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Odd-degree elements in the Morava $K(n)$ cohomology of finite groups

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Abstract

For each odd prime p, we construct a finite group P such that $K(n)^*(BP)$ has nontrivial odd-degree elements for all $n \geqslant 2$. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this note is to construct odd-degree elements in Morava K-theories for all $p > 2$ and all $n \geqslant 2$, thus generalizing a result of the first author [3] for the case $p = 3$ and $n = 2$. The group we consider is the p-Sylow subgroup of $GL_4(\mathbb{F}_p)$, which is the same as in [3]. The first author thought that his method [3] might work in the general case, but could not do all the calculations (a substantial difference is that, unlike the case of [3], the general case is infinite and hence cannot be verified by computer).

The second author found the correct formula for the odd-degree element in the general case, and proved that it was non-zero for $n = 2$. The case $n > 2$ turns out to require more difficult cohomological calculations, and was worked out jointly.

Throughout the note, p will be an odd prime. Define a group P by

$$P = (\mathbb{Z}/p)^2 \ltimes (\mathbb{Z}/p)^4 = \mathbb{Z}/p\{b_1, b_2\} \ltimes \mathbb{Z}/p\{a_{11}, a_{12}, a_{21}, a_{22}\}.$$

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This is the p-Sylow subgroup of $GL_4(\mathbb{F}_p)$. Our convention will be to write a semidirect product in the form $A \ltimes N$ where A acts on N. Thus, N is the normal subgroup. In the case of P, b_1 commutes with a_{2i}, b_2 commutes with a_{i2}, and

$$b_1^{-1} a_{1i} b_1 = a_{1i} a_{2i},$$
$$b_2^{-1} a_{i1} b_2 = a_{i1} a_{i2}.$$

Theorem 1. *For $p > 2$ and $n \geqslant 2$, $K(n)^{\mathrm{odd}} BP \neq 0$.*

Theorem 1 generalizes a result from [3] for $p = 3$ and $n = 2$.

For the proof of Theorem 1, we use integral Morava K-theory, $\widetilde{K}(n)$, a commutative associative complex-oriented ring spectrum satisfying

$$\widetilde{K}(n)_* = \mathcal{O}_K[v_n, v_n^{-1}],$$

where K is a degree n unramified extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ is its integer ring. For odd p, one can choose a formal group law F of $\widetilde{K}(n)$ satisfying

$$[p]_F x = px - x^{p^n}.$$

It can be shown that the total degree of the non-vanishing monomials of $F(x, y)$ are congruent to 1 modulo $p^n - 1$. If G is any finite group, the Lubin–Tate laws allow us to interpret $\widetilde{K}(n)^* BG \otimes \mathbb{Q}$ to be the ring of n-characters, or the set of class functions on conjugacy classes of n-tuples of commuting elements in G. The theory of n-characters was introduced by Hopkins, Kuhn and Ravenel [2]. Their characterization is a rational generalization of Atiyah's description of the ordinary K-theory of finite groups [1].

The initial steps for the proof for Theorem 1 use n-characters and the following result from [3].

Theorem 2. *Given a fibration*

$$F \to E \to B\mathbb{Z}/p,$$

suppose $K(n)^* E$ is concentrated in even dimensions. Then

$$H^1\big(\mathbb{Z}/p, \widetilde{K}(n)^{\mathrm{even}} F\big) = 0.$$

Theorem 1 follows from Theorem 2 if we can construct a non-zero element

$$\zeta \in H^1\big(\mathbb{Z}/p, \widetilde{K}(n)^*(BG)\big),$$

where G is a good subgroup of P and $G \to P \to \mathbb{Z}/p$ is a fibration. (Recall that a group is called *good* [2] if its Morava K-theory is additively generated by transfers of Euler classes.) This is what was done in [3] for $p = 3$ and $n = 2$. In that particular case, the $\mathbb{Z}[\mathbb{Z}/p]$-module $\widetilde{K}(n)^* BG$ is relatively small, and one can completely calculate its homology. For the general case, a more elaborate scheme is required. This is the main result of this paper. Some facts from [3] are reviewed in Section 2. Section 3 contains some preliminary computations. The element ζ is constructed in Section 4.

