makes it so that for any valuation $v \in U$, $|s(v)| \neq 0$. Huber proves that Huber rings have "enough valuations", so this forces s to be invertible in any Huber pair (B, B^+) under (A, A^+) with image $\subset U$. So we can start by taking $A[\frac{1}{s}]$.

Second, for any $v \in U$, $t \in T$, $|t(v)| \leq |s(v)|$, so if s is invertible, we can rephrase this as $|\frac{t}{s}(v)| \leq 1$. Again, because of a result of the form "there are enough valuations", this forces $\frac{t}{s} \in B^+ \subset B^\circ$. By Proposition 2.26, this means that there exists a ring of definition B_0 containing all the $\frac{t}{s}, t \in T$. This gives us a map $A_0[\frac{t}{s}, t \in T] \to B_0$, and we can endow the former [7] with the $IA_0[\frac{t}{s}, t \in T]$-adic topology.

One can show that this participates in a Huber topology on $A[\frac{1}{s}]$. We then let $A[\frac{1}{s}]^+$ denote the integral closure of $A^+[\frac{t}{s}, t \in T]$ inside $A[\frac{1}{s}]$, and finally we complete the Huber pair $(A[\frac{1}{s}], A[\frac{1}{s}]^+)$ to get a complete Huber pair, denoted

$$(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$$

The proof that this satisfies the desired universal property is an elaboration of the above construction.

Remark 3.3. In particular, the universal property shows that this ring only depends on the subset $U \subset \mathrm{Spa}(A, A^+)$ and not the presentation as a rational subset.

Remark 3.4. The restriction to complete Huber pairs is not a huge one. First, because the ideals of definition I are finitely generated, completion is fairly well-behaved. Second, Huber proves that completions of Huber pairs induce homeomorphisms of the relevant Spa's.

Definition 3.5. Let (A, A^+) be a Huber pair. We define a presheaf $(\mathcal{O}_{\mathrm{Spa}(A,A^+)}, \mathcal{O}^+_{\mathrm{Spa}(A,A^+)})$ on the space $\mathrm{Spa}(A, A^+)$ as follows. On rational subsets U, define $(\mathcal{O}_{\mathrm{Spa}(A,A^+)}(U), \mathcal{O}^+_{\mathrm{Spa}(A,A^+)}(U)) = (\mathcal{O}(U), \mathcal{O}(U)^+)$, and on arbitrary opens, define it via right Kan extension, that is :

$$\mathcal{O}_{\mathrm{Spa}(A,A^+)}(V) = \lim_{U \subset V \text{ rational}} \mathcal{O}_{\mathrm{Spa}(A,A^+)}(U)$$

and similarly for $\mathcal{O}^+$.

$\mathcal{O}^+$ can be alternatively characterized "stalkwise" - the following is an easy exercise:

Proposition 3.6. *Let (A, A^+) be a Huber pair, and $V \subset \mathrm{Spa}(A, A^+)$ an open subset. The subset $\mathcal{O}^+_{\mathrm{Spa}(A,A^+)}(V) \subset \mathcal{O}_{\mathrm{Spa}(A,A^+)}(V)$ consists of those elements f such that for all $v \in V, |f(v)| \leq 1$.*

Note that this criterion makes sense as any V is covered by rational subsets, and for U rational, a point in U is, by the final part of Theorem 3.2, a continuous valuation on $\mathcal{O}_{\mathrm{Spa}(A,A^+)}(U)$.

As an elaboration of that remark, let us point out that the stalks of $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ carry canonical valuations:

[7] Which is here viewed as a subring of $A[\frac{1}{s}]$.

Construction 3.7. Let $v \in \mathrm{Spa}(A, A^+)$. As rational subsets form a basis of the topology, we have $\mathcal{O}_{\mathrm{Spa}(A,A^+),v} \cong \mathrm{colim}_{U \ni v,\ \mathrm{rational}} \mathcal{O}_{\mathrm{Spa}(A,A^+)}(U)$ and the valuation v defined on A factors compatibly through each of the $\mathcal{O}_{\mathrm{Spa}(A,A^+)}(U)$.

Thus $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is really a presheaf of topological rings equipped with a(n equivalence class of) valuation(s) at each stalk, a valuation which sends the subsheaf $\mathcal{O}^+_{\mathrm{Spa}(A,A^+)}$ to something ≤ 1.

