CHAPTER 7: THE NEW STEMS (π_N^S, $46 \le N \le 64$)

1. Introduction

In Sections 2 through 5 we continue the computations of Chapter 6 to determine the next 19 stable stems. The tables of leaders in this chapter only include those leaders of degree less than 67. We are unable to determine three group extensions in degrees 54, 62 and 63. In particular we can not determine whether 2A[54,1] equals zero or ηA[8]D[45]. Whenever we deal with A[54,1] we take into account both possibilities for 2A[54,1]. In Section 6 we collect all the computations of tentative differentials which are determined by the leading differentials discovered in the previous sections. The results of this chapter are summarized in Appendices 1 to 4.

The computations of π_N^S, $46 \le N \le 59$, agree with the tentative computation of these stems by Mark Mahowald using the Adams spectral sequence [55]. Tables of the Adams spectral sequence through degree 64 are given in Appendix 6.

2. Computation of π_N^S, $46 \le N \le 50$.

In the tables of leaders below, all leaders of degree greater than 52 will have an asterisk at the left. They will be omitted from all other tables of leaders in this section except for the last one.

Recall from the table of leaders in Figure 6.3.7 that there are five leaders of degree 47 and four leaders of degree 48. Let $\phi = d^6(\eta A[40,1]M_1^3)$. We show that $\phi = 0$. By Theorem 2.4.2, $\phi \in \langle\eta^2, A[40,1], \nu\rangle = \langle\langle 2,\eta,2\rangle, A[40,1],\nu\rangle$ $= \langle 2,\langle\eta,2,A[40,1]\rangle,\nu\rangle + \langle 2,\eta,\langle 2,A[40,1],\nu\rangle\rangle$. Now $2\langle\eta,2,A[40,1]\rangle$ $= \langle 2,\eta,2\rangle A[40,1] = \eta^2 A[40,1] = 0$. Hence η^2 divides $\langle\eta,2,A[40,1]\rangle$, $0 \in \langle 2,\langle\eta,2,A[40,1]\rangle,\nu\rangle$ and $\phi \in \langle 2,\eta,\langle 2,A[40,1],\nu\rangle\rangle$. ($\phi$ is not divisible by ν

while the other elements of π^S_{46} are divisible by η and thus have order two.) Now $\eta\langle 2, A[40,1], \nu\rangle = \langle \eta, 2, A[40,1]\rangle\nu = 0$. Thus, $\langle 2, A[40,1], \nu\rangle = 2kC[44]$. Therefore, $\phi \in \langle 2, \eta, 2kC[44]\rangle = k\langle 2, \eta, 2\rangle C[44] = k\eta^2C[44]$, a contradiction, since ϕ is not divisible by η. Thus, $\phi = 0$ and $\eta A[40,1]M_1^3$ must bound. There is only one possibility: $d^8(B[34]M_1^4M_2) = \eta A[40,1]M_1^3$ and

$$\sigma B[34] = \eta A[40,1]. \qquad [7.1]$$

Clearly $C[44]M_1^2$ transgresses. Note that $2C[42]\bar{M}_2$ transgresses because $2\nu C[42] = 16D[45] = 0$. Observe that $2C[20]^2M_1\bar{M}_2$ equals $C[20]^2\bar{M}_2$ times the infinite cycle $2M_1$, and $d^6(C[20]^2\bar{M}_2) = A[45,2]$. Therefore $d^6(2C[20]^2M_1\bar{M}_2) = 2A[45,2]M_1 = 0$, and $2C[20]^2M_1\bar{M}_2$ transgresses. Thus, $\eta C[44]M_1$, $D[45]M_1$, $A[45,1]M_1$ and $A[45,2]M_1$ can not be boundaries. We have thus proved the following theorem.

THEOREM 7.2.1 $\pi^S_{46} = Z_2\eta^2C[44] \oplus Z_2\eta D[45] \oplus Z_2\eta A[45,1] \oplus Z_2\eta A[45,2]$.

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader
17	51	$\eta^2\gamma_1M_1^{17}$
32	50	$A[32,1](M_1^2M_3+M_2^3)$
38	50	$\eta\sigma A[30]M_1^3\bar{M}_2$, $B[38]M_1^6$
38	52	$2B[38]M_1^4\bar{M}_2$
39	51	$\eta B[38]M_1^3\bar{M}_2$
39	49	$A[39,1]M_1^2M_2$, $A[39,1]M_1^5$, $A[39,3]M_1^2\bar{M}_2$
40	52	$(\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2$, $A[40,1]M_1^6$
40	48	$2C[20]^2M_1\bar{M}_2$
42	48	$2C[42]M_1^3$
42	50	$\eta^2C[20]^2M_1\bar{M}_2$
44	48	$C[44]M_1^2$
45	49	$A[45,1]M_1^2$, $Z_8D[45]M_1^2$
45	51	$8D[45]M_1^3$, $A[45,2]\bar{M}_2$
46	48	$\eta^2C[44]M_1$, $\eta D[45]M_1$, $\eta A[45,1]M_1$, $\eta A[45,2]M_1$

FIGURE 7.2.1: Leaders from Rows 1 to 46 of Degree at Least 48

LEMMA 7.2.2 (a) $\sigma^2A[32,1] = 0$

(b) $\sigma A[39,3] = 0$

PROOF. (a) $\sigma^2 A[32,1] \in A[32,1]\langle\nu,\sigma,\nu\rangle = \langle A[32,1],\nu,\sigma\rangle\nu \subset \nu\cdot\pi^S_{43} = 0$.
(b) Since $A[39,3] \in \langle\nu,B[34],\eta\rangle$, $\sigma A[39,3] \in \sigma\langle\nu,B[34],\eta\rangle$
$= \langle\sigma,\nu,B[34]\rangle\eta = \langle\sigma,\nu,\langle\eta,2,A[32,1]\rangle\rangle\eta \supset \langle\sigma,\nu,\eta,2\rangle\eta A[32,1] = 0$ because $\langle\sigma,\nu,\eta,2\rangle \in \pi^S_{13} = 0$. Note that the four-fold Toda bracket is defined by Theorem 2.2.7(a) because $\langle\sigma,\nu,\eta\rangle = 0$ and $\langle\nu,\eta,2\rangle = 0$. Thus, $\sigma A[39,3] \in \eta(\mathrm{Indet}\ \langle\sigma,\nu,B[34]\rangle) = \{0,\eta\sigma B[38]\}$ since $\eta B[34] = 0$ and $\eta^2\sigma^2 A[30] = 0$. Therefore, $\sigma(A[39,3] + k\eta B[38]) = 0$. Note that $A[39,3]$ is defined as $d^6(B[34]M_1^3)$ and is thus only defined modulo $Z_2\eta B[38]$. Therefore, we can define $A[39,3]$ so that $\sigma A[39,3] = 0$. Alternatively, we shall see that $\sigma B[38] = 4D[45]$, and thus $0 = \eta\sigma B[38] = \sigma A[39,3]$. ∎

There are seven leaders of degree 48 and six leaders of degree 49. By Lemma 3.3.14, if $\eta^2 A[45,1]$ or $\eta^2 A[45,2]$ is nonzero then it must be divisible by two. Let $d^8(2C[20]^2 M_1\bar{M}_2) = B[47]$. Since $A[45,2] = d^6(C[20]^2\bar{M}_2)$, it follows from Theorem 2.4.2 that

$$B[47] \in \langle\eta,2,A[45,2]\rangle. \qquad [7.2]$$

Thus $2B[47] \in 2\langle\eta,2,A[45,2]\rangle = \langle 2,\eta,2\rangle A[45,2] = \eta^2 A[45,2]$. Let $A[47] = d^6(2C[42]\bar{M}_2)$. By Theorem 2.4.4(c),

$$A[47] \in \langle\eta,\nu,2C[42]\rangle. \qquad [7.3.]$$

Thus, $2A[47] \in 2\langle\eta,\nu,2C[42]\rangle = \langle 2,\eta,\nu\rangle 2C[42] = 0$. By 6.18, $2C[44] \in \langle\sigma,\nu,\nu A[30]\rangle$. Thus, $2\nu C[44] \in \nu\langle\sigma,\nu,\nu A[30]\rangle = \langle\nu,\sigma,\nu\rangle\nu A[30]$ $= \sigma^2\nu A[30] = 0$. Therefore, $B[47]$ is the only elements of CokJ_{47} which may not have order two. Thus, $\eta^2 A[45,1] = 0$, and $\eta A[45,1]M_1$ must be a boundary. In addition, $\eta^2 C[44]M_1$ must bound because $\eta^3 C[44] = 4\nu C[44] = \nu\sigma^2 A[30] = 0$. There are only three leaders of degree 49 which do not clearly transgress: $A[39,1]M_1^2M_2$, $A[39,1]M_1^5$ and $A[39,3]M_1^2\bar{M}_2$. If $d^8(A[39,1]M_1^2M_2)$ equals $\eta A[45,1]M_1$ or $\eta^2 C[44]M_1$ then r_{Δ_1} applied to $d^8(A[39,1]M_1^3M_2)$ produces a contradiction because there is no possibility for a hidden differential on $A[39,1]M_1^3M_2$.

Thus, $A[39,1]M_1^2M_2$ transgresses. Therefore, $2C[20]^2M_1\bar{M}_2$, $2C[42]\bar{M}_2$, $C[44]M_1^2$, $\eta A[45,2]M_1$ and $\eta D[45]M_1$ can not bound.

The only possibility for $\eta^2C[44]M_1$ and $\eta A[45,1]M_1$ to be boundaries is $\{d^8(A[39,1]M_1^5), d^8(A[39,3]M_1^2\bar{M}_2)\} = \{\eta^2C[44]M_1, \eta A[45,1]M_1\}$. Now $\sigma A[39,1] \in \sigma\langle\nu A[31],\nu,\eta\rangle = \langle\sigma,\nu A[31],\nu\rangle\eta$ and $\nu\langle\sigma,\nu A[31],\nu\rangle = \langle\nu,\sigma,\nu A[31]\rangle\nu$ $= \langle\nu,\sigma,\nu\rangle\nu A[31] = (\sigma^2)\nu A[31] = 0$. There is only one leader of degree 50 below the 34 row: $A[32,1](M_1^2M_3+M_2^3)$. As we shall see in the proof of Theorem 7.2.4, $A[39,1]M_1^2M_2$ must bound and the only possibility is $d^8(A[32,1](M_1^2M_3+M_2^3))$ $= A[39,1]M_1^2M_2$. Thus, $A[45,1]M_1^2$ can not bound and $\nu A[45,1]$ is nonzero. Therefore, $\langle\sigma,\nu A[31],\nu\rangle \neq A[45,1]$, $\sigma A[39,1] \neq \eta A[45,1]$ and $d^8(A[39,1]M_1^5) \neq \eta A[45,1]M_1$. Thus, $d^8(A[39,1]M_1^5) = \eta^2C[44]M_1$, $d^8(A[39,3]M_1^2\bar{M}_2) = \eta A[45,1]M_1$ and

$$\sigma A[39,1] = \eta^2C[44]. \qquad [7.4]$$

We have thus proved the following theorem.

THEOREM 7.2.3 $\pi_{47}^S = Z_4B[47] \oplus Z_2A[47] \oplus Z_2\eta^2D[45] \oplus Z_2\nu C[44] \oplus Z_{32}\gamma_5$ where $2B[47] = \eta^2A[45,2]$.

