MATHEMATICAL PROOF REVIEW

# The Kervaire invariant problem at the prime three

A review of the September 24, 2026 preprint

Codex, AI-assisted proof audit October 29, 2026

**Assessment.** In my judgment, the proof of Theorem 1.1 appears correct after making explicit a divisibility argument in Lemma 4.4. I checked both branches of the proof, reconstructed the finite calculations, and compared the theorem with published formulations of the problem. The proposed insertion is on page 7. Theorem 1.1.

The standard classes are $b_j \in \mathrm{Ext}_{\mathcal{A}_3}^{2,4\cdot 3^{j+1}}(\mathbb{F}_3,\mathbb{F}_3)$. Their sphere stem is $4\cdot 3^{j+1}-2$. Let $K$ be the set of indices surviving in the classical mod-three Adams spectral sequence and $K_3$ the set for which the detection coset contains a representative of additive order three. The conclusion under review is

$$K = K_3 = \{0, 2, 3\}.$$

| Adams class | Other indexing | Stem | Status and basis for the assessment |
|---|---|---|---|
| $b_0$ | $\theta_1$ | 10 | Classical order-three element $\beta_1$. |
| $b_1$ | $\theta_2$ | 34 | Classical non-survival, by Toda's differential. |
| $b_2$ | $\theta_3$ | 106 | Order-three survivor, independently checked on pages 19–20. |
| $b_3$ | $\theta_4$ | 322 | The preprint constructs a sphere class and proves both its detection and its exact order. |
| $b_j,\ j \geq 4$ | $\theta_{j+1}$ | $\geq 970$ | The preprint forces an outgoing differential on the auxiliary detector for every index. |

**How the two new conclusions are tested.** For the exclusion, the decisive question is whether any auxiliary class could *hit* a putative higher detector. An actual sphere image could die that way. Reconstructing the groups, transfers and differential constraints leaves four candidate lengths and excludes all four (page 11). For stem 322, the decisive question is whether there is an *actual sphere element* of order three. A corrected Moore-spectrum map supplies it (page 12). One obstruction vanishes by a two-layer filtration argument. The other is settled by the finite calculation (pages 13–15).

**Scope.** This is a review of a fixed manuscript and of the stated mathematical and computational inputs. The judgment is mine. Page 2 compares the theorem with what earlier authors formulated, proved, or proposed. Pages 17–18 identify two errors in the printed Belmont–Ray detection proof and distinguish the independent checks from the classical homotopy inputs.

Audited manuscript (PDF) Git revision `7ef5c0cc5ca4b0e072b7a050fca09ab24bc6fa3a`.

## The problem in the literature and in expert talks

### Which family and which sense of existence?

Ravenel defines the odd-primary Adams classes via the negative $p$-fold Massey power of $h_j$. Belmont and Ray give bidegree $(2, 2(p-1)p^{j+1})$. Substitution of $p = 3$ gives the manuscript's bidegree exactly. The conventional change of index is also explicit in Hill–Hopkins–Ravenel's 2011 account and Amelotte's §3.1. Their $\theta_i$ has detector $b_{i-1}$ in stem $2(p-1)p^i - 2$. Amelotte's strong condition asks for order $p$ in that stem. Thus the two sets $K, K_3$ separate two questions in the literature. Ravenel (1978), §1. Hill–Hopkins–Ravenel, eq. (1). Belmont–Ray, introduction. Amelotte, §3.1.

The older calculations also match at the delicate known endpoint. In stem 106 the surviving Adams–Novikov representative is a corrected combination $\beta_{9/9} \pm \beta_7$. The uncorrected summand denoted $\beta_{9/9}$ in Ravenel's §6.4 supports a differential. Ravenel records the corrected representative, and Amelotte records that the 3-local sphere group in stem 106 is $\mathbb{Z}/3$. This is why an exclusion above that stem must control every representative of an Adams detector. Ravenel, after Proposition 6.4.11. Amelotte, proof of Corollary 1.2.

### The Banff cutoff and the separate 322-stem

In the February 2012 Banff slides, Hill, Hopkins and Ravenel list $\theta_1, \theta_3$ as known and $\theta_2$ as absent. Slide 1.16 describes a proposed argument that would rule out $\theta_i$ for $i \geq 5$ and would leave $\theta_4$, in stem 322, open. In the preprint's indexing, these are precisely $b_j$ for $j \geq 4$ and the separate case $b_3$. The preprint reaches the proposed cutoff by finite-page differentials and answers the case left open by the slides through its Moore construction. Banff, slide 1.16.

The proposed methods must also be distinguished. The later Banff slide 1.26 discusses torsion-freeness in degree $-2$ for a proposed spectrum $\Omega$. Belmont–Ray formulate a 972-periodicity and a full degree-$-2$ vanishing conjecture specifically for $E_6^{hC_9}$. They state detection on $E_2$ and compute the coefficient page in positive filtration conditional on a coefficient-action model. Their printed detection proof requires the qualifications on page 17. The manuscript proves the model in its case and exhibits an infinite-order class in $\pi_{-2}(E_6^{hC_9})$. Thus it disagrees with the full-vanishing clause for that spectrum. Its classification uses neither that clause nor a proof of 972-periodicity. Banff, slide 1.26. Belmont–Ray, Conjecture 0.1 and Proposition 0.2. Theorem 0.6. Proposition 2.10.

Rin Ray's research entry identifies the September preprint as a claimed solution using the results and method of the joint work with Belmont. The historical comparisons here identify the problem being addressed. The assessment of the new arguments is based on the checks that follow. Ray, research entry on the prime-three problem.

### An earlier check on the subgroup differentials

Hill's 2007 Oberwolfach report on joint work with Hopkins and Ravenel gives formulas for the $C_p$ differentials assuming Hopkins's coefficient-action conjecture. In height six at three these give the pairs of lengths $(5, 9), (17, 33), (53, 105)$. All six agree with the lengths derived in the preprint. The earlier computation concerns the subgroup $C_3$ and has the stated hypothesis. The new argument still has to establish that hypothesis in this case, ascend to $C_9$, and settle stem 322. Hill, Oberwolfach Report 45/2007, pp. 2699–2700, Propositions 2–3.

## Detection and the deadline for an incoming differential

Write $G = C_9,\ H = C_3,\ Q = G/H$. The theory $E = E_6$ has coefficient ring $W(k)[[u_1, \dots, u_5]][u^{\pm 1}]$ for $k = \mathbb{F}_{3^6}$. The $G$- and $H$-Tate pages used below come from inverting a faithful representation Euler class in the homotopy fixed point spectral sequence. The proof uses a finite page at which the unit has died. This distinction matters because the localization allows incoming differentials into the image of a surviving sphere class.

### A nonzero symbol for an actual sphere image

Let $\alpha$ be detected by $b_j$ for $j \geq 2$. Comparing augmented towers bounds its ordinary $BP$-filtration above by two. The $BP$ internal degrees are divisible by four, so filtrations zero and one in this stem are absent. Thus $\alpha$ supplies a permanent two-line representative. The cyclotomic evaluation of any such representative has exactly one summand with unit intermediate residue. The outer Bockstein on the trivial line carries an element $c$ with unit residue to $3c$ in $\mathcal{O}/9$. It matters that the calculation retains this integral target.

The symbol lifts to stage two of the ordinary group-cochain tower. If its image in homotopy were zero, it would come from the boundary of the first two layers. Evenness makes every such boundary a degree-one cochain boundary, contradicting the nonzero class in $H^2$. The image in $E^{hG}$ is therefore nonzero and has finite order because the source stem is finite. Comparison with Tate cohomology is an isomorphism in positive filtration on $E_2$. On restriction to $H$ a nonzero scalar would have $d_5$-coefficient $1 - 4 \cdot 3^j = 1 \bmod 3$, with primitive target factor. A sphere image cannot support this differential. This identifies the one-dimensional restriction kernel tested on page 11. Filtration comparison. Two-layer lemma and Theorem 2.7. Detector coordinates. Belmont–Ray, Proposition 0.2.

### The nonzero group in degree $-2$

On the fraction field of $R = W(k)[[u_1, \dots, u_5]]$, the trace on $Ru^n$ is a weighted sum of distinct field automorphisms once one divides out the kernel of the action on $R$. The kernel acts trivially on $u$, since $R$ has no nontrivial ninth root of unity. Its cardinality contributes a nonzero integer. Independence of distinct field automorphisms gives a nonzero trace, and multiplying by an invariant denominator makes the input integral. The homotopy transfer lifts the trace to a class in $\pi_{-2n}(E^{hG})$ whose restriction has infinite order. This proves Proposition 2.10 for all $n$, including $n = 1$. In degree-zero Tate cohomology the quotient by additive traces kills these coefficient symbols. Proposition 2.10.

