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On the parameterized Tate construction

J. D. Quigley[1] | **Jay Shah**[2]

[1]Department of Mathematics, University of Virginia, Charlottesville, Virginia, USA

[2]Fachbereich Mathematik und Informatik, Universität Münster, Münster, Germany

Correspondence
Jay Shah, Fachbereich Mathematik und Informatik, Universität Münster, 48149 Münster, Germany.
Email: jayhshah@gmail.com

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Abstract

We introduce and study a genuine equivariant refinement of the Tate construction associated to an extension $\widehat{G}$ of a finite group G by a compact Lie group K, which we call the parameterized Tate construction $(-)^{t_G K}$. Our main theorem establishes the coincidence of three conceptually distinct approaches to its construction when K is also finite: one via recollement theory for the K-free $\widehat{G}$-family, another via parameterized ambidexterity for G-local systems, and the last via parameterized assembly maps. We also show that $(-)^{t_G K}$ uniquely admits the structure of a lax G-symmetric monoidal functor, thereby refining a theorem of Nikolaus and Scholze. Along the way, we apply a theorem of the second author to reprove a result of Ayala–Mazel-Gee–Rozenblyum on reconstructing a genuine G-spectrum from its geometric fixed points; our method of proof further yields a formula for the geometric fixed points of an $\mathcal{F}$-complete G-spectrum for any G-family $\mathcal{F}$.

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Contents

1 | INTRODUCTION

Let G be a finite group and M a G-module. The *Tate cohomology* $\widehat{H}^*(G;M)$ of G with coefficients in M was introduced by Tate in [80]. It combines information about three important invariants: group cohomology $H^*(G;M)$, group homology $H_*(G;M)$, and the additive norm map $\mathrm{Nm} : M_G \to M^G$ from coinvariants to invariants. Swan [79] used Tate cohomology to define an equivariant cohomology theory for G-spaces.

Greenlees [29] initiated the study of a vast generalization of Tate cohomology using equivariant homotopy theory. Let X be a spectrum with G-action. The *Tate construction* X^{tG} is defined by

$$X^{tG} := (\widetilde{EG} \wedge F(EG_+, X))^G,$$

where EG is a contractible space with free G-action and $\widetilde{EG}$ is the cofiber of the map $EG_+ \to S^0$ which collapses EG to the non-basepoint. This construction generalizes classical Tate cohomology by taking $X = HM$ to be the Eilenberg–MacLane spectrum for a G-module M, as

$$\pi_*(HM^{tG}) \cong \widehat{H}^{-*}(G;M).$$

The Tate construction was generalized to compact Lie groups and studied extensively by Greenlees and May in [30], where they constructed the primary computational tool for analyzing Tate constructions, the Tate spectral sequence.

The Tate construction has many applications in algebraic topology. In chromatic homotopy theory, the Tate spectrum often decreases chromatic complexity [3, 8, 23, 24, 30, 31, 45] and its vanishing controls descent [57, 58, 71]. In trace methods, the Tate construction arises when computing topological cyclic homology, an approximation to algebraic K-theory [36, 37, 67]. In equivariant stable homotopy theory, the Tate construction naturally arises in the construction of equivariant Adams spectral sequences [29, 46] and analysis of the Hill–Hopkins–Ravenel slice spectral sequence [40, 62, 81].

The input of the Tate construction is a spectrum with G-action and the output is an ordinary (nonequivariant) spectrum. The purpose of this paper is to study the *parameterized Tate construction*, a generalization of the Tate construction better suited for studying genuine[†] G-spectra. The input of the parameterized Tate construction will be a G-spectrum (for a finite group G) with a twisted action by a compact Lie group K and the output will be a G-spectrum.[‡] To make the idea of twisted action precise, we will use the formalism of *parameterized ∞-categories* as developed by Barwick–Dotto–Glasman–Nardin and the second author [10, 65, 66, 74, 75]. Let $\mathcal{O}_G$ denote the orbit category of G.

Definition 1.1. A *G-∞-category*, respectively, *G-space* is a cocartesian, respectively, left fibration $C \to \mathcal{O}_G^{\mathrm{op}}$. A *$G$-functor* $F : C \to D$ is then a morphism over $\mathcal{O}_G^{\mathrm{op}}$ that preserves cocartesian edges.

Remark 1.2. Under the straightening correspondence, a G-∞-category, respectively, G-space C is equivalently specified by a presheaf on $\mathcal{O}_G$ valued in ∞-categories, respectively, spaces, while a G-functor corresponds to a natural transformation of presheaves. By abuse of notation, we switch freely between these two perspectives.

Example 1.3. We have the G-∞-categories $\underline{\mathrm{Spc}}_G$ and $\underline{\mathrm{Sp}}^G$ of G-spaces and G-spectra.[§] Their respective fibers over an orbit $U \cong G/H$ are given by the ∞-categories $\mathrm{Spc}_H \simeq \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathrm{Spc})$ and Sp^H of H-spaces and H-spectra, and the cocartesian edges encode the functoriality of restriction and conjugation.

Now let

$$\psi = [1 \to K \to \widehat{G} \to G \to 1]$$

be a group extension, regard BK as a space with G-action via the Kan fibration $B\widehat{G} \to BG$, and let $B_G^{\psi}K$ be the G-space classified by the right Kan extension of BK along the inclusion $BG \subset \mathcal{O}_G^{\mathrm{op}}$.

Definition 1.4 (Definition 3.19). A *G-spectrum with ψ-twisted K-action* is a G-functor $X : B_G^{\psi}K \to \underline{\mathrm{Sp}}^G$.

Suppose for now that $\widehat{G}$ is also finite. The most expeditious definition of the parameterized Tate construction proceeds through the following theorem, a parameterized analogue of the canonical

[†] In this paper, we usually drop the qualifier "genuine" and simply refer to these as G-spectra.

[‡] Note the switch of notation: K has now taken the role that G used to play.

[§] We place the G differently to emphasize that $\underline{\mathrm{Sp}}^G$ is not the fiberwise stabilization of $\underline{\mathrm{Spc}}_G$.

embedding of spectra with G-action into G-spectra as the Borel complete (i.e., cofree) G-spectra. Recall that given a G-family $\mathcal{F}$ with universal G-space $E\mathcal{F}$, a G-spectrum X is *$\mathcal{F}$-torsion* if $E\mathcal{F}_+ \wedge X \xrightarrow{\simeq} X$ and *$\mathcal{F}$-complete* if $X \xrightarrow{\simeq} F(E\mathcal{F}_+, X)$, cf. Subsection 2.2.

Notation 1.5 (Notation 3.23). Let Γ_K be the $\widehat{G}$-family of *K-free* subgroups, that is, those $H \leqslant \widehat{G}$ such that $H \cap K = 1$.

Theorem A (Theorem 3.24). *Let*

$$\psi = [1 \to G \to \widehat{G} \to K \to 1]$$

be a group extension in which $\widehat{G}$ is finite. There exists a symmetric monoidal restriction functor[†]

$$j^* : \mathrm{Sp}^{\widehat{G}} \to \mathrm{Fun}_G\left(B^{\psi}_G K, \underline{\mathrm{Sp}}^G\right)$$

that participates in an adjoint triple

$$\begin{tikzcd} \mathrm{Fun}_G(B^{\psi}_G K, \underline{\mathbf{Sp}}^G) \arrow[r, hook, shift left=2.5ex, "j_!"] \arrow[r, hook, shift right=2.5ex, "j_*"'] & \mathbf{Sp}^{\widehat{G}} \arrow[l, "j^*"'] \end{tikzcd}$$

in which $j_!$ and j_* are fully faithful and embed $\mathrm{Fun}_G(B^{\psi}_G K, \underline{\mathrm{Sp}}^G)$ as the Γ_K-torsion and Γ_K-complete $\widehat{G}$-spectra, respectively.

Definition 1.6. The *parameterized Tate construction*

$$(-)^{t_K G} : \mathrm{Fun}_G\left(B^{\psi}_G K, \underline{\mathrm{Sp}}^G\right) \to \mathrm{Sp}^G$$

is the composite lax symmetric monoidal functor

$$\mathrm{Fun}_G\left(B^{\psi}_G K, \underline{\mathrm{Sp}}^G\right) \overset{j_*}{\hookrightarrow} \mathrm{Sp}^{\widehat{G}} \xrightarrow{-\wedge \widetilde{E\Gamma_K}} \mathrm{Sp}^{\widehat{G}} \xrightarrow{\Psi^K} \mathrm{Sp}^G,$$

where

(1) j_* is the embedding of Theorem A,
(2) $\widetilde{E\Gamma_K}$ is the cofiber of the map $E\Gamma_{K+} \to S^0$, and
(3) Ψ^K is the categorical K-fixed points.[‡]

Examples 1.7.

(1) If we take

$$\psi = [1 \to K \to K \to 1 \to 1],$$

then the parameterized Tate construction recovers the ordinary K-Tate construction.

[†] We endow $\mathrm{Fun}_G(B^{\psi}_G K, \underline{\mathrm{Sp}}^G)$ with the pointwise symmetric monoidal structure of Definition A.1.

[‡] We write this as Ψ^K to distinguish it from the spectrum-valued functor of categorical K-fixed points $(-)^K : \mathrm{Sp}^{\widehat{G}} \to \mathrm{Sp}$.

(2) Let μ_{p^n} be the group of p^nth roots of unity with C_2-action given by inversion and let

$$\psi = [1 \to \mu_{p^n} \to D_{2p^n} = \mu_{p^n} \rtimes C^2 \to C_2 \to 1].$$

The associated parameterized Tate construction $(-)^{t_{C_2}\mu_{p^n}}$ features extensively in the authors' work on real cyclotomic spectra [70]. We discuss this further in Remark 1.13.

(3) The case

$$\psi = [1 \to \mu_2^{\times 2^{n-1}} \to \mu_2^{\times 2^{n-1}} \rtimes C_{2^n} \to C_{2^n} \to 1],$$

where C_{2^n} acts by cyclically permuting the factors of $\mu_2^{\times 2^{n-1}}$, is analyzed in forthcoming work of the first author with Chatham–Li–Lorman. The $n = 1$ case was used in [50] to obtain a Tate splitting for real Johnson–Wilson theories and in [68] to understand the C_2-equivariant stable stems. These applications are outlined further in Remark 1.14.

Remark 1.8 (Extension to compact Lie groups). With suitable modifications, our proof of Theorem A (and consequently, Definition 1.6) also applies in the more general situation where K is a compact Lie group. The reason that we state and prove Theorem A only when K is finite is due to our choice of foundations for equivariant stable homotopy theory. In this paper, we adopt the foundations laid down by Bachmann and Hoyois [7, section 9], who attach to every profinite groupoid X a presentable, stable, and symmetric monoidal ∞-category $\mathrm{SH}(X)$ such that for $X = BG$, $\mathrm{SH}(BG)$ is equivalent to the underlying ∞-category Sp^G of the category of orthogonal G-spectra. To avoid mixing different approaches to the foundations of G-spectra, we then wish to avoid any mention of Sp^G for an infinite compact Lie group G. Instead, we will define the parameterized Tate construction when K is compact Lie via the machinery of parameterized assembly maps (cf. Theorem D).

To justify our claim that the parameterized Tate construction is a suitable genuine equivariant refinement of the Tate construction, we then have the following suite of basic results.

Observation 1.9 (Norm cofiber sequence). For the constant G-diagram functor

$$\delta : \mathrm{Sp}^G \to \mathrm{Fun}_G\left(B^{\psi}_G K, \underline{\mathrm{Sp}}^G\right),$$

define the *parameterized homotopy orbits* $(-)_{h_G K}$ to be its left adjoint and the *parameterized homotopy fixed points* to be its right adjoint.[†] The parameterized Tate construction is then designed to "measure the difference" between $(-)_{h_G K}$ and $(-)^{h_G K}$.

More precisely, we have that the adjoint triple $j_! \dashv j^* \dashv j_*$ of Theorem A is one half of the recollement on $\mathrm{Sp}^{\widehat{G}}$ determined by the idempotent E_∞-algebra $\widetilde{E\Gamma_K}$. Recollement theory then yields the cofiber sequence

$$j_!(-) \to j_*(-) \to j_*(-) \wedge \widetilde{E\Gamma_K}.$$

As Ψ^K is right adjoint to the inflation functor $\inf_G^{\widehat{G}}$ and $j^* \inf_G^{\widehat{G}} \simeq \delta$, we get that $\Psi^K j_* \simeq (-)^{h_G K}$. We also have the Adams-type isomorphism $\Psi^K j_! \simeq (-)_{h_G K}$ (Remark 4.30). Applying Ψ^K then yields

[†] These are the G-colimit and G-limit functors, respectively.

the *norm cofiber sequence*

$$(-)_{h_G K} \to (-)^{h_G K} \to (-)^{t_G K}.$$

Observation 1.10 (Geometric model). Let Y be a $\widehat{G}$-spectrum. By monoidal recollement theory, we have that $j_! j^* Y \simeq E\Gamma_{K+} \wedge Y$ and $j_* j^* Y \simeq F(E\Gamma_{K+}, Y)$. If we then consider $X = j^* Y$, we obtain a *geometric model* for the parameterized Tate construction[†]

$$X^{t_G K} \simeq \Psi^K (F(E\Gamma_{K+}, Y) \wedge \widetilde{E\Gamma_K}).$$

This identifies $(-)^{t_G K}$ as a special instance of the Greenlees–May generalized Tate construction for a $\widehat{G}$-family [30, part IV] and hence yields a spectral sequence with E_2-term given by certain Amitsur–Dress–Tate cohomology groups (Remark 4.33).

For the next observation, we write $\underline{\mathrm{Fun}}_G$ for the internal hom in G-∞-categories and recall that for a G-cocomplete G-∞-category such as $\underline{\mathrm{Sp}}^G$ and a (small) G-∞-category I, the G-colimit functor

$$\mathrm{colim}^G : \mathrm{Fun}_G(I, \underline{\mathrm{Sp}}^G) \to \mathrm{Sp}^G$$

always refines to a G-functor $\underline{\mathrm{Fun}}_G(I, \underline{\mathrm{Sp}}^G) \to \underline{\mathrm{Sp}}^G$ such that the fiber over G/H is given by the H-colimit. If $I = B^\psi_G K$, we then write $(-)_{\underline{h}_G K}$ for the G-colimit G-functor. Likewise, $\underline{\mathrm{Sp}}^G$ is also G-complete, and we write $(-)^{\underline{h}_G K}$ for the G-limit G-functor.

Observation 1.11 (Compatibility with restriction, Observation 4.34). Let X be a G-spectrum with ψ-twisted K-action, $H \leqslant G$ a subgroup, $\widehat{H} = \pi^{-1}(H)$ for the quotient map $\pi : \widehat{G} \to G$, and $\psi_H = [1 \to K \to \widehat{H} \to H \to 1]$. Regard X as a H-spectrum with ψ_H-twisted K-action via the restriction functor

$$\mathrm{Fun}_G \left(B^\psi_G K, \underline{\mathrm{Sp}}^G \right) \to \mathrm{Fun}_H \left(B^{\psi_H}_H K, \underline{\mathrm{Sp}}^H \right)$$

given by pulling back along $\mathcal{O}^{\mathrm{op}}_H \simeq (\mathcal{O}^{\mathrm{op}}_G)^{(G/H)/} \to \mathcal{O}^{\mathrm{op}}_G$. Then $\mathrm{res}^G_H X^{t_G K} \simeq X^{t_H K}$. In particular, the underlying spectrum of $X^{t_G K}$ is X^{tK}. More generally, $(-)^{t_G K}$ refines to a G-functor

$$(-)^{\underline{t}_G K} : \underline{\mathrm{Fun}}_G \left(B^\psi_G K, \underline{\mathrm{Sp}}^G \right) \to \underline{\mathrm{Sp}}^G,$$

and the norm cofiber sequence refines to a sequence of natural transformations of G-functors

$$(-)_{\underline{h}_G K} \to (-)^{\underline{h}_G K} \to (-)^{\underline{t}_G K}$$

whose fiber over G/H is given by the norm cofiber sequence

$$(-)_{h_H K} \to (-)^{h_H K} \to (-)^{t_H K}.$$

[†] As we have already seen, classically it is customary to conflate X and Y and simply write $X^{tK} \simeq (F(EK_+, X) \wedge \widetilde{EK})^K$.

Observation 1.12 (Residual action, Proposition 4.37). Let X as above. Suppose that $\widehat{G} \cong K \rtimes G$ for a G-action on K, and let $L \trianglelefteq K$ be a normal subgroup such that the inclusion is G-equivariant. Consider the group extensions

$$\psi' = [L \to L \rtimes G \to G], \quad \psi'' = [K/L \to K/L \rtimes G \to G],$$

and view X as a G-spectrum with ψ'-twisted L-action via restriction along $B^{\psi'}_G L \to B^{\psi}_G K$. Then $X^{t_G L}$ canonically acquires a "residual action" in the sense of lifting to a G-spectrum with ψ''-twisted K/L-action, and we have a fiber sequence of G-spectra[†]

$$(X_{h_G L})^{t_G K/L} \to X^{t_G K} \to (X^{t_G L})^{h_G K/L}.$$

Before stating our deeper results, we pause to explain some applications of the parameterized Tate construction.

1.1 | Motivation and applications

Our study of G-spectra with ψ-twisted K-action and the parameterized Tate construction $(-)^{t_K G}$ is motivated by a number of topics in equivariant stable homotopy theory. In particular, these concepts play a central role in our work on real algebraic K-theory, hyperreal oriented cohomology theories, and generalized Mahowald invariants. Our guiding philosophy is that the parameterized Tate construction should replace the ordinary Tate construction whenever a statement in classical stable homotopy theory might admit a genuine equivariant refinement. We discuss the implementation of this philosophy in the following remarks.

Remark 1.13 (Real algebraic K-theory). Hesselholt–Madsen [38], Schlichting [73], Spitzweck [77], Heine–Spitzweck–Verdugo [35], and Calmès et al. [15–17] have all defined *real algebraic K-theory*.[†] All approaches ultimately yield a genuine C_2-equivariant refinement of algebraic K-theory whose categorical C_2-fixed points are Grothendieck–Witt theory (i.e., hermitian K-theory) and whose geometric C_2-fixed points are some flavor of algebraic L-theory.

By the Dundas–Goodwillie–McCarthy theorem [26], the algebraic K-theory of connective ring spectra is closely approximated by topological cyclic homology, an invariant obtained from the *cyclotomic structure* on topological Hochschild homology. Cyclotomic structures were originally defined using genuine S^1-equivariant homotopy theory [12, 13, 36]. An incredible insight by Nikolaus–Scholze in [67] is that for bounded-below spectra, cyclotomic structures can be

[†] Note that $X_{h_G L}$ acquires a residual action by a more basic result of parameterized higher category theory. Namely, consider the G-left Kan extension $\rho_! X$ of X along $\rho : B^{\psi}_G K \to B^{\psi}_G K/L$. By definition, $\rho_! X$ is a G-spectrum with ψ''-twisted K/L-action, and as ρ has G-fiber $B^{\psi}_G L$ (with the basepoint of $B^{\psi}_G K/L$ given by the semidirect product splitting), the underlying G-spectrum of $\rho_! X$ is indeed $X_{h_G L}$.

[†] Hesselholt–Madsen work with exact categories with duality and weak equivalences, while Schlichting works with dg categories with weak equivalences and duality. Spitzweck constructed a Grothendieck–Witt space and connective real K-theory C_2-spectrum for stable ∞-categories with duality, and subsequently Heine–Spitzweck–Verdugo defined the real algebraic K-theory of Waldhausen ∞-categories with genuine duality. In a separate line of development, Calmès et al. define the real algebraic K-theory of Poincaré ∞-categories, which were introduced by Lurie [55] in his course on algebraic L-theory and surgery.

described completely using Borel S^1-equivariant homotopy theory. The Tate construction features prominently in their perspective: given a bounded-below spectrum X with S^1-action, a cyclotomic structure on X amounts to a collection of S^1-equivariant *Tate-valued Frobenius maps* $\varphi_p : X \to X^{tC_p}$, one for each prime p.

A genuine C_2-equivariant refinement of cyclotomic spectra, called *real cyclotomic spectra*, was introduced by Høgenhaven in [43]. This notion mirrors the original definition of cyclotomic spectra and requires the use of genuine $O(2)$-equivariant homotopy theory. In [70], we refine the approach of Nikolaus–Scholze to define real cyclotomic spectra using C_2-spectra with twisted S^1-action and the parameterized Tate construction as the receptacle of the real cyclotomic Frobenius, and prove that this recovers the genuine formulation provided that the underlying spectrum of the C_2-spectrum is bounded-below. This will be used to study real algebraic K-theory in future work; we refer the reader to the introduction of [70] for further discussion.

The following result is a relatively straightforward application of the results mentioned above. It extends [30, Theorem 16.1] to the parameterized setting.

Theorem B (Inverse limit formula for the parameterized Tate construction, Theorem 4.40). *Let $\psi = [K \to \widehat{G} \to G]$ be an extension with $K \subseteq H$ for each $H \notin \Gamma_K$. Suppose V is as in Lemma 4.38 for $\mathcal{F} = \Gamma_K$, that is, V is a G-representation with $V^H = 0$ for $H \notin \Gamma_K$ and $V^H \neq 0$ for $H \in \Gamma_K$. For any $X \in \mathrm{Sp}^{\Phi\Gamma_K} \simeq \mathrm{Sp}^G$, we have*

$$(j^* i_* X)^{t[\psi]} \simeq \lim_n \left(B_G^{\psi} K^{-nV} \otimes \Sigma X \right).$$

This formula for the parameterized Tate construction has several applications in equivariant stable homotopy theory.

Remark 1.14 (Tate blueshift). *Hyperreal oriented cohomology theories* are genuine C_{2^n}-spectra which generalize complex oriented cohomology theories, or cohomology theories with Thom isomorphisms for complex vector bundles. When $n = 1$, a hyperreal oriented cohomology theory is a real oriented cohomology theory as introduced by Hu–Kriz in [46]. Hyperreal oriented cohomology theories for $n > 1$ were first applied in the Hill–Hopkins–Ravenel solution to the Kervaire invariant one problem in geometric topology [40]. They have since been applied to study the stable homotopy groups of spheres [34, 41, 51], where their fixed points model the fixed points of Lubin–Tate spectra.

The ordinary Tate construction was shown to decrease the height of certain complex oriented cohomology theories by Ando–Morava–Sadofsky [3]. The key computational input to their work is the isomorphism of graded rings

$$\pi_*(E^{tC_p}) \cong E^*((x))/[p](x),$$

where $[p](x)$ is the p-series of the formal group law associated to E. This isomorphism arises from a geometric description of the Tate construction proven by Greenlees–May in [30, section 16]. The fact that it is an isomorphism of graded rings relies on the multiplicativity of the Tate construction (cf. [30, p. 7]).

The parameterized Tate construction was used to produce analogous results for real oriented cohomology theories in [50] and will be applied in forthcoming work of the first author with

Chatham–Li–Lorman to study hyperreal oriented cohomology theories. The starting point for these computations is the observation that the parameterized Tate construction, instead of the ordinary Tate construction, can be used to access formal group law techniques for hyperreal oriented cohomology. Theorem B is used to construct spectrum-level splittings of the parameterized Tate construction.

Remark 1.15 (Generalized Mahowald invariants). Theorem B and the generalized Segal conjecture (cf. [2] and Theorem 5.35) can be used to define a G-equivariant Mahowald invariant

$$M(-) : \pi^G_\star(\mathbb{S}) \rightsquigarrow \pi^G_\star(\mathbb{S})$$

whenever there exists a $\widehat{G}$-representation V as in Lemma 4.38. The classical Mahowald invariant may be used to produce Hopf invariant one elements in $\pi_*(\mathbb{S})$: one has $\eta = M(2)$, $\nu = M(\eta)$, and $\sigma = M(\nu)$. Similarly, it was shown in [68] that the C_2-equivariant Mahowald invariant gives rise to the C_2-equivariant Hopf invariant one elements in $\pi^{C_2}_\star(\mathbb{S})$. It could be interesting to produce analogous elements for other groups.

The Mahowald invariant can often be approximated by studying $E^{t\mu_p}$, where E is a ring spectrum equipped with a trivial μ_p-action. The multiplicativity of the ordinary Tate construction is often useful in these analyses.[†] The multiplicativity of the parameterized Tate construction should have similar consequences for computing approximations to generalized Mahowald invariants.

1.2 | Deeper results and main theorems

Definition 1.6 of the parameterized Tate construction may be viewed as a generalization of Greenlees' geometric model for the ordinary Tate construction. Another important abstract perspective on Tate constructions is afforded by Lurie's theory of *ambidexterity* [56, section 6.1.6]. Suppose C is any semiadditive ∞-category that admits limits and colimits indexed by finite groupoids, suppose K is a finite group, and let X be an object in C with K-action. Then one has an additive *norm map*

$$\mathrm{Nm} : X_{hK} \to X^{hK}$$

whose cofiber defines the Tate construction X^{tK}. The vanishing of Tate (e.g., in the $T(n)$ or $K(n)$-local category [22, 48]) then leads to a rich theory of *higher semiadditivity* [19–21, 44].

Using the formalism of Beck–Chevalley fibrations, Hopkins and Lurie have set up a very general framework for producing norm maps in [44, section 4.1-2]; to apply this theory in the above example, they then prove that the ∞-category $\mathrm{LocSys}(C)$ of local systems on C constitutes a Beck–Chevalley fibration over Spc [44, section 4.3]. We initiate the theory of *parameterized ambidexterity* in this paper by proving that for a suitably G-cocomplete G-∞-category C, the ∞-category $\mathrm{LocSys}^G(C)$ of G-local systems on C is a Beck–Chevalley fibration over Spc_G (Corollary 4.6). We then explicate the relevant π-finiteness and truncatedness conditions. These conditions turn out to be related to the notion of *G-semiadditivity* studied in [65].

Notation 1.16. Let C be a G-∞-category and let U be a finite G-set with orbit decomposition $U \simeq \coprod_{i=1}^n U_i$. We then write $C_U := \prod_{i=1}^n C_{U_i}$. Moreover, for a map of finite G-sets $f : U \to V$, we write $f^* : C_V \to C_U$ for the evident restriction functor.

† For instance, it implies that the inverse limit Adams spectral sequence for $E^{t\mu_p}$ is multiplicative.

Definition 1.17. Let C be a G-∞-category. Then C *admits finite G-coproducts* if for every map of finite G-sets $f : U \to V$, $f^* : C_V \to C_U$ admits a left adjoint $f_!$, and the Beck–Chevalley condition is satisfied, that is, for every pullback square of finite G-sets

$$\begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "g'"] & V' \arrow[d, "g"] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

the exchange transformation $f'_! g'^* \Rightarrow g^* f_!$ is an equivalence. Dually, we have the evident notion of when C admits finite G-products, and we denote the resulting right adjoints as f_*.

Suppose now that C admits finite G-products and G-coproducts. We say that C is *G-semiadditive* if C is fiberwise pointed and for all maps of finite G-sets $f : U \to V$, the canonical natural transformation $\chi : f_! \Rightarrow f_*$ is an equivalence.[†]

We now have that for a suitably G-bicomplete G-semiadditive G-∞-category C and any G-functor $X : B^{\psi}_{G} K \to C$, one has a *parameterized norm map* (Definition 4.24)

$$\mathrm{Nm} : X_{h_G K} \to X^{h_G K}.$$

Theorem C (Theorem 4.29). *Let*

$$\psi = [1 \to K \to \widehat{G} \to G \to 1]$$

be a group extension with $\widehat{G}$ finite. Suppose $C = \underline{\mathrm{Sp}}^G$. Then the two parameterized norm maps constructed via ambidexterity theory and recollement theory (Observation 1.9) coincide.

Remark 1.18. After Theorem C, we establish Observations 1.11 and 1.12 as a corollary of some general properties of norm maps established in [44, section 4.2].

There is yet a third approach to the Tate construction via *assembly maps* in the sense of Weiss–Williams [82], which was first studied by John Klein [47] and later taken up in the ∞-categorical context by Nikolaus and Scholze [67, section I.3-4], who used it together with the multiplicative theory of the Verdier quotient to prove that the Tate construction uniquely admits a lax symmetric monoidal structure [67, Theorem I.4.1]. We now let K be a compact Lie group and write $\mathbb{S}^{\mathfrak{a}}$ for Σ^∞ of the one-point compactification of the adjoint representation $\mathfrak{a}$ of K. Suppose also that $\widehat{G} \simeq K \rtimes G$ for some G-action on K,[‡] and note that $\mathbb{S}^{\mathfrak{a}}$ canonically admits the structure of a G-spectrum with ψ-twisted K-action (Observation 5.56).

Theorem D. *There exists a parameterized assembly map*

$$(\mathbb{S}^{\mathfrak{a}} \wedge -)_{\underline{h}_G K} \xrightarrow{\alpha} (-)^{\underline{h}_G K} : \underline{\mathrm{Fun}}_G\left(B^{\psi}_G K, \underline{\mathrm{Sp}}^G\right) \to \underline{\mathrm{Sp}}^G$$

[†] χ is constructed as in [65, 5.2] or Definition 4.15.

[‡] We make this assumption to identify the dualizing G-spectrum as $\mathbb{S}^{\mathfrak{a}}$; the parameterized assembly map itself exists for any G-space. We also do not need to suppose $\widehat{G} \simeq K \rtimes G$ if K is finite, as the dualizing G-spectrum is then necessarily the unit.

such that:

(1) *if K is finite, α coincides with the norm map $(-)_{\underline{h}_G K} \to (-)^{\underline{h}_G K}$ of Observation 1.11;*
(2) *α is terminal among all natural transformations $\alpha' : F \to (-)^{h_G K}$ from G-colimit preserving G-functors F;*
(3) *α is an equivalence when restricted to the full G-subcategory $\underline{\mathrm{Fun}}_G(B^{\psi}_G K, \underline{\mathrm{Sp}}^G)^{\omega}$ of compact objects.*

Moreover, properties (2) and (3) uniquely specify α.

Proof. Combine Theorem 5.48, Remark 5.51, and Proposition 5.57. □

Given Theorem D, we may unambiguously write $(-)^{\underline{t}_G K}$ for the cofiber of α. We now wish to understand the multiplicative properties of $(-)^{\underline{t}_G K}$ as a G-functor. Note that, at least if K is finite, Definition 1.6 already endows $(-)^{t_G K}$ with the structure of a lax symmetric monoidal functor. However, in the context of G-∞-categories, more elaborate multiplicative structure on $\underline{\mathrm{Sp}}^G$ in the form of the *Hill-Hopkins-Ravenel norm functors* is present [7, 39, 40], and these endow $\underline{\mathrm{Sp}}^G$ with the structure of a *G-symmetric monoidal ∞-category* (Definition 5.7). One also has a pointwise G-symmetric monoidal structure on $\underline{\mathrm{Fun}}_G(I, \underline{\mathrm{Sp}}^G)$ (Example 5.14). We then do not expect to be able to uniquely characterize the lax symmetric monoidal structure on $(-)^{t_G K}$, but rather only a to-be-defined *lax G-symmetric monoidal structure* on $(-)^{\underline{t}_G K}$. Indeed, by developing the theory of parameterized Verdier quotients, G-$\otimes$-ideals, and induced G-objects, we are able to prove:

Theorem E (Corollary 5.60). *The G-functor $(-)^{\underline{t}_G K}$ and natural transformation $\beta : (-)^{\underline{h}_G K} \to (-)^{\underline{t}_G K}$ uniquely admit the structure of a lax G-symmetric monoidal functor and morphism thereof.*[†]

Warning 1.19. The universal property of the map β in Theorem E is with respect to mapping $(-)^{\underline{h}_G K}$ into lax G-symmetric monoidal functors that vanish on "induced objects". This turns out to be a subtle notion in the parameterized setting: see Definition 5.38.

Remark 1.20. In applications to real trace methods, Theorem E is important because it implies the real topological cyclic homology of a C_2-E_∞-algebra to again be a C_2-E_∞-algebra.

Finally, it is imperative for applications to be able to understand the various geometric fixed points of the parameterized Tate construction in terms of computationally accessible spectra. To this end, we leverage recent advances in reconstructing G-spectra from their geometric fixed points [4, 28] in order to give an explicit formula for the geometric fixed points of an $\mathcal{F}$-complete G-spectrum, for G a finite group and $\mathcal{F}$ any family of subgroups of G. To explain our result, we need to introduce a few more definitions, cf. Subsection 2.3.

Definition 1.21. Given a preordered set $(S, <)$, we let $\mathrm{sd}(S)$ denote its *barycentric subdivision*. This is the category whose objects are *strings* $[x_0 < x_1 < \cdots < x_n]$ in S and where a morphism

$$\overline{\alpha} : [x_0 < x_1 \cdots < x_n] \to [y_0 < y_1 < \cdots < y_m]$$

[†] As with [67, Theorem I.4.1(vi)], the uniqueness assertion is really about β and not $(-)^{\underline{t}_G K}$ in isolation.

is the data of an injective map $\alpha : [n] \to [m]$ of totally ordered sets and a commutative diagram in S (regarded as a category)

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d, "\cong"] & x_1 \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & x_n \arrow[d, "\cong"] \\ y_{f(0)} \arrow[r] & y_{f(1)} \arrow[r] & \cdots \arrow[r] & y_{f(n)} \end{tikzcd}$$

where the vertical maps are isomorphisms.† By a slight abuse of terminology, we call such morphisms *string inclusions*.

Definition 1.22. Let $\mathfrak{S}$ be the preordered set whose objects are subgroups of G and where the relation is that of subconjugacy.

For any subgroup $H \leqslant G$, let $W_G H = N_G H/H$ denote the Weyl group, and recall that the geometric H-fixed points functor

$$(-)^{\phi H} : \mathrm{Sp}^G \to \mathrm{Sp}^{hW_G H} = \mathrm{Fun}(BW_G H, \mathrm{Sp})$$

admits a fully faithful right adjoint ι_H.

Definition 1.23. Let H_0 and H_1 be subgroups of G. The *generalized Tate functor* $\tau_{H_0}^{H_1}$ is defined to be the composite

$$\tau_{H_0}^{H_1} : \mathbf{Sp}^{hW_G H_0} \xhookrightarrow{\iota_{H_0}} \mathbf{Sp}^G \xrightarrow{(-)^{\phi H_1}} \mathbf{Sp}^{hW_G H_1}.$$

Example 1.24.

(1) If H_0 is not subconjugate to H_1, then $\tau_{H_0}^{H_1}$ is the zero functor. If H_0 is conjugate to H_1, then $\tau_{H_0}^{H_1}$ is an equivalence.

(2) If $H_0 = 1$ and $H_1 = G$, then τ^G is the *proper Tate construction*. This agrees with the ordinary Tate construction if G is cyclic of prime order, but not in general. For instance, if $G = C_{p^2}$, then

$$(-)^{\tau C_{p^2}} \simeq ((-)^{hC_p})^{tC_{p^2}/C_p}.$$

Observation 1.25. Given a sequence of properly subconjugate subgroups $H_0 < H_1 < H_2$, generalized Tate functors only *left laxly* (i.e., oplaxly) compose: that is, one has a canonical natural transformation

$$\mathrm{can} : \tau_{H_0}^{H_2} \Rightarrow \tau_{H_1}^{H_2} \circ \tau_{H_0}^{H_1}$$

which fails to be an equivalence in general. More generally, for any string inclusion

$$\overline{\alpha} : [H_0 < \cdots < H_n] \to [K_0 < \cdots < K_m]$$

† Note that $\mathrm{sd}(S)$ is then a category associated to a preordered set.

such that $\alpha(0) = 0$ and $\alpha(n) = m$, one has a canonical natural transformation[†]

$$\mathrm{can}_{\overline{\alpha}} : \tau^{H_n}_{H_{n-1}} \circ \cdots \circ \tau^{H_1}_{H_0} \Rightarrow \tau^{K_m}_{K_{m-1}} \circ \cdots \circ \tau^{K_1}_{K_0}.$$

Observation 1.26. For any string inclusion

$$\overline{\alpha} : [H_0 < \cdots < H_n] \to [K_0 < \cdots < K_m < H_0 < \cdots < H_n]$$

such that α is the inclusion of a convex subset, one has a natural transformation

$$\mathrm{can}_{\overline{\alpha}} : \tau^{H_n}_{H_{n-1}} \circ \cdots \circ \tau^{H_1}_{H_0} \circ (-)^{\phi H_0} \Rightarrow \tau^{H_n}_{H_{n-1}} \circ \cdots \circ \tau^{K_1}_{K_0} \circ (-)^{\phi K_0}$$

given by applying $\tau^{H_n}_{H_{n-1}} \circ \cdots \circ \tau^{H_1}_{H_0} \circ (-)^{\phi H_0}$ to the composite of unit maps

$$\mathrm{id} \Rightarrow \iota_{K_0}(-)^{\phi K_0} \Rightarrow \cdots \Rightarrow \iota_{K_n}(-)^{\phi K_n} \circ \cdots \circ \iota_{K_0}(-)^{\phi K_0}.$$

Notation 1.27. For a subgroup H not in $\mathcal{F}$, let $J_H \subset \mathrm{sd}(\mathfrak{S})$ be the full subcategory on strings $[K_0 < \cdots < K_n < H]$ such that $K_i \in \mathcal{F}$ for all $1 \leqslant i \leqslant n$.

Theorem F (Corollary 2.49). *Suppose that $X \in \mathrm{Sp}^G$ is an $\mathcal{F}$-complete G-spectrum. Then for each subgroup $H \notin \mathcal{F}$, the H-geometric fixed points $X^{\phi H} \in \mathrm{Sp}^{hW_G H}$ can be computed as a limit involving only the geometric fixed points $X^{\phi K}$ for subgroups $K \in \mathcal{F}$ and generalized Tate functors thereon. More precisely, we may functorially associate to X a functor $F : J_H \to \mathrm{Sp}^{hW_G H}$ such that $X^{\phi H} \simeq \lim_{J_H} F$ and*

(1) $F([K_0 < \cdots < K_n < H]) = \tau^H_{K_n} \tau^{K_n}_{K_{n-1}} \cdots \tau^{K_1}_{K_0}(X^{\phi K_0})$;
(2) *F sends a string inclusion $[K_0 < \cdots < K_n < H] \to [L_0 < \cdots < L_m < H]$ to the composite of the canonical maps of Observation 1.25 and Observation 1.26 associated to the factorization*

$$[K_0 < \cdots < K_n < H] \to [L_0 < \cdots < K_0 < \cdots < K_n < H] \to [L_0 < \cdots < L_m < H]$$

in which the first string inclusion is the inclusion of a convex subset.

Remark 1.28. The formula of Theorem F in the case where $G = D_{2p} = \mu_p \rtimes C_2$ and $\mathcal{F} = \Gamma_{\mu_p}$ plays an important role in our proof of the *dihedral Tate orbit lemma* in [70], which is the key computational input needed to show that genuine and Borel real cyclotomic spectra agree in the "underlying bounded-below" case.

Remark 1.29. Our proof of Theorem F is based off of an alternative proof of the Ayala–Mazel-Gee–Rozenblyum reconstruction theorem (Theorem 2.42) due to the second author. On the other hand, given their theorem and the appropriate cofinality arguments, it should not be difficult to derive Theorem F. We were motivated to take a slightly longer route in reproving their theorem in order to illustrate how a detailed understanding of the combinatorics of barycentric subdivision can substitute for any explicit usage of $(\infty, 2)$-category theory in the proof.[‡]

[†] Note that we implicitly make the identifications $\mathrm{Sp}^{hW_G H_0} \simeq \mathrm{Sp}^{hW_G K_0}$ and $\mathrm{Sp}^{hW_G H_n} \simeq \mathrm{Sp}^{hW_G K_m}$ using the conjugacy relations. All issues of conjugacy are dealt with rigorously in the main body of the paper.

[‡] Of course, using the theory of locally cocartesian fibrations may be thought of as implicitly using $(\infty, 2)$-category theory.

Remark 1.30. Shortly after the original appearance of this article, Ayala–Mazel-Gee–Rozenblyum published a greatly expanded version of their work in [6]. There, they derive a formula for the categorical fixed points of a G-spectrum in terms of its geometric fixed points [6, Observation 5.4.3] that should be contrasted with Theorem F.

Remark 1.31. In [5, Theorem A], Ayala–Mazel-Gee–Rozenblyum give a formula for the generalized Tate functors in terms of *proper* Tate constructions.

1.3 | Outline

In Section 2, we discuss background and the $\mathcal{F}$-recollement on Sp^G. In Subsection 2.1, we recall the ∞-category of genuine G-spectra Sp^G from Bachmann–Hoyois [7]. In Subsection 2.2, we recall the theory of families of subgroups of a finite group and describe the $\mathcal{F}$-recollement on Sp^G for any family of subgroups $\mathcal{F}$ of G. We describe the relationship between the $\mathcal{F}$-recollement and the acyclization, completion, and localization functors of Mathew–Naumann–Noel [59], as well as the canonical fracture of Ayala–Mazel-Gee–Rozenblyum [4]. In Subsection 2.3, we reprove Theorem A from [4] which identifies genuine G-spectra in terms of their geometric fixed points. We then prove Theorem F and apply it in the cases relevant for [70].

In Section 3, we specialize to the N-free family $\Gamma_N = \{H : H \cap N = 1\}$ of subgroups for a normal subgroup $N \leqslant G$. In Subsection 3.1, we define the ∞-category of N-naive G-spectra $\mathrm{Sp}^G_{N\text{-naive}}$ and discuss its fundamental properties. In Subsection 3.2, we define the ∞-category of N-Borel G-spectra $\mathrm{Sp}^G_{N\text{-Borel}}$. We study the forgetful functor $\mathcal{U}_b[N] : \mathrm{Sp}^G \to \mathrm{Sp}^G_{N\text{-Borel}}$, its adjoints, and the relationship to the Γ_N-recollement, proving Theorem A. As an application, we obtain the stable symmetric monoidal recollement describing $\mathrm{Sp}^{D_{2p^n}}$, which we use in [70] to analyze real cyclotomic spectra.

In Section 4, we define the parameterized Tate construction in the finite group case. In Subsection 4.1, we extend the ambidexterity theory of Hopkins–Lurie [44] to the parameterized setting. In Subsection 4.2, we use parameterized ambidexterity to define a parameterized norm map between parameterized homotopy orbits and parameterized homotopy fixed points for any finite group extension of G. The parameterized Tate construction is then defined as the cofiber of this map. Reconciling this definition with Definition 1.6, we prove Theorem C and deduce several useful properties mentioned above, such as Observations 1.9–1.12. We also prove Theorem B regarding the inverse limit formula for the parameterized Tate construction.

In Section 5, we define the parameterized Tate construction for an extension of a finite group by a compact Lie group by extending Klein's assembly map definition of the Tate construction (cf. [47] and [67, section I.4]) to the parameterized setting. In Subsection 5.1, we recall several notions related to G-symmetric monoidal structures used throughout the sequel. We then discuss parameterized Verdier quotients and G-$\otimes$-ideals in Subsection 5.2 and the parameterized notion of induced objects in Subsection 5.4. Our key result is that the G-subcategory of induced objects is a G-$\otimes$-ideal (Corollary 5.47). In Subsection 5.5, we apply all of these ideas to define the parameterized Tate construction for infinite groups (Definition 5.49) and prove Theorems D and E. We also discuss an example (Example 5.58) for $\widehat{G} = O(2)$ which plays an important role in our study of real cyclotomic spectra in [70]. Finally, we digress in the middle to upgrade the generalized Segal conjecture to a statement about Tambara functors (Theorem 5.35).

1.4 | Notation and terminology

We assume knowledge of the theory of recollements in this paper and refer to [76] as our primary reference. Given a stable ∞-category $\mathcal{X}$ decomposed by a stable recollement $(\mathcal{U}, \mathcal{Z})$, we will generically label the recollement adjunctions as

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook, shift left=2, "j_!"] \arrow[r, hook, shift right=2, "j_*"'] & \mathcal{X} \arrow[l, "j^*"'] \arrow[r, shift left=2, "i^*"] \arrow[r, shift right=2, "i^!"'] & \mathcal{Z} \arrow[l, hook', "i_*"'] \end{tikzcd}.$$

Here $j^* i_* = 0$ determines the directionality of the recollement.

