

FINITE LOCALIZATIONS

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This short note is a response to the articles [7] of Doug Ravenel and [3] of Mark Mahowald and Hal Sadofsky. I give cleaner and more general constructions of the “telescopic” or “finite localization,” which they write L_n' . I prefer to call this the *finite* $E(n)$ -localization and write L_n^f for it, because, as I shall show, it can be characterized in exactly the same terms as the Bousfield localization, but with the addition of a finiteness assumption. If B is a finite $E(n-1)$ -acyclic spectrum with a v_n -self-map $\phi: B \rightarrow \Sigma^{-q}B$ [2], then $L_n^f B$ is the mapping telescope of B ; so L_n^f is a generalization of this construction in that it can be applied to any spectrum X .

By the same method, a finite localization $L_{\mathcal{A}}^f$ can be defined for any set $\mathcal{A}$ of homotopy types of finite spectra. Of particular interest is the case in which $\mathcal{A}$ is the set of finite E -acyclic spectra for some spectrum E , and in this case we will write L_E^f for the corresponding finite localization. The construction of this localization is simpler than that of the Bousfield homology localization—one can work entirely in the homotopy category, and a countable telescope suffices for the construction. It turns out to be easy to show that $L_{\mathcal{A}}^f$ is always “smashing” (i.e., the natural map $X \rightarrow X \wedge L_{\mathcal{A}}^f S^0$ is an equivalence) and coincides with Bousfield localization with respect to the spectrum $L_{\mathcal{A}}^f S^0$.

For any spectrum E , there is a canonical map $L_E^f X \rightarrow L_E X$. The “telescope conjecture” for E (advertised for $E = E(n)$ by Ravenel in [4]) is the assertion that this map is an equivalence. It is equivalent to require that any E -acyclic spectrum has an exhaustive filtration whose associated quotients are wedges of *finite* E -acyclic spectra. This structural feature is exactly what Bousfield checks for K -theory, and by virtue of Ravenel’s computation [5] we now know that it fails for $E(2)$ at all primes. It would be very interesting to have invariants vanishing on finite $E(n)$ -acyclics and compatible with wedges and cofibrations, but not vanishing on all $E(n)$ -acyclics.

1. Finitely $\mathcal{A}$ -local spectra

Recall Bousfield’s definitions from [1]; here E is any spectrum.

Definition (1). 1. A spectrum W is E -local iff $[T, W] = 0$ for every E -acyclic spectrum T .

2. A map $X \rightarrow Y$ is an E -equivalence iff $E_* f$ is an isomorphism.

We have also the variant of this condition, which is clearly implied by it:

2'. Any map from X to an E -local W extends uniquely to a map from Y to W .

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Bousfield proves:

THEOREM (2). [1] *For any spectra E and X , there is an E -equivalence from X to an E -local spectrum.*

It is easy to see that this map is initial among maps from X to E -local spectra, and terminal among E -equivalences out of X . Either property shows that the map is unique up to canonical equivalence, and it is written $X \rightarrow L_E X$. Its existence shows that Condition 2.1 for a map f implies that the map is an E -equivalence, since the map $Z \rightarrow *$ from the mapping cone of f to a point, being initial among maps from Z to E -local spectra, must be E -localization and in particular an E -equivalence.

We now modify these definitions slightly, by testing only against *finite* E -acyclic spectra. In fact, the same methods work for any set $\mathcal{A}$ of homotopy types of finite spectra.

Definition (3). 1. A spectrum W is *finitely $\mathcal{A}$ -local* iff $[\Sigma^n A, W] = 0$ for every $A \in \mathcal{A}$ and every $n \in \mathbb{Z}$.

2. A spectrum Z is *finitely $\mathcal{A}$ -acyclic* iff $[Z, W] = 0$ for every finitely $\mathcal{A}$ -local spectrum W .

3. A map $f : X \rightarrow Y$ is a *finite $\mathcal{A}$ -equivalence* iff its mapping cone is finitely $\mathcal{A}$ -acyclic.

