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Explicit modular forms from the divided beta family

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ABSTRACT

We compute modular forms known to arise from the order 5 generators of the 5-local Adams-Novikov spectral sequence 2-line, generalizing and contextualizing previous computations of M. Behrens and G. Laures. We exhibit analogous computations at other primes and conjecture formulas for some of the modular forms arising in this way at arbitrary primes ≥ 5.

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1. Introduction

At an odd prime p, the 2-line $\mathrm{Ext}^{2,*}$ of the Adams-Novikov spectral sequence

$$\mathrm{Ext}^{s,t} := \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow (\pi_{t-s}S^0)_{(p)}$$

is generated additively by distinguished classes $\beta_{i/j,k} \in \mathrm{Ext}^{2,2i(p^2-1)-2j(p-1)}$ of order p^k for certain ordered triples of positive integers (i,j,k) [9]. These generators are known collectively as the p-primary divided beta family. Behrens [2] has shown that, for $p \geq 5$, each $\beta_{i/j,k}$ gives rise to a modular form $f_{i/j,k}$ over the integers satisfying conditions formulated in terms of i, j, k, and p, including both a congruence and non-congruence condition. The proof that these modular forms exist uses homotopical properties of spectra related to the $K(2)$-local sphere [1] and ultimately does not explicitly identify the $f_{i/j,k}$. The purpose of this paper is to compute the modular forms $f_{i/j,1}$ associated to order p divided beta family elements $\beta_{i/j,1}$ in several cases, including a complete computation of such forms at the prime 5.

Behrens' theorem [2, Theorem 1.3] says the following in the case $k = 1$. Let M_t be the space of weight t modular forms over $\mathbb{Z}$ for the full modular group $\Gamma_0(1) = SL_2(\mathbb{Z})$, and for a positive integer N, let $M_t(N)$ be the space of weight t modular forms over $\mathbb{Z}$ for the congruence subgroup $\Gamma_0(N) \subset \Gamma_0(1)$ [8]. Given an order p divided beta family element $\beta_{i/j,1}$ for $p \geq 5$, there exists a modular form $f_{i/j,1} \in M_{i(p^2-1)}$ satisfying the following conditions:

(C1) the Fourier expansion $f_{i/j,1}(q) \in \mathbb{Z}[[q]]$ is not congruent to 0 mod p,
(C2) the integer $12 \cdot \mathrm{ord}_q f_{i/j,1}(q)$ is either greater than $(p^2-1)i-(p-1)j$, or equal to $(p^2-1)i-(p-1)j-2$,
(C3) there does not exist $g \in M_t$ for $t < (p^2-1)i$ such that $f_{i/j,1}(q) \equiv g(q) \mod p$, and
(C4) for every prime $\ell \neq p$, there exists $g \in M_{(p^2-1)i-(p-1)j}(\ell)$ such that $(L_\ell f_{i/j,1})(q) \equiv g(q) \mod p$

where $L_N : M_t \to M_t(N)$ is a linear operator to be defined in Section 3.

There is ambiguity inherent in the computation of these modular forms. For example, if a modular form f satisfies conditions (C1) through (C4) for given i, j, and p, then so does $cf+h$, where $c \in \mathbb{Z}$ with $(c,p) = 1$ and h is a modular form of the same weight and q-order as f such that $h(q) \equiv 0 \mod p$. The precise extent to which a given $f_{i/j,k}$ fails to be unique will be the subject of future work. The goal here is to produce *at least one* candidate for any $f_{i/j,1}$ that we study, and an assertion of the form $f_{i/j,1} = f$ is understood to mean that f is one of many possible choices for $f_{i/j,1}$ in $M_{i(p^2-1)}$.

Our results generalize sample computer-aided computations of Behrens and Laures [3]. To state their results, and our generalizations thereof, requires additional notation. In the divided beta family, we write $\beta_{i/j}$ for $\beta_{i/j,1}$, we write β_i for $\beta_{i/1}$, and we abbreviate the corresponding $f_{i/j,k}$ similarly. In the space of modular forms, $\Delta \in M_{12}$ is the Ramanujan delta function and $E_4 \in M_4$ is the weight 4 Eisenstein series, whose definitions we will recall in Section 3. Behrens and Laures computed the following modular forms at the prime 5:

$$f_1 = \Delta^2, \quad f_2 = \Delta^4, \quad f_3 = \Delta^6, \quad f_4 = \Delta^8, \quad f_{5/5} = \Delta^{10}, \tag{1.1}$$

and

$$f_{25/29} = \Delta^{50} + 4\Delta^{42}E_4^{24} + 3\Delta^{41}E_4^{27}. \tag{1.2}$$

We shall see that the striking pattern of powers of Δ in (1.1) does indeed persist at the prime 5 and elsewhere.

The formula for $f_{25/29}$ in (1.2) is the catalyst for our 5-primary computations. This is despite the fact that the corresponding element $\beta_{25/29}$ in homotopy theory is actually not a member of the 5-primary divided beta

family (see Lemma 2.1), but rather can be properly interpreted as an element of a different, closely related spectral sequence (see [3]). The fact that $\Delta^{50} + 4\Delta^{42}E_4^{24} + 3\Delta^{41}E_4^{27}$ is indeed a modular form satisfying conditions (C1) through (C4) in the case $i = 25$, $j = 29$, and $p = 5$, is instructive. For example, while $\beta_{25/29}$ is not a 5-primary divided beta family element, $\beta_{50/29}$ is, and the way $f_{25/29}$ is built as the sum of a power of Δ and two "correction terms" serves as a model for how to compute $f_{50/29}$. In fact, it is by identifying appropriate analogs of these "correction terms" that we are able to compute all modular forms $f_{i/j}$ at the prime 5 that are not simply powers of Δ.

Before stating the main theorem let us define, for $n \geq 2$ and $r \geq 1$, modular forms

$$C_{0,n,r} = 4r\Delta^{42\cdot 5^{n-2}+2(r-1)5^n}E_4^{24\cdot 5^{n-2}},$$
$$D_{0,n,r} = 3r\Delta^{41\cdot 5^{n-2}+2(r-1)5^n}E_4^{27\cdot 5^{n-2}}$$

in $M_{24r\cdot 5^n}$, so that $C_{0,2,1} + D_{0,2,1} = 4\Delta^{42}E_4^{24} + 3\Delta^{41}E_4^{27}$ is precisely the summand appearing in (1.2). Moreover, for $n \geq 3$, $1 \leq m \leq n-2$, and $r \geq 1$, define

$$C_{m,n,r} = 3r\Delta^{8\cdot 5^{n-1}+2\cdot 5^{n-m-2}+2(r-1)5^n}E_4^{6\cdot 5^{n-1}-6\cdot 5^{n-m-2}},$$
$$D_{m,n,r} = r\Delta^{8\cdot 5^{n-1}+5^{n-m-2}+2(r-1)5^n}E_4^{6\cdot 5^{n-1}-3\cdot 5^{n-m-2}}$$

in $M_{24r\cdot 5^n}$. Finally, given a prime p and any integer $n \geq 0$, define

$$a_n = \begin{cases} 0, & \text{if } n = 0, \\ p^n + p^{n-1} - 1, & \text{if } n \geq 1. \end{cases}$$

Theorem 1.3. *Given a 5-primary divided beta family element $\beta_{i/j}$ of order 5, where $i = r\cdot 5^n$ with $(r,5) = 1$, the corresponding modular form is $f_{i/j} = \Delta^{2r\cdot 5^n}$ except when the inequalities $r > 1$ and $5^n + 1 \leq j \leq a_n$ hold simultaneously. For such r and j, there is a positive integer u, $1 \leq u \leq n-1$, such that*

$$5^n + 5^{n-1} - 5^{n-u} + 1 \leq j \leq 5^n + 5^{n-1} - 5^{n-u-1}$$

in which case

$$f_{i/j} = \begin{cases} \Delta^{2r\cdot 5^n} + \sum_{m=0}^{u-1}(C_{m,n,r} + D_{m,n,r}) & \text{if} \quad j > 5^n + 5^{n-1} - 5^{n-u} + 2\cdot 5^{n-u-1}, \\ \Delta^{2r\cdot 5^n} + \sum_{m=0}^{u-2}(C_{m,n,r} + D_{m,n,r}) + C_{u-1,n,r} & \text{if} \quad j \leq 5^n + 5^{n-1} - 5^{n-u} + 2\cdot 5^{n-u-1}. \end{cases}$$

In the case $r > 1$, Theorem 1.3 shows that the modular forms $f_{i/j}$ can all be recovered from f_{i/a_n} by trimming off some number of summands depending on the size of j. Example 1.6 below clearly exhibits this phenomenon. It is therefore worth noting that the formula for f_{i/a_n} in Theorem 1.3 has a recursive interpretation.