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
17	51	$\eta^2\gamma_1M_1^{17}$	44	52	$C[44]M_1^4$
32	50	$A[32,1](M_1^2M_3+M_2^3)$	45	49	$A[45,1]M_1^2$, $Z_8(D[45]M_1^2)$
38	50	$\eta\sigma A[30]M_1^3\bar{M}_2$, $B[38]M_1^6$	45	51	$8D[45]M_1^3$, $A[45,2]\bar{M}_2$
38	52	$2B[38]M_1^4\bar{M}_2$	46	52	$\eta A[45,1]M_1^3$
39	51	$\eta B[38]M_1^3\bar{M}_2$	*46	66	$\eta^2C[44]M_1^7\bar{M}_2$
39	49	$A[39,1]M_1^2M_2$	47	49	$\eta^2D[45]M_1$, $\eta^2A[45,2]M_1$, $A[47]M_1$, $B[47]M_1$
40	52	$(\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2$, $A[40,1]M_1^6$	47	51	$\nu C[44]M_1^2$
42	50	$\eta^2C[20]^2M_1\bar{M}_2$			

FIGURE 7.2.2: Leaders from Rows 1 to 47 of Degree at Least 49

There are nine leaders of degree 49 and four leaders of degree 50. Since $d^6(C[20]^2\bar{M}_2) = A[45,2]$, $d^6(\eta^2 C[20]^2 M_1\bar{M}_2) = \eta^2 A[45,2]M_1$. Note that $\xi = d^{10}(A[39,1]M_1^2M_2) = d^{10}(\sigma A[32,3]M_1^2M_2) \in \langle A[37],\sigma,\nu\rangle$ by Theorem 2.4.2. Assume that ξ is nonzero. Then ξM_1^2 can not bound, $\nu\xi \neq 0$ and $\nu\xi \notin (\eta)$. Thus, $\xi \in \langle\langle A[32,3],\eta,\nu\rangle,\sigma,\nu\rangle = \langle A[32,3],\langle\eta,\nu,\sigma\rangle,\nu\rangle + \langle A[32,3],\eta,\langle\nu,\sigma,\nu\rangle\rangle =$ $\langle A[32,3],0,\nu\rangle + \langle A[32,3],\eta,\sigma^2\rangle = \langle A[32,3],\eta,\sigma^2\rangle$ since ξ is not divisible by ν. Since $A[32,3] = d^{12}(\sigma^3 M_1^3\bar{M}_2) = d^{12}[(\sigma^2M_1^3)\sigma\bar{M}_2]$, Theorem 2.4.6(c) implies that

$$A[32,3] \in \langle A[19],\sigma,\nu,\eta\rangle. \qquad [7.5]$$

Then $\nu\xi \in \nu\langle A[32,3],\eta,\sigma^2\rangle = A[32,3]\langle\eta,\sigma^2,\nu\rangle = A[32,3]A[19]$
$\in A[19]\langle A[19],\sigma,\nu,\eta\rangle \subset \langle A[19]^2,\sigma,\nu,\eta\rangle = k\langle\eta\sigma A[30],\sigma,\nu,\eta\rangle$ since $\eta A[19]^2 = 0$;
$= k[d^8(\sigma^2 A[30]M_1\bar{M}_2) + \text{Indet}\ \langle\eta\sigma A[30],\sigma,\nu,\eta\rangle]$ by Theorem 2.4.6(c);
$= k[d^8(4C[44]M_1\bar{M}_2) + \text{Indet}\ \langle\eta\sigma A[30],\sigma,\nu,\eta\rangle]$;
$= \text{Indet}\ \langle\eta\sigma A[30],\sigma,\nu,\eta\rangle]$ since $4C[44]M_1\bar{M}_2$ is a d^8-boundary.
By Theorem 2.3.1(b), there is $\eta\zeta \in \pi^S_{46}$ such that $\nu\xi \in \langle\eta\sigma A[30],\beta_1,\eta\rangle + \langle\eta\zeta,\nu,\eta\rangle$. Then $\nu\xi \in \sigma A[30]\langle\eta,\beta_1,\eta\rangle + \nu^2\zeta = \sigma A[30](\nu\beta_1) + \nu^2\zeta = \nu^2\zeta$. Therefore, ν divides ξ, a contradiction. Thus, $A[39,1]M_1^2M_2$ must be a boundary. The only possibility is: $d^8(A[32,1](M_1^2M_3+M_2^3)) = A[39,1]M_1^2M_2$. Since $d^4(4D[45]M_1^2) =$ $4\nu D[45] = \eta^3 D[45] = 0$ in E^4, $4D[45]M_1^2$ must bound from below the 40 row. There is only one possibility: $d^8(B[38]M_1^6) = 4D[45]M_1^2$ and

$$\sigma B[38] = 4D[45]. \qquad [7.6]$$

Now $\eta^3 D[45] = 4\nu D[45] = \nu\sigma B[38] = 0$, and $\eta^2 D[45]M_1$ must be a boundary. There is only one possibility: $d^8(\eta\sigma A[30]M_1^3\bar{M}_2) = \eta^2 D[45]M_1$. We have thus proved the following theorem.

THEOREM 7.2.4 $\pi^S_{48} = Z_4\nu D[45] \oplus Z_2\nu A[45,1] \oplus Z_2\eta A[47] \oplus Z_2\eta B[47] \oplus Z_2\eta\gamma_5$.

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
17	51	$\eta^2\gamma_1M_1^{17}$	45	51	$8D[45]M_1^3$, $A[45,2]\bar{M}_2$
*32	62	$A[32,1]M_1^8\bar{M}_3$	46	52	$\eta A[45,1]M_1^3$
*38	62	$B[38]M_1^2\bar{M}_2\bar{M}_3$	*47	53	$\eta^2D[45]M_1^3$, $B[47]\bar{M}_2$
38	52	$2B[38]M_1^4\bar{M}_2$	*47	55	$\eta^2A[45,2]M_1\bar{M}_2$
39	51	$\eta B[38]M_1^3\bar{M}_2$, $A[39,1]M_1^3M_2$	*47	57	$A[47]M_1^2\bar{M}_2$
40	52	$(\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2$,	47	51	$\nu C[44]M_1^2$
		$A[40,1]M_1^6$	48	52	$\nu A[45,1]M_1^2$, $\nu D[45]M_1^2$
44	52	$C[44]M_1^4$	*48	54	$2\nu D[45]M_1^3$
*45	59	$2D[45]M_1\langle M_2^2\rangle$	48	50	$\eta A[47]M_1$, $\eta B[47]M_1$
*45	53	$4D[45]M_1M_2$			

FIGURE 7.2.3: Leaders from Rows 1 to 48 of Degree at Least 50

There are two leaders of degree 50 and six leaders of degree 51. Note that $B[47] \in \langle\eta,2,A[45,2]\rangle = \langle\eta,2,\langle\eta,\nu,C[20]^2\rangle\rangle$ $= \langle\eta,\langle 2,\eta,\nu\rangle,C[20]^2\rangle + \langle\langle\eta,2,\eta\rangle,\nu,C[20]^2\rangle = \langle\eta,0,C[20]^2\rangle + \langle 2\nu,\nu,C[20]^2\rangle$. Since $\sigma C[20] = 0$, $B[47] \in \langle 2\nu,\nu,C[20]^2\rangle$. Thus, $\eta B[47] \in \eta\langle 2\nu,\nu,C[20]^2\rangle$ $= \langle\eta,2\nu,\nu\rangle C[20]^2 = A[8]C[20]^2 = \langle\nu,\eta,\nu\rangle C[20]^2 = \nu\langle\eta,\nu,C[20]^2\rangle = \nu A[45,2]$ and

$$\eta B[47] = \nu A[45,2]. \qquad [7.7]$$

Therefore, $d^4(A[45,2]\bar{M}_2) = \eta B[47]M_1$. If $\eta^2A[47]$ were nonzero, it would be divisible by two. Since there is no other possibility for a nonzero element of CokJ_{49}, $\eta^2A[47] = 0$ and $\eta A[47]M_1$ must bound. Since $\eta^2\gamma_1M_1^{17} \in \text{Image } r_{\Delta_1}$, it can not hit $\eta A[47]M_1$. Let X represents $A[39,1]M_1^3M_2$ as an element of E^8 with $\partial X = (A[39,1] \wedge \sigma \wedge \mu_2) \cup (A[39,1] \wedge B_{\sigma\nu})$. Since $\sigma A[39,1] = \eta^2C[44]$, we can represent $A[39,1]M_1^3M_2$ by $X \cup (\eta C[44] \wedge \mu_{01})$ which transgresses. There remains only one possibility: $d^{10}(\eta B[38]M_1^3\bar{M}_2) = \eta A[47]M_1$. We have thus proved the following theorem.

THEOREM 7.2.5 $\pi_{49}^S = Z_2\alpha_6 \oplus Z_2\eta^2\gamma_5$.

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
17	51	$\eta^2\gamma_1 M_1^{17}$	45	51	$8D[45]M_1^3$
38	52	$2B[38]M_1^4\overline{M}_2$	46	52	$\eta A[45,1]M_1^3$
39	51	$A[39,1]M_1^3\overline{M}_2$	47	51	$\nu C[44]M_1^2$
40	52	$(\eta A[39,3]+\eta\sigma A[32,1])M_1^3\overline{M}_2$,	48	52	$\nu D[45]M_1^2$, $\nu A[45,1]M_1^2$
		$A[40,1]M_1^6$	*48	54	$\eta A[47]M_1^3$
44	52	$C[44]\langle M_1^4\rangle$	*48	56	$\eta B[47]M_1\overline{M}_2$

FIGURE 7.2.4: Leaders from Rows 1 to 49 of Degree at Least 51

We have four leaders of degree 51 and seven leaders of degree 52. Since $d^{12}(B[38]M_1^4\overline{M}_2) = 4D[45]M_1^3$, it follows that $d^{12}(2B[38]M_1^4\overline{M}_2) = 8D[45]M_1^3$. Clearly $A[50,1] = d^{34}(\eta^2\gamma_1 M_1^{17})$ and $A[50,2] = d^{12}(A[39,1]M_1^3\overline{M}_2)$ are nonzero. Recall that $C[44] = d^{14}(\eta A[30]M_1M_2^2)$ and note that $\eta A[30]M_1M_2^2$ is represented by $\mathcal{R} = (\mu_{02}\wedge \eta \wedge A[30] \wedge \mu_1) \cup (\mu_{02}\wedge B_{\eta A[30]\eta}) \cup (\mu_4 \wedge B_{\nu\eta}\wedge A[30] \wedge \mu_1)$ $\cup (B_{\langle\sigma,\nu,\eta\rangle}\wedge A[30] \wedge \mu_1) \cup (\mu_4\wedge A[31] \wedge \mu_{01}) \cup (\mu_4\wedge B_{A[31]\eta} \wedge \mu_2)$ $\cup (\mu_4 \wedge B_{\langle\nu,\eta,A[30]\eta\rangle\cup\langle A[31],\eta,\nu\rangle}) \cup (B_{\sigma A[31]}\wedge \mu_{01}) \cup (B_{\langle\sigma,A[31],\eta\rangle}\wedge \mu_1)$. Observe that $\mathcal{R}$, without the last term, shows that $0 = d^{12}(\eta A[30]M_1M_2^2)$ $= \langle\sigma,A[31],\eta\rangle M_1$. Thus, $\langle\sigma,A[31],\eta\rangle M_1$ is zero in E^{12}. Therefore, $\langle\sigma,A[31],\eta\rangle \in Z_2\eta A[39,2] \oplus Z_2\eta A[39,3] \oplus Z_2\eta\gamma_4 \subset \text{Indet}\ \langle\sigma,A[31],\eta\rangle$, and $0 \in \langle\sigma,A[31],\eta\rangle$. Now $\partial\mathcal{R} = (B_{\sigma\nu}\wedge B_{\eta A[30]\eta}) \cup (B_{\langle\sigma,\nu,\eta\rangle} \wedge A[30] \wedge \eta)$ $\cup (\sigma \wedge B_{\langle\nu,\eta,A[30]\eta\rangle\cup\langle A[31],\eta,\nu\rangle}) \cup (B_{\sigma A[31]}\wedge B_{\eta\nu}) \cup (B_{\langle\sigma,A[31],\eta\rangle}\wedge \nu)$ which represents an element of $\left\langle \sigma, \left[A[31],\nu\right], \begin{bmatrix} \eta & 0 \\ 0 & \eta \end{bmatrix}, \begin{bmatrix} \nu \\ A[30]\eta \end{bmatrix} \right\rangle$. Thus,

$$C[44] \in \left\langle \sigma, \left[A[31],\nu\right], \begin{bmatrix} \eta & 0 \\ 0 & \eta \end{bmatrix}, \begin{bmatrix} \nu \\ \eta A[30] \end{bmatrix} \right\rangle. \qquad [7.8]$$