We can now make two definitions:

Definition 3.8. A Huber pair (A, A^+) is sheafy if the structure presheaf is a sheaf.

Note that by the stalkwise description of the subpresheaf $\mathcal{O}^+_{\mathrm{Spa}(A,A^+)}$, we find that if $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is a sheaf, then so is $\mathcal{O}^+_{\mathrm{Spa}(A,A^+)}$.

Finally, we come to the main definition of the notes:

Definition 3.9. An adic space X is a topological space $|X|$ equipped with a sheaf of topological rings[8] $\mathcal{O}_X$ and a family of (equivalence classes of) valuations $|-|_x$ on the stalks $\mathcal{O}_{X,x}$ which is locally isomorphic to the topologically ringed space $(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}, |-|_v)$ for some sheafy Huber pair (A, A^+).

A morphism of such is a morphism of topologically locally[9] ringed spaces which is stalkwise compatible with the valuations in an obvious (strong) sense.

Our final goal in this section is now two-fold: firstly, describe a wide class of Huber pairs that are sheafy; and secondly explain one particular way to deal with the non-sheafy ones, as well as explaining one way to deal with the category of adic spaces.

For the sheafy Huber pairs, here is a theorem encompassing many examples [Hub94, Theorem 2.2]:

Theorem 3.10. *Let (A, A^+) be a complete Huber pair. In the following cases, it is sheafy:*

1. *If A is discrete. In particular, one can consider* $\mathrm{Spa}(A, A)$ *in this case;*
2. *A is finitely generated*[10] *over a* noetherian *ring of definition. In particular, one can consider formal schemes as adic spaces;*
3. *A is a Tate Huber ring and it is strongly noetherian, i.e. for any $n \geq 0$, $A\langle T_1, ..., T_n\rangle$ is noetherian.*

There is another source of sheafy Huber pairs, namely the *stably uniform ones* whose underlying Huber ring is *analytic*.

Definition 3.11. A Huber pair (A, A^+) is analytic if for all $v \in \mathrm{Spa}(A, A^+)$, $\mathrm{supp}(v)$ is *not* open[11].

[8] Different sources disagree about whether these are required to be complete

[9] It is a fun exercise to prove that the stalks of $\mathcal{O}^{(+)}_{\mathrm{Spa}(A,A^+)}$ are actually local and that a map of Huber pairs induces local maps on each stalk.

[10] I am a bit confused: both [Wei17, SW20] require A to be finitely generated over the ring of definition, but Huber [Hub94] seems not to require this condition.

[11] This intuitive definition turns out to be equivalent [SW20, Proposition 4.3.1] to the following generalization of the Tate condition: the ideal generated by topologically nilpotent elements is the whole ring.

Definition 3.12. A complete analytic Huber pair (A, A^+) is stably uniform if for all rational subsets $U \subset \mathrm{Spa}(A, A^+)$, $\mathcal{O}_{\mathrm{Spa}(A,A^+)}(U)$ is uniform.

We then have the following:

Theorem 3.13 ([SW20, Theorem 5.2.5][12])**.** *If a complete analytic Huber pair (A, A^+) is stably uniform, then it is sheafy.*

Remark 3.14. It turns out that perfectoid rings are examples of such things.

Now, as in classical geometry, we find :

Theorem 3.15 ([Hub94, Proposition 2.1])**.** *The functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ from sheafy complete Huber pairs to adic spaces is fully faithful.*

Since adic spaces are locally modelled on sheafy complete Huber pairs, we can use this to deduce:

Corollary 3.16. *The restricted Yoneda embedding, from the category of adic spaces to the category of presheaves on complete Huber pairs is fully faithful.*

We now use this to briefly describe preadic spaces:

Definition 3.17. Let CAff denote the category of complete Huber pairs. We equip $\mathrm{CAff}^{\mathrm{op}}$ with the topology generated by rational covers, i.e. families $(A, A^+) \to (B_i, B_i^+)$ of Huber pairs inducing a cover of $\mathrm{Spa}(A, A^+)$ by rational open subsets.

For $(A, A^+) \in \mathrm{CAff}$, let $\mathrm{Spa}^Y(A, A^+)$ denote the sheafification with respect to this topology of the representable presheaf. If (A, A^+) is sheafy, it follows that $\mathrm{Spa}^Y(A, A^+)$ is just the representable presheaf.