Corollary 1.4. *For $n \geq 1$ and $r \geq 2$, $f_{r\cdot 5^{n+1}/a_{n+1}} = (f_{r\cdot 5^n/a_n})^5 + C_{n-1,n+1,r} + D_{n-1,n+1,r}$.*

The formulas for a given f_{i/a_n} obtained from Theorem 1.3 and Corollary 1.4, respectively, will differ in M_{24i}, but they will be equivalent modulo 5.

Example 1.5. To preview the concepts and computational methods behind the proof of Theorem 1.3, let us employ them to outline a re-derivation of Equation (1.2). Since $f_{25/29} \in M_{600}$, it will follow from Proposition 3.1 and conditions (C1) and (C3) that

$$f_{25/29} = \Delta^{50} + \sum_{m=1}^{50} c_m \Delta^{50-m} E_4^{3m}$$

for integers $c_i \in \mathbb{Z}$. The demand on q-order given by condition (C2), together with Proposition 4.1(c), will imply that $c_{10} = c_{11} = \cdots = c_{50} = 0$.

To verify condition (C4) at a prime $\ell \neq 5$, Proposition 3.7 will imply that it suffices to show $L_\ell f_{25/29}$ is divisible by E_4^{29} in $M_*(\ell)_{\mathbb{Z}/5}$. Proposition 3.8 will imply that it suffices to do this for the case $\ell = 2$. In $M_*(2)_{\mathbb{Z}/5}$, we shall see that $L_2\Delta^{50}$ is divisible by E_4^{25} by Lemma 4.12. The E_4-divisibility of $L_2\Delta^{50-m}E_4^{3m}$ for $1 \leq m \leq 9$ can be computed directly in a similar manner using Lemma 4.11:

m	1	2	3	4	5	6	7	8	9
term	$L_2\Delta^{49}E_4^3$	$L_2\Delta^{48}E_4^6$	$L_2\Delta^{47}E_4^9$	$L_2\Delta^{46}E_4^{12}$	$L_2\Delta^{45}E_4^{15}$	$L_2\Delta^{44}E_4^{18}$	$L_2\Delta^{43}E_4^{21}$	$L_2\Delta^{42}E_4^{24}$	$L_2\Delta^{41}E_4^{27}$
E_4-div.	E_4^3	E_4^7	E_4^9	E_4^{13}	E_4^{15}	E_4^{19}	E_4^{21}	E_4^{25}	E_4^{27}

Thus, for $L_2 f_{25/29}$ to have the required E_4-divisibility, we much choose $c_1 = c_2 = \cdots = c_7 = 0$. From there, setting $c_8 = 4$ makes the term

$$L_2(\Delta^{50} + c_8\Delta^{42}E_4^{24})$$

divisible by E_4^{27} in $M_*(2)_{\mathbb{Z}/5}$, while all other choices for c_8 (modulo 5) keep its E_4-divisibility at E_4^{25}. Subsequently setting $c_9 = 3$ makes

$$L_2(\Delta^{50} + c_8\Delta^{42}E_4^{24} + c_9\Delta^{41}E_4^{27})$$

divisible by E_4^{29} in $M_*(2)_{\mathbb{Z}/5}$, and no other choice of c_9 (modulo 5) accomplishes this.

Example 1.6. Consider the conclusion of Theorem 1.3 in the case $n = 4$ and $r = 2$. The order 5 divided beta family elements $\beta_{i/j}$ with $i = 2 \cdot 5^4 = 1250$ are

$$\{\beta_{1250/j} : 1 \leq j \leq 749 \text{ and } j \neq 5, 10, 15, \ldots, 125\}$$

(see Lemma 2.1). By Theorem 1.3, $f_{1250/j} = \Delta^{2500}$ except when $626 \leq j \leq 749$, in which case $f_{1250/j} = \Delta^{2500} + \mathfrak{C}$ where

$$\mathfrak{C} = \begin{cases} 3\Delta^{2300}E_4^{600} + \Delta^{2275}E_4^{675} + \Delta^{2260}E_4^{720} + 2\Delta^{2255}E_4^{735} + \Delta^{2252}E_4^{744} + 2\Delta^{2251}E_4^{747} & \text{if } j = 748, 749, \\ 3\Delta^{2300}E_4^{600} + \Delta^{2275}E_4^{675} + \Delta^{2260}E_4^{720} + 2\Delta^{2255}E_4^{735} + \Delta^{2252}E_4^{744} & \text{if } j = 746, 747, \\ 3\Delta^{2300}E_4^{600} + \Delta^{2275}E_4^{675} + \Delta^{2260}E_4^{720} + 2\Delta^{2255}E_4^{735} & \text{if } 736 \leq j \leq 745, \\ 3\Delta^{2300}E_4^{600} + \Delta^{2275}E_4^{675} + \Delta^{2260}E_4^{720} & \text{if } 726 \leq j \leq 735, \\ 3\Delta^{2300}E_4^{600} + \Delta^{2275}E_4^{675} & \text{if } 676 \leq j \leq 725, \\ 3\Delta^{2300}E_4^{600} & \text{if } 626 \leq j \leq 675. \end{cases}$$

In each of the six cases, the term $\mathfrak{C}$ raises the E_4-divisibility of $L_2 f_{1250/j}$ in $M_*(2)_{\mathbb{Z}/5}$ as required by condition (C4), while keeping the q-order of $f_{1250/j}$ sufficiently large as required by condition (C2).

Remark 1.7. In addition to the modular forms in Equations (1.1) and (1.2), Behrens and Laures computed $f_{25/5,2} = \Delta^{50}$, where $\beta_{25/5,2}$ is the first order 25 element one encounters in the 5-primary divided beta family. Computation of the modular forms $f_{i/j,k}$ for $k > 1$ will be the subject of a future paper.

This paper is structured as follows. In Section 2 we recall how to enumerate the elements of order p in the p-primary divided beta family. In Section 3 we list results from the theory of modular forms relevant to our computations. Section 4 is the technical heart of the paper, where we compute at the prime 5 and prove Theorem 1.3. In Section 5 we exhibit our computational methods at the primes 7, 11, and 13, and for arbitrary primes $p \geq 5$ we conjecture which of the modular forms arising from Behrens' theorem are simply powers of the Ramanujan delta function.

2. The divided beta family

In this section we show how to enumerate the order p elements $\beta_{i/j}$ of the p-primary divided beta family. Given p, let a_n be defined as in Section 1. If we write $i = rp^n$ where $(r,p) = 1$, then [9, Theorem 2.6] implies that the entire p-primary divided beta family comprises the elements $\beta_{i/j,k}$ for ordered triples of positive integers (i,j,k) subject to the following rules:

i) if $r = 1$, then $1 \leq j \leq p^n$,
ii) $p^{k-1} | j \leq a_{n-k+1}$,
iii) if $p^k | j$, then $j > a_{n-k}$.

The order p divided beta family elements $\beta_{i/j}$ are therefore characterized as follows.

Lemma 2.1. *The divided beta family elements of order p of the form $\beta_{p^n/j}$ are*

$$\{\beta_{p^n/j} : 1 \leq j \leq p^n \text{ and } j \neq p, 2p, \ldots, a_{n-2}p\},$$

and those of the form $\beta_{rp^n/j}$ where $r > 1$ and $(r,p) = 1$ are

$$\{\beta_{rp^n/j} : 1 \leq j \leq a_n \text{ and } j \neq p, 2p, \ldots, a_{n-2}p\}.$$

Proof. Consider first the case $r = 1$. Rule i) allows j to range between 1 and p^n. Since $k = 1$, rule ii) does not apply any additional constraints. Rule iii) says that j cannot be both a multiple of p and $\leq a_{n-1} = p^{n-1} + p^{n-2} - 1$. Therefore, we must insist that

$$j \neq p, 2p, \ldots, p^{n-1} - p^{n-2} - 5 = a_{n-2}p.$$

In the case $r > 1$, rule i) does not apply and rule ii) allows j to range between 1 and a_n. Rule iii) disallows the same values of j as in the case $r = 1$. □

Example 2.2. Given p, the element $\beta_i = \beta_{i/1,1}$ is a divided beta family element of order p for any $i \geq 1$.

Example 2.3. Suppose $p = 5$. The divided beta family elements of the form $\beta_{5r/j}$ are

$$\beta_{5r/5}, \quad \beta_{5r/4}, \quad \beta_{5r/3}, \quad \beta_{5r/2}, \quad \beta_{5r/1} = \beta_{5r}.$$

Those of the form $\beta_{25/j}$ are

$$\beta_{25/25}, \quad \beta_{25/24}, \quad \ldots, \quad \beta_{25/6}, \quad \beta_{25/4}, \quad \ldots, \quad \beta_{25/1} = \beta_{25}$$

while those of the form $\beta_{25r/j}$ with $r > 1$ are

$$\beta_{25r/29}, \quad \beta_{25r/28}, \quad \dots, \quad \beta_{25r/6}, \quad \beta_{25r/4}, \quad \dots, \quad \beta_{25r/1} = \beta_{25r}.$$

In particular, as noted in Section 1, $\beta_{50/29}$ is a 5-primary divided beta family element, while $\beta_{25/29}$ is not.