As is already apparent from the introduction, we will also use concepts from parameterized higher category theory in this paper, mostly in Sections 4 and 5. Given any ∞-category $\mathcal{T}$, an *$\mathcal{T}$-∞-category* is a cocartesian fibration over $\mathcal{T}^{\mathrm{op}}$, and a *$\mathcal{T}$-functor* is a morphism of such preserving cocartesian edges. We then have attendant notions of $\mathcal{T}$-(co)limits and $\mathcal{T}$-Kan extensions. Note that the terminology G-∞-category, G-functor, and so on, is synonymous with $\mathcal{O}_G$-∞-category, $\mathcal{O}_G$-functor, and so on. Apart from the basic reference [74], we refer the reader to [75, section 2] for a quick overview of the theory of $\mathcal{T}$-(co)limits and $\mathcal{T}$-Kan extensions.[†] Let us also highlight the following example, which locates Definition 1.17 within the formalism of parameterized higher category theory:

Example 1.32 (Corepresentable $\mathcal{T}$-diagrams). Suppose that $\mathcal{T}$ admits *multipullbacks*, that is, the finite coproduct completion $\mathbb{F}_{\mathcal{T}}$ of $\mathcal{T}$ admits pullbacks. For example, $\mathcal{O}_G$ satisfies this condition. We call $\mathbb{F}_{\mathcal{T}}$ the ∞-category of *finite $\mathcal{T}$-sets* and $\mathcal{T} \subset \mathbb{F}_{\mathcal{T}}$ the *orbits*. For $U \in \mathbb{F}_{\mathcal{T}}$ with orbit decomposition $\coprod_{i \in I} U_i$, let

$$\underline{U} := \coprod_{i \in I} (\mathcal{T}^{/U_i})^{\mathrm{op}} \to \mathcal{T}^{\mathrm{op}}$$

be the corresponding *$\mathcal{T}$-∞-category of points*, and note that the assignment $U \mapsto \underline{U}$ is covariant in morphisms in $\mathbb{F}_{\mathcal{T}}$. Let $\alpha : U \to V$ be a morphism in $\mathbb{F}_{\mathcal{T}}$ such that V is an orbit. Let $x_i \in C_{U_i}$ be a set of objects for all $i \in I$ and write $(x_i) : \underline{U} \to C_{\underline{V}}$ for the $\mathcal{T}^{/V}$-functor determined by the x_i. Then the *$\mathcal{T}$-coproduct along α*

$$\coprod_{\alpha} x_i \in C_V$$

is defined to be the $\mathcal{T}^{/V}$-colimit of (x_i). A *finite $\mathcal{T}$-coproduct* is any $\mathcal{T}^{/V}$-colimit of this form. We have that C admits all finite $\mathcal{T}$-coproducts if and only if the following conditions obtain [74, Proposition 5.12].

(1) For all $V \in \mathcal{T}$, C_V admits finite coproducts, and for all morphisms $\alpha : V \to W$ in $\mathcal{T}$, the restriction functor $\alpha^* : C_W \to C_V$ preserves finite coproducts.
(2) For all morphisms $\alpha : V \to W$ in $\mathcal{T}$, α^* admits a left adjoint $\alpha_!$.

[†] In [74], we set $S = \mathcal{T}^{\mathrm{op}}$ and instead speak of S-∞-categories as cocartesian fibrations over S, S-functors as morphisms of cocartesian fibrations over S preserving cocartesian edges, and so on.

(3) Given $U \in \mathbb{F}_{\mathcal{T}}$ with orbit decomposition $\coprod_{i \in I} U_i$, let $C_U := \prod_{i \in I} C_{U_i}$ and extend α^* and $\alpha_!$ to be defined for all morphisms α in $\mathbb{F}_{\mathcal{T}}$ in the obvious way.[†] Then the *Beck–Chevalley conditions* hold: for every pullback square

$$\begin{tikzcd} U' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & V' \arrow[d, "\beta"] \\ U \arrow[r, "\alpha"] & V \end{tikzcd}$$

in $\mathbb{F}_{\mathcal{T}}$, the exchange transformation $(\alpha')_!(\beta')^* \Rightarrow \beta^*\alpha_!$ is an equivalence.

In this case, the $\mathcal{T}$-coproduct $\coprod_\alpha x_i$ above is computed by $\alpha_!(x_i)$.

Dually, C admits all finite $\mathcal{T}$-products if and only if the analogous conditions hold with respect to finite products in the fibers and right adjoints α_*.

We will also use the following terminology:

Definition 1.33 [74, Definition 8.3]. Let C, D be $\mathcal{T}$-∞-categories. Then an *$\mathcal{T}$-adjunction* is a relative adjunction

$$F : C \rightleftarrows D : G$$

in the sense of [56, Definition 7.3.2.2] such that F and G are both $\mathcal{T}$-functors.

2 | THE $\mathcal{F}$-RECOLLEMENT ON Sp^G

Let G be a finite group. In this section, we introduce and study recollements on the ∞-category Sp^G of G-spectra determined by a family $\mathcal{F}$ of subgroups of G. We then apply [76, Theorem 3.35] to reprove a theorem of Ayala, Mazel-Gee, and Rozenblyum that reconstructs Sp^G from its geometric fixed points (Theorem 2.46). As a corollary, we deduce a limit formula (Corollary 2.49) for the geometric fixed points of an $\mathcal{F}$-complete spectrum by means of the pointwise formula of [76, Theorem 3.29], which will play an important role in our proof of the dihedral Tate orbit lemma in [70] (cf. Example 2.51 and Example 2.53).

2.1 | Conventions on equivariant stable homotopy theory

At the outset, let us be clear about which foundations for equivariant stable homotopy theory are employed in this paper. In their monograph, Nikolaus and Scholze choose to work with the classical point-set model of orthogonal G-spectra [67, Definition II.2.3], then obtaining the ∞-category Sp^G of G-spectra via inverting equivalences [67, Definition II.2.5]. In contrast, we will use the foundations laid out by Bachmann and Hoyois in [7, section 9], which attaches to every profinite groupoid X a presentable, stable, and symmetric monoidal ∞-category $\mathrm{SH}(X)$ such that for $X = BG$, $\mathrm{SH}(BG)$ is equivalent to Sp^G as defined in [67] (cf. the remark prior to [7, Lemma 9.5]). In fact, we will only need the Bachmann–Hoyois construction for finite groupoids.

[†] For example, if $\alpha : U \to V$ is a map with V an orbit, then $\alpha_!(x_i) = \coprod_{i \in I} (\alpha_i)_!(x_i)$ for $\alpha_i : U_i \to V$ the restriction of α to U_i.

Definition 2.1. Let $\mathrm{Gpd}_{\mathrm{fin}}$ be the (2,1)-category of finite groupoids, and let

$$\mathscr{H}, \mathscr{H}_{\bullet}, \mathrm{SH} : \mathrm{Gpd}_{\mathrm{fin}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

denote the (restriction of the) functors constructed in [7, section 9.2]. For a map $f : X \to Y$ of finite groupoids, write f^* for the associated functor and f_* for its right adjoint.

Remark 2.2. Let $X = BG$. Then $\mathscr{H}(BG) \simeq \mathrm{Spc}_G := \mathrm{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathrm{Spc})$, the ∞-category of G-spaces defined as presheaves on the orbit category $\mathscr{O}_G$, and likewise $\mathscr{H}_{\bullet}(BG)$ is the ∞-category $\mathrm{Spc}_{G,*}$ of pointed G-spaces. As we already mentioned, $\mathrm{SH}(BG) \simeq \mathrm{Sp}^G$ is the ∞-category of G-spectra, defined as the filtered colimit taken in Pr^L

$$\mathrm{Spc}_{G,*} \xrightarrow{\Sigma^\rho} \mathrm{Spc}_{G,*} \xrightarrow{\Sigma^\rho} \mathrm{Spc}_{G,*} \xrightarrow{\Sigma^\rho} \cdots,$$

where ρ is the regular G-representation. In addition, by [7, Example 9.11] Sp^G is equivalent to the ∞-category of *spectral Mackey functors* on finite G-sets that was studied by Barwick [9] and Guillou–May [32].

Note that by definition, $f^* : \mathrm{SH}(Y) \to \mathrm{SH}(X)$ is the symmetric monoidal left Kan extension of $\mathscr{H}_{\bullet}(Y) \xrightarrow{f^*} \mathscr{H}_{\bullet}(X) \xrightarrow{\Sigma^\infty} \mathrm{SH}(X)$ along $\Sigma^\infty : \mathscr{H}_{\bullet}(Y) \to \mathrm{SH}(Y)$. Therefore:

(1) Suppose $f : BH \to BG$ is the map of groupoids induced by an injective group homomorphism $H \to G$. Then $f^* : \mathrm{Sp}^G \to \mathrm{Sp}^H$ is homotopic to the usual restriction functor, and $f_* : \mathrm{Sp}^H \to \mathrm{Sp}^G$ is homotopic to the usual induction functor. Instead of $f^* \dashv f_*$, we will typically write this adjunction as $\mathrm{res}^G_H \dashv \mathrm{ind}^G_H$. Note that this adjunction is ambidextrous and satisfies the projection formula (in fact, [7, Lemma 9.4(3)] establishes the projection formula for any finite covering map).
(2) Suppose $f : BG \to BG/N$ is the map of groupoids induced by a surjective group homomorphism $G \to G/N$. Then $f^* : \mathrm{Sp}^{G/N} \to \mathrm{Sp}^G$ is homotopic to the usual inflation functor, which we denote as inf^N. The right adjoint to inf^N is the *categorical fixed points* functor

$$\Psi^N : \mathrm{Sp}^G \to \mathrm{Sp}^{G/N}.$$

Now suppose $H \leqslant G$ is any subgroup and let $W_G H = N_G H/H$ be the Weyl group of H. Then we will also write

$$\Psi^H : \mathrm{Sp}^G \xrightarrow{\mathrm{res}^G_{N_G H}} \mathrm{Sp}^{N_G H} \xrightarrow{\Psi^H} \mathrm{Sp}^{W_G H}$$

Given a G-spectrum X, we introduce notation to distinguish the underlying spectrum of $\Psi^H X$.

Notation 2.3. For a G-spectrum X and subgroup $H \leqslant G$, we let $X^H = \mathrm{res}^{W_G H}\, \Psi^H(X)$.[†]

As the restriction functor $\mathrm{Sp}^{W_G H} \to \mathrm{Sp}$ lifts to $\mathrm{Fun}(BW_G H, \mathrm{Sp})$, the spectrum X^H also comes endowed with a $W_G H$-action.

[†] With respect to the description of Sp^G as spectral Mackey functors, X^H is given by evaluation at G/H.

Remark 2.4. By stabilizing the adjointability relations in [7, Lemma 9.4], it follows that that for any pullback square of finite groupoids

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow g \\ X & \xrightarrow{f} & Z, \end{array}$$

the canonical natural transformation $f^*g_* \to f_*g^*$ of functors $\mathrm{SH}(X) \to \mathrm{SH}(Y)$ is an equivalence. In particular, we have an equivalence $X^H \simeq \Psi^H \operatorname{res}^G_H(X)$.

We now turn to the *geometric fixed points* and *Hill–Hopkins–Ravenel norm* functors.

Definition 2.5. Let $\mathscr{H}^\otimes, \mathscr{H}_\bullet^\otimes, \mathrm{SH}^\otimes : \mathrm{Span}(\mathrm{Gpd}_{\mathrm{fin}}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{sift}})$ be the (restrictions of the) functors defined as in [7, section 9.2], which on the subcategory $\mathrm{Gpd}_{\mathrm{fin}}^{\mathrm{op}}$ restrict to the functors $\mathscr{H}, \mathscr{H}_\bullet, \mathrm{SH}$ of Definition 2.1. For a map of finite groupoids $f : X \to Y$, write $f_\otimes$ for the associated covariant functor.

Parallel to the discussion above, we note [7, Remark 9.9]:

(1) Suppose $f : BH \to BG$ for a subgroup $H \leqslant G$. Then $f_\otimes : \mathrm{Sp}^H \to \mathrm{Sp}^G$ is homotopic to the multiplicative norm functor N_H^G introduced by Hill, Hopkins, and Ravenel [40].
(2) Suppose $f : BG \to B(G/N)$. Then $f_\otimes : \mathrm{Sp}^G \to \mathrm{Sp}^{G/N}$ is homotopic to the usual geometric fixed points functor Φ^N. For $H \leqslant G$ any subgroup, we also write

$$\Phi^H : \mathrm{Sp}^G \xrightarrow{\operatorname{res}^G_{N_G H}} \mathrm{Sp}^{N_G H} \xrightarrow{\Phi^H} \mathrm{Sp}^{W_G H}.$$

Notation 2.6. For a G-spectrum X and subgroup $H \leqslant G$, we let $X^{\phi H} = \operatorname{res}^{W_G H} \Phi^H(X)$. Also let

$$\phi^H : \mathrm{Sp}^G \xrightarrow{\Phi^H} \mathrm{Sp}^{W_G H} \xrightarrow{\operatorname{res}} \mathrm{Fun}(BW_G H, \mathrm{Sp}).$$

Remark 2.7. Because $\mathrm{SH}^\otimes$ is defined on $\mathrm{Span}(\mathrm{Gpd}_{\mathrm{fin}})$, we have that for any pullback square of finite groupoids

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow g \\ X & \xrightarrow{f} & Z, \end{array}$$

there is a canonical equivalence $f^*g_\otimes \simeq f_\otimes g^*$ of functors $\mathrm{SH}(X) \to \mathrm{SH}(Y)$. In particular, we have an equivalence $X^{\phi H} \simeq \Phi^H \operatorname{res}^G_H X$.

Remark 2.8. We will use some additional features of these fixed points functors.

(1) For any subgroup $H \leqslant G$, the functor Ψ^H is colimit-preserving, as the inflation functors preserve dualizable and hence compact objects; indeed, by equivariant Atiyah duality [49, section III.5.1] every compact object in Sp^G is dualizable, and conversely, as the unit in Sp^G is compact, all dualizable objects in Sp^G are compact.

(2) The functors $\{(-)^H : H \leqslant G\}$ are jointly conservative, as the orbits $\Sigma^\infty_+ G/H$ corepresent $(-)^H$ and form a set of compact generators for Sp^G.
(3) The functors $\{\phi^H : H \leqslant G\}$ are jointly conservative, as the evaluation functors $\mathrm{ev}_{G/H}$ are jointly conservative for Spc_G, $\phi^H \Sigma^\infty_+ \simeq \Sigma^\infty_+ \mathrm{ev}_{G/H}$, and suspension spectra generate Sp^G under desuspensions and sifted colimits.

We will also need to use some aspects of the theory of G-∞-categories in this work.

Definition 2.9. Let $\omega_G : \mathbb{F}_G \to \mathrm{Gpd}_{\mathrm{fin}}$ be the functor that sends a finite G-set U to its action groupoid $U//G$.

Definition 2.10. Define the *G-∞-category of G-spectra* $\underline{\mathrm{Sp}}^G \to \mathcal{O}_G^{\mathrm{op}}$ to be the cocartesian fibration classified by $\mathrm{SH}\circ(\omega_G^{\mathrm{op}}|_{\mathcal{O}_G^{\mathrm{op}}})$. In addition, let $\underline{\mathrm{Sp}}^{G,\otimes} \to \mathcal{O}_G^{\mathrm{op}} \times \mathrm{Fin}_*$ be the cocartesian $\mathcal{O}_G^{\mathrm{op}}$-family of symmetric monoidal ∞-categories classified by $\mathrm{SH}\circ(\omega_G^{\mathrm{op}}|_{\mathcal{O}_G^{\mathrm{op}}})$ (when viewed as valued in $\mathrm{CMon}(\mathrm{Cat}_\infty)$).

Notation 2.11. If a chance for confusion arises, we write $\underline{C}$ to distinguish a G-∞-category from the ∞-category C given by the fiber of $\underline{C}$ over G/G.

Remark 2.12. For a subgroup H of G, let

$$\mathrm{ind}_H^G : \mathbb{F}_H \rightleftarrows \mathbb{F}_G : \mathrm{res}_H^G$$

denote the induction-restriction adjunction, where $\mathrm{ind}_H^G(U) = G \times_H U$. Then $\mathrm{ind}_H^G : \mathcal{O}_H \to \mathcal{O}_G$ factors as $\mathcal{O}_H \simeq (\mathcal{O}_G)_{/(G/H)} \to \mathcal{O}_G$. Moreover, $\omega_G \circ \mathrm{ind}_H^G$ and ω_H are canonically equivalent, so we have an equivalence of H-∞-categories

$$\underline{\mathrm{Sp}}^H \simeq \mathcal{O}_H^{\mathrm{op}} \times_{\mathcal{O}_G^{\mathrm{op}}} \underline{\mathrm{Sp}}^G.$$

Remark 2.13. Given a G-∞-category K, we may endow $\mathrm{Fun}_G(K, \underline{\mathrm{Sp}}^G)$ with the pointwise symmetric monoidal structure of Definition A.1 with respect to the construction $\underline{\mathrm{Sp}}^{G,\otimes}$ of Definition 2.10.

Finally, we will later need the G-symmetric monoidal structure on $\underline{\mathrm{Sp}}^G$ furnished by the Hill–Hopkins–Ravenel norms; we record this as Example 5.13.

2.2 | Basic theory of families

Definition 2.14. Given a finite group G, its *subconjugacy category* $\mathfrak{S}[G]$ is the category whose objects are subgroups H of G, and whose morphism sets are defined by

$$\mathrm{Hom}_{\mathfrak{S}[G]}(H,K) := \begin{cases} * & \text{if } H \text{ is subconjugate to } K, \\ \emptyset & \text{otherwise.} \end{cases}$$

We will also write $\mathfrak{S} = \mathfrak{S}[G]$ if the ambient group G is clear from context.

Definition 2.15. A *G-family* $\mathcal{F}$ is a sieve in $\mathfrak{S}$, that is, a full subcategory of $\mathfrak{S}$ whose set of objects is a set of subgroups of G closed under subconjugacy.

Remark 2.16. Abusing notation, we will also denote the set of objects of $\mathfrak{S}$ or a family $\mathcal{F}$ by the same symbol. If we view morphisms in $\mathfrak{S}$ as defining a binary relation $\leq$ on the set of subgroups of G, then $\mathfrak{S}$ is a preordered set, which is a poset if G is abelian. Although we generally reserve the expression $H \leqslant K$ for H a subgroup of K, when discussing strings in the preordered set $\mathfrak{S}$ we will also write $\leq$ for its binary relation, we trust the meaning to be clear from context.

Construction 2.17. Given a G-family $\mathcal{F}$, define G-spaces $E\mathcal{F}$ and $\widetilde{E\mathcal{F}}$ by the formulae

$$E\mathcal{F}^K = \begin{cases} \emptyset & \text{if } K \notin \mathcal{F}, \\ * & \text{if } K \in \mathcal{F}, \end{cases} \quad , \quad \text{and} \quad \widetilde{E\mathcal{F}}^K = \begin{cases} S^0 & \text{if } K \notin \mathcal{F}, \\ * & \text{if } K \in \mathcal{F}. \end{cases}$$

We have a cofiber sequence of pointed G-spaces

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}.$$

The unit map $S^0 \to \widetilde{E\mathcal{F}}$ exhibits $\widetilde{E\mathcal{F}}$ as an idempotent object [56, Definition 4.8.2.1] of $\mathrm{Spc}_{G,*}$ with respect to the smash product, hence $\widetilde{E\mathcal{F}}$ is a idempotent E_∞-algebra by [56, Proposition 4.8.2.9].[†] Let $E\mathcal{F}_+$ and $\widetilde{E\mathcal{F}}$ also denote Σ^∞ of the same pointed G-spaces. Then $\widetilde{E\mathcal{F}}$ is an idempotent E_∞-algebra in Sp^G, and hence by [76, Observation 2.36] defines a stable symmetric monoidal recollement

$$\begin{tikzcd} \mathbf{Sp}^{h\mathcal{F}} \arrow[r, shift right, hook, "j_*"'] & \mathbf{Sp}^G \arrow[l, shift right, "j^*"'] \arrow[r, shift left, "i^*"] & \mathbf{Sp}^{\Phi\mathcal{F}} \arrow[l, shift left, hook, "i_*"] \end{tikzcd}$$

such that $\mathrm{Sp}^{\Phi\mathcal{F}} \simeq \mathrm{Mod}_{\mathrm{Sp}^G}(\widetilde{E\mathcal{F}})$. By [76, Corollary 2.35], for any $X \in \mathrm{Sp}^G$ we have the *$\mathcal{F}$-fracture square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \otimes \widetilde{E\mathcal{F}} \arrow[d] \\ F(E\mathcal{F}_+, X) \arrow[r] & F(E\mathcal{F}_+, X) \otimes \widetilde{E\mathcal{F}}. \end{tikzcd}$$

Following standard terminology, we say that a G-spectrum X is *$\mathcal{F}$-torsion*, *$\mathcal{F}$-complete*, or *$\mathcal{F}^{-1}$-local* if it is in the essential image of $j_!$, j_*, or i_*, respectively. Note that for a G-spectrum X,

- X is $\mathcal{F}$-torsion if and only if $X \otimes E\mathcal{F}_+ \xrightarrow{\simeq} X$ or $X \otimes \widetilde{E\mathcal{F}} \simeq 0$;
- X is $\mathcal{F}$-complete if and only if $X \xrightarrow{\simeq} F(E\mathcal{F}_+, X)$ or $F(\widetilde{E\mathcal{F}}, X) \simeq 0$;
- X is $\mathcal{F}^{-1}$-local if and only if $X \xrightarrow{\simeq} X \otimes \widetilde{E\mathcal{F}}$ or $X \otimes E\mathcal{F}_+ \simeq 0$.

Notation 2.18. For a G-family $\mathcal{F}$, we have already set $\mathrm{Sp}^{h\mathcal{F}} \subset \mathrm{Sp}^G$ to be the full subcategory of $\mathcal{F}$-complete G-spectra and $\mathrm{Sp}^{\Phi\mathcal{F}} \subset \mathrm{Sp}^G$ to be the full subcategory of $\mathcal{F}^{-1}$-local G-spectra. We also let $\mathrm{Sp}^{\tau\mathcal{F}} \subset \mathrm{Sp}^G$ denote the full subcategory of $\mathcal{F}$-torsion G-spectra.

[†] This is also obvious because we are considering presheaves of sets.

In addition, if $\mathcal{F}$ is the trivial family $\{1\}$, we will also write $E\mathcal{F} = EG$, $\mathrm{Sp}^{h\mathcal{F}} = \mathrm{Sp}^{hG}$, and refer to $\mathcal{F}$-torsion or complete objects as *Borel* torsion or complete.[†] It is well-known that $\mathrm{Sp}^{hG} \simeq \mathrm{Fun}(BG, \mathrm{Sp})$ (see [59, Proposition 6.17] and [67, Theorem II.2.7]), we will later give two different generalizations of this fact (Lemma 2.36 and Theorem 3.24).

Remark 2.19. The functor $j_! j^* : \mathrm{Sp}^{h\mathcal{F}} \xrightarrow{\simeq} \mathrm{Sp}^{\tau\mathcal{F}}$ implements an equivalence between $\mathcal{F}$-complete and $\mathcal{F}$-torsion objects [11, Proposition 7].

Remark 2.20. The endofunctors $j_! j^*$, $j_* j^*$, and $i_* i^*$ of Sp^G attached to a family $\mathcal{F}$ agree with the $A_{\mathcal{F}}$-acyclization, $A_{\mathcal{F}}$-completion, and $A_{\mathcal{F}}^{-1}$-localization functors in [59] defined with respect to the E_∞-algebra

$$A_{\mathcal{F}} := \prod_{H \in \mathcal{F}} F(G/H_+, 1)$$

by [59, Propositions 6.5–6.6]. Moreover, the theory of A-torsion, A-complete, and A^{-1}-local objects for a dualizable E_∞-algebra A ([59, part 1] under the hypotheses [59, 2.26]) extends the more general monoidal recollement theory for the idempotent object $1 \to U_A$ of [59, Construction 3.12]. For example, the $\mathcal{F}$-fracture square for Sp^G given by [76, Corollary 2.12] agrees with the $A_{\mathcal{F}}$-fracture square given by [59, Theorem 3.20] (although we additionally consider the monoidal refinement [76, Theorem 2.30]).

As a separate consequence, we also have that $\mathrm{Sp}^{\tau\mathcal{F}} \subset \mathrm{Sp}^G$ is the localizing subcategory generated by the orbits $\{G/H_+ : H \in \mathcal{F}\}$. Also, G/H_+ is both $\mathcal{F}$-complete and $\mathcal{F}$-torsion.

In the remainder of this subsection, we collect some basic results concerning $\mathcal{F}$-recollements that we will need in the sequel. Classical references for this material are [49, section II] and [30, section 17], and other references include [59, section 6] and [4, section 2].

Lemma 2.21. *Let $\mathcal{F}$ be a G-family and let $X \in \mathrm{Sp}^G$.*

(1) *X is $\mathcal{F}^{-1}$-local if and only if $X^{\phi K} \simeq 0$ for all $K \in \mathcal{F}$.*
(2) *X is $\mathcal{F}$-torsion if and only if $X^{\phi K} \simeq 0$ for all $K \notin \mathcal{F}$.*

Therefore, for a map $f : X \to Y$ in Sp^G, f is a j^*-equivalence if and only if $f^{\phi K}$ is an equivalence for all $K \in \mathcal{F}$, and f is an i^*-equivalence if and only if $f^{\phi K}$ is an equivalence for all $K \notin \mathcal{F}$.

Proof. First note that for any $X \in \mathrm{Sp}^G$ and subgroup K of G,

$$(X \otimes E\mathcal{F}_+)^{\phi K} \simeq X^{\phi K} \otimes (E\mathcal{F}_+)^{\phi K} \simeq \begin{cases} 0 & \text{if } K \notin \mathcal{F}, \\ X^{\phi K} & \text{if } K \in \mathcal{F}, \end{cases}$$

$$(X \otimes \widetilde{E\mathcal{F}})^{\phi K} \simeq X^{\phi K} \otimes \widetilde{E\mathcal{F}}^{\phi K} \simeq \begin{cases} X^{\phi K} & \text{if } K \notin \mathcal{F}, \\ 0 & \text{if } K \in \mathcal{F}. \end{cases}$$

Thus, if X is $\mathcal{F}^{-1}$-local so that $X \simeq X \otimes \widetilde{E\mathcal{F}}$, then $X^{\phi K} \simeq 0$ for all $K \in \mathcal{F}$. Conversely, if $X^{\phi K} \simeq 0$ for all $K \in \mathcal{F}$, then $(X \otimes E\mathcal{F}_+)^{\phi K} \simeq 0$ for all subgroups K, so by the joint conservativity of the functors ϕ^K, $X \otimes E\mathcal{F}_+ \simeq 0$ and X is $\mathcal{F}^{-1}$-local. This proves (1), and the proof of (2) is similar. □

[†] Other authors refer to Borel torsion spectra as *free* and Borel complete spectra as *cofree*.

Remark 2.22. The identification of $\mathcal{F}^{-1}$-local objects in Sp^G above shows that if $N \trianglelefteq G$ is a normal subgroup such that $N \subseteq H$ for each $H \notin \Gamma_N$, then there is an equivalence of ∞-categories

$$\mathrm{Sp}^{\Phi\Gamma_N} \simeq \mathrm{Sp}^{G/N}.$$

Remark 2.23 (Extension to G-recollement). Suppose $\mathcal{F}$ is a G-family, and let $\mathcal{F}^H \subset \mathfrak{S}[H]$ denote the H-family obtained by intersecting $\mathcal{F}$ with $\mathfrak{S}[H] \subset \mathfrak{S}[G]$. For any map of G-orbits $f : G/H \to G/K$ with associated adjunction $f^* : \mathbf{Sp}^K \rightleftarrows \mathbf{Sp}^H : f_*$, note that

$$f^*(E\mathcal{F}^K_+ \to S^0 \to \widetilde{E\mathcal{F}^K}) \simeq E\mathcal{F}^H_+ \to S^0 \to \widetilde{E\mathcal{F}^H}.$$

By monoidality of f^*, it follows that f^* preserves $\mathcal{F}$-torsion and $\mathcal{F}^{-1}$-local objects. Furthermore, the projection formula implies that

$$f^*F(E\mathcal{F}^K_+, X) \simeq F(E\mathcal{F}^H_+, f^*X),$$

so f^* preserves $\mathcal{F}$-complete objects. Therefore, $\mathcal{F}$ defines a lift of the functor $\mathrm{SH} : \mathcal{O}_G^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}_\infty$ to $\mathrm{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$ [76, Definition 2.15], the ∞-category of stable recollements and strict morphisms thereof. Passing to Grothendieck constructions, let

$$\begin{tikzcd} \underline{\mathbf{Sp}}^{h\mathcal{F}} \arrow[r, shift right, "j_*"'] & \underline{\mathbf{Sp}}^{G} \arrow[l, shift right, "j^*"'] \arrow[r, shift left, "i^*"] & \underline{\mathbf{Sp}}^{\Phi\mathcal{F}} \arrow[l, shift left, "i_*"] \end{tikzcd}$$

denote the resulting diagram of G-adjunctions. By [76, Corollary 2.42], $\underline{\mathrm{Sp}}^{h\mathcal{F}}$ and $\underline{\mathrm{Sp}}^{\Phi\mathcal{F}}$ are G-stable G-∞-categories and all G-functors in the diagram are G-exact. We thereby obtain a *G-stable G-recollement* $(\underline{\mathrm{Sp}}^{h\mathcal{F}}, \underline{\mathrm{Sp}}^{\Phi\mathcal{F}})$ of $\underline{\mathrm{Sp}}^G$ in the sense of [76, Definition 2.43]. We also write $\underline{\mathrm{Sp}}^{\tau\mathcal{F}} \subset \underline{\mathrm{Sp}}^G$ for the essential image of $j_!$.

We may also consider $\mathcal{F}$-recollements of the ∞-category of G-spaces (indeed, of any ∞-category of $\mathcal{E}$-valued presheaves on $\mathcal{O}_G$).

Notation 2.24. Given a G-family $\mathcal{F}$, let $\mathcal{O}_{G,\mathcal{F}} \subset \mathcal{O}_G$ be the full subcategory on those orbits with stabilizer in $\mathcal{F}$, and let $\mathcal{O}^c_{G,\mathcal{F}}$ be its complement.

Construction 2.25 ($\mathcal{F}$-recollement of G-spaces). Given a G-family $\mathcal{F}$, we may define a functor $\pi : \mathcal{O}_G^{\mathrm{op}} \to \Delta^1$ such that $(\mathcal{O}_G^{\mathrm{op}})_1 = (\mathcal{O}_{G,\mathcal{F}})^{\mathrm{op}}$ and $(\mathcal{O}_G^{\mathrm{op}})_0 = (\mathcal{O}^c_{G,\mathcal{F}})^{\mathrm{op}}$. Let $\mathrm{Spc}_{h\mathcal{F}} = \mathrm{Fun}((\mathcal{O}_{G,\mathcal{F}})^{\mathrm{op}}, \mathrm{Spc})$ and $\mathrm{Spc}_{\Phi\mathcal{F}} = \mathrm{Fun}((\mathcal{O}^c_{G,\mathcal{F}})^{\mathrm{op}}, \mathrm{Spc})$. By [76, Example 3.6], we obtain a symmetric monoidal recollement with respect to the cartesian product on G-spaces

$$\begin{tikzcd} \mathbf{Spc}_{h\mathcal{F}} \arrow[r, shift right, "j_*"'] & \mathbf{Spc}_{G} \arrow[l, shift right, "j^*"'] \arrow[r, shift left, "i^*"] & \mathbf{Spc}_{\Phi\mathcal{F}} \arrow[l, shift left, "i_*"] \end{tikzcd}.$$

Moreover, if we instead take presheaves in Spc_*, we obtain a symmetric monoidal recollement with respect to the smash product of pointed G-spaces

$$\begin{tikzcd} \mathbf{Spc}_{h\mathcal{F},*} \arrow[r, shift right, "j_*"'] & \mathbf{Spc}_{G,*} \arrow[l, shift right, "j^*"'] \arrow[r, shift left, "i^*"] & \mathbf{Spc}_{\Phi\mathcal{F},*} \arrow[l, shift left, "i_*"] \end{tikzcd}.$$

where $\widetilde{E\mathcal{F}} \simeq i_* i^*(S^0)$ and the unit map exhibits $\widetilde{E\mathcal{F}}$ as the same idempotent object as above.

Given a map $f : X \to Y$ in Spc_G, by definition f is a j^*-equivalence if and only if $X^K \to Y^K$ is an equivalence for all $K \in \mathcal{F}$, and f is a i^*-equivalence if and only if $X^K \to Y^K$ is an equivalence for all $K \notin \mathcal{F}$. Therefore, by Lemma 2.21 and the compatibility of geometric fixed points with Σ^∞_+, the functor Σ^∞_+ is a morphism of recollements $(\mathrm{Spc}_{h\mathcal{F}}, \mathrm{Spc}_{\Phi\mathcal{F}}) \to (\mathrm{Sp}^{h\mathcal{F}}, \mathrm{Sp}^{\Phi\mathcal{F}})$, and likewise for Σ^∞. In particular, we get induced functors

$$\Sigma^\infty_+ : \mathrm{Spc}_{h\mathcal{F}} \to \mathrm{Sp}^{h\mathcal{F}}, \quad \Sigma^\infty_+ : \mathrm{Spc}_{\Phi\mathcal{F}} \to \mathrm{Sp}^{\Phi\mathcal{F}}.$$

On the other hand, Ω^∞ is not a morphism of recollements; indeed, if $X \in \mathrm{Sp}^G$ is $\mathcal{F}$-torsion, then we may have that $i^*\Omega^\infty X$ is nontrivial, so Ω^∞ does not preserve i^*-equivalences. However, if $f : X \to Y$ is a j^*-equivalence in Sp^G, so that $f^{\phi K}$ is an equivalence for all $K \in \mathcal{F}$, then $\mathrm{res}^G_H(f)$ is an equivalence for all $H \in \mathcal{F}$ because the functors ϕ^K for $K \leqslant H$ jointly detect equivalences in Sp^H. Therefore, $\Omega^\infty(f)$ is a j^*-equivalence, and the $\Sigma^\infty_+ \dashv \Omega^\infty$ adjunction induces an adjunction

$$\Sigma^\infty_+ : \mathbf{Spc}_{h\mathcal{F}} \rightleftarrows \mathbf{Sp}^{h\mathcal{F}} : \Omega^\infty.$$

Now suppose X is $\mathcal{F}^{-1}$-local, so that $X^{\phi K} \simeq 0$ for all $K \in \mathcal{F}$. Then $\mathrm{res}^G_H X \simeq 0$ for all $H \in \mathcal{F}$, so $(\Omega^\infty X)^H \simeq *$ for all $H \in \mathcal{F}$ and thus $\Omega^\infty X$ lies in the essential image of i_*. We thereby obtain an adjunction

$$\Sigma^\infty_+ : \mathbf{Spc}_{\Phi\mathcal{F}} \rightleftarrows \mathbf{Sp}^{\Phi\mathcal{F}} : \Omega^\infty.$$

To summarize the various compatibilities, we have that

(1) $j^*\Sigma^\infty_+ \simeq \Sigma^\infty_+ j^* : \mathrm{Spc}_G \to \mathrm{Sp}^{h\mathcal{F}}$ and $j_*\Omega^\infty \simeq \Omega^\infty j_* : \mathrm{Sp}^{h\mathcal{F}} \to \mathrm{Spc}_G$;
(2) $j^*\Omega^\infty \simeq \Omega^\infty j^* : \mathrm{Sp}^G \to \mathrm{Spc}_{h\mathcal{F}}$ and $j_!\Sigma^\infty_+ \simeq \Sigma^\infty j_! : \mathrm{Spc}_{h\mathcal{F}} \to \mathrm{Sp}^G$;
(3) $i^*\Sigma^\infty_+ \simeq \Sigma^\infty_+ i^* : \mathrm{Spc}_G \to \mathrm{Sp}^{\Phi\mathcal{F}}$ and $i_*\Omega^\infty \simeq \Omega^\infty i_* : \mathrm{Sp}^{\Phi\mathcal{F}} \to \mathrm{Spc}_G$.

Next, we study situations that arise in the presence of two G-families.

Remark 2.26. Let $\mathcal{F}$ and $\mathcal{G}$ be two G-families. Then their intersection $\mathcal{F} \cap \mathcal{G}$ is again a G-family. Note that $E(\mathcal{F} \cap \mathcal{G}) \simeq E\mathcal{F} \times E\mathcal{G}$ as G-spaces, so $E\mathcal{F}_+ \otimes E\mathcal{G}_+ \simeq E(\mathcal{F} \cap \mathcal{G})_+$. Consequently, for any $X \in \mathrm{Sp}^G$, the $\mathcal{G}$-fracture square for $F(E\mathcal{F}_+, X)$ yields a commutative diagram

$$\begin{tikzcd}
F(E\mathcal{F}_+, X) \otimes E\mathcal{G}_+ \arrow[r] \arrow[d, "\simeq"] & F(E\mathcal{F}_+, X) \arrow[r] \arrow[d] & F(E\mathcal{F}_+, X) \otimes \widetilde{E\mathcal{G}} \arrow[d] \\
F(E(\mathcal{F} \cap \mathcal{G})_+, X) \otimes E\mathcal{G}_+ \arrow[r] & F(E(\mathcal{F} \cap \mathcal{G})_+, X) \arrow[r] & F(E(\mathcal{F} \cap \mathcal{G})_+, X) \otimes \widetilde{E\mathcal{G}}
\end{tikzcd}$$

in which the right-hand square is a pullback square.

Lemma 2.27. *Let $\mathcal{F}$ and $\mathcal{G}$ be two G-families. Then $\mathrm{Sp}^{\Phi\mathcal{G}} \cap \mathrm{Sp}^{h\mathcal{F}} = \mathrm{Sp}^{\Phi(\mathcal{F}\cap\mathcal{G})} \cap \mathrm{Sp}^{h\mathcal{F}}$ and $\mathrm{Sp}^{\Phi\mathcal{G}} \cap \mathrm{Sp}^{h\mathcal{F}} = \mathrm{Sp}^{\Phi\mathcal{G}} \cap \mathrm{Sp}^{h(\mathcal{F}\cup\mathcal{G})}$.*

Proof. We prove the first equality, the proof of the second being similar. If X is $\mathcal{G}^{-1}$-local, then X is $(\mathcal{G}\cap\mathcal{F})^{-1}$-local by Lemma 2.28(2), so we have the forward inclusion. On the other hand, by Remark 2.26, for any $X \in \mathrm{Sp}^G$ we have that

$$F(E\mathcal{F}_+, X) \otimes E\mathcal{G}_+ \simeq F(E(\mathcal{F}\cap\mathcal{G})_+, X) \otimes E\mathcal{G}_+.$$

But $F(E(\mathcal{F}\cap\mathcal{G})_+, X) \simeq 0$ if X is $(\mathcal{F}\cap\mathcal{G})^{-1}$-local, and $X \simeq F(E\mathcal{F}_+, X)$ if X is $\mathcal{F}$-complete. Thus, if X is both $\mathcal{F}$-complete and $(\mathcal{F}\cap\mathcal{G})^{-1}$-local, then X is $\mathcal{G}^{-1}$-local. We thereby deduce the reverse inclusion. □

Lemma 2.28. *Suppose $\mathcal{G}$ is a subfamily of $\mathcal{F}$. Then*

(1) *if X is $\mathcal{G}$-torsion, then X is $\mathcal{F}$-torsion;*
(2) *if X is $\mathcal{F}^{-1}$-local, then X is $\mathcal{G}^{-1}$-local;*
(3) *if X is $\mathcal{G}$-complete, then X is $\mathcal{F}$-complete;*
(4) *if X is $\mathcal{G}^{-1}$-local, then its $\mathcal{F}$-completion $F(E\mathcal{F}_+, X)$ is again $\mathcal{G}^{-1}$-local;*
(5) *if X is $\mathcal{G}^{-1}$-local, then its $\mathcal{F}$-acyclization $X \otimes E\mathcal{F}_+$ is again $\mathcal{G}^{-1}$-local;*
(6) *if X is $\mathcal{G}$-complete and $\mathcal{F}^{-1}$-local, then $X \simeq 0$.*

Proof. (1) and (2) follow immediately from Lemma 2.21. For (3), to show X is $\mathcal{F}$-complete, we need to show that for all $\mathcal{F}^{-1}$-local Y, $\mathrm{Map}(Y, X) \simeq *$. But by (2), Y is $\mathcal{G}^{-1}$-local, so this mapping space is contractible because X is $\mathcal{G}$-complete by assumption. For (4), we need to show that for all $\mathcal{G}$-torsion Y, $\mathrm{Map}(Y, F(E\mathcal{F}_+, X)) \simeq *$. But

$$\mathrm{Map}(Y, F(E\mathcal{F}_+, X)) \simeq \mathrm{Map}(Y \otimes E\mathcal{F}_+, X) \simeq \mathrm{Map}(Y, X) \simeq *$$

as $Y \otimes E\mathcal{F}_+ \simeq Y$ by (1) and the assumption that X is $\mathcal{G}^{-1}$-local. The proof of (5) is similar: given $\mathcal{G}^{-1}$-local X and any $\mathcal{G}$-complete Y, we have that $\mathrm{Map}(X \otimes E\mathcal{F}_+, Y) \simeq *$ because Y is also $\mathcal{F}$-complete by (3), hence $X \otimes E\mathcal{F}_+$ is $\mathcal{G}^{-1}$-local. Finally, for (6) note that X is then $\mathcal{G}^{-1}$-local by (2), hence $X \simeq 0$. □

Supposing still that $\mathcal{G}$ is a subfamily of $\mathcal{F}$, by Lemma 2.28(1–3), the defining adjunctions of the $\mathcal{F}$ and $\mathcal{G}$-recollements on Sp^G restrict to adjunctions

$$(i_{\mathcal{F}})^* : \mathbf{Sp}^{\Phi\mathcal{G}} \rightleftarrows \mathbf{Sp}^{\Phi\mathcal{F}} : (i_{\mathcal{F}})_*, (j_{\mathcal{G}})^* : \mathbf{Sp}^{h\mathcal{F}} \rightleftarrows \mathbf{Sp}^{h\mathcal{G}} : (j_{\mathcal{G}})_*, (j'_{\mathcal{G}})_! : \mathbf{Sp}^{\tau\mathcal{G}} \rightleftarrows \mathbf{Sp}^{\tau\mathcal{F}} : (j'_{\mathcal{G}})^*.$$

By Lemma 2.28(4–5), the $\mathcal{F}$-completion adjunction restricts to

$$(j_{\mathcal{F}})^* : \mathbf{Sp}^{\Phi\mathcal{G}} \rightleftarrows \mathbf{Sp}^{h\mathcal{F}} \cap \mathbf{Sp}^{\Phi\mathcal{G}} : (j_{\mathcal{F}})_*$$

such that $(j_{\mathcal{F}})^*$ admits a left adjoint $(j_{\mathcal{F}})_!$ given by the inclusion of $\mathcal{F}$-torsion and $\mathcal{G}^{-1}$-local objects under the equivalence $\mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}} \simeq \mathrm{Sp}^{\tau\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$.

Next, let $(i_{\mathcal{G}})^* : \mathrm{Sp}^{h\mathcal{F}} \to \mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$ be the composite $\mathrm{Sp}^{h\mathcal{F}} \subset \mathrm{Sp}^G \xrightarrow{i^*} \mathrm{Sp}^{\Phi\mathcal{G}} \xrightarrow{(j_{\mathcal{F}})^*} \mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$. Then $(i_{\mathcal{G}})^*$ is left adjoint to the inclusion $(i_{\mathcal{G}})_*$. Likewise, define the left adjoint $(i'_{\mathcal{G}})^*$ to the inclusion $(i'_{\mathcal{G}})_* : \mathrm{Sp}^{\tau\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}} \to \mathrm{Sp}^{\tau\mathcal{F}}$. Finally, note that $\mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$ inherits a symmetric monoidal structure from the localization $(j_{\mathcal{F}})^* \dashv (j_{\mathcal{F}})_*$, with respect to which $(i_{\mathcal{G}})^*$ is symmetric monoidal. Under the equivalence of Remark 2.19, this transports to a symmetric monoidal structure on $\mathrm{Sp}^{\tau\mathcal{F}}$ and $\mathrm{Sp}^{\tau\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$ for which the adjunction $(i'_{\mathcal{G}})^* \dashv (i'_{\mathcal{G}})_*$ is symmetric monoidal.