A map of sheaves $\mathcal{F} \to \mathrm{Spa}^Y(A, A^+)$ is called open if there is an open subset U of $\mathrm{Spa}(A, A^+)$ such that

$$\mathcal{F} \cong \mathrm{colim}_{V \subset U \text{ rational}} \mathrm{Spa}^Y(\mathcal{O}_{\mathrm{Spa}(A,A^+)}(V), \mathcal{O}^+_{\mathrm{Spa}(A,A^+)}(V))$$

A map of sheaves is called an open immersion if it becomes one after pulling back to any $\mathrm{Spa}^Y(A, A^+)$.

Definition 3.18. A sheaf $\mathcal{F}$ on $\mathrm{CAff}^{\mathrm{op}}$ is called a pre-adic space if $\mathcal{F} \cong \mathrm{colim}_{\mathrm{Spa}^Y(A,A^+) \to \mathcal{F} \text{ open}} \mathrm{Spa}^Y(A, A^+)$.

The more general set-up of pre-adic spaces will be convenient when working out some examples, which is the purpose of the next section.

Remark 3.19. While it is clear that the category of sheaves admits pullbacks, this is actually not so clear for the category of pre-adic spaces. In [SW20, 4.1], the authors do state that pullbacks among adic spectra of Huber pairs (A, A^+) in which A is finitely generated over a ring of definition $A_0 \subset A^+$ do exist in pre-adic spaces.

[12] See the references therein.

4 More examples

4.1 Points in the closed unit disk

Recall that the unit disk D over a base C is given by the Huber pair $(C\langle T\rangle, C^\circ\langle T\rangle)$. In this section, we describe, **without proof**, the points of $D_C := \mathrm{Spa}(C\langle T\rangle, C^\circ\langle T\rangle)$, when C is an algebraically closed nonarchimedean field such as $\mathbb{C}_p$. Our reference, which we follow quite closely, is [Wei17, §1.7] We fix an absolute value $|-|$ inducing the topology on C.

Under additional assumptions on C, some of these points do not exist, but in general, all of them can appear. It turns out that there are five types of points, which we now list (types 2 and 3 can be described similarly, hence the single bullet point):

- Points α of C with $|\alpha| \leq 1$ give valuations on $C\langle T\rangle$ of the form $f \mapsto |f(\alpha)|$;
- Closed subdisks of the unit disk of C also give points. Let $\alpha \in C, |\alpha| \leq 1$ and $r \in (0,1]$, let $D(\alpha, r) = \{x \in C \mid |x-\alpha| \leq r\}$. The pair (α, r) gives a point $f \mapsto \sup_{x \in D(\alpha,r)} |f(x)|$. If $f = \sum_n a_n (T-\alpha)^n$ is a series expansion of f at α, this can be computed as $\sup_n |a_n| r^n$. These points turn out to have different behaviour depending on whether r can be attained in C or not: if $r \in |C|$, the point is of type 2, and if $r \notin |C|$ it is of type 3.
- If D_i is a dicreasing sequence of closed subdisks of the unit disk with empty intersection[13], the following also defines a valuation : $f \mapsto \inf_i \sup_{x \in D_i} |f(x)|$.
- Up until now, all the valuations had "rank 1", that is, they had values in $\mathbb{R}_{>0}$ - points of type 5 have rank 2. Fix (α, r) as in points of type 2, and a sign $\pm$ - the sign cannot be positive if $r = 1$, and fix $\gamma = \frac{1}{2}$ if the sign is negative, 2 if the sign is positive. The corresponding point is then $f \mapsto \sup_n (|a_n| r^n, \gamma^n) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ equipped with the lexicographic order, where $f = \sum_n a_n (T-\alpha)^n$. One thinks of (x, γ) as being infinitesimally larger than x if the sign is positive, and infinitesimally smaller if the sign is negative (which is why, when $r = 1$, the former is not allowed).

Type 2 points are not closed: the closure of $D(\alpha, r)$ contains $D(\alpha, r)^{\pm}$. All the other points are closed.

Note that the points of type 5 have higher rank, and they are needed for the closed disk to be connected. In particular, this is one of the reasons that higher rank valuations were considered in the definition of Spa. The disconnecting cover, without these rank 2 points, would be $U = \{|T| = 1\} = \{1 \leq |T| \neq 0\}$ (the boundary of the disk - note that $T \in C^\circ\langle T\rangle$, so every $v \in D_C$ satisfies $|T| \leq 1$) and $V = \bigcup_n U(\frac{T^n}{p})$ (which, we will see in a second, is the open unit disk).