2. Some facts about $\widetilde{K}(n)^*BH$ and $\widetilde{K}(n)^*BG$

In this section, we recall some results from [3]. We shall examine the groups

$$H = \mathbb{Z}/p\{b\} \ltimes \mathbb{Z}/p\{a_1, a_2\}, \quad \text{where } b^{-1}a_1b = a_1a_2,$$
$$G = \mathbb{Z}/p\{b_1\} \ltimes \mathbb{Z}/p\{a_{11}, a_{12}, a_{21}, a_{22}\} \subset P.$$

Both G and H are good groups; the Morava K-theory of G and H is concentrated in even dimensions. The Morava K-theory for H is calculated in [4]. The main result will come from the fibration

$$BG \to BP \to B\mathbb{Z}/p\{b_2\}.$$

Let $\alpha, \beta, \kappa \in \widetilde{K}(n)^*BH$ denote, respectively the Euler classes of the representations

$$\mu : H \to \mathbb{Z}/p\{a_1\} \subset S^1, \quad \nu : H \to \mathbb{Z}/p\{b\} \subset S^1,$$

and the representation ξ induced by the inclusion $\mathbb{Z}/p\{a_2, b\} \subset H$ from the representation

$$\sigma : \mathbb{Z}/p\{a_2, b\} \to \mathbb{Z}/p\{a_2\} \subset S^1.$$

The named generators in the source are mapped into the generators with the same name in the target, or to 0 if there is no generator with the same name in the target.

Now, consider the dimension 0 homogeneous summand of

$$\widetilde{K}(n)^*BH/\tau\big(\widetilde{K}(n)^*\big(\mathbb{Z}/p\{a_1, a_2\}\big)\big),$$

where τ denotes the transfer map. Let us call this $\widetilde{R}$. From [3], we have the following two lemmas.

Lemma 3. *In $\widetilde{R}$,*

$$\alpha^p = \beta^{p-1}\alpha.$$

Lemma 4. *In $\widetilde{R} \otimes \mathbb{Q}$,*

$$\sum_{i=0}^{n-1}{}^{F} \frac{\kappa^{p^i}}{\beta^{p^{i+1}-1}} = 0. \tag{1}$$

Corollary 5. *In $\widetilde{R}/p$,*

$$\sum_{i=0}^{n-1} \beta^{p^n - p^{i+1}} \kappa^{p^i} = 0.$$

Proof. Write

$$\nu = \frac{\kappa}{\beta^p}.$$

Note that we have $\nu^{p^n} = \nu$. Thus, the formula of Lemma 4 reads

$$\nu\beta +_F \nu^p\beta +_F \cdots +_F \nu^{p^{n-1}}\beta = 0. \tag{2}$$

But this also implies

$$\nu^{p^{n-1}}\beta = -_F(\nu\beta +_F \nu^p\beta +_F \cdots +_F \nu^{p^{n-2}}\beta). \tag{3}$$

Expanding the left hand side of (2) and substituting recursively (3) to all non-β-linear terms of (2) containing $\nu^{p^{n-1}}$ (note that the power of ν is at most p^{n-1} times the power of β in each term), we obtain

$$0 = \nu\beta + \nu^p\beta + \cdots + \nu^{p^{n-1}}\beta + \sum_{i \geqslant p^n} c_i \nu^{d_i}\beta^i$$

where $c_i \in \widetilde{K}(n)^*$, $d_i < p^{n-1}$. We see that all the terms after the $\sum$-sign are integral. □

The theory for G based upon our results for H. It can be seen that the $\mathbb{Z}/p\{b_1\}$-module $\widetilde{K}(n)^*B\mathbb{Z}/p\{a_{ij}\}$ is isomorphic to

$$\widetilde{K}(n)^*\big(B\mathbb{Z}/p\{a_1, a_2\}\big) \otimes_{\widetilde{K}(n)^*} \widetilde{K}(n)^*\big(B\mathbb{Z}/p\{a_1, a_2\}\big).$$

Consequently, the Tate cohomology of $\mathbb{Z}/p\{b_1\}$ with coefficients in $\widetilde{K}(n)^*B\mathbb{Z}/p\{a_{ij}\}$ is isomorphic to

$$\widehat{H}^0\big(\mathbb{Z}/p\{b\}, \widetilde{K}(n)^*\big(B\mathbb{Z}/p\{a_1, a_2\}\big)\big) \otimes \widehat{H}^0\big(\mathbb{Z}/p\{b\}, \widetilde{K}(n)^*\big(B\mathbb{Z}/p\{a_1, a_2\}\big)\big).$$

The two factors of the expression above are isomorphic. Now, there are two distinguished maps of groups $G \to H$. Both send b_1 to b while one sends the generators a_{i1} to a_i and the generators a_{i2} to 0; the other map sends generators a_{i2} to a_i and the generators a_{i1} to 0. Denote the pullbacks of the classes α, κ via these maps by α_1, κ_1, and respectively α_2, κ_2. Lemmas 3 and 4 imply that the 0-dimensional homogeneous summand S of $K(n)^*BG/\operatorname{Im}\tau$ (where τ is the transfer from $\mathbb{Z}/p\{a_{ij}\}$) is isomorphic to

$$\mathbb{Z}/p[\alpha_1, \alpha_2, \kappa_1, \kappa_2, \beta]/I, \quad \text{where}$$
$$I = \left(\beta^{p^n-1}, \sum_{i=0}^{n-1}\beta^{p^n-p^{i+1}}\kappa_1^{p^i}, \sum_{i=0}^{n-1}\beta^{p^n-p^{i+1}}\kappa_2^{p^i}, \alpha_1^p - \beta^{p-1}\alpha_1, \alpha_2^p - \beta^{p-1}\alpha_2\right). \tag{4}$$