THEOREM (4). *For any set $\mathcal{A}$ of finite spectra, and any spectrum X , there is a finite $\mathcal{A}$ -equivalence from X to a finitely $\mathcal{A}$ -local spectrum.*

The same considerations as above show that this map is initial among maps from X to finitely $\mathcal{A}$ -local spectra, and terminal among finite $\mathcal{A}$ -equivalences out of X . Again, either property shows that the map is unique up to canonical equivalence. We write $\eta : X \rightarrow L_{\mathcal{A}}^f X$ for it, and call it the *finite $\mathcal{A}$ -localization*.

Suppose $\mathcal{A}$ is the set of finite E -acyclic spectra, for some spectrum E , and write L_E^f for $L_{\mathcal{A}}^f$. Since any E -local spectrum is finitely $\mathcal{A}$ -local, or since any finite $\mathcal{A}$ -equivalence is an E -equivalence, we get a unique factorization

$$\begin{array}{ccc} & & L_E^f X \\ & \nearrow & \downarrow \\ X & & \\ & \searrow & \\ & & L_E X \end{array}$$

We will say that the *telescope conjecture* holds for E if $L_E^f X \rightarrow L_E X$ is an equivalence for all X .

The proof of Theorem (4) is extremely simple. Let $X_0 = X$, form the wedge $W_0 = \vee_f A$ over the set of all homotopy classes of maps $f : A \rightarrow X$ from all members of $\mathcal{A}$, and let X_1 be the mapping cone of the evident map. Continue the process to form a diagram

$$\begin{array}{ccccccc} X & = & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & W_0 & & W_1 & & W_2 & & \end{array}$$

Form the mapping telescope X_∞ . We claim that the map $X \rightarrow X_\infty$ is a finite $\mathcal{A}$ -localization. To see that the map is a finite $\mathcal{A}$ -equivalence, let W be finitely $\mathcal{A}$ -local. Since

$$[\bigvee A, W] = \prod [A, W] = 0,$$

we get an inverse system of isomorphisms

$$[X, W] \xleftarrow{\cong} [X_0, W] \xleftarrow{\cong} [X_1, W] \xleftarrow{\cong} \dots,$$

so the Milnor sequence implies that

$$[X, W] \xleftarrow{\cong} [X_\infty, W]$$

as needed. On the other hand, let A be any member of $\mathcal{A}$, and $f: A \rightarrow X_\infty$. f compresses through some X_n , and is then killed in X_{n+1} —so $f = 0$. This completes the proof.

2. Some properties of finite localizations

If we form the tower of fibers of the maps from X to the stages X_n in the construction of the finite $\mathcal{A}$ -localization, we obtain a sequence

$$\begin{array}{ccccccc} * & = & T_0 & \longrightarrow & T_1 & \longrightarrow & T_2 \longrightarrow \dots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & W_1 & & W_1 & & W_2 \end{array}$$

in which the fibers are the same (up to suspension) as those in the sequence of X_n 's. (The octahedral axiom [8] is used here.) The telescopes of the three sequences fit into a cofibration sequence. (Checking this is one place where a return to some underlying category of spectra seems essential; a map between cofibration sequences cannot generally be extended to a " 3×3 " diagram.) $T_\infty \rightarrow X$ might thus be called the finite $\mathcal{A}$ -acyclicization of X if such a thing were pronounceable. If X is finitely $\mathcal{A}$ -acyclic, then $T_\infty \xrightarrow{\cong} X$:

PROPOSITION (5) *A spectrum is finitely $\mathcal{A}$ -acyclic if and only if it is equivalent to the telescope of a sequence of cofibrations whose quotients are wedges of elements of $\mathcal{A}$.*

Thus the telescope conjecture for E is equivalent to the assertion that any E -acyclic spectrum can be expressed as such a telescope.