4.2 The open unit disk

Let us describe the open disk $\overset{\circ}{D}$.

[13] These can exist even in a complete situation, in the nonarchimedean world. The condition that they do not exist turns out to be equivalent to something called "spherical completeness".

We define it as follows: $\mathrm{Spa}(\mathbb{Z}_p[[T]])\times_{\mathrm{Spa}(\mathbb{Z}_p)}\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$. Note that $\mathrm{Spa}(\mathbb{Z}_p)$ has exactly two points, corresponding to the trivial valuation on $\mathbb{F}_p$ pulled back to $\mathbb{Z}_p$, and corresponding to the p-adic valuation - the latter is a generic point η, picked out exactly by $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$.

We have not proved that adic spaces have pullbacks, and indeed this seems to be a subtle point. Nonetheless we can define this pullback in preadic spaces and it sill makes sense.

Remark 4.1. One way to see the subtlety of pullbacks in this context is as follows: one would expect pullbacks of affinoids to be given by affinoids, in analogy with classical algebraic geometry. On the other hand, we shall see here that this pullback is simply not affinoid.

To understand this pullback, fix a complete Huber pair (A,A^+) and let us compute maps from $\mathrm{Spa}(A,A^+)$ into the pullback.

Firstly, maps to $\mathrm{Spa}(\mathbb{Z}_p)$ are the same as continuous maps $\mathbb{Z}_p\to A$, which are no data, but the condition that p be topologically nilpotent in A.

Second, maps to $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$ are the same as the above with the extra condition that p be invertible in A.

Finally, (given a map to $\mathrm{Spa}(\mathbb{Z}_p)$) maps to $\mathrm{Spa}(\mathbb{Z}_p[[T]])$ are given by a topologically nilpotent element T.

All in all, maps to the pullback exist if and only if p is invertible and topologically nilpotent, and they are then given by the choice of a topologically nilpotent element $a\in A$. Let a be such an element, and v a continuous valuation on A. By continuity of v, there is an n with $v(a)^n\leq v(p)$. Thus, since p is invertible, v is in $U(\frac{a^n}{p})$ for some n, and so the map $\mathrm{Spa}(A,A^+)\to\mathrm{Spa}(\mathbb{Z}_p[[T]])\times_{\mathrm{Spa}(\mathbb{Z}_p)}\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$ factors through $U(\frac{T^n}{p})$ for some n, and so $\mathrm{Spa}(\mathbb{Z}_p[[T]])\times_{\mathrm{Spa}(\mathbb{Z}_p)}\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)\cong\bigcup_n U(\frac{T^n}{p})$ (this union can be taken in preadic spaces, or as subspaces of $\mathrm{Spa}(\mathbb{Z}_p[[T]])$, since $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)\to\mathrm{Spa}(\mathbb{Z}_p)$ is an inclusion).

Since clearly none of them covers the whole pullback, we find that the pullback is *not* quasicompact, in particular not affinoid.

Let us point out that the above computation makes it clearer (at least to the author) that this really is an open unit disk.

We close this example by pointing out that an elaboration of this analysis can help describe $X=|\mathrm{Spa}(\mathbb{Z}_p[[T]])|$ fully. This adic spectrum has a stupid point $x_{\mathbb{F}_p}$ given by the trivial valuation on $\mathbb{F}_p$ pulled back along $\mathbb{Z}_p[[T]]\to\mathbb{F}_p$. Away from this, p and T are never simultaneously 0, and so for $v\in X\setminus\{x_{\mathbb{F}_p}\}$ following definition makes sense :

$$\kappa(v):=(\{\frac{m}{n}\in\mathbb{Q}_{>0}\mid |T(v)|^n\leq|p(v)|^m\},\{\frac{m}{n}\in\mathbb{Q}_{>0}\mid |T(v)|^n>|p(v)|^m\})$$

For any v, these two sets form a partition of $\mathbb{Q}_{>0}$. Since T is topologically nilpotent, and so is p, neither is empty.