Here we are using the naturality of transfer to conclude that, when pulling back by the map $G \to H$, $\operatorname{Im}\tau$ lands in $\operatorname{Im}\tau$. Also, note that the compositions of the two maps $G \to H$ with the projection $H \to \mathbb{Z}/p\{b\}$ coincide, and hence so do the two pullbacks of the representation ν, and its Euler class β.

Now we can examine $K(n)^*BP$. In order to obtain the necessary data, we need to reconstruct the action of $\mathbb{Z}/p\{b_2\}$ on S. The actions on α_2 and κ_2 are trivial. It is also obvious (from the action on representations) that the generator b_2 acts on α_1 by

$$\alpha_1 \mapsto \alpha_1 +_F \alpha_2.$$

As for the action on κ_1, we have the following result from [3].

Lemma 6. *Let $\widetilde{S}$ be the* 0*-dimensional homogeneous summand of $\widetilde{K}(n)^*BG/\operatorname{Im}\tau$. Then in $\widetilde{S}\otimes\mathbb{Q}$, b_2 has the following action*:

$$\frac{\kappa_1}{\beta^{p-1}} \mapsto \frac{\kappa_1}{\beta^{p-1}} +_F \frac{\kappa_2}{\beta^{p-1}}.$$

3. An explicit computation of the action of b_2

Before proceeding with our calculations, we shall first need a simple combinatorial identity. Recall that for $0 < r < p^n$, $p|\binom{p^n}{r}$.

Lemma 7. *For odd prime p, $n > 0$, and $0 < r < p^n$*

$$\frac{1}{p}\binom{p^n}{r} \equiv \begin{cases} 0 \pmod p & \text{if } p^{n-1}\nmid r, \\ (-1)^{j-1}j^{-1} \pmod p & \text{for } r = jp^{n-1},\ 1 \leqslant j \leqslant p-1. \end{cases} \tag{5}$$

Proof. We have

$$\binom{p^n}{r} = \frac{\prod_{i=0}^{r-1}(p^n - i)}{\prod_{i=1}^{r} i}.$$

Hence, the left hand side of (5) may be written as

$$\frac{1}{p}\binom{p^n}{r} = \frac{p^{n-1}}{r} \times \prod_{i=1}^{r-1}\frac{p^n - i}{i}.$$

Each i may be written in the form sp^m such that $p\nmid s$. Then

$$\frac{p^n - i}{i} = \frac{p^{n-m} - s}{s} \equiv \frac{-s}{s} \equiv -1 \pmod p.$$

Calculating modulo p, we have

$$\frac{1}{p}\binom{p^n}{r} \equiv (-1)^{r-1}\frac{p^{n-1}}{r} \pmod p. \tag{6}$$

If $p^{n-1}\nmid r$, then (6) is clearly 0, and the result in (5) clearly holds; it is also easily seen that the case for $p^{n-1}|r$ also follows from (6). □

The action of b_2 can now be expressed more explicitly. Recall that b_2 acts on α_1 by $\alpha_1 \mapsto \alpha_1 +_F \alpha_2$. We can show:

Lemma 8. *In S,* (*see* (4)) *the action of b_2 on α_1 is*:

$$\alpha_1 \mapsto \alpha_1 + \alpha_2 + \beta^{p^n-p}\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\alpha_1^j\alpha_2^{p-j}.$$

Proof. Because the total degrees of the monomials in the formal group law are congruent to 1 modulo $p^n - 1$, we observe that

$$\alpha_1 +_F \alpha_2 = \alpha_1 + \alpha_2 + \frac{1}{p}\big[(\alpha_1+\alpha_2)^{p^n} - \alpha_1^{p^n} - \alpha_2^{p^n}\big] + (\text{terms with total degree of } \alpha_1 \text{ and } \alpha_2) \geqslant \big(2(p^n-1)+1\big).$$

From (4), we have the relation $\alpha_i^p = \beta^{p-1}\alpha_i$. It can be shown by induction that $\alpha_i^{p^m} = \beta^{p^m-1}\alpha_i$. In the higher terms, either α_1 or α_2 is raised to a power at least p^n. But by (4), $\beta^{p^n-1} = 0$, so $\alpha_i^{p^n} = \beta^{p^n-1}\alpha_i = 0$. Hence we may ignore the higher terms. We thus have:

$$\alpha_1 +_F \alpha_2 = \alpha_1 + \alpha_2 + \sum_{i=1}^{p^n-1} \frac{1}{p}\binom{p^n}{i}\alpha_1^i \alpha_2^{p^n-i}.$$