Proposition 2.29. *Let $\mathcal{G}$ be a subfamily of $\mathcal{F}$. We have stable symmetric monoidal recollements*

$$\mathbf{Sp}^{h\mathcal{G}} \underset{(j_{\mathcal{G}})_*}{\overset{(j_{\mathcal{G}})^*}{\leftrightarrows}} \mathbf{Sp}^{h\mathcal{F}} \underset{(i_{\mathcal{G}})_*}{\overset{(i_{\mathcal{G}})^*}{\rightleftarrows}} \mathbf{Sp}^{h\mathcal{F}} \cap \mathbf{Sp}^{\Phi\mathcal{G}}, \quad \mathbf{Sp}^{\tau\mathcal{G}} \underset{(j'_{\mathcal{G}})_*}{\overset{(j'_{\mathcal{G}})^*}{\leftrightarrows}} \mathbf{Sp}^{\tau\mathcal{F}} \underset{(i'_{\mathcal{G}})_*}{\overset{(i'_{\mathcal{G}})^*}{\rightleftarrows}} \mathbf{Sp}^{\tau\mathcal{F}} \cap \mathbf{Sp}^{\Phi\mathcal{G}},$$

$$\mathbf{Sp}^{h\mathcal{F}} \cap \mathbf{Sp}^{\Phi\mathcal{G}} \underset{(j_{\mathcal{F}})_*}{\overset{(j_{\mathcal{F}})^*}{\leftrightarrows}} \mathbf{Sp}^{\Phi\mathcal{G}} \underset{(i_{\mathcal{F}})_*}{\overset{(i_{\mathcal{F}})^*}{\rightleftarrows}} \mathbf{Sp}^{\Phi\mathcal{F}}.$$

Furthermore, the equivalence $\mathrm{Sp}^{h\mathcal{F}} \xrightarrow{\simeq} \mathrm{Sp}^{\tau\mathcal{F}}$ of Remark 2.19 is an equivalence of recollements under which $(j_{\mathcal{G}})_!$ is the inclusion of $\mathcal{G}$-torsion objects into $\mathcal{F}$-torsion objects.

Proof. The defining properties of a stable symmetric monoidal recollement follow immediately from the same properties for the $\mathcal{F}$ and $\mathcal{G}$ recollements on Sp^G. For the last assertion, the equivalence of $\mathcal{F}$-complete and $\mathcal{F}$-torsion objects is implemented by $j_! j^*$, and as such clearly restricts to equivalences $\mathrm{Sp}^{h\mathcal{G}} \xrightarrow{\simeq} \mathrm{Sp}^{\tau\mathcal{G}}$ and $\mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}} \xrightarrow{\simeq} \mathrm{Sp}^{\tau\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$ compatibly with the adjunctions in view of Lemma 2.28(4–5). Finally, the claim about $(j_{\mathcal{G}})_!$ follows from a diagram chase of the right adjoints. $\square$

Remark 2.30 (Compact generation). Given a G-family $\mathcal{F}$, the $\mathcal{F}^{-1}$-local objects $\{G/H_+ \otimes \widetilde{E\mathcal{F}} : H \notin \mathcal{F}\}$ form a set of compact generators for $\mathrm{Sp}^{\Phi\mathcal{F}}$ because $\mathrm{Sp}^{\Phi\mathcal{F}} = \mathrm{Mod}_{\mathrm{Sp}^G}(\widetilde{E\mathcal{F}})$ and G/H_+ is $\mathcal{F}$-torsion for all $H \in \mathcal{F}$. Given two G-families $\mathcal{F}$ and $\mathcal{G}$, the essential image of $(j_{\mathcal{F}})_!$ is the localizing subcategory of $\mathrm{Sp}^{\Phi\mathcal{G}}$ generated by $\{G/H_+ \otimes \widetilde{E\mathcal{G}} : H \notin \mathcal{G}, H \in \mathcal{F}\}$.

Remark 2.31. The conclusions of Proposition 2.29 are also valid for the $\mathcal{F}$ and $\mathcal{G}$ recollements on the ∞-category of G-spaces. We likewise have the adjunction $\Sigma^\infty_+ : \mathbf{Spc}_{h\mathcal{F}} \cap \mathbf{Spc}_{\Phi\mathcal{G}} \rightleftarrows \mathbf{Sp}^{h\mathcal{F}} \cap \mathbf{Sp}^{\Phi\mathcal{G}} : \Omega^\infty$ and the same compatibility relations as in Construction 2.25.

Remark 2.32. Let us relate Proposition 2.29 to the "canonical fracture" of G-spectra studied in [4, section 2.4]. We say that a full subcategory $C_0 \subset C$ is *convex* if given any $x, z \in C_0$ such that there exists a 2-simplex $[x \to y \to z] \in C$, then $y \in C_0$. Let $\mathrm{Conv}(\mathfrak{S})$ denote the poset of convex subcategories of $\mathfrak{S}$ and let $\mathrm{Loc}(\mathrm{Sp}^G)$ denote the poset of reflective subcategories of Sp^G, with the order given by inclusion. Suppose $Q \in \mathrm{Conv}(\mathfrak{S})$ and write $Q = \mathcal{F} \setminus \mathcal{G}$ for some G-family $\mathcal{F}$ and subfamily $\mathcal{G}$. Then the assignment

$$\mathfrak{F}_G : \mathrm{Conv}(\mathfrak{S}) \to \mathrm{Loc}(\mathrm{Sp}^G)$$

of [4, Proposition 2.69] sends Q to $\mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$. Indeed, if we let $\mathcal{K}_H$ be the localizing subcategory of Sp^G generated by G/H_+ and examine [4, Notation 2.54], we see that $\mathcal{K}_{\leqslant Q} \simeq \mathrm{Sp}^{h\mathcal{F}}$ and $\mathcal{K}_{<Q} \simeq \mathrm{Sp}^{h\mathcal{G}}$ under the equivalence between torsion and complete objects. Thus, Sp^G_Q defined as the presentable quotient of $\mathcal{K}_{<Q} \to \mathcal{K}_{\leqslant Q}$ is equivalent to $\mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$ in view of Proposition 2.29.

Moreover, by inspection the functor $\rho : \mathrm{Sp}^G_Q \xrightarrow{\nu} \mathcal{K}_{\leqslant Q} \xrightarrow{i_R} \mathrm{Sp}^G$ in [4, Notation 2.54] exhibiting Sp^G_Q as a reflective subcategory embeds Sp^G_Q as $\mathcal{F}$-complete and $\mathcal{G}^{-1}$-local objects.

By [4, Proposition 2.69] the functor $\mathfrak{F}_G : \mathrm{Conv}(\mathfrak{S}) \to \mathrm{Loc}(\mathrm{Sp}^G)$ is a *fracture* in the sense of [4, Definition 2.32]. Thus, for any convex subcategory $Q = \mathcal{F} \setminus \mathcal{G}$ and sieve–cosieve decomposition of Q into $Q_0 = \mathcal{F}_0 \setminus \mathcal{G}_0$ and $Q_1 = \mathcal{F}_1 \setminus \mathcal{G}_1$, we obtain a recollement $(\mathrm{Sp}^{h\mathcal{F}_0} \cap \mathrm{Sp}^{\Phi\mathcal{G}_0}, \mathrm{Sp}^{h\mathcal{F}_1} \cap \mathrm{Sp}^{\Phi\mathcal{G}_1})$ of $\mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$. It is easily seen that these specialize to those considered in Proposition 2.29 in the case where Q is itself a sieve or a cosieve.

Notation 2.33. Given a subgroup H of G, let $\overline{H} = \mathfrak{S}_{\leqslant H}$ and $\partial\overline{H} = \mathfrak{S}_{<H}$ denote the G-family of subgroups that are subconjugate to H and properly subconjugate to H, respectively.[†] Let $\mathfrak{S}^c_{\geqslant H}$ denote the G-family of subgroups K such that H is *not* subconjugate to K.

Lemma 2.34. *Suppose $X \in \mathrm{Sp}^G$ is $\overline{H}$-complete and $(\partial\overline{H})^{-1}$-local. Then X is in addition $(\mathfrak{S}^c_{\geqslant H})^{-1}$-local, that is, for all subgroups K such that H is not subconjugate to K, $X^{\phi K} \simeq 0$.*

Proof. Note that $\partial\overline{H} = \overline{H} \cap \mathfrak{S}^c_{\geqslant H}$ and use Lemma 2.27. $\square$

The following two lemmas are explained in [4, Observation 2.11–14]) (and the first one also in [67, Proposition II.2.14]), so we will omit their proofs.

Lemma 2.35. *Let N be a normal subgroup of G. Then the geometric fixed points functor $\Phi^N : \mathrm{Sp}^G \to \mathrm{Sp}^{G/N}$ has fully faithful right adjoint with essential image $\mathrm{Sp}^{\Phi\mathfrak{S}^c_{\geqslant H}}$. Consequently, $\mathrm{Sp}^{G/N}$ is equivalent to the smashing localization $\mathrm{Mod}_{\mathrm{Sp}^G}(\widetilde{E\mathfrak{S}^c_{\geqslant N}})$.*

Lemma 2.36. *The geometric fixed points functor $\phi^H : \mathrm{Sp}^G \to \mathrm{Fun}(BW_G H, \mathrm{Sp})$ has fully faithful right adjoint with essential image $\mathrm{Sp}^{h\overline{H}} \cap \mathrm{Sp}^{\Phi(\partial\overline{H})} = \mathrm{Sp}^{h\overline{H}} \cap \mathrm{Sp}^{\Phi\mathfrak{S}^c_{\geqslant H}}$.*

2.3 | Reconstruction from geometric fixed points

We next aim to state the reconstruction theorem [4, Theorem A] of Ayala, Mazel-Gee, and Rozenblyum. For this, we need a few preliminary notions.

Definition 2.37. The *G-geometric locus*

$$\mathrm{Sp}^G_{\phi\text{-locus}} \subset \mathrm{Sp}^G \times \mathfrak{S}[G]$$

is the full subcategory on objects (X, H) such that $X \in \mathrm{Sp}^{h\overline{H}} \cap \mathrm{Sp}^{\Phi(\partial\overline{H})}$, that is, X is $\overline{H}$-complete and $(\partial\overline{H})^{-1}$-local (Notation 2.33).

Definition 2.38. Given H subconjugate to K, the *generalized Tate construction*

$$\tau^K_H : \mathrm{Fun}(BW_G H, \mathrm{Sp}) \to \mathrm{Fun}(BW_G K, \mathrm{Sp})$$

[†] This notation is consistent with viewing sieves as closed sets and cosieves as open sets for a topology on $\mathfrak{S}$.

is the functor given by the composition

$$\mathrm{Fun}(BW_GH, \mathrm{Sp}) \hookrightarrow \mathrm{Sp}^G \xrightarrow{\phi^K} \mathrm{Fun}(BW_GK, \mathrm{Sp})$$

where the first functor is the right adjoint to ϕ^H (which exists by Lemma 2.36). If $H = 1$, then we will write $\tau^K := \tau^K_1$.

If $G = C_p$ is a cyclic group of prime order, then $\tau^{C_p} \simeq t^{C_p}$ is the ordinary Tate construction [4, Remark 2.8], but not generally otherwise.

Remark 2.39. Evidently, the generalized Tate functors τ^K_H inherit some compatibility properties from the geometric fixed points functors. For example, for H a subgroup of K in G, the commutative diagrams

$$\begin{tikzcd} \mathbf{Sp}^K \arrow[r, "\Phi^H"] \arrow[d, "\mathrm{ind}"'] & \mathbf{Sp}^{W_KH} \arrow[r] \arrow[d, "\mathrm{ind}"'] & \mathrm{Fun}(BW_KH, \mathbf{Sp}) \arrow[d, "\mathrm{ind}"'] \\ \mathbf{Sp}^G \arrow[r, "\Phi^H"] & \mathbf{Sp}^{W_GH} \arrow[r] & \mathrm{Fun}(BW_GH, \mathbf{Sp}) \end{tikzcd}, \quad \begin{tikzcd} \mathbf{Sp}^G \arrow[r, "\Phi^K"] \arrow[d, "\mathrm{res}"] & \mathbf{Sp}^{W_KH} \arrow[d, "\mathrm{res}"] \\ \mathbf{Sp}^K \arrow[r, "\Phi^K"] & \mathbf{Sp} \end{tikzcd}$$

imply that the diagram of generalized Tate functors defined relative to G and K

$$\begin{tikzcd} \mathbf{Sp}^{hW_GH} \arrow[r, "\tau^K_H"] \arrow[d, "\mathrm{res}"] & \mathbf{Sp}^{hW_GK} \arrow[d, "\mathrm{res}"] \\ \mathbf{Sp}^{hW_KH} \arrow[r, "\tau^K_H"] & \mathbf{Sp} \end{tikzcd}$$

commutes. The notation is therefore unambiguous (or abusive) in the same sense as that for geometric fixed points.

Also, if $N_GK = N_GH$, then the composite (with the first functor right adjoint to Φ^H)

$$\mathrm{Sp}^{W_GH} \to \mathrm{Sp}^G \xrightarrow{\Phi^K} \mathrm{Sp}^{W_GK}$$

is homotopic to $\Phi^{K/H}$, and thus $\tau^K_H \simeq \tau^{K/H}$ for K/H regarded as a normal subgroup of W_GH.

Lemma 2.40. *The structure map $p : \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}} \to \mathfrak{S}$ is a locally cocartesian fibration such that for all $H \leqslant G$ subconjugate to $H \leqslant G$, the generalized Tate functors τ^K_H are the pushforward functors $(\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})_H \to (\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})_K$ encoded by p under the equivalence of Lemma 2.36.*

Proof. This is [4, Construction 2.38] applied to the fracture $\mathfrak{F}_G$ of Remark 2.32. To spell out a few more details, we need to show that for every edge $e : \Delta^1 \to \mathfrak{S}$ given by H subconjugate to K, the pullback $p|_e$ of p over Δ^1 is a cocartesian fibration. Let $C' \subset \mathrm{Sp}^G \times \Delta^1$ be the full subcategory on objects $\{(X, i)\}$ where if $i = 0$, then $X \in (\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})_H$. Then we have a factorization

$$\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}} \times_{\mathfrak{S},e} \Delta^1 \xrightarrow{i''} C' \xrightarrow{i'} \mathrm{Sp}^G \times \Delta^1.$$

Note that $C' \to \Delta^1$ is a sub-cocartesian fibration of $\mathrm{Sp}^G \times \Delta^1$ via i' (with cocartesian edges exactly those sent to equivalences via the projection to Sp^G). As for the fiber over 1, by definition we have that $(\mathrm{Sp}^G_{\phi\text{-locus}})_K$ is a localization of Sp^G. By an elementary lifting argument, this extends to a localization functor $L : C' \to C'$ whose essential image is $\mathrm{Sp}^G_{\phi\text{-locus}} \times_{\mathfrak{S},e} \Delta^1$. By [56, Lemma 2.2.1.11], we deduce that $p|_e$ is a cocartesian fibration. □

Recall the barycentric subdivision construction ([76, Definition 3.19] and [76, Observation 3.20]). Unwinding that definition in our situation of interest for a preordered set, we see that $\mathrm{sd}(\mathfrak{S})$ is the category whose objects are strings $\kappa = [H_0 < H_1 < \cdots < H_n]$ in $\mathfrak{S}$ with each H_i properly subconjugate to H_{i+1}, and where a morphism

$$\kappa = [H_0 < H_1 < \cdots < H_n] \to \lambda = [K_0 < K_1 < \cdots < K_m]$$

is the data of an injective map $\alpha : [n] \to [m]$ of totally ordered sets and a commutative diagram in $\mathfrak{S}$

$$\begin{tikzcd}
H_0 \arrow[r] \arrow[d] & H_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & H_n \arrow[d] \\
K_{\alpha(0)} \arrow[r] & K_{\alpha(1)} \arrow[r] & \cdots \arrow[r] & K_{\alpha(n)}
\end{tikzcd}$$

whose vertical morphisms are equivalences. Note that if a morphism $\kappa \to \lambda$ exists, then α and the commutative ladder are uniquely determined. Thus, the morphism sets in $\mathrm{sd}(\mathfrak{S})$ are either empty or singleton and $\mathrm{sd}(\mathfrak{S})$ is also a preordered set. Regard $\mathrm{sd}(\mathfrak{S})$ as a locally cocartesian fibration over $\mathfrak{S}$ via the functor which takes a string to its maximum element [76, Construction 3.21].

Remark 2.41. Given any locally cocartesian fibration $p : C \to \mathfrak{S}$ whose fibers C_H are stable ∞-categories and whose pushforward functors are exact, the right-lax limit $\mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}}(\mathrm{sd}(\mathfrak{S}), C)$ is a stable ∞-category by [76, Lemma 3.34].[†] Moreover, if the fibers are presentable and the pushforward functors are also accessible, then the right-lax limit is presentable by [76, Proposition 3.38].

We may now state [4, Theorem A], rewritten in our notation.

Theorem 2.42. *There is a canonical equivalence* $\mathrm{Sp}^G \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}}(\mathrm{sd}(\mathfrak{S}), \mathrm{Sp}^G_{\phi\text{-locus}})$.

Examining the proof of [4, Theorem 2.40], we see that this equivalence is implemented by the *right-lax* functor $\mathrm{Sp}^G \times \mathfrak{S} \dashrightarrow \mathrm{Sp}^G_{\phi\text{-locus}}$ that globalizes the left adjoints ϕ^H. This is not expressible as a functor $\mathrm{Sp}^G \times \mathfrak{S} \to \mathrm{Sp}^G_{\phi\text{-locus}}$; rather, its construction derives from an existence and uniqueness theorem on adjunctions in $(\infty, 2)$-categories (see [4, Lemma 1.34] and [27, Corollary 3.1.7]). However, by instead working with the defining inclusion $\mathrm{Sp}^G_{\phi\text{-locus}} \subset \mathrm{Sp}^G \times \mathfrak{S}$, we can avoid any explicit usage of $(\infty, 2)$-category theory and still define a comparison functor, as in the following construction.

[†] In agreement with [76, section 3], given a locally cocartesian fibration $p : C \to S$ we will regard the section ∞-category $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ as the *definition* of a right-lax limit. We refer to [4, sections 1.1–1.2] for a useful summary of the general theory of lax limits.

Construction 2.43. Let $\mathcal{F}$ be a G-family, $\mathcal{G}$ a subfamily, and $\mathcal{H} = \mathcal{F} \setminus \mathcal{G}$. Consider the composite functor

$$\Theta'_{\mathcal{H}} : \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{H}}\left(\mathrm{sd}(\mathcal{H}), \mathcal{H} \times_{\mathfrak{S}} \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}\right) \to \mathrm{Fun}(\mathrm{sd}(\mathcal{H}), \mathrm{Sp}^G) \xrightarrow{\lim} \mathrm{Sp}^G$$

where the first functor is postcomposition by the projection to Sp^G and the second takes the limit. Note that by Lemma 2.28, if $X \in (\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})_H$ for any $H \in \mathcal{F} \setminus \mathcal{G}$, then $X \in \mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}$. Therefore, $\Theta'_{\mathcal{H}}$ factors through the inclusion $\mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}} \subset \mathrm{Sp}^G$. Denote that functor by $\Theta_{\mathcal{H}}$.

In the case of $\mathcal{F} = \mathfrak{S}$ and $\mathcal{G} = \emptyset$, we also write Θ for the comparison functor.

Lemma 2.44. *Let $\mathcal{F}$ be a G-family, $\mathcal{G}$ a subfamily, and $\mathcal{H} = \mathcal{F} \setminus \mathcal{G}$. For every $H \in \mathcal{H}$, the composition*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{H}}\left(\mathrm{sd}(\mathcal{H}), \mathcal{H} \times_{\mathfrak{S}} \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}\right) \xrightarrow{\Theta'_{\mathcal{H}}} \mathrm{Sp}^G \xrightarrow{\phi^H} \mathrm{Fun}(BW_G H, \mathrm{Sp})$$

is homotopic to evaluation at $H \in \mathrm{sd}(\mathcal{H})$ under the equivalence $(\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})_H \simeq \mathrm{Fun}(BW_G H, \mathrm{Sp})$.

Proof. Let $f : \mathrm{sd}(\mathcal{H}) \to \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}$ be an object in $\mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{H}}(\mathrm{sd}(\mathcal{H}), \mathcal{H} \times_{\mathfrak{S}} \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})$, and let $f' : \mathrm{sd}(\mathcal{H}) \to \mathrm{Sp}^G$ denote the subsequent functor obtained by the projection to Sp^G. We need to produce a natural equivalence

$$\phi^H \lim f' \simeq f'(H).$$

As $\mathrm{sd}(\mathcal{H})$ is finite, it suffices instead to show $\lim \phi^H f' \simeq f'(H)$. Note that for any $X \in (\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})_K$, if K is not in $\overline{H}$ then $X^{\phi H} \simeq 0$; indeed, $\Phi^L(X) \simeq 0$ for all $L \in \mathfrak{S}^c_{\geqslant K}$ by definition. Therefore, if we let $J \subset \mathrm{sd}(\mathcal{H})$ be the full subcategory on those strings σ with $\max(\sigma) \leqslant H$, the functor $\phi^H f'$ is a right Kan extension of its restriction to J (for this, also note that if $\tau = [K_0 < \cdots < K_n] \in \mathrm{sd}(\mathcal{H})$ with $K_n \notin \overline{H}$, then $\mathrm{sd}(\mathcal{H})_{\tau/} \times_{\mathrm{sd}(\mathcal{H})} J = \emptyset$).

Next, let $I \subset J$ be the full subcategory on those strings σ with $\max(\sigma)$ conjugate to H. For a string $\tau = [K_0 < \cdots < K_n] \in J$ with K_n properly subconjugate to H, the unique string inclusion

$$e : [K_0 < \cdots < K_n] \to [K_0 < \cdots < K_n < H]$$

is sent to an equivalence by $\phi^H f'$ by definition of the locally cocartesian edges in $\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}$; indeed, $f'(e)$ is a unit map of the localization for the reflective subcategory $(\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})_H \subset \mathrm{Sp}^G$. Observe also that e is an initial object in $I \times_J J^{\tau/}$. We deduce that $\phi^H f'$ is a right Kan extension of its further restriction to I. Because H is an initial object of I, we conclude that $\lim \phi^H f' \simeq f'(H)$, as desired. □

For the next proposition, recall from [76, Theorem 3.35] the recollement of a right-lax limit defined by a sieve–cosieve decomposition of the base.

Proposition 2.45. *Let $\mathcal{F}$ be a G-family, $\mathcal{G}$ a subfamily, and $\mathcal{H} = \mathcal{F} \setminus \mathcal{G}$. The functor*

$$\Theta_{\mathcal{F}} : \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{F}}(\mathrm{sd}(\mathcal{F}), \mathcal{F} \times_{\mathfrak{S}} \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}) \to \mathrm{Sp}^{h\mathcal{F}}$$

is a strict morphism[†] *of stable recollements*

$$(\mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{G}}(\mathrm{sd}(\mathcal{G}), \mathcal{G} \times_{\mathfrak{S}} \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}), \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{H}}\left(\mathrm{sd}(\mathcal{H}), \mathcal{H} \times_{\mathfrak{S}} \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}\right)) \to (\mathrm{Sp}^{h\mathcal{G}}, \mathrm{Sp}^{h\mathcal{F}} \cap \mathrm{Sp}^{\Phi\mathcal{G}}).$$

Moreover, the resulting functors between the open and closed parts are equivalent to $\Theta_{\mathcal{G}}$ *and* $\Theta_{\mathcal{H}}$.

Proof. We need to show that $\Theta_{\mathcal{F}}$ sends the essential images of $j_!$, j_*, and i_* to $\mathcal{G}$-torsion[‡], $\mathcal{G}$-complete, and $\mathcal{G}^{-1}$-local objects, respectively. Let

$$f : \mathrm{sd}(\mathcal{F}) \to \mathcal{F} \times_{\mathfrak{S}} \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}$$

be a functor that preserves locally cocartesian edges. By [76, Proposition 3.32], if f is in the essential image of $j_!$, then $f(H) = 0$ for all $H \in \mathcal{H}$. By Lemma 2.44, we then have $\phi^H \Theta_{\mathcal{F}}(f) \simeq 0$ for all $H \in \mathcal{H}$, so $\Theta_{\mathcal{F}}(f)$ is $\mathcal{G}$-torsion. Similarly, using [76, Proposition 3.33] and Lemma 2.44 again, the same proof shows that if f is in the essential image of i_*, then $\phi^H \Theta_{\mathcal{F}}(f) \simeq 0$ for all $H \in \mathcal{G}$ and thus $\Theta_{\mathcal{F}}(f)$ is $\mathcal{G}^{-1}$-local. Finally, suppose that f is in the essential image of j_*. By [76, Theorem 3.29], f is a relative right Kan extension of its restriction to the subcategory $\mathrm{sd}(\mathcal{F})_0$ of strings whose minimums lie in $\mathcal{G}$. Because the inclusion $(\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})_H \subset \mathrm{Sp}^G$ of each fiber preserves limits, the further composition

$$f' : \mathrm{sd}(\mathcal{F}) \xrightarrow{f} \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}} \to \mathrm{Sp}^G$$

is then a right Kan extension of its restriction to $\mathrm{sd}(\mathcal{F})_0$ (in the nonrelative sense). Moreover, the inclusion $\mathrm{sd}(\mathcal{G}) \subset \mathrm{sd}(\mathcal{F})_0$ is right cofinal. Indeed, for every string $\sigma = [K_0 < \cdots < K_n]$ in $\mathrm{sd}(\mathcal{F})_0$, if we let σ' denote its maximal substring in $\mathrm{sd}(\mathcal{G})$, then σ' is a terminal object in $(\mathrm{sd}(\mathcal{F})_0)^{/\sigma} \times_{\mathrm{sd}(\mathcal{F})_0} \mathrm{sd}(\mathcal{G})$, so these slice categories are weakly contractible and we may thus apply Joyal's version of Quillen's theorem A [54, Theorem 4.1.3.1]. It follows that $\Theta_{\mathcal{F}}(f)$ is computed as a limit of $\mathcal{G}$-complete spectra and is hence itself $\mathcal{G}$-complete.

The two functors on the open and closed parts induced by the morphism of stable recollements are then definitionally $(j_{\mathcal{G}})^* \Theta_{\mathcal{F}} j_*$ and $(i_{\mathcal{G}})^* \Theta_{\mathcal{F}} i_*$. These are equivalent to $\Theta_{\mathcal{G}}$ and $\Theta_{\mathcal{H}}$ by the same cofinality arguments as above. □

Theorem 2.46 (cf. Theorem 2.42). *For every* G*-family* $\mathcal{F}$ *and subfamily* $\mathcal{G}$*, the functor* $\Theta_{\mathcal{F}\backslash\mathcal{G}}$ *is an equivalence of* ∞*-categories. In particular, we have an equivalence*

$$\Theta : \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}}\left(\mathrm{sd}(\mathfrak{S}), \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}\right) \xrightarrow{\simeq} \mathrm{Sp}^G.$$

Proof. Our strategy is to use Proposition 2.45 in conjunction with the fact that given a strict morphism $F : \mathcal{X} \to \mathcal{X}'$ of stable recollements $(\mathcal{U}, \mathcal{Z}) \to (\mathcal{U}', \mathcal{Z}')$, if F_U and F_Z are equivalences then F is an equivalence [76, Remark 2.7].[§] Let us first prove that $\Theta_{\mathcal{F}}$ is an equivalence for all families $\mathcal{F}$. We proceed by induction on the size of $\mathcal{F}$. For the base case, if $\mathcal{F} = \{1\}$ is the trivial family, then $\mathrm{sd}(\mathcal{F}) \cong \mathcal{F}$ and $\Theta_{\mathcal{F}}$ is definitionally an equivalence. Now suppose for the inductive hypothesis that $\Theta_{\mathcal{G}}$ is an equivalence for all proper subfamilies $\mathcal{G}$ of $\mathcal{F}$. Let $H \in \mathcal{F}$ be a maximal element and let

[†] A *strict morphism of recollements* is one that commutes with the gluing functors; cf. [76, Definition 2.6].

[‡] More precisely, $\mathcal{G}$-torsion with respect to the embedding of $\mathrm{Sp}^{h\mathcal{F}}$ in Sp^G as $\mathcal{F}$-torsion objects.

[§] This type of inductive argument is also used in the proof of [4, Theorem 2.40].

$\mathcal{G} \subset \mathcal{F}$ be the largest subfamily excluding H. Then $\mathcal{F} \setminus \mathcal{G} = \overline{H} \setminus \partial\overline{H}$, so $\Theta_{\mathcal{F}\setminus\mathcal{G}}$ is definitionally an equivalence. By Proposition 2.45, we deduce that $\Theta_{\mathcal{F}}$ is an equivalence.

Finally, to deal with the general case, we note that any strict morphism of stable recollements that is also an equivalence restricts to equivalences between the open and closed parts. Thus, having proven that $\Theta_{\mathcal{F}}$ is an equivalence, we further deduce that $\Theta_{\mathcal{F}\setminus\mathcal{G}}$ is an equivalence for any subfamily $\mathcal{G}$. □

Remark 2.47. The generalized Tate functors τ^K_H are lax symmetric monoidal, and the various natural transformations among these functors encoded by the locally cocartesian fibration are also lax symmetric monoidal. In [6, section 5], Ayala–Mazel-Gee–Rozenblyum explain how these data assemble to a symmetric monoidal structure on the right-lax limit $\mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}}(\mathrm{sd}(\mathfrak{S}), \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})$ such that the functor Θ of Theorem 2.46 is an equivalence of symmetric monoidal ∞-categories.

Theorem 2.46 and Proposition 2.45, along with the explicit description of the functor j_* given in [76, Theorem 3.29], gives the following formula for the geometric fixed points of an $\mathcal{F}$-complete spectrum in terms of a limit of generalized Tate constructions.

Definition 2.48. For $H \notin \mathcal{F}$, let $J_H \subset \mathrm{sd}(\mathfrak{S})$ be the full subcategory on strings $[K_0 < \cdots < K_n < H]$ such that $K_i \in \mathcal{F}$ for $1 \leqslant i \leqslant n$.

Corollary 2.49. *Let X be a G-spectrum and let $X^\bullet : \mathrm{sd}(\mathfrak{S}) \to \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}$ denote a lift of X under the equivalence Θ. Suppose that X is $\mathcal{F}$-complete. Then*

$$X^{\phi H} \simeq \lim_{J_H} X^\bullet,$$

with the limit taken in the fiber $\mathrm{Fun}(BW_G H, \mathrm{Sp}) \simeq (\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}})_H$.

Example 2.50. Suppose that $G = C_{p^2}$ and let $\mathcal{P}$ be the family of proper subgroups of G. Then $\mathrm{sd}(\mathcal{P}) \cong \mathrm{sd}(\Delta^1)$, so the data of a $\mathcal{P}$-complete spectrum X amount to

- A Borel C_{p^2}-spectrum X^1;
- A Borel C_{p^2}/C_p-spectrum $X^{\phi C_p}$;
- A $C_{p^2}/C_p \cong C_p$-equivariant map $\alpha : X^{\phi C_p} \to (X^1)^{tC_p}$.

The category $J_{C_{p^2}}$ as well as the functor $J_{C_{p^2}} \to \mathrm{Sp}$ is then identified as

$$\left(\begin{tikzcd} & {[C_p < C_{p^2}]} \arrow[d] \\ {[1 < C_{p^2}]} \arrow[r] & {[1 < C_p < C_{p^2}]} \end{tikzcd}\right) \rightsquigarrow \left(\begin{tikzcd} & X^{\phi C_p tC_p} \arrow[d, "\alpha^{tC_p}"] \\ (X^1)^{\tau C_{p^2}} \arrow[r, "\mathrm{can}"] & (X^1)^{tC_p tC_p} \end{tikzcd}\right),$$

where *can* is the canonical map encoded by the locally cocartesian fibration. Thus,

$$X^{\phi C_{p^2}} \simeq (X^1)^{\tau C_{p^2}} \times_{(X^1)^{tC_p tC_p}} X^{\phi C_p tC_p}.$$

Moreover, it is not difficult to see that $(-)^{\tau C_{p^2}} \simeq (-)^{hC_p tC_p}$; in fact, we will explain the identification $(-)^{\tau C_{p^n}} \simeq (-)^{hC_{p^{n-1}} tC_p}$ in [70].

Let us now turn to the examples of interest for the dihedral Tate orbit lemma that we will prove in [70].

Example 2.51. Suppose that $G = D_4 = C_2 \times \mu_2$ is the Klein four-group and let $\Gamma = \{1, C_2, \Delta\}$ for Δ the diagonal subgroup. The data of a Γ-complete spectrum X amount to

- A Borel D_4-spectrum X^1, Borel (D_4/C_2)-spectrum $X^{\phi C_2}$, and Borel (D_4/Δ)-spectrum $X^{\phi\Delta}$;
- A (D_4/C_2)-equivariant map $\alpha : X^{\phi C_2} \to (X^1)^{tC_2}$ and (D_4/Δ)-equivariant map $\beta : X^{\phi\Delta} \to (X^1)^{t\Delta}$.

As $J_{\mu_2} = \{[1 < \mu_2]\}$ and $J_{\mu_2} \to \mathrm{Fun}(B(D_4/\mu_2), \mathrm{Sp})$ is the pushforward of X^1 by $(-)^{t\mu_2}$, we see that $X^{\phi\mu_2} \simeq (X^1)^{t\mu_2}$. On the other hand, $J_{D_4} \to \mathrm{Sp}$ is given by

$$\left(\begin{tikzcd} {[\Delta < D_4]} \arrow[r] & {[1 < \Delta < D_4]} \\ {[1 < D_4]} \arrow[ur] \arrow[dr] & \\ {[C_2 < D_4]} \arrow[r] & {[1 < C_2 < D_4]} \end{tikzcd}\right) \rightsquigarrow \left(\begin{tikzcd} (X^{\phi\Delta})^{t(D_4/\Delta)} \arrow[r, "\beta^{t(D_4/\Delta)}"] & ((X^1)^{t\Delta})^{t(D_4/\Delta)} \\ (X^1)^{\tau D_4} \arrow[ur, "\mathrm{can}"'] \arrow[dr, "\mathrm{can}"] & \\ (X^{\phi C_2})^{t(D_4/C_2)} \arrow[r, "\alpha^{t(D_4/C_2)}"'] & ((X^1)^{tC_2})^{t(D_4/C_2)} \end{tikzcd}\right),$$

and $X^{\phi D_4}$ is the limit of this diagram.

To handle the case of the dihedral group D_{2p} of order $2p$ for p an odd prime, we first record a vanishing property of the generalized Tate construction.

Lemma 2.52. *Let G be a finite group and suppose $K \leqslant G$ is a nontrivial subgroup that is not a p-group. Then $\tau^K \simeq 0$.*

Proof. By the compatibility of the generalized Tate functors with restriction (Remark 2.39), we may suppose $K = G$ without loss of generality. Note that τ^G may be computed as the left Kan extension of $(-)^{hG}$ along the functor from Sp^{hG} to its Verdier quotient by orbits $\{G/H_+ : H < G\}$ with H proper [4, Remark 2.16]. If we let $\underline{All}$ be the family of subgroups H such that $|H| = p^n$ for some prime p and integer n as in [60, fig. 1.7], then $\underline{All}$ is a subfamily of the proper subgroups under our assumption. However, by [60, Theorem 4.25], the thick $\otimes$-ideal in Sp^G generated by $\{G/H_+ : H \in \underline{All}\}$ includes the Borel completion of the unit. Therefore, the Verdier quotient in question is the trivial category, and we deduce that $\tau^G \simeq 0$. □

Example 2.53. Let p be an odd prime, $G = D_{2p} = \mu_p \rtimes C_2$ the dihedral group of order $2p$, and Γ the family of subgroups H such that $H \cap \mu_p = 1$. Note that up to conjugacy, Γ consists of the subgroups 1 and C_2, and the Weyl group of C_2 is trivial. Thus, up to equivalence, the data of a Γ-complete spectrum X amount to

- A Borel D_{2p}-spectrum X^1 and a spectrum $X^{\phi C_2}$;
- A map $\alpha : X^{\phi C_2} \to (X^1)^{tC_2}$.

Using that $J_{\mu_p} = [1 < \mu_p]$, we compute $X^{\phi\mu_p} \simeq (X^1)^{t\mu_p}$. As for $X^{\phi D_{2p}}$, by Lemma 2.52 we have that $(X^1)^{\tau D_{2p}} \simeq 0$. We further claim that the generalized Tate functor $\tau^{D_{2p}}_{C_2}$ vanishes.

(∗) Let $\mathcal{H} = \{1, \mu_p\} = \mathfrak{S}^c_{\geqslant C_2}$. By Remark 2.23 applied to $\underline{\mathrm{Sp}}^{\Phi\mathcal{H}}$, the restriction and induction functors for $C_2 \subset D_{2p}$ descend to an adjunction

$$\mathrm{res}' : \mathbf{Sp}^{\Phi\mathcal{H}} \rightleftarrows \mathbf{Sp} : \mathrm{ind}',$$

where $(\underline{\mathrm{Sp}}^{\Phi\mathcal{H}})_{D_{2p}/C_2} \simeq \mathrm{Sp}$ because the restriction of $\mathcal{H}$ to a C_2-family yields the trivial family. Now consider the inclusion of the open fiber

$$j_* : \mathrm{Sp}^{hW_G C_2} \to \mathrm{Sp}^{\Phi\mathcal{H}}.$$

Because $W_{D_{2p}} C_2 \cong 1$, we have that $\mathrm{res}' \simeq j^*$, and we deduce that $j_! \simeq j_*$. Because $\mathcal{H}^c = \mathfrak{S}_{\geqslant C_2}$ consists only of the two subgroups C_2 and D_{2p} up to conjugacy, we may identify $\phi^{D_{2p}} : \mathrm{Sp}^{\Phi\mathcal{H}} \to \mathrm{Sp}$ with the restriction i^* to the closed complement of a recollement of $\mathrm{Sp}^{\Phi\mathcal{H}}$ with j_* as the inclusion of the open part. We then have $\tau^{D_{2p}}_{C_2} \simeq i^* j_*$. In view of the fiber sequence

$$j_! \xrightarrow{\simeq} j_* \to i_* i^* j_* \simeq 0,$$

we deduce that $\tau^{D_{2p}}_{C_2} \simeq 0$.

Using Corollary 2.49, we conclude that $X^{\phi D_{2p}} \simeq 0$.

We conclude this section by indicating how the comparison functor Θ is functorial in the group G with respect to restriction and geometric fixed points.

Construction 2.54 (Restriction functoriality for geometric loci). Let H be a subgroup of G and consider the map $i : \mathfrak{S}[H] \to \mathfrak{S}[G]$ that sends a subgroup K of H to the same K viewed as a subgroup of G. As i preserves the subconjugacy relation, i is a functor,[†] and also let $i : \mathrm{sd}(\mathfrak{S}[H]) \to \mathrm{sd}(\mathfrak{S}[G])$ denote the induced functor on barycentric subdivisions. Next, consider the functor $\mathrm{res}^G_H \times \mathrm{id} : \mathrm{Sp}^G \times \mathfrak{S}[H] \to \mathrm{Sp}^H \times \mathfrak{S}[H]$. As for any subgroup $K \leqslant H$, the restriction of the G-families $\mathfrak{S}[G]_{\leqslant K}$, $\mathfrak{S}[G]_{<K}$ to H yields H-families $\mathfrak{S}[H]_{\leqslant K}$, $\mathfrak{S}[H]_{<K}$, by Remark 2.23 we have an induced functor over $\mathfrak{S}[H]$

$$\mathrm{res}^G_H : \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}} \times_{\mathfrak{S}[G]} \mathfrak{S}[H] \to \mathrm{Sp}^H_{\phi\text{-}\mathrm{locus}}$$

that preserves locally cocartesian edges. Precomposition by i and postcomposition by res^G_H then defines a functor

$$\mathrm{res}^G_H : \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}[G]}\left(\mathrm{sd}(\mathfrak{S}[G]), \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}\right) \to \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}[H]}\left(\mathrm{sd}(\mathfrak{S}[H]), \mathrm{Sp}^H_{\phi\text{-}\mathrm{locus}}\right).$$

[†] However, as there may be additional conjugacy relations in G, i is not generally the inclusion of a subcategory.

We have a lax commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{cocart}_{/\mathfrak{S}[G]}(\mathrm{sd}(\mathfrak{S}[G]), \mathbf{Sp}^G_{\phi\text{-locus}}) \arrow[r, "\Theta_G", "\simeq"'] \arrow[d, "\mathrm{res}^G_H"'] & \mathbf{Sp}^G \arrow[d, "\mathrm{res}^G_H"] \\
\mathrm{Fun}^{cocart}_{/\mathfrak{S}[H]}(\mathrm{sd}(\mathfrak{S}[H]), \mathbf{Sp}^H_{\phi\text{-locus}}) \arrow[r, "\Theta_H", "\simeq"'] \arrow[ur, phantom, "\nearrow"] & \mathbf{Sp}^H
\end{tikzcd}$$

where the natural transformation $\eta : \mathrm{res}^G_H \circ \Theta_G \to \Theta_H \circ \mathrm{res}^G_H$ is defined using the contravariant functoriality of the limit for $i : \mathrm{sd}(\mathfrak{S}[H]) \to \mathrm{sd}(\mathfrak{S}[G])$.

We claim that η is an equivalence, so that this diagram commutes. Indeed, suppose given

$$f : \mathrm{sd}(\mathfrak{S}[G]) \to \mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}}$$

and let $g = \mathrm{res}^G_H f : \mathrm{sd}(\mathfrak{S}[H]) \to \mathrm{Sp}^H_{\phi\text{-}\mathrm{locus}}$. Let $f' : \mathrm{sd}(\mathfrak{S}[G]) \to \mathrm{Sp}^G$ and $g' : \mathrm{sd}(\mathfrak{S}[H]) \to \mathrm{Sp}^H$ be the functors obtained by postcomposition, so $g' = \mathrm{res}^G_H f' i$ by definition and η_f is the comparison map

$$\lim_{\mathrm{sd}(\mathfrak{S}[G])} \mathrm{res}^G_H f' \to \lim_{\mathrm{sd}(\mathfrak{S}[H])} \mathrm{res}^G_H f' i.$$

It suffices to check that for all subgroups $K \leqslant H$, $\phi^K(\eta_f)$ is an equivalence. But then by the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{Sp}^G \arrow[r, "\phi^K"] \arrow[d, "\mathrm{res}^G_H"'] & \mathrm{Fun}(BW_G K, \mathbf{Sp}) \arrow[d, "\mathrm{res}^{W_G K}_{W_H K}"] \\
\mathbf{Sp}^H \arrow[r, "\phi^K"] & \mathrm{Fun}(BW_H K, \mathbf{Sp}),
\end{tikzcd}$$

and under the equivalences $\phi^K \Theta_G \simeq \mathrm{ev}_K$ and $\phi^K \Theta_H \simeq \mathrm{ev}_K$ of Lemma 2.44, we see that $\phi^K(\eta_f)$ is an equivalence.

Example 2.55. Let $G = D_{2p} = \mu_p \rtimes C_2$ as in Example 2.53 and let $H = C_2$. The construction above defines a functor

$$\mathrm{res}^{D_{2p}}_{C_2} : \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}[D_{2p}]}(\mathrm{sd}(\mathfrak{S}[D_{2p}]), \mathrm{Sp}^{D_{2p}}_{\phi\text{-}\mathrm{locus}}) \to \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}[C_2]}(\mathrm{sd}(\mathfrak{S}[C_2]), \mathrm{Sp}^{C_2}_{\phi\text{-}\mathrm{locus}})$$

and we have an equivalence $\eta : \mathrm{res}^{D_{2p}}_{C_2} \circ \Theta_{D_{2p}} \xrightarrow{\sim} \Theta_{C_2} \circ \mathrm{res}^{D_{2p}}_{C_2}$.