Furthermore, if $|T(v)|^n\leq|p(v)|^m$ and $\frac{m'}{n'}\leq\frac{m}{n}$, i.e. $nm'\leq n'm$, we find $|T(v)|^{n'm}\leq|T(v)|^{nm'}\leq|p(v)|^{mm'}$ so that $|T(v)|^{n'}\leq|p(v)|^{m'}$ and so the first of these two sets is downwards closed[14]. Similarly, the second one is upwards closed.

With a tad more work, one can prove that this (almost) forms a Dedekind cut, i.e. fully determines a nonnegative real number (or ∞). Thus κ can be viewed as function $X\setminus\{x_{\mathbb{F}_p}\}\to\mathbb{R}_{\geq0}\cup\{\infty\}$. The set $\kappa^{-1}((0,\infty])$ is the set where there exists n such that $|T(v)|^n\leq|p(v)|$, so that it is exactly the open disk we discussed right before.

[14]We used implicitly that T is topologically nilpotent so that $|T(v)|\leq1$.

On the other hand $\kappa^{-1}(0)$ is exactly the set where $|p(v)| = 0$, so it is exactly $\mathrm{Spa}(\mathbb{F}_p[[T]]) \setminus \{x_{\mathbb{F}_p}\}$, and similarly $\kappa^{-1}(\infty) \setminus \{x_{\mathbb{F}_p}\} = \mathrm{Spa}(\mathbb{Z}_p)$. The latter we explained earlier consists of one point, and the same argument essentially works for the former, where there is one remaining point $x_{\mathbb{F}_p((t))}$ except for $x_{\mathbb{F}_p}$. There is a nice picture in [SW20, Figure 1].

4.3 The affine line

We saw that $\mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$ is supposed to be a closed unit disk, and that there is no reasonable way to make $\mathbb{Q}_p[[T]]$ into a Huber ring. Thus the definition of the affine line has to be slightly different. Specifically, let $f : \mathbb{Q}_p\langle T\rangle \to \mathbb{Q}_p\langle T\rangle$ send T to pT, and consider $\mathbb{A}^1_{\mathbb{Q}_p} := \mathrm{colim}_n \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$, where the colimit is taken along f^*.

Clearly, $\mathbb{A}^1_{\mathbb{Q}_p}$ is not quasicompact.

4.4 The projective line

The projective line $\mathbb{P}^1_{\mathbb{Q}_p}$ is built by gluing D with itself along its boundary $\{|T| = 1\}$, represented as $\mathrm{Spa}(\mathbb{Q}_p\langle T, T^{-1}\rangle)$, and specifically along $T \mapsto T^{-1}$.

A Totally ordered abelian groups

Definition A.1. A totally ordered abelian group is an abelian group Γ equipped with a total order $\leq$ such that for all $x, y, z \in \Gamma$, if $x \leq y$ then $xz \leq yz$.

We will typically denote them multiplicatively.

Example A.2. $\mathbb{R}_{>0}$ equipped with the obvious ordering and the multiplicative structure is a totally ordered abelian group, isomorphic (via log and exp) to $\mathbb{R}$ with addition.

Lemma A.3. *A totally ordered abelian group is torsion-free.*

In particular, for $x, y \in \Gamma$, $x \leq y$ if and only if $x^n \leq y^n$.

Proof. Suppose x is torsion, so $x^n = 1$. If $x < 1$, then $x^2 < x$ and by induction $x^n < 1$, which is a contradiction. The same argument works if $x > 1$. All in all, $x = 1$.

If $x \leq y$ then $x^n \leq y^n$, so the totality of the order guarantees the result. ☐

We briefly discuss ranks.

Definition A.4. Let $H \subset \Gamma$ be a subgroup of a totally ordered abelian group. H is called convex if it is an interval, i.e. if $h, h' \in H, \gamma \in \Gamma$ satisfy $h \leq \gamma \leq h'$ then $\gamma \in H$.

Note that convexity follows from the same condition in the special case $h = 1$.

Lemma A.5. *Suppose $H, H' \subset \Gamma$ are convex subgroups of a totally ordered abelian group. Either $H \subset H'$ or $H' \subset H$.*

Proof. Suppose that for all $h \in H_{\geq 1}$, there exists $h' \in H'$ such that $h \leq h'$.

In this case, convexity of H' implies that $H_{\geq 1} \subset H'$ and thus $H \subset H'$.

