

Eilenberg Mac Lane spectra as p -cyclonic Thom spectra

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Abstract

Hopkins and Mahowald gave a simple description of the mod p Eilenberg Mac Lane spectrum \mathbb{F}_p as the free \mathbb{E}_2 -algebra with an equivalence of p and 0. We show for each faithful 2-dimensional representation λ of a p -group G that the G -equivariant Eilenberg Mac Lane spectrum \mathbb{F}_p is the free \mathbb{E}_λ -algebra with an equivalence of p and 0. This unifies and simplifies recent work of Behrens, Hahn, and Wilson, and extends it to include the dihedral 2-subgroups of $O(2)$. The main new idea is that \mathbb{F}_p has a simple description as a p -cyclonic module over $\mathrm{THH}(\mathbb{F}_p)$. We show that our result is the best possible one in that it gives all groups G and representations V such that \mathbb{F}_p is the free \mathbb{E}_V -algebra with an equivalence of p and 0.

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1 | INTRODUCTION

Consider the nontrivial virtual bundle on the circle, $S^1 \rightarrow \mathrm{BO}$. We can extend this to a double loop map $\mu_2 : \Omega^2 S^3 \rightarrow \mathrm{BO}$. Mahowald proved the following result [17].

Theorem 1.1 (Mahowald). *The Thom spectrum $(\Omega^2 S^3)^{\mu_2}$ is equivalent as an \mathbb{E}_2 -ring to the Eilenberg Mac Lane spectrum \mathbb{F}_2 .*

This result and its generalizations have since found great use, for example, giving a conceptual explanation of Bokstedt's computation of $\mathrm{THH}(\mathbb{F}_p)$ [7, 8].

Hopkins observed that Mahowald's theorem generalizes to odd primes. We can consider the p -local stable spherical fibration on the circle $S^1 \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)})$ representing the class $1 - p$ in π_1 . Extending this to a double loop map $\mu_p : \Omega^2 S^3 \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)})$, he showed the following theorem.

Theorem 1.2 (Hopkins [18, Theorem 4.18]). *The Thom spectrum $(\Omega^2 S^3)^{\mu_p}$ is equivalent as an \mathbb{E}_2 -ring to the Eilenberg Mac Lane spectrum \mathbb{F}_p .*

Remark 1.2.1. The Thom spectrum in Theorems 1.1 and 1.2 can be interpreted as the free \mathbb{E}_2 -ring with $p \simeq 0$ [2, Theorem 5.1]. This means that an \mathbb{E}_2 -algebra map $(\Omega^2 S^3)^{\mu_p} \rightarrow R$ is the same as a nulhomotopy of p in R .

Behrens–Wilson proved an equivariant generalization of Mahowald's result [6], which we now explain.

Construction 1.2.1. Let G be a group and $V : G \rightarrow \mathrm{O}(n)$ be any representation. We can consider the V -loop map $\mu_{V,p} : \Omega^V S^{1+V} \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)})$ extending the map $1 - p \in \pi_1^G(\mathrm{BGL}_1(\mathbb{S}_{(p)}))$ where \mathbb{S} is the G -equivariant sphere spectrum. Then $X_{V,p}$ is defined to be the Thom spectrum of $\mu_{V,p}$, which is an \mathbb{E}_V -algebra.

The Thom spectra $X_{V,p}$ are equivariant generalizations of the Thom spectra considered by Hopkins and Mahowald. For example, as in Remark 1.2.1, $X_{V,p}$ is the free \mathbb{E}_V -algebra with $p \simeq 0$.

Theorem 1.3 (Behrens–Wilson). *Let ρ be the regular representation of the group C_2 . Then there is an equivalence of \mathbb{E}_ρ -algebras $X_{\rho,2} \cong \mathbb{F}_2$.*

Their proof relies on studying \mathbb{E}_ρ power operations, similarly to how Mahowald and Hopkins' results rely on \mathbb{E}_2 power operations.

Hahn–Wilson generalized the result to all cyclic p -subgroups of $\mathrm{O}(2)$ [12].

Theorem 1.4 (Hahn–Wilson). *Let $\lambda : G \rightarrow \mathrm{O}(2)$ be a faithful representation of a cyclic p -group. Then there is an equivalence of \mathbb{E}_λ -algebras $X_{\lambda,p} \cong \mathbb{F}_p$.*

Notably, their proof did not involve studying \mathbb{E}_λ power operations, but rather used the structure on $X_{\lambda,p}$ as a module over its norm $N_*^G(X_{\lambda,p})$, the nonequivariant Hopkins–Mahowald result, and an exotic Thom ring structure on the Thom spectrum as inputs.

We extend the result of Hahn–Wilson to the dihedral 2-subgroups of $\mathrm{O}(2)$ and give a simpler proof in the case of $C_{p^n} \subset \mathrm{SO}(2)$, removing the need for exotic Thom ring structures.

Theorem A. *Let $\lambda : G \rightarrow \mathrm{O}(2)$ be a faithful representation of a p -group. Then there is an equivalence of \mathbb{E}_λ -algebras $X_{\lambda,p} \cong \mathbb{F}_p$.*

There is a concise way to capture all of the structure present in Theorem A. Recall [21, Definition 2.10] that a *cyclonic* spectrum is one that has a Borel S^1 -action genuine with respect to the finite subgroups of S^{1^\dagger} . If it is only genuine with respect to the finite p -subgroups of S^1 , it is said to be *p-cyclonic*.

In this language, for $p > 2$, Theorem A says that $X_{\lambda,p} \cong \mathbb{F}_p$ as p -cyclonic \mathbb{E}_λ -algebras.[‡]

At the prime 2, Theorem A gives more than a 2-cyclonic equivalence since it is a genuine equivalence on the finite dihedral 2-groups D_{2^n} as well. Accordingly, we can say that a *real 2-cyclonic* spectrum is a Borel $\mathrm{O}(2)$ -spectrum that is genuine on the finite 2-subgroups of $\mathrm{O}(2)$. Then Theorem A says for $p = 2$ that $X_{\lambda,2} \cong \mathbb{F}_2$ as real 2-cyclonic \mathbb{E}_λ -algebras.

Theorem A answers a question of Hahn and Wilson [12, Question 9.4], asking if \mathbb{F}_p can be a Thom spectrum for any group G that is not a cyclic p -group. It is also the best possible result in that it gives all groups G and representations V of G such that the free \mathbb{E}_V -algebra with $p \simeq 0$ is \mathbb{F}_p .

Theorem B. *Let G be a finite group and $V : G \rightarrow \mathrm{O}(n)$ a representation. $X_{V,p} \cong \mathbb{F}_p$ if and only if $n = 2$, G is a p -group, and V is faithful.*

The main new idea in the proof of Theorem A is to use a module structure that exists on both \mathbb{F}_p and $X_{\lambda,p}$ that is constructed via equivariant factorization homology.

Definition 1.5. Let G be a finite group and $V : G \rightarrow \mathrm{O}(n)$ a representation. For an \mathbb{E}_V -algebra X , we define $N^V X$ to be the genuine equivariant factorization homology $\int_{\mathbb{R}^V - \mathbb{R}^V}^G X$ as in [15].

Some important properties of N^V are as follows.

- (N1) Any \mathbb{E}_V -algebra R is naturally an \mathbb{E}_{V^G} -algebra over the \mathbb{E}_{V^G+1} -ring $N^V R$.
- (N2) For any map f of \mathbb{E}_V -algebras such that $f^{\Phi H}$ is an equivalence for any proper subgroup $H \subset G$, $N^V f$ is an equivalence.

In the proof of Theorem A, we construct a map of $\mathrm{O}(2)$ - \mathbb{E}_λ -algebras $f : X_{\lambda,p} \rightarrow \mathbb{F}_p$, and prove by induction on the finite p -subgroups $G \subset \mathrm{O}(2)$ that $f^{\Phi G}$ is an equivalence. By (N2), in the induction step we know that $(N^\lambda f)^{\Phi G}$ is an equivalence, so that by (N1), $f^{\Phi G}$ is a $N^\lambda \mathbb{F}_p$ -module map. The following result is key in showing that $f^{\Phi G}$ is an equivalence.

Theorem C. *Let $\lambda : G \rightarrow \mathrm{O}(2)$ be a faithful irreducible representation of a p -group. Then $X_{\lambda,p}^{\Phi G}$ and $\mathbb{F}_p^{\Phi G}$ are rank 2 free modules over $(N^\lambda X_{\lambda,p})^{\Phi G}$ and $(N^\lambda \mathbb{F}_p)^{\Phi G}$, respectively, generated in degrees 0 and 1.*

Because $f^{\Phi G}$ is a module map, Theorem C reduces showing that $f^{\Phi G}$ is an equivalence to just showing that it is 1-connected.

[†] This terminology was introduced in [5].