Using (4) and Lemma 7, we observe

$$\begin{aligned}
\alpha_1 \mapsto \alpha_1 + \alpha_2 + \sum_{j=1}^{p-1}\frac{1}{p}\binom{p^n}{jp^{n-1}}\alpha_1^{jp^{n-1}}\alpha_2^{p^n-jp^{n-1}} \\
= \alpha_1 + \alpha_2 + \sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\big(\alpha_1^{p^{n-1}}\big)^j\big(\alpha_2^{p^{n-1}}\big)^{p-j} \\
= \alpha_1 + \alpha_2 + \sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\big(\beta^{p^{n-1}-1}\alpha_1\big)^j\big(\beta^{p^{n-1}-1}\alpha_2\big)^{p-j}
\end{aligned}$$

and the result follows. □

We shall now construct the corresponding explicit description of the action of b_2 on κ_1.

Lemma 9. *In S, the action of b_2 on κ_1 is*:

$$\kappa_1 \mapsto \kappa_1 + \kappa_2 + \beta^{p-1}\sum_{j=1}^{p-1}(-1)^j j^{-1}\kappa_1^{jp^{n-2}}\kappa_2^{(p-j)p^{n-2}} \pmod{\beta^p}.$$

Proof. By Lemma 6, we have the action

$$\frac{\kappa_1}{\beta^{p-1}} \mapsto \frac{\kappa_1}{\beta^{p-1}} +_F \frac{\kappa_2}{\beta^{p-1}}$$

in $\widetilde{S}\otimes\mathbb{Q}$. Similarly as in [3], there is no torsion in $\widetilde{S}$. When multiplied by β^{p-1}, the formula of Lemma 6 involves elements of $\widetilde{S}$ only. Using similar reasoning to that used for Lemma 8, we can write the formula of Lemma 6 as:

$$\frac{\kappa_1}{\beta^{p-1}} \mapsto \frac{\kappa_1}{\beta^{p-1}} + \frac{\kappa_2}{\beta^{p-1}} + \sum_{i=1}^{p^n-1}\frac{1}{p}\binom{p^n}{i}\left(\frac{\kappa_1}{\beta^{p-1}}\right)^i\left(\frac{\kappa_2}{\beta^{p-1}}\right)^{p^n-i} + \left(\text{terms with total degree of } \tfrac{\kappa_1}{\beta^{p-1}} \text{ and } \frac{\kappa_2}{\beta^{p-1}} \geqslant 2(p^n-1)+1\right).$$

This can be simplified modulo p using n-characters. In $\widetilde{S}$, we have $\beta^{p^n-1} = p$ and $\kappa_i^{p^n} = p^p \kappa_i$ (see [3]). These facts imply that the terms with total degree $\geqslant 2(p^n - 1) + 1$ in the above expression will vanish modulo p, β^p.

In fact,

$$\left(\frac{\kappa_i}{\beta^{p-1}}\right)^{p^n} = \frac{p^p}{p^{p-1}} \frac{\kappa_i}{\beta^{p-1}} = p \frac{\kappa_i}{\beta^{p-1}}. \tag{7}$$

Moreover, for some $y \in \widetilde{S}$,

$$\begin{aligned}\left(\frac{\kappa_i}{\beta^{p-1}}\right)^{p^{n-1}} &= \frac{-\sum_{i=0}^{n-2} \beta^{p^n - p^{i+1}} \kappa_i^{p^i} + py}{\beta^{p^n - p^{n-1}}} \\ &= -\sum_{i=0}^{n-2} \beta^{p^{n-1} - p^{i+1}} \kappa_i^{p^i} + \beta^{p^{n-1}-1} y\end{aligned}$$

(by (4)), so

$$\left(\frac{\kappa_i}{\beta^{p-1}}\right)^{p^{n-1}} \in \widetilde{S}. \tag{8}$$