If not, there exists $h \in H_{\geq 1}$ such that for all $h' \in H', h > h'$. Thus convexity of H implies that $H'_{\geq 1} \subset H$ and so $H' \subset H$. ☐

Definition A.6. The rank of a totally ordered abelian group is the cardinality of the set of its nontrivial convex subgroups.

Remark A.7. Since this set is totally ordered, if it is finite, then its rank characterizes the total order of this set of convex subgroups.

More generally, the order type of this set of subgroups is probably a better invariant.

The following is an instructive exercise:

Example A.8. The rank of $\mathbb{R}^n_{>0}$ equipped with the lexicographic ordering is n, and its convex subgroups are exactly $\{0\}^k \times \mathbb{R}^{n-k}_{>0}, k \in \{0, ..., n-1\}$ ($k = n$ is the trivial subgroup).

Construction A.9. Let $H \subset \Gamma$ be a convex subgroup of a totally ordered abelian group. The quotient Γ/H acquires a canonical order via setting $\overline{g} \leq \overline{g}'$ if and only if $g \leq g'$. The point of convexity here is that this does not depend on the choice of lifts g, g' : if $g \leq g'$ and $g' \leq gh$, then $1 \leq g'g^{-1} \leq h$ so that $g'g^{-1} \in H$, and so $\overline{g} = \overline{g}'$.

B Specializations among valuations

Recall the following definition:

Definition B.1. Let X be a topological space. A point x is a specialization of x', written $x' \rightsquigarrow x$ if x is in the closure of $\{x'\}$. x' is called a generalization of x.

For a general topological space, understanding specializations may not say much about the topology, but for example, for a finite space, the specialization poset describes the whole topology (and in fact establishes an equivalence between finite preordered sets and finite topological spaces).

For spaces that "look" like finite ones, such as spectral spaces, the specialization order may give a lot of information, though not all of it. This is the case for quasi-compact schemes, and for $\mathrm{Spa}(A, A^+)$ by Theorem 2.31. We discuss specializations in that space.

Recall that there is a canonical map $\mathrm{supp} : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A)$ sending a valuation to its support (where it vanishes). There are two important types of specializations: the vertical ones, happening within the fibers of supp, and the horizontal ones, happening above the ones in $\mathrm{Spec}(A)$. We now describe them.

Construction B.2. Let $v : A \to \Gamma \cup \{0\}$ be a (continuous) valuation on a (Huber) ring A, and let $H \subset \Gamma$ be a convex subgroup.

Define $v_{/H}$ to be the composite $A \to \Gamma \cup \{0\} \to (\Gamma/H) \cup \{0\}$.

Note that $\mathrm{supp}(v_{/H}) = \mathrm{supp}(v)$, so they lie in the same fiber. Note:

Lemma B.3. *For any valuation $v : A \to \Gamma \cup \{0\}$, the valuation $v_{/H}$ is a generalization of v, so $v_{/H} \rightsquigarrow v$. If v is continuous, so is $v_{/H}$.*

Proof. Let $\{w \mid |f(w)| \leq |g(w)| \neq 0\}$ be an open containing v. That is, $f, g \in A$ are such that $v(f) \leq v(g) \neq 0$. Both of these properties are clearly preserved under composition with $\Gamma \cup \{0\} \to (\Gamma/H) \cup \{0\}$.

The continuity property is also clear. $\square$

Definition B.4. We call these vertical generalizations, or vertical specializations.

The second type of specialization changes the support.

Definition B.5. Let $v : A \to \Gamma \cup \{0\}$ be a specialization, and let $c\Gamma_v$ denote the smallest convex subgroup of Γ containing $\Gamma_{\geq 1} \cap v(A)$. It is called the characteristic subgroup of v.

Construction B.6. Let $v : A \to \Gamma \cup \{0\}$ be a (continuous) valuation on a (Huber) ring A, and let $H \subset \Gamma$ be a convex subgroup containing the characteristic subgroup $c\Gamma_v$.

Define $v_{|H} : A \to H \cup \{0\}$ as $v(a)$ if $v(a) \in H$, and 0 else.

Note that it is a priori not obvious that this is in fact a valuation but the conditions on H guarantee that it is.

Its support consist of the a's in A such that $v(a) \notin H$ which certainly contains the support of v, i.e. $\mathrm{supp}(v) \subset \mathrm{supp}(v_{|H})$ so $\mathrm{supp}(v_{|H})$ is a specialization of $\mathrm{supp}(v)$ in $\mathrm{Spec}(A)$. We claim that this is already so in $\mathrm{Spa}(A, A^+)$.

