

Communicated by Michael Atiyah, November 22, 1972

Fix an algebraically closed field k . A (one-parameter, commutative) formal group law over k is an element $F \in k[[X, Y]]$ such that

1. $F(X, Y) = F(Y, X) = X + Y + \text{higher order terms}$,
2. $F(F(X, Y), Z) = F(X, F(Y, Z))$.

Let Λ denote the set of such formal group laws. By a theorem of Lazard [4], there exists a ring L carrying a universal formal group law; consequently Λ can be identified with the set of ring homomorphisms $L \rightarrow k$. This may be used to enrich the structure of the set Λ ; it gains a Zariski topology, and a sheaf of rings making it into a ringed space; in fact with this data, Λ would be an algebraic space [7] in the sense of Serre, except that it is not noetherian.

AMS (MOS) subject classifications (1970). Primary 14L05, 55G10, 55G25, 57D90; Secondary 14L25, 14D20, 14M15.

Key words and phrases. Landweber-Novikov algebra, Steenrod algebra, formal group, algebraic group.

Now let Γ denote the group of formal power series $f \in k[[T]]$ of the form $f(T) = uT + \text{higher order terms}$, $u \neq 0$, the group operation being composition. This is a (nonnoetherian) algebraic group, which acts on Λ by changing coordinates: if $f \in \Gamma$, $F \in \Lambda$, then we define $F^f(X, Y) = f^{-1}F(fX, fY)$. It is not hard to see that $(f, F) \mapsto F^f$ defines a morphism $\Gamma \times \Lambda \rightarrow \Lambda$ of (pro-)algebraic spaces of Serre. In this note we discuss this group action when the characteristic of k is positive (the char 0 case being trivial). In a succeeding note we will apply these results to the study of unitary cobordism, via Quillen's theorem, which identifies the ring L with the unitary cobordism ring of a point [6]. We remark here that Γ is the algebraic group underlying the Landweber-Novikov algebra of operations for cobordism.

THEOREM 1. Λ is stratified into orbits, $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \dots \cup \Lambda_n$, such that

- (a) $\Lambda_n = \bigcup_{m \geq n} \Lambda_m$; Λ_1 is open, and Λ_n is closed.
- (b) Λ_n is a complete intersection of hyperplanes.
- (c) Λ_n is homogeneous under Γ ; for finite n , there exists a p -adic Lie group G_n such that $\Gamma|_{G_n} \supset \Lambda_n$.
- (d) The normal bundle of Λ_n in Λ is given by an $(n-1)$ -dimensional representation of G_n over k .

REMARK. The Λ_n can be described explicitly in terms of Milnor's generators of the unitary cobordism ring: Λ_n is the locus where $p = p_1 = \dots = p_{n-1} = 0$ and p_n is a unit, where p_i is a Milnor generator of dimension $2(p_i - 1)$. A theorem of Landweber [5] is a corollary: the ideals $(p, p_1, \dots, p_n)$ in the unitary cobordism ring are the only prime ideals invariant under the Landweber-Novikov algebra.

To complete the description of the orbit structure we must identify the groups G_n and the representations in (d). For this we recall that central simple division algebras over the p -adic numbers $\mathbb{Q}_p$ are completely classified by the rank (as $\mathbb{Q}_p$ -vector spaces) and Brauer invariant, which lies in $\mathbb{Q}'/\mathbb{Z}$.

Let D_n be such a division algebra of rank n^2 and Brauer invariant $1/n$. There is a natural valuation on D_n , $v: D_n^* \rightarrow \mathbb{Q}'/\mathbb{Z}$, the former arrow being the norm, and the latter being p -adic valuation. Let $E_n = \{x \in D_n | v(x) \geq 0\}$; denote the ring of integers of D_n .

Now consider the twisted polynomial algebra $k\langle\delta\rangle$ defined by the relation $\delta k = \delta k$, $k \in k$; we abbreviate by M_n the quotient ring modulo the (two-sided) ideal generated by δ^n . It can be shown [1, p. 80] that E_n/pE_n is isomorphic to $F_q\langle\delta\rangle/(\delta^n)$, where $q = p^n$ and $F_q\langle\delta\rangle$ is defined like $k\langle\delta\rangle$. Consequently, M_n is a right E_n -module, whenever $n \geq m$.