Construction 2.56 (Geometric fixed points functoriality for geometric loci). Let N be a normal subgroup of G. Then we may embed $\mathfrak{S}[G/N]$ as a cosieve in $\mathfrak{S}[G]$ via the functor $i : \mathfrak{S}[G/N] \to \mathfrak{S}[G]$ that sends M/N to M. We also let $i : \mathrm{sd}(\mathfrak{S}[G/N]) \to \mathrm{sd}(\mathfrak{S}[G])$ denote the induced functor on barycentric subdivisions, which is a cosieve inclusion. By Lemma 2.35, $\Phi^N : \mathrm{Sp}^G \to \mathrm{Sp}^{G/N}$ has fully faithful right adjoint with essential image given by the $(\mathfrak{S}[G] \setminus \mathfrak{S}[G/N])^{-1}$-local objects. Therefore, Φ^N implements an equivalence over $\mathfrak{S}[G/N]$

$$\mathrm{Sp}^G_{\phi\text{-}\mathrm{locus}} \times_{\mathfrak{S}[G]} \mathfrak{S}[G/N] \xrightarrow{\simeq} \mathrm{Sp}^{G/N}_{\phi\text{-}\mathrm{locus}}.$$

Define $\Phi^N : \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}[G]}(\mathrm{sd}(\mathfrak{S}[G]), \mathrm{Sp}^G_{\phi\text{-locus}}) \to \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}[G/N]}(\mathrm{sd}(\mathfrak{S}[G/N]), \mathrm{Sp}^{G/N}_{\phi\text{-locus}})$ to be the functor obtained by i^* under that equivalence. Then because Θ is a morphism of recollements, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{cocart}_{/\mathfrak{S}[G]}(\mathrm{sd}(\mathfrak{S}[G]), \mathbf{Sp}^G_{\phi\mp\mathrm{locus}}) \arrow[r, "\Theta_G", "\simeq"'] \arrow[d, "\Phi^N"'] & \mathbf{Sp}^G \arrow[d, "\Phi^N"] \\
\mathrm{Fun}^{cocart}_{/\mathfrak{S}[G/N]}(\mathrm{sd}(\mathfrak{S}[G/N]), \mathbf{Sp}^{G/N}_{\phi\mp\mathrm{locus}}) \arrow[r, "\Theta_{G/N}", "\simeq"'] & \mathbf{Sp}^{G/N}.
\end{tikzcd}$$

Example 2.57. Let $G = D_{2p} = \mu_p \rtimes C_2$ as in Example 2.53 and let $N = \mu_p$ so $G/N = C_2$. The construction above defines a functor

$$\Phi^{\mu_p} : \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}[D_{2p}]}\left(\mathrm{sd}(\mathfrak{S}[D_{2p}]), \mathrm{Sp}^{D_{2p}}_{\phi\text{-locus}}\right) \to \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{S}[C_2]}\left(\mathrm{sd}(\mathfrak{S}[C_2]), \mathrm{Sp}^{C_2}_{\phi\text{-locus}}\right)$$

and we have an equivalence $\Phi^{\mu_p} \circ \Theta_{D_{2p}} \simeq \Theta_{C_2} \circ \Phi^{\mu_p}$.

3 | THEORIES OF G-SPECTRA RELATIVE TO A NORMAL SUBGROUP N

In classical approaches to equivariant stable homotopy theory [49, 61], one attaches to every G-universe $\mathcal{V}$ a corresponding theory of G-spectra indexed with respect to $\mathcal{V}$; upon inverting the weak equivalences, this yields a stable ∞-category $\mathrm{Sp}^G_{\mathcal{V}}$. For the complete G-universe $\mathcal{V}$, one obtains *genuine* G-spectra $\mathrm{Sp}^G_{\mathcal{V}} \simeq \mathrm{Sp}^G$, whereas for the trivial G-universe $\mathcal{U}^G$, one obtains *naive* G-spectra $\mathrm{Sp}^G_{\mathcal{V}^G} \simeq \mathrm{Fun}(\mathcal{O}^{\mathrm{op}}_G, \mathrm{Sp})$.[†] Interpolating between genuine and naive G-spectra, for every normal subgroup $N \trianglelefteq G$, one has the fixed points G-universe $\mathcal{V}^N$ [61, chapter XVI, section 5] and the associated ∞-category $\mathrm{Sp}^G_{\mathcal{V}^N}$. In this section, we will revisit these notions from a different and intrinsically ∞-categorical perspective that makes no reference to representation theory. Using the language of parameterized ∞-category theory, we define ∞-categories $\mathrm{Sp}^G_{N\text{-naive}}$ and $\mathrm{Sp}^G_{N\text{-Borel}}$ of *N-naive* and *N-Borel* G-spectra (Definition 3.5 and Definition 3.19). We then show $\mathrm{Sp}^G_{N\text{-Borel}}$ admits two canonical embeddings into Sp^G as the Γ_N-torsion and Γ_N-complete G-spectra for Γ_N the N-free G-family (Theorem 3.24).

Warning 3.1. From this section until the end of Section 4, the role of G is different from its role in the introduction. In particular, G is always viewed as an extension of G/N by a normal subgroup N throughout this section, whereas G was the quotient $\widehat{G}/K$ in the introduction. This change is made to emphasize the fact that G is finite throughout the paper.

To begin with, we will need a lemma on cartesian fibrations arising from adjunctions.

Lemma 3.2.

(1) *Let* $L : C \rightleftarrows D : R$ *be an adjunction such that for all* $c \in C$, $d \in D$, *and* $f : d \to Lc$ *the natural map*

$$L(Rd \times_{RLc} c) \to d$$

[†] We identify the ∞-category as the ordinary stabilization of G-spaces $\mathrm{Spc}_G = \mathrm{Fun}(\mathcal{O}^{\mathrm{op}}_G, \mathrm{Spc})$.

adjoint to the projection $Rd \times_{RLc} c \to Rd$ is an equivalence. Then L is an essentially cartesian fibration[†] *with L-cartesian edges given by $Rd \times_{RLc} c \to c$, and hence a cartesian fibration if L is assumed to be a categorical fibration.*

(2) *Let*

$$L'' = L' \circ L : C \underset{R}{\overset{L}{\rightleftarrows}} C' \underset{R'}{\overset{L'}{\rightleftarrows}} D : R \circ R' = R''$$

be a diagram of adjunctions such that $L \dashv R$, $L' \dashv R'$, and $L'' \dashv R''$ all satisfy the assumption in (1). Then L sends L''-cartesian edges to L'-cartesian edges.

Proof. For (1), under our assumption, we need only show that $Rd \times_{RLc} c \to c$ is a L-cartesian edge. But for this, for any $c' \in C$ we have the pullback square of spaces

$$\begin{tikzcd} \mathrm{Map}_C(c', Rd \times_{RLc} c) \arrow[r] \arrow[d] & \mathrm{Map}_C(c', Rd) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Map}_D(Lc', d) \arrow[d] \\ \mathrm{Map}_C(c', c) \arrow[r] & \mathrm{Map}_C(c', RLc) \arrow[r, "\simeq"] & \mathrm{Map}_D(Lc', Lc), \end{tikzcd}$$

and the assertion follows from the definition of L-cartesian edge and a simple diagram chase.

For (2), let $c \in C$, $(f : d \to L''c) \in D$, and consider the L''-cartesian edge $R''d \times_{R''L''c} c \to c$ (this case suffices because all L''-cartesian edges are of this form up to equivalence). Note that the unit map for $L'' \dashv R''$ factors as the composition $R''L''c \simeq RR'L'Lc \to RLc \to c$ of unit maps for $L' \dashv R'$ and $L \dashv R$. Thus, we have

$$L(R''d \times_{R''L''c} c) \simeq L(R\big(R'd \times_{R'L''c} Lc\big) \times_{RLc} c) \xrightarrow{\simeq} R'd \times_{R'L''c} Lc$$

by our assumption on $L \dashv R$, and this equivalence respects the projection map to $L(c)$. But our assumption on $L' \dashv R'$ ensures that $R'(d) \times_{R'L''(c)} L(c) \to L(c)$ is a L'-cartesian edge. □

3.1 | N-naive G-spectra

We begin with a treatment of N-naive G-spectra. Although we will mostly work with N-Borel G-spectra (Subsection 3.2) in the sequel, we include this simpler case as a warm-up and for the sake of completeness.

Observation 3.3. Let N be a normal subgroup of G and let $\pi : G \to G/N$ denote the quotient map. We have the adjunction

$$r_N : \mathbb{F}_G \rightleftarrows \mathbb{F}_{G/N} : \iota_N$$

where $r_N(U) = U/N$ and $\iota_N(V) = V$ with V regarded as a G-set via π. Note that $r_N \iota_N(V) = V$, so ι_N is fully faithful. For $U \in \mathbb{F}_G$, $V \in \mathbb{F}_{G/N}$, and a G/N-map $f : V \to U/N$, we also have

$$(V \times_{U/N} U)/N \cong V,$$

[†] An *essentially cartesian fibration* is the ∞-categorical analogue of a Street fibration [78] obtained by replacing categories by ∞-categories and isomorphisms by equivalences in the definition of a Street fibration.

so r_N is a cartesian fibration by Lemma 3.2. Note also that the adjunction $r_N \dashv \iota_N$ restricts on orbits to

$$r_N : \mathcal{O}_G \rightleftarrows \mathcal{O}_{G/N} : \iota_N$$

and $V \times_{U/N} U$ is transitive if V and U are, hence r_N remains a cartesian fibration when restricted to $\mathcal{O}_G$. Given a G-orbit G/H and a G/N-map $f : \frac{G/N}{K/N} \cong G/K \to \frac{G/N}{HN/N} \cong G/HN$, we may identify the pullback

$$G/H \times_{G/HN} G/K \cong G/(H \cap K),$$

and thus an r_N-cartesian edge lifting f is given by $G/(H \cap K) \to G/H$.

Convention 3.4. For N a normal subgroup of G, we will regard $\mathcal{O}_G^{\mathrm{op}}$ as a G/N-category via r_N^{op}.

Definition 3.5. Let $\mathrm{Sp}^G_{N\text{-naive}} := \mathrm{Fun}_{G/N}(\mathcal{O}_G^{\mathrm{op}}, \underline{\mathrm{Sp}}^{G/N})$ be the ∞-category of *naive G-spectra relative to N*, or *N-naive G-spectra*.

For example, if $N = G$ we have the usual ∞-category $\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Sp})$ of naive G-spectra, and if $N = 1$ we instead have the ∞-category Sp^G itself.

Remark 3.6. Although we expect the ∞-category $\mathrm{Sp}^G_{N\text{-naive}}$ to be equivalent to $\mathrm{Sp}^G_{\mathcal{U}^N}$, we will not give a precise comparison in this paper.

Construction 3.7. We define a "forgetful" functor

$$\mathcal{U}[N] : \mathrm{Sp}^G \to \mathrm{Sp}^G_{N\text{-naive}} = \mathrm{Fun}_{G/N}\left(\mathcal{O}_G^{\mathrm{op}}, \underline{\mathrm{Sp}}^{G/N}\right).$$

First, let

$$q_N : \omega_G \circ \iota_N \Rightarrow \omega_{G/N} : \mathbb{F}_{G/N} \to \mathrm{Gpd}_{\mathrm{fin}}$$

be the natural transformation defined on objects $U \in \mathbb{F}_{G/N}$ by the functor $U//G \to U//(G/N)$ which sends objects $x \in U$ to the same $x \in U$ and morphisms $[g : x \to g \cdot x = \pi(g) \cdot x]$ to $[\pi(g) : x \to \pi(g) \cdot x]$.

Recall the functor $\mathrm{SH} : \mathrm{Gpd}_{\mathrm{fin}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ of Definition 2.1. For any ∞-category C, the adjunction $r_N \dashv \iota_N$ induces an adjunction

$$(r_N^{\mathrm{op}})^* : \mathrm{Fun}(\mathbb{F}_{G/N}^{\mathrm{op}}, C) \rightleftarrows \mathrm{Fun}(\mathbb{F}_G^{\mathrm{op}}, C) : (\iota_N^{\mathrm{op}})^*$$

where we may identify $(r_N^{\mathrm{op}})^*$ with the left Kan extension along ι_N^{op}. Let

$$\underline{\inf}[N] : \mathrm{SH}\omega_{G/N}^{\mathrm{op}} r_N^{\mathrm{op}} \to \mathrm{SH}\omega_G^{\mathrm{op}}$$

be the natural transformation adjoint to $\mathrm{SH}q_N^{\mathrm{op}}$ and let

$$\underline{\inf}[N] : \mathcal{O}_G^{\mathrm{op}} \times_{r_N^{\mathrm{op}}, \mathcal{O}_{G/N}^{\mathrm{op}}} \underline{\mathrm{Sp}}^{G/N} \to \underline{\mathrm{Sp}}^G$$

also denote the associated G-functor given by unstraightening. Note that for a G-orbit G/H, $\underline{\inf}[N]_{G/H}$ is given by the inflation functor $\inf^{H\cap N} : \mathrm{Sp}^{H/H\cap N} \to \mathrm{Sp}^H$. By the dual of [56, Proposition 7.3.2.6], $\underline{\inf}[N]$ admits a relative right adjoint

$$\widehat{\Psi}[N] : \underline{\mathrm{Sp}}^G \to \mathcal{O}_G^{\mathrm{op}} \times_{r_N^{\mathrm{op}}, \mathcal{O}_{G/N}^{\mathrm{op}}} \underline{\mathrm{Sp}}^{G/N}$$

over $\mathcal{O}_G^{\mathrm{op}}$ that does *not* preserve cocartesian edges; rather, for a map of G-orbits $f : G/K \to G/H$ we have a lax commutative square

$$\begin{tikzcd} \mathbf{Sp}^H \arrow[r, "\gamma_*"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\swarrow"] & \mathbf{Sp}^{H/H\cap N} \arrow[d, "f^*"] \\ \mathbf{Sp}^K \arrow[r, "\gamma_*"] & \mathbf{Sp}^{K/K\cap N} \end{tikzcd}$$

for the square of G-orbits

$$\begin{tikzcd} G/K \arrow[r, "\gamma"] \arrow[d, "f"] & (G/N)/(KN/N) \arrow[d, "f"] \\ G/H \arrow[r, "\gamma"] & (G/N)/(HN/N). \end{tikzcd}$$

However, for a map of G/N-orbits $f : \frac{G/N}{K/N} \to \frac{G/N}{HN/N}$ we have a homotopy commutative square

$$\begin{tikzcd} \mathbf{Sp}^H \arrow[r, "\gamma_*"] \arrow[d, "f^*"] & \mathbf{Sp}^{H/(H\cap N)} \arrow[d, "f^*"] \\ \mathbf{Sp}^{K\cap H} \arrow[r, "\gamma_*"] & \mathbf{Sp}^{(K\cap H)/(H\cap N)} \end{tikzcd}$$

for the pullback square

$$\begin{tikzcd} G/(K\cap H) \arrow[r, "f"] \arrow[d, "\gamma"] & G/H \arrow[d, "\gamma"] \\ G/K \arrow[r, "f"] & G/HN \end{tikzcd}$$

and hence the further composition $\underline{\Psi}[N] := \mathrm{pr} \circ \widehat{\Psi}[N] : \underline{\mathrm{Sp}}^G \to \underline{\mathrm{Sp}}^{G/N}$ *does* preserve cocartesian edges over $\mathcal{O}_{G/N}^{\mathrm{op}}$, where we regard $\underline{\mathrm{Sp}}^G$ as a G/N-∞-category via r_N^{op}. We also have the unit $\eta : \mathrm{id} \to \iota_N r_N$ which by precomposition yields the G-functor

$$\underline{\mathrm{res}}[N] : \mathcal{O}_G^{\mathrm{op}} \times_{(\iota_N r_N)^{\mathrm{op}}, \mathcal{O}_G^{\mathrm{op}}} \underline{\mathrm{Sp}}^G \to \underline{\mathrm{Sp}}^G,$$

where for a G-orbit G/H, $\underline{\mathrm{res}}[N]_{G/H}$ is given by the restriction functor $\mathrm{res}^{HN}_H : \mathrm{Sp}^{HN} \to \mathrm{Sp}^H$. The composite

$$\underline{\Psi}[N] \circ \underline{\mathrm{res}}[N] : \mathcal{O}_G^{\mathrm{op}} \times_{(\iota_N r_N)^{\mathrm{op}}, \mathcal{O}_G^{\mathrm{op}}} \underline{\mathrm{Sp}}^G \simeq \mathcal{O}_G^{\mathrm{op}} \times_{r_N^{\mathrm{op}}, \mathcal{O}_{G/N}^{\mathrm{op}}} \left(\mathcal{O}_{G/N}^{\mathrm{op}} \times_{\iota_N^{\mathrm{op}}, \mathcal{O}_G^{\mathrm{op}}} \underline{\mathrm{Sp}}^G \right) \to \underline{\mathrm{Sp}}^{G/N}$$

is then a G/N-functor. This defines by adjunction the G/N-functor[†]

$$\widetilde{\mathcal{U}}[N] : \mathcal{O}^{\mathrm{op}}_{G/N} \times_{\iota_N^{\mathrm{op}}, \mathcal{O}^{\mathrm{op}}_G} \underline{\mathrm{Sp}}^G \to \underline{\mathrm{Fun}}_{G/N}\left(\mathcal{O}^{\mathrm{op}}_G, \underline{\mathrm{Sp}}^{G/N}\right).$$

Define $\mathcal{U}[N]$ to be the fiber of $\widetilde{\mathcal{U}}[N]$ over $(G/N)/(G/N)$.

Observation 3.8 (Monoidality of forgetful functor). In Construction 3.7, the monoidality of inflation and restriction implies that with respect to $\underline{\mathrm{Sp}}^{G,\otimes}$ and $\underline{\mathrm{Sp}}^{G/N,\otimes}$, the G-functors $\underline{\mathrm{inf}}[N]$ and $\underline{\mathrm{res}}[N]$ are symmetric monoidal and the G/N-functor $\underline{\Psi}[N]$ is lax symmetric monoidal. Therefore, $\mathcal{U}[N]$ is lax symmetric monoidal with respect to the pointwise symmetric monoidal structure on $\mathrm{Sp}^G_{N\text{-naive}}$.

Observation 3.9 (Extension to G-∞-category). For any subgroup H of G, consider the commutative diagram of restriction functors

$$\begin{tikzcd}
\mathbb{F}_H & \mathbb{F}_G \arrow[l, "\mathrm{res}^G_H"'] \\
\mathbb{F}_{H/(H\cap N)} \arrow[u, "\iota_{H\cap N}"] & \mathbb{F}_{G/N} \arrow[l, "\mathrm{res}^{G/N}_{H/(H\cap N)}"] \arrow[u, "\iota_N"']
\end{tikzcd}$$

that yields by adjunction

$$\begin{tikzcd}
\mathbb{F}_H \arrow[r, "\mathrm{ind}^G_H"] \arrow[d, "r_{N\cap H}"] & \mathbb{F}_G \arrow[d, "r_N"] \\
\mathbb{F}_{H/(H\cap N)} \arrow[r, "\mathrm{ind}^{G/N}_{H/(H\cap N)}"'] & \mathbb{F}_{G/N}.
\end{tikzcd}$$

Precomposition by $(\mathrm{ind}^G_H)^{\mathrm{op}} : \mathcal{O}^{\mathrm{op}}_H \to \mathcal{O}^{\mathrm{op}}_G$ yields functors

$$\mathrm{res}^G_H : \mathrm{Fun}_{G/N}\left(\mathcal{O}^{\mathrm{op}}_G, \underline{\mathrm{Sp}}^{G/N}\right) \to \mathrm{Fun}_{H/(H\cap N)}\left(\mathcal{O}^{\mathrm{op}}_H, \underline{\mathrm{Sp}}^{H/(H\cap N)}\right)$$

that assemble to the data of a functor $\mathcal{O}^{\mathrm{op}}_G \to \mathrm{Cat}_\infty$ and thereby define a G-∞-category $\underline{\mathrm{Sp}}^G_{N\text{-naive}}$. Furthermore, $\mathcal{U}[N]$ extends to a G-functor $\underline{\mathcal{U}}[N] : \underline{\mathrm{Sp}}^G \to \underline{\mathrm{Sp}}^G_{N\text{-naive}}$, given on the fiber over G/H by $\mathcal{U}[N\cap H]$.

Observation 3.10 (Evaluation functors). For any subgroup H of G, evaluation on the orbit G/H yields a functor

$$s^*_H : \mathrm{Fun}_{G/N}\left(\mathcal{O}^{\mathrm{op}}_G, \underline{\mathrm{Sp}}^{G/N}\right) \to \mathrm{Sp}^{H/(H\cap N)}.$$

[†] The *ad hoc* notation $\widetilde{\mathcal{U}}[N]$ for this G/N-functor is employed so as not to conflict with the G-functor $\underline{\mathcal{U}}[N]$ in Observation 3.9.

By construction, this fits into a commutative diagram

$$\begin{tikzcd} \mathbf{Sp}^G \arrow[r, "\mathcal{U}[N]"] \arrow[d, "\mathrm{res}^G_H"'] & \mathrm{Fun}_{G/N}(\mathcal{O}_G^{\mathrm{op}}, \underline{\mathbf{Sp}}^{G/N}) \arrow[d, "s_H^*"] \\ \mathbf{Sp}^H \arrow[r, "\Psi^{H\cap N}"] & \mathbf{Sp}^{H/(H\cap N)}. \end{tikzcd}$$

Because $\underline{\mathrm{Sp}}^{G/N}$ is a G/N-presentable G/N-stable G/N-∞-category, the same holds for $\underline{\mathrm{Fun}}_{G/N}(\mathcal{O}_G^{\mathrm{op}}, \underline{\mathrm{Sp}}^{G/N})$ with G/N-limits and colimits computed as in [74, Proposition 9.17]. Thus, $\mathrm{Fun}_{G/N}(\mathcal{O}_G^{\mathrm{op}}, \underline{\mathrm{Sp}}^{G/N})$ is a presentable stable ∞-category such that the s_H^* form a set of jointly conservative functors that preserve and detect limits and colimits. As both the restriction and categorical fixed points functors preserve limits and colimits, it follows that $\mathcal{U}[N]$ preserves limits and colimits and therefore admits both left and right adjoints.

We also have a partial compatibility relation as H varies. Namely, given H, if K is a subgroup such that $N \cap H \leqslant K \leqslant H$ (so $K \cap N = H \cap N$), then

$$\mathrm{res}^{H/(N\cap H)}_{K/(N\cap H)} \circ s_H^* \simeq s_K^* : \mathrm{Fun}_{G/N}\left(\mathcal{O}_G^{\mathrm{op}}, \underline{\mathrm{Sp}}^{G/N}\right) \to \mathrm{Sp}^{K/N\cap H}.$$

Let us separate out the conclusion of Observation 3.10 into a definition.

Definition 3.11. Let $\mathcal{F}[N], \mathcal{F}^\vee[N] : \mathrm{Sp}^G_{N\text{-naive}} \to \mathrm{Sp}^G$ denote the left, respectively, right adjoints to $\mathcal{U}[N]$.

Example 3.12. If $G = N = C_p$, then the functor

$$\mathcal{U}[C_p] : \mathrm{Sp}^{C_p} \to \mathrm{Sp}^{C_p}_{C_p\text{-naive}} = \mathrm{Fun}\left(\mathcal{O}_{C_p}^{\mathrm{op}}, \mathrm{Sp}\right)$$

forgets transfers and the functors

$$\mathcal{F}[C_p], \mathcal{F}^\vee[C_p] : \mathrm{Sp}^{C_p}_{C_p\text{-naive}} \to \mathrm{Sp}^{C_p}$$

freely and cofreely adjoin transfers.

More precisely, writing $Y \in \mathrm{Sp}^{C_p}_{C_p\text{-naive}}$ as $[X^{C_p} \in \mathrm{Sp}, X \in \mathrm{Sp}^{BC_p}, X^{C_p} \to X]$, one can show that

$$\mathcal{F}[C_p](Y) = [X^{C_p} \times_{X^{tC_p}} X^{hC_p} \leftrightarrows X],$$

where $X^{C_p} \to X^{tC_p}$ is the composite $X^{C_p} \to X^{hC_p} \to X^{tC_p}$ and restriction is the composite of the projection onto X^{C_p} with the structure map $X^{C_p} \to X$ in Y. To describe transfers, it suffices to describe $\mathcal{F}[C_p](Y)$ in terms of geometric fixed points, where we have $\mathcal{F}[C_p](Y) = [X, X^{C_p} \to X^{tC_p}]$ with $X^{C_p} \to X^{tC_p}$ as described above.

One may also show that

$$\mathcal{F}^\vee[C_p](Y) = [X^{hC_p} \times_{F(\mathbb{S}^{C_p}, X^{hC_p})} F(\mathbb{S}^{C_p}, X^{C_p}) \leftrightarrows X \times_{X^{hC_p}} X^{C_p}],$$

where the structure maps are determined using the following observations. First, if $Z = [W^{hC_p} \xrightarrow{\mathrm{res}} W]$ is Borel, then we have $\mathcal{F}^\vee[C_p](Z) = [W^{hC_p} \leftrightarrows W]$ with transfer given by the composite $W \xrightarrow{\text{can}} W_{hC_p} \xrightarrow{\text{Nm}} W^{hC_p}$. Second, if $Z = [W \to 0]$, then $\mathscr{F}^\vee[C_p](Z) = [F(\mathbb{S}^{C_p}, W) \leftrightarrows W]$, where the transfer and restriction are induced by the structure maps $\mathrm{res}_{\mathbb{S}}$ and $\mathrm{tr}_{\mathbb{S}}$ in $i(\mathbb{S}) = [\mathbb{S}^{C_p} \leftrightarrows \mathbb{S}]$ for $i : \mathrm{Sp} \to \mathrm{Sp}^{C_p}$ the left adjoint to the forgetful functor $\mathrm{Sp}^{C_p} \to \mathrm{Sp}$. Third, we compute the general form of $\mathscr{F}^\vee[C_p](Y)$ from writing $Y = [W^{C_p} \to W]$ as a pullback $[W^{hC_p} \to W] \times_{[W^{hC_p} \to 0]} [W^{C_p} \to 0]$ and applying the prior two observations along with functoriality and exactness of $\mathscr{F}^\vee[C_p]$.

Observation 3.13 (Interaction with G-spaces). By repeating the construction of $\mathscr{U}[N]$ for G-spaces and using the compatibility of restriction and categorical fixed points with Ω^∞, we obtain a commutative diagram

$$\begin{tikzcd}
\mathbf{Sp}^G \arrow[r, "\mathscr{U}[N]"] \arrow[d, "\Omega^\infty"'] & \mathrm{Fun}_{G/N}(\mathscr{O}_G^{\mathrm{op}}, \underline{\mathbf{Sp}}^{G/N}) \arrow[d, "\Omega^\infty"] \\
\mathbf{Spc}_G \arrow[r, "\mathscr{U}'[N]"] & \mathrm{Fun}_{G/N}(\mathscr{O}_G^{\mathrm{op}}, \underline{\mathbf{Spc}}_{G/N})
\end{tikzcd}$$

where the right-hand Ω^∞ functor denotes postcomposition by the G/N-functor $\Omega^\infty : \underline{\mathrm{Sp}}^{G/N} \to \underline{\mathrm{Spc}}_{G/N}$. Moreover, a diagram chase reveals that under the equivalence

$$\mathrm{Fun}_{G/N}(\mathscr{O}_G^{\mathrm{op}}, \underline{\mathrm{Spc}}_{G/N}) \simeq \mathrm{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathrm{Spc}) = \mathrm{Spc}_G$$

of [74, Proposition 3.10], $\mathscr{U}'[N]$ is an equivalence.

To understand the compact generation of N-naive G-spectra, we need the following lemma.

Lemma 3.14. *Let C and $\{C_i : i \in I\}$ be presentable stable ∞-categories (with I a small set) such that each C_i has a (small) set $\{x_{i\alpha} : \alpha \in \Lambda_i\}$ of compact generators. Suppose we have functors $U_i : C \to C_i$ that preserve limits and colimits and are jointly conservative. Let F_i be left adjoint to U_i. Then C has a (small) set of compact generators given by $\{F_i x_{i\alpha} : i \in I, \alpha \in \Lambda_i\}$. In particular, C is compactly generated.*

Proof. We check directly that the indicated set generates C. Let $c \in C$ be any object and suppose that $\mathrm{Hom}_C(\Sigma^n F_i x_{i\alpha}, c) \cong 0$ for all choices of indices. Then by adjunction, $\mathrm{Hom}_{C_i}(\Sigma^n x_{i\alpha}, U_i c) \cong 0$, hence $U_i c \simeq 0$ for all $i \in I$. Invoking the joint conservativity of the U_i, we deduce that $c \simeq 0$. As for compactness, note that the assumption that each U_i preserves colimits ensures that its left adjoint F_i preserves compact objects. $\square$

Corollary 3.15. *The ∞-category $\mathrm{Fun}_{G/N}(\mathscr{O}_G^{\mathrm{op}}, \underline{\mathrm{Sp}}^{G/N})$ has a set of compact generators given by*

$$\{(s_H)_!(1) : H \leqslant G\},$$

where $(s_H)_!$ denotes the left adjoint to the functor $(s_H)^*$ of Observation 3.10.

Proof. By applying Lemma 3.14 to the functors s_H^*, we deduce that

$$\{(s_H)_! \left(\frac{H/(H \cap N)}{K/(H \cap N)_+} \right) : H \leqslant G\}$$

is a set of compact generators for $\mathrm{Fun}_{G/N}(\mathcal{O}_G^{\mathrm{op}}, \underline{\mathrm{Sp}}^{G/N})$. Because $\mathrm{res}^{H/(N \cap H)}_{K/(N \cap H)} s_H^* \simeq s_K^*$, we may eliminate redundant expressions and reduce to the set $\{(s_H)_!(1) : H \leqslant G\}$. □

3.2 | *N*-Borel *G*-spectra

We next consider Borel G-spectra relative to N. Let ψ denote the extension $N \to G \to G/N$. Our main goal is to prove Theorem 3.24, which plays a key role in the sequel.

Definition 3.16. Let $B^{\psi}_{G/N} N \subset \mathcal{O}_G^{\mathrm{op}}$ be the full subcategory on those G-orbits that are N-free.

Note that $B^{\psi}_{G/N} N$ is a cosieve in $\mathcal{O}_G^{\mathrm{op}}$: this amounts to the observation that if U is N-free and $f : V \to U$ is a G-equivariant map of G-sets, then V is N-free.

Lemma 3.17. *The cocartesian fibration $r_N^{\mathrm{op}} : \mathcal{O}_G^{\mathrm{op}} \to \mathcal{O}_{G/N}^{\mathrm{op}}$ restricts to a left fibration*

$$\rho_N : B^{\psi}_{G/N} N \to \mathcal{O}_{G/N}^{\mathrm{op}}.$$

Proof. Because $B^{\psi}_{G/N} N$ is a cosieve, the inclusion $B^{\psi}_{G/N} N \subset \mathcal{O}_G^{\mathrm{op}}$ is stable under r_N^{op}-cocartesian edges, so ρ_N is a cocartesian fibration such that the inclusion preserves cocartesian edges. Furthermore, if $f : U \to V$ is a G-equivariant map of N-free G-sets such that $\overline{f} : U/N \to V/N$ is an isomorphism, then it is easy to check that f is an isomorphism. Because ρ_N is conservative, we deduce that ρ_N is in addition a left fibration. □

Remark 3.18. If N yields a product decomposition $G \cong G/N \times N$, then $B^{\psi}_{G/N} N$ is spanned by those orbits of the form G/Γ_ϕ where Γ_ϕ is the graph of a homomorphism $\phi : M \to G/N$ for $M \leqslant N$. As a G/N-space, $B^{\psi}_{G/N} N$ then is the classifying G/N-space for G/N-equivariant principal N-bundles, which is usually denoted as $B_{G/N} N$. We thus think of $B^{\psi}_{G/N} N$ as a twisted variant of $B_{G/N} N$ for nontrivial extensions ψ. Many other authors have also studied equivariant classifying spaces in varying levels of generality, for example, see [33, 52, 53, 61].

Definition 3.19. Let $\mathrm{Sp}^G_{N\text{-}\mathrm{Borel}} := \mathrm{Fun}_{G/N}(B^{\psi}_{G/N} N, \underline{\mathrm{Sp}}^{G/N})$ be the ∞-category of *Borel G-spectra relative to N*, or *N-Borel G-spectra*. We will also refer to G/N-functors

$$X : B^{\psi}_{G/N} N \to \underline{\mathrm{Sp}}^{G/N}$$

as *G/N-spectra with ψ-twisted N-action*.

Notation 3.20. Given an abelian group A, we will use $B^t_{C_2}A$ as preferred alternative notation in lieu of $B^\psi_{C_2}A$ for the defining extension $\psi = [A \to A \rtimes C_2 \to C_2]$ of the semidirect product, where C_2 acts on A by the inversion involution. We will also refer to C_2-functors $X : B^t_{C_2}A \to \underline{\mathrm{Sp}}^{C_2}$ as *C_2-spectra with twisted A-action*, leaving ψ implicit.

Definition 3.21. Define the forgetful functor

$$\mathcal{U}_b[N] : \mathrm{Sp}^G \to \mathrm{Sp}^G_{N\text{-}\mathrm{Borel}} = \mathrm{Fun}_{G/N}(B^\psi_{G/N}N, \underline{\mathrm{Sp}}^{G/N})$$

to be the composition of $\mathcal{U}[N]$ with restriction along the G/N-functor $i : B^\psi_{G/N}N \subset \mathcal{O}^{\mathrm{op}}_G$. Also let

$$\mathcal{F}_b[N], \mathcal{F}^\vee_b[N] : \mathrm{Sp}^G_{N\text{-}\mathrm{Borel}} \to \mathrm{Sp}^G$$

denote the left, respectively, right adjoints to $\mathcal{U}_b[N]$ (cf. Observation 3.22(1)).

Parallel to the above discussion of $\mathcal{U}[N]$, we now enumerate some of the properties of $\mathcal{U}_b[N]$.

Observation 3.22 (Properties of $\mathcal{U}_b[N]$).

(1) Because both $\mathcal{U}[N]$ and restriction along i preserve limits and colimits, $\mathcal{U}_b[N]$ preserves limits and colimits and thus admits left and right adjoints $\mathcal{F}_b[N]$ and $\mathcal{F}^\vee_b[N]$ that factor through $\mathcal{F}[N]$ and $\mathcal{F}^\vee[N]$.

(2) For all $G/H \in B^\psi_{G/N}N$ we have $H \cap N = 1$. Therefore, the smaller collection of functors

$$\{s^*_H : \mathrm{Sp}^G_{N\text{-}\mathrm{Borel}} \to \mathrm{Sp}^H : H \cap N = 1\}$$

is jointly conservative and preserves and detects limits and colimits. Moreover, from Observation 3.10 we get that $s^*_H \circ \mathcal{U}_b[N] \simeq \mathrm{res}^G_H$.

(3) As $\mathcal{U}[N]$ is lax symmetric monoidal by Observation 3.8 and restriction is symmetric monoidal, we get that $\mathcal{U}_b[N]$ is a lax symmetric monoidal functor. However, because each $s^*_H \circ \mathcal{U}_b[N]$ for $H \cap N = 1$ is now symmetric monoidal, $\mathcal{U}_b[N]$ is in fact symmetric monoidal.

(4) As in Construction 3.7, the functor $\mathcal{U}_b[N]$ is the fiber over $(G/N)/(G/N)$ of a G/N-functor

$$\widetilde{\mathcal{U}}_b[N] : \mathcal{O}^{\mathrm{op}}_{G/N} \times_{\mathcal{O}^{\mathrm{op}}_G} \underline{\mathrm{Sp}}^G \to \underline{\mathrm{Fun}}_{G/N}\left(\mathcal{O}^{\mathrm{op}}_G, \underline{\mathrm{Sp}}^{G/N}\right).$$

Also, as in Observation 3.9, $\mathrm{Sp}^G_{N\text{-}\mathrm{Borel}}$ extends to a G-∞-category $\underline{\mathrm{Sp}}^G_{N\text{-}\mathrm{Borel}}$ and $\mathcal{U}_b[N]$ extends to a G-functor $\underline{\mathcal{U}}_b[N] : \underline{\mathrm{Sp}}^G \to \underline{\mathrm{Sp}}^G_{N\text{-}\mathrm{Borel}}$, given on the fiber over G/H by

$$\mathcal{U}_b[N \cap H] : \mathrm{Sp}^H \to \mathrm{Fun}_{H/(N\cap H)}(B^{\psi_H}_{H/(H\cap N)}(N \cap H), \underline{\mathrm{Sp}}^{H/(N\cap H)})$$

for the restricted extension $\psi_H := [N \cap H \to H \to H/(N \cap H)]$.

(5) As with N-naive G-spectra, we have a commutative diagram

$$\begin{tikzcd} \mathbf{Sp}^G \arrow[r, "\mathscr{U}_b[N]"] \arrow[d, "\Omega^\infty"'] & \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathbf{Sp}}^{G/N}) \arrow[d, "\Omega^\infty"] \\ \mathbf{Spc}_G \arrow[r, "\mathscr{U}'_b[N]"] \arrow[d, equal] & \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathbf{Spc}}_{G/N}) \arrow[d, "\simeq"] \\ \mathrm{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathbf{Sp}) \arrow[r, "i^*"] & \mathrm{Fun}(B^{\psi}_{G/N}N, \mathbf{Spc}). \end{tikzcd}$$

where now we may identify $\mathscr{U}'_b[N]$ with restriction along i. Consider the transposed lax commutative diagram

$$\begin{tikzcd} \mathbf{Sp}^G \arrow[r, "\mathscr{U}_b[N]"] & \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathbf{Sp}}^{G/N}) \\ \mathbf{Spc}_G \arrow[u, "\Sigma^\infty_+"] \arrow[r, "\mathscr{U}'_b[N]"] \arrow[ur, phantom, "\nwarrow"] & \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathbf{Spc}}_{G/N}). \arrow[u, "\Sigma^\infty_+"'] \end{tikzcd}$$

For any subgroup H such that $H \cap N = 1$, we may extend this diagram to

$$\begin{tikzcd} \mathbf{Sp}^G \arrow[r, "\mathscr{U}_b[N]"] & \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathbf{Sp}}^{G/N}) \arrow[r, "\mathrm{ev}_H"] & \mathbf{Sp}^H \\ \mathbf{Spc}_G \arrow[u, "\Sigma^\infty_+"] \arrow[r, "\mathscr{U}'_b[N]"] \arrow[ur, phantom, "\nwarrow"] & \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathbf{Spc}}_{G/N}) \arrow[u, "\Sigma^\infty_+"'] \arrow[r, "\mathrm{ev}_H"] & \mathbf{Spc}_H \arrow[u, "\Sigma^\infty_+"'] \end{tikzcd}$$

where the horizontal composites are given by the restriction functor res^G_H because $H/(H \cap N) \cong H$ (cf. Observation 3.10 and the analogous setup for G-spaces). The right-hand square commutes by definition, and the outer square commutes by the compatibility of restriction with Σ^∞_+. As the ev_H are jointly conservative, it follows that the left-hand square commutes.

Notation 3.23. Let Γ_N be the *N-free* G-family consisting of subgroups H such that $H \cap N = 1$.

Recall the definitions of $\mathcal{F}$-torsion and $\mathcal{F}$-complete G-spectra from Construction 2.17.

Theorem 3.24. *The functors $\mathscr{F}_b[N]$ and $\mathscr{F}_b^\vee[N]$ are fully faithful with essential image the Γ_N-torsion and Γ_N-complete G-spectra, respectively.*

Proof. We first check that the unit $\eta : \mathrm{id} \to \mathscr{U}_b[N]\mathscr{F}_b[N]$ is an equivalence to show that the left adjoint $\mathscr{F}_b[N]$ is fully faithful.

Because $\Omega^\infty \mathscr{U}_b[N] \simeq i^*\Omega^\infty$ and $\Sigma^\infty_+ i^* \simeq \mathscr{U}_b[N]\Sigma^\infty_+$ by Observation 3.22(5), we have an equivalence of left adjoints $\Sigma^\infty_+ i_! \simeq \mathscr{F}_b[N]\Sigma^\infty_+$, and for $X \in \mathrm{Fun}(B^{\psi}_{G/N}N, \mathrm{Spc})$ we may identify $\eta_{\Sigma^\infty_+ X}$ with $\Sigma^\infty_+ \eta'_X$, where $\eta' : \mathrm{id} \to i^* i_!$ is the unit of the adjunction $i_! \dashv i_*$. But η' is an equivalence because $i_!$ is left Kan extension along the inclusion of a full subcategory. Thus, η is an equivalence on all suspension spectra. In view of the commutative diagram for $H \in \Gamma_N$

$$\begin{tikzcd} \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathbf{Sp}}^{G/N}) \arrow[r, "s_H^*"] \arrow[d, "\Omega^\infty"] & \mathbf{Sp}^H \arrow[d, "\Omega^\infty"] \\ \mathrm{Fun}(B^{\psi}_{G/N}N, \mathbf{Spc}) \arrow[r, "{s'_H}^*"] & \mathbf{Spc}_H \end{tikzcd}$$

where $s'_H = (\mathrm{ind}_H^G)^{\mathrm{op}} : \mathcal{O}_H^{\mathrm{op}} \to B^{\psi}_{G/N}N \subset \mathcal{O}_G^{\mathrm{op}}$, we have an equivalence of left adjoints $s_{H!}\Sigma^\infty_+ \simeq \Sigma^\infty_+ s'_{H!}$, so in particular $s_{H!}(1)$ is a suspension spectrum. Elaborating upon Corollary 3.15, we observe that the set $\{s_{H!}(1) : H \in \Gamma_N\}$ constitutes a set of compact generators for $\mathrm{Sp}^G_{N\text{-}\mathrm{Borel}}$. Because both the domain and codomain of η commute with colimits, we conclude that η is an equivalence. Moreover, because $(s_H)^* \circ \mathcal{U}_b[N] \simeq \mathrm{res}^G_H$ as noted in Observation 3.10, by adjunction $\mathcal{F}_b[N] \circ (s_H)_! \simeq \mathrm{ind}^G_H$ and thus the essential image of $\mathcal{F}_b[N]$ is the localizing subcategory generated by the set $\{G/H_+ : H \in \Gamma_N\}$. This equals the full subcategory of Γ_N-torsion G-spectra. Because we already have the stable recollement

$$\begin{tikzcd} \mathbf{Sp}^{h\Gamma_N} \arrow[r, shift right, "j_*"'] & \mathbf{Sp}^G \arrow[l, shift right, "j^*"'] \arrow[r, shift left, "i^*"] & \mathbf{Sp}^{\Phi\Gamma_N}, \arrow[l, shift left, "i_*"] \end{tikzcd}$$

it follows that $\mathcal{F}_b^\vee[N]$ is fully faithful with essential image the Γ_N-complete G-spectra. In more detail,

- the composite $i^*\mathcal{F}_b[N] \simeq 0$ because $\mathcal{F}_b[N](X) \in j_!(\mathrm{Sp}^{h\Gamma_N})$ and $i^* j_! \simeq 0$. Passing to adjoints, we get $\mathcal{U}_b[N]i_* \simeq 0$. Then for all $X \in \mathrm{Sp}^G_{N\text{-}\mathrm{Borel}}$, $\mathcal{F}_b^\vee[N](X)$ is Γ_N-complete by the equivalence

$$\mathrm{Map}(i_*Z, \mathcal{F}_b^\vee[N](X)) \simeq \mathrm{Map}(\mathcal{U}_b[N]i_*Z, X) \simeq 0;$$

- using that $\mathcal{F}_b[N]$ is an equivalence onto its essential image, we see that the composite $\mathcal{U}_b[N]j_!$ is an equivalence from $\mathrm{Sp}^{h\Gamma_N}$ to $\mathrm{Sp}^G_{N\text{-}\mathrm{Borel}}$. Its right adjoint $j^*\mathcal{F}_b^\vee[N]$ is thus an equivalence;
- combining these two assertions, we have that the composite

$$\begin{tikzcd} \mathbf{Sp}^G_{N\mp\mathrm{Borel}} \arrow[r, "\mathcal{F}_b^\vee{[N]}"] \arrow[rr, bend right, "\simeq"'] & \mathbf{Sp}^G \arrow[r, "j^*"] & \mathbf{Sp}^{h\Gamma_N} \arrow[r, "j_*"] & \mathbf{Sp}^G \end{tikzcd}$$

is equivalent to $\mathcal{F}_b^\vee[N]$ via the unit $\mathcal{F}_b^\vee[N] \xrightarrow{\simeq} j_*j^*\mathcal{F}_b^\vee[N]$ and is fully faithful onto Γ_N-complete G-spectra. □

Remark 3.25 (Monoidality). Because $\mathcal{U}_b[N]$ is symmetric monoidal by Observation 3.22(3), its right adjoint $\mathcal{F}_b^\vee[N]$ is lax symmetric monoidal. Therefore, the equivalence

$$\mathcal{F}_b^\vee[N] : \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathrm{Sp}}^{G/N}) \xrightarrow{\simeq} \mathrm{Sp}^{h\Gamma_N}$$

of Theorem 3.24 is one of symmetric monoidal ∞-categories with respect to the pointwise symmetric monoidal structure on the left-hand side and the symmetric monoidal structure induced by the Γ_N-symmetric monoidal recollement on the right-hand side.