[‡] The equivalence is a map of genuine $\mathrm{O}(2)$ spectra (though it is only an equivalence on the p -subgroups).

Our proof of Theorem C is a calculation, which for the cyclic groups $G = C_{p^n}$ is made easier by the fact that we identify $N^\lambda \mathbb{F}_p$ as a G -spectrum with the topological Hochschild homology $\mathrm{THH}(\mathbb{F}_p)$. In fact, for the dihedral 2-subgroups $\lambda : D_{2^n} \rightarrow \mathrm{O}(2)$, $N^\lambda \mathbb{F}_2$ agrees with the real topological Hochschild homology $\mathrm{THR}(\mathbb{F}_2)$.

Remark 1.5.1. Theorem C can also be concisely stated for all G at once using cyclonic spectra. At odd primes, it says that $X_{\lambda,p}^{\Phi C_p}$ and $\mathbb{F}_p^{\Phi C_p}$ are free rank 2 p -cyclonic $\mathrm{THH}(\mathbb{F}_p)$ -modules in degrees 0 and 1, and for $p = 2$, the statement is the same, but with real 2-cyclonic instead of p -cyclonic.

In section 5, we also identify $X_{\lambda,p}$ when λ is a faithful representation of a cyclic group C_q for q a prime not equal to p . The underlying G -spectrum is algebraic: it is a sum of suspensions of Mackey functors. More generally, we can ask for $X_{\lambda,p}$ to be algebraic in a weaker sense: the underlying G -spectrum, viewed as a spectral Mackey functor as in [11] or [4], can be lifted to a Mackey functor with values in \mathbb{F}_p -modules.

Question 1.5.1. Under what conditions does the G -spectrum $X_{V,p}$ lift to a Mackey functor with values in \mathbb{F}_p -modules?

A necessary condition is that $\dim(V) \geq 2$, and in section 5, we explain why it is plausible this is sufficient.

In section 6, we show that when $p = 2$, the Thom spectra in Theorem A admit extra structure.

Theorem D. *Let λ be the standard 2-dimensional representation of $\mathrm{O}(2)$. The map $\mu_{\lambda,2} : \Omega^\lambda S^{1+\lambda} \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(2)})$ admits a refinement to a $\lambda + \det \lambda$ -loop map.*

Corollary 1.6. *When $p = 2$, the equivalence in Theorem A is one of $\mathbb{E}_{\lambda+\det \lambda}$ -algebras.*

It is known that Theorem 1.1 refines to an \mathbb{E}_3 -equivalence, coming from the equivalence $\Omega^2 S^3 = \Omega^3 \mathrm{BSp}(1)$. In [12], it was shown that the equivalence in Theorem 1.4 is an $\mathbb{E}_{\lambda+1}$ equivalence for G a cyclic 2-subgroup of $\mathrm{O}(2)$. Corollary 1.6 extends these results to include the dihedral subgroups D_{2^n} of $\mathrm{O}(2)$.

2 | MODULE STRUCTURES FOR \mathbb{E}_V -ALGEBRAS

An important tool in the proof of Theorem A is the use of various module and algebra structures associated to \mathbb{E}_V -algebras. In this section, we explain how these structures can be obtained via genuine equivariant factorization homology as in [15].

Equivariant factorization homology is a construction taking as input a V -framed G -manifold M and an \mathbb{E}_V -algebra X , and giving an object $\int_M X$ as output. For definitions and details, see [15], but we recall some properties here.

$$(F1) \quad \int_{\mathbb{R}^V} X = X.$$

(F2) It is a G -symmetric monoidal functor, meaning that it is natural with respect to restriction to subgroups and takes G -set indexed disjoint unions of manifolds to G -set indexed tensor products (i.e., the Hill–Hopkins–Ravenel norm in G -spectra).

- (F3) If a G -framed manifold M is equipped with a collar, that is an identification $M \simeq M' \otimes (-1, 1)$ as a framed G -manifold, this naturally gives $\int_{M' \otimes (-1, 1)} X$ an \mathbb{E}_1 -algebra structure [15, Construction 5.2.1].
- (F4) (G - \otimes -excision) Given a V -framed manifold M with a decomposition $M = M_{-1} \cup_{N \times (-1, 1)} M_1$ into open submanifolds M_{-1}, M_1 glued along a collar $N \times (-1, 1)$, $\int_{M_{-1}} X$ and $\int_{M_1} X$ are naturally left and right modules respectively over $\int_{N \times (-1, 1)} X$, and $\int_M X = \int_{M_{-1}} X \otimes_{\int_{N \times (-1, 1)} X} \int_{M_1} X$.

From these properties, we extract general results about \mathbb{E}_V -algebras.

Proposition 2.1. *Let U be an open subset of $S(V)$, the unit sphere of \mathbb{R}^V , and X an \mathbb{E}_V -algebra. Let S_U be the preimage of U under the projection $\mathbb{R}^V - 0 \rightarrow S(V)$. Then X is a module over the \mathbb{E}_1 -ring $\int_{S_U} X$, and there is a canonical map of $\int_{S_U} X$ modules $\int_{S_U} X \rightarrow X$ that is compatible with unit maps $\mathbb{S} \rightarrow X$ and $\mathbb{S} \rightarrow \int_{S_U} X$. This module structure is natural in U .*

Proof. The V -framed G -manifold \mathbb{R}^V decomposes as $\mathbb{R}^V \cup_{S_U} S_U$, and S_U admits a collar $S_U \cong U \times (-1, 1)$. Thus the first claim follows from property (F4) of equivariant factorization homology above. The second claim follows by observing that there is a tautological decomposition $S_U = S_U \cup_{S_U} S_U$ compatible with the decomposition of $\mathbb{R}^V - 0$, so by naturality of property (F4), we obtain our map of modules. To see that this map is unital, we simply observe that the unit maps are the maps induced on factorization homology from the inclusion of the empty manifold ϕ . Naturality in U follows from naturality in (F3), (F4). \square

When we set $U = S(V) - S(V^G)$ in Proposition 2.1 $S_U = \mathbb{R}^V - \mathbb{R}^{V^G}$, so we obtain the following.

Corollary 2.2. *Given an \mathbb{E}_V -algebra X , X is naturally a module over $N^V X$, and receives a unital $N^V X$ -module map from $N^V X$.*

Remark 2.2.1. Corollary 2.2 says that X is an \mathbb{E}_0 -algebra in $N^V X$ -modules. In fact, it is true that $N^V X$ is an \mathbb{E}_{V^G+1} -algebra and X is canonically an \mathbb{E}_{V^G} -algebra in $N^V X$ -modules, and this refines the \mathbb{E}_{V^G} -algebra in G -spectra.

We also need the following module structure, where $N_*^P X$ is the indexed smash product (i.e., HHR norm).

Proposition 2.3. *Let $P \subset S(V)$ be a discrete G -subset. Then P determines a natural module structure of X over $N_*^P X$ as well as a unital $N_*^P X$ -module map $N_*^P X \rightarrow X$.*

Proof. Choose disjoint G -equivariant disks around each point p in S^V in a G -invariant way, and let U be the union of these disks. For each orbit $O \subset P$ with isotropy group H , the union of the preimages of the disks around O in S_U is isomorphic to $G \times_H \mathbb{R}^{V|H}$. It then follows from properties (F1) and (F2) of equivariant factorization homology that $\int_{S_U} X \cong N_*^P X$. The result then follows by applying Proposition 2.1 to U . \square

The following facts about $N^\lambda X$ are important to the proof of the main theorem.

Lemma 2.4. *Let $\lambda : G \rightarrow \mathrm{O}(2)$ be a faithful representation. If $X \rightarrow Y$ is a map of \mathbb{E}_λ -algebras that is an equivalence on H fixed points for H any proper subgroup of G , then the map $N^\lambda X \rightarrow N^\lambda Y$ is an equivalence of \mathbb{E}_1 -algebras.*

Proof. In the case that $G \rightarrow \mathrm{O}(2)$ factors through $\mathrm{SO}(2)$, we can break up $\mathbb{R}^\lambda - 0$ into two copies of $\mathbb{R}^2 \times G$ glued along the collared manifold $(\mathbb{R}^1 \cup \mathbb{R}^1) \times G \times (-1, 1)$. By G - \otimes -excision (F4) and the fact that induced manifolds are sent to norms (F2), $\int_{\mathbb{R}^2 \times G} X = N_*^G X$, which depends on the underlying spectrum of X , giving the result.

In the other case, $G = D_{2n}$ is a dihedral subgroup of $\mathrm{O}(2)$ for $n \geq 1$. If $n = 1$, then $N^\lambda X = N_*^{C_2} X$, which again only depends on the underlying spectrum.