3. Modular forms

In this section we record facts from the theory of modular forms required for our computations. Much of this can be found in [8] unless otherwise noted.

If $t \geq 4$ is an even integer and $q = e^{2\pi iz}$, the *weight t Eisenstein series* $E_t \in M_t$ is given by the formula

$$E_t(z) = \frac{1}{2} \sum_{(m,n)=1} \frac{1}{(mz+n)^t} = 1 - \frac{2t}{B_t} \sum_{n=1}^{\infty} \sigma_{t-1}(n) q^n$$

where B_t is the t-th Bernoulli number and $\sigma_{t-1}(n)$ is the sum of the $(t-1)$st powers of the divisors of n. The *Ramanujan Delta function* Δ is a modular form of weight 12 given by the formula

$$\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728} = \sum_{n=1}^{\infty} \tau(n) q^n = q + \cdots.$$

The graded (by weight) ring M_* of all modular forms over $\mathbb{Z}$ for the full modular group is, by a result of Deligne [4, Proposition 6.1],

$$M_* = \frac{\mathbb{Z}[E_4, E_6, \Delta]}{1728\Delta = E_4^3 - E_6^2}.$$

Proposition 3.1. *[8, Theorem 10.4.3] If* $t \equiv 0 \mod 4$, *then* $\{\Delta^a E_4^b : 0 < a, b \in \mathbb{Z}, 12a + 4b = t\}$ *is a* $\mathbb{Z}$*-basis for* M_t.

Let $M_*(2)[1/2]$ denote the graded ring of modular forms for $\Gamma_0(2)$ with 2 inverted. The structure of this ring is well-known (see, e.g., [5, Appendix I]).

Proposition 3.2. *There exist modular forms* $\delta \in M_2(2)[1/2]$ *and* $\varepsilon \in M_4(2)[1/2]$ *with*

$$\delta(q) = 4^{-1} + 6q + 6q^2 + 24q^3 + 6q^4 + 36q^5 + 24q^6 + \cdots,$$
$$\varepsilon(q) = 16^{-1} - q + 7q^2 - 28q^3 + 71q^4 - 126q^5 + 196q^6 - \cdots,$$

and such that $M_*(2)[1/2] = \mathbb{Z}[1/2][\delta, \varepsilon]$.

Given an integer $N \geq 1$, we define two linear operators on modular forms: the first is

$$\iota_N : M_* \to M_*(N)$$
$$f \mapsto f$$

obtained by regarding $f \in M_*$ as a modular form for $\Gamma_0(N)$; the second is the *Verschiebung*

$$V_N : M_* \to M_*(N)$$
$$f(q) \mapsto f(q^N)$$

which satisfies $V_N(fg) = V_N(f)V_N(g)$ since $q \mapsto q^N$ is a ring endomorphism of $\mathbb{Z}[[q]]$. In particular, we will often write $V_N f^m$ for $1 \leq m \in \mathbb{Z}$, which is unambiguous since $V_N(f^m) = (V_N f)^m$. The linear operator appearing in condition (C4) of Behrens' theorem is

$$L_N = V_N - \iota_N.$$

We will not distinguish between f and $\iota_N f$ and so we will write $L_N f = V_N f - f$.

It will be convenient to define

$$\mu = \delta^2 - \varepsilon \in M_4(2)[1/2]$$

and direct computations in $M_*(2)[1/2]$ yield the identities

$$E_4 = 64\mu + 16\varepsilon, \tag{3.3}$$

$$V_2 E_4 = 4\mu + 16\varepsilon, \tag{3.4}$$

$$\Delta = 64\mu\varepsilon^2, \tag{3.5}$$

$$V_2\Delta = \mu^2\varepsilon. \tag{3.6}$$

Let $M_t(N)_{\mathbb{Z}/p}$ be the weight t modular forms for $\Gamma_0(N)$ defined over $\mathbb{Z}/p$. As long as N is invertible in $\mathbb{Z}/p$,

$$M_t(N)_{\mathbb{Z}/p} = M_t(N) \otimes \mathbb{Z}/p$$

(see [2, Equation (1.1)]). We will not draw a notational distinction between $f \in M_t(N)$ and its mod p reduction $f \in M_t(N)_{\mathbb{Z}/p}$ as the meaning will always be clear from the context.

The following result of Serre is a generalization of the well known congruence $E_{p-1}(q) \equiv 1 \mod p$, and dictates exactly when congruences between modular forms of different weights modulo p can occur (see [2, Theorem 10.2] or [7, Corollary 4.4.2]).

Proposition 3.7 *(Serre). Let $f_1 \in M_{t_1}(N)_{\mathbb{Z}/p}$ and $f_2 \in M_{t_2}(N)_{\mathbb{Z}/p}$ with $t_1 < t_2$. Then*

$$f_1(q) = f_2(q) \in \mathbb{Z}/p[[q]]$$

(that is, $f_1 \equiv f_2 \mod p$) if and only if $t_1 \equiv t_2 \mod (p-1)$ and $f_2 = E_{p-1}^{\frac{t_2-t_1}{p-1}} f_1$.

Finally, the following rigidity result implies that checking condition (C4) for a given $f_{i/j}$ often reduces to a computation with $\Gamma_0(\ell)$ modular forms at a *single* prime $\ell \neq p$ (see [2, Theorem 1.5]).

Proposition 3.8 *(Behrens). If the prime ℓ_0 is a topological generator of $\mathbb{Z}_p^\times$ and f is a modular form of weight $t \equiv (p-1)$ satisfying conditions (C1) through (C3), as well as condition (C4) for $\ell = \ell_0$, then f satisfies condition (C4) for all primes $\ell \neq p$.*

4. Computations at the prime 5

In this section we fix $p = 5$ and prove Theorem 1.3. The first goal is to work toward a proof that the modular forms identified as $f_{i/j}$ in Theorem 1.3 satisfy conditions (C1), (C2), and (C3) of Behrens' theorem.

Proposition 4.1. *Suppose $f \in M_t$ with $t \equiv 0 \mod 12$.*

(a) There exist integers $c_0, \dots, c_{t/12}$ such that

$$f = c_0\Delta^{t/12} + c_1\Delta^{(t/12)-1}E_4^3 + \cdots + c_{(t/12)-1}\Delta E_4^{(t/4)-3} + c_{t/12}E_4^{t/4}.$$

(b) If $c_0 \not\equiv 0 \mod 5$ then $f(q) \not\equiv 0 \mod 5$ and f is not divisible by E_4 in $(M_*)_{\mathbb{Z}/5}$.
(c) If $c_m \not\equiv 0$ and $c_{m+1}, c_{m+2}, \dots, c_{t/12} = 0$ for some m, $0 \le m \le t/12$, then

$$\operatorname{ord}_q f = \frac{t}{12} - m.$$

Proof. Part (a) is a special case of Proposition 3.1. Part (b) follows from the fact that the integral basis for M_t given by Proposition 3.1 remains a basis for the vector space $(M_t)_{\mathbb{Z}/5}$ (see, e.g., Section 1 of [6]). Since $\Delta(q) = q + \cdots$ and $E_4(q) = 1 + \cdots$, the value of $\operatorname{ord}_q(f)$ is the smallest power of Δ appearing in the expansion of f given in Part (a), and Part (c) follows. □

Corollary 4.2. *The modular forms identified as $f_{i/j}$ in Theorem 1.3 satisfy conditions (C1) and (C3) of Behrens' theorem.*

Proof. The modular form $f_{i/j}$ has weight $24i$ and so has an expansion as in Proposition 4.1(a). The formulas given in Theorem 1.3 show that each such expansion has leading coefficient $c_0 = 1$ by construction. Thus $f_{i/j}(q) \not\equiv 0 \mod 5$ by Proposition 4.1(b), and so $f_{i/j}$ satisfies condition (C1). Condition (C3) is also satisfied by Propositions 4.1(b) and 3.7. □

Lemma 4.3. *At the prime 5, condition (C2) is equivalent to* $\operatorname{ord}_q f_{i/j} > 2i - \dfrac{j}{3}$.

Proof. At the prime 5, condition (C2) says that either $\operatorname{ord}_q f_{i/j} > (24i - 4j)/12$ or $\operatorname{ord}_q f_{i/j} = (24i - 4j - 2)/12$. But $24i - 4j - 2$ is never divisible by 12 for integer values of i and j, so condition (C2) reduces to the inequality only. □

Proposition 4.4. *The modular forms $f_{i/j}$ as identified in Theorem 1.3 all satisfy condition (C2) of Behrens' theorem.*

Proof. Suppose $i = r \cdot 5^n$ with $(r, 5) = 1$. Consider first the case $j \le p^n$, so that $f_{r\cdot 5^n/j} = \Delta^{2r\cdot 5^n}$. By Lemma 4.3, verifying condition (C2) is equivalent to verifying the inequality

$$\operatorname{ord}_q \Delta^{2r\cdot 5^n} > 2r \cdot 5^n - \frac{j}{3}, \tag{4.5}$$

but $\operatorname{ord}_q \Delta^{2r\cdot 5^n} = 2r \cdot 5^n$ by Proposition 4.1(c), so (4.5) clearly holds for $j \ge 1$.