Now the terms of

$$x_1 +_F x_2$$

of total degree $\geqslant 2(p^n - 1) + 1$ in

$$x_i = \frac{\kappa_i}{\beta^{p-1}}$$

contain at least one factor (7), zero or more factors (8), and

$$\left(\frac{\kappa_1}{\beta^{p-1}}\right)^s \left(\frac{\kappa_2}{\beta^{p-1}}\right)^t \tag{9}$$

with $s, t < p^{n-1}$. Now if the degree in x_i of our term is exactly $2(p^n - 1) + 1$, then

$$s + t \equiv -1 \bmod p^{n-1},$$

and hence

$$s + t = p^{n-1} - 1 \tag{10}$$

(as $s + t = 2p^{n-1} - 1$ implies $s \geqslant p^{n-1}$ or $t \geqslant p^{n-1}$). But the denominator of (9) with (10) is

$$\beta^{(p-1)(p^{n-1}-1)} = \beta^{p^n - p^{n-1} - p + 1}.$$

Now since the numerator contains a multiple of p (note that we have a factor (7) and are also multiplying the entire formula of Lemma 6 by β^{p-1}), we are left with a numerator containing a multiple of

$$\beta^{p^n - 1 - (p^n - p^{n-1} - p + 1)} = \beta^{p^{n-1} + p - 2},$$

which vanishes modulo β^p (as $p^{n-1} + p - 2 \geqslant p$).

If our term has degree $\geqslant 3(p^n-1)+1$, we know that at least two factors of the form (7) occur, so the numerator contains a multiple of

$$\frac{p^2}{\beta^{p-1}} = \beta^{2p^n-p-1}.$$

Then

$$s,t \leqslant p^{n-1}-1$$

implies that the denominator contains at most the power

$$\beta^{2(p-1)(p^{n-1}-1)} = \beta^{2p^n-2p^{n-1}-2p+2}.$$

Similarly as above, we are left with a factor of at least the power

$$\beta^{(2p^n-p-1)-(2p^n-2p^{n-1}-2p+2)} = \beta^{2p^{n-1}+p-3}$$

in the numerator, which is divisible by β^p, as $2p^{n-1}+p-3 \geqslant p$.

Now Lemma 7 allows us to simplify the terms with total degree p^n; most of these terms vanish. Since b_2 respects β, we can multiply through by β^{p-1}. As a result, we observe that the action of b_2 in S is,

$$\kappa_1 \mapsto \kappa_1 + \kappa_2 + \beta^{p-1}\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\left(\frac{\kappa_1}{\beta^{p-1}}\right)^{jp^{n-1}}\left(\frac{\kappa_1}{\beta^{p-1}}\right)^{(p-j)p^{n-1}}.$$

In S, we have the following identity from (4),

$$\kappa_i^{p^{n-1}} = -\sum_{i=0}^{n-2}\beta^{p^n-p^{i+1}}\kappa_i^{p^i}.$$

Thus, we observe that the action of b_2 must be

$$\begin{aligned}
\kappa_1 \mapsto{}& \kappa_1+\kappa_2 \\
&+\beta^{p-1}\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\left(\frac{-\sum_{i=0}^{n-2}\beta^{p^n-p^{i+1}}\kappa_1^{p^i}}{\beta^{p^n-p^{n-1}}}\right)^{j}\left(\frac{-\sum_{i=0}^{n-2}\beta^{p^n-p^{i+1}}\kappa_2^{p^i}}{\beta^{p^n-p^{n-1}}}\right)^{p-j} \\
={}& \kappa_1+\kappa_2+\beta^{p-1}\sum_{j=1}^{p-1}(-1)^{j}j^{-1}\left(\sum_{i=0}^{n-2}\beta^{p^{n-1}-p^{i+1}}\kappa_1^{p^i}\right)^{j}\left(\sum_{i=0}^{n-2}\beta^{p^{n-1}-p^{i+1}}\kappa_2^{p^i}\right)^{p-j}.
\end{aligned}$$

In the summation, nearly all the terms will have some factor of β. Thus when computing modulo β^p, only those terms involving $\kappa_i^{p^{n-2}}$ need to be considered, and the result follows. □

4. A non-zero element of $K(n)^{\mathrm{odd}}BP$

With the action of b_2 explicitly defined, we observe that in S,

$$(1-b_2)(\alpha_2\kappa_1-\alpha_1\kappa_2) = \beta^{p-1}\alpha_2\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\kappa_1^{jp^{n-2}}\kappa_2^{(p-j)p^{n-2}} \bmod (\beta^p).$$

It follows that in $\widetilde{S}$,

$$(1-b_2)\big(\beta(\alpha_2\kappa_1-\alpha_1\kappa_2)\big) = \beta^p\alpha_2\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\kappa_1^{jp^{n-2}}\kappa_2^{(p-j)p^{n-2}} \bmod (\beta^{p+1}, p\beta), \tag{11}$$

and since $p\beta = \beta^{p^n}$, we can conclude that $(1-b_2)(\alpha_2\kappa_1-\alpha_1\kappa_2)$ differs from the summation on the right hand side of (11) by some sum B of multiples of β^{p+1} in $\widetilde{S}$. We thus have

$$(1-b_2)\big(\beta(\alpha_2\kappa_1-\alpha_1\kappa_2)\big) - \left(\beta^p\alpha_2\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\kappa_1^{jp^{n-2}}\kappa_2^{(p-j)p^{n-2}}\right) + B \in \operatorname{Im}\tau \subset \widetilde{K}(n)^*BG. \tag{12}$$

Now, $1-b_2$ respects the β-filtration of $\widetilde{K}(n)^*BG$. Hence, $(1-b_2)(\beta(\alpha_2\kappa_1-\alpha_1\kappa_2))$ is in filtration $\geqslant 1$, and it follows that the left hand side of (12) is also in β-filtration $\geqslant 1$.