Let $\mathfrak{C}$ denote the mapping cone of $A[40,1]$ with $\rho: S \longrightarrow \mathfrak{C}$ the canonical map. In the Atiyah-Hirzebruch spectral sequence for $\mathfrak{C}_*BP$, $\rho(\eta A[30]M_1^3M_2^2)$ survives to

E^{12}. Since $\eta A[40,1] \neq 0$ and $\eta^2 A[40,1] = 0$, $\rho: \pi_{42}^S \longrightarrow \mathfrak{C}_{42}$ is an isomorphism. The only possibility for $d^{12}(\rho(\eta A[30]M_1^3M_2^2))$ to be nonzero is $d^{12}(\rho(\eta A[30]M_1^3M_2^2)) = \rho(2C[42]M_1^3)$ which implies that $d^{12}(\eta A[30]M_1M_2^2) = 2C[42]M_1$ which contradicts that $\eta A[30]M_1M_2^2$ survives to E^{14}. Thus, $\rho(\eta A[30]M_1^3M_2^2)$ survives to E^{14} and $r_{2\Delta_1}$ shows that $d^{14}(\rho(\eta A[30]M_1^3M_2^2)) = \rho(C[44]M_1^2)$. Therefore $\rho(\nu C[44]) = 0$ and $A[40,1]$ divides $\nu C[44]$. The only possibility is

$$\sigma A[40,1] = \nu C[44]. \qquad [7.9]$$

Thus, $d^8(A[40,1]M_1^6) = \nu C[44]M_1^2$. Clearly $2A[50,2] = 0$. We shall see in Section 7.3 that $\eta A[50,2] \neq 0$. Therefore $2A[50,1] = 0$. We have thus proved the following theorem.

THEOREM 7.2.6 $\pi_{50}^S = Z_2A[50,1] \oplus Z_2A[50,2] \oplus Z_2\eta\alpha_6$.

The computations in Section 6 show that we have the following leaders. Since this is the final table of leaders of this section, we include the leaders of all degrees.

Row	Degree	Leader	Row	Degree	Leader
9	63	$\eta^2\sigma M_1^{21}M_2^2$	40	58	$A[40,2]M_1^6\bar{M}_2$
11	57	$4\beta_1(M_1^7\bar{M}_2^3\bar{M}_3+M_1^{10}\bar{M}_2^2\bar{M}_3+M_1^{14}\bar{M}_2^3)$	40	52	$(\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2$
17	59	$\eta^2\gamma_1M_1^{15}\bar{M}_2^2$	44	52	$C[44]<M_1^4>$
18	64	$4C[18]M_1^7\bar{M}_2M_2^2\bar{M}_3$	44	56	$2C[44]M_1^6$
19	55	$\beta_2M_1^{18}$	44	58	$4C[44]M_1^7$
21	53	$\nu C[18]M_1^6\bar{M}_2\bar{M}_3$	45	59	$2D[45]M_1<M_2^2>$
22	62	$\nu A[19]M_1^7M_2^2<M_3>$	45	53	$4D[45]M_1M_2$
23	63	$\gamma_2M_1^{20}$	46	52	$\eta A[45,1]M_1^3$
24	60	$\eta A[23]M_1^{15}\bar{M}_2$	46	66	$\eta^2C[44]M_1^7\bar{M}_2$
30	60	$A[30]<M_4>$	47	55	$\nu C[44]M_1M_2, \eta^2A[45,2]M_1\bar{M}_2$

32	62	$A[32,1]M_1^8\bar{M}_3$
34	64	$2B[34]M_1^5\bar{M}_2\bar{M}_3$
36	54	$A[36]M_1^2\langle M_3\rangle$
38	62	$B[38]M_1^2\bar{M}_2\bar{M}_3$
39	53	$\sigma A[32,1]M_1^4\bar{M}_2$
39	61	$A[39,1]M_1M_2M_3$
40	64	$\eta\sigma A[32,1]M_1^5\bar{M}_3$
40	60	$A[40,1]\bar{M}_2\bar{M}_3$
47	53	$\eta^2D[45]M_1^3$, $B[47]\bar{M}_2$
47	57	$A[47]M_1^2\bar{M}_2$
48	52	$\nu D[45]M_1^2$, $\nu A[45,1]M_1^2$
48	54	$2\nu D[45]M_1^3$, $\eta A[47]M_1^3$
48	56	$\eta B[47]M_1\bar{M}_2$
50	56	$A[50,1]M_1^3$, $A[50,1]M_2$
50	52	$A[50,2]M_1$

FIGURE 7.2.5: Leaders from Rows 1 to 50 of Degree at Least 52

3. Computation of π_N^S, $51 \le N \le 55$.

We continue the computations of Section 2. In the tables of leaders below, all leaders of degree greater than 57 will have an asterisk at the left. They will be omitted from all tables of leaders in this section except for the last one.

From Figure 7.2.5, we see that there are six leaders of degree 52 and five leaders of degree 53. Observe that η, ν times $4D[45]$ and η, ν times $\eta^2D[45]$ and ν times $B[47]$ are zero. Therefore, $4D[45]M_1M_2$, $\eta^2D[45]M_1^3$ and $B[47]\bar{M}_2$ must transgress. Assume that $\nu A[45,1]M_1^2$ is not a boundary. Then $\nu^2A[45,1]$ is nonzero and not divisible by η. In particular, there is no way for $\eta A[50,2]M_1$ to bound. Therefore, $\eta^2A[50,2]$ is nonzero. By Lemma 3.3.14, $\eta^2A[50,2]$ must be divisible by two. This is impossible because all the other elements of π_{52}^S have order two. Thus, $\nu A[45,1]M_1^2$ must be a boundary.

We show that $(\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2$ must be a boundary. Assume that $\xi = d^{12}((\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2)$ is nonzero. The only leader of degree 54 or 56 below the 40 row is $A[36]M_1^2\bar{M}_3$ which we shall see must bound $4D[45]M_1M_2$. Thus, ξM_1 and ξM_1^2 can not bound. Therefore $\eta\xi$, $\nu\xi$ are nonzero and $\nu\xi$ is not divisible by η. If $\eta^2\xi$ is nonzero then $\eta^2\xi M_1$ must be a boundary because $\eta^3\xi = 4\nu\xi = 0$. There are only four leaders of degree 56 below row 52: $A[50,1]M_2$, $C[44]\langle M_2^2\rangle$, $2C[44]\langle M_2^2\rangle$ and $\eta B[47]M_1\bar{M}_2$. Now $A[50,1]M_2$ transgresses. and $d^6(\eta B[47]M_1\bar{M}_2) = \eta A[52,2]M_1$. Since $\langle M_2^2\rangle$ has a representative with boundary $(\sigma \wedge \mu_2) \cup B_{\sigma\nu}$, $C[44]\langle M_2^2\rangle$ hits $\sigma C[44]M_1^2$ if $\sigma C[44] \neq 0$ or transgresses if $\sigma C[44] = 0$. Then $2C[44]\langle M_2^2\rangle$ hits $2\sigma C[44]M_1^2$ if $2\sigma C[44] \neq 0$ or transgresses to an element of $\langle 2,\sigma C[44],\nu\rangle$ if $2\sigma C[44] = 0$. Thus, $\eta^2\xi M_1$ can not be a boundary, $\eta^2\xi = 0$ and $\eta\xi M_1$ must be a boundary. Assume that $d^6(\eta^2 A[45,2]M_1\bar{M}_2) = \eta\xi M_1$. Then $\eta\xi \in \langle\nu,\eta,\eta^2A[45,2]\rangle \subset \langle\nu,\eta^3,A[45,2]\rangle = \langle\nu,4\nu,A[45,2]\rangle \supset 2\langle\nu,2\nu,A[45,2]\rangle = 0$ and $\eta\xi \in \text{Indet}\,\langle\nu,4\nu,A[45,2]\rangle = \nu\cdot\pi_{49}^S + A[45,2]\cdot\pi_7^S = \sigma A[45,2] \in \sigma\langle C[20]^2,\nu,\eta\rangle = \langle\sigma,C[20]^2,\nu\rangle\eta$ and $\xi \in \langle\sigma,C[20]^2,\nu\rangle$. Then $\nu\xi \in \nu\langle\sigma,C[20]^2,\nu\rangle = \langle\nu,\sigma,C[20]^2\rangle\nu$, $\xi \in \langle\nu,\sigma,C[20]^2\rangle$ and $\eta\xi \in \eta\langle\nu,\sigma,C[20]^2\rangle = \langle\eta,\nu,\sigma\rangle C[20]^2 = 0$, a contradicition. Thus, $d^6(\eta^2A[45,2]M_1\bar{M}_2)$ can not equal $\eta\xi M_1$. If $\beta_2M_1^{18}$ hits $\eta\xi M_1$ then there is a hidden differential on $\beta_2M_1^{19}$ which can only hit $2C[44]M_1^6$. However, there is no possibility for a hidden differential on $\beta_2M_1^{19}M_2$, $r_{3\Delta_1}(\beta_2M_1^{19}M_2) = \beta_2M_1^{19}$ and $2C[44]M_1^6 \notin \text{Image}\; r_{3\Delta_1}$. There remains only one possibility: $d^6(\nu C[44]M_1M_2) = \eta\xi M_1$. Then $\eta\xi \in \langle\nu,\eta,\nu C[44]\rangle = \langle\nu,\eta,\nu\rangle C[44] = A[8]C[44] \in \langle\eta,\nu,2\nu\rangle C[44] = \eta\langle\nu,2\nu,C[44]\rangle$ and $\xi \in \langle\nu,2\nu,C[44]\rangle$. Thus, $\nu\xi \in \nu\langle\nu,2\nu,C[44]\rangle \subset \langle\nu^2,2\nu,C[44]\rangle \supset \langle\nu^2,2,\nu C[44]\rangle \supset \langle\langle\eta,\nu,\eta\rangle,2,\nu C[44]\rangle \supset \eta\langle\nu,\eta,2,\nu C[44]\rangle$. This Toda bracket is defined by Theorem 2.2.7(a) because $\langle\nu,\eta,2\rangle = 0$ and $\langle\eta,2,\nu C[44]\rangle$ contains an element of $\text{CokJ}_{49} = 0$. Since $\nu\xi$ is not divisible by η, $\nu\xi \in \text{Indet}\,\langle\nu^2,2\nu,C[44]\rangle = \nu^2\cdot\pi_{48}^S + \sigma\cdot\pi_{47}^S$. Thus, $\nu\xi \in \{\sigma A[47],\sigma B[47]\}$, and $\nu\xi M_1^2 \in \{d^8(A[47]\langle M_2^2\rangle), d^8(B[47]\langle M_2^2\rangle)\}$. This contradicts the fact that both $A[47]\langle M_2^2\rangle$ and $B[47]\langle M_2^2\rangle$ are boundaries. There remains no possiblity for $\eta\xi M_1$ to bound. This contradicition implies that ξ must be zero, and $(\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2$ must bound from below the 37 row. There is only one possibility: $d^{20}(\nu C[18]M_1^6\bar{M}_2\bar{M}_3) = (\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2$.