If $n \geq 2$, we similarly use G - \otimes excision to see that $N^\lambda X$ is identified the tensor product $N^{G/H_0} X \otimes_{N^{G_X}} N^{G/H_1} X$ where H_0 are two conjugate maximal subgroups of D_{2n} . This formula again only depends on X^H for proper subgroups of G . \square

Remark 2.4.1. More generally, the factorization homology \int_M depends on the H fixed points when H is contained in an isotropy group appearing in M . In particular $N^V X$ only depends on X^H for H a proper subgroup of G .

When λ is a two-dimensional representation of G , N^λ can be identified with a twisted version of THR. We only use the following special case.

Lemma 2.5. *Let $\lambda : G \rightarrow \mathrm{SO}(2)$ be a faithful representation, and let X be a G -commutative algebra such that the G -action on the underlying algebra is trivial. Then there is an identification $N^\lambda X \simeq \mathrm{THH}(X_{un})$, where THH has the standard genuine G action.*

Proof. Consider the covering map $\mathbb{R}^\lambda - 0 \rightarrow \mathbb{R}^2 - 0$ which (ignoring framings) is a $|G|$ -fold cover. We can break $\mathbb{R}^2 - 0$ into two copies of \mathbb{R}^2 glued along a collared $(\mathbb{R} \cup \mathbb{R}) \times (-1, 1)$, and this induces a decomposition of $\mathbb{R}^\lambda - 0$ into $\mathbb{R}^2 \times G$ and $(\mathbb{R} \times \mathbb{R}) \times G \times (-1, 1)$. We can choose a framing on $(\mathbb{R} \times \mathbb{R}) \times G \times (-1, 1)$ so that this gives a collar decomposition. The choice of framing does not matter because of commutativity.

By G - \otimes -excision, this decomposition realizes $N^\lambda X$ as $N^G X \otimes_{N^{G_X} \otimes N^{G_X}} N^G X$. Since the action on the underlying spectrum of X is trivial, this tensor product agrees with the formula in [1, Theorem 4.4] of the genuine G -action on $\mathrm{THH}(X)$. \square

In the next section we use the following.

Lemma 2.6. *Let V be a representation of G of dimension ≥ 2 . Then $(N^V X_{V,p})^{\Phi G}$ and $X_{V,p}^{\Phi G}$ are \mathbb{F}_p -modules.*

Proof. Let N be the kernel of V . Then $S(V) - S(V^G)$ contains an orbit of the form G/N . Thus we obtain by Proposition 2.3, Corollary 2.2, and the naturality in Proposition 2.1 an algebra map $N_N^G X_{V,p} \rightarrow N^V X_{V,p}$. We can apply geometric fixed points and use [14, Proposition 2.57] to obtain an algebra map $(X_{V,p})^{\Phi N} \rightarrow (N^V X_{V,p})^{\Phi G}$. But N acts trivially on V , so it acts trivially on $\Omega^V S^{V+1}$ and we can identify $(X_{V,p})^{\Phi N}$ with the nonequivariant spectrum $X_{V,p}$. But this underlying spectrum is an \mathbb{F}_p -algebra by Theorem 1.1 and Theorem 1.2 since $\dim V \geq 2$.

Via Corollary 2.2, $X_{V,p}^{\Phi G}$ is a module over $(N^V X_{V,p})^{\Phi G}$, so it is also an \mathbb{F}_p -module. \square

3 | COMPUTING GEOMETRIC FIXED POINTS

Here, we study the geometric fixed points of \mathbb{F}_{-p} and $X_{\lambda,p}$, the objects of study in Theorem A. Often $X_{\lambda,p}$ will be shortened to just X_λ . The results of this section together prove the following theorem.

Theorem C. Let $\lambda : G \rightarrow \mathrm{O}(2)$ be a faithful irreducible representation of a p -group. Then $X_{\lambda,p}^{\Phi G}$ and $\mathbb{F}_{-p}^{\Phi G}$ are rank 2 free modules over $(N^\lambda X_{\lambda,p})^{\Phi G}$ and $(N^\lambda \mathbb{F}_{-p})^{\Phi G}$, respectively, generated in degrees 0 and 1.

Proof. This follows immediately from Proposition 3.4, Proposition 3.8, and Proposition 3.2. \square

Thom spectra

In this subsection, we study X_λ as an $N^\lambda X_\lambda$ -module. The key fact allowing us to do so is that the module structure arises from taking the Thom spectrum of an action on the level of spaces.

Lemma 3.1. Let λ be a 2-dimensional faithful irreducible representation of G .

Then $\mathrm{Map}^G(S(\lambda), S^{\lambda+1})$ is simply connected.

Proof. For any such λ , $S(\lambda)$ has a cell decomposition with a 1-cell with isotropy G and zero cells with nontrivial isotropy group. Let P be the 0-skeleton of such a cell decomposition. Then there is a pushout square

$$\begin{array}{ccc} (S^0 \times G) & \longrightarrow & P \\ \downarrow & \lrcorner & \downarrow \\ G & \longrightarrow & S(\lambda), \end{array}$$

which upon applying $\mathrm{Map}^G(-, S^{\lambda+1})$ yields the pullback square

$$\begin{array}{ccc} \mathrm{Map}^G(S(\lambda), S^{\lambda+1}) & \longrightarrow & S^3 \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}^G(P, S^{\lambda+1}) & \longrightarrow & S^3 \times S^3. \end{array}$$

Since P has no fixed points, $\mathrm{Map}^G(P, S^{\lambda+1})$ is a product of spheres of dimension ≥ 2 so is simply connected. Applying the exact sequence on homotopy groups to the above pullback square, we obtain that $\mathrm{Map}^G(S(\lambda), S^{\lambda+1})$ is simply connected. \square

Proposition 3.2. Let λ be a rank 2 irreducible representation of G . Then $X_{\lambda,p}^{\Phi G}$ is a rank 2 free module over $(N^\lambda X_{\lambda,p})^{\Phi G}$, generated in degrees 0 and 1.

Proof. For an \mathbb{F}_V -space X and a V -framed manifold M , equivariant nonabelian Poincaré duality says that $\int_M X = \mathrm{Map}_*(M^+, X)$, where M^+ is the 1-point compactification of M [16, Theorem 2.2].

This shows that $N^\lambda(\Omega^\lambda S^{\lambda+1})$ agrees with $\text{Map}_*((\mathbb{R}^\lambda - 0)^+, S^{\lambda+1})$, where $(\mathbb{R}^\lambda - 0)^+$ can also be described as either the cofiber of the inclusion $S^0 \rightarrow S^\lambda$ or $\Sigma S(\lambda)_+$.

Recall that by Corollary 2.2, X_λ is a module over $N^\lambda X$. By the naturality of equivariant non-abelian Poincaré duality in the manifold M , the $N^\lambda(\Omega^\lambda S^{\lambda+1})$ -module structure on $\Omega^\lambda S^{\lambda+1}$ comes from applying $\text{Map}_*(-, S^{\lambda+1})$ on the natural coaction of the cogroup $(\mathbb{R}^\lambda - 0)^+ = \Sigma S(\lambda)^+$ on $(\mathbb{R}^\lambda)^+$. This coaction arises from the coaction of the suspension on the cofiber in the cofiber sequence

$$S(\lambda)^+ \xrightarrow{\epsilon} *^+ \rightarrow (\mathbb{R}^\lambda)^+ \rightarrow (\mathbb{R}^\lambda - 0)^+.$$

Thus the action of $N^\lambda(\Omega^\lambda S^{\lambda+1})$ is the action of loop space on the fiber in the sequence

$$N^\lambda(\Omega^\lambda S^{\lambda+1}) \rightarrow \Omega^\lambda S^{\lambda+1} \rightarrow \text{Map}_*(S^0, S^{\lambda+1}) \xrightarrow{\epsilon^*} \text{Map}_*(S(\lambda)^+, S^{\lambda+1}).$$

Let R denote the \mathbb{E}_1 -ring $(N^\lambda X_{\lambda,p})^{\Phi G}$. The geometric fixed points of a Thom spectrum are given by the Thom spectrum of the fixed points, so the R -module structure of $X_{\lambda,p}$ arises from applying the Thom spectrum to the G -fixed points of the action of the first two maps in the above fiber sequence.

By Lemma 3.1, ϵ^* is nulhomotopic after taking G -fixed points since $(S^{\lambda+1})^G = S^1$. It follows that $\Omega^\lambda S^{\lambda+1}$ is the product $(N^\lambda(\Omega^\lambda S^{\lambda+1}))^G \times S^1$, and $N^\lambda(\Omega^\lambda S^{\lambda+1})^G$ acts only on the first component.