We now claim that an element of $\operatorname{Im}\tau$ which is in β-filtration $\geqslant 1$ is 0. Recall that the β-filtration of $\widetilde{K}(n)^*BG$ is the one associated to the Hochschild–Serre spectral sequence of the fibration

$$B\mathbb{Z}/p\{a_{11}, a_{12}, a_{21}, a_{22}\} \to BG \to B\mathbb{Z}/p\{b_1\}.$$

The E_2-term of this spectral sequence is

$$H^*\big(B\mathbb{Z}/p\{b_1\}, \widetilde{K}(n)^*\big(B\mathbb{Z}/p\{a_{ij}\}\big)\big).$$

In this term, an element $\tau(q)$ is weakly represented in filtration 0 by the element $(1+b_1+\cdots+b_1^{p-1})q$. Since, in our case, $\tau(q)$ is in filtration >0, we have

$$(1+b_1+\cdots+b_1^{p-1})q = 0.$$

Since $H^1(\mathbb{Z}/p\{b_1\}, \widetilde{K}(n)^*B\mathbb{Z}/p\{a_{ij}\}) = 0$, for any $q \in \widetilde{K}(n)^*B\mathbb{Z}\{a_{ij}\}$, $(1+b_1+\cdots+b_1^{p-1})q = 0$ implies $q \in \operatorname{Im}(1-b_1)$, which implies $\tau(q) = 0$. Thus, an element in $\operatorname{Im}(\tau)$ represented in filtration $\geqslant 1$ in $\widetilde{K}(n)^*BG$ is zero, as claimed (see also [3]).

We conclude that

$$(1-b_2)\big(\beta(\alpha_2\kappa_1-\alpha_1\kappa_2)\big) - \left(\beta^p\alpha_2\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\kappa_1^{jp^{n-2}}\kappa_2^{(p-j)p^{n-2}}\right) + B = 0 \in \widetilde{K}(n)^*BG. \tag{13}$$

Now, select ζ such that

$$\beta^{p-1}\zeta = \left(\beta^p\alpha_2\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\kappa_1^{jp^{n-2}}\kappa_2^{(p-j)p^{n-2}}\right) - B.$$

We observe that ζ is in the kernel of $\sum_{i=0}^{p-1} b_2^i$: since ζ is a multiple of β, we have

$$\begin{aligned} p\left(\sum_{i=0}^{p-1} b_2^i\right)\zeta &= \beta^{p^n-1}\left(\sum_{i=0}^{p-1} b_2^i\right)\zeta \\ &= \beta^{p^n-p}\left(\sum_{i=0}^{p-1} b_2^i\right)\left(\left(\beta^p\alpha_2\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\kappa_1^{jp^{n-2}}\kappa_2^{(p-j)p^{n-2}}\right)-B\right) \\ &= 0 \in \widetilde{K}(n)^*BG. \end{aligned}$$

But $\widetilde{K}(n)^*BG$ is torsion free; we hence observe that

$$\zeta \in H^1\big(\mathbb{Z}/p\{b_2\}, \widetilde{K}(n)^*BG\big).$$

Now, we have

$$\zeta = \beta\alpha_2\sum_{j=1}^{p-1}(-1)^{j-1}j^{-1}\kappa_1^{jp^{n-2}}\kappa_2^{(p-j)p^{n-2}} - (\text{terms with higher powers of } \beta).$$

To finish the proof, we only need to establish the following result.

Theorem 10. $0 \neq \zeta \in H^1(\mathbb{Z}/p\{b_2\}, \widetilde{K}(n)^*BG)$.

Proof. We only need to show that the statement is true with $\widetilde{K}(n)^*BG$ replaced with S. Consider the decreasing filtration of S by powers of the ideal (β). In the associated graded module E_0S, ζ is represented by an element in filtration degree 1. By (4) and Lemmas 8 and 9, the structure of S is the same as the structure of E_0S modulo error terms with powers of β which are $\geqslant 2$. If an element in filtration degree 1 is in $\mathrm{Im}(1-b_2) \subset S$, then it is also in $\mathrm{Im}(1-b_2) \subset E_0S$. The reason is that in the spectral sequence

$$H^*(\mathbb{Z}/p, E_0S) \Rightarrow H^*(\mathbb{Z}/p, S)$$

corresponding to the filtration of S by powers of β, we have

$$d_1 = 0$$

(because the error terms to the structure formulas in S are in filtration degrees $\geqslant 2$). We hence only need to show that $0 \neq \zeta \in H^1(\mathbb{Z}/p\{b_2\}, E_0S)$.