Corollary 3.26 (Compatibility with restriction). *The left and right adjoints $\mathcal{F}_b[N]$ and $\mathcal{F}_b^\vee[N]$ extend to G-left and right adjoints*

$$\underline{\mathcal{F}}_b[N],\ \underline{\mathcal{F}}_b^\vee[N] : \underline{\mathrm{Sp}}^G_{N\text{-}\mathrm{Borel}} \to \underline{\mathrm{Sp}}^G.$$

Proof. Combine Remark 2.23 and Theorem 3.24. □

We conclude this section by applying Theorem 3.24 to decompose the ∞-category of D_{2p^n}-spectra; this example plays a crucial role in our study of real cyclotomic spectra in [70].

Example 3.27. Let $\Gamma = \Gamma_{\mu_{p^n}}$ be the D_{2p^n}-family that consists of those subgroups H such that $H \cap \mu_{p^n} = 1$. Note that $H \notin \Gamma$ if and only if $\mu_p \leqslant H$. Therefore, $\mathrm{Sp}^{\Phi\Gamma} \simeq \mathrm{Sp}^{D_{2p^{n-1}}}$ for $D_{2p^{n-1}}$ viewed as the quotient D_{2p^n}/μ_p. Together with Theorem 3.24, we obtain a stable symmetric monoidal recollement

$$\begin{tikzcd} \mathrm{Fun}_{C_2}(B^t_{C_2}\mu_{p^n}, \underline{\mathbf{Sp}}^{C_2}) \arrow[r, shift right, "j_*=\mathcal{F}_b^\vee[\mu_{p^n}]"'] & \mathbf{Sp}^{D_{2p^n}} \arrow[l, shift right, "j^*=\mathcal{U}_b[\mu_{p^n}]"'] \arrow[r, shift left, "i^*=\Phi^{\mu_p}"] & \mathbf{Sp}^{D_{2p^{n-1}}}. \arrow[l, shift left, "i_*"] \end{tikzcd}$$

Furthermore, using Remark 2.23, this extends to a C_2-stable C_2-recollement

$$\begin{tikzcd} \underline{\mathrm{Fun}}_{C_2}(B^t_{C_2}\mu_{p^n}, \underline{\mathbf{Sp}}^{C_2}) \arrow[r, shift right, "\widetilde{\mathcal{F}}_b^\vee[\mu_{p^n}]"'] & \mathcal{O}^{\mathrm{op}}_{C_2} \times_{\mathcal{O}^{\mathrm{op}}_{D_{2p^n}}} \underline{\mathbf{Sp}}^{D_{2p^n}} \arrow[l, shift right, "\widetilde{\mathcal{U}}_b[\mu_{p^n}]"'] \arrow[r, shift left, "\Phi^{\mu_p}"] & \mathcal{O}^{\mathrm{op}}_{C_2} \times_{\mathcal{O}^{\mathrm{op}}_{D_{2p^{n-1}}}} \underline{\mathbf{Sp}}^{D_{2p^{n-1}}} \arrow[l, shift left, "i_*"] \end{tikzcd}$$

whose fiber over $C_2/1$ is the stable symmetric monoidal recollement

$$\begin{tikzcd} \mathrm{Fun}(B\mu_{p^n}, \mathbf{Sp}) \arrow[r, shift right, "\mathcal{F}_b^\vee[\mu_{p^n}]"'] & \mathbf{Sp}^{\mu_{p^n}} \arrow[l, shift right, "\mathcal{U}_b[\mu_{p^n}]"'] \arrow[r, shift left, "\Phi^{\mu_p}"] & \mathbf{Sp}^{\mu_{p^{n-1}}}, \arrow[l, shift left, "i_*"] \end{tikzcd}$$

with the C_2-action induced by the inversion action of C_2 on μ_{p^n}.

4 | PARAMETERIZED NORM MAPS AND AMBIDEXTERITY

In this section, we construct *parameterized norm maps* that will permit us to define the parameterized Tate construction (Definition 4.24). Our strategy is to mimic Lurie's construction of the norm maps [56, section 6.1.6] in a parameterized setting over a base ∞-category $\mathcal{T}$, eventually specializing to $\mathcal{T} = \mathcal{O}_G$. We first collect a few necessary aspects of the theory of $\mathcal{T}$-colimits, limits, and Kan extensions from [74].

Warning 4.1. As mentioned in Subsection 1.4, in [74] we set $S = \mathcal{T}^{\mathrm{op}}$ and label notions of parameterized category theory using S instead of $\mathcal{T}$ (for example, S-colimits versus $\mathcal{T}$-colimits). We will implicitly convert between the two notations when citing results from [74].

Observation 4.2. Let K, L, and C be $\mathcal{T}$-∞-categories with $p : L \to \mathcal{T}^{\mathrm{op}}$ the structure map, and let $\phi : K \to L$ be a $\mathcal{T}$-cocartesian fibration [74, Definition 7.1]. We are interested in computing the left adjoint $\phi_!$ to the restriction functor

$$\phi^* : \mathrm{Fun}_{\mathcal{T}}(L, C) \to \mathrm{Fun}_{\mathcal{T}}(K, C)$$

as a (pointwise) $\mathcal{T}$-left Kan extension [74, Definition 10.1 or Definition 9.13]. In [74, Theorem 9.15 and Proposition 10.8], the second author gave a pointwise existence criterion and formula for $\phi_!$ of a $\mathcal{T}$-functor $F : K \to C$. Namely, for all objects $x \in L$, if we let

$$\underline{x} := \{x\} \times_{L,\mathrm{ev}_0} \mathrm{Ar}^{\mathrm{cocart}}(L),$$

then the point inclusion $\{x\} \subset L$ canonically extends to a $\mathcal{T}$-functor $i_x : \underline{x} \to L$ given by evaluation at the target (cf. [74, Notation 2.29]). Let $t = p(x)$ and note that

$$q = (\mathrm{id}, \mathrm{Ar}(p)) : \underline{x} \xrightarrow{\sim} (\mathcal{T}^{\mathrm{op}})^{t/} \cong (\mathcal{T}^{/t})^{\mathrm{op}}$$

is a trivial fibration (cf. [74, Lemma 12.10]), so we may think of $\underline{x}$ as an $\mathcal{T}$*-point* of L. Let

$$F_x = (q \circ \mathrm{pr}_{\underline{x}}, F \circ \mathrm{pr}_K) : K_{\underline{x}} := \underline{x} \times_L K \to C_{\underline{t}} := (\mathcal{T}^{/t})^{\mathrm{op}} \times_{\mathcal{T}^{\mathrm{op}}} C$$

denote the resulting $\mathcal{T}^{/t}$-functor. Then $\phi_! F$ exists if the $\mathcal{T}^{/t}$-colimit of F_x exists for all $x \in L$, after which $(\phi_! F)|_{\underline{x}}$ is computed as that $\mathcal{T}^{/t}$-colimit.

We will also say that C *admits the relevant* $\mathcal{T}$*-colimits* with respect to $\phi : K \to L$ if for all $x \in L$ with $t = p(x)$, $C_{\underline{t}}$ admits all $\mathcal{T}^{/t}$-colimits indexed by $K_{\underline{x}}$. In this case, the left adjoint $\phi_!$ to ϕ^* exists by [74, Corollary 9.16].

Now suppose instead that $\phi : K \to L$ is an $\mathcal{T}$-cartesian fibration [74, Definition 7.1]. In view of the discussion of vertical opposites in [74, section 5] and the observation that the formation of vertical opposites exchanges $\mathcal{T}$-cocartesian and $\mathcal{T}$-cartesian fibrations, we may dualize the above discussion to see that the $\mathcal{T}$-right Kan extension $\phi_* F$ exists if the $\mathcal{T}^{/t}$-limit of F_x exists for all $x \in L$, after which $\phi_* F|_{\underline{x}}$ is computed as that $\mathcal{T}^{/t}$-limit. Likewise, we have the dual notion of C admitting the relevant $\mathcal{T}$-limits with respect to ϕ, in which case the right adjoint ϕ_* exists.

Finally, suppose that K and L are $\mathcal{T}$-spaces. Using the cocartesian model structure on $s\mathrm{Set}^+_{/\mathcal{T}^{\mathrm{op}}}$ and the description of the fibrations between fibrant objects [56, Proposition B.2.7], up to equivalence we may replace any $\mathcal{T}$-functor $\phi : K \to L$ by a categorical fibration. But a categorical fibration between left fibrations over $\mathcal{T}$ is necessarily both an $\mathcal{T}$-cocartesian and $\mathcal{T}$-cartesian fibration, hence both of the above formulae apply to compute $\phi_!$ and ϕ_*.

Observation 4.3. We can also consider the $\mathcal{T}$-∞-category $\underline{\mathrm{Fun}}_{\mathcal{T}}(K, C) \to \mathcal{T}^{\mathrm{op}}$ whose cocartesian sections are $\mathrm{Fun}_{\mathcal{T}}(K, C)$. Let $\phi : K \to L$ be an $\mathcal{T}$-cocartesian fibration and suppose that for every $t \in \mathcal{T}$, the $\mathcal{T}^{/t}$-∞-category $C_{\underline{t}}$ admits the relevant $\mathcal{T}^{/t}$-colimits with respect to $\phi_{\underline{t}} : K_{\underline{t}} \to L_{\underline{t}}$, so that the restriction functor

$$\phi_{\underline{t}}^* : \mathrm{Fun}_{\mathcal{T}^{/t}}(L_{\underline{t}}, C_{\underline{t}}) \to \mathrm{Fun}_{\mathcal{T}^{/t}}(K_{\underline{t}}, C_{\underline{t}})$$

admits a left adjoint $(\phi_{\underline{t}})_!$ computed as above. Then using the built-in compatibility of $\mathcal{T}$-left Kan extension with restriction, by [56, Proposition 7.3.2.11] these fiberwise left adjoints assemble to yield an $\mathcal{T}$-adjunction [74, Definition 8.3]

$$\underline{\phi}_! : \underline{\mathrm{Fun}}_{\mathcal{T}}(K, C) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(L, C) : \underline{\phi}^*$$

(also see [74, Corollary 9.16 and Theorem 10.5]). In particular, upon forgetting the structure maps[†] we have an ordinary adjunction $\underline{\phi}_! \dashv \underline{\phi}^*$. Similarly, for ϕ an $\mathcal{T}$-cartesian fibration we can consider the $\mathcal{T}$-adjunction

$$\underline{\phi}^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(L, C) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(K, C) : \underline{\phi}_*.$$

For $\phi : X \to Y$ a map of $\mathcal{T}$-spaces, we will consider $\underline{\phi}_! \dashv \underline{\phi}^* \dashv \underline{\phi}_*$.

The key result that enables the construction of norm maps is the following lemma on adjointability. Note for the formulation of the statement that $\mathcal{T}$-(co)cartesian fibrations are stable under pullback, and the property that C admits the relevant $\mathcal{T}$-(co)limits with respect to ϕ is stable under pullbacks in the ϕ variable.

Lemma 4.4. *Let C be an $\mathcal{T}$-∞-category and let*

$$\begin{tikzcd} K' \arrow[r, "f'"] \arrow[d, "\phi'"'] & K \arrow[d, "\phi"] \\ L' \arrow[r, "f"] & L \end{tikzcd}$$

be a pullback square of $\mathcal{T}$-∞-categories. Consider the resulting commutative square of $\mathcal{T}$-functor categories and restriction functors

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{T}}(K', C) & \mathrm{Fun}_{\mathcal{T}}(K, C) \arrow[l, "f'^*"'] \\ \mathrm{Fun}_{\mathcal{T}}(L', C) \arrow[u, "\phi'^*"] & \mathrm{Fun}_{\mathcal{T}}(L, C). \arrow[l, "f^*"'] \arrow[u, "\phi^*"'] \end{tikzcd}$$

1. *If ϕ is an $\mathcal{T}$-cocartesian fibration and C admits the relevant $\mathcal{T}$-colimits, then the square is left adjointable, that is, the natural map $\phi'_! f'^* \to f^* \phi_!$ is an equivalence.*
2. *If ϕ is an $\mathcal{T}$-cartesian fibration and C admits the relevant $\mathcal{T}$-limits, then this square is right adjointable, that is, the natural map $f^* \phi_* \to \phi'_* f'^*$ is an equivalence.*
3. *If K, L, K', L' are $\mathcal{T}$-spaces, then the square is both left and right adjointable provided that C admits the relevant $\mathcal{T}$-colimits and $\mathcal{T}$-limits.*

Likewise, we have the same results for the commutative square of $\mathcal{T}$-functor $\mathcal{T}$-∞-categories

$$\begin{tikzcd} \underline{\mathrm{Fun}}_{\mathcal{T}}(K', C) & \underline{\mathrm{Fun}}_{\mathcal{T}}(K, C) \arrow[l, "\underline{f'}^*"'] \\ \underline{\mathrm{Fun}}_{\mathcal{T}}(L', C) \arrow[u, "\underline{\phi'}^*"] & \underline{\mathrm{Fun}}_{\mathcal{T}}(L, C). \arrow[l, "\underline{f}^*"'] \arrow[u, "\underline{\phi}^*"'] \end{tikzcd}$$

Proof. Let $F : K \to C$ be an $\mathcal{T}$-functor. For (1), as ϕ is an $\mathcal{T}$-cocartesian fibration, $\phi_!$ and $\phi'_!$ exist and can be computed by the below colimits by Observation 4.2 and Observation 4.3. We need to

[†] In other words, a relative adjunction yields an adjunction between the Grothendieck constructions.

check that $\phi'_! f'^* F \to f^* \phi_! F$ is an equivalence of $\mathcal{T}$-functors. It suffices to evaluate on $\mathcal{T}$-points $\underline{x}$ in L', and we then have the map

$$\operatorname{colim}^{\mathcal{T}^{/t}} (F \circ f')_x \to \operatorname{colim}^{\mathcal{T}^{/t}} F_{f(x)}$$

where x lies over t. But as $\underline{x} \times_{L'} K' \simeq \underline{f(x)} \times_L K$ as $\mathcal{T}^{/t}$-∞-categories, these $\mathcal{T}^{/t}$-colimits are equivalent under the comparison map. The proof of (2) is similar. For (3), we replace ϕ by a categorical fibration and then use (1) and (2). For the corresponding assertion about $\underline{\mathrm{Fun}}_{\mathcal{T}}(-, C)$, it suffices to check that the natural transformations of interest are equivalences fiberwise, upon which we reduce to the prior assertion for $\mathrm{Fun}_{\mathcal{T}^{/t}}(-, C_{\underline{t}})$ (ranging over all $t \in \mathcal{T}$). □

4.1 | Ambidexterity of parameterized local systems

In this subsection, we extend Hopkins and Lurie's study of ambidexterity for local systems [44, section 4.3] to the parameterized setting. The following definition generalizes [44, Definition 4.3].

Definition 4.5. Let C be an $\mathcal{T}$-∞-category. The ∞-category of *$\mathcal{T}$-local systems* on C

$$\mathrm{LocSys}^{\mathcal{T}}(C) \to \mathrm{Spc}_{\mathcal{T}}$$

is the cartesian fibration classified by the composite

$$\mathrm{Fun}_{\mathcal{T}}(-, C) : (\mathrm{Spc}_{\mathcal{T}})^{\mathrm{op}} \subset \mathrm{Cat}^{\mathrm{op}}_{\infty,\mathcal{T}} \to \mathrm{Cat}_{\infty}.$$

The *$\mathcal{T}$-∞-category of $\mathcal{T}$-local systems* on C

$$\underline{\mathrm{LocSys}}^{\mathcal{T}}(C) \to \mathrm{Spc}_{\mathcal{T}}$$

is the cartesian fibration classified by the composite

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(-, C) : (\mathrm{Spc}_{\mathcal{T}})^{\mathrm{op}} \subset \mathrm{Cat}^{\mathrm{op}}_{\infty,\mathcal{T}} \to \mathrm{Cat}_{\infty,\mathcal{T}} \xrightarrow{U} \mathrm{Cat}_{\infty}.$$

where U forgets the structure map of a cocartesian fibration.

Corollary 4.6. *Suppose that for all $t \in \mathcal{T}$, $C_{\underline{t}}$ admits all $\mathcal{T}^{/t}$-colimits indexed by $\mathcal{T}^{/t}$-spaces. Then* $\mathrm{LocSys}^{\mathcal{T}}(C)$ *and* $\underline{\mathrm{LocSys}}^{\mathcal{T}}(C)$ *are Beck–Chevalley fibrations [44, Definition 4.1.3].*

Proof. Note that by the same argument as [74, Remark 5.14], the hypothesis ensures that C admits all $\mathcal{T}$-colimits indexed by $\mathcal{T}$-spaces. The corollary is then immediate from Lemma 4.4. □

Remark 4.7. The natural transformation

$$\theta : \underline{\mathrm{Fun}}_{\mathcal{T}}(-, C) \Rightarrow \mathrm{const}_{\mathcal{T}} : (\mathrm{Spc}_{\mathcal{T}})^{\mathrm{op}} \to \mathrm{Cat}_{\infty},$$

given objectwise by the structure map $\theta_X : \underline{\mathrm{Fun}}_{\mathcal{T}}(X, C) \to \mathcal{T}^{\mathrm{op}}$, unstraightens to define a cocartesian fibration

$$\underline{\mathrm{LocSys}}^{\mathcal{T}}(C) \to \mathcal{T}^{\mathrm{op}},$$

so $\underline{\mathrm{LocSys}}^{\mathcal{T}}(C)$ is indeed an $\mathcal{T}$-∞-category.

We now have the general theory of ambidexterity [44, section 4.1-2] for a Beck–Chevalley fibration, along with the attendant notions of ambidextrous and weakly ambidextrous morphisms [44, Construction 4.1.8 and Definition 4.1.11] in $\mathrm{Spc}_{\mathcal{T}}$. For the reader's convenience, let us recall the relevance of these notions for constructing norm maps, referring to [44, §4.1] for greater detail and precise definitions.

Observation 4.8. Suppose $\mathcal{E} \to \mathrm{Spc}_{\mathcal{T}}$ is a Beck–Chevalley fibration, $f : X \to Y$ is a map of $\mathcal{T}$-spaces, the left and right adjoints $f_!$ and f_* to f^* exist, and we wish to construct a norm map $\mathrm{Nm}_f : f_! \to f_*$. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta"] \arrow[drr, bend left, "="] \arrow[ddr, bend right, "="'] & & \\
& X \times_Y X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

and suppose we have already constructed a norm map $\mathrm{Nm}_\delta : \delta_! \to \delta_*$ and shown it to be an equivalence. If $\mathrm{Nm}_\delta^{-1} : \delta_* \xrightarrow{\simeq} \delta_!$ is a choice of inverse, then we have a natural transformation

$$\mathrm{pr}_1^* \xrightarrow{\eta_\delta} \delta_* \delta^* \mathrm{pr}_1^* \simeq \delta_* \xrightarrow{\mathrm{Nm}_\delta^{-1}} \delta_! \simeq \delta_! \delta^* \mathrm{pr}_2^* \xrightarrow{\epsilon_\delta} \mathrm{pr}_2^* .$$

By adjunction and using the Beck–Chevalley property, we obtain a map

$$f^* f_! \simeq (\mathrm{pr}_1)_! \, \mathrm{pr}_2^* \to \mathrm{id} .$$

Finally, we may adjoint this map in turn to define

$$\mathrm{Nm}_f : f_! \to f_* .$$

Thus, for an inductive construction of norm maps, we may single out a class of "ambidextrous" morphisms for which a norm map has been constructed and shown to be an equivalence, and then define "weakly ambidextrous" morphisms to be those morphisms $f : X \to Y$ whose diagonal $\delta : X \to X \times_Y X$ is ambidextrous.

Continuing our study, we henceforth suppose that $C_{\underline{t}}$ also admits all $\mathcal{T}^{/t}$-limits indexed by $\mathcal{T}^{/t}$-spaces, so that the right adjoints f_*, $\underline{f}_*$ exist for all maps f of $\mathcal{T}$-spaces. Then by [44, Remark 4.1.12], for $\mathrm{LocSys}^{\mathcal{T}}(C)$ a map $f : X \to Y$ in $\mathrm{Spc}_{\mathcal{T}}$ is ambidextrous if and only if the norm map $\mathrm{Nm}_{f'} : f'_! \to f'_*$ is an equivalence for all pullbacks $f' : X' \to Y'$ of f, and similarly for $\underline{\mathrm{LocSys}}^{\mathcal{T}}(C)$.

To simplify the following discussion, we will phrase all of our statements for $\mathrm{LocSys}^{\mathcal{T}}(C)$. However, such statements have obvious implications for $\underline{\mathrm{LocSys}}^{\mathcal{T}}(C)$ via checking fiberwise.

Lemma 4.9. *Let C be an $\mathcal{T}$-∞-category with the property that $C_{\underline{t}}$ admits all $\mathcal{T}^{/t}$-limits and $\mathcal{T}^{/t}$-colimits indexed by $\mathcal{T}^{/t}$-spaces for all $t \in \mathcal{T}$.*

(1) *Let $f : X \to Y$ be a weakly ambidextrous morphism. Then f is ambidextrous if and only if for all $y \in Y$, the norm map Nm_{f_y} for the pullback $f_y : X_{\underline{y}} \to \underline{y}$ is an equivalence.*
(2) *$f : X \to Y$ is weakly ambidextrous if and only if for all $y \in Y$, the pullback $f_y : X_{\underline{y}} \to \underline{y}$ is weakly ambidextrous.*

Proof. For (1), first note that the maps f_y are weakly ambidextrous by [44, Proposition 4.1.10(3)], so the statement is well-posed. The “only if” direction holds by definition. For the “if” direction, suppose given a pullback square of $\mathscr{T}$-spaces

$$\begin{tikzcd} X' \arrow[r, "\phi'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\phi"] & Y. \end{tikzcd}$$

For any point $y' \in Y'$, if we let $y = \phi(y')$ then we have an equivalence $X'_{\underline{y'}} \simeq X_{\underline{y}} \to \underline{y'} \simeq \underline{y}$. Therefore, without loss of generality it suffices to prove that $\mathrm{Nm}_f : f_! \to f_*$ is an equivalence. Let $y \in Y$ and denote the inclusion of the $\mathscr{T}$-point as $i_y : \underline{y} \to Y$ and the $\mathscr{T}$-fiber as $j_y : X_{\underline{y}} \to X$. By [44, Remark 4.2.3] and Lemma 4.4, we have an equivalence

$$i_y^* \mathrm{Nm}_f \simeq \mathrm{Nm}_{f_y} j_y^* : \mathrm{Fun}_{\mathscr{T}}(X, C) \to \mathrm{Fun}_{\mathscr{T}}(\underline{y}, C) \simeq C_t$$

(where y covers t), which by assumption is an equivalence. Because the evaluation functors i_y^* are jointly conservative, it follows that Nm_f is an equivalence.

For (2), we only need to prove the “if” direction. We will show that the diagonal $\delta : X \to X \times_Y X$ is ambidextrous. Let δ_y denote the diagonal for f_y. For all y we have a pullback square

$$\begin{tikzcd} X_{\underline{y}} \arrow[r, "j_y"] \arrow[d, "\delta_y"] & X \arrow[d, "\delta"] \\ X_{\underline{y}} \times_{\underline{y}} X_{\underline{y}} \arrow[r, "{(j_y, j_y)}"] & X \times_Y X. \end{tikzcd}$$

Given any object $(x, x') \in X \times_Y X$ with $f(x) = f(x') = y$,[†] the inclusion of the $\mathscr{T}$-fiber

$$i_{(x,x')} : \underline{(x, x')} \to X \times_Y X$$

factors through $X_{\underline{y}} \times_{\underline{y}} X_{\underline{y}}$. Therefore, if δ_y is ambidextrous, the norm map for $X_{\underline{(x,x')}} \to \underline{(x, x')}$ is an equivalence. By statement (1) of the lemma, we conclude that δ is ambidextrous. □

Recall from [44, Proposition 4.1.10(6)] that given a weakly n-ambidextrous morphism $f : X \to Y$ and $-2 \leqslant m \leqslant n$, f is weakly m-ambidextrous if and only if f is m-truncated [54, Definition 5.5.6.1]. To identify the n-truncated maps in $\mathrm{Spc}_{\mathscr{T}}$, we have the following result.

Lemma 4.10. *Let $X : \mathscr{T}^{\mathrm{op}} \to \mathrm{Spc}$ be an $\mathscr{T}$-space. Then X is n-truncated as an object of $\mathrm{Spc}_{\mathscr{T}}$ if and only if for each $t \in \mathscr{T}$, $X(t)$ is an n-truncated space. Similarly, for a map $f : X \to Y$ of $\mathscr{T}$-spaces, f is n-truncated if and only if $f(t)$ is an n-truncated map of spaces for all $t \in \mathscr{T}$.*

[†] We write an equality here because we are implicitly modeling f as a categorical fibration of left fibrations over $\mathscr{T}$.

Proof. We repeat the argument of [54, 5.5.8.26] for the reader's convenience. It suffices to prove the result for maps. Let $j : \mathcal{T} \to \mathrm{Spc}_{\mathcal{T}}$ denote the Yoneda embedding. Then if f is n-truncated, for any $t \in \mathcal{T}$,

$$f(t) \simeq \mathrm{Map}(j(t), f) : \mathrm{Map}(j(t), X) \simeq X(t) \to \mathrm{Map}(j(t), Y) \simeq Y(t)$$

is n-truncated. Conversely, suppose each $f(t)$ is n-truncated. The collection of $\mathcal{T}$-spaces Z for which $\mathrm{Map}(Z, f)$ is n-truncated is stable under colimits, because limits of n-truncated spaces and maps are again n-truncated. As the representable functors $j(t)$ generate $\mathrm{Spc}_{\mathcal{T}}$ under colimits, it follows that f itself is n-truncated. □

Let us now consider the $n = -1$ case.

Definition 4.11. Let C be an $\mathcal{T}$-∞-category. C is $\mathcal{T}$*-pointed* if for every $t \in \mathcal{T}$, C_t is pointed, and for every $\alpha : u \to t$, the pullback functor $\alpha^* : C_t \to C_u$ preserves the zero object. If $\mathcal{T} = \mathcal{O}_G$, we also say that C is *G-pointed*.

Lemma 4.12. *Let C be a $\mathcal{T}$-∞-category and suppose that for all objects $t \in \mathcal{T}$, C_t admits initial and final objects, and for all morphisms $[\alpha : u \to t] \in \mathcal{T}$, the pullback functor $\alpha^* : C_t \to C_u$ preserves initial and final objects. Then C is $\mathcal{T}$-pointed if and only if for every $t \in \mathcal{T}$, the weakly* (-1)*-ambidextrous morphism $0_t : \emptyset \to \underline{t}$ is ambidextrous.*

Proof. For all $t \in \mathcal{T}$, under our assumptions the norm map Nm_{0_t} exists and identifies as the canonical map between the initial and final object in C_t. Moreover, for all morphisms α in $\mathcal{T}$, these norm maps are preserved by the pullback functors α^*. Now to apply Lemma 4.9, note that for morphisms $[\alpha : u \to t]$ of $\mathcal{T}$ viewed as objects of $\underline{t} = (\mathcal{T}^{/t})^{\mathrm{op}}$, the map $\underline{\alpha} \to \underline{t}$ is homotopic to the canonical $\mathcal{T}$-functor $j(\alpha) : \underline{u} \to \underline{t}$ induced by α under the Yoneda embedding $j : \mathcal{T} \subset \mathrm{Cat}_{\infty,\mathcal{T}}$. Then as pullback of 0_t along $j(\alpha)$ is 0_u, we deduce that 0_t is ambidextrous if and only if C_t admits a zero object, and for all $\alpha : u \to t$, the pullback functor $\alpha^* : C_t \to C_u$ preserves the zero object. The conclusion then follows from considering all t. □

Warning 4.13. In contrast to the nonparameterized case [56, Proposition 6.1.6.7], if C is $\mathcal{T}$-pointed then we may have weakly (-1)-ambidextrous morphisms that fail to be ambidextrous. For example, let $\mathcal{T} = \mathcal{O}_{C_2}$ and let $p : J \to \mathcal{O}_{C_2}^{\mathrm{op}}$ be the C_2-functor given by the inclusion of the full subcategory on the free transitive C_2-sets (so $J \simeq BC_2$). Then J is a C_2-space via p, and for any C_2-∞-category C, we have that $\mathrm{Fun}_{C_2}(J, C) \simeq (C_{C_2/1})^{hC_2}$ for the C_2-action on the fiber $C_{C_2/1}$ encoded by the cocartesian fibration. On the one hand, the C_2-diagonal functor $\delta : J \to J \times_{\mathcal{O}_{C_2}^{\mathrm{op}}} J$ is an equivalence, so J is a (-1)-truncated C_2-space. On the other hand, for $C = \underline{\mathrm{Sp}}^{C_2}$, the restriction p^* may be identified with

$$j^* : \mathrm{Sp}^{C_2} \simeq \mathrm{Fun}_{C_2}(\mathcal{O}_{C_2}^{\mathrm{op}}, \underline{\mathrm{Sp}}^{C_2}) \to \mathrm{Fun}_{C_2}(J, \underline{\mathrm{Sp}}^{C_2}) \simeq \mathrm{Fun}(BC_2, \mathrm{Sp})$$

which we saw has left and right adjoints $j_!$ and j_* such that the norm map $j_! \to j_*$ is *not* an equivalence.

We next consider the $n = 0$ case. Recall from [64, Definition 4.1] that an ∞-category $\mathcal{T}$ is said to be *atomic orbital* if its finite coproduct completion $\mathbb{F}_{\mathcal{T}}$ admits pullbacks and $\mathcal{T}$ has no nontrivial retracts (i.e., every retract is an equivalence). For example, $\mathcal{O}_G$ is atomic orbital. We now assume that $\mathcal{T}$ is atomic orbital, and we regard $V \in \mathcal{T}$ as "orbits" and $U \in \mathbb{F}_{\mathcal{T}}$ as "finite $\mathcal{T}$-sets". Recall the notation $\underline{U}$ from Example 1.32.

Lemma 4.14. *Suppose C is $\mathcal{T}$-pointed. Then for any finite $\mathcal{T}$-set U, orbit V and morphism in* $\mathrm{Spc}_{\mathcal{T}}$

$$f : \underline{U} \to \underline{V}$$

(necessarily specified by a morphism $f : U \to V$ in $\mathbb{F}_{\mathcal{T}}$), the diagonal $\delta : \underline{U} \to \underline{U} \times_{\underline{V}} \underline{U}$ is ambidextrous. Consequently, if $g : X \to Y$ is a morphism between finite coproducts of representables, then g is weakly 0-ambidextrous.

Proof. By our assumption on $\mathcal{T}$, $\underline{U} \times_{\underline{V}} \underline{U}$ decomposes as a finite disjoint union of representables $\coprod_{i\in I} \underline{V_i}$. Moreover, because $\mathcal{T}$ admits no nontrivial retracts, for some $J \subset I$ we have that $\underline{U} \simeq \coprod_{j\in J} \underline{V_j}$ with matching orbits, and δ is a summand inclusion $\underline{U} \to (\coprod_{j\in J} \underline{V_j}) \sqcup (\coprod_{i\in I-J} \underline{V_i})$. δ is then ambidextrous by Lemma 4.9 and Lemma 4.12. The final consequence also follows by Lemma 4.9. □

By Lemma 4.14, the following definition is well-posed.

Definition 4.15. Let C be $\mathcal{T}$-pointed. We say that C is *$\mathcal{T}$-semiadditive* if for each morphism $f : U \to V$ in $\mathbb{F}_T$, the norm map Nm_f for $f : \underline{U} \to \underline{V}$ is an equivalence. If $\mathcal{T} = \mathcal{O}_G$, we will instead say that C is *G-semiadditive*.

Equivalently, in Definition 4.15 we could demand only that the norm maps for $f : U \to V$ with V an orbit are equivalences.

Remark 4.16. Unwinding the definition of the norm maps produced via our setup and in [64, Construction 5.2], one sees that Definition 4.15 is the same as the notion of $\mathcal{T}$-semiadditive given in [64, Definition 5.3]. In particular, for $\mathcal{T} = \mathcal{O}_G$, $\underline{\mathrm{Sp}}^G$ is an example of a G-semiadditive G-∞-category. This amounts to the familiar fact that for each orbit G/H, Sp^H is semiadditive, and for each map of orbits $f : G/H \to G/K$, the left and right adjoints to the restriction functor $f^* : \mathrm{Sp}^K \to \mathrm{Sp}^H$ given by induction and coinduction are canonically equivalent.

In the remainder of this subsection, we further specialize to the case $\mathcal{T} = \mathcal{O}_G$ for G a finite group. We have already encountered a potential problem in Warning 4.13 with developing a useful theory of G-ambidexterity. The issue is essentially due to the presence of fiberwise discrete G-spaces that do not arise from G-sets. To remedy this, we will restrict our attention to the Borel subclass of G-spaces.

Definition 4.17. Suppose that $X \to \mathcal{O}_G^{\mathrm{op}}$ is a G-space. Then X is *Borel* if the functor $\mathcal{O}_G^{\mathrm{op}} \to \mathrm{Spc}$ classifying X is a right Kan extension along the inclusion of the full subcategory $BG \subset \mathcal{O}_G^{\mathrm{op}}$.

Remark 4.18. Definition 4.17 is equivalent to the following condition on a G-space X: if we let

$$X_H := \mathcal{O}_H^{\mathrm{op}} \times_{(\mathrm{ind}_H^G)^{\mathrm{op}}, \mathcal{O}_G^{\mathrm{op}}} X$$

denote the restriction of X to an H-space, then for every subgroup $H \leqslant G$, the natural map

$$X_{G/H} \simeq \mathrm{Map}^{\mathrm{cocart}}_{/\mathcal{O}_H^{\mathrm{op}}}(\mathcal{O}_H^{\mathrm{op}}, X_H) \to \mathrm{Map}_{/BH}(BH, BH \times_{\mathcal{O}_H^{\mathrm{op}}} X_H) \simeq (X_{G/1})^{hH}$$

is an equivalence.

Remark 4.19. Limits and coproducts of Borel G-spaces are Borel. Moreover, for every G-set U, the G-space $\underline{U}$ is Borel. Indeed, this amounts to the observation that $\mathrm{Hom}_G(G/H, U) \cong U^H$ for all subgroups $H \leqslant G$. In particular, as representables are Borel, the Borel property is stable under passage to G-fibers.

Definition 4.20. Let $f : X \to Y$ be a map of Borel G-spaces and let $f_0 : X_0 \to Y_0$ denote the underlying map of spaces. Then by Lemma 4.10, f is n-truncated if and only if f_0 is n-truncated. Furthermore, because every G-orbit is a finite set, the following two conditions are equivalent.

(1) For every $y \in Y_0$, the homotopy fiber $(X_0)_y$ is a finite n-type [44, Definition 4.4.1].
(2) For every $y \in Y$, the underlying space of the homotopy G-fiber $X_{\underline{y}}$ is a finite n-type.

In this case, we say that f is *π-finite n-truncated*. Note that if f is π-finite n-truncated, then its diagonal is π-finite $(n-1)$-truncated; indeed, this property can be checked for the underlying spaces.

More generally, if $f : X \to Y$ is a map of G-spaces such that for every $y \in Y$, $X_{\underline{y}}$ is Borel, then we say that f is *(π-finite) n-truncated* if the above conditions hold for all y and $X_{\underline{y}}$.

Lemma 4.21. *Let $f : X \to Y$ be a map of G-spaces such that for all $y \in Y$, $X_{\underline{y}}$ is Borel.*

(1) *Suppose that C is G-pointed.*
   (1.1) *If f is (-1)-truncated, then f is (-1)-ambidextrous.*
   (1.2) *If f is 0-truncated, then f is weakly 0-ambidextrous.*
(2) *Suppose in addition that C is G-semiadditive.*
   (2.1) *If f is π-finite 0-truncated, then f is 0-ambidextrous.*
   (2.2) *If f is π-finite 1-truncated, then f is weakly 1-ambidextrous.*

Proof. By Lemma 4.9 and under our hypothesis on the parameterized fibers, we may suppose in the proof that $Y = \underline{G/H}$ and X is Borel. For (1), if f is (-1)-truncated then the underlying space of X is a discrete set that injects into G/H. But if X is nonempty then f must also be a surjective map of G-sets because the G-action on G/H is transitive. Thus, either $X = \emptyset$ or f is an equivalence, so by Lemma 4.12, f is (-1)-ambidextrous. If f is 0-truncated, then the (-1)-truncated diagonal $X \to X \times_Y X$ is (-1)-ambidextrous as just shown, so f is weakly 0-ambidextrous.

For (2), we employ the same strategy. If f is π-finite 0-truncated, then X is necessarily a finite G-set, so f is 0-ambidextrous by hypothesis. If f is π-finite 1-truncated, then the diagonal $X \to X \times_Y X$ is π-finite 0-truncated and hence 0-ambidextrous, so f is weakly 1-ambidextrous. ☐

To apply the parameterized ambidexterity theory to our situation of interest, we need the following lemma.

Lemma 4.22. *The G/N-space $B^{\psi}_{G/N}N$ of Definition 3.16 is Borel.*

Proof. For any subgroup K/N of G/N, $(B^{\psi}_{G/N}N)_{K/N} \simeq (B^{\psi'}_{K/N}N)$ for $\psi' = [N \to K \to K/N]$. Therefore, without loss of generality it suffices to prove that the map of groupoids

$$\chi : \mathrm{Map}^{\mathrm{cocart}}_{/\mathcal{O}^{\mathrm{op}}_{G/N}}(\mathcal{O}^{\mathrm{op}}_{G/N}, B^{\psi}_{G/N}N) \to \mathrm{Map}_{/B(G/N)}(B(G/N), (B^{\psi}_{G/N}N) \times_{\mathcal{O}^{\mathrm{op}}_{G/N}} B(G/N))$$

is an equivalence. The fiber E of $B^{\psi}_{G/N}N$ over the terminal G/N-set $*$ is spanned by those N-free G-orbits U such that $U/N \cong *$, and an explicit inverse to the evaluation map

$$\mathrm{Map}^{\mathrm{cocart}}_{/\mathcal{O}^{\mathrm{op}}_{G/N}}(\mathcal{O}^{\mathrm{op}}_{G/N}, B^{\psi}_{G/N}N) \xrightarrow{\simeq} E$$

is given by sending U to the cocartesian section $s_U = (- \times U) : \mathcal{O}^{\mathrm{op}}_{G/N} \to B^{\psi}_{G/N}N$ that sends V to $V \times U$: this follows from our identification of the cocartesian edges in Lemma 3.2. Then

$$\chi(s_U) : B(G/N) \to (B^{\psi}_{G/N}N) \times_{\mathcal{O}^{\mathrm{op}}_{G/N}} B(G/N)$$

is the section which sends G/N to the free transitive G-set $G/N \times U$. Let us now select a basepoint to identify $U \cong G/H$ for H a subgroup such that $H \cap N = 1$ and $G = NH$. We have $W_G H \cong \mathrm{Aut}_G(G/H)$, where a coset $\overline{a} \in W_G H$ gives an automorphism $\theta_{\overline{a}}$ of G/H that sends $1H$ to the well-defined coset aH, and under χ this is sent to the automorphism $\mathrm{id} \times \theta_{\overline{a}}$ of the section $\chi(s_{G/H})$.

By elementary group theory, each coset in G/N has a unique representative xN with $x \in H$, and each coset in G/H has a unique representative yH with $y \in N$. Moreover, the inclusion $N_G(H) \cap N \to N_G(H)$ yields an isomorphism $N_G(H) \cap N \cong W_G(H)$; the map is an injection because $N \cap H = 1$ and a surjection because $G = NH$. The two surjections $G/1 \to G/N$ and $G/1 \to G/H$ sending $1G$ to $1N$ and $1H$ define an isomorphism $G/1 \xrightarrow{\cong} G/N \times G/H$ for which an explicit inverse sends (xN, yH) to $x \cdot y$. Under this isomorphism, $\mathrm{id} \times \theta_{\overline{a}}$ is sent to the unique element $a \in N_G(H) \cap N$ that is a representative for $\overline{a}$.

On the other hand, $(B^{\psi}_{G/N}N) \times_{\mathcal{O}^{\mathrm{op}}_{G/N}} B(G/N) \simeq BG$ where we select $G/1$ to be the unique object of BG. We compute the groupoid of maps $\mathrm{Map}_{/B(G/N)}(B(G/N), BG)$ to have objects given by splittings $\tau : G/N \to G$ of the surjection $\pi : G \to G/N$ and morphisms $\tau \to \tau'$ given by $n \in N$ such that for every coset bN, $n\tau(bN)n^{-1} = \tau'(bN)$. In particular, $\mathrm{Aut}(\tau) = N \cap C_G(H)$ for $H = \tau(G/N)$. However, if $nhn^{-1} = h' \in H$, then $\pi(nhn^{-1}h^{-1}) = 1$ shows that $nhn^{-1}h^{-1} \in N \cap H = 1$, so in fact $nh = hn$ and thus $\mathrm{Aut}(\tau) = N \cap N_G(H)$. Combining this with the explicit understanding of the comparison map given above, we deduce that χ is fully faithful. Essential surjectivity is also clear by the bijection between splittings of π and subgroups H with $N \cap H = 1$ and $G = NH$. We conclude that χ is an equivalence. □

4.2 | The parameterized Tate construction

In view of Lemma 4.22, we may define the parameterized Tate construction (Definition 4.24) by applying the parameterized ambidexterity theory to the Beck–Chevalley fibration

$$\mathrm{LocSys}^{G/N}(\underline{\mathrm{Sp}}^{G/N}) \to \mathrm{Spc}_{G/N}.$$

Let $\rho_N : B^{\psi}_{G/N}N \to \mathcal{O}^{\mathrm{op}}_{G/N}$ be the structure map as in Lemma 3.17, and let

$$\rho_N{}^* : \mathrm{Sp}^{G/N} \simeq \mathrm{Fun}_{G/N}(\mathcal{O}^{\mathrm{op}}_{G/N}, \underline{\mathrm{Sp}}^{G/N}) \to \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathrm{Sp}}^{G/N})$$

be the functor given by restriction along ρ_N. We first introduce alternative notation for its left and right adjoints $(\rho_N)_!$ and $(\rho_N)_*$.

Notation 4.23. Given $X \in \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathrm{Sp}}^{G/N})$, we write

$$X_{h[\psi]} := (\rho_N)_!(X) \text{ and } X^{h[\psi]} := (\rho_N)_* X$$

for the parameterized homotopy orbits and fixed points functors, respectively.

Definition 4.24. The G/N-functor ρ_N has as its underlying map of spaces $BN \to *$, which is π-finite 1-truncated. By Lemma 4.21, ρ_N is weakly 1-ambidextrous, so we can construct the *norm map*

$$\mathrm{Nm}_{\rho_N} : (\rho_N)_! \to (\rho_N)_*.$$

The *ambidextrous Tate construction*

$$(-)^{t[\psi]} : \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathrm{Sp}}^{G/N}) \to \mathrm{Sp}^{G/N}$$

is the cofiber of Nm_{ρ_N}.

Notation 4.25. When the group extension ψ is clear from context, we will also write

$$X_{h_{G/N}N} = X_{h[\psi]}, \quad X^{h_{G/N}N} = X^{h[\psi]}, \quad X^{t_{G/N}N} = X^{t[\psi]}.$$

Observation 4.26 (The norm vanishes on induced objects). For $H \in \Gamma_N$ (so that $H \cap N = 1$), let $U = \rho_N(G/H) \cong \frac{G/N}{HN/N}$ be the G/N-orbit and $s_H : \underline{U} \to B^{\psi}_{G/N}N$ be the unique G/N-functor that selects G/H, so that the functor s_H^* of Observation 3.10 is obtained by restriction along s_H. Note that the map of Borel G/N-spaces s_H is π-finite 0-truncated because its underlying map of spaces is $U \to BN$ with U a finite discrete set. By Lemma 4.21, we see that $\mathrm{Nm}_{s_H} : s_{H!} \xrightarrow{\simeq} (s_H)_*$.

Now consider the composite map $p_U = \rho_N \circ (s_H)$, which is also π-finite 0-truncated. On the one hand, the associated norm map Nm_{p_U} is an equivalence (explicitly, between induction and coinduction from Sp^H to $\mathrm{Sp}^{G/N}$ for $H \cong HN/N$ viewed as a subgroup of G/N). On the other hand, by [44, Remark 4.2.4], we have that Nm_{p_U} is homotopic to the composite $((\rho_N)_* \mathrm{Nm}_{s_H}) \circ (\mathrm{Nm}_{\rho_N}(s_H)_!)$. We deduce that Nm_{ρ_N} is an equivalence on the image of $(s_H)_!$. This extends the observation that the ordinary norm map $X_{hG} \to X^{hG}$ is an equivalence on objects induced from Sp to $\mathrm{Fun}(BG, \mathrm{Sp})$.