COROLLARY (6). *The class of finitely $\mathcal{A}$ -acyclic spectra is the smallest class of spectra which is closed under cofibers and all wedges and which contains $\mathcal{A}$.*

Remark (7). These closure conditions on a class $\mathcal{C}$ imply that $\mathcal{C}$ is also closed under retracts. For, it is closed under formation of mapping telescopes (since

these are cofibers of maps between wedges), and C is the telescope of the sequence

$$\begin{array}{ccccccc} C \vee D & \longrightarrow & C \vee \Sigma C & \longrightarrow & C \vee \Sigma^2 D & \longrightarrow & C \vee \Sigma^3 C \longrightarrow \dots \\ \uparrow \alpha & & \uparrow \beta & & \uparrow \alpha & & \uparrow \beta \\ C \vee D & & \Sigma C \vee \Sigma D & & \Sigma^2 C \vee \Sigma^2 D & & \Sigma^3 C \vee \Sigma^3 D \end{array}$$

where $\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

COROLLARY (8). *If Z is a finitely $\mathcal{A}$ -acyclic spectrum and X is any spectrum, then $X \wedge Z$ is again finitely $\mathcal{A}$ -acyclic.*

PROPOSITION (9). *For any spectrum X , the natural map*

$$X \cong X \wedge S^0 \xrightarrow{1 \wedge \eta} X \wedge L_{\mathcal{A}}^f S^0$$

is a finite $\mathcal{A}$ -localization.

Proof. We show that this map satisfies the defining conditions. To see that the target is finitely $\mathcal{A}$ -local, let $A \in \mathcal{A}$ and $f : A \rightarrow X \wedge L_{\mathcal{A}}^f S^0$ a map. There is a finite subspectrum X_{α} of X , and a stage S_n^0 in the directed system constructing $L_{\mathcal{A}}^f S^0$, such that f compresses through $X_{\alpha} \wedge S_n^0$. Form the adjoint map $A \wedge DX_{\alpha} \rightarrow S_n^0$. The evident induction over skelata of DX_{α} shows that the composite $A \wedge DX_{\alpha} \rightarrow S_{n+k}^0$ is null for some k , and hence that $f = 0$.

To see that the map is a finite $\mathcal{A}$ -equivalence, notice that by Corollary (8), each map in the inverse system

$$[X \wedge S^0, W] = [X \wedge S_0^0, W] \longleftarrow [X \wedge S_1^0, W] \longleftarrow \dots$$

is bijective. The Milnor sequence then implies that

$$[X, W] \xleftarrow{\cong} [X \wedge L_{\mathcal{A}}^f S^0, W].$$

COROLLARY (10). *The telescope conjecture holds for E if and only if E is smashing (i.e., the map $X \cong X \wedge S^0 \rightarrow X \wedge L_E S^0$ is an equivalence) and the natural map $L_E^f S^0 \rightarrow L_E S^0$ is an equivalence.*

COROLLARY (11). *Finite $\mathcal{A}$ -localization is Bousfield localization with respect to the spectrum $L_{\mathcal{A}}^f S^0$.*

Proof. Take for X the spectrum $R = L_{\mathcal{A}}^f S^0$. Since R is finitely $\mathcal{A}$ -local, the map

$$R \cong R \wedge S^0 \xrightarrow{1 \wedge \eta} R \wedge R$$

is an equivalence. Its inverse gives R the structure of a ring-spectrum for which the multiplication map is an equivalence. Standard arguments ([7],

proof of 3.7) show that if R is any such ring-spectrum and any spectrum X , the map

$$X \cong X \wedge S^0 \longrightarrow X \wedge R$$

is an R -localization, and this proves the corollary.

3. The examples

We recall a basic fact about the Morava K -theories $K(n)$.

THEOREM (12). [4] *Any $K(n)$ -acyclic finite complex is $K(n-1)$ -acyclic.*

It follows that the finite localization with respect to $K(n)$ coincides with the finite localization with respect to the spectrum

$$K(\leq n) = K(0) \vee K(1) \vee \cdots \vee K(n).$$

Write L_n^f for this localization functor; this coincides with the finite localization with respect to the spectrum $E(n)$. The spectrum $K(n)$ is not smashing unless $n = 0$, but a deep theorem of Hopkins and Ravenel (see [6]) asserts that $K(\leq n)$ is smashing. Write L_n for the Bousfield localization with respect to $K(\leq n)$. The traditional telescope conjecture is the telescope conjecture for this theory $K(\leq n)$. Bousfield [1] used computations of Mark Mahowald and the author to verify it for $n = 1$, and observed that it gave a filtration of any $K(\leq 1)$ -acyclic spectrum whose quotients are wedges of certain finite $K(\leq n)$ -acyclic spectra; see Corollary (16) below.