The Thom isomorphism in \mathbb{F}_p -homology identifies the action on homology with the action of $N^\lambda X_\lambda$ on X_λ in homology. But the action on \mathbb{F}_p -homology is a rank 2 free module generated in degrees 0 and 1 by the homology of S^1 . By Lemma 2.6, $X_\lambda, N^\lambda X_\lambda$ are \mathbb{F}_p -modules, so it follows that X_λ is a rank 2 free module generated in degrees 0, 1. □

Remark 3.2.1. A more careful analysis in Lemma 3.1 shows that when λ is a cyclic subgroup, $(\Omega^\lambda S^{\lambda+1})^G$ is equivalent to $\Omega S^3 \times S^1 \times \Omega^2 S^3$. When λ is a dihedral group D_{2^n} for $n \geq 2$, it is equivalent to $\Omega S^2 \times \Omega S^2 \times S^1 \times \Omega^2 S^3$. It is interesting that the proof of Proposition 3.2 does not require explicitly identifying these spaces. We do carry this more careful analysis out for the representation ρ in Lemma 3.3.

We now identify $X_{\rho,2}^{\Phi C_2}$, where ρ is the regular representation of C_2 . Since an \mathbb{E}_ρ -algebra is in particular an \mathbb{E}_1 -algebra as $1 \subset \rho$, and $(-)^{\Phi C_2}$ is symmetric monoidal, $X_{\rho,2}^{\Phi C_2}$ is an \mathbb{E}_1 -algebra.

Lemma 3.3. $X_{\rho,2}^{\Phi C_2}$ is the \mathbb{E}_1 -algebra $\mathbb{F}_2 \otimes \Sigma_+^\infty \Omega S^2$.

Proof. $X_{\rho,2}^{\Phi C_2}$ is a Thom spectrum of $(\Omega^\rho S^{\rho+1})^{C_2}$. To compute these fixed points, we apply $\text{Map}_*^{C_2}(-, S^{\rho+1})$ to the cofiber sequence $\Sigma(C_2)_+ \rightarrow S^1 \rightarrow S^\rho \rightarrow \Sigma^2(C_2)_+$ to get a fiber sequence of loop spaces

$$\Omega^2 S^3 \rightarrow (\Omega^\rho S^{\rho+1})^{C_2} \rightarrow \Omega S^2 \rightarrow \Omega S^3.$$

Since the second map is a loop map, it is nulhomotopic, so $(\Omega^\rho S^{\rho+1})^{C_2} = \Omega(S^2 \times \Omega S^3)$ as a loop space, and $\Omega^2 S^3$ acts trivially on the ΩS^2 component. Because the Thom spectrum functor is

G -symmetric monoidal and takes fixed points to geometric fixed points, the Thom spectrum of the composite $\Omega^2 S^3 = \text{Map}(\Sigma^2(C_2)_+, S^{\rho+1})^{C_2} \rightarrow (\Omega^\rho S^{\rho+1})^{C_2} \rightarrow \text{BGL}_1(\mathbb{S}_{(2)})$ is $(N_*^{C_2} X_{\rho,2})^{\Phi C_2}$. By [14, Proposition 2.57], this is the underlying spectrum of $X_{\rho,2}$, which is $X_{2,2} = \mathbb{F}_2$ by Theorem 1.1, so it follows that $X_{\rho,2}^{\Phi C_2}$ is the \mathbb{F}_1 - \mathbb{F}_2 -algebra $\mathbb{F}_2 \otimes \Sigma_+^\infty \Omega S^2$. \square

Mackey functors

Now we turn to understanding \mathbb{F}_p . First we study $\mathbb{F}_p^{\Phi G}$ as a $N^\lambda \mathbb{F}_p^{\Phi G}$ -module for the cyclic p -subgroups of $\text{SO}(2)$, using the identification of $N^\lambda \mathbb{F}_p$ with $\text{THH}(\mathbb{F}_p)$.

Proposition 3.4. *Let G be a p -subgroup of $\text{SO}(2)$. Then $\mathbb{F}_p^{\Phi G}$ is a rank 2 free module over $(N^\lambda \mathbb{F}_p)^{\Phi G}$ generated in degrees 0 and 1.*

Proof. Let H be the maximal proper subgroup of G . We have a commutative square of ring maps

$$\begin{array}{ccc} (N^\lambda \mathbb{F}_p)^{\Phi G} & \longrightarrow & \mathbb{F}_p^{\Phi G} \\ \downarrow & & \downarrow \\ ((N^\lambda \mathbb{F}_p)^{\Phi H})^{tG/H} & \longrightarrow & (\mathbb{F}_p^{\Phi H})^{tG/H}. \end{array}$$

$N^\lambda \mathbb{F}_p$ can be identified with $\text{THH}(\mathbb{F}_p)$ as a G -spectrum by Lemma 2.5. Since $\text{THH}(\mathbb{F}_p)$ is p -cyclotomic, the homotopy ring of $(N^\lambda \mathbb{F}_p)^{\Phi G} = \text{THH}(\mathbb{F}_p)^{\Phi G}$ agrees with that of $\text{THH}(\mathbb{F}_p)$, and so is a polynomial algebra $\mathbb{F}_p[x]$ where $|x| = 2$. By [19, Corollary IV.4.13], the element x becomes a unit in $\pi_* \text{THH}(\mathbb{F}_p)^{tC_p}$.

This map $\text{THH}(\mathbb{F}_p)^{\Phi C_p} \rightarrow \text{THH}(\mathbb{F}_p)^{tC_p}$ is identified with the left vertical map in the diagram above, so we learn that $x \in \pi_2(N^\lambda \mathbb{F}_p^{\Phi G})$ is sent to a unit via that map. A unit cannot be in the kernel of any nonzero ring map, so the image of x is not in the kernel of lower horizontal map. Thus x has nonzero image in $(\mathbb{F}_p^{\Phi H})^{tG/H}$, and thus also in $\mathbb{F}_p^{\Phi G}$.

The ring $\pi_*(\mathbb{F}_p^{\Phi G})$ is well known to be $\mathbb{F}_2[t]$ for $|t| = 1$ when $p = 2$, and $\mathbb{F}_p[b] \otimes \Lambda(s)$ with $|b| = 2$, $|s| = 1$ for $p > 2$ (see, for example, [14, Proposition 3.18] or [12, Lemma 8.2]). In either case, it is a rank 2 free module over the subalgebra generated by any nonzero class in degree 2, with module generators in degrees 0, 1. Thus the claim follows. \square

Remark 3.4.1. It is possible to identify $N^\lambda \mathbb{F}_2$ for the dihedral groups D_{2^n} with the genuine action of D_{2^n} on $\text{THR}(\mathbb{F}_2)$ (see [9, 13–20]). An approach similar to that in Proposition 3.4 should prove the analogous result for the dihedral groups.

Next, we study $\mathbb{F}_p^{\Phi G}$ where G is one of the dihedral groups D_{2^n} for $n \geq 2$. First we will review a few different ways of thinking about the isotropy separation sequence [14, Section 2.5.2], which for a G -spectrum X is the cofiber sequence

$$X_{hP} \xrightarrow{\text{tr}} X^G \rightarrow X^{\Phi G},$$

where $X_{hP} = (X \otimes EP_+)^G$, with EP the classifying space for proper subgroups of G , and tr the map induced from $EP_+ \rightarrow * = S^0$.

Let Orb'_G be the category of nontrivial orbits of G . One model for EP_+ is the colimit $\text{colim}_{G/H \in \text{Orb}'_G} G/H_+$, showing that an alternative formula for X_{hP} is $\text{colim}_{G/H \in \text{Orb}'_G} X^H$, and the natural map to X^G is the colimit of the transfer maps. From this description, we see the functor $(-)^{\Phi G}$ factors through certain fixed point functors.

Lemma 3.5. *Let G be a group, and let N be a normal subgroup contained in all the maximal proper subgroups of G . Then for any G -spectrum X , $X^{\Phi G} = (X^N)^{\Phi(G/N)}$, where X^N is given the structure of a genuine G/N spectrum.*

Proof. The category of orbits for proper subgroups of G contains the category of orbits of proper subgroups of G/N as the subcategory of orbits with isotropy containing N . We claim that this inclusion is final, which implies that the two isotropy separation sequences computing $X^{\Phi G}$ and $(X^N)^{\Phi(G/N)}$ agree, giving the result.

To see finality, we need to check for any orbit $G/H \in \text{Orb}'_G$, the category maps to orbits G/H' with $H' \supset N$ are nonempty and connected. It is nonempty since N contains all maximal proper subgroups. To see it is connected, any two maps $G/H \rightarrow G/H', G/H''$, choose basepoints on the G -sets so H, H', H'' are the isotropy groups of the basepoints. Then the projections $G/H \rightarrow G/H', G/H''$ factor through G/N as pointed G -sets. \square

Note that the above lemma is useful for understanding $\mathbb{F}_p^{\Phi G}$ since \mathbb{F}_p^N agrees with \mathbb{F}_p for the group G/N .