### Why a finite-page bound is available

For $K' = H, G$, use the degree-52 coefficient of $x^{27}$ in the change of two $MU$ orientations. Its indexed norm has zero geometric $H$-fixed image even with the residual action, since the two orientations coincide before the remaining quotient norm is taken. Isotropy separation lifts the class to the free part, where connectivity kills its multiple by $a_\lambda^d$ for $52|K'| - 2d < 0$. Its underlying image in Morava theory is a unit, so multiplication by it is an equivalence on the cofree spectrum. Taking $d = 79, 235$, respectively, gives null powers whose filtration-$2d$ symbols are initially nonzero. The finite-tower boundary formula gives an incoming by length $2d$, and parity improves the bound to $157, 469$, respectively. After Euler inversion the unit must die within these bounds. Proposition 2.13.

## The coefficient action on the height-six deformation

Proposition 3.3 must lift a residual period to a period $x_0$ of internal degree $-2$ with $\mathrm{Tr}_H(x_0) = 0$ exactly. Its orbit gives the proposed generators and the three exact trace relations of the coefficient algebra. Write $\sigma = g^3$. At first order $g - 1$ lowers the parameter index by one and $\sigma - 1$ lowers it by three. The coefficient $a$ of the first residual nonlinear term of $\sigma$ satisfies $aa^{27} = -1$ by $(\sigma - 1)^2 = -3\sigma$ in the endomorphism algebra. Thus if $y_0$ is the symbol of $\sigma(u)/u - 1$, then $l_0 = (\sigma - 1)y_0 = -[3]$. This cancels the first-order trace obstruction.

In the graded coefficient complex each of the three tangent blocks has coordinates $y, l$ with $\sigma y = y + l$. Removing powers of its norm $y^3 - yl^2$, the remaining homogeneous part in degree $d \geq 2$ has $k$-basis $l^d, yl^{d-1}, y^2l^{d-2}$. The action on this space is one Jordan block of length three, giving a free $k[H]$-module of rank one. This leaves the exterior classes in Lemma 3.2. The only degree-two trace obstructions for $u$ have extra weights $2, 8, 10$ modulo 26, none zero. Correcting the trace through this order and taking $g$-differences gives the stated $d_1$ on the other generators.

There is a direct check of the crucial auxiliary $d_2$. Put $\lambda = \sigma(u)/u - 1$. Let $E_i$ be the elementary symmetric functions of its three conjugates. Their product after each is increased by one telescopes to one. Hence $E_1 + E_2 + E_3 = 0$, with $E_1 \in \mathfrak{m}^2$. The cochain $s = \lambda - \lambda^2/2$ has leading class $z_0$ and

$$\mathrm{Tr}_H(s) = E_1 + E_2 - \tfrac{1}{2}E_1^2 \equiv -E_3 \pmod{\mathfrak{m}^4}.$$

The degree-three symbol of $E_3$ is $N_0 = y_0^3 - y_0l_0^2$, nonzero on the auxiliary $\mathcal{E}_2$ page. This gives $d_2(z_0) = -bN_0$ up to the cyclic convention, independently of the paper's CM specialization. Proposition 3.3.

After this $d_2$, each remaining odd algebra generator has period exponent one modulo three and has no even target of that exponent. The even generators lift by norms or cyclic periodicity. The possible lifts of $eu$ of each finite accuracy form nested nonempty closed sets in a compact module, so an exact trace-zero lift $x_0$ exists. The three trace relations now hold for $x_i = g^i x_0$. The first five positive $g$-differences of $x_0$, together with 3, span the cotangent space in triangular order. Formal inversion gives the completed $G$-ring isomorphism. The $H$-cohomology algebra can also be obtained from permutation monomials before taking the three trace quotients. Each quotient adds a degree-one exterior factor by the connecting sequence. Localization and flat completion preserve this calculation. Belmont–Ray compute the cohomology conditional on the coefficient model. Theorem 3.4 and Proposition 3.5. Belmont–Ray, Theorem 0.6 and §2.

### What agreement with Belmont–Ray means here

The complete ring is thus obtained from a polynomial algebra on the cyclic orbit, with the three trace relations, inversion of its norm and completion at the residual point. The trace-zero lift and the cotangent isomorphism supply the group action on the *actual* Morava coefficients needed to use the cohomology presentation in the topological spectral sequence. Comparing only the tangent representation would leave this identification open. Theorem 3.4. Belmont–Ray, Conjecture 0.3.

## Fixed deformations and maps of the coefficient lines

The subgroup calculation needs two pieces of geometric information. The first identifies the scalar coordinate ring after restriction to the fixed deformation. The second prevents certain equivariant maps from the exterior plane to a scalar line. Both are used to rule out candidate differentials in Section 4.

In the fixed ungraded deformation write $r = \sigma x_0/x_0$. The equations on ratios imply $r^2 + r + 1 = 0$, including over Artinian test rings. Rigidity follows by applying the finite-height special-fibre 3-series to the first nonzero ideal layer of the difference of two endomorphisms. Fixing the structural root $r = \zeta_3$ gives the strict $\mathbb{Z}_3[\zeta_3]$-module deformation. In the reduced fibre $\bar{D} = k[[r_1, r_2]]$, the evaluated $q_i/q$ are $(1 + r_i)^3$. Thus the scalar ring $K$ is $\bar{D}^3$. The first strict coefficient $h_1$ and then the second, $h$, give successive regular parameters in a Lie trivialization by the uniformizer series. For $c$ centralizing $\sigma$, the cocycle identity gives $ct_1(\sigma) - t_1(\sigma) = (\sigma - 1)t_1(c)$. This verifies invariance of its cohomology class $\theta$. Theorem 3.8. Strauch, equation (1.1).

On the curve $D' = k[[\tau]]$ cut out by $h_1$, set $S_0 = k[[\tau^3]]$. The two basis evaluations of the quotient plane $W$ have orders 0 and 1. They are independent over $k((\tau^3))$, since their leading orders differ modulo 3. This is the scalar field needed in Theorem 3.12. An open subgroup of the centralizer $\mathcal{C}$ cannot act trivially on the curve. Its lifts would identify nine closed-point endomorphisms independent over $\mathbb{Q}_3(\zeta_3)$ with endomorphisms in a generic algebra of dimension four. A generic relation clears denominators and specializes. The extension of complete Laurent-series fields defined by a nonconstant invariant (after removing its constant term and taking its inverse if needed) is finite. Hence such a Laurent function has a finite stabilizer in the group of disc automorphisms, so invariant Laurent functions are constant.

For the determinant, the Hodge quotient of $M_0 = \mathcal{M}/\epsilon\mathcal{M}$ gives $\det \Omega^1_{\bar{D}/k} = (\det M_0)w^{-3}$ by the deformation theorem. The crystalline operator of underlying rank five has determinant $\epsilon$ times a unit over $\bar{D}[\epsilon]/\epsilon^2$. Indeed its underlying determinant is the square of its constant term, hence zero on the reduced base, and its closed elementary divisors are $1, 1, \epsilon$. Dividing gives a Frobenius-fixed determinant line with sign character $\chi$ of weight $2 + 8 + 3 = 13 \bmod 26$. First jets on the curve give $\det W|_{D'} = \Omega^1_{D'/k} w^{-4} = w^{-9}\chi$ using conormal $w^2$. An equivariant $W \to P_n$ would give an invariant in $w^{7+3n}\chi$. Taking eighth powers and multiplying by $h^{7+3n}$ forces the remaining factor to be constant, incompatible with the nontrivial sign. The two distinct leading orders also justify division by $\tau^{3a}$ inside $W$ when $h^{1+3a}$ occurs. Lemma 3.11 and Theorem 3.12. Chai–Oort, Theorem 2.4.

## The subgroup differentials and their possible lengths

In the manuscript's local notation $K = k[[n_1, n_2]]$ is a coefficient ring. The subgroup page is

$$E_2(H) = K[q^{\pm 1}, b^{\pm 1}] \otimes_K \Lambda_K(V), \qquad |q| = (0, -6), \quad |b| = (2, 0), \quad |V| = (1, 4).$$

Here $q = \mathrm{Nm}_H(x_0)$, $V$ has rank three, and $\theta \in V$ is the invariant strict-coordinate cocycle. Write $A = qb^{-2}$, $\Theta = b^{-1}\theta$, $F = q^2 h_1^3$, and $W = (V/K\theta) \otimes_K K/(F)$. The ring $K$ and the unit $A$ consist of cycles on the nonzero pages. In particular all scalars in the quotient $S_0 = K/(F)$ used on the next page are cycles. Lemma 3.6 on scalar cycles. The subgroup page and its generators.