Now the only possibility for $\nu A[45,1]M_1^2$ to be a boundary is $d^{10}(\sigma A[32,1]M_1^4\bar{M}_2)$ $= \nu A[45,1]M_1^2$. Then $\eta A[45,1]M_1^3$, $C[44]M_1^4$, $\nu D[45]M_1^2$ and $A[50,2]M_1$ can not be boundaries. Therefore $\lambda = d^6(\eta A[45,1]M_1^3)$, $\sigma C[44]$, $\nu^2 D[45]$ and $\eta A[50,2]$ are nonzero. We shall see that $d^8(2C[44]M_1^6) = \lambda M_1^2$. Therefore, $2\sigma C[44] = \lambda$ and

$$2\sigma C[44] \in \langle \eta, \eta A[45,1], \nu \rangle. \qquad [7.10]$$

If $2\lambda = k\eta A[50,2] + \nu^2 D[45]$ then $d^8(4C[44]M_1^7) = \nu^2 D[45]M_1^3$ since $\eta A[50,2]M_1^3$ $= d^4(\nu A[45,1]M_1^2 M_2)$. This contradicts $d^4(\nu^2 D[45]M_1^3) = \eta A[8]D[45]M_1 \neq 0$. Thus, $2\lambda = k\eta A[50,2]$. Assume that $k = 1$. We will see that $\nu^2 A[50,2]$ is nonzero and not divisible by two. Then $\nu^2 A[50,2] \in \langle \eta, \nu, \eta \rangle A[50,2] \subset \langle \eta, \nu, \eta A[50,2] \rangle$ $= \langle \eta, \nu, 4\sigma C[44] \rangle \supset 4C[44]\langle \eta, \nu, \sigma \rangle = 0$. Thus, $\nu^2 A[50,2] \in$ Indet $\langle \eta, \nu, 4\sigma C[44] \rangle =$ $= (\eta)$, a contradiction. Therefore, $k = 0$ and $4\sigma C[44] = 0$. We have thus proved the following theorem.

THEOREM 7.3.1 $\pi_{51}^S = Z_4 \sigma C[44] \oplus Z_2 \eta A[50,2] \oplus Z_2 \nu^2 D[45] \oplus Z_8 \beta_6$

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
11	57	$4\beta_1(M_1^7\bar{M}_2^3\bar{M}_3 + M_1^{10}\bar{M}_2^2\bar{M}_3 + M_1^{14}\bar{M}_2^3)$	47	57	$A[47]M_1^2\bar{M}_2$
19	55	$\beta_2 M_1^{18}$	48	54	$2\nu D[45]M_1^3$, $\nu A[45,1]M_1^3$, $\eta A[47]M_1^3$
36	54	$A[36]M_1^2\bar{M}_3$			
*39	59	$\sigma A[32,1]M_1^4 M_2^2$	48	56	$\eta B[47]M_1\bar{M}_2$
*40	66	$\eta A[39,3]M_1^3\bar{M}_2\langle M_3\rangle$	50	56	$A[50,1]M_1^3$, $A[50,1]M_2$
44	56	$2C[44]M_1^6$	50	54	$A[50,2]M_1^2$
45	53	$4D[45]M_1 M_2$	51	53	$\sigma C[44]M_1$, $\eta A[50,2]M_1$
47	55	$\nu C[44]M_1 M_2$, $\eta^2 A[45,2]M_1\bar{M}_2$	51	55	$2\sigma C[44]M_1^2$
47	53	$\eta^2 D[45]M_1^3$, $B[47]\bar{M}_2$	51	57	$\nu^2 D[45]M_1^3$

FIGURE 7.3.1: Leaders from Rows 1 to 51 of Degree at Least 53

There are five leaders of degree 53 and five leaders of degree 54. Let $<\mu_{001}>$ represent $<M_3>$ such that $\partial <\mu_{001}> = (\sigma \wedge \bar{\mu}_{01}) \cup (B_{\sigma\nu} \wedge \mu_1) \cup B_{<\nu,\sigma,\eta>}$. Since $\sigma A[36] = 0$ and $\nu A[36] = \eta B[38]$, $A[36]M_1^2<M_3>$ can be represented by $[\mu_2 \wedge B_{A[36]\sigma} \wedge \bar{\mu}_{01}] \cup [((\mu_2 \wedge A[36]) \cup (\mu_1 \wedge B[38])) \wedge <\mu_{001}>]$ which has boundary $(\mu_1 \wedge B[38]] \wedge \sigma \wedge \bar{\mu}_{01})$ union elements of filtration degree six. Thus, $d^{10}(A[36]M_1^2<M_3>) = \sigma B[38]M_1\bar{M}_2 = 4D[45]M_1\bar{M}_2$. Let $A[52,1] = d^6(\eta^2 D[45]M_1^3)$. Then

$$A[52,1] \in <\eta, \eta^2 D[45], \nu>. \qquad [7.11]$$

Thus, $2A[52,1] \in 2<\eta,\eta^2 D[45],\nu> = <2,\eta,\eta^2 D[45]>\nu \in \nu \cdot \pi_{49}^S = 0$. Let $A[52,2] = d^6(B[47]\bar{M}_2)$. Then

$$A[52,2] \in <\eta, \nu, B[47]>. \qquad [7.12]$$

Thus, $2A[52,2] \in 2<\eta,\nu,B[47]> = <2,\eta,\nu>B[47] = 0$. By Lemma 3.3.14, $\eta^2 A[50,2]$ must be divisible by two. However, there is no possibility for an element of order four in π_{52}^S. Thus, $\eta^2 A[50,2] = 0$, and $\eta A[50,2]M_1$ must bound. Clearly $A[50,2]M_1^2$, $\eta A[47]M_1^3$ and $2\nu D[45]M_1^3$ transgress. Thus, $d^4(\nu A[45,1]M_1^3) = \eta A[50,2]M_1$ and

$$\nu^2 A[45,1] = \eta A[50,2]. \qquad [7.13]$$

Now $\eta\sigma C[44]$, $A[52,1]$ and $A[52,2]$ are nonzero. Observe that $A[52,1]$ can not be divisible by σ because we shall see that $\nu A[52,1] \neq 0$. If σ times $A[45,1]$, $A[45,2]$ or $D[45]$ were $A[52,2]$ then $A[52,2]\bar{M}_2$ would be $d^8(A[45,1]<M_3>)$, $d^8(A[45,2]<M_3>)$ or $d^8(D[45]<M_3>)$. This would contradict that $A[45,1]<M_3>$, $A[45,2]<M_3>$ and $D[45]<M_3>$ are boundaries. Note that $A[45,1]$, $A[45,2]$ and $D[45]$ are only defined modulo $\eta C[44]$. Thus, we can define $A[45,1]$, $A[45,2]$ and $D[45]$ such that

$$\sigma A[45,1] = \sigma A[45,2] = \sigma D[45] = 0. \qquad [7.14]$$

We have thus proved the following theorem.

THEOREM 7.3.2 $\pi_{52}^S = Z_2 A[52,1] \oplus Z_2 A[52,2] \oplus Z_2 \eta\sigma C[44]$

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
11	57	$4\beta_1(M_1^7\bar{M}_2^3\bar{M}_3+M_1^{10}\bar{M}_2^2\bar{M}_3+M_1^{14}\bar{M}_2^3)$	50	56	$A[50,1]M_1^3$, $A[50,1]M_2$
19	55	$\beta_2M_1^{18}$	50	54	$A[50,2]M_1^2$
*36	60	$A[36]M_1^6\bar{M}_2^2$	*51	59	$\sigma C[44]M_1^4$
44	56	$2C[44]M_1^6$	51	55	$2\sigma C[44]M_1^2$
45	57	$4D[45]M_1^3M_2$	*51	63	$\eta A[50,2]M_1^3\bar{M}_2$
47	55	$\nu C[44]M_1M_2$, $\eta^2A[45,2]M_1\bar{M}_2$	51	57	$\nu^2D[45M_1^3$
47	57	$A[47]M_1^2\bar{M}_2$	52	54	$\eta\sigma C[44]M_1$, $A[52,2]M_1$
48	54	$2\nu D[45]M_1^3$, $\eta A[47]M_1^3$	52	56	$A[52,1]M_1^2$
48	56	$\eta B[47]M_1\bar{M}_2$			

FIGURE 7.3.2: Leaders from Rows 1 to 52 of Degree at Least 54

There are five leaders of degree 54 and five leaders of degree 55. Clearly $\sigma C[44]M_1^2$ and $2\sigma C[44]M_1^2$ transgress. We show that $\eta\sigma C[44]M_1$ must be a boundary. Observe that $\eta^3\sigma C[44] = 4\nu\sigma C[44] = 0$. If $\eta^2\sigma C[44] \neq 0$ then $\eta^2\sigma C[44]M_1$ must be a boundary. There are only three leaders of degree 56 below row 52: $A[50,1]M_2$, $2C[44]<M_2^2>$ and $\eta B[47]M_1\bar{M}_2$. Now $A[50,1]M_2$ transgresses, $d^8(C[44]<M_2^2>) = 2\sigma C[44]M_1^2$ and $d^6(\eta B[47]M_1\bar{M}_2) = \eta A[52,2]M_1$ since $A[52,2] = d^6(B[47]\bar{M}_2)$. (If $\eta A[52,2] = 0$ then $\eta B[47]M_1\bar{M}_2$ transgresses.) Thus, $\eta^2\sigma C[44]M_1$ can not be a boundary, $\eta^2\sigma C[44] = 0$ and $\eta\sigma C[44]M_1$ must be a boundary.

Assume that $\xi = d^6(\eta A[47]M_1^3)$ is nonzero. By Theorem 2.4.2, $\xi \in <\eta^2, A[47], \nu>$

$= <<2,\eta,2>, A[47], \nu> = <2,<\eta,2,A[47]>,\nu> + <2,\eta,<2.A[47],\nu>>$

$= <2,0,\nu> + <2,\eta,<\nu,A[47],2>>$ since $CokJ_{49} = 0$;

$\equiv <2,<\eta,\nu,A[47]>,2> + <<2,\eta,\nu>,A[47],2>$ modulo $(2,\nu)$;

$\equiv k\eta<\eta,\nu,A[47]> + <0,A[47],2>$ modulo $(2,\nu)$;

$\subset (2,\eta,\nu)$. Since ξ is nonzero in E^6, ξ must divisible by two. However, there is no possibility for ξ to be divisible by two. Thus, $\xi = 0$ and $\eta A[47]M_1^3$ must bound. If $d^{30}(\beta_2 M_1^{18}) = \eta A[47]M_1^3$ then $d^{30}(\beta_2 M_1^{18} M_2) = \eta A[47]M_1^3\bar{M}_2$. Since $r_{3\Delta_1}(\beta_2 M_1^{18} M_2) = \beta_2 M_1^{18}$, $r_{3\Delta_1}(\eta A[47]M_1^3\bar{M}_2) = \eta A[47]M_1^3$ not $\eta A[47]\bar{M}_2$, a contradiction. Therefore the only way for $\eta A[47]M_1^3$ to bound is by a hidden differential which replaces a tentative differential which originates below the 39 row and lands above the 48 row. The only possibility is $d^{32}(\eta^2\gamma_1 M_1^{19}) = \eta A[47]M_1^3$.