THEOREM 2. The stabilizer G_n is isomorphic to the group of units of E_n . The representation of (1d) has M_{n-1} as underlying k -vector space, with G_n -action given by

$$g(v) = vg^{-1}, \quad v \in M_{n-1}, g \in G_n \subset E_n.$$

REMARKS. G_n is a profinite group over F_p , but it can be given a p -adic analytic structure. As such it is a form, in the sense of Galois cohomology, of $\mathrm{GL}(n, \mathbb{Z}_p)$.

From the description of the normal bundle of Λ_n in Λ just given, it is possible to read off the normal representation of Λ_n in Λ_n , $n \leq m$. It is also possible to identify the stabilizer of the infinite stratum $\Lambda_\infty = \Gamma^* G_\infty$, where G_∞ is the group of units of $k\langle\langle \mathcal{G} \rangle\rangle$, considered as a proalgebraic group over F_p . This is just the group underlying the reduced Steenrod algebra (at the prime p).

NOTES ON THE PROOFS. Theorem 1(a) is just a restatement of a theorem of Lazard [4]: two formal group laws over an algebraically closed field are isomorphic iff they are of the same height. Thus Λ_n is the moduli variety of formal groups of height n , and part (b) is proved by applying Lazard's techniques to the moduli functor of formal groups of height n . Part (d) is trivial.

To prove (c), we use a theorem of Grothendieck [2, III, §3]. Let G, X be respectively a group-scheme and a scheme upon which G acts, both noetherian over k , with X smooth. Then X is a homogeneous space of G if $G(k)$ acts transitively on $X(k)$. (Here $G(k)$ and $X(k)$ are the k -valued points of G, X .)

Now the Λ_n are represented by localizations of polynomial rings and are smooth, but neither they nor Γ are noetherian. However the above result can be extended to proalgebraic group actions of a certain kind. Thus we let $\Lambda_n(\deg r)(A)$ denote the set of r -bits of a formal group over A , of height n ($0 \leq n \leq p^n$). Then $\Lambda_n = \mathrm{proj} \lim \Lambda_n(\deg r)$, the maps being surjections for any A ; it is not hard to see that $\Lambda_n(\deg r)$ is a smooth, noetherian scheme. Similarly, $\Gamma(A)$ is the set of invertible series in $A[[T]]/(T^p - 1)$, and acts on $\Lambda_n(\deg r)$, compatibly with truncations. Using these approximations systematically, we prove (c).

The identification of G_n is due to Dieudonné and Lubin; see also [1, Theorem 3, p. 72]. To identify the normal representation, we show by direct computation, following [5], that the tangent space to Λ at F is the group $Z^2(F; k)$ of 2-cocycles of F , while the tangent space to Λ_n at F , n being the height of F , is the group $B^2(F; k)$ of 2-coboundaries. Thus the normal bundle is given by the 2-cohomology representation $H^2_*(F; k)$. A basis for this group is approximately known, and one checks directly that the representation is as indicated.

BIBLIOGRAPHY

1. A. Fröhlich, *Formal groups*, Lecture Notes in Math. vol. 74, Springer, New York, 1963. MR 39 #4164.
2. P. Gabriel and M. Demazure, *Groupe algébriques*, Tom. I, North-Holland, Amsterdam, 1970.
3. P. S. Landweber, *Annihilator ideals and primitive elements in complex cobordism*, Rutgers University, (preprint).
4. M. Lazard, *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Math. France 83 (1955), 251-274. MR 17, 593.
5. J. Lubin and J. Tate, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France 94 (1966) 49-59. MR 39 #214.
6. D. Quillen, *Elementary proofs of some results in cobordism theory I*, Advances in Math. 7 (1971).
7. J. E. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) 61 (1955), 197-278. MR 16, 952.