First, consider the $\mathbb{Z}/p[\mathbb{Z}/p]$-module

$$K(a,b) = \mathbb{Z}/p[a,b]/(a^p, b^p),$$

where the generator b_2 acts by

$$a \mapsto a+b, \quad b \mapsto b. \tag{14}$$

Put $T = 1-b_2$. Recall that as usual, $T^p = 0$. Similarly as in [3], we have

Lemma 11.

$$K(a,b) \cong \left(\bigoplus_{k=1}^{p-1}\big(\mathbb{Z}/p[T]/T^k\big)\otimes\{a^{k-1}, a^{p-1}b^{p-k}\}\right)\oplus\big(\mathbb{Z}/p[T]/T^p\otimes\{a^{p-1}\}\big).$$

Put, more generally,

$$K_m(a,b) = \mathbb{Z}/p[a,b]/(a^{p^m}, b^{p^m}).$$

Thus, $K_1(a,b) = K(a,b)$. It is immediate that

$$E_0S \cong K(\alpha_1,\alpha_2) \otimes K_{n-1}(\kappa_1,\kappa_2) \otimes \mathbb{Z}/p[\beta]/\beta^{p-1}$$

(where $b_2\beta = \beta$).

Now for $n = 2$, the structure of E_0S is entirely determined by Lemma 11. We have

$$\zeta = \beta\alpha_2 \sum_{j=1}^{p-1} (-1)^{j-1} j^{-1} \kappa_1^j \kappa_2^{p-j} \in K(\alpha_1,\alpha_2) \otimes K(\kappa_1,\kappa_2)\{\beta\}.$$

Note that, as $\mathbb{Z}/p[\mathbb{Z}/p]$-modules,

$$K(a,b) = \bigoplus_{t=0}^{2p-2} K^t(a,b),$$

where $K^t(a,b)$ is the submodule generated by monomials of total degree t. Now we see that, for $n = 2$, neglecting the generator β (which plays no role),

$$\begin{aligned}\zeta &\in K(\alpha_1,\alpha_2) \otimes K^p(\kappa_1,\kappa_2)\\ &= \mathbb{Z}/p[T]/T^2\{\alpha_1\} \otimes \mathbb{Z}/p[T]/T^{p-1}\{\kappa_1^{p-1}\kappa_2\} =: Q.\end{aligned}$$

Now note that

$$T(\kappa_1^{p-1}\kappa_2) = \kappa_1^{p-1}\kappa_2 - (\kappa_1+\kappa_2)^{p-1}\kappa_2$$

contains no monomial with a factor of κ_1^{p-1}. Next, we have a submodule

$$Q \supset Q_0 = \mathbb{Z}/p\{\alpha_2\} \otimes K^p(\kappa_1,\kappa_2).$$

We have

$$\begin{aligned}&T(\alpha_1 T^i(\kappa_1^{p-1}\kappa_2))\\ &\quad = \alpha_1 T^i(\kappa_1^{p-1}\kappa_2) - (\alpha_1+\alpha_2) b_2 T^i(\kappa_1^{p-1}\kappa_2)\\ &\quad = \alpha_1 T^i(\kappa_1^{p-1}\kappa_2) - (\alpha_1+\alpha_2)(1-T) T^i(\kappa_1^{p-1}\kappa_2)\\ &\quad = \alpha_1 T^{i+1}(\kappa_1^{p-1}\kappa_2) + x,\end{aligned}$$

where $x \in Q_0$ and, moreover, x contains no monomials with a factor of κ_1^{p-1} if $i > 0$. Consequently, polynomials in $\mathrm{Im}(T) \cap Q_0$ contain no monomials with a factor of κ_1^{p-1}. Therefore, $\zeta \in Q_0$ does not belong to $\mathrm{Im}(T)$. This concludes the argument for $n = 2$.