At the fixed deformation the integral cyclic Bockstein for a coefficient line with trivial $H$-action gives $b$ times the value of the intermediate 1-cocycle. Reduction by $\zeta_3 - 1$ gives $v_1 = 0$ and $v_2 = -t_1(\sigma)^4$. The three $v_1$-Bocksteins for $v_2, v_2^3, v_2^4$ give $t_1^3, t_1^9, \pm t_1^{15}$, respectively. Thus the three Greek-letter images are $bJ, bJ^3, \pm bJ^5$, where $J = h_1^3$. The classical $d_5$ on $\beta_{3/3}$ forces $d_5 b = \theta b^3$. The $d_9$ on $\alpha_1\beta_4$ forces $A^2 d_9(\Theta) = cF$ for $F = q^2 J$ and $c \neq 0$. The factor cancelled here is regular on the scalar $E_6$ summand. Lemmas 4.1–4.2. Belmont–Shimomura, Lemma 2.5.

Higher-order errors in $d_5$ on the complementary plane matter. With the paper's normalization, all coefficients of the error matrix $N$ are in $\mathfrak{n}^2$. In exponent sector $-\mu$, multiplication by $\Theta$ induces the natural $\ker N \to \operatorname{coker} N$. For $\alpha \in \operatorname{coker} N$, applying the derivation $A^2 d_9$ to $\Theta\alpha = 0$ shows $cF\alpha$ is in its image. This gives two columns in $\ker N$ congruent to the columns of $cFI$ modulo $\mathfrak{n}^2$. Their determinant has nonzero initial term $c^2F^2$, whence $N = 0$. There is no need to lift $d_9$ to representatives before $d_5$-homology. Here $\mu$ is the residual coefficient of $b^{-3}d_5$ on the normalized complementary plane. The case $\mu = 0$ is ruled out by equivariance modulo $\mathfrak{n}^2$ using the residual three-step Jordan block. For $\mu \notin \mathbb{F}_3$, the complementary complexes are invertible and would leave only scalars after $d_9$, in conflict with the Euler bound. Lemma 4.3.

After $E_{10}$ the allowed lengths up to 157 from degrees alone are

| $\mu$ | exterior degree increases | exterior degree decreases |
|---|---|---|
| 1 | $17, 53, 89, 125$ | $33, 69, 105, 141$ |
| $-1$ | $29, 65, 101, 137$ | $21, 57, 93, 129.$ |

The centralizer calculation excludes an initial decreasing map and forces the first increase to be 17 or 65, respectively. In the $\mu = -1$ branch a possible $d_{93}$ to scalars vanishes after passing to the fraction field, and its target is torsion-free. The remaining potential incoming on the unit is $d_{129}$. At its source, $n = 22$, the two invariant coefficient orders are $6, 10 \pmod{13}$. An invariant representative is divisible by $Z^6$.

Here $Z = q^8 h^3$ generates the completed scalar ring $S_0 = k[[Z]]$ on the curve, in the chosen trivialization. The divisibility of an invariant expression is visible already on the unquotiented plane. Page 7 checks that it gives divisibility by *cycles* on the later pages. Lemma 4.4.

## The insertion in Lemma 4.4

The issue is localized to the $\mu = -1$ case. The source under consideration would have to hit the unit at length 129. The weight congruences show that an invariant representative in the ambient plane is divisible by $Z^6$. A subquotient can contain a cycle $Z^6 e$ even if $e$ is not a cycle. Thus before the printed proof works with $e$ on $E_{66}(H)$, one must establish that $e$ actually defines a class there. The weight calculation and the passage to the quotient.

### Proposed insertion after the two weight congruences

Lift the invariant class to a $d_{65}$-cycle $z$ on $E_{65}(H)$ and average this lift over $T$. This is possible because $|T| = 26$ is invertible. The weight congruences on its two coefficients give $z = Z^6 e$ in the plane on $E_{65}(H)$. In bidegree $(s, t) = (-129, -128)$, the $d_{65}$-target is $(-64, -64)$, of exterior degree two and scalar index 12. It is free of rank one over $S_0 = k[[Z]]$ because $E_{65} = E_{10}$ in this branch. Since $Z$ is a cycle, $0 = d_{65} z = Z^6 d_{65} e$ implies $d_{65} e = 0$. Therefore $[z] = Z^6 [e]$ already on $E_{66}(H)$.

### Why this suffices at the next two lengths

The outgoing $d_{93}$ on the plane is zero and incoming maps only form a quotient. Hence $[e]$ still gives a representative on $E_{94}$. The exterior-degree-two target for $d_{101}$ is now a submodule of its $E_{66}$ predecessor. Its torsion is annihilated by $Z$ because the earlier $d_{65}$ matrices are $Z$ times constant matrices. If $Z^6[e]$ survives $d_{101}$, it follows that $Z\, d_{101}[e] = 0$. Thus $Z[e]$ is a cycle and the putative $d_{129}$ source still has a factor $Z^5$. Its differential cannot be the scalar unit, which is indivisible by $Z$. This contradiction rules out $\mu = -1$. The insertion uses freeness on the earlier page and bounded torsion on the later one, exactly where each is available. Remainder of Lemma 4.4.

### Finishing the $C_3$ calculation

For $\mu = 1$ the two $d_{17}$ coefficients are $p$ and $p - 1$ up to normalization, where $p$ is the exponent of $b^3$ in the sector. The competing sector supports either exclude every possible unit incoming or leave only scalars. Contraction by $d_{33}$ kills the parameter $Z$. Afterward only $53, 105$ can occur. The sole unit source $A^{18} C^{-2} U$ survives $d_{53}$ for either value of its coefficient. When the $d_{53}$ coefficient is nonzero, the possible boundary coefficient is $-3 = 0$. The bound forces $d_{105}$ to hit 1. Section 5 must still establish $d_{53} \neq 0$. Lemma 4.5, Proposition 4.6 and Theorem 4.7.

## Passing to the group of order nine

The scalar group at $G$ is an extension of $P_n^Q$ by one $k$ at $3 \mid n$, with the same description in exterior degree three. One must keep this integral extension. In the polynomial model, the unequal block multidegrees are induced. In a repeated block of degree $d$, the nonprojective $H$-lattice is respectively trivial, cyclotomic, or zero for $d = 0, 1, 2 \bmod 3$. This follows from the trace-quotient exact sequence and the decomposition of torsion-free $W(k)[C_3]$-lattices. Tensoring the three quotient sequences gives a threefold stable suspension for tensor induction of the cyclotomic lattice. Its exceptional Tate group at $G$ is $W(k)/9$, in odd degree, and the repeated trivial block gives the same in even degree. Restriction reduces modulo 3 and transfer takes a generator to its nonzero three-multiple. The volume generator identifies the two descriptions. Finiteness over the invariant ring and flatness of its completion preserve the cyclic complexes. As a separate check the integral matrix computation recovers these patterns for all cyclic orbits of block degrees 0 through 3 and the additional orbit $(4, 4, 4)$. Lemma 5.3.

Write $D = g - 1$, $N = 1 + g + g^2$, and $r_0 = w_0 - j \bmod 3$. The $d_5$ complexes have successive maps $r_0 \operatorname{res}$, $\pm(r_0 + 1)D$, $(r_0 + 2) \operatorname{tr}$. The last map is determined by transfer, including its invisible value in the restriction kernel. For $d_9$, put $S_n = P_n/JP_{n+2}$, $W_n = S_n \otimes (k[Q]/N)$. In exterior degree two, write $J = NL_1$. The map from a volume class restricting to $f$ is $[L_1 f]$. In the case where restriction fails to determine it, every source is a transfer and $[L_1 N f'] = [Jf']$. In index $n = 1 \bmod 3$ the connecting map on the new norm kernel sends $[L_1 f]$ to $[f]$ because $N(L_1 f) = Jf$. This kills exactly that kernel. In degree one, the reduction of the preceding plane's $\widehat{H}^{-1}$ kills the norm kernel precisely at $3 \mid n$. The resulting transfer modules are

$$\mathcal{T}_n = \begin{cases} NS_n & n = 1 \bmod 3, \\ (S_n)_Q & \text{otherwise}, \end{cases} \qquad \mathcal{W}_n = \begin{cases} NW_n & n = 0 \bmod 3, \\ (W_n)_Q & \text{otherwise}. \end{cases}$$

There is a further scalar restriction lift at $3 \mid n$, together with the exceptional scalar and volume lines stated in Proposition 5.5. Lemma 5.4 and Proposition 5.5.

At $d_{13}$ consider the scalar lift in weight $w_0 = 3p$, $3 \mid n$. Its possible logarithmic coefficient is $p\varepsilon + (i - 2n/3)\varepsilon'$. A zero with $p \neq 0 \bmod 3$ would leave its restriction to support a primitive $h$ at $d_{17}$. But the $G$-target restricts into a norm, and that norm vanishes in the residual two-step Jordan block. Semilinear Frobenius makes the ratio of the two coefficients, when defined, lie in $\mathbb{F}_3$. Varying $i$ forces $\varepsilon \neq 0, \varepsilon' = 0$. The extra scalar can still survive in a family $\rho = 2, 5, 8 \bmod 9$.