Now $A[50,1]M_1^2$ becomes a new leader of degree 54 which can not bound since there are no leaders of degree 55 below the 17 row. Since $A[50,2]M_1^2$ can only bound from below the 39 row, only $\beta_2 M_1^{18}$ could hit it. In that case we can argue as above to show that $r_{3\Delta_1}(A[50,2]M_1^2 M_2) = A[50,2]M_1^2$, a contradicition. Thus, $A[50,2]M_1^2$ can not bound. If $\beta_2 M_1^{18}$ hits $\eta\sigma C[44]M_1$, $A[52,2]M_1$ or $2\nu D[45]M_1^3$ then there must be a hidden differential on $\beta_2 M_1^{16}\bar{M}_2$ which lands below the 52 row. There is no possibility for such a hidden differential because $A[50,1]M_2$ can only bound from below the 17 row. Thus, $\beta_2 M_1^{18}$ must transgress. Assume that $d^6(\eta^2 A[45,2]M_1\bar{M}_2) = \eta\sigma C[44]M_1$. Then $\eta\sigma C[44] \in \langle\nu,\eta,\eta^2 A[45,2]\rangle = \langle\nu,\eta,2B[47]\rangle \supset \langle\nu,\eta,2\rangle B[47] = 0$, and $\eta\sigma C[44]$ is divisible by ν, a contradiction. There remains only one possibility: $d^6(\nu C[44]M_1 M_2) = \eta\sigma C[44]M_1$. Now $\eta A[52,2]$, $\nu A[50,1]$, $\nu A[50,2]$ and $d^6(2\nu D[45]M_1^3) = A[8]D[45]$ must be nonzero. We have thus proved the following theorem.

THEOREM 7.3.3 $\pi_{53}^S = Z_2 A[8]D[45] \oplus Z_2 \nu A[50,1] \oplus Z_2 \nu A[50,2] \oplus Z_2 \eta A[52,2]$

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
11	57	$4\beta_1(M_1^7\bar{M}_2^3\bar{M}_3+M_1^{10}\bar{M}_2^2\bar{M}_3+M_1^{14}\bar{M}_2^3)$	50	56	$A[50,1]M_2$
*17	59	$\eta^2\gamma_1M_1^{15}\bar{M}_2^2$	51	55	$2\sigma C[44]M_1^2$
19	55	$\beta_2M_1^{18}$	51	57	$\nu^2D[45]M_1^3$
44	56	$2C[44]M_1^6$	52	56	$A[52,1]M_1^2$
45	57	$4D[45]M_1^3M_2$	*52	58	$A[52,2]\bar{M}_2$
47	55	$\eta^2A[45,2]M_1\bar{M}_2$ or $\nu C[44]M_1M_2$	*52	60	$\eta B[51]M_1\bar{M}_2$
*47	65	$\nu C[44]M_1^3\langle M_2^2\rangle$	53	55	$\eta A[52,2]M_1$, $A[8]D[45]M_1$
47	57	$A[47]M_1^2\bar{M}_2$	53	59	$\nu A[50,1]M_1^3$
48	56	$\eta B[47]M_1\bar{M}_2$	53	57	$\nu A[50,2]M_1^2$

FIGURE 7.3.3: Leaders from Rows 1 to 53 of Degree at Least 55

There are four leaders of degree 55 and four leaders of degree 56. Clearly $A[52,1]M_1^2$ and $A[50,1]M_2$ transgress. Since $A[52,2] = d^6(B[47]\bar{M}_2)$, $\eta A[52,2]M_1 = d^6(\eta B[47]M_1\bar{M}_2)$. Clearly $d^8(2C[44]\langle M_2^2\rangle) = 2\sigma C[44]M_1^2$. Now $A[54,1] = d^{36}(\beta_2M_1^{18})$, $A[54,2] = d^8(\eta^2A[45,2]M_1\bar{M}_2)$ and $\eta A[8]D[45] = d^2(A[8]D[45]M_1)$ must be nonzero. By Theorem 2.4.5(a),

$$A[54,2] \in \langle\nu,\eta,\eta^2A[45,2],\eta\rangle. \qquad [7.15]$$

Note that $d^6(B[47]\bar{M}_2) = A[52,2]$ and $d^8(2B[47]M_1\bar{M}_2) = d^8(\eta^2A[45,2]M_1\bar{M}_2) = A[54,2]$. By Theorem 2.4.2,

$$A[54,2] \in \langle\eta,2,A[52,2]\rangle. \qquad [7.16]$$

Therefore, $2A[54,2] \in 2\langle\eta,2,A[52,2]\rangle = \langle 2,\eta,2,\rangle A[52,2] = \eta^2A[52,2] = 0$.

We show that $2A[54,1]$ can only be a multiple of $\eta A[8]D[45]$. Assume that $2A[54,1] = A[54,2] + \lambda\eta A[8]D[45]$. Then $d^{36}(2\beta_2M_1^{16}\bar{M}_2) = A[54,2]M_1$, $\nu A[54,1] = 0$ and $A[54,1]\bar{M}_2$ will transgress to define an element $\xi \in \pi_{59}^S$ such that $2\xi = d^6(A[54,2]\bar{M}_2) = A[59,1]$. By Theorem 2.4.4(c), $\xi \in \langle\eta,\nu,A[54,1]\rangle$ and $2\xi \in 2\langle\eta,\nu,A[54,1]\rangle = \langle 2,\eta,\nu\rangle A[54,1] = 0$, a contradiction. Thus, $2A[54,1] \neq A[54,2]$. Thus, $2A[54,1]$

$= \lambda\eta A[8]D[45]$. We are unable to determine whether or not λ is zero because, as we shall see, the value of λ makes no significant difference in the computations through degree 64 in our spectral sequence.

Now $\sigma A[47] \in \sigma\langle\eta, 2C[42], \nu\rangle \supset \sigma\langle\eta, 2, \nu C[42]\rangle = \sigma\langle\eta, 2, 8D[45]\rangle = \sigma\langle\eta, 2, 2\sigma B[38]\rangle$
$\supset \sigma^2\langle\eta, 2, 2B[38]\rangle = \langle\sigma^2, \eta, 2\rangle 2B[38] = 0$ since $\eta\langle\sigma^2, \eta, 2\rangle = 2\langle\eta, \sigma^2, \eta\rangle = 2\nu\sigma^2 = 0$ while $\eta A[16] \neq 0$. Thus, $\sigma A[47] \in \text{Indet}\ \sigma\langle\eta, 2C[42], \nu\rangle = \eta\sigma\cdot\pi^S_{46} = \eta^2\sigma\cdot\pi^S_{45} = 0$ and

$$\sigma A[47] = 0. \qquad [7.17]$$

Since $B[47] = d^8(2C[20]^2 M_1\bar{M}_2)$, Theorem 2.4.5(a) implies that

$$B[47] \in \langle\nu, \eta, 2C[20]^2, \eta\rangle. \qquad [7.18]$$

Then $\sigma B[47] \in \sigma\langle\nu, \eta, 2C[20]^2, \eta\rangle \supset \sigma\langle\nu, \eta, 2, \eta C[20]^2\rangle \subset \langle\langle\sigma, \nu, \eta\rangle, 2, \eta C[20]^2\rangle$
$= \langle 0, 2, \eta C[20]^2\rangle = \eta C[20]^2\cdot\pi^S_{13} = 0$ and $\sigma B[47] \in \text{Indet}\ \sigma\langle\nu, \eta, 2C[20]^2, \eta\rangle$
$= \sigma\langle\nu, \eta, \pi^S_{42}\rangle + \sigma\langle\nu, X, \eta\rangle$ with $X = 2hC[42] + k\eta^2 C[20]^2$ by Theorem 2.3.1(b);
$= \langle\sigma, \nu, \eta\rangle\cdot\pi^S_{42} + h\sigma\langle\nu, 2C[42], \eta\rangle + k\sigma\langle\nu, \eta^2 C[20]^2, \eta\rangle = h\sigma A[47] + k\sigma\langle\nu, \eta, \eta^2 C[20]^2\rangle$
$= k\langle\sigma, \nu, \eta\rangle\eta^2 C[20]^2 = 0$. Thus,

$$\sigma B[47] = 0. \qquad [7.19]$$

We have thus proved the following theorem.

THEOREM 7.3.4 π^S_{54} has a composition series

$$Z_2 A[54,1] \oplus Z_2 A[54,2],\ Z_2\eta A[8]D[45]$$

where $2A[54,1] = \lambda\eta A[8]D[45]$ and $2A[54,2] = 0$.

We now have the following table of leaders.

Row	Degree	Leader	Row	Degree	Leader
11	57	$4\beta_1(M_1^7\bar{M}_2^3\bar{M}_3 + M_1^{10}\bar{M}_2^2\bar{M}_3 + M_1^{14}\bar{M}_2^3)$	52	56	$A[52,1]M_1^2$
19	57	$2\beta_2 M_1^{16}\bar{M}_2$	*53	61	$\eta A[52,2]M_1\bar{M}_2$
*44	66	$2C[44]M_1^2 M_2^3$	53	57	$\nu A[50,2]M_1^2$

45	57	$4D[45]M_1^3M_2$	*53	63	$A[8]D[45]M_1^2\overline{M}_2$
47	57	$A[47]M_1^2\overline{M}_2$	*54	60	$A[54,1]M_1^3$
51	57	$\nu^2D[45]M_1^3$	54	56	$A[54,2]M_1, \eta A[8]D[45]M_1$

FIGURE 7.3.4: Leaders from Rows 1 to 54 of Degree at Least 56

There are four leaders of degree 56 and six leaders of degree 57. Since $\nu^3 = \eta A[8]$, $d^4(\nu^2D[45]M_1^3) = \eta A[8]D[45]M_1$. Assume that $A[52,1]M_1^2$ is not a boundary. Then $\nu A[52,1]$ is nonzero and not divisible by η. Since $A[52,1] = d^6(\eta^2D[45]M_2)$, $A[52,1] \in \langle\nu,\eta,\eta^2D[45]\rangle \subset \langle\nu,\eta^3,D[45]\rangle = \langle\nu,4\nu,D[45]\rangle$. Then $\nu A[52,1] \in \nu\langle\nu,4\nu,D[45]\rangle \subset \langle\nu^2,4\nu,D[45]\rangle = \langle\langle\eta,\nu,\eta\rangle,4\nu,D[45]\rangle \supset \eta\langle\nu,\eta,4\nu,D[45]\rangle$. Note that this Toda bracket is defined by Theorem 2.2.7(a) because $\langle\nu,\eta,4\nu\rangle = 4A[8] + \nu\cdot\pi_5^S = 0$ and $\langle\eta,4\nu,D[45]\rangle \supset \langle\eta,\nu,4D[45]\rangle = \langle\eta,\nu,\sigma B[38]\rangle \supset \langle\eta,\nu,\sigma\rangle B[38] = 0$. Thus, $\nu A[52,1] \in \text{Indet}\,\langle\nu^2,4\nu,D[45]\rangle = \nu^2\cdot\pi_{49}^S + D[45]\cdot\pi_{10}^S = \eta\alpha_1D[45]$, a contradiction. Thus, $A[52,1]M_1^2$ must be a boundary. Now $A[52,1]M_1^2$ can only bound from below the 47 row, and we shall see that $A[52,1]M_1M_2$ must also bound. There is only one possibility: $d^8(4D[45]M_1^3M_2) = A[52,1]M_1^2$ and $A[52,1]M_1M_2$ bounds from below the 45 row.