We now explain how the nilpotence theorem can be used to identify certain L_n^f localizations with more explicit telescopes.

THEOREM (13). [2] 1. *If B is a $K(n-1)$ -acyclic finite complex then for some q there is a map $\phi : B \rightarrow \Sigma^{-q}B$ inducing an isomorphism in $K(n)$ and nilpotent maps in $K(m)$ for all $m \neq n$. (Such a map is called a v_n -self map.)*

2. *If A and B are $K(n-1)$ -acyclic finite complexes with v_n -self-maps ψ and ϕ , and $f : A \rightarrow B$ is any map, then there are positive integers i and j for which the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \psi^i & & \downarrow \phi^j \\ \Sigma^{-r}A & \xrightarrow{\Sigma^{-r}f} & \Sigma^{-r}B \end{array}$$

commutes.

Given any self-map $\phi : B \rightarrow \Sigma^{-q}B$, let $\text{tel}(\phi)$ be the corresponding mapping telescope of the sequence

$$\begin{array}{ccccccc} B & \xrightarrow{\phi} & \Sigma^{-q}B & \xrightarrow{\Sigma^{-q}\phi} & \cdots \\ \uparrow & & \uparrow & & \\ C & & \Sigma^{-q}C & & \end{array},$$

in which C is the desuspension of the mapping cone of ϕ . Note that C is $K(n)$ -acyclic.

PROPOSITION (14). *If B is a $K(n-1)$ -acyclic finite complex, with v_n -self-map $\phi : B \rightarrow \Sigma^{-q}B$, then the map $B \rightarrow \text{tel}(\phi)$ is a finite $K(n)$ -localization.*

Proof. First, the map is a finite $K(n)$ -equivalence: Let W be finitely $K(n)$ -local. Since C is $K(n)$ -acyclic, each map in the sequence

$$[B, W] \xleftarrow{[\phi, 1]} [\Sigma^{-q}B, W] \xleftarrow{[\Sigma^{-q}\phi, 1]} \dots$$

is an isomorphism. The Milnor sequence shows that $[B, W] \leftarrow [\text{tel}(\phi), W]$ is bijective.

Next, $\text{tel}(\phi)$ is finitely $K(n)$ -local: Let A be any finite $K(n)$ -acyclic spectrum, and $f : A \rightarrow \text{tel}(\phi)$ any map. f compresses through a map $g : A \rightarrow \Sigma^{-kq}B$ for some k . The trivial map is a v_n -self-map of A , so for some j the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & \Sigma^{-nq}B \\ \downarrow * & & \downarrow \Sigma^{-nq}\phi^j \\ \Sigma^{-jq}A & \xrightarrow{\Sigma^{-jq}g} & \Sigma^{-(n+j)q}B \end{array}$$

commutes. Thus $f = 0$.

Finally, we note that there is a criterion for a spectrum W to be finitely $K(n)$ -local in terms of homotopy with suitable coefficients:

PROPOSITION (15). *Let A be any finite spectrum with $K(n+1)_*A \neq 0$ and $K(n)_*A = 0$. A spectrum W is finitely $K(n)$ -local if and only if*

- (i) $\pi_*(W; \mathbb{Z}/l) = 0$ for all primes $l \neq p$, and
- (ii) $[\Sigma^k A, W] = 0$ for all $k \in \mathbb{Z}$.

Proof. Let $\mathcal{A}$ be the set consisting of A and $S^0 \cup_l e^1$ for all $l \neq p$. The set of finite finitely $\mathcal{A}$ -acyclic spectra is closed under cofibers and retracts (by Remark (7)), so it must be the class of finite $K(m)$ -acyclics for some m , by the main theorem of [2]. It also consists entirely of $K(n)$ -acyclics (since its generators are $K(n)$ -acyclic), but contains the $K(n+1)$ -nonacyclic A . Hence $m = n$.

COROLLARY (16). *Let A be as in the proposition. Then any finitely $K(n)$ -acyclic spectrum is the telescope of a sequence of spectra having wedges of suspensions of A and mod l -Moore spaces for l prime to p as cofibers.*

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