At $d_{17}$ the transfer chain is $\mathcal{T}_n \to \mathcal{W}_{n-2} \to \mathcal{T}_{n-4}$, scaled by $p, p + 1$. Its unscaled version is exact. The surviving scalar lift can support a further map to one norm-kernel line with coefficient $V(t, u) = t\nu + u\nu'$. It is retained as an unknown. At $21, 25, 29, 33$ one must also retain any differential into a restriction kernel. Reducing every residue case gives the 15 tridegrees of Table 1 for each $\rho$, including both possible layers of $\mathsf{A}$. Page 9 gives a calculation of the underlying modules independent of the paper's valuation comparison. Lemmas 5.6–5.8.

## An exact calculation over the complete curve ring

An exact arithmetic model tests the $d_{17}$ and $d_{33}$ quotients. It starts with the preceding Tate dimension $\dim_k \widehat{H}^0(Q; S_0) = 1$ and the primitive vector $h$ already forced by Section 4. The calculation retains the complete valuation ring because finite truncations of a power-series module can introduce spurious kernels.

Let $E = \operatorname{Frac} S_0$, $F_* = E^Q$ and normalize the valuation of $E$ to have value group $\mathbb{Z}$. The action on the disc is nontrivial, since a trivial action would give a trivial action on its scalar lines (there are no order-three units of the characteristic-three domain). That would contradict the finite Tate dimensions. The extension has degree three and ramification index three. Indeed the orbit product of a uniformizer has valuation three and its power-series ring gives a rank-three subring with the same residue field. There is an $X \in E$ with $gX = X + 1$. Choose an element of field trace one and apply $g - 1$ to it, using $(g-1)^2 = N$. By shifting $X$ by elements of $F_*$, minimize its pole order $m$. It is positive, since $g$ acts trivially on the residue field, and $3 \nmid m$, since an invariant uniformizer has valuation three and an invariant leading term could otherwise be subtracted.

In the basis $1, X, X^2$ the three coefficient terms have distinct valuations modulo three. For an invariant uniformizer $T$, an integral element $\sum a_j X^j$ satisfies $v_{F_*}(a_j) \geq \lceil jm/3 \rceil$. Its trace is $-a_2$. Consequently $\dim \widehat{H}^0(Q; S_0) = \lceil 2m/3 \rceil = 1$ forces $m = 1$. Now take $T = (X^3 - X)^{-1}$. Put $L = h^3$, the invariant *scalar* Hasse section of index $-8$ and valuation one, and let $s = L\Delta^3$, of index 1. Dividing scalar sections of index $n$ by $s^n$ realizes $S_n$ as $I_n = \{z : v_E(z) \geq -n\}$. Over $k[[T]]$ it has explicit basis

$$T^{\lceil (j-n)/3 \rceil} X^j, \qquad 0 \leq j \leq 2.$$

Write $u, v$ for a Jordan basis of the constant plane, $gu = u,\ gv = v + u$. The invariant vector defining $h$ in $I_{-2} \otimes \langle u, v \rangle$ has minimum coefficient valuation two by primitivity. Every invariant vector has shape $au + b(Xu - v)$, $a, b \in F_*$. Distinct valuation residues imply $v_{F_*}(a) \geq 1,\ v_{F_*}(b) = 1$. An invariant unit rescaling and an integral shear of $v$ therefore put the vector in the form $T(Xu - v)$. Wedge with it gives the successive maps

$$f \longmapsto (TXf, -Tf), \qquad (a, b) \longmapsto T(a + Xb)$$

on $I_n, I_{n-2}^{\oplus 2}, I_{n-4}$. Multiplication by $L$, after the invertible $\Delta^3$ shift, is the inclusion $I_n \subset I_{n+1}$.

The supplied Smith calculations impose $\operatorname{im}(g - 1)$ for coinvariants and $\ker N$ for norm images. The unscaled $d_{17}$ chain is exact for each $n \bmod 3$, including at both ends. For the $d_{33}$ contraction the resulting pairs are

| $n \bmod 3$ | 0 | 1 | 2 |
|---|---|---|---|
| $(\ker, \operatorname{coker})$ | $(k, 0)$ | $(0, k)$ | $(0, k)$. |

The first kernel and last cokernel are the respective norm kernels. Earlier removal of the latter by $d_{25}$ or $d_{29}$ cannot create a new $d_{33}$ kernel, since the image of the injective map meets that line trivially. The cokernels have representatives nonzero at the closed point in their original scalar lines. These are polynomial matrices extended flatly from $\mathbb{F}_3[T]$. The calculation uses no cutoff in $T$. Lemma 5.8.

## Forcing the norm and transfer differentials

In the 15 entries of the finite table, the scalar slot $\mathsf{A}$ can retain a transfer kernel as well as a restriction lift. The symbols $\mathsf{x}, \mathsf{X}$ denote degree-one transfer lines, $\mathsf{y}$ a residual norm-kernel line, and $\mathsf{f}$ a volume class. The later detector occupies the extra scalar line $\mathsf{p}$. The table's labels describe these subquotients before one knows which differentials are nonzero. Table 1 and its qualifications.

Use the coordinates $s = 2m + j,\ t_{\mathrm{int}} = 4j - 6n,\ w_0 = m + 2n + 2j$. The first two determine the exterior degree $j \in \{0, 1, 2, 3\}$ uniquely by parity and residue modulo six. The symmetry conjugating $g$ to $g^{-1}$ has residual scalar $d_0$ with $d_0^2 = -1$. On the quotient of the Jordan block the coefficient weight $-2$ and the sign from inverting an odd cyclic cocycle cancel. Down the residual flag the characters are $1, -1, 1$, by $g - 1 \mapsto -(g-1) + (g-1)^2$. This gives the finite-label character $d_0^{2m+3n+\phi}$, with $\phi = 0$ on scalars and $\mathsf{x}, \mathsf{X}$ and $\phi = 2$ on the other labels, up to a common twist at fixed $i$. Preserving this character and shifting $(s, t_{\mathrm{int}})$ by $(r, r-1)$ reproduces the arrow lists. Lemma 5.9.

For the three $\rho$ values, every possible incoming on $\mathsf{A}$ by the Euler bound comes from $\mathsf{f}$. The length lists are

$$\rho = 2:\ 25, 241, 457 \qquad \rho = 5:\ 169, 385 \qquad \rho = 8:\ 97, 313.$$

There is a subgroup scalar which is a $d_{105}$ target and whose transfer is the nonzero kernel at $\mathsf{A}$ on the earlier page. Its $d_{105}$ source has zero transfer at $G$. Naturality forces the transferred scalar to disappear *before* $E_{105}$. It cannot support its own earlier differential. At $\rho = 2$, the 25 arrow vanishes by $\mathsf{f} = \mathsf{vd}$ and the vanishing of products of extra scalars. This forces $\rho = 8$ and the 97 incoming. If $e_{53} = 0$ at $H$, this reasoning consumes every $\mathsf{f}$ at 97. That map lands in the transfer kernel, leaving the unit with no possible incoming by 469. Thus $e_{53} \neq 0$. Restriction at 53 then forces $V(t, u) = 0$ only at $3 \mid t$, so $\nu \neq 0, \nu' = 0$. Proposition 5.10.

A second coefficient $\omega$ needs an independent nonvanishing proof. Take the *ordinary* $H$-orientation $Y = u_{\lambda|H}^{-9}$. Comparison into Tate is injective at each earlier outgoing target, giving a lift of this specified filtration-zero symbol over the 52-skeleton. The norm has actual virtual degree $9 \operatorname{Ind} \lambda|H - 18(1 + \lambda')$. Its three faithful sphere factors are identified using prime-to-three power maps. The first obstruction of the norm is the transfer of the single-factor obstructions because the reduced diagonal from $S^{52}$ to two reduced factors $S^{104}$ is stably zero. Equivalently this is the same-page norm-transfer theorem on the ordinary homotopy fixed point tower. The conjugate symbols agree here because the representation extends and the coefficient is one. Multiply by the actual faithful Euler power $a_\lambda^{27}$ and use the projection formula to get $\mathrm{tr}(C^2 d_{53} C)$. Its degree is $j = 1, m = 53, n = -8, w_0 = 39, i = 18$. This is $\mathsf{x}$ at $(4, 24)$, whose constant representative survives the $d_{33}$ quotient and has no earlier incoming. The target is nonzero on $E_{53}$, proving $\omega \neq 0$. Lemma 5.12. Duan et al., Theorem 4.6 and Proposition 5.4.

Finally the two possible $d_{25}$'s, $\mathsf{b} \to \mathsf{v}$ and $\mathsf{d} \to \mathsf{r}$, have matching zero status in every copy, by applying Leibniz to $\mathsf{bd} = 0$ and using the nonzero mixed products. If both vanished, a required $\mathsf{f}_2 = \mathsf{vd}$ would have zero $d_{97}$ by Leibniz, contrary to the preceding transfer argument. Thus all four endpoints are removed. This step excludes a possible volume source for the higher detectors. Lemma 5.13.