Another model for EP_+ is $S(\infty V)_+$, where V is any G -representation such that $V^G = 0$ and $V^H \neq 0$ for any proper subgroup of G . There is a cofiber sequence $S(\infty V)_+ \rightarrow S^0 \rightarrow S^{\infty V}$, showing that $X^{\Phi G}$ can also be described as $(X[a_V^{-1}])^G$ where a_V is the element in $\pi_{-V}^G \mathbb{S}$ arising from the inclusion $S^0 \rightarrow S^V$.

As a warm up, we will compute $\pi_*(\mathbb{F}_2^{\Phi G})$ when G is the group C_{2^n} . This is a well-known computation, but we present it to demonstrate the strategy and set up notation for computing the geometric fixed points for the dihedral groups D_{2^n} .

Lemma 3.6. *Let $G = C_{2^n}$. Then the homotopy ring $\pi_*(\mathbb{F}_2^{\Phi G})$ is a polynomial algebra on a generator u_σ in degree 1.*

Proof. Let σ be the nontrivial 1-dimensional representation of G , and let b be a choice of cyclic generator of G . Since σ^H is nontrivial for all proper subgroups of G , $\mathbb{F}_2^{\Phi G} = \mathbb{F}_2[a_\sigma^{-1}] = \text{colim}_n \mathbb{F}_2 \otimes S^{n\sigma}$. There is a minimal cell structure on $S^{n\sigma}$ with a cell called “1” in dimension 0 with isotropy group C_{2^n} and a cell called x_i with isotropy group $C_{2^{n-1}}$ in dimension i for $1 \leq i \leq n$. Taking the fixed points of the \mathbb{F}_2 cellular chain complex, we obtain a chain complex over \mathbb{F}_2 generated by 1 and $x_i + bx_i$, with zero differential, showing that $\pi_* \mathbb{F}_2 \otimes S^{n\sigma}$ is \mathbb{F}_2 in dimensions 0 to n . Taking the limit as $n \rightarrow \infty$, this computes $\pi_*(\mathbb{F}_2^{\Phi G})$ additively.

To obtain the multiplicative structure, we observe that there is another choice of cells on $S^{n\sigma}$ coming from the standard cell structure on S^σ and the decomposition $S^{n\sigma} \cong (S^\sigma)^{\otimes n}$. The isomorphism can be chosen so that it sends $1 \mapsto 1^{\otimes n}$ and $x_i \mapsto 1^{\otimes i} \otimes x_1 \otimes (x_1 + bx_1)^{\otimes n-i-1}$. From this, the multiplication sends $x_i \otimes x_j \rightarrow x_{i+j}$, so the description as a polynomial algebra follows. \square

Next, we study the dihedral groups D_{2^n} for $n \geq 2$.

Proposition 3.7. *Let $G = D_{2^n}$ for $n \geq 2$. Then there is an isomorphism of rings*

$$\pi_*(\mathbb{F}_2^{\Phi G}) = \mathbb{F}_2[u_{\sigma_1}, u_{\sigma_2}, u_{\sigma'}] / (u_{\sigma_1}u_{\sigma_2} + u_{\sigma_1}u_{\sigma'} + u_{\sigma_2}u_{\sigma'}).$$

Proof. The intersection of the maximal proper subgroups of D_{2^n} is a normal subgroup of index 4 with quotient $D_4 = C_2 \times C_2$. Thus by Lemma 3.5, we can assume $G = C_2 \times C_2$. Suppose that G is generated by b_1, b_2 , let σ_i be the 1-dimensional representation of G whose kernel is b_i , and let σ' be $\sigma_1 \otimes \sigma_2$.

The element $\sigma_1 + \sigma_2 + \sigma'$ has no fixed points and its restriction to all proper subgroups is nontrivial. Thus $X^{\Phi G} = X[a_{\sigma_1 + \sigma_2 + \sigma'}^{-1}]$ for all G -spectra X . Moreover, $a_{\sigma_1 + \sigma_2 + \sigma'} = a_{\sigma_1}a_{\sigma_2}a_{\sigma'}$.

Thus $\mathbb{F}_2^{\Phi G} = \text{colim}_{n,m,p \rightarrow \infty} \mathbb{F}_2 \otimes S^{n\sigma_1 + m\sigma_2 + p\sigma'}$. We can take the minimal cell structures used for $S^{\infty\sigma_i}$ and $S^{\infty\sigma'}$ used in Lemma 3.6 and tensor them together, where we use the names x_i, y_i, z_i for the cells in $\sigma_1, \sigma_2, \sigma'$, respectively. In the \mathbb{F}_2 cellular chain complex, the elements $(x_1 + b_2x_1) \otimes 1 \otimes 1, 1 \otimes (y_1 + b_1y_1) \otimes 1, 1 \otimes 1 \otimes (z_1 + b_1z_1)$ represent the classes $u_{\sigma_1}, u_{\sigma_2}, u_{\sigma'}$, respectively. The products work just as in Lemma 3.6 for the same reason.

By applying the cellular differential to the chain $\sum_{g \in G} g(x_1 \otimes y_1 \otimes z_1)$, we obtain the relation $u_{\sigma_1}u_{\sigma_2} + u_{\sigma_1}u_{\sigma'} + u_{\sigma_2}u_{\sigma'} = 0$ in the cohomology of the cellular complex. It is straightforward to check that the classes $u_{\sigma_1}, u_{\sigma_2}, u_{\sigma'}$ generate the homology of the chain complex as an algebra, and that there are no more relations (for example, by counting the dimension of the homology). \square

Proposition 3.8. *Let $G = D_{2^n}$ for $n \geq 2$, and λ the standard inclusion to $O(2)$. $\mathbb{F}_2^{\Phi G}$ is a rank 2 free module over $N^\lambda \mathbb{F}_2$ generated in degrees 0 and 1.*

Proof. We first reduce to the case $n = 2$. Present G as $\langle a, b | a^2, abab, b^{2^{n-1}} \rangle$, and consider subgroups $H_0 = \langle a \rangle, H_1 = \langle ab \rangle$. G - \otimes -excision (F4) lets us identify $N^\lambda \mathbb{F}_2$ with the tensor product $N^{G/H_0} \mathbb{F}_2 \otimes_{N^G \mathbb{F}_2} N^{G/H_1} \mathbb{F}_2$, so applying $(-)^{\Phi G}$, we get $\mathbb{F}_2^{\Phi H_0} \otimes_{\mathbb{F}_2} \mathbb{F}_2^{\Phi H_1}$. By factoring the functor $(-)^{\Phi G}$ as $((-)^{\Phi(b^2)})^{\Phi G / \langle b^2 \rangle}$, we are reduced to the case $n = 2$, so our group is $C_2 \times C_2$.

We compute the maps $\mathbb{F}_2^{\Phi H_i} \cong (N^{G/H_i} \mathbb{F}_2)^{\Phi G} \xrightarrow{N^{\Phi G}} \mathbb{F}_2^{\Phi G}$ on π_* for $i = 0, 1$. By symmetry we can assume $i = 0$. The map $\mathbb{F}_2 \rightarrow \beta \mathbb{F}_2 := \text{Map}(EG_+, \mathbb{F}_2)$ induces a commutative square:

$$\begin{array}{ccc} \mathbb{F}_2^{\Phi H_0} & \xrightarrow{N^{\Phi G}} & \mathbb{F}_2^{\Phi G} \\ \downarrow & & \downarrow \\ \mathbb{F}_2^{\tau H_0} & \xrightarrow{N^{\tau G}} & \mathbb{F}_2^{\tau G} \end{array}$$

The element u_σ in $\pi_1(\mathbb{F}_2^{\Phi H_0})$ that gets sent to an invertible element in $\pi_1(\mathbb{F}_2^{\tau H_0})$. It follows that its image in $\mathbb{F}_2^{\tau G}$, a nonzero ring, cannot be 0, so u_σ has nonzero image in $\mathbb{F}_2^{\Phi G}$ too.

To figure out what the image in $\mathbb{F}_2^{\Phi G}$ is, we observe that u_σ comes from applying geometric fixed points to an element in $\pi_{1-\sigma}^{H_0} \mathbb{F}_2$, where σ is the nontrivial 1-dimensional representation of H_0 , so its norm is an element in $\pi_{\text{Ind}_{H_0}^G (1-\sigma)} \mathbb{F}_2$. If σ_1 is the nontrivial representation of G fixing H_1 , σ_0 is the nontrivial representation fixing H_0 , and σ' is the tensor product, $\text{Ind}_{H_0}^G (1-\sigma) = 1 + \sigma_0 - \sigma_1 - \sigma'$.

Composing with a_{σ_0} , we get a nonzero element in $\pi_{1-\sigma_1-\sigma'}^G \mathbb{F}_2$ in the image of the norm map. By the proof of Proposition 3.7, this element is a nonzero linear combination of $u_{\sigma'}$ and u_{σ_1} , but there is an automorphism fixing H and swapping σ' and σ_1 , so by naturality of $N^{\Phi G}$, it must be the sum $u_{\sigma_1} + u_{\sigma'}$. Symmetrically, the generator for $\pi_1(\mathbb{F}_2^{\Phi H_1})$ gets sent to $u_{\sigma_0} + u_{\sigma'}$.