By Theorem 3.24, we also have a norm map $\mathrm{Nm}' : \mathcal{F}_b[N] \to \mathcal{F}_b^{\vee}[N]$ arising from the Γ_N-recollement of Sp^G, with functors $\mathcal{F}_b[N]$ and $\mathcal{F}_b^{\vee}[N]$ as in Observation 3.22. We now proceed to show that the two norm maps $\Psi^N \mathrm{Nm}'$ and Nm_{ρ_N} are equivalent.

Lemma 4.27. *The functor ρ_N^* is homotopic to the composite*

$$\mathcal{U}_b[N]\circ\mathrm{inf}^N : \mathrm{Sp}^{G/N} \to \mathrm{Sp}^G \to \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathrm{Sp}}^{G/N}).$$

Proof. For the proof, we work in the setup of Construction 3.7. Let

$$\underline{\mathrm{inf}}'[N] : \underline{\mathrm{Sp}}^{G/N} \to \mathcal{O}^{\mathrm{op}}_{G/N} \times_{i_N^{\mathrm{op}},\mathcal{O}_G^{\mathrm{op}}} \underline{\mathrm{Sp}}^G$$

be the G/N-functor defined by unstraightening the natural transformation

$$\mathrm{SH}q_N^{\mathrm{op}} : \mathrm{SH}\omega^{\mathrm{op}}_{G/N} \to \mathrm{SH}\omega^{\mathrm{op}}_G \iota^{\mathrm{op}}_N,$$

so for a G/N orbit $V = \frac{G/N}{K/N}$, the fiber $\underline{\mathrm{inf}}'[N]_V : \mathrm{Sp}^{K/N} \to \mathrm{Sp}^K$ is given by the inflation functor inf^N. By definition, the composite

$$\underline{\mathbf{Sp}}^{G/N} \xrightarrow{\underline{\mathrm{inf}}'[N]} \mathcal{O}^{\mathrm{op}}_{G/N} \times_{i_N^{\mathrm{op}},\mathcal{O}_G^{\mathrm{op}}} \mathbf{Sp}^G \xrightarrow{\widetilde{\mathcal{U}}[N]} \underline{\mathrm{Fun}}_{G/N}(\mathcal{O}_G^{\mathrm{op}}, \underline{\mathbf{Sp}}^{G/N})$$

is adjoint to the composite (abusively denoting $\underline{\mathrm{inf}}'[N]$ for its pullback along r_N^{op})

$$\mathcal{O}_G^{\mathrm{op}} \times_{r_N^{\mathrm{op}},\mathcal{O}_{G/N}^{\mathrm{op}}} \underline{\mathbf{Sp}}^{G/N} \xrightarrow{\underline{\mathrm{inf}}'[N]} \mathcal{O}_G^{\mathrm{op}} \times_{(i_N r_N)^{\mathrm{op}},\mathcal{O}_G^{\mathrm{op}}} \underline{\mathbf{Sp}}^G \xrightarrow{\underline{\mathrm{res}}[N]} \underline{\mathbf{Sp}}^G \xrightarrow{\widehat{\Psi}[N]} \mathcal{O}_G^{\mathrm{op}} \times_{r_N^{\mathrm{op}},\mathcal{O}_{G/N}^{\mathrm{op}}} \underline{\mathbf{Sp}}^{G/N} \xrightarrow{\mathrm{pr}} \underline{\mathbf{Sp}}^{G/N}.$$

For a G-orbit G/H, the fiber of $\widehat{\Psi}[N]\circ\underline{\mathrm{res}}[N]\circ\underline{\mathrm{inf}}'[N]$ over G/H is given by the composition

$$\mathbf{Sp}^{HN/N} \xrightarrow{\mathrm{inf}^N} \mathbf{Sp}^{HN} \xrightarrow{\mathrm{res}^{HN}_H} \mathbf{Sp}^H \xrightarrow{\Psi^{H\cap N}} \mathbf{Sp}^{H/(H\cap N)}.$$

Using that $HN/N \cong H/(H\cap N)$, the composition $\mathrm{res}^{HN}_H \circ \mathrm{inf}^N$ is homotopic to $\mathrm{inf}^{H\cap N}$. Therefore, if $H \in \Gamma_N$ so that $H \cap N = 1$, the entire composite is trivial. We deduce that the composite

$$B^{\psi}_{G/N}N \times_{\rho_N,\mathcal{O}^{\mathrm{op}}_{G/N}} \underline{\mathbf{Sp}}^{G/N} \xrightarrow{\underline{\mathrm{inf}}'[N]} B^{\psi}_{G/N}N \times_{\rho_N,\mathcal{O}^{\mathrm{op}}_{G}} \underline{\mathbf{Sp}}^{G} \xrightarrow{\underline{\mathrm{res}}[N]} \underline{\mathbf{Sp}}^G \xrightarrow{\widehat{\Psi}[N]} B^{\psi}_{G/N}N \times_{\rho_N,\mathcal{O}^{\mathrm{op}}_{G/N}} \underline{\mathbf{Sp}}^{G/N}$$

is homotopic to the identity, which proves the claim. □

Observation 4.28. Passing to right adjoints in Lemma 4.27, we get that $(\rho_N)_* \simeq \Psi^N \mathcal{F}_b^\vee[N]$. Let

$$(-)^{t'[\psi]} : \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathrm{Sp}}^{G/N}) \to \mathrm{Sp}^{G/N}$$

be the cofiber of $\Psi^N\,\mathrm{Nm}'$. As the orbits $G/H_+ \in \mathrm{Sp}^G$ for $H \in \Gamma_N$ are both Γ_N-torsion and Γ_N-complete, $\mathrm{Nm}' \circ (s_H)_!(1)$ is an equivalence for all $H \in \Gamma_N$. Therefore, $(-)^{t'[\psi]}$ vanishes on each $(s_H)_!(1)$. Because $\{(s_H)_!(1) : H \in \Gamma_N\}$ is a set of compact generators for $\mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathrm{Sp}}^{G/N})$ and $(\rho_N)_!$ is a colimit preserving functor, the composite

$$(-)_{h[\psi]} = (\rho_N)_! \xrightarrow{\mathrm{Nm}_{\rho_N}} (-)^{h[\psi]} = (\rho_N)_* \simeq \Psi^N\mathcal{F}_b^\vee[N] \to (-)^{t'[\psi]}$$

is null-homotopic. We thereby obtain a natural transformation $\nu : (-)^{t[\psi]} \to (-)^{t'[\psi]}$. Taking fibers, we also have a natural transformation $\mu : (\rho_N)_! \to \Psi^N \mathscr{F}_b[N]$. All together, for all $X \in \mathrm{Sp}^G_{N\text{-}\mathrm{Borel}}$ we have a morphism of fiber sequences of G/N-spectra

$$\begin{tikzcd} X_{h[\psi]} \arrow[r, "(\mathrm{Nm}_{\rho_N})_X"] \arrow[d, "\mu_X"] & X^{h[\psi]} \arrow[r] \arrow[d, "\simeq"] & X^{t[\psi]} \arrow[d, "\nu_X"] \\ \Psi^N \mathscr{F}_b[N](X) \arrow[r, "\Psi^N \mathrm{Nm}'_X"] & \Psi^N \mathscr{F}_b^\vee[N](X) \arrow[r] & X^{t'[\psi]}. \end{tikzcd}$$

Theorem 4.29. *The natural transformations μ and ν are equivalences. In particular, the ambidextrous Tate construction is canonically equivalent to that defined by the parameterized recollement theory.*

Proof. It suffices to show that μ is an equivalence. By Observation 4.26, Nm_{ρ_N} is an equivalence on $(s_H)_!(1)$ for every $H \in \Gamma_N$, and we just saw the same property for Nm'. Therefore, μ is an equivalence on each $(s_H)_!(1)$ by the two-out-of-three property of equivalences. As both $(\rho_N)_!$ and $\Psi^N \mathscr{F}_b[N]$ preserve colimits and the $(s_H)_!(1)$ form a set of compact generators, we conclude that μ is an equivalence. □

Remark 4.30 (∞-categorical Adams isomorphism). By Theorem 4.29, for $X \in \mathrm{Sp}^G_{N\text{-}\mathrm{Borel}}$, we have an equivalence of G/N-spectra $X_{h[\psi]} \simeq \Psi^N \mathscr{F}_b[N](X)$. Viewing X as an "N-free" G-spectrum via the embedding $\mathscr{F}_b[N]$, this amounts to the Adams isomorphism for a normal subgroup N of a *finite* group G in our context (compare [61, chapter XVI, Theorem 5.4]).

We also note that Sanders [72] has recently introduced a different formal framework for producing the Adams isomorphism, in the more general situation of a closed normal subgroup of a compact Lie group. It would be interesting to understand the relationship between his results and ours.

Remark 4.31. In view of Theorem 4.29, we could have defined the parameterized Tate construction as $(-)^{t'[\psi]}$ to begin with. However, we still need the Adams isomorphism to identify the fiber term $\Psi^N \mathscr{F}_b[N]$ of $(\rho_N)_* \to (-)^{t'[\psi]}$ as the parameterized orbits functor $(\rho_N)_!$.

Remark 4.32 (Point-set models). Let $X \in \mathrm{Sp}^G_{N\text{-}\mathrm{Borel}}$ and consider the fiber sequence of G/N-spectra

$$X_{h[\psi]} \to X^{h[\psi]} \to X^{t[\psi]}.$$

By Theorem 4.29 and the monoidal recollement theory for Γ_N, this fiber sequence is obtained as Ψ^N of the fiber sequence of G-spectra

$$\mathscr{F}_b^\vee[N](X) \otimes E\Gamma_{N+} \to \mathscr{F}_b^\vee[N](X) \to \mathscr{F}_b^\vee[N](X) \otimes \widetilde{E\Gamma}_N.$$

If we let $X = \mathscr{U}_b[N](Y)$ for $Y \in \mathrm{Sp}^G$, then we may also write this as

$$Y \otimes E\Gamma_{N+} \simeq F(E\Gamma_{N+}, Y) \otimes E\Gamma_{N+} \to F(E\Gamma_{N+}, Y) \to F(E\Gamma_{N+}, Y) \otimes \widetilde{E\Gamma}_N.$$

Remark 4.33 (Parameterized Tate spectral sequence). The point-set model for the parameterized Tate construction given in Remark 4.32 allows us to apply [30, Theorem 22.6] to define the *parameterized Tate spectral sequence*

$$E^2_{p,q} = H^p_{\Gamma_N}(N; E_q) \Rightarrow \pi_{q-p} E^{t_{G/N}N}$$

to study the G/N-parameterized N-Tate construction of $E \in \mathrm{Sp}^G_{N\text{-Borel}}$. The E_2-term is given by the Γ_N*-Amitsur–Dress–Tate cohomology* of N with coefficients in E_* as defined in [30, Definition 21.1]. See also [60, sections 2.3, 3.2].

An analogous spectral sequence also exists in $RO(G/N)$-graded form, where we replace $q \in \mathbb{Z}$ by $\star \in RO(G/N)$. We will use the $RO(C_2)$-graded parameterized Tate spectral sequence in [70] to compute the real topological cyclic homology of $H\underline{\mathbb{F}_p}$, where $H\underline{\mathbb{F}_p}$ is the Eilenberg–MacLane C_2-spectrum for the constant Mackey functor on $\mathbb{F}_p$ with p odd.

Observation 4.34 (Compatibility with restriction). Note that the norm map Nm' also extends to a natural transformation of G-functors

$$\left(\underline{\mathrm{Nm}'} : \underline{\mathscr{F}_b}[N] \Rightarrow \underline{\mathscr{F}^\vee_b}[N]\right) : \underline{\mathrm{Sp}}^G_{N\text{-}\mathrm{Borel}} \to \underline{\mathrm{Sp}}^G.$$

Postcomposing with the functor $\widehat{\Psi}[N] : \underline{\mathrm{Sp}}^G \to \mathscr{O}^{\mathrm{op}}_G \times_{\mathscr{O}^{\mathrm{op}}_{G/N}} \underline{\mathrm{Sp}}^{G/N}$ defined in Construction 3.7 and taking the cofiber, we may extend $(-)^{t[\psi]}$ to a functor over $\mathscr{O}^{\mathrm{op}}_G$

$$(-)^{\widehat{t}[\psi]} : \underline{\mathrm{Sp}}^G_{N\text{-}\mathrm{Borel}} \to \mathscr{O}^{\mathrm{op}}_G \times_{\mathscr{O}^{\mathrm{op}}_{G/N}} \underline{\mathrm{Sp}}^{G/N}$$

that over an orbit G/H is given by

$$(-)^{t[\psi_H]} : \mathrm{Fun}_{H/(N\cap H)}(B^{\psi_H}_{H/(N\cap N)}(N \cap H), \underline{\mathrm{Sp}}^{H/(N\cap H)}) \to \mathrm{Sp}^{H/(N\cap H)}.$$

However, because $\underline{\Psi}[N]$ is not typically a G-functor, $(-)^{\widehat{t}[\psi]}$ may also fail to be a G-functor. If instead we precompose by the inclusion

$$\underline{\mathrm{Fun}}_{G/N}(B^\psi_{G/N}N, \underline{\mathrm{Sp}}^{G/N}) \simeq \mathscr{O}^{\mathrm{op}}_{G/N} \times_{i^{\mathrm{op}}_N, \mathscr{O}^{\mathrm{op}}_G} \underline{\mathrm{Sp}}^G_{N\text{-}\mathrm{Borel}} \to \underline{\mathrm{Sp}}^G_{N\text{-}\mathrm{Borel}}$$

and postcompose by the projection to $\underline{\mathrm{Sp}}^{G/N}$, then we obtain a G/N-functor

$$(-)^{\underline{t}[\psi]} : \underline{\mathrm{Fun}}_{G/N}(B^\psi_{G/N}N, \underline{\mathrm{Sp}}^{G/N}) \to \underline{\mathrm{Sp}}^{G/N}.$$

By checking fiberwise, it is easy to verify that $(-)^{\underline{t}[\psi]}$ is equivalent to the cofiber of the norm map $\mathrm{Nm}_{\rho_N} : (\underline{\rho_N})_! \to (\underline{\rho_N})_*$ produced by the ambidexterity theory for the other Beck–Chevalley fibration

$$\underline{\mathrm{LocSys}}^{G/N}(\underline{\mathrm{Sp}}^{G/N}) \to \mathrm{Spc}_{G/N}.$$

In any case, we obtain a compatibility between the parameterized Tate construction and restriction. For example, given a G/N-functor $X : B^\psi_{G/N}N \to \underline{\mathrm{Sp}}^{G/N}$, we see that the underlying spectrum of $X^{t[\psi]}$ is $X^{tN} := (\mathrm{res}^{G/N} X)^{tN}$ for the underlying functor $\mathrm{res}^{G/N} X : BN \to \mathrm{Sp}$.

A useful consequence of Theorem 4.29 is that it enables us to endow the functor $(-)^{t[\psi]}$ and the natural transformation $(-)^{h[\psi]} \to (-)^{t[\psi]}$ with lax symmetric monoidal structures.

Corollary 4.35. *The functor $(-)^{t[\psi]}$ and the natural transformation $(-)^{h[\psi]} \to (-)^{t[\psi]}$ are lax symmetric monoidal with respect to the pointwise symmetric monoidal structure on $\mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathrm{Sp}}^{G/N})$ and the smash product on $\mathrm{Sp}^{G/N}$.*

Proof. The cofiber of Nm' is the lax symmetric monoidal map $j_* \to i_* i^* j_*$ of the Γ_N-recollement of Sp^G, where we use Remark 3.25 to relate the pointwise symmetric monoidal structure on the domain $\mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathrm{Sp}}^{G/N})$ to the symmetric monoidal recollement. As the categorical fixed points functor Ψ^N is also lax symmetric monoidal, we deduce that $\Psi^N \mathcal{F}^{\vee}_b[N](-) \to (-)^{t'[\psi]}$ is lax symmetric monoidal. The conclusion now follows from Theorem 4.29. $\square$

On the other hand, one practical benefit of defining the parameterized Tate construction via the ambidexterity theory is that we may exploit the general naturality properties of norms as detailed in [44, section 4.2]. To state our next result, which involves two normal subgroups $M \trianglelefteq N$ of G, we first require a preparatory lemma.

Lemma 4.36. *Let $M \trianglelefteq N \trianglelefteq G$ be two normal subgroups of G and let*

$$\psi = [N \to G \to G/N], \quad \psi' = [N/M \to G/M \to G/N]$$

denote the extensions.

(1) *$\Psi^M : \mathrm{Sp}^G \to \mathrm{Sp}^{G/M}$ sends Γ_N-torsion spectra to $\Gamma_{N/M}$-torsion spectra.*
(2) *Let $r_M : \mathbb{F}_G \to \mathbb{F}_{G/M}$, $r_M(U) = U/M$ be as in Observation 3.3, and regard $\mathbb{F}_G$, $\mathbb{F}_{G/M}$ as cartesian fibrations over $\mathbb{F}_{G/N}$ via r_N, $r_{N/M}$, respectively. Then r_M preserves cartesian edges, so the restricted functor $r_M^{\mathrm{op}} : \mathcal{O}_G^{\mathrm{op}} \to \mathcal{O}_{G/M}^{\mathrm{op}}$ is a G/N-functor. Moreover, r_M^{op} further restricts to a G/N-functor*

$$\rho_M : B^{\psi}_{G/N}N \to B^{\psi'}_{G/N}N/M.$$

(3) *We have a commutative diagram*

$$\begin{tikzcd}
\mathbf{Sp}^G_{N\mp\mathrm{Borel}} = \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathbf{Sp}}^{G/N}) & & \mathbf{Sp}^G \arrow[ll, "\mathcal{U}_b{[N]}"'] \\
\mathbf{Sp}^{G/M}_{N)M\mp\mathrm{Borel}} = \mathrm{Fun}_{G/N}(B^{\psi'}_{G/N}(N/M), \underline{\mathbf{Sp}}^{G/N}) \arrow[u, "(\rho_M)^*"] & & \mathbf{Sp}^{G/M} \arrow[ll, "\mathcal{U}_b{[N/M]}"'] \arrow[u, "\mathrm{inf}^G_{G/M}"']
\end{tikzcd}$$

that yields a commutative diagram of right adjoints

$$\begin{tikzcd}
\mathbf{Sp}^G_{N\mp\mathrm{Borel}} = \mathrm{Fun}_{G/N}(B^{\psi}_{G/N}N, \underline{\mathbf{Sp}}^{G/N}) \arrow[rr, "\mathcal{F}^{\vee}_b{[N]}"] \arrow[d, "(\rho_M)_*"'] & & \mathbf{Sp}^G \arrow[d, "\Psi^M"] \\
\mathbf{Sp}^{G/M}_{N)M\mp\mathrm{Borel}} = \mathrm{Fun}_{G/N}(B^{\psi'}_{G/N}(N/M), \underline{\mathbf{Sp}}^{G/N}) \arrow[rr, "\mathcal{F}^{\vee}_b{[N/M]}"] & & \mathbf{Sp}^{G/M}
\end{tikzcd}$$

where the left-hand vertical functor is computed by the G/N-right Kan extension along r_M^{op}.

Proof. For (1), note that if G/H is a N-free G-orbit, then G/H is also M-free. Thus, we may compute $\Psi^M(G/H_+)$ as by taking the quotient by the M-action to obtain $\frac{G/M}{HM/M}_+$, which is N/M-free and thus $\Gamma_{N/M}$-torsion. Because the subcategory of Γ_N-torsion spectra is the localizing subcategory generated by such G/H_+ and Ψ^M preserves colimits, the statement follows. (2) is a direct consequence of Lemma 3.2(2). (3) is a relative version of Lemma 4.27, and also follows by an elementary diagram chase after unpacking the various definitions. $\square$

Now suppose G is a semidirect product of N and G/N, so we have chosen a splitting $G/N \to G$ of the quotient map such that $G \cong N \rtimes G/N$, and with respect to the G/N-action on N, the inclusion $M \subset N$ is G/N-equivariant. Then $M \rtimes G/N$ is a subgroup of G, and we let

$$\psi'' = [M \to M \rtimes G/N \to G/N].$$

Also regard $B^{\psi'}_{G/N} N/M$ as a based G/N-space via the splitting. Then we have a homotopy pullback square of G/N-spaces

$$\begin{tikzcd} B^{\psi''}_{G/N} M \arrow[r] \arrow[d, "\rho_M"] & B^{\psi}_{G/N} N \arrow[d, "\rho_M"] \\ \mathcal{O}^{\mathrm{op}}_{G/N} \arrow[r] & B^{\psi'}_{G/N} N/M \end{tikzcd}$$

that arises from the fiber sequence $BM \to BN \to BN/M$ of spaces with G/N-action.

Proposition 4.37. *Suppose $X \in \mathrm{Sp}^G_{N\text{-}\mathrm{Borel}}$ and also write X for its restriction to $\mathrm{Sp}^G_{M\text{-}\mathrm{Borel}}$. Then $X^{t[\psi'']}$ canonically acquires a "residual action" by lifting to an object in $\mathrm{Sp}^G_{N/M\text{-}\mathrm{Borel}}$, and we have a fiber sequence of G/N-spectra*

$$(X_{h[\psi'']})^{t[\psi']} \to X^{t[\psi]} \to (X^{t[\psi'']})^{h[\psi']}$$

that restricts to a fiber sequence of Borel G/N-spectra

$$(X_{hM})^{t(N/M)} \to X^{tN} \to (X^{tM})^{h(N/M)}.$$

Proof. We apply [44, Remark 4.2.3] to the pullback square above to deduce the first assertion. For the second assertion, we apply [44, Remark 4.2.4] to the factorization of ρ_N as $\rho_{N/M} \circ \rho_M$ to obtain a commutative diagram of G/N-spectra

$$\begin{tikzcd} X_{h[\psi]} \arrow[r, "\mathrm{Nm}_{\rho_{N/M}} \circ (\rho_M)_!"] \arrow[d] & (X_{h[\psi'']})^{h[\psi']} \arrow[r, "(\rho_{N/M})_* \circ \mathrm{Nm}_{\rho_M}"] \arrow[d] & X^{h[\psi]} \arrow[d] \\ 0 \arrow[r] & (X_{h[\psi'']})^{t[\psi']} \arrow[r] \arrow[d] & X^{t[\psi]} \arrow[d] \\ & 0 \arrow[r] & (X^{t[\psi'']})^{h[\psi']} \end{tikzcd}$$

in which every rectangle is a homotopy pushout (using the two-out-of-three property of homotopy pushouts to show this for the upper right-hand square and then the lower right-hand square). The final assertion follows from Observation 4.34. ☐

4.3 | Inverse limit formula for the parameterized Tate construction

We prove that the parameterized Tate construction can be expressed as a limit involving certain Thom spectra, generalizing [30, Theorem 16.1]. This identification is used to prove parameterized Tate blueshift in [50] and to define C_2-equivariant Mahowald invariants in [68].

Lemma 4.38. *Let $\mathcal{F}$ be a family of subgroups of G. Suppose V be a G-representation with $V^H = 0$ for $H \notin \mathcal{F}$ and $V^H \neq 0$ for $H \in \mathcal{F}$. Then $S(V^{\oplus\infty}) \simeq E\mathcal{F}$ and $S^{\infty V} \simeq \widetilde{E\mathcal{F}}$.*

The following lemma generalizes [30, Lemma 2.6].

Lemma 4.39. *Let $\mathcal{F}$ be a family of subgroups of G. There is an equivalence of G-spectra*

$$\widetilde{E\mathcal{F}} \otimes F(E\mathcal{F}_+, X) \simeq F(\widetilde{E\mathcal{F}}, E\mathcal{F}_+ \otimes \Sigma X).$$

Proof. We may identify

$$\widetilde{E\mathcal{F}} \otimes F(E\mathcal{F}_+, X) \simeq i_* i^* j_* j^*(X),$$

$$F(\widetilde{E\mathcal{F}}, E\mathcal{F}_+ \otimes \Sigma X) \simeq i_* i^! j_! j^*(\Sigma X).$$

Writing $X = [U, Z \to \phi(U)]$ for the recollement decomposition of X, we find that $i_* i^* j_* j^*(X) = [0, \phi(U) \xrightarrow{=} \phi(U)]$. We also have $j_! j^* \Sigma X = [\Sigma U, 0 \to \Sigma\Phi(U)]$, so $i_* i^! j_! j^*(\Sigma X) = [0, \Phi(U) \xrightarrow{=} \Phi(U)]$ as desired. ☐

Theorem 4.40. *Let $\psi = [N \to G \to G/N]$ be an extension with $N \subseteq H$ for each $H \notin \Gamma_N$. Suppose V is as in Lemma 4.38 for $\mathcal{F} = \Gamma_N$. For any $X \in \mathrm{Sp}^{\Phi\Gamma_N} \simeq \mathrm{Sp}^{G/N}$, we have*

$$(j^* i_* X)^{t[\psi]} \simeq \lim_n (B^{\psi}_{G/N} N^{-nV} \otimes \Sigma X).$$

Proof. We compute:

$$\begin{aligned}
F(j^* i_* X)^{t[\psi]} &\simeq \Psi^N(\widetilde{E\Gamma_N} \otimes F(E\Gamma_{N+}, j^* i_* X)) && (\textit{Remark } 4.32)\\
&\simeq \Psi^N(F(\widetilde{E\Gamma_N}, E\Gamma_{N+} \otimes \Sigma j^* i_* X)) && (\textit{Lemma } 4.39)\\
&\simeq \Psi^N(F(S^{\infty V}, E\Gamma_{N+} \otimes \Sigma j^* i_* X)) && (\textit{Lemma } 4.38)\\
&\simeq \Psi^N(F(\lim_n S^{nV}, E\Gamma_{N+} \otimes \Sigma j^* i_* X))\\
&\simeq \Psi^N(\lim_n F(S^{nV}, E\Gamma_{N+} \otimes \Sigma j^* i_* X))\\
&\simeq \Psi^N(\lim_n S^{-nV} \otimes E\Gamma_{N+} \otimes \Sigma j^* i_* X)
\end{aligned}$$

$$\simeq \lim_n (\Psi^N(S^{-nV} \otimes E\Gamma_{N+}) \otimes \Psi^N(\Sigma j^* i_* X))$$

$$\simeq \lim_n ((S^{-nV})_{h_{G/N}N} \otimes \Sigma X) \qquad (\textit{Theorem } 4.29)$$

$$\simeq \lim_n (B^{\psi}_{G/N} N^{-nV} \otimes \Sigma X). \qquad \square$$

Remark 4.41. Although we have restricted to finite G in this section, the results of the next section can be used to show that the evident analogue of Theorem 4.40 holds for G a compact Lie group. This agrees with the level of generality in the nonparameterized result of Greenlees–May [30, Theorem 16.1].

5 | PARAMETERIZED ASSEMBLY

Suppose K is a compact Lie group. Then if K is infinite, the Tate construction $(-)^{tK}$ cannot be defined via the Hopkins–Lurie ambidexterity theory because BK is not π-finite. Instead, one defines $(-)^{tK}$ to be the cofiber of an *assembly map*

$$(\mathbb{S}^{\mathfrak{a}} \otimes -)_{hK} \to (-)^{hK}$$

where $\mathbb{S}^{\mathfrak{a}}$ is Σ^∞ of the one-point compactification of the adjoint representation of K (cf. [47] and [67, section I.4]). This construction is of particular interest when $K = S^1$. For example, given a cyclotomic spectrum X (e.g., THH of a ring spectrum A), one defines the *topological periodic homology* $\mathrm{TP}(X)$ to be X^{tS^1}, and this term (or rather, its profinite completion) participates in a fiber sequence computing the topological cyclic homology $\mathrm{TC}(X)$.

As we are ultimately interested in applications to trace methods for real algebraic K-theory, we are thus motivated to develop a parameterized refinement of the above picture. More precisely, let G be a finite group and $\psi : G \to \mathrm{Aut}(K)$ a group homomorphism (where $\mathrm{Aut}(K)$ denotes the group of continuous automorphisms of K).

Definition 5.1. The G-space $B^{\psi}_G K$ is the Borel G-space obtained via right Kan extension along $BG \subset \mathcal{O}_G^{\mathrm{op}}$ of BK regarded as a space with G-action via ψ.

Remark 5.2. By Lemma 4.22, this is consistent with our earlier definition of $B^{\psi}_G K$ when K is finite.

Warning 5.3. The role of G in this section is the same as its role in the introduction, but different from its role in the previous two sections. In particular, G now plays the role of the quotient group $G = \widehat{G}/K$. Again, this is to emphasize that G is finite; the notation for the normal subgroup has changed from N to K to emphasize that K can be infinite, while N was always finite.

In this section, we will construct a *parameterized assembly map* (Theorem 5.48 and Proposition 5.57)

$$(\mathbb{S}^{\mathfrak{a}} \otimes -)_{h_G K} \to (-)^{h_G K}$$

and then define the parameterized Tate construction $(-)^{t_G K}$ to be its cofiber (Definition 5.49). We unpack the example of $K = S^1$ with $G = C_2$ acting by complex conjugation as Example 5.58.

We will also equip $(-)^{t_G K}$ with a lax G-symmetric monoidal structure and thereby show that it preserves G-commutative algebras (Corollary 5.60); when K is finite, this improves upon Corollary 4.35.

We will use the following notation heavily in this section. Recall the notation $\underline{U}$ for a finite G-set U from Example 1.32 and note that $\underline{U}^{\mathrm{op}} \simeq (\mathcal{O}_G)^{/U}$.[†] Now let $\mathrm{Cat}_{\infty,\underline{U}}$ denote the ∞-category of $(\mathcal{O}_G)^{/U}$-∞-categories, that is, cocartesian fibrations $C \to \underline{U}$.

Notation 5.4. Let $f : U \to V$ be a map of finite G-sets and consider the pullback functor

$$f^* : \mathrm{Cat}_{\infty,\underline{V}} \to \mathrm{Cat}_{\infty,\underline{U}}, \quad C \mapsto C_{\underline{U}}.$$

We write

$$\begin{tikzcd} \mathbf{Cat}_{\infty,\underline{U}} \arrow[r, shift left=2, "f_!"] \arrow[r, shift right=2, "f_*"'] & \mathbf{Cat}_{\infty,\underline{V}} \arrow[l, "f^*" description] \end{tikzcd}$$

for the adjoint triple.

Remark 5.5. Let

$$\begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "g'"] & V' \arrow[d, "g"] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

be a pullback square of finite G-sets. Then the exchange transformations

$$g^* f_* \Rightarrow f'_* g'^*,\ f'_! g'^* \Rightarrow g^* f_! : \mathrm{Cat}_{\infty,\underline{U}} \to \mathrm{Cat}_{\infty,\underline{V'}}$$

are equivalences.

Warning 5.6. In this section, when we speak of $\underline{U}$-∞-categories, $\underline{U}$-functors, and so on, we mean this as shorthand for $(\mathcal{O}_G)^{/U}$-∞-categories, $(\mathcal{O}_G)^{/U}$-functors, and so on. Unfortunately, $\underline{U}$ also refers to the ∞-category $((\mathcal{O}_G)^{/U})^{\mathrm{op}}$. We warn the reader to not confuse $\underline{U}$-∞-categories with presheaves on $\underline{U}$, that is, cocartesian fibrations over $\underline{U}^{\mathrm{op}} \simeq (\mathcal{O}_G)^{/U}$.

5.1 | Recollections on G-symmetric monoidal structures

One of our main goals in this section is to explain how (for an extension ψ of a finite group G by a compact Lie group K) the parameterized Tate construction

$$(-)^{t_G K} : \mathrm{Fun}_G\left(B^{\psi}_G K, \underline{\mathrm{Sp}}^G\right) \to \mathrm{Sp}^G$$

[†] $(\mathcal{O}_G)^{/U}$ is defined to be the full subcategory of $(\mathbb{F}_G)^{/U}$ on those morphisms with orbit source.

refines to a *lax G-symmetric monoidal functor*. We begin by recalling the necessary terminology and concepts from the theory of G-symmetric monoidal ∞-categories, which will be explained in more detail in [66].[†]

Definition 5.7. A *G-symmetric monoidal ∞-category* $C^{\otimes}$ is a cocartesian fibration over $\mathrm{Span}(\mathbb{F}_G)$ whose straightening

$$F_{C^{\otimes}} : \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_{\infty}$$

is a product-preserving functor. The *underlying G-∞-category* of $C^{\otimes}$ is the restriction $C = C^{\otimes}|_{\mathcal{O}_G^{\mathrm{op}}}$ over the subcategory $\mathcal{O}_G^{\mathrm{op}} \subset \mathrm{Span}(\mathbb{F}_G)$. Conversely, given a G-∞-category C, a *G-symmetric monoidal structure* $C^{\otimes}$ on C is such an extension of C over $\mathrm{Span}(\mathbb{F}_G)$. When the G-symmetric monoidal structure is understood, we refer to C itself as a G-symmetric monoidal ∞-category.

For a map of finite G-sets $f : U \to V$, we define the (multiplicative) *norm functor*

$$f_{\otimes} : C^{\otimes}_U \to C^{\otimes}_V$$

as the pushforward functor associated to the span $[U \xleftarrow{=} U \xrightarrow{f} V]$.

A (strong) *G-symmetric monoidal functor* $F : C^{\otimes} \to D^{\otimes}$ is a map of cocartesian fibrations over $\mathrm{Span}(\mathbb{F}_G)$. If we only suppose that F preserves cocartesian edges over $\mathbb{F}_G^{\mathrm{op}}$, then F is said to be *lax G-symmetric monoidal*.

Remark 5.8. Definition 5.7 is an ∞-categorical version of the G-symmetric monoidal categories introduced by Hill and Hopkins in [39].

Given a G-symmetric monoidal ∞-category $C^{\otimes}$, a *G-commutative algebra A* in $C^{\otimes}$ is a section $A^{\otimes} : \mathrm{Span}(\mathbb{F}_G) \to C^{\otimes}$ whose restriction $A^{\otimes}|_{\mathbb{F}_G^{\mathrm{op}}}$ preserves cocartesian edges. Though we would not discuss the notion of G-commutative algebra much in this paper, the reader should bear in mind that one of the main points of the theory of G-symmetric monoidal ∞-categories is to streamline arguments involving G-commutative algebras. For example, a lax G-symmetric monoidal functor necessarily preserves G-commutative algebras.

Remark 5.9. Let $C^{\otimes}, D^{\otimes}$ be G-symmetric monoidal ∞-categories and let $F : C^{\otimes} \to D^{\otimes}$ by a G-symmetric monoidal functor whose underlying G-functor is G-left adjoint. Then an easy argument with relative adjunctions shows that its G-right adjoint R canonically refines to a lax G-symmetric monoidal functor.

Note that if $V \cong \coprod_{j \in J} V_j$ for orbits $\{V_j\}_{j \in J}$ and we write $f_j : U_j := U \times_V V_j \to V_j$ for the fiber, then we have

$$f_{\otimes} \simeq \prod_{j \in J} (f_j)_{\otimes} : \prod_{j \in J} C^{\otimes}_{U_j} \to \prod_{j \in J} C_{V_j}.$$

Because of this, we may think of a G-symmetric monoidal structure as being defined by the collection of functors $\{f_{\otimes}\}$ for those maps $[f : U \to V] \in \mathbb{F}_G$ with V an orbit, subject to relations encoded by $\mathrm{Span}(\mathbb{F}_G)$.

[†] Another basic reference is [7], but note that there is some work involved in translating between their setup and ours.

Remark 5.10. Given a G-symmetric monoidal ∞-category $C^{\otimes}$, for every finite G-set V the fiber $C_V^{\otimes}$ is a symmetric monoidal ∞-category via restriction of $C^{\otimes}$ along the map

$$\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}(\mathbb{F}_G), \quad n \mapsto V^{\sqcup n}.$$

In other words, the fold maps for V define the symmetric monoidal structure on $C_V^{\otimes}$. Moreover, for any map $f : U \to V$ the norm functor $f_{\otimes} : C_U^{\otimes} \to C_V^{\otimes}$ is then symmetric monoidal.

Remark 5.11. Suppose $C^{\otimes}$ is a G-symmetric monoidal ∞-category and $f : U \to V$ is a map of finite G-sets. Then in view of the compatibility between norms and restriction as encoded by $\mathrm{Span}(\mathbb{F}_G)$, the norm functor $f_{\otimes} : C_U^{\otimes} \to C_V^{\otimes}$ canonically lifts to a *norm $\underline{V}$-functor*

$$f_{\otimes} : f_* f^* C_{\underline{V}}^{\otimes} = f_* C_{\underline{U}}^{\otimes} \to C_{\underline{V}}^{\otimes}.$$

We next discuss the main examples of G-symmetric monoidal ∞-categories.

Example 5.12. Suppose C is a G-∞-category that admits finite G-products (e.g., $\underline{\mathrm{Spc}}_G$). Then by applying Barwick's unfurling construction [9, section 11] to the Beck–Chevalley fibration $\widetilde{C} \to (\mathbb{F}_G)^{\mathrm{op}}$ that encodes the functoriality of restriction and coinduction, we may define the *G-cartesian* G-symmetric monoidal structure $C^{\times} \to \mathrm{Span}(\mathbb{F}_G)$ such that the norm functors are given by coinduction.

Example 5.13. The usual symmetric monoidal structures on $\{\mathrm{Sp}^H\}_{H \leqslant G}$ together with the Hill–Hopkins–Ravenel norm functors furnish a G-symmetric monoidal structure on $\underline{\mathrm{Sp}}^G$. Indeed, we may define this via restricting $\mathrm{SH}^{\otimes}$ (Definition 2.5) along the map $\omega_G : \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Span}(\mathrm{Gpd}_{\mathrm{fin}}), U \mapsto U//G$.

Example 5.14 (Pointwise G-symmetric monoidal structure). Let C be a G-symmetric monoidal ∞-category and let I be a G-∞-category. Then we have a *pointwise* G-symmetric monoidal structure on $\underline{\mathrm{Fun}}_G(I, C)$ such that for a map $f : U \to V$ of finite G-sets, the norm functor

$$f_{\otimes} : \mathrm{Fun}_{\underline{U}}(I_{\underline{U}}, C_{\underline{U}}) \to \mathrm{Fun}_{\underline{V}}(I_{\underline{V}}, C_{\underline{V}})$$

sends a $\underline{U}$-functor $[F : I_{\underline{U}} = f^* I_{\underline{V}} \to C_{\underline{U}} = f^* C_{\underline{V}}]$ to the $\underline{V}$-functor

$$f_{\otimes} F : I_{\underline{V}} \xrightarrow{\eta} f_* f^* I_{\underline{V}} \xrightarrow{f_* F} f_* f^* C_{\underline{V}} \xrightarrow{f_{\otimes}} C_{\underline{V}},$$

where η is the unit map and $f_{\otimes} : f_* f^* C_{\underline{V}} \to C_{\underline{V}}$ is the norm $\underline{V}$-functor defined by $C^{\otimes}$. We will construct this in [66] as the cotensor in G-∞-operads.[†] If we write $\underline{\mathrm{CAlg}}_G(C)$ for the G-∞-category of G-commutative algebras, we then have $\underline{\mathrm{CAlg}}_G(\underline{\mathrm{Fun}}_G(I, C)) \simeq \underline{\mathrm{Fun}}_G(I, \underline{\mathrm{CAlg}}_G(C))$.

[†] We note that this will supersede Definition A.1 in all examples of interest in this paper. However, as the theory of parameterized ∞-operads imposes strong restrictions upon the base ∞-category $\mathcal{T}$ (namely, that $T = S^{\mathrm{op}}$ is atomic orbital), Definition A.1 itself is not superseded by the work done in [66].

We end this subsection by introducing the concept of G-*distributivity*, which will play an important role in establishing the existence and uniqueness of a lax G-symmetric monoidal structure on $(-)^{t_G K}$ (Corollary 5.60).

Definition 5.15. Let $f : U \to V$ be a map of finite G-sets, let C be a $\underline{U}$-∞-category, and let D be a $\underline{V}$-∞-category. Let $F : f_* C \to D$ be a $\underline{V}$-functor. Then we say that F is $\underline{V}$-*distributive* if for every pullback square

$$\begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "g'"] & V' \arrow[d, "g"] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

of finite G-sets and $\underline{U'}$-colimit diagram $\overline{p} : K^{\trianglerighteq} \to g'^* C$, the $\underline{V'}$-functor

$$(f'_* K)^{\trianglerighteq} \xrightarrow{\text{can}} f'_*(K^{\trianglerighteq}) \xrightarrow{f'_* \overline{p}} f'_* g'^* C \simeq g^* f_* C \xrightarrow{g^* F} g^* D$$

is a $\underline{V'}$-colimit diagram.[†]

Remark 5.16. In the situation of Definition 5.15, suppose that C is $\underline{U}$-cocomplete and D is $\underline{V}$-cocomplete. In view of the pointwise formula for parameterized Kan extensions [Shah, Theorem 10.3], we deduce the following apparently stronger conclusion.

(∗) Suppose that

$$\begin{tikzcd} K \arrow[r, "p"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\Downarrow\eta"] & (g')^* C \\ L \arrow[ur, "\phi_! p"'] & \end{tikzcd}$$

is a $\underline{U'}$-left Kan extension diagram. Then

$$\begin{tikzcd} f'_* K \arrow[r, "f'_* p"] \arrow[d, "f'_* \phi"'] \arrow[dr, phantom, "\Downarrow\eta'"] & f'_* g'^* C \arrow[r, "g^* F"] & g^* D \\ f'_* L \arrow[urr, "g^* F \circ f'_*(\phi_! p)"'] & & \end{tikzcd}$$

is a $\underline{V'}$-left Kan extension diagram, where η' is given by whiskering $f'_* \eta$ by $g^* F$.

The following definition generalizes the notion of a cocomplete symmetric monoidal ∞-category in which the tensor product distributes over all colimits.

Definition 5.17. Let $C^{\otimes}$ be a G-symmetric monoidal ∞-category. Then $C^{\otimes}$ is G-*distributive* if:

[†] Given a $\underline{U'}$-∞-category K, we write $K^{\trianglerighteq}$ for the parameterized join $K \star_{\underline{U'}} \underline{U'}$ [74, Definition 4.1], and likewise $(f'_* K)^{\trianglerighteq} = (f'_* K) \star_{\underline{V'}} \underline{V'}$ (so the notation $(-)^{\trianglerighteq}$ implicitly involves the base). Using the compatibility of the parameterized join with restriction [74, Lemma 4.4], the canonical map $(f'_* K)^{\trianglerighteq} \xrightarrow{\text{can}} f'_*(K^{\trianglerighteq})$ is then defined to be the adjoint to $\epsilon^{\trianglerighteq} : (f'^* f'_* K)^{\trianglerighteq} \to K^{\trianglerighteq}$.

(1) C is G-cocomplete;
(2) for every map $f : U \to V$ of finite G-sets, the norm $\underline{V}$-functor $f_\otimes$ is $\underline{V}$-distributive (Definition 5.15).

Remark 5.18. Definition 5.15 is formulated so that the pullback of a $\underline{V}$-distributive functor along a map $g : V' \to V$ is again $\underline{V'}$-distributive. As condition (2) of Definition 5.17 is checked against *all* maps of finite G-sets, and the pullback along g of a norm $\underline{V}$-functor is the corresponding $\underline{V'}$-norm functor, this base-change condition is superfluous in formulating Definition 5.17.

Example 5.19. Nardin proved in his thesis [65] that the G-symmetric monoidal ∞-category $\underline{\mathrm{Sp}}^G$ is G-distributive. In fact, he constructs the G-symmetric monoidal structure on $\underline{\mathrm{Sp}}^G$ as the initial G-distributive G-presentable G-stable G-symmetric monoidal ∞-category along the lines of Lurie's construction of the tensor product on spectra [56, section 4.8.2]; G-distributivity is thus essential to prove the *uniqueness* of the G-symmetric monoidal structure on $\underline{\mathrm{Sp}}^G$. One may also establish G-distributivity by proving separately that the norm functors preserve sifted colimits and distribute over finite G-coproducts (cf. [7, section 5.2]).

Example 5.20. If C is a G-distributive G-symmetric monoidal ∞-category, then the pointwise G-symmetric monoidal structure on $\underline{\mathrm{Fun}}_G(I, C)$ is also G-distributive.