## Why every higher detector must support a differential

The following deduction from Lemma 5.13 and the detector coordinates checks the higher-index conclusion independently of the final matching. Write

$$\mathcal{L}_\kappa = \{(t,u) : t + 3\kappa u = 0 \bmod 9\}.$$

The $\mathsf{x}$ support at $E_{54}$ is $(1,6)+\mathcal{L}_\kappa$ and the $\mathsf{y}$ support has $t = 0 \bmod 3$. The earlier pages give no incoming to the detector label $\mathsf{p}$ through 53. Solving the two bidegree equations and the normalizer-character equation for every surviving label, up to the finite Euler bound, gives the following four possibilities.

| $r$ | source | $(\Delta t, \Delta u)$ | necessary target support |
|---|---|---|---|
| 57 | $\mathsf{x}$ | $(1,6)$ | $(2,12)+\mathcal{L}_\kappa$ |
| 93 | $\mathsf{y}$ | $(2,10)$ | $t = 2 \bmod 3$ |
| 273 | $\mathsf{x}$ | $(5,30)$ | $(6,36)+\mathcal{L}_\kappa$ |
| 309 | $\mathsf{y}$ | $(6,34)$ | $t = 0 \bmod 3$ |

For instance, for exterior-degree-one sources write $r = 12l + 9$. With target $\mathsf{p}$, the $w_0$ condition gives $l = 4 \bmod 9$ for $\mathsf{x}, \mathsf{X}$ or 7 mod 9 for $\mathsf{y}$. The $n$ residue excludes $\mathsf{X}$. The character gives $l = 4, 7 \bmod 18$, respectively. The only possible exterior-three source has the removed $\mathsf{d}$ label. This verifies exhaustiveness through 469. Even the candidate 57, independently zero by transfer, may be retained for this argument. Degree rule and copy shifts.

A sphere survivor gives an ordinary filtration-two class and therefore its Tate image has zero outgoing on every page where it remains. An incoming in Tate is possible. The known $b_2$ detector at $(-4,12)$ must be hit, since the Tate unit dies by 469. Only 93 can do so. It uses the one-dimensional $\mathsf{y}$ at $(-6,2)$. Multiplication by $C^9, u'^3$ and their inverses gives all translations in $9\mathbb{Z} \times 3\mathbb{Z}$ through this differential because the logarithmic $d_{53}$'s on $C^3, u'$ are killed by 3 and there is no intervening arrow on the unit slots. Hence every such translate of this $\mathsf{y}$ is consumed at 93.

For $j \geq 4$ the proposed detector has $(t_j, u_j) = (-4 \cdot 3^{j-2}, 4 \cdot 3^{j-1}) \in 9\mathbb{Z} \times 9\mathbb{Z}$. Its $t_j$ excludes 57 and 93. Its $t_j + 3\kappa u_j = 0 \bmod 9$ excludes 273 for every $\kappa$. Its only possible source at 309, $(t_j - 6, u_j - 34)$, differs from $(-6,2)$ by $(t_j, u_j - 36) \in 9\mathbb{Z} \times 3\mathbb{Z}$. That source was used at 93. Thus an incoming is impossible. The Euler bound forces an outgoing, which rules out a sphere survivor. The index $j = 3$ has first coordinate $-12$, outside $9\mathbb{Z}$, so it falls outside this exclusion. The paper's full matching further places this outgoing at 85 or 157 in the auxiliary $C_9$ sequence. The classical Adams sequence enters through the comparison on page 3. Detector coordinates. Proof of Theorem 5.1.

I also checked the paper's longer route. Products with the permanent Euler copy and the $b_2$ detector exclude $\mathsf{a}_0, \mathsf{a}_2$ after $d_{49}$. Either an earlier boundary on a factor would kill a required $\mathsf{f}_2$ product, or the product would be a $d_{97}$-cycle. The $d_{97}$ on $\mathsf{A}$ hits its transfer kernel, preserving the restriction units in the quotient. Their lattice survives to $E_{313}$, where $\mathsf{f}$ must hit the unit. Enumerating the larger graph before the paper fixes the $\mathsf{p}_1$ partners gives 150 allowed matchings for $\kappa = -1$, zero for $\kappa = 0, 1$. The reduced graph gives 135. These count possibilities subject to necessary constraints. The known $b_2$ and every $b_j$ with $j \geq 4$ have the same respective fates in all allowed matchings. Lemma 5.14 through Proposition 5.17.

# Constructing the class in stem 322

### Identifying the class

Write $S \xrightarrow{3} S \xrightarrow{i} M \xrightarrow{q} \Sigma S$ for the Moore cofiber sequence. For $m = 3^{j+1}$, the strong criterion tests whether some $a : S^{4m-1} \to M$ has a nonzero operation $\mathcal{P}^m : H^0(C_a; \mathbb{F}_3) \to H^{4m}(C_a; \mathbb{F}_3)$. The associated sphere class is $\theta = \Sigma^{-1} qa$. To check this identification, dualize the three-cell cone. The bottom restriction of the resulting map $\Sigma^{4m-2} M \to S$ is $\theta$ up to a unit, since $DM \simeq \Sigma^{-1} M$ interchanges the inclusion and quotient. Conjugation takes $\mathcal{P}^m$ to $-\mathcal{P}^m$ modulo products of positive reduced powers, whose intermediate degrees are absent. The primary test thus passes to the dual. Amelotte explicitly identifies its middle-cell attaching map as the Kervaire element. The published indexing assigns it the ordinary Adams detector $b_j$. Lemma 6.2, Selick, pp. 431–432, Amelotte, proof of Theorem 1.4.

### The input in stem 106

Amelotte's proof of Corollary 1.2 records $b_1$ supporting a differential, $b_2$ detecting $\theta_3$, and $\pi_{106} S_{(3)} \cong \mathbb{Z}/3$. Hence one can choose $a_0 : S^{107} \to M$ with nonzero $\mathcal{P}^{27}$ and $qa_0 = \Sigma x$ for an order-three $x$. Its primary operation forces filtration at most one and $BP_{107} M = 0$ excludes filtration zero. In the manuscript's notation, the $t_1^{27}$ coefficient of the leading cobar class is a unit. After rescaling, Lemma 7.2 identifies this class as $y + \mu v_1^{26} a$, with $a = [t_1]$. The reduced integral Bockstein $B$ kills $v_1^{26} a$ because $B(v_1) = \pm a$, $B(a) = 0$, and $a^2 = 0$. The symbol of $ix^3$ in filtration six is consequently $\pm B(y)^3 = 0$. Since $318 \equiv 2 \pmod 4$, the next possible filtration is ten. This argument uses a representative of the known homotopy class and the filtered Moore boundary. Proposition 6.4, Amelotte, proof of Corollary 1.2.

### The extended cube

The truncated cube produces $F : \Sigma^{322} M \to T$ in a triangle $M \to T \xrightarrow{\pi_T} \Sigma^4 M \to \Sigma M$, with $\pi_T F = x^3 1_M$. Three points in this construction affect the conclusion. First, the orbit of a single upper Moore cell has attaching degree three, because taking coinvariants identifies the three generators. Second, in the truncated lens-space construction the orientation twist cancels the inversion sign on the mod-three two-cell. The Euler class of the faithful $C_3$-line therefore pairs nontrivially with it. Third, the diagonal cone carries the degree-324 class to a unit times $u^{\otimes 3}$, for $u$ the degree-108 generator in $C_{a_0}$. The Cartan formula gives $\mathcal{P}_*^{81}(u^{\otimes 3}) = (\mathcal{P}_*^{27} u)^{\otimes 3} \neq 0$. Comparison of the tested bottom and top degrees yields the nonzero operation on $C_F$. Lemmas 6.5–6.6, Proposition 6.7, Toda, §1, Theorem 2.

## Lifting to the Moore spectrum and proving exact order

### Why the topological correction exists

Let $F_f$ denote the $f$th stage of the $BP$-Adams tower. The section of $F(M,M) \to M$ takes $i$ to $1_M$. A lift of $ix^3$ in filtration ten thus supplies a lift of $x^3 1_M$ at stage two. The obstruction to lifting it on to $F_2T$ has two components. The component in $\pi_{321}F_2M$ vanishes after forgetting filtration because $F$ exists. That forgetful map is injective since the first two layers have zero $\pi_{322}$. The component in $\pi_{322}F_2M$ is induced by a map $\Sigma^4 M \to M$, hence acts on $BP_*M$ by a scalar multiple of $v_1$. Lemma 7.2 gives zero on $E_6^{10,328} \to E_6^{10,332}$. Lemma 6.9 transfers this calculation to the truncated tower at filtration ten. Sparseness puts the remaining obstruction in filtration at least fourteen. There the relative and absolute $E_{10}$ terms agree and vanish in stem 322. Connectivity makes the filtration finite. This sequence of comparisons allows for a full-tower $d_9$ entering filtration ten from filtration one. Lemma 6.9, Proposition 6.10.