Lemma B.7. *Let $v : A \to \Gamma \cup \{0\}$ be a valuation on a ring A, and let $H \subset \Gamma$ be a convex subgroup containing the characteristic subgroup $c\Gamma_v$.*

There is a specialization $v \rightsquigarrow v_{|H}$. If v is continuous, so is $v_{|H}$.

Proof. Let $\{w \mid |f(w)| \leq |g(w)| \neq 0\}$ be an open containing $v_{|H}$, so $v_{|H}(f) \leq v_{|H}(g) \neq 0$.

The second condition guarantees that $v_{|H}(g) = v(g) \neq 0$. Now suppose $v(g) < v(f)$. If $v(f) \geq 1$, then $v(f) \in H$ and so $v(f) = v_{|H}(f)$, which is a contradiction. So $v(f) < 1$. But then $v(g) < v(f) < 1$ and the convexity of H guarantee that $v(f) \in H$, which implies again that $v(f) = v_{|H}(f)$.

So $v(f) \leq v(g)$, and so v is also in this open, as claimed.

Continuity is left to the reader. $\square$

Definition B.8. We call horizontal specializations those of the form $v \rightsquigarrow v_{|H}$ for H a convex subgroup of Γ_v containing the characteristic subgroup $c\Gamma_v$.

There is now a result that puts vertical and horizontal specializations at the forefront of the topology of $\mathrm{Spa}(A, A^+)$, namely:

Proposition B.9. *Any specialization $v \rightsquigarrow w$ in $\mathrm{Spa}(A, A^+)$ can be written as a vertical specialization followed by a horizontal ones.*

If either $v(A) \not\subset \Gamma_{\leq 1}$ or $v(A \setminus \mathrm{supp}(w)) \neq 1$, then it can also be written as a horizontal specialization followed by a vertical one.

We can now draw $\mathrm{Spa}(\mathbb{Z})$ lying over $\mathrm{Spec}(\mathbb{Z})$:

$$\begin{tikzcd}
 & & \cdots & & & & & \\
 & & v_7 \arrow[rrrrddd, rightsquigarrow] & & & & & \\
 & \eta \arrow[ru, rightsquigarrow] \arrow[r, rightsquigarrow] \arrow[rd, rightsquigarrow] \arrow[rdd, rightsquigarrow] & v_5 \arrow[rrrdd, rightsquigarrow] & & & & & \\
 \mathrm{Spa}(\mathbb{Z}) & & v_3 \arrow[rrd, rightsquigarrow] & & & & & \\
 & & v_2 \arrow[r, rightsquigarrow] & x_{\mathbb{F}_2} & x_{\mathbb{F}_3} & x_{\mathbb{F}_5} & x_{\mathbb{F}_7} & \cdots \\
 & & \eta \arrow[rd, rightsquigarrow] \arrow[rrd, rightsquigarrow] \arrow[rrrd, rightsquigarrow] \arrow[rrrrd, rightsquigarrow] & & & & & \\
 \mathrm{Spec}(\mathbb{Z}) & & & 2 & 3 & 5 & 7 & \cdots
\end{tikzcd}$$

where: the leftmost specializations are vertical (despite the drawing) and lie in the fiber over the generic point $\eta \in \mathrm{Spec}(\mathbb{Z})$, the ones of the form $v_p \to x_{\mathbb{F}_p}$ are horizontal and lie above the specializations $\eta \rightsquigarrow p$.

References

[BSSW24] Tobias Barthel, Tomer M Schlank, Nathaniel Stapleton, and Jared Weinstein. On the rationalization of the $K(n)$-local sphere. *arXiv preprint arXiv:2402.00960*, 2024.

[CS] B Conrad and Seminaire Scholze. Stanford, number theory learning seminar 2014–2015.

[Hub93] Roland Huber. Continuous valuations. *Math. Z*, 212(3):455–477, 1993.

[Hub94] Roland Huber. A generalization of formal schemes and rigid analytic varieties. *Mathematische Zeitschrift*, 217:513–551, 1994.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley Lectures on p-adic Geometry:(AMS-207)*. Princeton University Press, 2020.

[Wei17] Jared Weinstein. Arizona winter school 2017: Adic spaces. *Lecture Notes*, 2017.