Assume that $\xi = d^6(A[50,1]M_2)$ is nonzero. Then ξM_1 can not bound because we will show that the only leader of degree 58 below the 50 row must hit $A[47]M_1^2\overline{M}_2$. Thus, $\eta\xi \neq 0$. By Theorem 2.4.4(b), $\xi \in \langle\nu,\eta,A[50,1]\rangle$ and $\eta\xi \in \eta\langle\nu,\eta,A[50,1]\rangle = \langle\eta,\nu,\eta\rangle A[50,1] = \nu^2A[50,1]$. Thus, $\eta\xi M_1 = d^4(\nu A[50,1]M_1^3)$. However, we shall see in the derivation of π_{58}^S that $\nu A[50,1]M_1^3$ must be a boundary. Thus, $\xi = 0$ and $A[50,1]M_2$ must bound from below the 17 row. There is only one possibility: $A[50,1]M_2 = d^{40}(4\beta_1M_1^7\overline{M}_2^3\overline{M}_3)$.

Clearly $2\beta_2M_1^{19} = 2M_1\cdot2\beta_2M_1^{18}$ survives to E^{36} and hits $2A[54,1]M_1 = 0$, i.e. $2\beta_2M_1^{19}$ transgresses. Since $A[54,2]M_1$ can only bound from below the 47 row, $\eta A[54,2]$ is nonzero. We have thus proved the following theorem.

THEOREM 7.3.5 $\pi^S_{55} = Z_2\eta A[54,2] \oplus Z_{16}\gamma_6$

The computations of Section 6 show that we have the following leaders. This table contains the leaders of all degrees.

Row	Degree	Leader
9	63	$\eta^2\sigma M_1^{21}M_2^2$
11	65	$\beta_1(5M_1^{20}\bar{M}_3+4M_1^6\bar{M}_2^7)$
17	59	$\eta^2\gamma_1 M_1^{15}\bar{M}_2^2$
18	64	$4C[18]M_1^7\bar{M}_2 M_2^2\bar{M}_3$
19	57	$2\beta_2 M_1^{16}\bar{M}_2$
22	62	$\nu A[19]M_1^7M_2^2\langle M_3\rangle$
23	63	$\gamma_2 M_1^{20}$
24	60	$\eta A[23]M_1^{15}\bar{M}_2$
30	60	$A[30]\langle M_4\rangle$
32	62	$A[32,1]M_1^8\bar{M}_3$
34	64	$2B[34]M_1^5\bar{M}_2\bar{M}_3$
36	60	$A[36]M_1^6\bar{M}_2^2$
38	62	$B[38]M_1^2\bar{M}_2\bar{M}_3$
39	61	$A[39,1]M_1M_2M_3$
39	66	$\eta A[39,3]M_1^3\bar{M}_2\langle M_3\rangle$
39	59	$\sigma A[32,1]M_1^4M_2^2$
40	64	$\eta\sigma A[32,1]M_1^5\bar{M}_3$
40	60	$A[40,1]\bar{M}_2\bar{M}_3$
40	58	$A[40,2]M_1^6\bar{M}_2$
44	66	$2C[44]M_1^2M_2^3$
44	58	$4C[44]M_1^7$
45	59	$2D[45]M_1\langle M_2^2\rangle$
45	63	$4D[45M_1^6M_2$
46	66	$\eta^2C[44]M_1^7\bar{M}_2$
47	65	$\nu C[44]M_1^3\langle M_2^2\rangle$
47	57	$A[47]M_1^2\bar{M}_2$
51	63	$\eta A[50,2]M_1^3\bar{M}_2$
51	59	$\sigma C[44]M_1^4$
52	58	$A[52,1]M_1^3$, $A[52,2]\bar{M}_2$
52	60	$\eta\sigma C[44]M_1\bar{M}_2$
53	61	$\eta A[52,2]M_1\bar{M}_2$
53	59	$\nu A[50,1]M_1^3$
53	57	$\nu A[50,2]M_1^2$
53	63	$A[8]D[45]M_1^2\bar{M}_2$
54	60	$A[54,1]M_1^3$, $A[54,2]\bar{M}_2$
54	66	$\eta A[8]D[45]M_1^3\bar{M}_2$
55	57	$\eta A[54,2]M_1$

FIGURE 7.3.5: Leaders from Rows 1 to 55 of Degree at Least 57

4. Computation of π_N^S, $56 \le N \le 60$.

We continue the computations of Section 3. In the tables of leaders below, all leaders of degree greater than 62 will have an asterisk at the left. They will be omitted from all tables of leaders in this section except for the last one.

From Figure 7.3.5, we see that there are four leaders of degree 57 and four leaders of degree 58. Note that $\nu A[52,2] \in \nu\langle\eta,\nu,B[47]\rangle = \langle\nu,\eta,\nu\rangle B[47]$ $= A[8]B[47] = \langle\eta,2\nu,\nu\rangle B[47] = \eta\langle 2\nu,\nu,B[47]\rangle = \eta\langle\langle\eta,2,\eta\rangle,\nu,B[47]\rangle$ $= \eta\langle\eta,\langle 2,\eta,\nu\rangle,B[47]\rangle + \eta\langle\eta,2,\langle\eta,\nu,B[47]\rangle\rangle = \eta\langle\eta,0,B[47]\rangle + \eta\langle\eta,2,A[52,2]\rangle$ $= \eta A[54,2]$ since $\eta^2\cdot\pi_{53}^S = 0$ and $\eta\sigma B[47] = 0$. We will show in the derivation of π_{61}^S that $\nu A[52,1]$ equals 0 and does not equal $\nu A[52,2]$. Therefore,

$$\nu A[52,2] = \eta A[54,2] \text{ and } \nu A[52,1] = 0. \qquad [7.20]$$

Thus, $d^4(A[52,2]\bar{M}_2) = \eta A[54,2]M_1$. Observe that $4C[44]M_1^7$ survives to E^{10} since $4\sigma C[44] = 0$. Assume that $d^{10}(4C[44]M_1^7) = \nu A[50,2]M_1^2$. Then $\nu A[50,2] \in \langle\eta,4C[44],\sigma\rangle = \langle\eta,\sigma^2A[30],\sigma\rangle$ and $\nu A[50,2] \in \langle\eta,\sigma^2,\sigma A[30]\rangle$ because $\sigma\cdot\pi_{46}^S = \eta\sigma\cdot\pi_{45}^S$ which can not contain $\nu A[50,2]$. Thus, $\nu A[50,2] \in \langle\eta,\sigma^3,A[30]\rangle$ $= \langle\eta,\nu C[18],A[30]\rangle$ and $\nu A[50,2] \in \langle\eta,\nu,C[18]A[30]\rangle$. Then $\nu A[50,2]M_1^2 =$ $d^6(C[18]A[30]M_1^2M_2)$, a contradiction. Therefore, $4C[44]M_1^7$ transgresses. (Note that $A[30]\cdot\pi_{23}^S = \{\sigma A[16]A[30], \nu C[20]A[30], A[30]\gamma_2\}$. Now $A[16]A[30]$ is divisible by η. Also, $A[30]\gamma_2 \in A[30]\langle 2\gamma_1,16,\sigma\rangle = \langle A[30],2\gamma_1,16\rangle\sigma$ which is divisible by η. In addition, $A[30]A[14] \in A[30]\langle 2,A[8],\nu,\eta\rangle$ $\subset \langle\langle A[30],2,A[8]\rangle,\nu,\eta\rangle$ which projects to zero in E^8. However, π_{44}^S projects monomorphicly into E^8. Thus, $A[30]A[14] = 0$. Now $C[20]A[30] \in$ $A[30]\langle A[14],2,\eta,\nu\rangle \subset \langle\langle A[30],A[14],2\rangle,\eta,\nu\rangle$. Thus, $C[20]A[30]$ is zero in E^8. Since π_{50}^S projects monomorphically into E^8, $C[20]A[30] = 0$.) We will use the following lemma to continue our analysis of the leaders of degree 57.

LEMMA 7.4.1 (a) $\sigma A[40,2] = A[47]$

(b) $\nu^2 A[50,1] = 0$

(c) $\nu^2 A[50,2]$ is not divisible by η.

PROOOF. (a) By 6.20, $2C[42] \in \langle\eta,2,A[40,2]\rangle$. Thus, $A[47] \in \langle\eta,\nu,2C[42]\rangle \subset \langle\eta,\nu,\langle\eta,2,A[40,2]\rangle\rangle = \langle\langle\eta,\nu,\eta\rangle,2,A[40,2]\rangle + \langle\eta,\langle\nu,\eta,2\rangle,A[40,2]\rangle = \langle\nu^2,2,A[40,2]\rangle + \langle\eta,0,A[40,2]\rangle$ and $A[47] \in \langle\nu^2,2,A[40,2]\rangle \subset \langle\nu,2\nu,A[40,2]\rangle \supset \langle 2\nu,\nu,A[40,2]\rangle = \langle\langle\eta,2,\eta\rangle,\nu,A[40,2]\rangle \supset \eta\langle 2,\eta,\nu,A[40,2]\rangle$. Note that this four-fold Toda bracket is defined by Theorem 2.2.7(a) because $\langle 2,\eta,\nu\rangle = 0$ and $0 = d^6(A[40,2]\bar{M}_2) \in \langle\eta,\nu,A[40,2]\rangle$. Therefore, $A[47] \in \mathrm{Indet}\ \langle\nu^2,2,A[40,2]\rangle = \nu^2\cdot\pi^S_{41} + A[40,2]\cdot\pi^S_7 = \{\sigma A[40,2]\}$ because $\nu\cdot\pi^S_{41} = 0$. Thus, $\sigma A[40,2] = A[47]$.

(b) The hidden differential from the 17 row which hits $\eta A[47]M_1\bar{M}_2$ instead of $A[50,1]\bar{M}_2$ shows that these elements have homologous representatives. Therefore, representatives of $0 = \nu\eta A[47]M_1\bar{M}_2$ and $\nu A[50,1]\bar{M}_2$ are also homologous. Hence $\nu A[50,1]\bar{M}_2$ must bound. In addition, $d^4(A[50,1]M_1M_2) = \nu A[50,1]M_1^2$. Thus, $\nu^2 A[50,1] = 0$.

(c) By (a), $d^8(A[40,2]M_1^6\bar{M}_2) = A[47]M_1^2\bar{M}_2$. There remains no possibility for $\nu A[50,2]M_1^2$ to be a boundary. Thus, $\nu^2 A[50,2]$ is nonzero and is not divisible by η. ∎

By Lemma 7.4.1(a), $A[47]M_1^2\bar{M}_2 = d^8(A[40,2]M_1^6\bar{M}_2)$. Thus, $\nu A[50,2]M_1^2$, $2\beta_2M_1^{19}$ can not bound and $\nu^2 A[50,2]$, $A[56] = d^{38}(2\beta_2M_1^{19})$ are nonzero. Since $2A[54,1] = \lambda\eta A[8]D[45]$ and $d^{36}(\beta_2M_1^{18}) = A[54,1]$, Theorem 2.4.2 implies that

$$A[56] \in \langle\eta,(2,\eta A[8]),(A[54,1],\lambda D[45])^T\rangle. \tag{7.21}$$

Then $2A[56] \in 2\langle\eta,(2,\eta A[8]),(A[54,1],\lambda D[45])^T\rangle = \langle 2,\eta,(2,\eta A[8])\rangle(A[54,1],\lambda D[45])^T = \langle 2,\eta,2\rangle A[54,1] + \lambda\langle 2,\eta,\eta A[8]\rangle D[45] = \eta^2 A[54,1] + 2\lambda'\beta_1 D[45]$ since $\langle 2,\eta,\eta A[8]\rangle = \langle 2,\eta,\nu^3\rangle \supset \langle 2,\eta,\nu\rangle\nu^2 = 0$. Now $\eta^2 A[54,1] = 0$. Moreover, $\lambda' = 0$ because $\eta A[56] \neq 0$ so that $A[56] \neq \lambda'\beta_1 D[45]$. Thus, $2A[56] = 0$. We have now proved the following theorem.