Subtracting the resulting $F_2T$-correction from $F$ gives a map lifting to $Z : \Sigma^{322}M \to M$. Its primary test survives because the first ordinary Adams extension is additive and vanishes on maps of $BP$-filtration at least two. The section component of $Z$ also has filtration at least two, by the support of the first two layers in stem 322. After removing it, the remaining map factors as $a_1 \circ \Sigma^{322}q$ with $a_1 : S^{323} \to M$, and the cone comparison is an isomorphism in degrees 0 and 324. Thus $C_{a_1}$ passes the $\mathcal{P}^{81}$ test. The strong criterion identifies the nonzero $b_3$-detected $\theta = \Sigma^{-1}qa_1$ in stem 322. The actual composite $(\Sigma 3)qa_1 = 0$ proves its additive order is exactly three. Proof of Theorem 6.1.

### Where the finite computations are used

Three facts enter this geometric construction. First, the leading $BP$ symbol of $ix^3$ is zero in the particular cohomology group where the cube lies. Second, a $v_1$-map is zero on the sixth page in the precise bidegree of the second obstruction. Third, the tenth page vanishes in stem 322 in every possible filtration at least fourteen. Pages 14–15 give the arithmetic checks and the uniform argument that covers the remaining filtrations. Lemma 7.2.

## The finite calculation and the cubic boundary

Write $M = S/3$ and $D_{f,d} = E_2^{f,4d}(M)$, so the stem is $4d - f$. Three conclusions of Lemma 7.2 enter the construction. A normalized permanent class $z \in D_{1,27}$ has $B(z) = B(y)$, where $y$ is the corrected divided coboundary of $v_2^9 - v_1^8 v_2^7$ and $B$ is the reduced integral Bockstein. Its cube $B(y)^3$ vanishes. Multiplication by $v_1$ on $E_6^{10,328}(M)$ is zero into $E_6^{10,332}(M)$, and $E_{10}^{f,322+f}(M)$ vanishes in every possible filtration $f \geq 14$. The cubic identity raises the filtration of the reduced cube of the 106-stem class from at least six to at least ten. The last two dispose of the remaining obstruction in the Moore-spectrum correction. Lemma 7.2 and Proposition 6.10.

### From the Hopf algebroid to the finite matrices

The audit reran the digit-resolution and signed Yoneda-product algorithms from fresh intermediates. Through the weight-92 cutoff it reconstructed all 2,269 Ext groups, of total dimension 6,341, and all 15,453 multiplication matrices. All 39,517 columns agreed entrywise with the reference data. Repeating an algorithm tests its reproducibility. A separate implementation reconstructed the operator products it needed from the rational Hazewinkel logarithms, checked the prescribed leading differential, and checked $\partial^2 = 0$ on every one of the 56,591 positive-filtration generators through weight 93 and filtration 33. To check exactness in Appendix A, take the least-order part of a cycle in the filtered resolution. It is a cycle for the additive leading differential. The additive contraction gives a preimage for that part. Subtracting its full boundary strictly raises the least order, and only finitely many orders occur below a fixed cutoff. This inductively gives the desired preimages and exactness. In the overlap with the separate Wang–Belmont calculations through internal degree 156, all 547 Ext dimensions and 1,550 multiplication ranks tested agreed. Proposition A.2 and Wang–Belmont data.

### The cubic identity

In $D_{6,81}$ the cochain dimension is 11,419 and the incoming and outgoing ranks are 5,256 and 6,156. This leaves a seven-dimensional cohomology group, so the particular cube matters. Direct substitution verified that the archived 2,642-term cochain has differential equal to the 76-term representative of $B(y)^3$. An independent normalized-bar cup product constructed a different representative with 265 terms and solved for a 2,627-term preimage. Applying the differential verified this second identity as well. As a shorter check, multiplication by $y$ is zero on all four generators of $D_{4,54}$. Thus $yB(y)^2 = 0$, and applying the Bockstein derivation gives $B(y)^3 = 0$ since $B^2 = 0$. Table 3 and its cochain interpretation and Appendix A, arithmetic checkpoints.

## Arithmetic constraints on the fifth and ninth differentials

The binary Moore-spectrum pairing gives the Leibniz rule. It suffices to normalize $L = d_5$ by a unit and impose constraints which every actual $L$ satisfies. At the weight-60 cutoff, of the nine possible coefficient pairs $(k, e)$, only $(1, 0)$ has coefficient and augmented ranks both 2,667. Each other augmented rank is 2,668. This eliminates the $t_7'$ term of $z$. The remaining ambiguity, a multiple of $v_1^{26}[t_1]$, has zero Bockstein. At the weight-92 cutoff, independent elimination of 137,471 scalar equations on 23,384 entries gives rank 23,119. The successive affine consequences of $L^2 = 0$ lower the dimension from 265 to 227 to 226. Every actual $d_5$ lies in this affine space. The elementary kernel and image bounds are uniform on the space. Even if it includes nonsquare-zero points, the actual differential is among the candidates and its homology satisfies the resulting estimates. The calculation recovered all sixteen enclosure rows, both cancellation triples and the three injection tests. Tables 4–7.

For example, write a projected matrix as $A(u) = A_0 + \sum_{i>0} u_i A_i$. Its kernel lies between $\bigcap_{i\geq 0} \ker A_i$ and $A_0^{-1}(\sum_{i>0} \operatorname{im} A_i)$. Its image contains $A_0(\bigcap_{i>0} \ker A_i)$ and lies in $\sum_{i\geq 0} \operatorname{im} A_i$. These inclusions hold even when parameters are shared between different maps. They explain how the finite calculation can certify vanishing without determining all 226 coordinates. Lemma 7.4 and the enclosure formulas.

Table 8 consists of *five* maps with 29 source basis columns altogether. A separate bar-product implementation checked that the given representatives form bases and compared each column with its Yoneda calculation modulo boundaries. The ranks before and after adjoining the product columns are displayed below. Table 8.

| $(f, d)$ of target | $(22, 91)$ | $(17, 90)$ | $(19, 93)$ | $(26, 87)$ | $(31, 88)$ |
|---|---|---|---|---|---|
| boundary rank | 4491 | 7540 | 7589 | 1868 | 1000 |
| rank with products | 4499 | 7546 | 7595 | 1872 | 1004 |
| rank of multiplication | 8 | 6 | 6 | 4 | 4 |

The enclosure at filtration ten gives the asserted zero $v_1$-map. In stem 322 the low filtrations requiring checks are $14, 18, 22, 26$. The enclosures kill the 22 slot already on $d_5$-homology. At the other three, the cancellation and injection tests, product ranks, and the classical $d_9(P) = \epsilon b^6$, with $\epsilon$ a unit, express every $d_9$-cycle as a boundary. For all $f \geq 30$ with $f = 2 \bmod 4$, the additive-digit estimate gives surjectivity of $b^6$ onto the relevant $d_5$-homology and injectivity into the outgoing $d_9$-target. Thus $d_9 c = 0$ for $c = b^6 x$ implies $d_9 x = 0$ and $d_9(Px) = \epsilon c$. The inequalities which justify the comparison with $d_5$-homology reduce to

$$17f > 464, \qquad 17f > 396, \qquad 17f > 328, \qquad 17f > 396.$$

All hold already at $f = 30$. This is the step extending the finite computation to every remaining filtration. Lemma 7.5 and conclusion of Lemma 7.2.

## The unstable corollary and the review record

### From an order-three representative to the splitting

For $n > 1$, write $S^{2n+1}\{3\}$ for the homotopy fibre of the degree-three map. In the primary-operation criterion, $n = 3^i$ and the stable degree is $4n - 2$. Exact order three in Theorem 1.1 therefore picks out $i = 1, 3, 4$, equivalently $n = 3, 27, 81$. At $i = 4$, Amelotte's Theorem 1.1 gives the following 3-local $H$-space equivalences. The two dimensions follow from $2 \cdot 81 + 1 = 163$ and $2 \cdot 243 + 1 = 487$.

$$\Omega S^{163}\{3\} \simeq T^{163}(3) \times \Omega T^{487}(3), \qquad BW_{81} \simeq \Omega T^{487}(3).$$

Looping yields $W_{81} \simeq \Omega^2 T^{487}(3)$, with $W_n$ the fibre of the double suspension. This checks that it is the established *strong* form of the Kervaire problem that the 322-stem construction solves. Amelotte, Theorem 1.1. Corollary 1.2.