Example 5.21. Suppose C is a presentable and cartesian closed ∞-category, so that the cartesian symmetric monoidal structure on C is distributive. Then the G-cartesian symmetric monoidal structure on the G-∞-category $\underline{C}_G$ of G-objects in C is G-distributive.[†]

We may further unwind the meaning of G-distributivity in the case that $C = \mathrm{Cat}_\infty$. For a map of finite G-sets $f : U \to V$, write

$$f_\times : f^* f_* \underline{\mathrm{Cat}}_{\infty,\underline{V}} \to \underline{\mathrm{Cat}}_{\infty,\underline{V}}$$

for the norm $\underline{V}$-functor associated to the G-cartesian symmetric monoidal structure; over the fiber $W \xrightarrow{\pi} V$, we have that

$$f_\times : [C \to \underline{W \times_V U}] \mapsto [f'_* C \to \underline{W}],$$

where $f' : W \times_V U \to W$ is the pullback of f along π. Let $\alpha : A \to U$ be a map of finite G-sets, C an $\underline{A}$-∞-category, and $F_C : \underline{A} \to \underline{\mathrm{Cat}}_{\infty,\underline{U}}$ the $\underline{U}$-functor determined by C. Then

$$\mathrm{colim}^{\underline{U}} F_C = \alpha_! C \in \mathrm{Cat}_{\infty,\underline{U}}$$

and G-distributivity implies a formula for $f_* \alpha_! C \in \mathrm{Cat}_{\infty,\underline{V}}$. Namely, let $f^* \dashv f_*$ also denote the adjunction

$$f^* : (\mathbb{F}_G)^{/V} \rightleftarrows (\mathbb{F}_G)^{/U} : f_*$$

[†] Note that $\underline{C}_G$ is G-cocomplete and G-complete by [74, Propositions 5.5 and 5.6].

and consider the $\underline{V}$-functor

$$\underline{f_*A} \simeq f_*\underline{A} \xrightarrow{f_*(F_C)} f_*f^*\underline{\mathrm{Cat}}_{\infty,\underline{V}} \xrightarrow{f_\times} \underline{\mathrm{Cat}}_{\infty,\underline{V}}.$$

By G-distributivity, $f_*\alpha_! C \simeq \mathrm{colim}^{\underline{V}}(f_\times \circ f_*(F_C))$. Moreover, the $\underline{V}$-functor $f_*(F_C)$ is equivalent data to the $\underline{f^*f_*A}$-∞-category $\mathrm{ev}^* C$, where $\mathrm{ev} : f^*f_*A \to A$ denotes the counit of the adjunction $f^* \dashv f_*$ (which may be concretely described as evaluation). Now let

$$\mathrm{pr} : f^*f_*A = (f_*A) \times_V U \to f_*A$$

denote the projection. Under the correspondence between $\underline{f_*A}$-points in $\underline{\mathrm{Cat}}_{\infty,\underline{V}}$ and $\underline{f_*A}$-∞-categories, one then identifies $f_\times \circ f_*(F_C)$ with $\mathrm{pr}_* \,\mathrm{ev}^* C$. Finally, let $g : f_*A \to V$ denote the structure map. Summing up, given the commutative diagram

$$\begin{tikzcd} & f^*f_*A \arrow[r, "\mathrm{pr}"] \arrow[dl, "\mathrm{ev}"'] \arrow[d] & f_*A \arrow[d, "g"] \\ A \arrow[r, "\alpha"] & U \arrow[r, "f"] & V, \end{tikzcd}$$

we obtain the equivalence

$$f_*\alpha_! C \simeq g_! \,\mathrm{pr}_* \,\mathrm{ev}^* C.$$

5.2 | Parameterized Verdier quotients

In this section, we set up a parameterized generalization of the theory of Verdier quotients. We first consider the G-Verdier quotient of a G-stable G-∞-category by a full G-stable G-subcategory (Theorem 5.24), and then the G-Verdier quotient of a G-stable G-symmetric monoidal ∞-category by a G-$\otimes$-ideal (Theorem 5.29). We will use this theory to prove that the parameterized Tate construction uniquely admits a lax equivariant symmetric monoidal structure (cf. Proposition 5.54).

For the definition of G-Verdier quotient, recall from [42] that given a cocartesian fibration $\pi : C \to S$ and a class of fiberwise edges $\mathscr{W} \subset C$ closed under cocartesian pushforward, one may form the Dwyer–Kan localization $\pi^W : C[\mathscr{W}^{-1}] \to S$ as the fibrant replacement of the marked simplicial set $(C, \mathscr{W})$ in the cocartesian model structure on $s\mathrm{Set}^+_{/S}$. For every $s \in S$, one then has $(C[\mathscr{W}^{-1}])_s \simeq C_s[\mathscr{W}_s^{-1}]$; more generally, for every ∞-category $K \to S$, if we let $\mathscr{W}_K \subset C_K = K \times_S C$ denote the restriction of $\mathscr{W}$, then $C_K[\mathscr{W}_K^{-1}] \simeq K \times_S C[\mathscr{W}^{-1}]$.

Definition 5.22. Let C be a G-stable G-∞-category and $D \subset C$ a full G-stable G-subcategory. We define the *G-Verdier quotient* C/D to be the Dwyer–Kan localization $C[\mathscr{W}^{-1}]$, where $\mathscr{W} \subset C$ is the class of fiberwise edges given over an orbit V by those edges $x \to y$ with cofiber in D_V.

Remark 5.23. Suppose C is a small G-∞-category that admits all finite G-colimits. Then the fiberwise Ind-completion $\underline{\mathrm{Ind}}_G(C)$ is fiberwise presentable, G-cocomplete and G-complete,[†] and the

[†] Given a G-∞-category E that is fiberwise presentable, to check that E admits all small G-limits and G-colimits it suffices to verify that all restriction functors $f^* : E_V \to E_U$ admit left and right adjoints that satisfy the Beck–Chevalley conditions.

fiberwise Yoneda embedding $j^\omega_G : C \to \underline{\mathrm{Ind}}_G(C)$ preserves finite G-colimits. Moreover, by [75, Theorem 9.11] $\underline{\mathrm{Ind}}_G(C)$ identifies with the full G-subcategory

$$i : \underline{\mathrm{Fun}}^{\mathrm{lex}}_G(C^{\mathrm{vop}}, \underline{\mathrm{Spc}}_G) \subset \underline{\mathrm{P}}_G(C)$$

whose fiber over an orbit $V \cong G/H$ is spanned by those H-presheaves that strongly preserve H-finite H-limits, and the G-Yoneda embedding $j_G : C \to \underline{\mathrm{P}}_G(C)$ factors as

$$C \xhookrightarrow{j^\omega_G} \underline{\mathbf{Ind}}_G(C) \xhookrightarrow{i} \underline{\mathbf{P}}_G(C).$$

If we further suppose that C is G-stable, then as j^ω_G also preserves all finite G-limits (as j_G does and i is a right G-adjoint), it follows that $\underline{\mathrm{Ind}}_G(C)$ is G-stable and j^ω_G is G-exact.

We have the following parameterized generalization of [67, Theorem I.3.3].

Theorem 5.24. *Let C be a small G-stable G-∞-category and $D \subset C$ a full G-stable G-subcategory.*

(1) *The G-Verdier quotient C/D is G-stable and the quotient G-functor $p : C \to C/D$ is G-exact. Moreover, for any G-stable G-∞-category E, the restriction functor*

$$p^* : \mathrm{Fun}^{\mathrm{ex}}_G(C/D, E) \to \mathrm{Fun}^{\mathrm{ex}}_G(C, E)$$

is fully faithful with essential image spanned by those G-exact G-functors $F : C \to E$ such that $F|_D \simeq 0_G$. In particular, passing to mapping spaces we see that C/D is the cofiber of the inclusion $D \subset C$ taken in the ∞-category of G-stable G-∞-categories.

(2) *The restricted G-Yoneda G-functor $C/D \to \underline{\mathrm{P}}_G(C)$ factors through a G-exact G-functor*

$$r : C/D \to \underline{\mathrm{Ind}}_G(C)$$

given fiberwise by the formula of [67, Theorem I.3.3(ii)].

(3) *The quotient G-functor $p : C \to C/D$ prolongs to a G-functor*

$$p_! : \underline{\mathrm{Ind}}_G(C) \to \underline{\mathrm{Ind}}_G(C/D)$$

that preserves all G-colimits and admits a fully faithful right G-adjoint p^* given by restriction along p, or equivalently as the prolongation of r. Moreover, p^* also preserves all G-colimits and admits a right G-adjoint.

(4) *Let E be a G-stable G-cocomplete G-∞-category. The restriction functor*

$$p^* : \mathrm{Fun}^{\mathrm{ex}}_G(C/D, E) \to \mathrm{Fun}^{\mathrm{ex}}_G(C, E)$$

then admits a left adjoint L, which sends a G-exact G-functor $F : C \to E$ to the composite

$$C/D \xrightarrow{r} \underline{\mathbf{Ind}}_G(C) \xrightarrow{\underline{\mathbf{Ind}}_G(F)} \underline{\mathbf{Ind}}_G(E) \xrightarrow{\mathrm{colim}^G} E.$$

Moreover, this adjunction globalizes to a G-adjunction

$$L : \underline{\mathrm{Fun}}^{\mathrm{ex}}_G(C, E) \rightleftarrows \underline{\mathrm{Fun}}^{\mathrm{ex}}_G(C/D, E) : p^*$$

Proof.

(1) As $(C/D)_V \simeq C_V/D_V$, we already know that C/D is fiberwise stable and p is fiberwise exact. To see that C/D is moreover G-stable and p is G-exact, observe that as the inclusion G-functor $D \subset C$ is G-exact by assumption, it follows by the universal property of the Verdier quotient that the induction and coinduction functors $f_!$ and f_* descend to C/D (so then intertwine with p). Moreover, in view of the fiberwise surjectivity of p, the functors $\{f_!, f_*\}_{f \in \mathrm{Ar}(\mathbb{F}_G)}$ continue to satisfy the Beck–Chevalley condition for pullback squares in $\mathbb{F}_G$, and the G-semiadditivity equivalence $f_! \simeq f_*$ in C/D is implied by that in C.

For the second assertion, by the universal property of the Dwyer–Kan localization we already know that

$$p^* : \mathrm{Fun}_G^{\text{fib-ex}}(C/D, E) \to \mathrm{Fun}_G^{\text{fib-ex}}(C, E)$$

is fully faithful with essential image spanned by those fiberwise exact G-functors $F : C \to E$ such that $F|_D \simeq 0_G$. As we proved that p is G-exact, it follows that after passage to subcategories of G-exact functors on both sides, p^* remains fully faithful with essential image as described - note that if $F' : C/D \to E$ is a G-functor such that $F' \circ p$ is G-exact, then it follows from the fiberwise surjectivity of p that F' is G-exact.

(2) Note first that $\underline{\mathrm{Ind}}_G(C)$ is G-stable by Remark 5.23. In addition, under the identification

$$\underline{\mathrm{Ind}}_G(C) \simeq \underline{\mathrm{Fun}}_G^{\mathrm{lex}}(C^{\mathrm{vop}}, \underline{\mathrm{Spc}}_G)$$

of Remark 5.23 (and ditto for C/D), the G-functor

$$p^* : \mathrm{P}_G(C/D) \to \mathrm{P}_G(C) \quad \text{restricts to} \quad p^* : \underline{\mathrm{Ind}}_G(C/D) \to \underline{\mathrm{Ind}}_G(C).$$

The assertion then follows from [67, Theorem I.3.3(ii)].

(3) In view of the universal property of $\underline{\mathrm{Ind}}_G$ [75, Theorem 9.11], we have a G-adjunction

$$p_! : \underline{\mathbf{Ind}}_G(C) \rightleftarrows \underline{\mathbf{Ind}}_G(C/D) : p^*$$

that fiberwise restricts to the adjunction of [67, Proposition I.3.5]. It follows that p^* is fully faithful from the fiberwise assertion. Moreover, as the inclusion $\underline{\mathrm{Ind}}_G(C/D) \subset \underline{\mathrm{P}}_G(C)$ preserves filtered G-colimits [ref] and restriction between functor G-categories preserves all G-colimits, p^* preserves all filtered G-colimits. As p^* is also G-exact, it thus preserves all G-colimits. As $\underline{\mathrm{Ind}}_G(-)$ is in addition fiberwise presentable, we then see that p^* admits a right G-adjoint.

(4) By part (3), we have an adjunction

$$(p^*)^* : \mathrm{Fun}_G^L(\underline{\mathbf{Ind}}_G(C), E) \rightleftarrows \mathrm{Fun}_G^L(\underline{\mathbf{Ind}}_G(C/D), E) : (p_!)^*$$

in which $(p_!)^*$ is fully faithful. By the universal property of $\underline{\mathrm{Ind}}_G$, the restriction functor

$$(j_G^\omega)^* : \mathrm{Fun}_G^L(\underline{\mathrm{Ind}}_G(C), E) \xrightarrow{\simeq} \mathrm{Fun}_G^{\mathrm{ex}}(C, E)$$

is an equivalence, and ditto for C/D. Under these equivalences, p^* then corresponds to $(p_!)^*$ and $(p^*)^*$ yields the localization functor L with the indicated formula. Finally, both assignments L and p^* are clearly compatible with base-change, hence the adjunction globalizes to a G-adjunction. $\square$

Next, given a G-symmetric monoidal structure on C, we would like to descend that structure to the G-Verdier quotient C/D. To do so, we need to suppose in addition that $D \subset C$ is a *G-$\otimes$-ideal* in the sense of the following definition.

Definition 5.25. Let C be a G-stable G-symmetric monoidal ∞-category and let $D \subset C$ be a G-stable G-subcategory. We say that D is a *G-$\otimes$-ideal* if

(1) for every orbit V, $D_V \subset C_V$ is a $\otimes$-ideal;
(2) for every map $f : U \to V$ between orbits,[†] if $x \in D_U$, then $f_\otimes x \in D_V$.

Remark 5.26. In Definition 5.25, conditions (1) and (2) may be combined as follows.

($*$) Suppose V is an orbit, U is a finite G-set with orbit decomposition $U \simeq \coprod_{i=1}^n U_i$, and $f : U \to V$ is a map. Then D is a G-$\otimes$-ideal if and only if for all n-tuples $x = (x_1, \dots, x_n) \in C_U \simeq \prod_{i=1}^n C_{U_i}$ such that $x_i \in D_{U_i}$ for some $1 \leqslant i \leqslant n$, we have that $f_\otimes x \in D_V$.

Example 5.27. Let $\mathcal{F}$ be a G-family. Then we claim that the G-stable G-subcategory $\underline{\mathrm{Sp}}^{\Phi\mathcal{F}} \subset \underline{\mathrm{Sp}}^G$ of Remark 2.23 is a G-$\otimes$-ideal. Indeed, as this is a fiberwise $\otimes$-ideal, it suffices to show stability under norms for any map of orbits $f : U \to V$. Without loss of generality, suppose $U = G/H$ and $V = G/G$. Given $X \in \mathrm{Sp}^H$, we need to show that if $\Phi^K(X) = 0$ for all $K \in \mathcal{F}^H$, then $\Phi^L(N_H^G X) = 0$ for all $L \in \mathcal{F}$. But using that $N_H^G(E\mathcal{F}_+) \simeq \Sigma^\infty_+(\mathrm{Coind}_H^G E\mathcal{F})$ where Coind_H^G denotes right Kan extension along $\mathcal{O}_H^{\mathrm{op}} \simeq (\mathcal{O}_G^{\mathrm{op}})^{G/H} \to \mathcal{O}_G^{\mathrm{op}}$, this follows from evaluating $(\mathrm{Coind}_H^G E\mathcal{F})^L = *$ using the pointwise formula for right Kan extension.

On the other hand, note that $\underline{\mathrm{Sp}}^{\tau\mathcal{F}} \subset \underline{\mathrm{Sp}}^G$ is generally not a G-$\otimes$-ideal. Indeed, suppose $\mathcal{F}$ is a proper nonempty G-family (and G is nontrivial). Then $\mathrm{Sp}^{\tau\mathcal{F}^e} = \mathrm{Sp}$, but $N^G : \mathrm{Sp} \to \mathrm{Sp}^G$ is split by Φ^G and hence doesn't send $\mathrm{Sp}^{\tau\mathcal{F}^e}$ into $\mathrm{Sp}^{\tau\mathcal{F}}$.

To then understand the interaction of G-Verdier quotients with G-symmetric monoidal structures, we will need the following lemma, a G-version of [67, Proposition A.5].

Lemma 5.28. *Let C be a G-symmetric monoidal ∞-category. Suppose given classes of edges $\{\mathcal{W}_V \subset C_V\}_{V \in \mathcal{O}_G}$ with each $\mathcal{W}_V$ containing all the equivalences in C_V and subject to the following condition.*

($*$) *For every map of finite G-sets $f : U \to V$ with orbit decomposition $U \simeq \coprod_{i=1}^n U_i$ and edges $\{\alpha_i \in (\mathcal{W}_{U_i})_1\}_{1 \leqslant i \leqslant n}$, we have that $f_\otimes(\alpha_1, \dots, \alpha_n) \in \mathcal{W}_V$.*[‡]

Define $\mathcal{W}^\otimes \subset C^\otimes$ to be the class of those fiberwise edges given over a finite G-set U with orbit decomposition $U \simeq \coprod_{i=1}^n U_i$ by

$$\mathcal{W}_U^\otimes := \mathcal{W}_{U_1} \times \dots \times \mathcal{W}_{U_n}.$$

[†] Recall our convention that orbits are nonempty, so we are not supposing that $1 \in D_V$.

[‡] To obtain the Dwyer–Kan localization $C^\otimes[(\mathcal{W}^\otimes)^{-1}]$, it would suffice to require the apparently weaker condition in which we suppose that all but one of the α_i are identity morphisms.

Then the Dwyer–Kan localization $C^{\otimes}[(\mathscr{W}^{\otimes})^{-1}]$ exists and comes equipped with a map to $\mathrm{Span}(\mathbb{F}_G)$ *that renders it a G-symmetric monoidal ∞-category. Moreover, $C^{\otimes}[(\mathscr{W}^{\otimes})^{-1}]$ has the following properties.*

(1) *The underlying G-∞-category of $C^{\otimes}[(\mathscr{W}^{\otimes})^{-1}]$ is equivalent to $C[\mathscr{W}^{-1}]$,[†] and its fiber over an orbit V is given by $C_V[\mathscr{W}_V^{-1}]$. More generally, the procedure of Dwyer–Kan localization is stable with respect to pullback in the base.*
(2) *The localization functor $L^{\otimes} : C^{\otimes} \to C^{\otimes}[(\mathscr{W}^{\otimes})^{-1}]$ is a G-symmetric monoidal functor and restricts to the localization functor $L : C \to C[\mathscr{W}^{-1}]$ over $\mathcal{O}_G^{\mathrm{op}}$. More generally, $L^{\otimes}$ intertwines with pullback in the base.*
(3) *For every G-symmetric monoidal ∞-category D, the pullback along the functor $L^{\otimes}$ induces fully faithful functors*

$$\mathrm{Fun}_G^{\otimes}(C[\mathscr{W}^{-1}], D) \to \mathrm{Fun}_G^{\otimes}(C, D), \quad \mathrm{Fun}_G^{\mathrm{lax}}(C[\mathscr{W}^{-1}], D) \to \mathrm{Fun}_G^{\mathrm{lax}}(C, D)$$

with essential image spanned by those (lax) G-symmetric monoidal functors $F : C \to D$ that send $\mathscr{W}$ to equivalences.

Proof. We may proceed exactly as in [67, Proposition A.5] once we observe that Hinich's work on Dwyer–Kan localization [42] applies to the general context of working over a base ∞-category with a subclass of "inert" edges that distinguishes "strong" from "lax" morphisms; we apply his theory here to $(\mathrm{Span}(\mathbb{F}_G), \mathbb{F}_G^{\mathrm{op}})$. □

We now can prove the parameterized analogue of [67, Theorem I.3.6].

Theorem 5.29. *Let C be a small G-stable G-symmetric monoidal ∞-category and $D \subset C$ a G-⊗-ideal.*

(1) *The G-Verdier quotient C/D uniquely inherits a G-symmetric monoidal structure from C such that the projection G-functor $p : C \to C/D$ is G-symmetric monoidal. Moreover, for any G-symmetric monoidal ∞-category E, the restriction functor*

$$p^* : \mathrm{Fun}_G^{\mathrm{ex},(\otimes\text{ or lax})}(C/D, E) \to \mathrm{Fun}_G^{\mathrm{ex},(\otimes\text{ or lax})}(C, E)$$

is fully faithful with essential image spanned by those (lax) G-symmetric monoidal G-exact G-functors $F : C \to E$ such that $F|_D \simeq 0_G$.
(2) *Let E be a G-stable G-distributive G-symmetric monoidal ∞-category. Then*

$$p^* : \mathrm{Fun}_G^{\mathrm{ex,lax}}(C/D, E) \to \mathrm{Fun}_G^{\mathrm{ex,lax}}(C, E)$$

admits a left adjoint L that sends a G-exact lax G-symmetric monoidal functor $F : C \to E$ to the composite

$$C/D \xrightarrow{r} \underline{\mathbf{Ind}}_G(C) \xrightarrow{\underline{\mathbf{Ind}}_G(F)} \underline{\mathbf{Ind}}_G(E) \xrightarrow{\mathrm{colim}^G} E$$

as in Theorem 5.24(4). Moreover, the adjunction $L \dashv p^*$ globalizes to a G-adjunction.

[†] Here, $\mathscr{W} \subset C$ denotes the class of those fiberwise edges given over an orbit V by $\mathscr{W}_V$.

Proof.

(1) For an orbit V, let $\mathscr{W}_V$ be the class of edges $\{\alpha : x \to y,\ \mathrm{cof}(\alpha) \in D_V\}$ in C_V. Under our assumption that D is a G-$\otimes$-ideal, Lemma 5.28 then applies to prove all of the claims.

(2) We first show that all the G-functors in the composite are lax G-symmetric monoidal. In [66], the second author showed that for a lax G-symmetric monoidal functor $\phi : I \to J$, the restriction functor

$$\phi^* : \mathrm{Fun}_G^{\mathrm{lax}}(J, E) \to \mathrm{Fun}_G^{\mathrm{lax}}(I, E)$$

admits a left adjoint $\phi_!$ given by G-operadic left Kan extension, such that $\phi_!$ is computed on underlying G-functors as left G-Kan extension. He also showed that $\underline{\mathrm{P}}_G(C)$ is a G-distributive G-symmetric monoidal ∞-category with respect to G-Day convolution and the G-Yoneda embedding $j : C \to \underline{\mathrm{P}}_G(C)$ is a G-symmetric monoidal functor. Using G-distributivity it then follows that $\underline{\mathrm{Ind}}_G(C) \subset \underline{\mathrm{P}}_G(C)$ is a G-symmetric monoidal subcategory and j_G^ω is also G-symmetric monoidal. We thus see that:

(i) $r = p^* \circ j_G^\omega$ is lax G-symmetric monoidal. Here we use that with respect to G-Day convolution, precomposition along a (strong or lax) G-symmetric monoidal functor is lax G-symmetric monoidal. Alternatively, as in the proof of [67, Theorem I.3.6] we could observe that the prolongation of the G-symmetric monoidal functor p to $p_! : \underline{\mathrm{Ind}}_G(C) \to \underline{\mathrm{Ind}}_G(C/D)$ is actually strong G-symmetric monoidal in view of G-distributivity, and hence its G-right adjoint p^* is lax G-symmetric monoidal.

(ii) $\overline{F} = \mathrm{colim}^G \circ \underline{\mathrm{Ind}}_G(F)$, which is the left G-Kan extension of F along j_G^ω, canonically inherits the structure of a lax G-symmetric monoidal functor.

The functor $L : \mathrm{Fun}_G^{\mathrm{ex,lax}}(C, E) \to \mathrm{Fun}_G^{\mathrm{ex,lax}}(C/D, E)$ is thus well-defined. Moreover, we may define the unit transformation $\eta : \mathrm{id} \to p^* L$ to be that given objectwise by the map

$$F = \overline{F} \circ j_G^\omega \to \overline{F} \circ r \circ p = \overline{F} \circ p^* \circ j_G^\omega \circ p \simeq \overline{F} \circ p^* p_! \circ j_G^\omega$$

induced by the unit of the adjunction

$$p_! : \underline{\mathbf{Ind}}_G(C) \rightleftarrows \underline{\mathbf{Ind}}_G(C/D) : p^*,$$

where as noted above $p_!$ is strong G-symmetric monoidal and p^* is lax G-symmetric monoidal. As η by definition covers the unit transformation of the adjunction in Theorem 5.24(4), we see that $L \dashv p^*$ (for the other labeled p^*). Finally, the globalization assertion follows from similar ones established for all the functors used to construct L. $\square$

Remark 5.30. Roughly speaking, Definition 5.25 may be viewed as a categorification of the Tambara ideals introduced by Nakaoka in [63, Definition 2.1]. Theorem 5.29 can then be viewed as a categorification of the fact that quotients by Tambara ideals are Tambara functors [63, Proposition 2.6].

Example 5.31. Applying Theorem 5.29 to the G-$\otimes$-ideal $\underline{\mathrm{Sp}}^{\Phi\mathcal{F}} \subset \underline{\mathrm{Sp}}^G$ we get an induced G-symmetric monoidal structure on $\underline{\mathrm{Sp}}^{h\mathcal{F}}$.[†] Consequently, in view of Remark 5.9 given a G-commutative algebra A in $\underline{\mathrm{Sp}}^G$, the unit map $A \to F(E\mathcal{F}_+, A)$ canonically refines to a morphism of G-commutative algebras.

[†] We may get around the smallness assumption in Theorem 5.29 by using that the inclusion is fiberwise accessible.

5.3 | Digression on the Segal conjecture

As an application of Theorem 5.29, we upgrade the generalized Segal conjecture to a statement about incomplete Tambara functors. Recall that the Segal conjecture (cf. [18]) states that the map

$$(\pi_G^*\mathbb{S})_I^\wedge \to \pi^*(\Sigma_+^\infty BG)$$

between the G-equivariant stable cohomotopy groups of a point completed at the augmentation ideal of the Burnside ring and the stable cohomotopy groups of BG is an isomorphism. Adams, Haeberly, Jackowski, and May extended the Segal conjecture to families in [2].

Definition 5.32 [2, Introduction]. Let G be a finite group, let $\mathcal{F}$ be a class of subgroups of G, and let C be a category. A functor $h : \mathrm{Spc}_G \to C$ is $\mathcal{F}$*-invariant* if it carries $\mathcal{F}$-equivalences to isomorphisms in C.

Remark 5.33 [2, Remark 1.2]. If $\mathcal{F}$ is a family of subgroups of G, a functor $h : \mathrm{Spc}_G \to C$ is $\mathcal{F}$-invariant if and only if the map $h(X) \to h(E\mathcal{F} \times X)$ induced by the projection $E\mathcal{F} \times X \to X$ is an isomorphism for each X.

Theorem 5.34 (Generalized Segal conjecture, [2, Theorem 1.5 and Corollary 1.6]). *For any family $\mathcal{F}$ the pro-group valued functor $\pi_G^*(-)_{I(\mathcal{F})}^\wedge$, given by equivariant cohomotopy completed at*

$$I(\mathcal{F}) := \bigcap_{H\in\mathcal{F}} \ker(A(G) \to A(H)),$$

is $\mathcal{F}$-invariant. In particular, there is a pro-isomorphism

$$\pi_G^*(X)_{I(\mathcal{F})}^\wedge \xrightarrow{\cong} \pi_G^*(E\mathcal{F} \times X) \tag{5.1}$$

natural in the G-space X.

Taking $X = *$ to be a point and $\mathcal{F} = \Gamma_N$ to be the N-free G-family, we have a pro-isomorphism

$$\pi_G^*(\mathbb{S})_{I(\Gamma_N)}^\wedge \xrightarrow{\cong} \pi_G^*(E\Gamma_N).$$

The source can be identified with the $I(\Gamma_N)$-completion of $\pi_*(\mathbb{S}^G)$ by Spanier–Whitehead duality, while the point-set model for parameterized homotopy fixed points (Remark 4.32) gives

$$\pi_G^*(E\Gamma_N) \cong \pi_*^G F(E\Gamma_{N+}, \mathbb{S}) \cong \pi_* \mathbb{S}^{h_{G/N}N}.$$

Therefore, the map (5.1) can be obtained by applying $\pi_*^G(-)$ and completing the source of the vertical map in the commutative diagram

$$\begin{tikzcd}
E\Gamma_{N+} \arrow[r] \arrow[d, "\simeq"] & \mathbb{S} \arrow[r] \arrow[d, "\alpha"] & \widetilde{E\Gamma_N} \arrow[d, "\tilde{\alpha}"] \\
F(E\Gamma_{N+}, \mathbb{S}) \otimes E\Gamma_{N+} \arrow[r] & F(E\Gamma_{N+}, \mathbb{S}) \arrow[r] & F(E\Gamma_{N+}, \mathbb{S}) \otimes \widetilde{E\Gamma_N}.
\end{tikzcd}$$

The map $\alpha : \mathbb{S} \to F(E\Gamma_{N+}, \mathbb{S})$ is a map of G-commutative algebras (Example 5.31) and π_0 of a G-commutative algebra is a Tambara functor [14], so we obtain the following multiplicative refinement of the generalized Segal conjecture:

Theorem 5.35 (Multiplicative generalized Segal conjecture). *The functor $\pi^*_{(-)}(-)_{\underline{I(\Gamma_N)}}$ valued in pro-Tambara functors, given by equivariant cohomotopy completed at the Tambara ideal $\underline{I}$ defined by*

$$\underline{I}(G/K) := \bigcap_{H \in i_K^* \Gamma_N} \ker(A(K) \to A(H)),$$

is Γ_N-invariant. In particular, the pro-isomorphism of pro-groups (5.1) refines to a pro-isomorphism of pro-Tambara functors

$$\pi^0_{(-)}(X)^\wedge_{\underline{I}} \xrightarrow{\cong} \pi^0_{(-)}(E\Gamma_N \times X)$$

natural in the G-space X.

Example 5.36. Taking $G = D_4$ and $N = \mu_2$ recovers the C_2-equivariant analogue of Lin's Theorem [68, Theorem A.22] proven using the C_2-equivariant Adams spectral sequence, that is, the map

$$\mathbb{S} \to \mathbb{S}^{t_{C_2}\mu_2}$$

is a 2-adic equivalence of C_2-spectra. More generally, taking $G = D_{2p}$ and $N = \mu_p$ yields a C_2-equivariant generalization of Gunawardena's Theorem [1], that is, the map

$$\mathbb{S} \to \mathbb{S}^{t_{C_2}\mu_p}$$

is a p-adic equivalence of genuine C_2-spectra.

Remark 5.37. The Tate diagram implies that the Segal conjecture for $N = G = C_p$ is equivalent to showing that the map

$$\pi_*(\mathbb{S}) \to \pi_*(\mathbb{S}^{tC_p})$$

is an isomorphism after I-completion of the source, where $I = \ker(A(C_p) \to \mathbb{Z})$ is the augmentation ideal. Elementary commutative algebra shows that I-completion coincides with p-completion, so the Segal conjecture for $N = G = C_p$ is equivalent to showing that the map

$$\mathbb{S} \to \mathbb{S}^{tC_p}$$

is a p-adic equivalence.†

More generally, however, the Segal conjecture is *not* equivalent to showing that the map $\mathbb{S} \to \mathbb{S}^{tG}$ is a p-adic equivalence, even if G is a p-group. For instance,

$$X^{tC_{p^2}} \simeq X^{tC_p hC_p} \simeq (\mathbb{S}^\wedge_p)^{hC_p} \not\simeq \mathbb{S}^\wedge_p.$$

† The latter statement is occasionally referred to as the Segal conjecture for C_p.

5.4 | Induced objects

Let S_0 be a space. One may define the full subcategory of *induced objects* $\mathrm{Fun}(S_0, \mathrm{Sp})_{\mathrm{ind}}$ in $\mathrm{Fun}(S_0, \mathrm{Sp})$ to be the thick subcategory[†] generated by the set $\{s_! E : E \in \mathrm{Sp}\}_{s \in S_0}$. It then follows from the projection formula $s_!(E) \otimes F \simeq s_!(E \otimes s^* F)$ that $\mathrm{Fun}(S_0, \mathrm{Sp})_{\mathrm{ind}}$ is a thick $\otimes$-ideal (cf. [67, Lemma I.3.8(ii)]). Coupled with the multiplicative theory of the Verdier quotient [67, Theorem I.3.6] and vanishing of the Tate construction on induced objects [67, Lemma I.3.8(i)], we then deduce that the Tate construction uniquely admits a lax symmetric monoidal structure [67, Theorem I.3.1].

More generally, let S be a G-space. To ultimately apply Theorem 5.29 in the context of the parameterized Tate construction, we now want a full G-stable G-subcategory

$$\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)_{\mathrm{ind}} \subset \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$$

of induced objects which is a (fiberwise thick) G-$\otimes$-ideal in the sense of Definition 5.25. Because such a full G-subcategory is necessarily closed under restriction, induction, and norms for the pointwise G-symmetric monoidal structure on $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$, the correct formulation of "induced objects" becomes considerably more involved in the parameterized setting. We begin by specifying the set of induced generators for every fiber and then show that this yields first a G-stable G-subcategory (Definition 5.43) and subsequently a G-$\otimes$-ideal (Corollary 5.47).

Definition 5.38. Let S be a G-space. Consider maps of finite G-sets

$$U \xrightarrow{f} V \xrightarrow{p} W$$

where W is an orbit. Let $s \in S_U$ be a point[‡] and also write $s : \underline{U} \to S_{\underline{U}}$ for the unique $\underline{U}$-functor that selects $s \in S_U$. Let $s_f : \underline{V} \to f_* f^* S_{\underline{V}}$ be the $\underline{V}$-functor adjoint to s.[§] Let $\eta_f : S_{\underline{V}} \to f_* f^* S_{\underline{V}}$ denote the unit $\underline{V}$-functor and define the $\underline{V}$-∞-category $\mathrm{fib}_s^f(\eta)$ to be the pullback

$$\begin{tikzcd}
\mathrm{fib}_s^f(\eta) \arrow[r, "\varphi_s^f"] \arrow[d, "\pi_s^f"'] & S_{\underline{V}} \arrow[d, "\eta_f"] \\
\underline{V} \arrow[r, "s_f"] & f_* f^* S_{\underline{V}}.
\end{tikzcd}$$

Now let $\epsilon_p : p_! p^* S_{\underline{W}} \to S_{\underline{W}}$ denote the counit and consider the composite of $\underline{W}$-functors

$$p_! \,\mathrm{fib}_s^f(\eta) \xrightarrow{p_! \varphi_s^f} p_! p^* S_{\underline{W}} \xrightarrow{\epsilon_p} S_{\underline{W}}.$$

We let

$$\mathcal{I}_W(f, p, s) := \{(\epsilon_p \circ p_! \varphi_s^f)_!(X) \,:\, X \in \mathrm{Fun}_{\underline{W}}(\mathrm{fib}_s^f(\eta), (\underline{\mathrm{Sp}}^G)_{\underline{W}})\} \subset \mathrm{Fun}_{\underline{W}}(S_{\underline{W}}, (\underline{\mathrm{Sp}}^G)_{\underline{W}})$$

[†] One could also take the minimal stable subcategory as in [67, Definition I.3.7].

[‡] If U has orbit decomposition $\coprod_{i \in I} U_i$, then $S_U = \prod_{i \in I} S_{U_i}$ and s is given as a tuple (s_i).

[§] s_f is also given by $f_*(s)$ because f_* preserves terminal objects.

where $(\epsilon_p \circ p_! \varphi^f_s)_!$ denotes $\underline{W}$-left Kan extension. Now let $\mathcal{I}_W := \bigcup_{f,p,s} \mathcal{I}_W(f,p,s)$ and define the full subcategory of *induced $\underline{W}$-objects*

$$\underline{\mathrm{Fun}}_{\underline{W}}(S_{\underline{W}}, (\underline{\mathrm{Sp}}^G)_{\underline{W}})_{\mathrm{ind}} \subset \underline{\mathrm{Fun}}_{\underline{W}}(S_{\underline{W}}, (\underline{\mathrm{Sp}}^G)_{\underline{W}})$$

to be the thick subcategory generated by the set $\mathcal{I}_W$.

Remark 5.39. In terms of cocartesian fibrations, $p_!$ is implemented by postcomposition with the structure map $p : \underline{V} \to \underline{W}$, after which the counit $\epsilon_p : S_{\underline{V}} \simeq \underline{V} \times_{\underline{W}} S_{\underline{W}} \to S_{\underline{W}}$ identifies with the projection to $S_{\underline{W}}$.

To show that the induced $\underline{W}$-objects assemble to define a full G-stable G-subcategory of induced objects, we first show that the induced generators are stable under restriction and induction.

Lemma 5.40. *The collection $\{\mathcal{I}_W\}_{W \in \mathcal{O}_G}$ of induced generators is stable under restriction and induction (along maps of orbits) in $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$.*

Proof. We first show closure under restriction. Suppose

$$\begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "g''"] & V' \arrow[r, "p'"] \arrow[d, "g'"] & W' \arrow[d, "g"] \\ U \arrow[r, "f"] & V \arrow[r, "p"] & W \end{tikzcd}$$

is a commutative diagram of pullback squares of finite G-sets in which W and W' are orbits. In the setup of Definition 5.38, consider the commutative diagram of $\underline{W}$-∞-categories

$$\begin{tikzcd} p_! \, \mathrm{fib}^f_s(\eta) \arrow[r, "p_!\varphi^f_s"] \arrow[d, "p_!\pi^f_s"'] & p_! S_{\underline{V}} \arrow[r, "\epsilon_p"] \arrow[d, "p_!\eta_f"] & S_{\underline{W}} \\ \underline{V} \arrow[r, "p_! s_f"] & p_! f_* f^* S_{\underline{V}}. & \end{tikzcd}$$

Let $s' = g''^*(s)$. Applying g^* then yields the commutative diagram

$$\begin{tikzcd} p'_! \, \mathrm{fib}^{f'}_{s'}(\eta) \arrow[r, "p'_!\varphi^{f'}_{s'}"] \arrow[d, "p'_!\pi^{f'}_{s'}"'] & p'_! S_{\underline{V'}} \arrow[r, "\epsilon_{p'}"] \arrow[d, "p'_!\eta_{f'}"] & S_{\underline{W'}} \\ \underline{V'} \arrow[r, "p'_! s'_{f'}"] & p'_! f'_* f'^* S_{\underline{V'}} & \end{tikzcd}$$

in view of Remark 5.5. Using the built-in compatibility of parameterized left Kan extension with restriction, we thus see that $g^*(\mathcal{I}_W(f,p,s)) \subset \mathcal{I}_{W'}(f',p',s')$.

As for closure under induction, suppose that $h : W \to W''$ is a map of orbits. Then by Remark 5.41, we have that $h_!(\mathcal{I}_W(f,p,s)) \subset \mathcal{I}_{W''}(f, h\circ p, s)$. $\square$

Remark 5.41. Suppose C is a G-∞-category that admits finite G-coproducts, $p : V \to W$ is a map of orbits, and $F : K \to C_{\underline{V}}$ is a $\underline{V}$-functor. Let $F' = \epsilon_p \circ p_! F : p_! K \to C_{\underline{W}}$ be the $\underline{W}$-functor adjoint to F. Then we have a canonical equivalence

$$\mathrm{colim}^{\underline{W}} F' \simeq p_!(\mathrm{colim}^{\underline{V}} F) \in C_W$$

provided that the parameterized colimits exist. Indeed, this follows from the commutative diagram

$$\begin{tikzcd} C_W \arrow[r, "p^*"] \arrow[d, "\delta"] & C_V \arrow[d, "\delta"] \\ \mathrm{Fun}_{\underline{W}}(p_!K, C_{\underline{W}}) \arrow[r, "\simeq"] & \mathrm{Fun}_{\underline{V}}(K, C_{\underline{V}}). \end{tikzcd}$$

More generally, let L be a $\underline{W}$-∞-category. Then $\mathrm{Fun}_{\underline{W}}(L, C_{\underline{W}})$ admits finite $\underline{W}$-coproducts; write

$$p_! : \mathrm{Fun}_{\underline{V}}(L_{\underline{V}}, C_{\underline{V}}) \rightleftarrows \mathrm{Fun}_{\underline{W}}(L, C_{\underline{W}}) : p^*$$

for the resulting adjunction. Let $\varphi : K \to L_{\underline{V}}$ be a $\underline{V}$-functor and write $\varphi' : p_!K \to L$ for the $\underline{W}$-functor adjoint to φ. Suppose that $\varphi_! F$ exists. Then we have a canonical equivalence

$$\varphi'_! F' \simeq p_!(\varphi_! F) : L \to C_{\underline{W}}.$$

The next lemma highlights the relevance of Lemma 5.40.

Lemma 5.42. *Let C be a G-stable G-∞-category and suppose $\{\mathcal{J}_W \subset C_W\}_{W \in \mathcal{O}_G}$ is a collection of objects stable under restriction and induction. Let $D_W \subset C_W$ be the thick subcategory generated by the set $\mathcal{J}_W$. Then for every map of orbits $f : V \to W$, the adjunction*

$$f_! : C_V \rightleftarrows C_W : f^*$$

restricts to an adjunction between D_V and D_W. Consequently, the collection $\{D_W\}_{W \in \mathcal{O}_G}$ assembles to define a G-stable G-subcategory $D \subset C$.

Proof. It suffices to check that the collection $\{D_W\}_{W \in \mathcal{O}_G}$ is stable under restriction and induction; closure under restriction ensures that $\{D_W\}_{W \in \mathcal{O}_G}$ assembles to a G-subcategory D, and closure under induction together with the assumption that C is G-stable then ensures that D is G-stable and the inclusion $D \subset C$ is G-exact. So, let $f : V \to W$ be a map of orbits. Let $D^0_W \subset D_W$ be the full stable subcategory consisting of objects $x \in D_W$ such that $f^*(x) \in D_V$. By assumption, $f^*(\mathcal{J}_W) \subset \mathcal{J}_V$, so $\mathcal{J}_W \subset D^0_W$. As f^* is exact, it follows that D^0_W is a thick subcategory and thus $D^0_W = D_W$. The same argument shows that $f_!(D_V) \subset D_W$. □

We are now ready to state the main definition of this subsection.

Definition 5.43. By Lemma 5.40 and Lemma 5.42, the full subcategories of induced $\underline{W}$-objects ranging over all orbits W assemble to define a G-stable full G-subcategory

$$\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)_{\mathrm{ind}} \subset \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$$

of *G-induced objects.*

Our remaining goal in this subsection is to show that $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)_{\mathrm{ind}}$ is a G-$\otimes$-ideal. First, we show that the induced generators are closed under norms.