THEOREM 7.4.2 $\pi^S_{56} = Z_2A[56] \oplus Z_2\nu^2A[50,2] \oplus Z_2\eta\gamma_6$

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
17	59	$\eta^2\gamma_1M_1^{15}\overline{M}_2^2$	44	58	$4C[44]M_1^7$
19	61	$\beta_2M_1^{14}\overline{M}_3$	45	59	$2D[45]M_1\langle M_2^2\rangle$
22	62	$\nu A[19]M_1^7M_2^2\langle M_3\rangle$	51	59	$\sigma C[44]M_1^4$
24	60	$\eta A[23]M_1^{15}\overline{M}_2$	52	58	$A[52,1]M_1^3$
30	60	$A[30]\langle M_4\rangle$	52	60	$\eta\sigma C[44]M_1\overline{M}_2$
32	62	$A[32,1]M_1^8$	53	61	$\eta A[52,2]M_1\overline{M}_2$
36	60	$A[36]M_1^6\overline{M}_2^2$	53	59	$\nu A[50,1]M_1^3$
38	62	$B[38]M_1^2\overline{M}_2\overline{M}_3$	54	60	$A[54,1]M_1^3$, $A[54,2]\overline{M}_2$
39	61	$A[39,1]M_1M_2M_3$	*55	63	$\eta A[54,2]M_1\overline{M}_2$
39	59	$\sigma A[32,1]M_1^4M_2^2$	56	58	$A[56]M_1$
40	60	$A[40,1]\overline{M}_2\overline{M}_3$	56	62	$\nu^2A[50,2]M_1^3$

FIGURE 7.4.1: Leaders from Rows 1 to 56 of Degree at Least 58

There are three leaders of degree 58 and five leaders of degree 59. Clearly $\sigma C[44]M_1^4$ transgresses. Since $D[45]\langle M_1^4\rangle$ bounds from the 18 row, $2D[45]$ bounds from the 34 row and $2M_2$ is an infinite cycle, $2D[45]M_2\langle M_1^4\rangle$ must bound from the r-row for some $18 \le r < 34$. There are two such leader of degree 60: $A[30]\langle M_4\rangle$ and $\eta A[23]M_1^{15}\overline{M}_2$. In the derivation of π^S_{58}, we will show that $d^{16}(A[30]\langle M_4\rangle) = \sigma A[32,1]M_1^4M_2^2$. Thus, we must have $d^{22}(\eta A[23]M_1^{15}\overline{M}_2) = 2D[45]M_2\langle M_1^4\rangle$. By Lemma 7.4.1(b), $\nu A[50,1]M_1^3$ transgresses. Since $\sigma^2A[32,1] = 0$, $\sigma A[32,1]\langle M_1^4\rangle\langle M_2^2\rangle$ survives to E^{16}. Since $A[56]M_1$ can only bound from below the 19 row, $\sigma A[32,1]\langle M_1^4\rangle\langle M_2^2\rangle$ transgresses.

We show that $A[52,1]M_1^3$ is a boundary. Assume that $\xi = d^6(A[52,1]M_1^3) \neq 0$. There is no possibility for ξM_1 to bound from below the 52 row, and thus

$\eta\xi \neq 0$. (We shall see that $A[40,1]\bar{M}_2\bar{M}_3$, $A[36]M_1^6\bar{M}_2^2$, $A[30]<M_4>$ and $\eta A[23]M_1^{15}\bar{M}_2$ are needed to bound other elements.) By Theorem 2.4.4(a), $\xi \in <\eta, A[52,1], \nu>$ and $\eta\xi \in \eta<\eta, A[52,1],\nu> \subset <\eta^2, A[52,1], \nu> = <<2,\eta,2>, A[52,1],\nu>$ $\supset 2<\eta,2,A[52,1],\nu> \subset 2\cdot\pi_{58}^S$ which, as we shall see, must be zero. Thus, $\eta\xi \in$ Indet $<\eta^2, A[52,1],\nu>$ and $\eta\xi \in \nu\cdot\pi_{55}^S = 0$. Therefore, $\xi = 0$ and $A[52,1]M_1^3$ must be a boundary. The only possibility is $d^{36}(\eta^2\gamma_1 M_1^{15}\bar{M}_2^2) = A[52,1]M_1^3$. (Use Theorem 2.2.7(e) to show that the four-fold Toda bracket is defined. By Theorem 2.4.2, $<\eta,2,A[52,1]>$ contains $d^8(2\eta^2 D[45]M_1M_2) = 0$. Therefore, $<\eta,2,A[52,1]>$ is zero in E^8, and the only possibility is $<\eta,2,A[52,1]>$ $= \eta A[8]D[45] \in$ Indet $<\eta,2,A[52,1]>$. Thus, $<\eta,2,A[52,1]>$ contains 0. Now $\eta<2,A[52,1],\nu> = \nu<\eta,2,A[52,1]> = <\nu,\eta,2>A[52,1] = 0$. Observe that if $\eta A[56] = 0$ then $A[56]M_1 = d^{40}(\eta^2\gamma_1 M_1^{15}\bar{M}_2^2)$, $\xi \neq 0$ and $\eta\xi \neq 0$ as we remarked above. Since $2\xi \in 2<\eta,\nu,A[52,1]> = <2,\eta,\nu>A[52,1] = 0$, $\eta^2\xi$ is divisible by two which we shall see is impossible. Thus, $\eta^2\xi = 0$ and $\eta\xi M_1$ must bound. The only possibility is $d^6(\eta A[52,2]M_1\bar{M}_2) = \eta\xi M_1$, and $\pi_{60}^S = Z_2\eta A[59,2]$. However, we shall see that $\eta A[59,2]$ is divisible by two, a contradiciton. Therefore, $\eta A[56] \neq 0$ and $<2,A[52,1],\nu> \subset Z_2\nu^2 A[50,2] \subset$ Indet $<2,A[52,1],\nu>$. Thus, $0 \in <2,A[52,1],\nu>$. Finally, π_{53}^S is generated by $A[8]D[45]$, $\eta A[52,2]$, $\nu A[50,1]$ and $\nu A[50,2]$. Note that ν times $A[8]D[45]$ and $\eta A[52,2]$ are zero while η times $\nu A[50,1]$ and $\nu A[50,2]$ are zero.)

Now $A[56]M_1$ and $4C[44]M_1^7$ can not be boundaries. Thus, $\eta A[56]$ and $A[57] = d^{14}(4C[44]M_1^7)$ are nonzero. Since $<\sigma,4C[44],\nu> \in \mathrm{CokJ}_{55} = \nu\cdot\pi_{52}^S$, $<\sigma,4C[44],\nu>$ contains 0. Also $0 \in <4C[44],\nu,\eta>$ because $\mathrm{CokJ}_{49} = 0$. Since $\sigma\pi_{48}^S = 0$, $<\sigma,4C[44],\nu,\eta>$ is defined by Theorem 2.2.7(c). By Theorem 2.4.6(c),

$$A[57] \in <\sigma,4C[44],\nu,\eta>. \qquad [7.22]$$

Then $A[57] \in <\sigma,\sigma^2 A[30],\nu,\eta> \supset <\sigma^2 A[30],\sigma,\nu,\eta> = <4C[44],\sigma,\nu,\eta>$ and

$2A[57] \in 2<\eta,\nu,\sigma,4C[44]> + 2\cdot$Indet $<\sigma,\sigma^2 A[30],\nu,\eta>$;

$\subset <<2,\eta,\nu>,\sigma,4C[44]> + 2[<\sigma,X,\eta> + <Y,\nu,\eta>]$ by Theorem 2.3.1(b);

$= \langle 0,\sigma,4C[44]\rangle + \sigma\langle X,\eta,2\rangle + Y\langle \nu,\eta,2\rangle = \sigma\langle X,\eta,2\rangle$. If $\nu X = 0$ then $\nu\langle X,\eta,2\rangle = \langle \nu,X,\eta\rangle 2 = 0$ and $\langle X,\eta,2\rangle = 0$ since multiplication by ν is a monomorphism on CokJ_{50}. Thus, $X = h\nu D[45]+k\nu A[45,1]$ and $2A[57] = \sigma\langle h\nu D[45]+k\nu A[45,1],\eta,2\rangle = \sigma(hD[45]+kA[45,1])\langle \nu,\eta,2\rangle = 0$. Since A[57] bounds from the 44 row which is below the 50 row, A[57] can not be divisible by σ. Thus, $\sigma A[50,2] = k\eta A[56]$ and

$$\eta\sigma A[50,2] = 0. \qquad [7.23]$$

We have thus proved the following theorem.

THEOREM 7.4.4 $\pi^S_{57} = Z_2\eta A[56] \oplus Z_2 A[57] \oplus Z_2\alpha_7 \oplus Z_2\eta^2\gamma_6$

We now have the following table of leaders.

Row	Degree	Leader	Row	Degree	Leader
19	61	$\beta_2 M_1^{14}\bar{M}_3$	40	60	$A[40,1]\bar{M}_2\bar{M}_3$
22	62	$\nu A[19]M_1^7 M_2^2\langle M_3\rangle$	45	59	$2D[45]M_1\langle M_2^2\rangle$
24	60	$\eta A[23]M_1^{15}\bar{M}_2$	51	59	$\sigma C[44]M_1^4$
30	60	$A[30]\langle M_4\rangle$	52	60	$A[52,1]M_1M_2$, $\eta\sigma C[44]M_1\bar{M}_2$
32	62	$A[32,1]M_1^8\bar{M}_3$	53	61	$\eta A[52,2]M_1\bar{M}_2$
36	60	$A[36]M_1^6\bar{M}_2^2$	53	59	$\nu A[50,1]M_1^3$
38	62	$B[38]M_1^2\bar{M}_2\bar{M}_3$	54	60	$A[54,1]M_1^3$, $A[54,2]\bar{M}_2$
39	61	$A[39,1]M_1M_2M_3$	56	62	$\nu^2 A[50,2]M_1^3$
39	59	$\sigma A[32,1]M_1^4 M_2^2$	57	59	$A[57]M_1$, $\eta A[56]M_1$

FIGURE 7.4.2: Leaders from Rows 1 to 57 of Degree at Least 59

There are six leaders of degree 59 and eight leaders of degree 60. We digress to show that $A[32,1] \in \langle \eta,2,A[30]\rangle$. Note that $d^8(4M_1^2\langle M_1^4\rangle^2\bar{M}_3) \equiv 4\sigma M_1^{13}$ modulo (8σ). In addition, if $4M_1^2\langle M_1^4\rangle^2\bar{M}_3$ survived to E^{10} it would hit $\eta^2\sigma M_1^5\bar{M}_3$. Thus, A[32,1] is represented by the boundary of a representative of

$\eta^2\sigma M_1^5\bar{M}_3$ which is homologous to the boundary of a representative of $4\sigma M_1^{13}$. By Theorem 2.4.2, the boundary of a representative of $4\sigma M_1^{13} = M_1\cdot 2\cdot 2\sigma M_1^{12}$ is an element of $<\eta, 2, A[30]>$ and

$$A[32,1] \in <\eta, 2, A[30]>. \qquad [7.24]$$

Recall that $d^8<M_4> = 2\sigma M_1\bar{M}_2<M_3>$, and μ_1, $\bar{\mu}_{01}$, $<\mu_{001}>$ is a representative of M_1, $\bar{M}_2$, $<M_3>$, respectively. Then $A[30]<M_4>$ is represented by $A[30]<M_4> \cup (\sigma \wedge B_{A[30]\cdot 2} \wedge \mu_1 \wedge \bar{\mu}_{01} \wedge <\mu_{001}>)$ which has boundary $(\sigma A[30] \wedge B_{2\eta} \wedge \bar{\mu}_{01} \wedge <\mu_{001}>) \cup (\sigma \wedge B_{A[30]\cdot 2} \wedge \eta \wedge \bar{\mu}_{01} \wedge <\mu_{001}>)$ $= \sigma<A[30], 2, \eta> \wedge \bar{\mu}_{01} \wedge <\mu_{001}>$ modulo elements of filtration degree 18. Thus, $d^{10}(A[30]<M_4>) = \sigma A[32,1]\bar{M}_2<M_3> = \sigma A[32,1]M_1^4M_2^2 + d^8(A[32,1]M_1^4\bar{M}_2\bar{M}_3)$ $= \sigma A[32,1]M_1^4M_2^2$ in E^{10}.