Next, suppose $j > p^n$. Let u be the positive integer between 1 and $n - 1$ such that

$$5^n + 5^{n-1} - 5^{n-u} + 1 \le j \le 5^n + 5^{n-1} - 5^{n-u-1}.$$

Assume $j > 5^n + 5^{n-1} - 5^{n-u} + 2 \cdot 5^{n-u-1}$. By the formulas given in Theorem 1.3, we must verify condition (C2) for

$$f_{r\cdot 5^n/j} = \Delta^{2r\cdot 5^n} + \sum_{m=0}^{u-1}(C_{m,n,r} + D_{m,n,r}),$$

a form whose q-order is the power of Δ occurring in $D_{u-1,n,r}$ by Proposition 4.1(c). Therefore, by Lemma 4.3, we must verify the inequality

$$8 \cdot 5^{n-1} + 5^{n-u-1} + 2(r-1)5^n > 2r \cdot 5^n - \frac{j}{3}. \tag{4.6}$$

It suffices to verify (4.6) for the smallest possible value of j, which in this case is

$$j = 5^n + 5^{n-1} - 5^{n-u} + 2 \cdot 5^{n-u-1} + 1.$$

At this value of j, (4.6) becomes

$$8 \cdot 5^{n-1} + 5^{n-u-1} + 2(r-1)5^n > 2r \cdot 5^n - \frac{5^n + 5^{n-1} - 5^{n-u} + 2 \cdot 5^{n-u-1} + 1}{3}$$

which is equivalent to $1/3 > 0$. So (4.6) holds.

Continuing with the case $j > p^n$, we now assume $j \leq 5^n + 5^{n-1} - 5^{n-u} + 2 \cdot 5^{n-u-1}$. By the formulas given in Theorem 1.3, we must verify condition (C2) for

$$f_{r \cdot 5^n/j} = \Delta^{2r \cdot 5^n} + \sum_{m=0}^{u-2} (C_{m,n,r} + D_{m,n,r}) + C_{u-1,n,r}$$

whose q-order is the power of Δ occurring in $C_{u-1,n,r}$ by Proposition 4.1(c). Therefore, by Lemma 4.3, the inequality we must verify is

$$8 \cdot 5^{n-1} + 2 \cdot 5^{n-u-1} + 2(r-1)5^n > 2r \cdot 5^n - \frac{j}{3}. \tag{4.7}$$

The smallest possible value of j is now $j = 5^n + 5^{n-1} - 5^{n-u} + 1$, so it suffices to verify

$$8 \cdot 5^{n-1} + 2 \cdot 5^{n-u-1} + 2(r-1)5^n > 2r \cdot 5^n - \frac{5^n + 5^{n-1} - 5^{n-u} + 1}{3}$$

which is equivalent to

$$\frac{5^{n-u-1} + 1}{3} > 0.$$

So (4.7) holds, and we have shown condition (C2) is satisfied in all cases. □

Lemma 4.8. *If a modular form $f_{i/j}$ identified in Theorem 1.3 satisfies conditions (C1) through (C3), then it satisfies condition (C4) if and only if $L_2 f_{i/j}$ is divisible by E_4^j in $M_*(2)_{\mathbb{Z}/5}$.*

Proof. Since 2 is a topological generator of $\mathbb{Z}_5^\times$, checking $f_{i/j}$ satisfies condition (C4) is equivalent to showing there exists $g \in M_{24i-4j}(2)$ such that $(L_2 f_{i/j})(q) \equiv g(q) \mod 5$ by Proposition 3.8. The lemma then follows from Proposition 3.7. □

The remaining goal of this section is to complete the proof of Theorem 1.3 by showing that the modular forms $f_{i/j}$ as identified in the theorem satisfy condition (C4). To begin, we establish a method for computing the E_4-divisibility required by Lemma 4.8.

Proposition 4.9. *If $f \in M_t$, then $L_2 f \in M_t(2)_{\mathbb{Z}/5}$ is expressible as a homogeneous element of $\mathbb{Z}/5[\mu, \varepsilon]$ of degree $t/4$, and if $y = 4\mu/\varepsilon + 1$, then*

$$L_2 f = \varepsilon^{t/4} P(y) \tag{4.10}$$

for an inhomogeneous polynomial $P(y) \in \mathbb{Z}/5[y]$. Moreover, $L_2 f$ is divisible by E_4^j in $M_*(2)_{\mathbb{Z}/5}$ if and only if $P(y) = O(y^j)$ (that is, $P(y)$ is divisible by y^j) in $\mathbb{Z}/5[y]$.

Proof. By Proposition 3.1, $f \in M_t$ is a $\mathbb{Z}$-linear combination of terms of the form $\Delta^a E_4^b$ for non-negative integers a, b with $12a + 4b = t$. From the basic properties of L_N outlined in Section 1, as well as Equations (3.3)-(3.6), it follows that

$$L_2(\Delta^a E_4^b) = V_2 \Delta^a V_2 E_4^b - \Delta^a E_4^b = (\mu^2 \varepsilon)^a (4\mu + 16\varepsilon)^b - (64\mu\varepsilon^2)^a (64\mu + 16\varepsilon)^b \in M_t(2)$$

which is a homogeneous polynomial in μ and ε over the integers of degree $3a+b = t/4$. Passing to $M_t(2)_{\mathbb{Z}/5} = M_t(2) \otimes \mathbb{Z}/5$ yields an expression of $L_2(\Delta^a E_4^b)$ as a homogeneous polynomial of the same degree over $\mathbb{Z}/5$. Thus, $L_2 f \in M_t(2)_{\mathbb{Z}/5}$ is a sum of such homogeneous polynomials by the linearity of L_2. If we put $x = \mu/\varepsilon$, then $L_2 f = \varepsilon^{t/4} P(x)$ where $P(x) \in \mathbb{Z}/5[x]$, and making the change of variable $x = 4y+1$ (so that $y = 4x+1$) yields Equation (4.10).

By Equation (3.3), $L_2 f$ is divisible by E_4^j in $M_*(2)_{\mathbb{Z}/5}$ if and only if the corresponding homogeneous element of $\mathbb{Z}/5[\mu, \varepsilon]$ is divisible by $(4\mu + \varepsilon)^j$. But this is equivalent to $P(y)$ being divisible by y^j since

$$(4\mu + \varepsilon)^j = \varepsilon^j (4x + 1)^j = \varepsilon^j y^j. \quad \square$$

The next four lemmas produce equations of the form (4.10) for various modular forms f.

Lemma 4.11. *For $0 \leq a, b \in \mathbb{Z}$, $L_2(\Delta^a E_4^b) = \varepsilon^{3a+b} y^b ((4y + 1)^{2a} - (-1)^a (4y + 1)^a)$ in $M_*(2)_{\mathbb{Z}/5}$.*

Proof. Equation (3.4) implies that $V_2 E_4 = E_4$ in $M_*(2)_{\mathbb{Z}/5}$. Therefore, using the notation from the proof of Proposition 4.9,

$$\begin{aligned} L_2(\Delta^a E_4^b) &= V_2 \Delta^a V_2 E_4^b - \Delta^a E_4^b \\ &= E_4^b (V_2 \Delta^a - \Delta^a) \\ &= (4\mu + \varepsilon)^b (\mu^{2a} \varepsilon^a - (-1)^a \mu^a \varepsilon^{2a}) \\ &= \varepsilon^{3a+b} (4x + 1)^b (x^{2a} - (-1)^a x^a) \\ &= \varepsilon^{3a+b} y^b ((4y - 1)^{2a} - (-1)^a (4y + 1)^a) \end{aligned}$$

in $M_*(2)_{\mathbb{Z}/5}$. $\square$

Lemma 4.12. *For integers $r \geq 1$ and $n \geq 0$, $L_2 \Delta^{2r \cdot 5^n} = \varepsilon^{6r \cdot 5^n} (3r y^{5^n} + O(y^{a_n}))$ in $M_*(2)_{\mathbb{Z}/5}$.*

Proof. By Lemma 4.11,

$$\begin{aligned} L_2(\Delta^{2r \cdot 5^n}) &= \varepsilon^{6 \cdot 5^n} ((4y + 1)^{4r \cdot 5^n} - (4y + 1)^{2r \cdot 5^n}) \\ &= \varepsilon^{6 \cdot 5^n} ((4y^{5^n} + 1)^{4r} - (4y^{5^n} + 1)^{2r}) \\ &= \varepsilon^{6 \cdot 5^n} (1 + 4r \cdot 4y^{5^n} + O(y^{2 \cdot 5^n}) - (1 + 2r \cdot 4y^{5^n} + O(y^{2 \cdot 5^n}))) \\ &= \varepsilon^{6 \cdot 5^n} (3r y^{5^n} + O(y^{a_n})). \quad \square \end{aligned}$$

Lemma 4.13. *For integers $n \geq 2$ and $r \geq 1$,*

$$\begin{aligned} L_2C_{0,n,r} &= r\varepsilon^{6r\cdot 5^n}(2y^{5^n} + 4y^{27\cdot 5^{n-2}} + 4y^{28\cdot 5^{n-2}} + 3y^{29\cdot 5^{n-2}} + O(y^{a_n})), \\ L_2D_{0,n,r} &= r\varepsilon^{6r\cdot 5^n}(y^{27\cdot 5^{n-2}} + y^{28\cdot 5^{n-2}} + 3y^{29\cdot 5^{n-2}} + O(y^{a_n})), \quad \textit{and} \\ L_2D_{n-2,n,r} &= r\varepsilon^{6r\cdot 5^n}(2y^{6\cdot 5^{n-1}-3} + 2y^{6\cdot 5^{n-1}-2} + O(y^{a_n})), \end{aligned}$$

in $M_*(2)_{\mathbb{Z}/5}$.