Lemma 5.44. *The collection $\{\mathcal{I}_W\}_{W \in \mathcal{O}_G}$ is stable under norms (taken along maps of orbits) in the pointwise G-symmetric monoidal structure on $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$.*

Proof. Suppose

$$U \xrightarrow{f} V \xrightarrow{p} W \xrightarrow{\gamma} W'$$

is a sequence of maps of finite G-sets in which W and W' are orbits, and

$$\begin{tikzcd}
\mathrm{fib}_s^f(\eta) \arrow[r, "\varphi_s^f"] \arrow[d, "\pi_s^f"'] & S_{\underline{V}} \arrow[d, "\eta_f"] \\
\underline{V} \arrow[r, "s_f"] & f_* f^* S_{\underline{V}}
\end{tikzcd}$$

is a pullback square of $\underline{V}$-∞-categories as in Definition 5.38. Let $X : p_! \,\mathrm{fib}_s^f(\eta) \to (\underline{\mathrm{Sp}}^G)_{\underline{W}}$ be a $\underline{W}$-functor and consider the induced $\underline{W}$-object

$$Y = (\epsilon_p \circ p_!(\varphi_s^f))_! X : S_{\underline{W}} \to (\underline{\mathrm{Sp}}^G)_{\underline{W}}.$$

We want to show that $Y' = \gamma_\otimes((\epsilon_p \circ p_!(\varphi_s^f))_! X)$ is an induced $\underline{W'}$-object. To do so, we will need to consider the commutative diagram of finite G-sets

$$\begin{tikzcd}
& U' \arrow[r, "f'"] \arrow[dl, "\mathrm{ev}'"'] \arrow[rr, bend left, "g"] & \gamma^*\gamma_* V \arrow[r, "\gamma'"] \arrow[dl, "\mathrm{ev}"'] \arrow[d] & V' = \gamma_* V \arrow[d, "q"] \\
U \arrow[r, "f"] & V \arrow[r, "p"] & W \arrow[r, "\gamma"] & W'
\end{tikzcd}$$

in which both parallelograms are pullback squares.[†]

First note that by G-distributivity of $(\underline{\mathrm{Sp}}^G, \otimes)$, if we let Y'' be the $\underline{W'}$-left Kan extension of

$$\gamma_* p_! \,\mathrm{fib}_s^f(\eta) \xrightarrow{\gamma_* X} \gamma_* \gamma^* (\underline{\mathrm{Sp}}^G)_{\underline{W'}} \xrightarrow{\gamma_\otimes} (\underline{\mathrm{Sp}}^G)_{\underline{W'}}$$

along

$$\gamma_* p_! \,\mathrm{fib}_s^f(\eta) \xrightarrow{\gamma_* p_! \varphi_s^f} \gamma_* p_! S_{\underline{V}} \xrightarrow{\gamma_* \epsilon_p} \gamma_* S_{\underline{W}} \simeq \gamma_* \gamma^* S_{\underline{W'}},$$

[†] Here, notation is as in Example 5.21; we write $\gamma_* : (\mathbb{F}_G)^{/W} \to (\mathbb{F}_G)^{/W'}$ for the right adjoint to pullback along γ and $\mathrm{ev} : \gamma^*\gamma_* \to \mathrm{id}$ for the counit of the adjunction.

then we may identify Y' as $Y'' \circ \eta_\gamma$. We then need to identify the pullback of $\gamma_* p_! \varphi^f_s$ and $\gamma_* \epsilon_p$ along η_γ. To deal with $\gamma_* \epsilon_p$, note that applying γ_* to the pullback square of $\underline{W}$-∞-categories

$$\begin{tikzcd} p_! p^* S_{\underline{W}} \arrow[r, "\epsilon_p"] \arrow[d] & S_{\underline{W}} \arrow[d] \\ \underline{V} \arrow[r, "p"] & \underline{W} \end{tikzcd}$$

yields a pullback square of $\underline{W'}$-∞-categories

$$\begin{tikzcd} q_! q^*(\gamma_* \gamma^* S_{\underline{W'}}) \arrow[r, "\epsilon_q"] \arrow[d] & \gamma_* S_{\underline{W}} = \gamma_* \gamma^* S_{\underline{W'}} \arrow[d] \\ \underline{V'} \arrow[r, "q"] & \underline{W'} \end{tikzcd}$$

identifying $\gamma_* \epsilon_p$ with ϵ_q. We then have the commutative diagram

$$\begin{tikzcd} q_! q^* S_{\underline{W'}} \arrow[r, "q_! q^* \eta_q"] \arrow[d, "\epsilon_q"] & q_! q^* \gamma_* \gamma^* S_{\underline{W'}} \arrow[r] \arrow[d, "\epsilon_q"] & \underline{V'} \arrow[d, "q"] \\ S_{\underline{W'}} \arrow[r, "\eta_q"] & \gamma_* \gamma^* S_{\underline{W'}} \arrow[r] & \underline{W'} \end{tikzcd}$$

in which the outer rectangle and the right-hand square are pullback squares, hence the left-hand square is a pullback square. Furthermore, under the equivalence $q^* \gamma_* \gamma^* S_{\underline{W'}} \simeq \gamma'_* \gamma'^* S_{\underline{V'}}$ we get that $q_! q^* \eta_q$ identifies with

$$q_! \eta_{\gamma'} : q_! S_{\underline{V'}} \to q_! \gamma'_* \gamma'^* S_{\underline{V'}}.$$

We next have the sequence of equivalences

$$\gamma_* p_! f_* f^* S_{\underline{V}} \simeq q_! \gamma'_* \operatorname{ev}^* f_* f^* S_{\underline{V}} \simeq q_! g_* g^* S_{\underline{V'}}$$

where we use Example 5.21 for the first equivalence and Remark 5.5 for the second equivalence. Moreover, a diagram chase reveals that under this equivalence, $\gamma_* p_! s_f$ identifies with $q_! s'_g$, where $s' = \operatorname{ev}'^*(s) \in S_{U'}$. Putting these observations together, we obtain a commutative diagram

$$\begin{tikzcd} q_! \operatorname{fib}^g_{s'}(\eta) \arrow[r, "\rho"] \arrow[d, "q_! \varphi^g_{s'}"'] & \gamma_* p_! \operatorname{fib}^f_s(\eta) \arrow[r] \arrow[d] & \underline{V'} \arrow[d, "q_! s'_g"] \\ q_! S_{\underline{V'}} \arrow[r, "q_! \eta_{\gamma'}"] & q_! \gamma'_* \gamma'^* S_{\underline{V'}} \arrow[r] & q_! g_* g^* S_{\underline{V'}} \end{tikzcd}$$

in which the right-hand square is $\gamma_* p_!$ of the pullback square defining $\operatorname{fib}^f_s(\eta)$, hence a pullback square, and the outer rectangle is $q_!$ of the pullback square defining $\operatorname{fib}^g_{s'}(\eta)$, hence also a pullback square. It follows that the left-hand square is a pullback square. Finally, if we let $X' = \gamma_\otimes \circ \gamma_* X \circ \rho$, then by Lemma 4.4(3) we get that $Y' \simeq (\epsilon_q \circ q_! \varphi^g_{s'})_! X'$. We conclude that $\gamma_\otimes(\mathcal{I}_W(f, p, s)) \subset \mathcal{I}_{W'}(g, q, s')$ and thus the collection $\{\mathcal{I}_W\}_{W \in \mathcal{O}_G}$ is stable under norms. $\square$

On the other hand, to show that $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)_{\mathrm{ind}}$ is fiberwise a $\otimes$-ideal, we will need the following projection formula.

Lemma 5.45. *Suppose $\varphi : I \to J$ is a map of G-spaces and let C be a G-distributive G-symmetric monoidal ∞-category.*[†] *Then*

$$\varphi_! : \mathrm{Fun}_G(I, C) \to \mathrm{Fun}_G(J, C)$$

satisfies the projection formula*: that is, for every pair of G-functors $X : I \to C$ and $Y : J \to C$, the canonical map*

$$\chi : \varphi_!(X \otimes \varphi^* Y) \to \varphi_! X \otimes Y$$

adjoint to $X \otimes \varphi^* Y \xrightarrow{\eta \otimes \mathrm{id}} \varphi^* \varphi_! X \otimes \varphi^* Y$ is an equivalence.

Proof. It suffices to check that χ is an equivalence on all points of J. Let $j \in J_V$ and also write $j : \underline{V} \to J$ for the corresponding G-functor. We then want to show that $j^*\chi$ is an equivalence of $\underline{V}$-functors. As $\varphi_!$, φ^*, and $\otimes$ are all compatible with restriction along $\underline{V} \to \mathcal{O}_G^{\mathrm{op}}$, we may suppose $V = G/G$ without loss of generality. Now consider the pullback square of G-spaces

$$\begin{tikzcd} I_j \arrow[r, "\iota"] \arrow[d, "\pi"] & I \arrow[d, "\varphi"] \\ *_G \arrow[r, "j"] & J. \end{tikzcd}$$

By Lemma 4.4(3), we have the base-change equivalence $j^*\pi_! \simeq \varphi_! \iota^*$, so we reduce to showing the map

$$j^*\chi : \pi_! \iota^*(X \otimes \varphi^* Y) \simeq \pi_!(\iota^* X \otimes \pi^* j^* Y) \to \pi_! \iota^* X \otimes j^* Y$$

is an equivalence. For this, we can use the G-distributivity of $(C, \otimes)$. Let $p : U = (G/G)^{\sqcup 2} \to G/G$ be the fold map and consider the left G-Kan extension diagram

$$\begin{tikzcd} I_j \coprod *_G \arrow[r, "\iota^* X \coprod j^* Y"] \arrow[d, "\pi \coprod \mathrm{id}"'] \arrow[dr, phantom, "\Downarrow"] & C \coprod C \\ *_G \coprod *_G \arrow[ur, bend right, "\pi_! \iota^* X \coprod j^* Y"'] & \end{tikzcd}$$

as a diagram of $\underline{U}$-∞-categories. Then we obtain a left G-Kan extension diagram

$$\begin{tikzcd} I_j \arrow[r, "(\iota^* X, \pi^* j^* Y)"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\Downarrow"] & C \times C = p_* p^* C \arrow[r, "\otimes"] & C \\ *_G, \arrow[urr, bend right, "\pi_! \iota^* X \otimes j^* Y"'] & & \end{tikzcd}$$

[†] In fact, one only needs that for fold maps $p : V^{\sqcup n} \to V$, the norm $\underline{V}$-functor $p_* p^* C_{\underline{V}} \to C_{\underline{V}}$ is distributive. However, this is stronger than C_V being distributive as a symmetric monoidal ∞-category.

and thus a canonical equivalence $\pi_!(\iota^*X \otimes \pi^* j^* Y) \xrightarrow{\simeq} \pi_!\iota^*X \otimes j^*Y$. A chase of the definitions shows this equivalence to be implemented by $j^*\chi$. □

The following lemma now suffices to show that $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)_{\mathrm{ind}}$ is a G-$\otimes$-ideal.

Lemma 5.46. *Let C be a G-stable G-distributive G-symmetric monoidal ∞-category and suppose $\{\mathcal{J}_W \subset C_W\}_{W\in\mathcal{O}_G}$ is a collection of objects stable under restriction, induction, and norms. Let $D_W \subset C_W$ be the thick subcategory generated by $\mathcal{J}_W$. Suppose in addition that each D_W is a $\otimes$-ideal. Then for every map of orbits $f : V \to W$, $f_\otimes : C_V \to C_W$ restricts to a functor $f_\otimes : D_V \to D_W$.*

Proof. Let $D_V^0 \subset D_V$ be the full stable subcategory consisting of objects $x \in D_V$ such that $f_\otimes(x) \in D_W$. We claim that D_V^0 is a thick subcategory, for which it suffices to show that D_V^0 is closed under desuspensions, finite colimits, and retracts.† We deal with these cases in turn:

(1) Suppose $x \in D_V^0$. Then $\Sigma^{-1}x = S^{-1} \otimes x$ where $S^{-1} = \Sigma^{-1}(1)$ is the desuspension of the unit in C_V. As $f_\otimes$ is symmetric monoidal (Remark 5.10), we have $f_\otimes(\Sigma^{-1}x) \simeq f_\otimes(S^{-1}) \otimes f_\otimes(x)$. But by assumption $f_\otimes(x) \in D_W$ and $D_W \subset C_W$ is a $\otimes$-ideal. It follows that $f_\otimes(\Sigma^{-1}x) \in D_W$ and thus $\Sigma^{-1}x \in D_V^0$.
(2) To show that D_V^0 is stable under finite colimits, in order to use the G-distributivity of $(C, \otimes)$ it is convenient to establish a more general assertion regarding parameterized colimits. To this end, let $D \subset C$ be the G-stable full G-subcategory as in Lemma 5.42. For any map of orbits $W' \to W$, consider the pullback square

$$\begin{tikzcd} V' \arrow[r, "f'"] \arrow[d] & W' \arrow[d] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

and let $D_{V'}^0 \subset D_{V'}$ be the full subcategory on those objects $x \in D_{V'}$ such that $f'_\otimes(x) \in D_{W'}$. Then using the compatibility of norms with restriction, we see that the collection

$$\{D_{V'}^0 \subset D_{V'} : V' \cong V \times_W W'\}_{W' \in (\mathbb{F}_G)^{/W}}$$

assembles to define a full $\underline{W}$-subcategory $(f_* D_{\underline{V}})^0$ of $f_* D_{\underline{V}}$.

Now suppose that K is a $\underline{V}$-finite $\underline{V}$-∞-category and $\varphi : K \to D_{\underline{V}}$ is a $\underline{V}$-functor such that $f_*\varphi : f_*K \to f_*D_{\underline{V}}$ factors through $(f_*D_{\underline{V}})^0$; if $K \simeq L \times \underline{V}$ for a finite ∞-category L, then this hypothesis is equivalent to supposing that $\varphi_V : L \to D_V$ factors through D_V^0, so this is the case of interest for us. We aim to show that $\mathrm{colim}^{\underline{V}}\varphi \in D_V^0$. But using G-distributivity, we see that

$$f_\otimes(\mathrm{colim}^{\underline{V}}\varphi) \simeq \mathrm{colim}^{\underline{W}}(f_\otimes \circ f_*\varphi : f_*L \to f_*D_{\underline{V}} \subset f_*f^*C_{\underline{W}} \to C_{\underline{W}}).$$

As $f_*\varphi$ factors through $(f_*D_{\underline{V}})^0$, $f_\otimes \circ f_*\varphi$ factors through $D_{\underline{W}}$. Then as f_*L is $\underline{W}$-finite and the inclusion $D_{\underline{W}} \subset C_{\underline{W}}$ is $\underline{W}$-exact, we get that $\mathrm{colim}^{\underline{W}}(f_\otimes \circ f_*\varphi) \in D_W$, as desired.
(3) As $f_\otimes$ preserves retract diagrams and D_W is stable under retracts, it follows that D_V^0 is stable under retracts.

† Of course, a simple argument as in Lemma 5.42 no longer works because $f_\otimes$ is not an exact functor.

We conclude that D_V^0 is a thick subcategory. As $\mathcal{J}_W \subset D_V^0$ by assumption, it follows that $D_V^0 = D_V$, as desired. $\square$

Corollary 5.47. *Let S be a G-space. Then the full G-subcategory $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)_{\mathrm{ind}}$ of induced objects in $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$ is a G-$\otimes$-ideal.*

Proof. By a routine thick subcategory argument, Lemma 5.45 implies that $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)_{\mathrm{ind}}$ is fiberwise a $\otimes$-ideal. Using Lemma 5.44, Lemma 5.46 then applies to show that $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$ is a G-$\otimes$-ideal. $\square$

5.5 | Main results

We are now ready to define a genuine equivariant refinement of the Tate construction for infinite groups. Our discussion is parallel to that in [67, section I.4]. First, we generalize work of Weiss–Williams [82] by constructing parameterized assembly maps.

Theorem 5.48. *Let G be a finite group, S a G-space, and let $p : S \to *_G$ be the projection to the terminal G-space.*

(1) *The G-stable G-∞-category $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$ has compactly generated fibers, and taking compact objects fiberwise yields a G-stable full G-subcategory $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)^\omega$ such that*

$$\underline{\mathrm{Ind}}_G\left(\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)^\omega\right) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G).$$

(2) *For every G-exact G-functor $F_1 : \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G) \to \underline{\mathrm{Sp}}^G$, there exists a G-colimit preserving G-functor F_0 equipped with a natural transformation $\alpha : F_0 \to F_1$, which is an "assembly map" in the sense of being terminal among all such natural transformations. Moreover, α is an equivalence when restricted to the full G-subcategory $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)^\omega$ of compact objects, and this uniquely specifies α.*
(3) *Every G-colimit preserving functor $F : \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G) \to \underline{\mathrm{Sp}}^G$ is of the form*

$$F(-) \simeq p_!(D \otimes -)$$

for a uniquely determined G-functor $D : S \to \underline{\mathrm{Sp}}^G$.
(4) *For $F = p_*$, the assembly map takes the form $p_!(D_S \otimes -) \to p_*(-)$ for*

$$D_S : S \xrightarrow{j} \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Spc}}_G) \xrightarrow{\Sigma^\infty_+} \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G) \xrightarrow{p_*} \underline{\mathrm{Sp}}^G$$

where j is the G-Yoneda embedding (under the equivalence $S \simeq S^{\mathrm{vop}}$).

Proof.

(1) As G-colimits and G-limits are computed pointwise [Shah, Proposition 9.17], $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$ is G-cocomplete and G-complete; it moreover has presentable fibers by [54, Proposition 5.4.7.11]. Now for every orbit $V \cong G/K$ and point $s \in S_V$, also write $s : \underline{V} \to S$ for the corresponding

G-functor that selects s. We then have an adjunction

$$s_! : \mathbf{Sp}^K \simeq \mathrm{Fun}_G(\underline{V}, \underline{\mathbf{Sp}}^G) \rightleftarrows \mathrm{Fun}_G(S, \underline{\mathbf{Sp}}^G) : s^*,$$

and by the same argument as in the proof of [67, Theorem I.4.1(i)], the set $\{s_!(\mathbb{S}) : s \in S_V\}$ furnishes a set of compact generators for $\mathrm{Fun}_G(S, \underline{\mathrm{Sp}}^G)$. The argument for compact generation of the other fibers is identical.

Taking compact objects fiberwise then yields a G-stable full G-subcategory because restriction and induction are strongly cocontinuous, and the last claim is clear by definition of $\underline{\mathrm{Ind}}_G$ as fiberwise Ind-completion.

(2) For any small G-stable G-∞-category C and G-cocomplete G-stable G-∞-category D, as G-colimits decompose in terms of fiberwise filtered colimits and finite G-colimits, one has a colocalization adjunction [75, Theorem 9.11]

$$\mathrm{Fun}^{\mathrm{ex}}_G(C, D) \simeq \mathrm{Fun}^L_G(\underline{\mathbf{Ind}}_G C, D) \rightleftarrows \mathrm{Fun}^{\mathrm{ex}}_G(\underline{\mathbf{Ind}}_G C, D)$$

with right adjoint given by restriction along the fiberwise Yoneda embedding and left adjoint given by fiberwise left Kan extension (which agrees in this case with G-left Kan extension). If we then let $C = \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)^\omega$ and $D = \underline{\mathrm{Sp}}^G$, we may define the assembly map to be the counit of the adjunction, after which its claimed properties are immediate.

(3) We have the equivalences

$$\begin{aligned}\mathrm{Fun}^L_G(\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G), \underline{\mathrm{Sp}}^G) &\simeq \mathrm{Fun}^L_G(\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Spc}}_G), \underline{\mathrm{Sp}}^G)\\ &\simeq \mathrm{Fun}_G(S, \underline{\mathrm{Sp}}^G)\end{aligned}$$

implemented by restriction along Σ^∞_+ and the G-Yoneda embedding, respectively; the first equivalence follows from [Nardin, Theorem 7.4] and the second equivalence from the parameterized Yoneda lemma [Shah, Theorem 11.5]. The claim then follows by observing that $p_!(D \otimes -)$ restricts to D under this equivalence.

(4) The proof is identical to that of [67, Theorem I.4.1(iv)]. □

Definition 5.49. In the situation of Theorem 5.48, we define p^T_* by extending the assembly map to a cofiber sequence

$$p_!(D_S \otimes -) \xrightarrow{\alpha} p_*(-) \xrightarrow{\beta} p^T_*.$$

If $S = B^\psi_G K$ for a group extension ψ of the finite group G by a (possibly infinite) group K, then we will also write $(-)^{\underline{t}[\psi]}$ or $(-)^{\underline{t}_G K}$ for the G-functor p^T_* and $(-)^{t[\psi]}$ or $(-)^{t_G K}$ for the fiber of p^T_* over G/G.

Remark 5.50 (Compatibility with restriction). Suppose that $V \cong G/H$ is an orbit and let $\psi' = \psi|_H$. Then $(B^\psi_G K)_{\underline{V}} \simeq B^{\psi'}_H K$ and the fiber of $(-)^{\underline{t}_G K}$ over V identifies with $(-)^{t_H K}$.

Remark 5.51 (Agreement of definitions). Suppose that ψ is an extension of G by a *finite* group K. Then the G-functor $(-)^{\underline{t}[\psi]}$ of Definition 5.49 coincides with that in Observation 4.34 and hence the functor $(-)^{t[\psi]}$ coincides with that defined prior in Definition 4.24. To see this, by

Theorem 5.48(2) it suffices to show that the norm map $(-)_{\underline{h}[\psi]} \to (-)^{\underline{h}[\psi]}$ constructed via the parameterized ambidexterity theory (for the Beck–Chevalley fibration $\underline{\mathrm{LocSys}}^G(\underline{\mathrm{Sp}}^G) \to \mathrm{Spc}_G$) is an equivalence on compact objects. This follows from Observation 4.26 and the description of the compact generators given in Theorem 5.48(1).

Remark 5.52 (Point-set models). As when all groups were finite, the parameterized Tate construction for a compact Lie group K admits a point-set model using equivariant universal spaces, cf. Remark 4.32.

For the following pair of propositions, we refer to p_*^T defined with respect to a fixed G-space S. First, in direct analogy to [67, Theorem I.4.1(iii)], we have the following universal property of p_*^T.

Proposition 5.53. *The natural transformation $\beta : p_* \to p_*^T$ is initial among those natural transformations from p_* to G-exact G-functors which vanishes on compact objects fiberwise.*

Proof. We first note that p_* is fiberwise κ-accessible for some fixed regular cardinal κ by the adjoint functor theorem [HTT, Corollary 5.5.2.9]; indeed, for every orbit V, $(p_*)_V$ is κ_V-accessible as a right adjoint between presentable ∞-categories, and we may then let $\kappa = \sup_{V \in \mathcal{O}_G} \kappa_V$. As the fiber of β preserves all G-colimits by definition, it follows that p_*^T is also fiberwise κ-accessible.

Now by Theorem 5.24(4) applied to $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)^\omega \subset \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)^\kappa$, there exists a natural transformation $\beta' : p_* \to L(p_*)$ of (fiberwise κ-accessible) G-exact $\underline{G}$-functors with the indicated universal property. We then have a map of fiber sequences

$$\begin{tikzcd} R(p_*) \arrow[r, "\alpha'"] \arrow[d, "\alpha''"] & p_* \arrow[r, "\beta'"] \arrow[d, "="] & L(p_*) \arrow[d, "\beta''"] \\ p_!(D_S \otimes -) \arrow[r, "\alpha"] & p_* \arrow[r, "\beta"] & p_*^T. \end{tikzcd}$$

We want to show that β'' is an equivalence, for which it suffices to show that α'' is an equivalence. As $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$ is fiberwise compactly generated and α'' is an equivalence when restricted to fiberwise compact objects, this will follow once we show that $R(p_*)$ preserves fiberwise filtered colimits. But this follows as in the proof of [67, Theorem I.4.1(iii)]. □

We next discuss the interaction of parameterized assembly with G-symmetric monoidal structures. Note first that with respect to the pointwise G-symmetric monoidal structure of Example 5.14 on $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$, $p^* : \underline{\mathrm{Sp}}^G \to \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)$ is a G-symmetric monoidal functor, hence its right G-adjoint p_* is lax G-symmetric monoidal. For p_*^T, we then have the following proposition.

Proposition 5.54. *Suppose that p_*^T restricts to the zero functor on the full G-subcategory of induced objects. Then p_*^T and the natural transformation $\beta : p_* \to p_*^T$ uniquely inherit the structure of a lax G-symmetric monoidal functor and morphism thereof.*

Proof. As in the proof of Proposition 5.53, we may as well ignore size-theoretic issues in our proof. Now by Theorem 5.29 applied with respect to the G-$\otimes$-ideal of induced objects (cf. Corollary 5.47), there exists a natural transformation $\beta' : p_* \to L(p_*)$ of lax G-symmetric monoidal functors such that $L(p_*)$ vanishes on G-induced objects and β' is the initial map with respect to this property. Moreover, as $\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)^\omega \subset \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)_{\mathrm{ind}}$ by Lemma 5.55, after forgetting G-symmetric monoidal structure we obtain a comparison map $p_*^T \to L(p_*)$ compatible with the maps from p_*. By our assumption on p_*^T, we likewise obtain a map $L(p_*) \to p_*^T$, and these are easily seen to be mutually inverse equivalences. We may thus equip β (and hence p_*^T) with the indicated lax G-symmetric monoidal structure. Finally, the uniqueness assertion follows from the universal property of β'. □

Lemma 5.55. *We have an inclusion*

$$\underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)^\omega \subset \underline{\mathrm{Fun}}_G(S, \underline{\mathrm{Sp}}^G)_{\mathrm{ind}}.$$

Proof. We will show this inclusion on the fiber over G/G; the argument for the other fibers will then be identical. As both sides are thick subcategories of $\mathrm{Fun}_G(S, \underline{\mathrm{Sp}}^G)$, it suffices to check that the compact generators described in the proof of Theorem 5.48(1) are induced. So, let V be an orbit and $s \in S_V$ a point. Then if we let $f = \mathrm{id}_V$ and $p : V \to G/G$ be the unique map, the G-functor

$$\epsilon_p \circ p_!(\varphi_s^f) : p_! \operatorname{fib}_s^f(\eta) \to p_! p^* S \to S$$

in Definition 5.38 coincides with $s : \underline{V} \to S$. Therefore, $s_!(\mathbb{S})$ is also an induced generator. □

We now specialize our discussion to the main case of interest. Let K be a compact Lie group, G a finite group, $\psi : G \to \mathrm{Aut}(K)$ a group homomorphism, and $S = B_G^\psi K$.

Observation 5.56 (Parameterized actions on K and its Lie algebra). Let G act on $K \times K$ by $\psi \times \psi$ and consider the unique point-set action of $(K \times K) \rtimes G$ on K that extends the $K \times K$-action given by $(k, k') \cdot x = kxk'^{-1}$ and the G-action given by ψ, so that $((k, k'), g) \cdot x = k\psi_g(x)k'^{-1}$.[†] This defines K first as an object in

$$\mathrm{Spc}_{(K\times K)\rtimes G} \simeq \mathrm{Fun}(\mathcal{O}^{\mathrm{op}}_{(K\times K)\rtimes G}, \mathrm{Spc})$$

and then as an object in

$$\mathrm{Fun}_G(B_G^\psi(K \times K), \underline{\mathrm{Spc}}_G) \simeq \mathrm{Fun}(B_G^\psi(K \times K), \mathrm{Spc})$$

via restriction along $B_G^\psi(K \times K) \subset \mathcal{O}^{\mathrm{op}}_{(K\times K)\rtimes G}$. Let

$$\Sigma^\infty_+ K \in \mathrm{Fun}_G(B_G^\psi(K \times K), \underline{\mathrm{Sp}}^G)$$

denote the infinite G-suspension of K. Then the dualizing object D_S of Theorem 5.48(4) identifies with $(\Sigma^\infty_+ K)^{h_G(K\times 1)}$.

Now let $\mathfrak{a}$ be the Lie algebra of K and let $\mathbb{S}^{\mathfrak{a}}$ be Σ^∞ of the one-point compactification $S^{\mathfrak{a}}$. We have that the $K \rtimes G$-action on K, uniquely specified as the action extending $k \cdot x = kxk^{-1}$ and ψ, defines a point-set $K \rtimes G$-action on $S^{\mathfrak{a}}$ and subsequently $\mathbb{S}^{\mathfrak{a}}$ as an object in $\mathrm{Fun}_G(B_G^\psi K, \underline{\mathrm{Sp}}^G)$, where (abusing notation) we also write $\mathbb{S}^{\mathfrak{a}}$ for the infinite G-suspension of $S^{\mathfrak{a}}$, which restricts to the prior spectrum $\mathbb{S}^{\mathfrak{a}}$.

[†] This action is well-defined because $(1, g) \cdot ((k, k'), 1) \cdot x = \psi_g(kxk'^{-1}) = \psi_g(k)\psi_g(x)\psi_g(k')_{-1} = ((\psi_g(k), \psi_g(k')), g) \cdot x$.

Let S_0 be the underlying space of S and recall that John Klein identified the dualizing spectrum D_{S_0} with $\mathbb{S}^{\mathfrak{a}}$ as a spectrum with K-action [47, Theorem 10.1].

Proposition 5.57. *Let K be a compact Lie group, G a finite group, $\psi : G \to \mathrm{Aut}(K)$ a group homomorphism, $S = B^{\psi}_G K$, and $\mathfrak{a}$ the Lie algebra of K, with actions as above. Klein's equivalence promotes to an equivalence*

$$D_S = (\Sigma^\infty_+ K)^{h_G(K\times 1)} \simeq \mathbb{S}^{\mathfrak{a}} \in \mathrm{Fun}_G\left(B^{\psi}_G K, \underline{\mathrm{Sp}}^G\right).$$

Proof. This follows by inspecting the equivariance of the maps in Klein's proof along with invoking equivariant Atiyah duality in Sp^G. In more detail, first note that Klein's point-set $K \times K$-equivariant maps[†] $\alpha : K_+ \to F(K_+, S^{\mathfrak{a}})$ and its adjoint $\widehat{\alpha} : (K\times K)_+ \to S^{\mathfrak{a}}$ are in fact $(K \times K) \rtimes G$-equivariant.[‡] Suspending $\widehat{\alpha}$ and taking its adjoint, we then obtain a map

$$\Sigma^\infty_+ K \to F(\Sigma^\infty_+ K, \mathbb{S}^{\mathfrak{a}}) \in \mathrm{Fun}_G(B^{\psi}_G(K\times K), \underline{\mathrm{Sp}}^G)$$

which is an equivalence by equivariant Atiyah duality [49, section III.5]. Taking G-parameterized homotopy fixed points with respect to the factor $K \times 1$ then shows the claim. $\square$

Example 5.58 (Complex conjugation on the circle). Suppose that $K = S^1$ with $G = C_2$ acting by complex conjugation, so that $K \rtimes G \cong O(2)$. Write $S = B^t_{C_2} S^1$ for the corresponding C_2-space. As $BS^1 = \Omega^\infty H\mathbb{Z}[2]$, a group cohomology computation shows that $(BS^1)^{hC_2} \simeq B\mu_2$ and the map $(BS^1)^{hC_2} \to BS^1$ identifies with the delooping of the inclusion $\mu_2 = \{\pm 1\} \subset S^1$. Note also that we have a canonical isomorphism $\mu_2 \cong W_{O(2)}C_2$ of μ_2 with the Weyl group of C_2.[§] We then see that a C_2-functor $X : B^t_{C_2} S^1 \to \underline{\mathrm{Sp}}^{C_2}$ (with underlying C_2-spectrum X) is equivalent data to:

(1) an $O(2)$-action on the underlying spectrum X^e;
(2) a μ_2-action on the geometric C_2-fixed points $X^{\phi C_2}$ such that the gluing map $X^{\phi C_2} \to (X^e)^{tC_2}$ is μ_2-equivariant.

In particular, if we let D_S be the dualizing C_2-spectrum and σ denote the sign C_2-representation, then under the equivalence of Proposition 5.57 we see that $D_S = \mathbb{S}^\sigma$ with *trivial* C_2-parameterized action, that is, so that as a C_2-functor, D_S factors as

$$D_S : B^t_{C_2} S^1 \to *_{C_2} \xrightarrow{\mathbb{S}^\sigma} \underline{\mathrm{Sp}}^{C_2}.$$

Therefore, we may write $D_S \otimes X$ as the C_2-tensor of X by the C_2-spectrum S^σ. It follows that we may write the parameterized assembly map as

$$(D_S \otimes X)_{h_{C_2} S^1} \simeq \mathbb{S}^\sigma \otimes X_{h_{C_2} S^1} = \Sigma^\sigma X_{h_{C_2} S^1} \to X^{h_{C_2} S^1}.$$

The appearance of $\mathbb{S}^\sigma$ here is closely related to the suspensions by the signed sphere appearing explicitly in [43, Theorem C] and implicitly in [25, Theorem B].

[†] Note that Klein writes G and not K for the compact Lie group.

[‡] By choosing an invariant metric, we may assume that G acts by isometries on $\mathfrak{a}$.

[§] We may choose any section $C_2 \to O(2)$ of the determinant map to view C_2 as a subgroup of $O(2)$, but also note that under the isomorphism $O(2) = S^1 \rtimes C_2$ one has a preferred section.

Finally, we establish lax equivariant symmetric monoidality of the parameterized Tate construction by showing that it vanishes on induced objects.

Theorem 5.59. *Let K be a compact Lie group, G a finite group, and $\psi : G \to \mathrm{Aut}(K)$ a group homomorphism. Then $(-)^{\underline{t}_G K}$ vanishes on the full G-subcategory of induced objects.*

Proof. We first recall the proof that $(-)^{tK}$ vanishes on induced objects (cf. [47, Corollary 10.2]). As $(-)^{tK}$ is an exact functor, by a thick subcategory argument it suffices to show that $(-)^{tK}$ vanishes on induced generators. Given a point $s : * \to BK$ and writing $p : BK \to *$ for the unique functor to a point, we thus need to show that the assembly map

$$\alpha_E : p_!(\mathbb{S}^{\mathfrak{a}} \otimes s_! E) \to p_* s_! E$$

is an equivalence for all spectra E. By definition, α_E is an equivalence when E is a compact spectrum, so it suffices to show that both sides commute with filtered colimits. For the left-hand side, this is obvious, while for the right-hand side, we note that

$$p_* s_!(-) = (\Sigma^\infty_+ K \otimes -)^{hK} \simeq F(DK_+, -)^{hK} \simeq F(K_+, \Sigma^{\mathfrak{a}} -)^{hK} \simeq \mathbb{S}^{\mathfrak{a}} \otimes -$$

(invoking Atiyah duality for the middle equivalence), so $p_* s_!$ commutes with all colimits.

Now let $S = B^{\psi}_G K$ and write $\rho : S \to *_G$ for the unique G-functor (we don't use p to avoid confusion with the notation of Definition 5.38). As $(-)^{\underline{t}_G K}$ is fiberwise exact, it suffices to show that $(-)^{\underline{t}_G K}$ vanishes on induced generators. Moreover, as $(-)^{\underline{t}_G K}$ is G-exact, for any map of orbits $h : W \to W'$ and $X \in \mathrm{Fun}_{\underline{W}}(S_{\underline{W}}, (\underline{\mathrm{Sp}}^G)_{\underline{W}})$, if $\underline{(X)}^{t_G K} = 0$ then $\underline{(h_! X)}^{t_G K} = 0$. Therefore, considering induced generators as in the setup of Definition 5.38, we may suppose that $p = \mathrm{id}_V$ there.[†] Furthermore, without loss of generality we may suppose $V = G/G$. So, let U be a finite G-set, let $s \in S_U$ be a point, and write $f : U \to G/G$ for the unique map. Consider the pullback square of G-∞-categories

$$\begin{tikzcd}
\mathrm{fib}^f_s(\eta) \arrow[r, "\varphi"] \arrow[d, "\pi"] & S \arrow[d, "\eta"] \\
*_G \arrow[r, "s'"] & f_* f^* S.
\end{tikzcd}$$

We need to show that for any G-functor $X : \mathrm{fib}^f_s(\eta) \to \underline{\mathrm{Sp}}^G$, the parameterized assembly map

$$\alpha_X : \rho_!(\mathbb{S}^{\mathfrak{a}} \otimes \varphi_! X) \to \rho_* \varphi_! X$$

is an equivalence in Sp^G. We first note that by definition α_X is an equivalence if X is compact. As $\mathrm{Fun}_G(\mathrm{fib}^f_s(\eta), \underline{\mathrm{Sp}}^G)$ is compactly generated by Theorem 5.48(1), it suffices to check that both sides commute with filtered colimits as functors $\mathrm{Fun}_G(\mathrm{fib}^f_s(\eta), \underline{\mathrm{Sp}}^G) \to \mathrm{Sp}^G$. For the left-hand side this is obvious, while for the right-hand side we note that as K is compact Lie it follows that $\mathrm{fib}^f_s(\eta)$ is a *finite* G-space, hence

$$\eta_* \varphi_! \simeq s'_! \pi_* : \mathrm{Fun}_G(\mathrm{fib}^f_s(\eta), \underline{\mathrm{Sp}}^G) \to \mathrm{Fun}_G(f_* f^* S, \underline{\mathrm{Sp}}^G),$$

[†] The case where V is not an orbit also follows because that only involves in addition taking fiberwise coproducts.

and writing $\rho' : f_* f^* S \to *_G$ for the unique G-functor, we get that

$$\rho_* \varphi_! \simeq \rho'_* \eta_* \varphi_! \simeq \rho'_* s'_! \pi_* : \mathrm{Fun}_G(\mathrm{fib}^f_s(\eta), \underline{\mathrm{Sp}}^G) \to \mathrm{Sp}^G.$$

Now again using that $\mathrm{fib}^f_s(\eta)$ is finite, we get that π_* commutes with filtered colimits, and it remains to check that $\rho'_* s'_!$ commutes with filtered colimits. For this, note first that if $U = G/G$, then by equivariant Atiyah duality applied as in Proposition 5.57, we would have

$$\rho'_* s'_!(-) \simeq \mathbb{S}^{\mathfrak{a}} \otimes (-).$$

In general, we may identify the indexed product $f_* f^* S = \prod_U B^\psi_G K$ as $B^{\psi_U}_G(\prod_{|U|} K)$ for the group homomorphism

$$\psi_U : G \to \mathrm{Aut}\left(\prod_{|U|} K\right), \quad (\psi_U)_g : (k_i)_{i \in U} \mapsto (\psi_g(k_{g^{-1}.i}))_{i \in U}$$

and thereby reduce to the logic of the prior case because $\prod_{|U|} K$ is still compact Lie. □

Corollary 5.60. *Let K be a compact Lie group, G a finite group, and $\psi : G \to \mathrm{Aut}(K)$ a group homomorphism. Then the G-functor $(-)^{\underline{t}_G K}$ and natural transformation $(-)^{\underline{h}_G K} \to (-)^{\underline{t}_G K}$ uniquely inherit the structure of a lax G-symmetric monoidal functor and morphism thereof.*

Proof. Combine Proposition 5.54 and Theorem 5.59. □

APPENDIX: POINTWISE MONOIDAL STRUCTURE

In this appendix, we construct the *pointwise* symmetric monoidal structure on the $\mathcal{T}$-functor ∞-category $\mathrm{Fun}_{\mathcal{T}}(K, C)$, given appropriate structure on C. For a quick reminder on how to construct the pointwise symmetric monoidal structure on the usual functor ∞-category, see [76, Construction 2.23].

Suppose C is an $\mathcal{T}$-∞-category classified by a functor $\mathcal{T}^{\mathrm{op}} \to \mathrm{CMon}(\mathrm{Cat}_\infty)$ valued in symmetric monoidal ∞-categories and symmetric monoidal functors thereof. In terms of fibrations, such functors correspond to *cocartesian $\mathcal{T}$-families of symmetric monoidal ∞-categories* $C^\otimes \to \mathcal{T}^{\mathrm{op}} \times \mathrm{Fin}_*$ [56, Definition 4.8.3.1].† Let $\mathfrak{P}_{\mathcal{T}}$ be the categorical pattern

$$(\mathrm{All}, \mathrm{All}, \{\lambda_{t,n} : (\langle n \rangle^\circ)^{\triangleleft} \to \{t\} \times \mathrm{Fin}_* \subset \mathcal{T}^{\mathrm{op}} \times \mathrm{Fin}_* : t \in \mathcal{T}\})$$

on $\mathcal{T}^{\mathrm{op}} \times \mathrm{Fin}_*$, where $\lambda_{t,n}$ is the usual map appearing in the definition of the model structure on preoperads that sends the cone point v to $\langle n \rangle$, $i \in \langle n \rangle^\circ$ to $\langle 1 \rangle$, and the unique morphism $v \to i$ to the inert morphism $\rho^i : \langle n \rangle \to \langle 1 \rangle$ in Fin_* that selects $i \in \langle n \rangle^\circ$. Then cocartesian $\mathcal{T}$-families of symmetric monoidal ∞-categories are by definition $\mathfrak{P}$-fibered [56, Definition B.0.19] and hence are the fibrant objects for the model structure on $s\mathrm{Set}^+_{/\mathcal{T}^{\mathrm{op}} \times \mathrm{Fin}_*}$ defined by $\mathfrak{P}$ [56, Theorem B.0.20].

Definition A.1. Suppose $C^\otimes \to \mathcal{T}^{\mathrm{op}} \times \mathrm{Fin}_*$ is a cocartesian $\mathcal{T}$-family of symmetric monoidal ∞-categories and $q : K \to \mathcal{T}^{\mathrm{op}}$ is an $\mathcal{T}$-∞-category. Consider the span of marked simplicial sets

$$(\mathbf{Fin}_*)^\sharp \xleftarrow{\mathrm{pr}} {}_\natural K \times (\mathbf{Fin}_*)^\sharp \xrightarrow{q \times \mathrm{id}} (\mathcal{T}^{\mathrm{op}})^\sharp \times (\mathbf{Fin}_*)^\sharp.$$

† More precisely, we have an equivalence of ∞-categories $(\mathrm{Cat}_\infty)^{\mathrm{cocart}}_{/\mathcal{T}^{\mathrm{op}} \times \mathrm{Fin}_*} \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Cat}_\infty))$ under which a cocartesian $\mathcal{T}$-family of symmetric monoidal ∞-categories corresponds to a functor valued in commutative monoid objects.

Define the *pointwise symmetric monoidal structure* on $\mathrm{Fun}_{\mathcal{T}}(K, C)$ to be

$$\mathrm{Fun}_{\mathcal{T}}(K, C)^{\otimes} := \mathrm{pr}_*(q \times \mathrm{id})^*({}_{\natural}C^{\otimes})$$

regarded as a simplicial set over Fin_*.

Note that the fiber of $\mathrm{Fun}_{\mathcal{T}}(K, C)^{\otimes} \to \mathrm{Fin}_*$ over $\langle 1 \rangle$ is $\mathrm{Fun}_{\mathcal{T}}(K, C)$.

Lemma A.2. *With respect to the categorical patterns $\mathfrak{P} = \mathfrak{P}_*$ on Fin_* and $\mathfrak{P}_{\mathcal{T}}$ on $\mathcal{T}^{\mathrm{op}} \times \mathrm{Fin}_*$, the span of marked simplicial sets in Definition A.1 satisfies the hypotheses of [56, Theorem B.4.2], so $\mathrm{Fun}_{\mathcal{T}}(K, C)^{\otimes}$ is a symmetric monoidal ∞-category.*

Proof. The projection map pr is both a cartesian and cocartesian fibration where an edge e is (co)cartesian if and only if its projection to K is an equivalence. Using also the basic stability property of cocartesian edges in K [54, Lemma 2.4.2.7], it is then easy to verify conditions (1)-(8) of [56, Theorem B.4.2]. By [56, Theorem B.4.2], $\mathrm{pr}_* q^* : s\mathrm{Set}^+_{/\mathfrak{P}_{\mathcal{T}}} \to s\mathrm{Set}^+_{/\mathfrak{P}}$ is right Quillen, which shows that $\mathrm{Fun}_{\mathcal{T}}(K, C)^{\otimes}$ is a fibrant object in $s\mathrm{Set}^+_{/\mathfrak{P}}$ and hence a symmetric monoidal ∞-category. □

Remark A.3. An $\mathcal{T}$-functor $f : L \to K$ yields a morphism of spans

$$\begin{tikzcd} & {}_{\natural}L \times (\mathbf{Fin}_*)^{\sharp} \arrow[dl] \arrow[d, "f\times\mathrm{id}"] \arrow[dr] & \\ (\mathbf{Fin}_*)^{\sharp} & {}_{\natural}K \times (\mathbf{Fin}_*)^{\sharp} \arrow[l] \arrow[r] & S^{\sharp} \times (\mathbf{Fin}_*)^{\sharp} \end{tikzcd}$$

and therefore induces a map $f^* : {}_{\natural}\mathrm{Fun}_{\mathcal{T}}(K, C)^{\otimes} \to {}_{\natural}\mathrm{Fun}_{\mathcal{T}}(L, C)^{\otimes}$ of marked simplicial sets over Fin_*. In other words, restriction along f is a symmetric monoidal functor.

Variant A.4. Consistent with the symmetric monoidality of restriction, the hypotheses of [56, Theorem B.4.2] also apply to the span

$$(\mathcal{T}^{\mathrm{op}})^{\sharp} \times (\mathbf{Fin}_*)^{\sharp} \xleftarrow{q\times\mathrm{id}} {}_{\natural}K \times (\mathbf{Fin}_*)^{\sharp} \xrightarrow{q\times\mathrm{id}} (\mathcal{T}^{\mathrm{op}})^{\sharp} \times (\mathbf{Fin}_*)^{\sharp}.$$

We then define

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(K, C)^{\otimes} := (q \times \mathrm{id})_*(q \times \mathrm{id})^*({}_{\natural}C^{\otimes})$$

as a pointwise symmetric monoidal enhancement of $\underline{\mathrm{Fun}}_{\mathcal{T}}(K, C)$.

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This work is an expansion of [69, sections 3–5]. The main changes are as follows.

(1) We extended the theory to compact Lie groups.
(2) We added the material on parameterized assembly.
(3) We proved that the parameterized Tate construction uniquely admits the structure of a lax G-symmetric monoidal functor.
(4) We added an application to the generalized Segal conjecture.
(5) We added an inverse limit formula for the parameterized Tate construction.

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