Finally, we observe that $\pi_*(\mathbb{F}_2^{\Phi G}) = \mathbb{F}_2[u_{\sigma_0}, u_{\sigma_1}, u_{\sigma'}]/(u_{\sigma_1}u_{\sigma_0} + u_{\sigma_1}u_{\sigma'} + u_{\sigma_0}u_{\sigma'})$ is freely generated in degrees 0 and 1 as a module over $\mathbb{F}_2[u_{\sigma_1} + u_{\sigma'}, u_{\sigma_0} + u_{\sigma'}]$. \square

4 | PROOF OF THEOREM A

In this section we prove Theorem A. Let λ be the standard 2-dimensional representation of $O(2)$. We begin by producing a map of genuine $O(2)$ - \mathbb{E}_λ -algebras from $X_{\lambda,p} \rightarrow \mathbb{F}_p$. As before, $X_{\lambda,p}$ will often be shortened to X_λ .

Lemma 4.1. *There is a map $f_\lambda : X_\lambda \rightarrow \mathbb{F}_p$ of genuine $O(2)$ - \mathbb{E}_λ -algebras.*

Proof. Since X_λ is an \mathbb{E}_λ Thom spectrum, to give such a map f_λ , it suffices by [10, X.6.4] to show that the composite $S^{\lambda+1} \rightarrow B^{\lambda+1} GL_1(\mathbb{S}_{(p)}^0) \rightarrow B^{\lambda+1} GL_1(\mathbb{F}_p)$ is null.

The map $\mu_{\lambda,p}$ represents the class $1 - p$ in $\pi_0^G(GL_1(\mathbb{F}_p))$, but $1 - p = 1$ is the basepoint. \square

Remark 4.1.1. In fact, the space of nulhomotopies in Lemma 4.1 is contractible, so the map f_λ is essentially unique.

Restricting f_λ to the p -subgroups of $O(2)$, we obtain our map of interest. There are two kinds of p -subgroups of $O(2)$. One family consists of the cyclic subgroups $C_{p^n} \subset SO(2)$, and the other family consists of the dihedral subgroups D_{2^n} , $n \geq 1$. By convention, D_{2^n} has order 2^n , so that D_2 is the cyclic group of order 2 with its regular representation into $O(2)$.

We prove Theorem A by inducting on the finite subgroups of $O(2)$. This means that the base case is when G is the trivial group, where the result is the nonequivariant result of Hopkins–Mahowald (Theorems 1.1 and 1.2). This shows that Borel equivariantly, f_λ is an equivalence, and the rest of the proof amounts to turning this into an equivalence of genuine G -spectra. In the inductive step, we have a finite p -subgroup $G \subset O(2)$, and we assume that f_λ^H is an equivalence for all proper subgroups of G .

The first step is to compute π_0 of X_λ . The main ingredient is to use maps from norms of X_λ to itself to show that certain elements of the Burnside ring are zero.

Proposition 4.2. *When λ is a faithful rank 2 representation of a p -group, $\pi_0(X_\lambda) = \mathbb{F}_p$.*

Proof. $p = 0$ in $\pi_0^G X_\lambda$ by construction: the Thom spectrum of the composite $S^1 \rightarrow \Omega^\lambda S^{\lambda+1} \rightarrow BGL_1(\mathbb{S}_{(p)})$ is the Moore spectrum \mathbb{S}/p , so there is a unital map $\mathbb{S}/p \rightarrow X_\lambda$, exhibiting the nulhomotopy of p in X_λ .

By induction on the subgroups of G , we can assume that the two Mackey functors $\mathbb{F}_p, \pi_0 X_\lambda$ agree on proper subgroups of G . The unit map $\mathbb{S} \rightarrow X_\lambda$ is surjective on π_0^G because X_λ has only one cell in dimension 0 and is connective, since $\Omega^\lambda S^{\lambda+1}$ is connected. As remarked already, $p = 0$

in $\pi_0^G(X_\lambda^G)$. $\pi_0^G(\mathbb{S})$ is the Burnside ring $A(G)$, so it suffices to check that the classes $[G/H]$ are zero for every subgroup $H \leq G$.

If H is a nonmaximal proper subgroup, then $[G/H]$ is zero by the induction hypothesis: we can choose $H \leq H' \leq G$, so that the class $[G/H]$ factors through the class $[H'/H]$, which we know to be zero in $X_\lambda^{H'}$.

To finish the proof, we will break into cases.

If G is the group C_{p^n} , any choice of a free orbit on $S(\lambda)$ gives by Proposition 2.3 a unital map $N_*^G X \rightarrow X$. Since $p = 0$, the composite map

$$\mathbb{S} = N_*^G \mathbb{S} \xrightarrow{N_*^G p} N_*^G X \rightarrow X$$

is also zero. Since the map $N_*^G X \rightarrow X$ is unital, the diagram

$$\begin{array}{ccccc} N_*^G \mathbb{S} & \xrightarrow{N_*^G p} & N_*^G X & \longrightarrow & X \\ \parallel & & \uparrow N_*^G 1 & & \uparrow 1 \\ N_*^G \mathbb{S} & \xrightarrow{N_*^G p} & N_*^G \mathbb{S} & = & \mathbb{S} \end{array}$$

commutes. By the distributive law for the indexed tensor product [14, A.3.3, Lemma A.36], the lower horizontal map corresponds in the Burnside ring to the G -set $\text{Hom}(G, p)$. In the Burnside ring, $[\text{Hom}(G, p)] \equiv [G/C_{p^{n-1}}] \pmod{p, [C_{p^{n-i}}], i \geq 2}$ [12, Lemma 6.6], so it follows that $[G/C_{p^{n-1}}] = 0$.

Next we consider the case $G = D_{2n}$ for $n \geq 2$. We present D_{2n} as $\langle a, b | a^2, b^{2^{n-1}}, abab \rangle$ so that in the representation λ , $b \in \text{SO}(2)$, and a acts as a reflection. There are three maximal proper subgroups of G : $\langle a, b^2 \rangle$, $\langle ab, b^2 \rangle$, $\langle b \rangle$.

\mathbb{R}^λ has orbits of the form $P_0 = G/\langle a \rangle$, $P_1 = G/\langle ab \rangle$, and G , so by Proposition 2.3 we obtain three unital maps from $N_*^G X_\lambda$, $N_*^{P_0} X_\lambda$, $N_*^{P_1} X_\lambda$ to X . As in the case of cyclic groups, this implies that $[\text{Hom}(G, 2)]$, $[\text{Hom}(P_0, 2)]$, $[\text{Hom}(P_1, 2)]$ are all 0. To see what this implies about $\pi_0^G(X_\lambda)$, it suffices to compute each of these in the Burnside ring modulo the ideal generated by 2 and $[G/H]$ for nonmaximal proper subgroups H .

For each of the G -sets $S = G, P_0, P_1$, there are two fixed points in $\text{Hom}(S, 2)$. Since maximal subgroups of G contain $\langle b^2 \rangle$, we just need to count surjections in $\text{Hom}(S, 2)$ that are fixed by left multiplication by $\langle b^2 \rangle$.

For $S = P_0$, it follows that $[\text{Hom}(P_0, 2)] \equiv [G/\langle a, b^2 \rangle]$ since the double coset $\langle b^2 \rangle \backslash G / \langle a \rangle$ consists of two elements. For the same reason, $[\text{Hom}(P_1, 2)] \equiv [G/\langle ab, b^2 \rangle]$. For $S = G$, there is a contribution for each maximal proper subgroup, giving $[\text{Hom}(G, 2)] \equiv [G/\langle ab, b^2 \rangle] + [G/\langle a, b^2 \rangle] + [G/\langle b \rangle]$. Thus all of the desired classes in the Burnside ring are zero. \square

Below we compare the isotropy separation sequences for the map $f_\lambda : X_\lambda \rightarrow \mathbb{F}_p$.

$$\begin{array}{ccccc} (X_\lambda)_{hP} & \longrightarrow & X_\lambda^G & \longrightarrow & X_\lambda^{\Phi G} \\ \downarrow (f_\lambda)_{hP} & & \downarrow f_\lambda^G & & \downarrow f_\lambda^{\Phi G} \\ (\mathbb{F}_p)_{hP} & \longrightarrow & \mathbb{F}_p^G & \longrightarrow & \mathbb{F}_p^{\Phi G} \end{array}$$

Lemma 4.3. Suppose that f_λ^H is an equivalence for all proper subgroups H of G . Then the map $f_\lambda^{\Phi G} : X_\lambda^{\Phi G} \rightarrow \mathbb{F}_p^{\Phi G}$ is 1-connected.