Proof. To begin, we compute $L_2C_{0,n,r}$. By Lemma 4.11,

$$\begin{aligned} L_2C_{0,n,r} &= 4rL_2(\Delta^{42\cdot 5^{n-2}+2(r-1)5^n}E_4^{24\cdot 5^{n-2}}) \\ &= 4r\varepsilon^{6r\cdot 5^n}y^{24\cdot 5^{n-2}}((4y+1)^{5^{n-2}(100r-16)} - (4y+1)^{5^{n-2}(50r-8)}) \\ &= 4r\varepsilon^{6r\cdot 5^n}y^{24\cdot 5^{n-2}}((4y^{5^{n-2}}+1)^{25(4r-1)+5+4} - (4y^{5^{n-2}}+1)^{25(2r-1)+5(3)+2}). \end{aligned} \tag{4.14}$$

In $\mathbb{Z}/5[y]$,

$$\begin{aligned} (4y^{5^{n-2}}+1)^{25(4r-1)+5+4} &= (4y^{5^n}+1)^{4r-1}(4y^{5^{n-1}}+1)(4y^{5^{n-2}}+1)^4 \\ &= (4y^{5^n}+1)^{4r-1}(4y^{5^{n-1}}+1)(1+y^{5^{n-2}}+y^{2\cdot 5^{n-2}}+y^{3\cdot 5^{n-2}}+y^{4\cdot 5^{n-2}}) \\ &= (4y^{5^n}+1)^{4r-1}(1+y^{5^{n-2}}+y^{2\cdot 5^{n-2}}+y^{3\cdot 5^{n-2}}+y^{4\cdot 5^{n-2}}+4y^{5^{n-1}}+O(y^{6\cdot 5^{n-2}})) \\ &= 1+y^{5^{n-2}}+y^{2\cdot 5^{n-2}}+y^{3\cdot 5^{n-2}}+y^{4\cdot 5^{n-2}}+4y^{5^{n-1}}+O(y^{6\cdot 5^{n-2}}) \end{aligned}$$

and

$$\begin{aligned} (4y^{5^{n-2}}+1)^{25(2r-1)+5(3)+2} &= (4y^{5^n}+1)^{2r-1}(4y^{5^{n-1}}+1)^3(4y^{5^{n-2}}+1)^2 \\ &= (4y^{5^n}+1)^{2r-1}(1+2y^{5^{n-1}}+3y^{2\cdot 5^{n-1}}+4y^{3\cdot 5^{n-1}})(1+3y^{5^{n-2}}+y^{2\cdot 5^{n-2}}) \\ &= (4y^{5^n}+1)^{2r-1}(1+3y^{5^{n-2}}+y^{2\cdot 5^{n-2}}+2y^{5^{n-1}}+O(y^{6\cdot 5^{n-2}})) \\ &= 1+3y^{5^{n-2}}+y^{2\cdot 5^{n-2}}+2y^{5^{n-1}}+O(y^{6\cdot 5^{n-2}}) \end{aligned}$$

which, when combined with (4.14), yield

$$\begin{aligned} L_2C_{0,n,r} &= 4r\varepsilon^{6r\cdot 5^n}y^{24\cdot 5^{n-2}}(3y^{5^{n-2}}+y^{3\cdot 5^{n-2}}+y^{4\cdot 5^{n-2}}+2y^{5^{n-1}}+O(y^{6\cdot 5^{n-2}})) \\ &= r\varepsilon^{6r\cdot 5^n}(2y^{5^n}+4y^{27\cdot 5^{n-2}}+4y^{28\cdot 5^{n-2}}+3y^{29\cdot 5^{n-2}}+O(y^{a_n})) \end{aligned}$$

in $M_*(2)_{\mathbb{Z}/5}$.

Next, we compute $L_2D_{0,n,r}$. By Lemma 4.11,

$$\begin{aligned} L_2D_{0,n,r} &= 3rL_2(\Delta^{41\cdot 5^{n-2}+2(r-1)5^n}E_4^{27\cdot 5^{n-2}}) \\ &= 3r\varepsilon^{6r\cdot 5^n}y^{27\cdot 5^{n-2}}((4y+1)^{5^{n-2}(100r-18)} + (4y+1)^{5^{n-2}(50r-9)}) \\ &= 3r\varepsilon^{6r\cdot 5^n}y^{27\cdot 5^{n-2}}((4y^{5^{n-2}}+1)^{25(4r-1)+5+2} + (4y^{5^{n-2}}+1)^{25(2r-1)+5(3)+1}). \end{aligned} \tag{4.15}$$

In $\mathbb{Z}/5[y]$,

$$\begin{aligned}(4y^{5^{n-2}}+1)^{25(4r-1)+5+2} &= (4y^{5^n}+1)^{4r-1}(4y^{5^{n-1}}+1)(4y^{5^{n-2}}+1)^2\\ &= (4y^{5^n}+1)^{4r-1}(4y^{5^{n-1}}+1)(1+3y^{5^{n-2}}+y^{2\cdot 5^{n-2}})\\ &= 1+3y^{5^{n-2}}+y^{2\cdot 5^{n-2}}+O(y^{3\cdot 5^{n-2}})\end{aligned}$$

and

$$\begin{aligned}(4y^{5^{n-2}}+1)^{25(2r-1)+5(3)+1} &= (4y^{5^n}+1)^{2r-1}(4y^{5^{n-1}}+1)^3(4y^{5^{n-2}}+1)\\ &= 1+4y^{5^{n-2}}+O(y^{3\cdot 5^{n-2}})\end{aligned}$$

which, when combined with (4.15), yield

$$\begin{aligned}L_2D_{0,n,r} &= 3r\varepsilon^{6\cdot 5^n}y^{27\cdot 5^{n-2}}(2+2y^{5^{n-2}}+y^{2\cdot 5^{n-2}}+O(y^{3\cdot 5^{n-2}}))\\ &= r\varepsilon^{6r\cdot 5^n}(y^{27\cdot 5^{n-2}}+y^{28\cdot 5^{n-2}}+3y^{29\cdot 5^{n-2}}+O(y^{a_n}))\end{aligned}$$

in $M_*(2)_{\mathbb{Z}/5}$.

Finally, we compute $L_2D_{n-2,n,r}$. By Lemma 4.11,

$$\begin{aligned}L_2D_{n-2,n,r} &= rL_2(\Delta^{8\cdot 5^{n-1}+1+2(r-1)5^n}E_4^{6\cdot 5^{n-1}-3})\\ &= r\varepsilon^{6r\cdot 5^n}y^{6\cdot 5^{n-1}-3}((4y+1)^{5^{n-1}(20r-4)+2}+(4y+1)^{5^{n-1}(10r-2)+1})\\ &= r\varepsilon^{6r\cdot 5^n}y^{6\cdot 5^{n-1}-3}((4y^{5^{n-1}}+1)^{20r-4}(4y+1)^2+(4y^{5^{n-1}}+1)^{10r-2}(4y+1))\\ &= r\varepsilon^{6r\cdot 5^n}y^{6\cdot 5^{n-1}-3}(2+2y+O(y^2))\\ &= r\varepsilon^{6r\cdot 5^n}(2y^{6\cdot 5^{n-1}-3}+2y^{6\cdot 5^{n-1}-2}+O(y^{a_n}))\end{aligned}$$

in $M_*(2)_{\mathbb{Z}/5}$. □

Lemma 4.16. *For* $1\le m\le n-2$,

$$L_2C_{m,n,r} = r\varepsilon^{6r\cdot 5^n}(4y^{6\cdot 5^{n-1}-5^{n-m-1}}+3y^{6\cdot 5^{n-1}-3\cdot 5^{n-m-2}}+3y^{6\cdot 5^{n-1}-2\cdot 5^{n-m-2}}+O(y^{a_n}))$$

and for $1\le m\le n-3$,

$$L_2D_{m,n,r} = r\varepsilon^{6r\cdot 5^n}(2y^{6\cdot 5^{n-1}-3\cdot 5^{n-m-2}}+2y^{6\cdot 5^{n-1}-2\cdot 5^{n-m-2}}+y^{6\cdot 5^{n-1}-5^{n-m-2}}+O(y^{a_n}))$$

in $M_*(2)_{\mathbb{Z}/5}$.