Next, we observe that the above argument also proves that

$$\alpha_2 \otimes \xi \notin \mathrm{Im}(T)$$

for every $\xi \in K^p(\kappa_1,\kappa_2)$ such that

$$\xi \notin \mathrm{Im}(T).$$

Thus, for every $\mathbb{Z}/p[\mathbb{Z}/p]$-module M and every element $\xi \in M$ which generates a direct $\mathbb{Z}/p[\mathbb{Z}/p]$-summand isomorphic to $\mathbb{Z}/p[T]/T^{p-1}$, we have

$$\alpha_2 \otimes \xi \notin \mathrm{Im}(T) \quad \text{in } K^1(\alpha_1, \alpha_2) \otimes M.$$

Now we turn to the case $n > 2$. In this case, Lemma 11 is replaced by the following slightly more complicated

Lemma 12. *Let*

$$\omega = a^p - ab^{p-1} \in K_m(a, b).$$

Then, of course, $T\omega = 0$. *Moreover,*

$$K_m(a,b) = \left(\mathbb{Z}/p[\omega]/\big(\omega^{p^{m-1}}\big) \otimes \left(\bigoplus_{k=1}^{p-1} \big(\mathbb{Z}/p[T]/T^k\big) \otimes \big\{a^{k-1}, a^{p-1}b^{p^m-k}\big\}\right)\right) \oplus \left(\bigoplus_{k=0}^{p^m-p} \big(\mathbb{Z}/p[T]/T^p\big) \otimes \{a^{p-1}b^k\}\right).$$

Similarly as above, $K_m(a, b)$ is graded by total power of a, b. Let

$$K_m^k(a, b)$$

be the sub-$\mathbb{Z}/p[\mathbb{Z}/p]$-module spanned by monomials of total degree k. Note that, by Lemma 12,

$$K_m^{p^m} \cong F \oplus \mathbb{Z}/p[T]/T^{p-1}, \tag{15}$$

where F is a free $\mathbb{Z}/p[\mathbb{Z}/p]$-module. Now consider

$$\zeta = \alpha_2 \otimes \xi \in K^1(\alpha_1, \alpha_2) \otimes K_{n-1}^{p^{n-1}}(\kappa_1, \kappa_2),$$

where

$$\xi = \sum_{j=1}^{p-1} (-1)^{j-1} j^{-1} \kappa_1^{jp^{n-2}} \kappa_2^{(p-j)p^{n-2}}.$$

By the above arguments, we will be done if we can show that ξ spans a direct $\mathbb{Z}/p[\mathbb{Z}/p]$-summand isomorphic to $\mathbb{Z}/p[T]/T^{p-1}$ in $K_{n-1}^{p^{n-1}}(\kappa_1, \kappa_2)$.

From now on, we shall work in $K_m^{p^m}(a, b)$. Note that, because of the homogeneous degree, we can substantially simplify notation by setting $b = 1$, without losing information.

Lemma 13. *In* $K_2^{p^2}(a, 1)$,

$$a^{jp} \in \mathrm{Im}(T) \quad \textit{for } j < p - 1. \tag{16}$$

Further,

$$a^{p(p-1)} - \varepsilon a^{p-1} \in \mathrm{Im}(T) \quad \textit{for some } 1 \leqslant \varepsilon \leqslant p - 1. \tag{17}$$

Note that then, we can deduce our result as follows: identify κ_1 with a and κ_2 with 1. Then for every $0 \leqslant j \leqslant n-3$, we have a map of $\mathbb{Z}/p[\mathbb{Z}/p]$-modules

$$\psi_j = (?)^{p^j} : K_2^{p^2}(a,1) \to K_{n-1}^{p^{n-1}}(a,1).$$

For $j = n-3$, (16) of Lemma 13 then implies that ξ is $\varepsilon_{n-1} a^{p^{n-2}(p-1)}$ modulo $\mathrm{Im}(T)$. Similarly now, using ψ_j, (17) of Lemma 13 implies that $a^{p^{j+1}(p-1)} = \varepsilon_j a^{p^j(p-1)}$ modulo $\mathrm{Im}(T)$ $(1 \leqslant \varepsilon_j \leqslant p-1)$.

Finally, a^{p-1} spans a summand isomorphic to $\mathbb{Z}/p[T]/T^{p-1}$ by Lemma 12. Hence, so does ξ by (15).

Proof of Lemma 13. We will first show by induction that

$$a^{i+jp} \in \mathrm{Im}(T) \quad \text{for } 0 \leqslant i,\ j < p,\ i+j < p-1. \tag{18}$$

In effect, we have

$$T\left(a^{(i+1)+jp}\right) = a^{i+1+jp} - (a+1)^{i+1}(a^p+1)^j.$$

The right hand side is a sum of a^{i+jp} and terms $a^{i'+j'p}$ where $j' < j$ and $i' \leqslant i+1$ or $j' = j$ and $i' < i$. Hence, induction.

Next, we see in the same way that if

$$i + j = p - 1,$$

then

$$T\left(a^{(i+1)+jp}\right) = \varepsilon_1 a^{i+jp} + \varepsilon_2 a^{(i+1)+(j-1)p}$$

modulo terms of the form (18) $(0 < \varepsilon_i < p)$. Thus,

$$\varepsilon_1 a^{i+jp} + \varepsilon_2 a^{(i+1)+(j-1)p} \in \mathrm{Im}(T).$$

Adding these formulas, we obtain (17). □

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