Proof. Since f_λ^H is an equivalence for all proper subgroups of G , the map $(f_\lambda)_{hP}$ is an equivalence, since $(f_\lambda)_{hP} = \operatorname{colim}_{G/H \in \operatorname{Orb}'_G} f_\lambda^H$.

By Proposition 4.2, f_λ^G is an isomorphism on $\pi_{\leq 0}$. In addition, $\pi_1^G(\mathbb{F}_p) = 0$, so f_λ^G is 1-connected. Thus by the isotropy separation sequence, $f_\lambda^{\Phi G}$ is too. \square

We now finish the proof of Theorem A.

Theorem A. Let $\lambda : G \rightarrow \operatorname{O}(2)$ be a faithful representation of a p -group. Then there is an equivalence of \mathbb{E}_λ -algebras $X_{\lambda,p} \cong \mathbb{F}_p$.

Proof. By induction, we can assume that the map $f_\lambda : X_\lambda \rightarrow \mathbb{F}_p$ induces an equivalence on fixed points for all proper subgroups. It suffices then to show that $f_\lambda^{\Phi G} : X_\lambda^{\Phi G} \rightarrow \mathbb{F}_p^{\Phi G}$ is an equivalence. The base case is Theorem 1.1 and Theorem 1.2.

First consider the case when λ is the regular representation ρ of the group C_2 . Then $X_\lambda^{\Phi C_2} \rightarrow \mathbb{F}_2^{\Phi C_2}$ is an \mathbb{E}_1 -algebra map. The codomain has homotopy ring $\mathbb{F}_2[x]$, $|x| = 1$, and the domain does too by Lemma 3.3. By Lemma 4.3, we learn that the generators are sent to the generators, so $f_\lambda^{\Phi C_2}$ is an equivalence.

In all other cases, λ is irreducible. For such λ , the induction hypothesis and Lemma 2.4 show that $N^\lambda X_\lambda^{\Phi G} \simeq N^\lambda \mathbb{F}_p^{\Phi G}$.

Thus $f_\lambda^{\Phi G}$ is a map of $N^\lambda X_\lambda^{\Phi G} \simeq N^\lambda \mathbb{F}_p^{\Phi G}$ -modules, and Theorem C shows that both sides are rank 2 free modules generated in degrees 0 and 1. By Lemma 4.3, both module generators of $\mathbb{F}_p^{\Phi G}$ are hit by $f_\lambda^{\Phi G}$. Thus the map is surjective on homotopy groups, but must also be injective since both sides are rank 2 free modules. \square

Remark 4.3.1. The category Sp_G of G -spectra contains a full reflective subcategory Sp_G^P of G -spectra of the form $R^{P\beta} := \operatorname{Map}(EP, R)$, where EP is the classifying space for proper subgroups of G . Any \mathbb{E}_V -algebra in $\operatorname{Sp}_G R$ can be presented by giving the data of $R^{P\beta}$ as an \mathbb{E}_V -algebra in Sp_G^P along with an $\mathbb{E}_{VG} \cdot N^V R$ -algebra map $R^{\Phi G} \rightarrow (R^{P\beta})^{\Phi G}$.

The proof of Theorem A uses all this data, as we now explain. Fix λ, G as in Theorem A. In the proof, the inductive hypothesis gave that $(X_{\lambda,p})^{P\beta} \cong (\mathbb{F}_p)^{P\beta}$, and we used our understanding of $X_{\lambda,p}^{\Phi G}, \mathbb{F}_p^{\Phi G}$ as $\mathbb{E}_{\lambda G} \cdot N^\lambda \mathbb{F}_p^{\Phi G}$ -algebras to see that $X_{\lambda,p} \cong \mathbb{F}_p$.

Remark 4.3.2. The proof of Theorem A in the case of the regular representation ρ of C_2 is essentially the same as that in [12]. This is explained by the fact that $N^\rho = N_*^{C_2}$ is just the HHR norm. In all other cases overlapping with Theorem 1.4, the proof of Theorem A improves the previously known proof, especially at odd primes, where an exotic Thom ring structure on $X_{\lambda,p}$, as well as an explicit computation of $\pi_1^G(X_{\lambda,p})$ is used in [12].

5 | OTHER THOM SPECTRA

In this section we study other $X_{\lambda,p}$, showing that the choices of G, λ, p in Theorem A give all $X_{\lambda,p}$ isomorphic to \mathbb{F}_p . Then we discuss the algebraicity of $X_{\lambda,p}$ for other G, λ, p .

Theorem B. Let G be a finite group and $V : G \rightarrow \mathrm{O}(n)$ a representation. $X_{V,p} \cong \mathbb{F}_p$ if and only if $n = 2$, G is a p -group, and V is faithful.

Proof. The “if” part of the result is Theorem A.

The underlying spectrum of $X_{V,p}$ is a Thom spectrum of $\Omega^n S^{n+1}$ which only has the right mod p homology if $\dim V = 2$, so that is a necessary condition.

Note that if $X_{V,p} \cong \mathbb{F}_p$ holds for V, p , it must hold for $V|_H, p$ for all subgroups H . Suppose that some element c of order p acts trivially on V . Restricting to $H = \langle c \rangle$, we have the trivial representation on the cyclic group of order p . Since the action is trivial, there is an identification of $X_{V|_H,p}^{\Phi H}$ with the underlying Thom spectrum, which is \mathbb{F}_p , and $\pi_*(\mathbb{F}_p) \neq \pi_*(\mathbb{F}_p^{\Phi H})$, as the latter is \mathbb{F}_p in each nonnegative degree (see the discussion in Proposition 3.4).

Next, suppose that some element of prime order $q \neq p$ acts trivially on V . Let H be the subgroup generated by this element so $V|_H$ is trivial. Then $X_{V|_H,p} \not\cong \mathbb{F}_p$ because $(\mathbb{F}_p)^{\Phi H} = 0$, but $X_{V|_H,p}^{\Phi H} = \mathbb{F}_p$ by Theorems 1.2 and 1.1.

Finally, suppose that V is injective, and its image contains an element of prime order $q \neq p$. Then restricting to the subgroup generated by that element, Theorem 5.1 (proven below) shows that $X_{V,p}$ does not agree with \mathbb{F}_p when restricted to that subgroup. Thus the conditions on V in Theorem A are necessary. \square

For a finite group G and an abelian group A , let \tilde{A} be the Mackey functor whose value on G/G is A and otherwise 0. Given a representation V , recall that a_V is the element in $\pi_{-V}^G \mathbb{S}$ given by the inclusion $S^0 \rightarrow S^V$.

Theorem 5.1. Let $G = C_p$ be a cyclic group of prime order p , $\lambda : G \rightarrow \mathrm{O}(2)$ an embedding, and $p \neq q$. Then as a G -spectrum, $X_{\lambda,q} \cong \mathbb{F}_q \oplus \bigoplus_0^\infty \Sigma^i \tilde{\mathbb{F}}_q$.

Proof. Let $X = X_{\lambda,q}$. As a C_p spectrum, there is a fracture square

$$\begin{array}{ccc} X & \xrightarrow{\quad} & a_\lambda^{-1} X \\ \downarrow & \lrcorner & \downarrow \\ X_{a_\lambda}^\wedge & \longrightarrow & a_\lambda^{-1} (X_{a_\lambda}^\wedge) \end{array}$$

which upon applying G -fixed points becomes the Tate square. By Lemma 2.6 and either Remark 3.2.1 or [12, Proposition 3.9], $X^{\Phi G} = a_\lambda^{-1} X^G$ is an \mathbb{F}_q -module whose homotopy groups are \mathbb{F}_q in each nonnegative degree. Thus $a_\lambda^{-1} X^G \cong \bigoplus_0^\infty \Sigma^i \tilde{\mathbb{F}}_q$. Furthermore, since the underlying action on X is trivial on homotopy groups, and $p \neq q$, $X_{a_\lambda}^\wedge$ has fixed points equal to the homotopy fixed points, so is \mathbb{F}_q .

The G -spectrum $a_\lambda^{-1} (X_{a_\lambda}^\wedge)$ is zero, since its underlying spectrum is 0 and the homotopy groups of its fixed points are the Tate cohomology of \mathbb{F}_q with a trivial C_p action. Thus the above pullback square gives the decomposition claimed. \square

Remark 5.1.1. Using Proposition 3.2 and Remark 4.3.1, it is easy to see that the proof of Theorem 5.1 gives a complete description of the \mathbb{E}_λ -algebra structure.

The $X_{\lambda,p}$ in Theorem 5.1 are sums of generalized Eilenberg Mac Lane spectra, but this is not always the case. For example, take λ to be a trivial 2-dimensional representation for a

nontrivial group, and let p be any prime. Then $X_{2,p}$ is *not* a generalized Eilenberg Mac Lane spectrum. Rather, it is the algebra $\mathbb{S} \otimes \mathbb{F}_p$, so that modules over it are the same as \mathbb{F}_p -module valued Mackey functors. In this sense, it can still be considered algebraic.