Proof. First, we compute $L_2C_{m,n,r}$ for $1\le m\le n-2$. By Lemma 4.11,

$$\begin{aligned}L_2C_{m,n,r} &= 3rL_2(\Delta^{8\cdot 5^{n-1}+2\cdot 5^{n-m-2}+2(r-1)5^n}E_4^{6\cdot 5^{n-1}-6\cdot 5^{n-m-2}})\\ &= 3r\varepsilon^{6\cdot 5^n}y^{6\cdot 5^{n-1}-6\cdot 5^{n-m-2}}((4y+1)^{16\cdot 5^{n-1}+4\cdot 5^{n-m-2}+4(r-1)5^n}\\ &\qquad -(4y+1)^{8\cdot 5^{n-1}+2\cdot 5^{n-m-2}+2(r-1)5^n}).\end{aligned}\tag{4.17}$$

In $\mathbb{Z}/5[y]$,

$$\begin{aligned}(4y+1)^{16\cdot 5^{n-1}+4\cdot 5^{n-m-2}+4(r-1)5^n} &= (4y+1)^{(4r-1)5^n+5^{n-1}+4\cdot 5^{n-m-2}}\\ &= (4y^{5^n}+1)^{4r-1}(4y^{5^{n-1}}+1)(4y^{5^{n-m-2}}+1)^4\\ &= 1+y^{5^{n-m-2}}+y^{2\cdot 5^{n-m-2}}+y^{3\cdot 5^{n-m-2}}+y^{4\cdot 5^{n-m-2}}+O(y^{6\cdot 5^{n-m-2}})\end{aligned}$$

and

$$\begin{aligned}(4y+1)^{8\cdot 5^{n-1}+2\cdot 5^{n-m-2}+2(r-1)5^n} &= (4y+1)^{(2r-1)5^n+3\cdot 5^{n-1}+2\cdot 5^{n-m-2}}\\ &= (4y^{5^n}+1)^{2r-1}(4y^{5^{n-1}}+1)^3(4y^{5^{n-m-2}}+1)^2\\ &= 1+3y^{5^{n-m-2}}+y^{2\cdot 5^{n-m-2}}+O(y^{6\cdot 5^{n-m-2}})\end{aligned}$$

which, when combined with (4.17), yield

$$\begin{aligned}L_2C_{m,n,r} &= 3r\varepsilon^{6\cdot 5^n}y^{6\cdot 5^{n-1}-6\cdot 5^{n-m-2}}(3y^{5^{n-m-2}}+y^{3\cdot 5^{n-m-2}}+y^{4\cdot 5^{n-m-2}}+O(y^{6\cdot 5^{n-m-2}}))\\ &= r\varepsilon^{6\cdot 5^n}(4y^{6\cdot 5^{n-1}-5^{n-m-1}}+3y^{6\cdot 5^{n-1}-3\cdot 5^{n-m-2}}+3y^{6\cdot 5^{n-1}-2\cdot 5^{n-m-2}}+O(y^{a_n}))\end{aligned}$$

in $M_*(2)_{\mathbb{Z}/5}$.

Next, we compute $L_2D_{m,n,r}$ for $1\le m\le n-3$. By Lemma 4.11,

$$\begin{aligned}L_2D_{m,n,r} &= rL_2(\Delta^{8\cdot 5^{n-1}+\cdot 5^{n-m-2}+2(r-1)5^n}E_4^{6\cdot 5^{n-1}-3\cdot 5^{n-m-2}})\\ &= r\varepsilon^{6\cdot 5^n}y^{6\cdot 5^{n-1}-3\cdot 5^{n-m-2}}((4y+1)^{16\cdot 5^{n-1}+2\cdot 5^{n-m-2}+4(r-1)5^n}\\ &\qquad +(4y+1)^{8\cdot 5^{n-1}+5^{n-m-2}+2(r-1)5^n}).\end{aligned}\tag{4.18}$$

In $\mathbb{Z}/5[y]$,

$$\begin{aligned}(4y+1)^{16\cdot 5^{n-1}+2\cdot 5^{n-m-2}+4(r-1)5^n} &= (4y+1)^{(4r-1)5^n+5^{n-1}+2\cdot 5^{n-m-2}}\\ &= (4y^{5^n}+1)^{4r-1}(4y^{5^{n-1}}+1)(4y^{5^{n-m-2}}+1)^2\\ &= 1+3y^{5^{n-m-2}}+y^{2\cdot 5^{n-m-2}}+O(y^{3\cdot 5^{n-m-2}})\end{aligned}$$

and

$$\begin{aligned}(4y+1)^{8\cdot 5^{n-1}+5^{n-m-2}+2(r-1)5^n} &= (4y+1)^{(2r-1)5^n+3\cdot 5^{n-1}+5^{n-m-2}}\\ &= (4y^{5^n}+1)^{2r-1}(4y^{5^{n-1}}+1)^3(4y^{5^{n-m-2}}+1)\\ &= 1+4y^{5^{n-m-2}}+O(y^{3\cdot 5^{n-m-2}})\end{aligned}$$

which, when combined with (4.18), yield

$$\begin{aligned}L_2D_{m,n,r} &= r\varepsilon^{6\cdot 5^n}y^{6\cdot 5^{n-1}-3\cdot 5^{n-m-2}}(2+2y^{5^{n-m-2}}+y^{2\cdot 5^{n-m-2}}+O(y^{3\cdot 5^{n-m-2}}))\\ &= r\varepsilon^{6r\cdot 5^n}(2y^{6\cdot 5^{n-1}-3\cdot 5^{n-m-2}}+2y^{6\cdot 5^{n-1}-2\cdot 5^{n-m-2}}+y^{6\cdot 5^{n-1}-5^{n-m-2}}+O(y^{a_n}))\end{aligned}$$

in $M_*(2)_{\mathbb{Z}/5}$. Note that the assumption $m\le n-3$ is essential, because if $m=n-2$,

$$6\cdot 5^{n-1}-5^{n-m-2}=6\cdot 5^{n-1}-1=a_n$$

and so the term $y^{6\cdot 5^{n-1}-5^{n-m-2}} = y^{a_n}$ would not appear. This is why the computation of $L_2D_{n-2,n,r}$ is handled separately in Lemma 4.13. □

Theorem 4.19. *The modular forms $f_{r\cdot 5^n/j}$ as identified in Theorem 1.3 satisfy condition (C4) of Behrens' theorem.*

Proof. By Corollary 4.2, Proposition 4.4, and Lemma 4.8, it suffices to show that $L_2 f_{r\cdot 5^n/j}$ is divisible by E_4^j in $M_*(2)_{\mathbb{Z}/5}$.

Consider first the case $j \leq 5^n$, so that $f_{r\cdot 5^n/j} = \Delta^{2r\cdot 5^n}$. Lemma 4.12 implies that

$$L_2\Delta^{2r\cdot 5^n} = \varepsilon^{6\cdot 5^n}(3ry^{5^n} + O(y^{a_n})) \in M_*(2)_{\mathbb{Z}/5}$$

showing divisibility by $E_4^{5^n}$ and verifying condition (C4) in this case.

Next, suppose $j > 5^n$, and let u be the positive integer between 1 and $n-1$ such that

$$5^n + 5^{n-1} - 5^{n-u} + 1 \leq j \leq 5^n + 5^{n-1} - 5^{n-u-1}.$$

Assume first that $j > 5^n + 5^{n-1} - 5^{n-u} + 2\cdot 5^{n-u-1}$, so that the modular form in question is

$$f_{r\cdot 5^n/j} = \Delta^{2r\cdot 5^n} + \sum_{m=0}^{u-1}(C_{m,n,r} + D_{m,n,r}).$$

To show that $L_2 f_{r\cdot 5^n/j}$ is divisible by E_4^j in $M_*(2)_{\mathbb{Z}/5}$, it suffices to show divisibility by $E_4^{5^n+5^{n-1}-5^{n-u-1}}$ since $5^n + 5^{n-1} - 5^{n-u-1}$ is the largest possible j-value in this case. By Lemmas 4.12 and 4.13,

$$L_2(\Delta^{2r\cdot 5^n} + C_{0,n,r} + D_{0,n,r}) = \varepsilon^{6r\cdot 5^n}(ry^{29\cdot 5^{n-2}} + O(y^{a_n})) \tag{4.20}$$

in $M_*(2)_{\mathbb{Z}/5}$. Lemmas 4.13 and 4.16 imply

$$L_2(C_{n-2,n,r} + D_{n-2,n,r}) = \varepsilon^{6r\cdot 5^n}(4ry^{6\cdot 5^{n-1}-5} + O(y^{a_n})) \in M_*(2)_{\mathbb{Z}/5} \tag{4.21}$$

and for $1 \leq m \leq n-3$, Lemma 4.16 implies

$$L_2(C_{m,n,r} + D_{m,n,r}) = \varepsilon^{6r\cdot 5^n}(4ry^{6\cdot 5^{n-1}-5^{n-m-1}} + ry^{6\cdot 5^{n-1}-5^{n-m-2}} + O(y^{a_n})) \in M_*(2)_{\mathbb{Z}/5}. \tag{4.22}$$