Selick's atomicity criterion applies for every $n > 1$ in the absence of a strong class, including $n$ that are not powers of three. Atomicity rules out a nontrivial product of the underlying space. Together with the positive splittings, this gives exactly $n = 3, 27, 81$. Amelotte's Theorem 1.4 gives the same three $n$ for existence of a homotopy-associative multiplication on $T^{2n+1}(3)$. For the multiplication supplied by the new splitting, homotopy commutativity and nullity of the third-power map pass to the $H$-space retracts, as in the proof of Amelotte's Corollary 3.3. The power-map conclusions concern the induced multiplications on the two factors. Selick, Theorem 3.3 and Corollary 3.4. Amelotte, Theorem 1.4. Proofs of Theorem 1.4 and Corollary 3.3.

### What the computational comparisons establish

Every manuscript link pins the same Git revision. All eight section files used in the review matched the public source by Git object ID. The finite calculation also uses code from a predecessor implementation. I treated its outputs as reference data. The rebuild through weight 60 reproduced 7,777 named multiplication matrices and eight inconsistency certificates. The weight-92 rebuild compared every column with the reference data. The independent formula, cochain, and matrix checks are identified on pages 14–15. A separate public data set from Wang and Belmont gives a basis-independent check through internal degree 156. The checked overlap comprises 547 dimensions, 1,550 multiplication ranks, 2,797 composite ranks and 2,012 dimensions of sums of images. Each comparison agreed. Wang–Belmont data.

The companion `verification-files.zip` contains scripts, compact inputs, comparison results and a guide to running the checks. All 16 compact commands in the guide passed from a freshly extracted copy. For the larger weight-92 and weight-93 runs, the archive includes the regeneration procedures and results. The topological constructions, the formal-deformation arguments, convergence or finite-tower hypotheses, and the applicability of published theorems were reviewed mathematically. An arithmetic replay starts only after such inputs have supplied its modules and constraints. This distinction is especially important for the norm obstruction on page 10 and the two filtration comparisons on page 13.

**Conclusion.** With the insertion on page 7, I found an argument for each implication needed by the classification and its unstable corollary. The remark attributed to Ravenel, as relayed to me, supplied no specific mathematical step to test. The assessment above concerns the mathematical claims and the specific version linked on page 1.

# External inputs I. The printed detection argument

Belmont–Ray's Proposition 6.2 concerns the image of a two-line class in the *integral* group $H^2(C_9;\pi_T E_6)$, $T = 4\cdot 3^{j+1}$. Their printed proof in version 1 has two problems. Each can be tested in their coordinates. The first also explains why keeping the integral target on page 3 is essential. Belmont–Ray, Proposition 6.2 and proof.

### The final reduction kills the class

In the cyclotomic evaluation let $\mathcal{O}$ be the coefficient ring on the relevant degree line, suppressing its invertible grading generator. The $C_9$-action on this line is trivial in the degrees under discussion, and $\mathcal{O}$ is torsion-free. Consequently the two maps in question are

$$H^1(C_9;\mathcal{O}/3) \xrightarrow{\delta} H^2(C_9;\mathcal{O}) \longrightarrow H^2(C_9;\mathcal{O}/3), \qquad \bar{c} \longmapsto 3c \bmod 9 \longmapsto 0.$$

Indeed, the cyclic resolution has norm 9 on this line. Lifting $\bar{c}$ to $c$ and dividing its coboundary $9c$ by three gives the indicated connecting homomorphism. Belmont–Ray first ask for a nonzero composite into reduced coefficients. Their subsequent Bockstein diagram instead establishes nonvanishing in the integral target, provided the input has unit residue. The two tests give different answers. The integral one suffices for the detection statement. Belmont–Ray, pp. 26–27. Integral Bockstein in September Lemma 2.4.

### A missing term before specialization

At $j = 2$, work modulo three. Put $v = v_1$, $A = t_1^3 - v^2 t_1$, and $d = \eta_R - \mathrm{id}$, so that $\eta_R(v_2) = v_2 + vA$. The exact expansion gives

$$\begin{aligned} v^{-9} d(v_2^9 - v^8 v_2^7) &= A^9 - \frac{(v_2 + vA)^7 - v_2^7}{v} \\ &\equiv t_1^{27} - v_2^6 t_1^3 \pmod{v}. \end{aligned}$$

The second term comes from the binomial coefficient $7 \equiv 1 \bmod 3$. Thus the printed congruence to $t_1^{27}$ fails even modulo $v$, a weaker test than their claimed modulus $v^6$. At the cyclotomic point both $v_1, v_2$ have zero residue while $t_1(g)$ has nonzero residue. In fact, each term of the correction fraction contains six factors in the ideal $(v_1, v_2)$, whose image lies in $(\zeta_9 - 1)$. The cyclotomic uniformizer satisfies $(\zeta_9 - 1)^6\mathcal{O} = 3\mathcal{O}$. Thus the needed unit survives specialization. Belmont–Ray, Lemma 6.1 and the displayed calculation in Proposition 6.2.

**Effect on the September proof.** Lemma 2.2 constructs an explicit uncorrected basis using the odd-primary two-line theorem. Lemma 2.4 evaluates that basis, finds a unique unit-residue contribution when the Thom image is $b_j$, and takes its Bockstein in $\mathcal{O}/9$. The finite calculation in the other branch also keeps the correction explicitly in $\delta_9(v_2^9) = y + t_7'$. The two errors above affect citing the earlier proof as verification. I found no resulting gap in the September classification. September Lemma 2.2. Lemma 2.4. Appendix A, corrected divided coboundary.

# External inputs II. What is still imported

### The 106-stem calculation is shared by both branches

The higher-index exclusion uses permanence of $b_2$ to force a particular $d_{93}$ pairing. The constructive branch starts with an order-three $b_2$-detected sphere element. Ravenel's digital third edition (July 2026) identifies the surviving Adams–Novikov combination $\beta_{9/9} \pm \beta_7$, gives the calculation in §7.5, and lists the 3-primary group in stem 106 as $\mathbb{Z}/3$. Amelotte's Corollary 1.2 and Belmont–Shimomura's Lemma 4.3 also use that stem. Both cite Ravenel. The Wang–Belmont chart documentation expressly says that its higher sphere differentials through stem 108 were inferred from Ravenel's table. An early draft by Behrens et al. expresses concern about the range 104–108, without locating an error in stem 106. The calculation on pages 19–20 recovers the group, its order and the Adams detector directly from the algebraic $E_2$ data and Toda's lower differential. Behrens et al., early draft, p. 5. Ravenel, after Proposition 6.4.11. Theorem 7.5.3. Table A3.2. Chart provenance.

The algebraic classification in Miller–Ravenel–Wilson, Theorem 2.6, includes $p = 3$. Ravenel's stronger all-lifts differential argument for $p \geq 5$ has an explicit prime restriction. The September proof supplies a new higher-index exclusion at three. It does import the classical sphere differentials $d_5(\beta_{3/3}) = \pm\alpha_1\beta_1^3$ and $d_9(\alpha_1\beta_4) = \pm\beta_1^6$, used in the two branches. I verified their statements and use, without redoing their classical proofs. Miller–Ravenel–Wilson, Theorem 2.6. Ravenel, Lemma 6.4.13. Belmont–Shimomura, Lemma 2.5.

### The translates in the norm differential

Theorem 4.6 of Duan et al. gives a formula with conjugate factors and then simplifies those factors using centrality of $H$. Centrality alone does not imply that $G/H$ fixes an $H$-level class $Y$. For example, a cyclic quotient can permute three factors in a coefficient ring. For the $C_3 \subset C_9$ calculation, September Lemma 5.12 derives

$$d_{53}\,\mathrm{Nm}(Y) = \mathrm{tr}\big((d_{53}Y)(gY)(g^2Y)\big)$$

on free cells. It verifies $gY = Y = g^2Y$ for this particular orientation symbol because its coefficient is one and its representation extends to $G$. This supplies the hypothesis for the simplification actually used. This qualification is my reading of the displayed general formula. Duan et al., Theorem 4.6. September Lemma 5.12, calculation with translates.

### A published erratum on the unstable side

Gray–Theriault published a replacement proof of Theorem 4.3(l) in their construction of Anick's fibration. Amelotte uses the prime-three construction in the splitting theorem invoked for Corollary 1.2. The stable classification is established before this application. The stable positive branch separately uses the three-cell primary-operation criterion and constructs its own witness. The discussion on page 12 traces the particular attaching map. I checked the scope of the erratum and the use of the corrected theorem, without independently redoing the replacement induction. Gray–Theriault, 2013 erratum. Amelotte, Anick's fibration. Amelotte, proof of Theorem 1.4.