In Question 1.5.1, it was asked what sufficient conditions are for $X_{\lambda,p}$ to be algebraic, that is, a \mathbb{F}_p -module-valued Mackey functor. It was also suggested that the only condition for $X_{\lambda,p}$ to be algebraic is that $\dim \lambda \geq 2$. The arguments of this paper are show that this is not far from being true. Lemma 2.6 showed that $X_{\lambda,p}^{\Phi G}$ is always an \mathbb{F}_p -module in this case. By filtering $X_{\lambda,p}$ by its various isotropy separation sequences, we can use this to learn that $X_{\lambda,p}$ has a finite filtration whose associated graded objects are algebraic. Moreover, a refinement of the arguments of Theorem 5.1 and Theorem B show that $X_{\lambda,p}$ is algebraic whenever G is a finite abelian group and $\dim \lambda \geq 2$.

6 | ADDITIONAL STRUCTURE

At the prime 2, Theorem A can be sharpened to an equivalence of $\mathbb{E}_{\lambda+\det \lambda}$ -algebras. To show this, one only needs to show that the map $\mu_{\lambda,2} : \Omega^\lambda S^{\lambda+1} \rightarrow \mathrm{BGL}_1 \mathbb{S}_{(2)}$ can be refined to a $\lambda + \det \lambda$ -loop map. The goal of this section is to explain how to construct this refinement. We closely follow the strategy in [12, Section 4], except we bypass an unnecessary argument there. From now on, λ denotes the standard 2-dimensional representation of $O(2)$, and σ denotes the representation $\det \lambda$.

The ability to upgrade $\mu_{\lambda,2}$ to an $\mathbb{E}_{\lambda+\sigma}$ -loop map stems from the fact that $S^{1+\lambda}$ can be σ -delooped to $\mathbb{H}\mathbb{P}^\infty$, and equivariant Bott periodicity gives a natural σ -delooping of $\mu_{\lambda,2}$. First, we explain the delooping $\Omega^\sigma \mathbb{H}\mathbb{P}^\infty = S^{1+\lambda}$.

The conjugation action of $\mathrm{Sp}(1)$ on \mathbb{H} makes \mathbb{H} into an $\mathrm{SO}(3) \cong \mathrm{Sp}(1)/\{\pm 1\}$ -equivariant algebra, which induces an action of $\mathrm{SO}(3)$ on $\mathbb{H}\mathbb{P}^\infty$. Restricting this action to $O(2)$, \mathbb{H} is equivariantly isomorphic to the Clifford algebra $\mathrm{Cl}(\lambda)$ where λ is the standard representation of $O(2)$ with a negative definite form. As a representation, $\mathbb{H} = \mathrm{Cl}(\lambda)$ is isomorphic to $1 + \lambda + \det \lambda = 1 + \lambda + \sigma$.

Via the inclusion $O(2) \rightarrow \mathrm{SO}(3)$, we can view $\mathbb{H}\mathbb{P}^\infty$ as a $O(2)$ -space. Observe that $(\mathbb{H}\mathbb{P}^\infty)^G = \mathbb{C}\mathbb{P}^\infty$ when G is a cyclic subgroup of $O(2)$, and that $(\mathbb{H}\mathbb{P}^\infty)^G = \mathbb{R}\mathbb{P}^\infty$ when G is a dihedral subgroup D_{2n} for $n \geq 2$.

Lemma 6.1. *Restricting the action of $\mathrm{SO}(3)$ on $\mathbb{H}\mathbb{P}^\infty$ to $O(2)$, there is an equivalence of $O(2)$ -spaces $S^{\lambda+1} \cong \Omega^\sigma \mathbb{H}\mathbb{P}^\infty$.*

Proof. As an $O(2)$ -space, $\mathbb{H}\mathbb{P}^1 \cong S^{\mathbb{H}} \cong S^{\lambda+\sigma+1}$, and so the inclusion $\mathbb{H}\mathbb{P}^1 \rightarrow \mathbb{H}\mathbb{P}^\infty$ is adjoint to a map $S^{\lambda+1} \rightarrow \Omega^\sigma \mathbb{H}\mathbb{P}^\infty$. We show that this adjoint map is an equivalence on G fixed points for every subgroup $G \subset O(2)$. For the cyclic subgroups of $O(2)$, this is [12, Proposition 3.4,3.6], so it suffices to show that $S^1 \cong (\Omega^\sigma \mathbb{H}\mathbb{P}^\infty)^{D_{2n}}$ for $n \geq 2$.

There is a cofiber sequence

$$(D_{2n}/C_n)_+ \rightarrow S^0 \rightarrow S^\sigma.$$

Applying $\mathrm{Map}(-, \mathbb{H}\mathbb{P}^\infty)$, we obtain a fiber sequence

$$\Omega^\sigma \mathbb{H}\mathbb{P}^\infty \rightarrow \mathbb{R}\mathbb{P}^\infty \rightarrow \mathbb{C}\mathbb{P}^\infty.$$

The fiber of the map $\mathrm{BC}_2 \rightarrow \mathrm{BS}^1$ is S^1 , and which is $(S^{\lambda+\sigma+1})^{D_{2n}} = S^1$. □

Theorem D. Let λ be the standard 2-dimensional representation of $O(2)$. The map $\mu_{\lambda,2} : \Omega^\lambda S^{1+\lambda} \rightarrow \text{BGL}_1(\mathbb{S}_{(2)})$ admits a refinement to a $\lambda + \det \lambda$ -loop map.

Proof. The construction of our candidate refinement is exactly that in [12, Section 4] which we refer to for more details. The difference between their arguments and what is done here is that we prove directly that the constructed map refines $\mu_{\lambda,2}$, bypassing the argument in their Proposition 4.12.

Let $\text{BGL}(\mathbb{H})$ be the space classifying $G - \mathbb{H}$ -bundles, that is, G -equivariant real vector bundles $E \rightarrow X$ with a G -equivariant algebra map $\mathbb{H} \rightarrow \text{End}(E)$.

There is an identification $\Omega^\infty \Sigma^{\mathbb{H}} \text{KO}_G \simeq \mathbb{Z} \times \text{BGL}(\mathbb{H})$ [12, Theorem 4.5], where the equivalence is given by the Bott map: the tensor product of a bundle with the virtual $G - \mathbb{H}$ bundle γ on the pair $(\mathbb{H}, \mathbb{H} - \{0\})$ defined by the complex of bundles $\mathbb{H} \rightarrow \mathbb{H}$ given at a point $v \in \mathbb{H}$ by multiplication by v (see [3, Section 4]). Our delooping is the composite

$$\mathbb{H}\mathbb{P}^\infty \xrightarrow{\mathcal{O}(-1)-1} \text{BGL}(\mathbb{H}) \rightarrow \Omega^\infty \Sigma^{\mathbb{H}} \text{KO}_G \xrightarrow{\eta} \Omega^\infty \Sigma^{\lambda+\sigma} \text{KO}_G.$$

To see that this does deloop $\mu_{\lambda,2}$, it suffices to identify the virtual $G - \mathbb{H}$ -bundle $\mathcal{O}(-1) - 1$ restricted to $\mathbb{H}\mathbb{P}^1$ with the class coming from γ . For then the element in $\pi_{\lambda+\sigma+1}^{O(2)} \Omega^\infty \Sigma^{\lambda+\sigma} \text{KO}_G = \pi_1^{O(2)} \text{BO}_G$ is η , which is the class defining $\mu_{\lambda,2}$.

To see this identification, we first analyze γ . Replacing the pair $(\mathbb{H}, \mathbb{H} - 0)$ with the pair $(\mathbb{H}, \mathbb{H} - D^4)$ the data of the complex $\mathbb{H} \rightarrow \mathbb{H}$ defines a bundle γ' over $Y = \mathbb{H}_0 \cup_{\mathbb{H}-D^4} \mathbb{H}_1$. There is a retraction $r : Y \rightarrow \mathbb{H}_1$, and the corresponding virtual bundle is the difference of this bundle with the bundle on Y that is the pullback of the composite $Y \xrightarrow{r} \mathbb{H}_1 \rightarrow Y$, thought of as a relative class of the pair $(Y, \mathbb{H}_1) \simeq (\mathbb{H}, \mathbb{H} - D^{\mathbb{H}})$. However, this pullback is trivial since it factors through a contractible space, so after adding 1 to the virtual bundle corresponding to γ , we just get the bundle γ' . There is an equivalence $Y \simeq \mathbb{H}\mathbb{P}^1$ identifying the copies of \mathbb{H} with hemispheres, under which γ' corresponds to the bundle with the tautological clutching function $S(\mathbb{H}) \rightarrow \text{GL}_1(\mathbb{H})$, which is $\mathcal{O}(-1)$. \square

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