Therefore, for $1 \leq u \leq n-2$,

$$\begin{aligned} L_2 f_{r\cdot 5^n/j} &= \varepsilon^{6r\cdot 5^n}\left(ry^{29\cdot 5^{n-2}} + \sum_{m=1}^{u-1}(4ry^{6\cdot 5^{n-1}-5^{n-m-1}} + ry^{6\cdot 5^{n-1}-5^{n-m-2}}) + O(y^{a_n})\right) \\ &= \varepsilon^{6r\cdot 5^n}(ry^{6\cdot 5^{n-1}-5^{n-u-1}} + O(y^{a_n})) \in M_*(2)_{\mathbb{Z}/5} \end{aligned} \tag{4.23}$$

by Equations (4.20) and (4.22), showing divisibility by $E_4^{5^n+5^{n-1}-5^{n-u-1}}$. For the case $u = n-1$, we obtain

$$L_2 f_{r\cdot 5^n/j} = \varepsilon^{6r\cdot 5^n}(ry^{6\cdot 5^{n-1}-5} + 4ry^{6\cdot 5^{n-1}-5} + O(y^{a_n})) = \varepsilon^{6r\cdot 5^n}\cdot O(y^{a_n})$$

from Equations (4.21) and (4.23), showing divisibility by $E_4^{5^n+5^{n-1}-5^{n-(n-1)-1}} = E_4^{a_n}$. Thus, for $j > 5^n + 5^{n-1} - 5^{n-u} + 2\cdot 5^{n-u-1}$, condition (C4) is verified.

Continuing with the case $j > 5^n$, assume now that $j \leq 5^n + 5^{n-1} - 5^{n-u} + 2 \cdot 5^{n-u-1}$, so that the modular form in question is

$$f_{r\cdot 5^n/j} = \Delta^{2r\cdot 5^n} + \sum_{m=0}^{u-2} (C_{m,n,r} + D_{m,n,r}) + C_{u-1,n,r}.$$

In this case, it suffices to show divisibility by $E_4^{5^n+5^{n-1}-5^{n-u}+2\cdot 5^{n-u-1}}$ since $5^n + 5^{n-1} - 5^{n-u} + 2 \cdot 5^{n-u-1}$ is the largest possible value of j. For $u = 1$, Lemmas 4.12 and 4.13 together imply

$$\begin{aligned} L_2 f_{r\cdot 5^n/j} &= L_2(\Delta^{2r\cdot 5^n} + C_{0,n,r}) \\ &= \varepsilon^{6r\cdot 5^n}(3ry^{5^n} + 2ry^{5^n} + 4ry^{27\cdot 5^{n-2}} + 4ry^{28\cdot 5^{n-2}} + 3ry^{29\cdot 5^{n-2}} + O(y^{a_n})) \\ &= \varepsilon^{6r\cdot 5^n}(4ry^{27\cdot 5^{n-2}} + 4ry^{28\cdot 5^{n-2}} + 3ry^{29\cdot 5^{n-2}} + O(y^{a_n})) \in M_*(2)_{\mathbb{Z}/5} \end{aligned}$$

showing divisibility by $E_4^{5^n+5^{n-1}-5^{n-1}+2\cdot 5^{n-2}} = E_4^{27\cdot 5^{n-2}}$. For $2 \leq u \leq n-1$, we obtain

$$\begin{aligned} L_2 f_{r\cdot 5^n/j} = \varepsilon^{6r\cdot 5^n} \Bigg(& ry^{29\cdot 5^{n-2}} + \sum_{m=1}^{u-2} (4ry^{6\cdot 5^{n-1}-5^{n-m-1}} + ry^{6\cdot 5^{n-1}-5^{n-m-2}}) \\ & + 4ry^{6\cdot 5^{n-1}-5^{n-u}} + 3ry^{6\cdot 5^{n-1}-3\cdot 5^{n-u-1}} + 3ry^{6\cdot 5^{n-1}-2\cdot 5^{n-u-1}} + O(y^{a_n}) \Bigg) \\ = \varepsilon^{6r\cdot 5^n}(3ry^{6\cdot 5^{n-1}-3\cdot 5^{n-u-1}} & + 3ry^{6\cdot 5^{n-1}-2\cdot 5^{n-u-1}} + O(y^{a_n})) \in M_*(2)_{\mathbb{Z}/5} \end{aligned}$$

from Equations (4.20), (4.21), and (4.22), showing divisibility by

$$E_4^{5^n+5^{n-1}-5^{n-u}+2\cdot 5^{n-u-1}} = E_4^{6\cdot 5^{n-1}-3\cdot 5^{n-u-1}}.$$

Thus, condition (C4) is verified for this last remaining case. □

Theorem 1.3 follows from Corollary 4.2, Proposition 4.4, and Theorem 4.19.

5. Computations at other primes

5.1. The prime 7

In the case $p = 7$ we are hunting for $f_{i/j} \in M_{48i}$ associated to each order 7 generator $\beta_{i/j}$. The demand on q-order in condition (C2) says

$$\operatorname{ord}_q f_{i/j} > \frac{48i - 6j}{12} \quad \text{or} \quad \operatorname{ord}_q f_{i/j} = \frac{48i - 6j - 2}{12}$$

which is equivalent to $\operatorname{ord}_q f_{i/j} > 4i - \frac{1}{2}j$. In particular, if $j = 1$, the inequality becomes

$$\operatorname{ord}_q f_i \geq 4i. \tag{5.1}$$

When combined with parts (a) and (c) of Proposition 4.1, the inequality (5.1) forces

$$f_i = c\Delta^{4i}$$

for some integer c prime to 7. Taking $c = 1$ and recalling that β_i is a 7-primary divided beta family element for all integers $i \geq 1$ (see Example 2.2) yields the following theorem.

Theorem 5.2. *For any integer $i \geq 1$, the 7-primary divided beta family element β_i has corresponding modular form $f_i = \Delta^{4i}$.*

Remark 5.3. Proposition 3.8 does not apply in the case $\ell_0 = 2$ and $p = 7$ since 2 is not a topological generator of $\mathbb{Z}_7^\times$. Further computations at the prime 7 could be done by working with $\Gamma_0(3)$ modular forms instead, since $\ell = 3$ *is* a topological generator of $\mathbb{Z}_7^\times$.

5.2. The prime 11

We shall compute $f_{i/j} \in M_{120i}$ at the prime 11 in the case $1 \leq j \leq 11^n$, where $i = r \cdot 11^n$ with $(r, 11) = 1$. Since 2 is a topological generator of $\mathbb{Z}_{11}^\times$, Proposition 3.8 applies, and so it suffices to compute with $\Gamma_0(2)$ modular forms. In $M_*(2)[1/2]$,

$$E_{10} = -32768\delta\mu^2 - 4096\delta\mu\varepsilon + 1024\delta\varepsilon^2,$$
$$V_2E_{10} = -32\delta\mu^2 + 128\delta\mu\varepsilon + 1024\varepsilon^2$$

and so in $M_*(2)_{\mathbb{Z}/11}$,

$$E_{10} = V_2E_{10} = \delta\mu^2 + 7\delta\mu\varepsilon + \delta\varepsilon^2 = \delta(\mu + 3\varepsilon)(\mu + 4\varepsilon). \tag{5.4}$$

Equations (3.5) and (3.6) imply the identities

$$\Delta = 9\mu\varepsilon^2, \tag{5.5}$$
$$V_2\Delta = \mu^2\varepsilon \tag{5.6}$$

in $M_*(2)_{\mathbb{Z}/11}$.

Proposition 5.7. *If $f \in M_t$, then $L_2f \in M_t(2)_{\mathbb{Z}/11}$ is expressible as a homogeneous element of $\mathbb{Z}/11[\mu, \varepsilon]$ of degree $t/4$, and if $x = \mu/\varepsilon$, then*

$$L_2f = \varepsilon^{t/4}P(x) \tag{5.8}$$

for an inhomogeneous polynomial $P(x) \in \mathbb{Z}/11[x]$. Moreover, if $P(x)$ is divisible by $(x+1)^j(x+3)^j(x+4)^j$ in $\mathbb{Z}/11[x]$, then L_2f is divisible by E_{10}^j in $M_*(2)_{\mathbb{Z}/11}$.

Proof. This proposition is analogous to Proposition 4.9, and the proof of (5.8) is similar to the proof of (4.10).