## An independent calculation of the 3-primary 106-stem

Write $(n, s)$ for (sphere stem, Adams–Novikov filtration), so $t = n + s$. Write $D = d_5$. The prime-three $BP$-cobar complex is supported in internal degrees divisible by four. Thus $E_2 = E_5$ and a differential changes $(n, s)$ to $(n - 1, s + r)$, with $r \equiv 1 \bmod 4$. Each normalized cobar factor has internal degree at least four, so $t \geq 4s$. In stems 105–107 this bounds $s$ by 35 and $t$ by 142. The published resolution of length 40 covers all needed bidegrees. I used the algebraic basis, multiplication-by-three table and product files in the pinned Wang–Belmont data. The argument reconstructs the topological arrows from the Leibniz rule and Toda's lower differential. Wang–Belmont source. Belmont–Shimomura, Lemma 2.5.

| Stem | Nonzero filtrations $s$ (dimension) | | | |
|---|---|---|---|---|
| 105 | 3 (1) | 7 (1) | 11 (1) | 15 (1) |
| 106 | 2 (2) | 6 (1) | 10 (1) | 18 (1) |

All groups in these two rows have exponent three. In filtration two the computational basis is $\{u, k\}$, where $u =$ `[2-7]` has algebraic Novikov filtration zero and $k =$ `v1^4v3^1[2-4]` has positive algebraic Novikov filtration. Thus $k$ reduces to zero under the Thom map. The image of the two-line group is spanned by $b_2$. Indeed, the uncorrected double connecting class of $v_2^9$ has Thom image the standard Bockstein of $t_1^{27}$, which represents $b_2$. This proves that $u$ has nonzero Thom image regardless of which multiple of $k$ is included in a lift. Ravenel distinguishes the uncorrected representative here from the Miller–Ravenel–Wilson convention. Ravenel, Lemma 6.4.12 and eq. (6.4.16). September manuscript, Thom reduction.

### The four nonzero arrows

The product calculations on page 20 give the following ranks over $\mathbb{F}_3$.

| Source $(n, s)$ | Target $(n, s)$ | Rank | Consequence |
|---|---|---|---|
| $(106, 2)$ | $(105, 7)$ | 1 | $Dk \neq 0$, so the kernel maps onto $\langle b_2 \rangle$. |
| $(106, 6)$ | $(105, 11)$ | 1 | Clears the possible $d_9$ target. |
| $(106, 10)$ | $(105, 15)$ | 1 | Clears the possible $d_{13}$ target. |
| $(107, 13)$ | $(106, 18)$ | 1 | Clears the last higher-filtration class. |

There is no incoming Novikov differential to filtration two. All later outgoing targets there vanish by the displayed groups and sparseness. Thus the 106-stem of $E_\infty$ is a single $\mathbb{F}_3$ in filtration two. The filtration is finite, which gives $(\pi_{106}^S)_{(3)} \cong \mathbb{Z}/3$ with no hidden additive extension. The Thom map compares the two spectral sequences and induces the identity on sphere homotopy. The image $b_2$ cannot support a differential by naturality. The only possible incoming ordinary Adams differential would be a $d_2$ from $\mathrm{Ext}^{0,107} = 0$. Hence the sphere generator is detected by $b_2$.

## The products forcing the four differentials

Use the Wang–Belmont choices of basis. Set $a = \alpha_1,\ b = \beta_1,\ c = \beta_2,\ h = \beta_{3/3},\ f = \beta_4$. The $D$-targets of $a, b, c$ vanish in their respective bidegrees. Toda's differential says $Dh = \epsilon ab^3$, where $\epsilon \in \mathbb{F}_3^\times$. The proof below keeps this unit and uses the sign of a derivation of odd degree. The table's other generators needed here are $e =$ `[3-5]` in $(57, 3)$, $q =$ `[5-6]` in $(75, 5)$, and $z =$ `v3^1[6-3]` in $(106, 6)$. The equalities of products below and the indicated nonvanishing tests are read from the raw algebraic multiplication tables. In particular the mixed bidegrees used for the signed equalities have exponent three. Basis and multiplication conventions.

### Why the surviving line has nonzero Thom image

The signed products are $c^3 = -bh^2 - b^2 f$ and $f^2 = -bk$. Applying $D$ to the first gives $b^2 Df = -2\epsilon ab^4 h$. The target of $Df$ is the line spanned by $ab^2 h$, and $ab^4 h \neq 0$, so $Df = -2\epsilon ab^2 h$. Also $fab^2 h \neq 0$, so applying $D$ to the second equality yields $-bDk = 2fDf \neq 0$. This determines that the kernel avoids $k$ without having to choose a sign in a named $\beta_{9/9}$ convention. There is also an external check on this nonzero differential. Belmont–Shimomura's Lemma 6.3 gives the elliptic image of $\beta_7$ as a unit times $\Delta^4\beta$. Bauer's $d_5(\Delta) = \pm\alpha\beta^2$ gives a nonzero differential on that image. Belmont–Shimomura, Lemma 6.3. Bauer, eq. (6.1).

### Why the other three groups disappear

The raw data give three signed relations

$$ae = -ch, \qquad aq = c^3, \qquad az = hq + c^2 e.$$

Applying $D$ to the first gives $aDe = \epsilon ab^3 c$. The target of $De$ is generated by $b^3 c$, with $ab^3 c \neq 0$, hence $De = \epsilon b^3 c$. The only possible target of $Dq$ is spanned by $b^4 h$, whose own differential is $\epsilon ab^7 \neq 0$. Thus $D^2 = 0$ forces $Dq = 0$. The third identity now gives

$$-aDz = \epsilon ab^3 q + \epsilon b^3 c^3 = 2\epsilon b^3 c^3 \neq 0.$$

This forces the arrow out of filtration six. For the two remaining arrows the product rule gives

$$D(b^2 c^2 h) = \epsilon ab^5 c^2, \qquad D(b^5 e) = \epsilon b^8 c.$$

The source and target of each equality generate their respective one-dimensional groups.

### What the verification assumes

The initial algebra comes from Wang's resolution as computed by Belmont, together with the standard Thom comparison. No sphere homotopy result in stems 104–108 is needed. The only nonzero topological differential supplied to this calculation is Toda's in stem 34. The companion command `python verify-stem106.py ANSS_data` checks the input hashes, evaluates the identities and tests multiplication by three in their target groups. I also regenerated the tables with the authors' pinned C++ resolution program, using cutoff 145, and repeated the product checks on that output. Both runs gave the same local ranks and identities. They share one resolution implementation. Resolution source.

## Selected primary sources

The links in the body lead to the particular results used. The following references identify the historical publications and talks, including the status of the two Hill–Hopkins–Ravenel presentations.

**Classical Adams family and low stems.** D. C. Ravenel, *The non-existence of odd primary Arf invariant elements in stable homotopy*, Mathematical Proceedings of the Cambridge Philosophical Society **83** (1978), 429–443, §1. His book *Complex Cobordism and Stable Homotopy Groups of Spheres*, second edition, AMS Chelsea (2004), gives the corrected beta combination after Proposition 6.4.11. 1978 article. Book, p. 220.

**The subgroup prediction.** M. A. Hill, *Recent computational work on $EO_n$*, report on joint work with M. J. Hopkins and D. C. Ravenel, Oberwolfach Reports **4** (2007), 2698–2701. Propositions 2 and 3 give a coefficient description and the conditional $C_p$ differential pattern. Report 45/2007.

**The proposed higher exclusion.** M. A. Hill, M. J. Hopkins and D. C. Ravenel, *On the 3-primary Arf–Kervaire invariant problem*, work in progress, September 14, 2011. The same authors' Banff workshop presentation, *The 3-primary Arf–Kervaire invariant problem*, February 16, 2012, is particularly explicit about the possible cutoff and the question it leaves in stem 322. 2011 manuscript. 2012 Banff slides.

**The height-six formulation.** E. Belmont and R. Ray, *Towards the $p = 3$ Kervaire Invariant Problem*, arXiv:2507.10157v1 (2025). The introduction states the detection result separately from the conjectures concerning coefficients, periodicity and degree $-2$. The issues in the printed proof of Proposition 6.2 and the separate integral proof are discussed on page 17. For the classical differentials used at the subgroup, see E. Belmont and K. Shimomura, *Beta families arising from a $v_2^9$ self map on $S/(3, v_1^8)$*, Algebraic & Geometric Topology **23** (2023), 4349–4378, Lemma 2.5. Belmont–Ray. Belmont–Shimomura.

**Moore-spectrum and unstable formulations.** H. Toda, *Extended $p$-th powers of complexes and applications to homotopy theory*, Proceedings of the Japan Academy **44** (1968), 198–203. P. S. Selick, *A reformulation of the Arf invariant one mod $p$ problem and applications to atomic spaces*, Pacific Journal of Mathematics **108** (1983), 431–450. S. Amelotte, *The fibre of the degree 3 map, Anick spaces and the double suspension*, Proceedings of the Edinburgh Mathematical Society **63** (2020), 830–843. Toda. Selick. Amelotte.

**Independent low-degree arithmetic.** G. Wang and E. Belmont's published table files and the code that emits them were used to interpret the Moore-spectrum dimensions and products within the common degree range. Both snapshots are pinned. Table files. Table emitter.