If $P(x)$ is divisible by $(x+1)^j(x+3)^j(x+4)^j$ in $\mathbb{Z}/11[x]$, then the corresponding homogeneous element of $\mathbb{Z}/11[\mu, \varepsilon]$ is divisible by $(\mu+\varepsilon)^j(\mu+3\varepsilon)^j(\mu+4\varepsilon)^j$. This in turn implies divisibility by E_{10}^j since

$$(\mu+\varepsilon)^j(\mu+3\varepsilon)^j(\mu+4\varepsilon)^j = \delta^j E_{10}^j$$

by Equation (5.4) and the identity $\delta^2 = \mu + \varepsilon$. □

Theorem 5.9. *Given an 11-primary divided beta family element $\beta_{r\cdot 11^n/j}$ with $(r, 11) = 1$ and $1 \leq j \leq 11^n$, the corresponding modular form is $f_{r\cdot 11^n/j} = \Delta^{10r\cdot 11^n} \in M_{120r\cdot 11^n}$.*

Proof. The modular form $\Delta^{10r\cdot 11^n}$ has nonzero Fourier expansion modulo 11, and its q-order is

$$\operatorname{ord}_q \Delta^{10r\cdot 11^n} = 10r\cdot 11^n > \frac{120r\cdot 11^n - 10j}{12}$$

for $j \geq 1$. Any power of $\Delta = E_4^3 + 10E_6^2$ is not divisible by $E_{10} = E_4E_6$ in $(M_*)_{\mathbb{Z}/11}$. Thus, conditions (C1), (C2), and (C3) are satisfied.

Using the notation from Proposition 5.7, Equations (5.5) and (5.6) imply

$$\begin{aligned} L_2\Delta^{10r\cdot 11^n} &= V_2\Delta^{10r\cdot 11^n} - \Delta^{10r\cdot 11^n} \\ &= (\mu^{20r}\varepsilon^{10r} - 9^{10r}\mu^{10r}\varepsilon^{10r})^{11^n} \\ &= \varepsilon^{30r\cdot 11^n}x^{10r\cdot 11^n}(x^{10r}-1)^{11^n} \\ &= \varepsilon^{30r\cdot 11^n}x^{10r\cdot 11^n}(x^{10}-1)^{11^n}(x^{10(r-1)} + x^{10(r-2)} + \cdots + x^{10} + 1)^{11^n}. \end{aligned} \tag{5.10}$$

Moreover, in $\mathbb{Z}/11[x]$,

$$x^{10} - 1 = (x+1)(x+2)(x+3)(x+4)(x+5)(x+6)(x+7)(x+8)(x+9)(x+10)$$

and so Equation (5.10) has the form $L_2\Delta^{10r\cdot 11^n} = \varepsilon^{30r\cdot 11^n}P(x)$ where $P(x)$ is divisible by

$$(x+1)^{11^n}(x+3)^{11^n}(x+4)^{11^n}.$$

By Proposition 5.7, this implies $L_2\Delta^{10r\cdot 11^n}$ is divisible by $E_{10}^{11^n}$ (and hence by E_{10}^j for $1 \leq j \leq 11^n$) in $M_*(2)_{\mathbb{Z}/11}$. This verifies condition (C4) in the case $\ell = 2$ by Proposition 3.7. □

5.3. The prime 13

We shall establish the analog of Theorem 5.9 at the prime 13. The proof will follow the structure of Subsection 5.2 but will be made more concise. Since 2 is a topological generator of $\mathbb{Z}_{13}^\times$, it suffices to compute with $\Gamma_0(2)$ modular forms when checking condition (C4).

Theorem 5.11. *Given a 13-primary divided beta family element $\beta_{r\cdot 13^n/j}$ with $(r, 13) = 1$ and $1 \leq j \leq 13^n$, the corresponding modular form is $f_{r\cdot 13^n/j} = \Delta^{14r\cdot 13^n} \in M_{168r\cdot 13^n}$.*

Proof. The modular form $\Delta^{14r\cdot 13^n}$ has nonzero Fourier expansion modulo 11, and its q-order is

$$\operatorname{ord}_q \Delta^{14r\cdot 13^n} = 14r\cdot 13^n > \frac{168r\cdot 13^n - 12j}{12}$$

for $j \geq 1$. Any power of $\Delta = 12E_4^3 + E_6^2$ is not divisible by $E_{12} = 6E_4^3 + 8E_6^2$ in $(M_*)_{\mathbb{Z}/13}$. Thus, conditions (C1), (C2), and (C3) are satisfied.

In $M_*(2)_{\mathbb{Z}/13}$,

$$\Delta = 12\mu\varepsilon^2, \tag{5.12}$$

$$V_2\Delta = \mu^2\varepsilon, \tag{5.13}$$

and

$$E_{12} = V_2E_{12} = 12\mu^3 + 9\mu^2\varepsilon + 4\mu\varepsilon^2 + \varepsilon^3 = 12(\mu + 12\varepsilon)(\mu^2 + 5\mu\varepsilon + \varepsilon^2). \tag{5.14}$$

Using the notation from Proposition 5.7, Equations (5.12) and (5.13) imply

$$\begin{aligned} L_2\Delta^{14r\cdot 13^n} &= V_2\Delta^{14r\cdot 13^n} - \Delta^{14r\cdot 13^n} \\ &= (\mu^{28r}\varepsilon^{14r} - \mu^{14r}\varepsilon^{28r})^{13^n} \\ &= \varepsilon^{42r\cdot 13^n} x^{14r\cdot 13^n}(x^{14r}-1)^{13^n} \\ &= \varepsilon^{42r\cdot 13^n} x^{14r\cdot 13^n}(x^{14}-1)^{13^n}(x^{14(r-1)} + x^{14(r-2)} + \cdots + x^{14} + 1)^{13n}. \end{aligned} \tag{5.15}$$

Moreover, in $\mathbb{Z}/13[x]$,

$$x^{14}-1 = (x+1)(x+12)(x^2+3x+1)(x^2+5x+1)(x^2+6x+1)(x^2+7x+1)(x^2+8x+1)(x^2+10x+1)$$

and so Equation (5.15) has the form $L_2\Delta^{14r\cdot 13^n} = \varepsilon^{42r\cdot 13^n}P(x)$ where $P(x)$ is divisible by

$$(x+12)^{13^n}(x^2+5x+1)^{13^n}.$$

Using Equation (5.14) and arguing as in the proof of Proposition 5.7 shows this is equivalent to $L_2\Delta^{14r\cdot 13^n}$ being divisible by $E_{12}^{13^n}$ (and hence by E_{12}^j for $1 \le j \le 13^n$) in $M_*(2)_{\mathbb{Z}/13}$. This verifies condition (C4) in the case $\ell = 2$ by Proposition 3.7. □

Remark 5.16. The author has verified that analogs of Theorems 5.9 and 5.11 hold at larger primes p for which 2 is a topological generator of $\mathbb{Z}_p^\times$ (e.g., $p = 677$) using computer algebra software.

5.4. A conjecture

Theorems 1.3, 5.2, 5.9, and 5.11 all give evidence for the following conjecture.

Conjecture 5.17. *Let $p \ge 5$ be a prime. If $1 \le i \in \mathbb{Z}$ and $i = rp^n$ with $(r,p) = 1$, then*

$$f_{i/j} = \Delta^{i(p^2-1)/12}$$

for $1 \le j \le p^n$, and $f_{i/j} \ne \Delta^{i(p^2-1)}$ for all other values of j allowed by Lemma 2.1.

In theory, one possible approach to Conjecture 5.17 would be to obtain a closed formula for the Eisenstein series $E_{p-1} \in M_{p-1}(2)_{\mathbb{Z}/p}$ as a polynomial in μ and ε (and δ if $p \equiv 3 \mod 4$). Specific instances of this include Equations (5.4) and (5.14), each obtainable by employing the recursive formula

$$(t-3)(2t-1)(2t+1)\frac{B_{2t}}{(2t)!}E_{2t} = -3\sum_{a+b=t}\frac{(2a-1)(2b-1)B_{2a}B_{2b}}{(2a)!(2b)!}E_{2a}E_{2b}$$

with $t = (p-1)/2$ and then reducing modulo p for $p = 11$ and 13, respectively. Another approach would be to find a more conceptual reason why $E_{p-1}^{p^n}$ should always divide $L_\ell\Delta^{i(p^2-1)/12}$ in $M_*(\ell)_{\mathbb{Z}/p}$.

The next logical step beyond Conjecture 5.17 is to generalize Theorem 1.3 by computing all modular forms $f_{i/j}$ at arbitrary primes $p \ge 7$. We are hopeful that the "correction terms" used in this paper at the prime 5 somehow have natural p-primary analogs. In particular, one can first try to generalize Equation (1.2) by producing modular forms f_{p^2/a_2} satisfying conditions (C1) through (C4) for $i = p^2$ and $j = a_2$.

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