Since $\nu A[54,2] \in \nu<\eta, 2, A[52,2]> = <\nu, \eta, 2>A[52,2] = 0$, $A[54,2]\bar{M}_2$ must transgress. If $d^6(\eta\sigma C[44]M_1\bar{M}_2) = \eta A[56]M_1$ then $d^6(\sigma C[44]\bar{M}_2) = A[56]$, a contradiction. Since $A[57]M_1$ can only bound from below the 44 row, $\eta\sigma C[44]M_1\bar{M}_2$ must transgress. In the derivation of π_{59}^S, we show that $A[52,1]M_1M_2$ is a boundary. Since there is no possibility for an element of order four in π_{58}^S, $\eta A[56]M_1$ must be a boundary. Since $\sigma A[40,1] = \nu C[44]$, a representative $\mathcal{R}$ of $A[40,1]\bar{M}_2<M_3>$ has boundary $(C[44] \wedge \nu \wedge <\mu_{02}>) \cup (C[44] \wedge B_{\nu\sigma} \wedge \mu_2)$. Then $\mathcal{R} \cup (C[44] \wedge \mu_2 \wedge <\mu_{02}>)$ has boundary $C[44] \wedge \mu_2 \wedge \sigma \wedge \mu_2$ union elements of filtration degree 4. Thus, $d^{18}(A[40,1]\bar{M}_2<M_3>) = \sigma C[44]M_1^4$. Recall that in the derivation of π_{57}^S we showed that $d^{22}(\eta A[23]M_1^{15}\bar{M}_2) = 2D[45]<M_1^4>M_2$. In the proof of Lemma 7.4.1(b) we showed that $\nu A[50,1]\bar{M}_2$ must be a boundary. Since $\nu A[50,1]\bar{M}_2 = \nu A[50,1]M_1^3$ in E^4, $\nu A[50,1]M_1^3$ must bound from below the 50 row. The only possibility is $d^{18}(A[36]M_1^6\bar{M}_2^2) = \nu A[50,1]M_1^3$. There remains only one way for $\eta A[56]M_1$ to bound: $d^4(A[54,1]M_1^3) = \eta A[56]M_1$. Thus,

$$\nu A[54,1] = \eta A[56] \text{ and } \nu A[54,2] = 0. \qquad [7.25]$$

Now $A[57]M_1$ can not be a boundary. We have thus proved the following theorem.

THEOREM 7.4.5 $\pi^S_{58} = Z_2\eta A[57] \oplus Z_2\eta\alpha_7$

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
19	61	$\beta_2 M_1^{14}\bar{M}_3$	51	61	$\sigma C[44]M_1^2\bar{M}_2$
22	62	$\nu A[19]M_1^7M_2^2\langle M_3\rangle$	52	60	$A[52,1]M_1M_2$, $\eta\sigma C[44]M_1\bar{M}_2$
32	62	$A[32,1]M_1^8\bar{M}_3$	53	61	$\eta A[52,2]M_1\bar{M}_2$
38	62	$B[38]M_1^2\bar{M}_2\bar{M}_3$	*53	63	$\nu A[50,1]M_1^2M_2$
*39	63	$\sigma A[32,1]M_1^6M_2^2$	54	60	$A[54,2]\bar{M}_2$
39	61	$A[39,1]M_1M_2M_3$	56	62	$\nu^2A[50,2]M_1^3$
45	61	$2D[45]M_1\langle M_3\rangle$	58	60	$\eta A[57]M_1$

FIGURE 7.4.3: Leaders from Rows 1 to 58 of Degree at Least 60

There are four leaders of degree 60 and five leaders of degree 61. Since $\eta A[56] = \nu A[54,1]$, $d^4(A[54,1]M_1^3) = \eta A[56]M_1$. Since $D[45]\langle M_3\rangle$ is a d^{28}-boundary and $2M_1$ is an infinite cycle, $2M_1D[45]\langle M_3\rangle$ must be a boundary. Let $A[59,2] = d^6(A[54,2]\bar{M}_2)$. By Theorem 2.4.4(c),

$$A[59,2] \in \langle\eta,\nu,A[54,2]\rangle. \qquad [7.26]$$

Thus, $2A[59,2] \in 2\langle\eta,\nu,A[54,2]\rangle = \langle 2,\eta,\nu\rangle A[54,2] = 0$. Let $A[59,1] = d^6(\eta\sigma C[44]M_1\bar{M}_2)$. By Theorem 2.4.5(a),

$$A[59,1] \in \langle\eta,\eta\sigma C[44],\eta,\nu\rangle. \qquad [7.27]$$

Thus, $2A[59,1] \in 2\langle\eta,\eta\sigma C[44],\eta,\nu\rangle \subset \langle\langle 2,\eta,\eta\sigma C[44]\rangle,\eta,\nu\rangle = \langle\langle\eta\sigma C[44],\eta,2\rangle,\eta,\nu\rangle$ $= \langle\eta\sigma C[44],\langle\eta,2,\eta\rangle,\nu\rangle + \langle\eta\sigma C[44],\eta,\langle 2,\eta,\nu\rangle\rangle = \langle\eta\sigma C[44],2\nu,\nu\rangle + \langle\eta\sigma C[44],\eta,0\rangle$ $= A[8](\sigma C[44]) + \nu\cdot\pi^S_{56} = \{\nu^3A[50,2],\nu A[56]\} = \{\eta A[8]A[50,2]\}$ because $\nu A[56] \in \nu\langle\eta,(2,\eta A[8]),(A[54,1],\lambda D[45])^T\rangle = \langle\nu,\eta,(2,\eta A[8])\rangle(A[54,1],\lambda D[45])^T$ $= \langle\nu,\eta,2\rangle A[54,1] + \lambda\langle\nu,\eta,\eta A[8]\rangle D[45] = \lambda\langle\nu,\eta,\nu^3\rangle D[45] = \lambda\nu^2A[8]D[45] = 0$. Observe that $\langle\eta^2,\nu,\eta,2\rangle$ is defined by Theorem 2.2.7(a) since $\langle\eta^2,\nu,\eta\rangle$ $= \langle\nu,\eta,2\rangle = 0$. Then $\eta\langle\eta^2,\nu,\eta,2\rangle = \langle\eta^2,\nu,\langle\eta,2,\eta\rangle\rangle = \langle\eta^2,\nu,2\nu\rangle = \eta A[8]$. Since multiplication by η is a monomorphism on π_8^S,

$$A[8] \in \langle \eta^2, \nu, \eta, 2\rangle. \qquad [7.28]$$

Thus, $A[50,2]A[8] = A[50,2]\langle \eta^2,\nu,\eta,2\rangle \subset \langle\langle A[50,2],\eta^2,\nu\rangle,\eta,2\rangle$. Now $\langle A[50,2],\eta^2,\nu\rangle \supset \langle \eta A[50,2],\eta,\nu\rangle$ which contains $d^6(\eta A[50,2]M_2) = 0$. Therefore, $A[50,2]A[8] = k\langle \nu^2 A[50,2],\eta,2\rangle = k\nu A[50,2]\langle \nu,\eta,2\rangle = 0$ since $2\cdot\pi_{58}^S = 0$. Thus, $2A[59,1] = 0$ and

$$\nu^3 A[50,2] = \nu A[56] = 0. \qquad [7.29]$$

We show that $A[52,1]M_1M_2$ must be a boundary. Assume that $\xi = d^8(A[52,1]M_1M_2)$ is not zero. Let $\mathfrak{C}$ denote the mapping cone of $2\nu D[45]$ with $\rho: S \longrightarrow \mathfrak{C}$ the canonical map. In the Atiyah-Hirzebruch spectral sequence for $\mathfrak{C}_*BP$, $\rho(2D[45])M_1^5M_2$ clearly survives to E^6. The only possibility for $d^6(\rho(2D[45])M_1^5M_2)$ is $\partial^{-1}(\eta)M_1^5$. In that case, $d^6(\rho(2D[45])M_1M_2) = \partial^{-1}(\eta)M_1$ and $0 = \eta\partial^{-1}(\eta) = \partial^{-1}(\eta^2)$, a contradiction. Thus, $\rho(2D[45])M_1^5M_2$ survives to E^8 and $d^8(\rho(2D[45])M_1^5M_2) = \rho(A[52,1])M_1M_2$. Then $\rho(\xi)$ must be zero in E^8. If $\rho(\xi) = d^2(\partial^{-1}(h\eta^2\sigma+k\eta A[8])M_1)$ then in π_*^S, $\xi \in \langle \eta, h\eta^2\sigma+k\eta A[8], 2\nu D[45]\rangle$ $\supset \langle \eta, h\eta^2\sigma+k\eta A[8],2\rangle\nu D[45] = 0$, a contradiction. If $\rho(\xi) = d^4(\partial^{-1}(\sigma)M_1^2)$ then $\xi \in \langle \nu,\sigma,2\nu D[45]\rangle \supset \langle \nu,\sigma,\nu\rangle 2D[45] = \sigma^2(2D[45]) = 0$, a contradiction. Note that $\rho(\xi)$ can not be in Image d^6 because $\pi_5^S = 0$. Thus, $\xi \in 2\nu D[45]\cdot\pi_{11}^S = 0$, and $A[52,1]M_1M_2$ must bound from below the 47 row. Note that if $d^{34}(\beta_2 M_1^{14}\bar{M}_3)$ $= A[52,1]M_1M_2$ then $r_{\Delta_2}\circ d^{34}(\beta_2 M_1^{15}M_2^3) = A[52,1]M_1M_2$, an impossibility. There is is only one remaining possibility: $d^{14}(A[39,1]M_1M_2M_3) = A[52,1]M_1M_2$.

Since $\sigma C[44]M_1^2\bar{M}_2$ survives to E^6, it must transgress. Observe that the argument above that $\beta_2 M_1^{14}\bar{M}_3$ can not hit $A[52,1]M_1M_2$ also shows that $\beta_2 M_1^{14}\bar{M}_3$ can not hit $A[54,2]\bar{M}_2$ nor $\eta\sigma C[44]M_1\bar{M}_2$. Thus, $A[54,2]\bar{M}_2$, $\eta\sigma C[44]M_1\bar{M}_2$ can not be boundaries, and $A[59,1]$, $A[59,2]$ are nonzero. By Lemma 3.3.14, $\eta^2 A[57]$ must be zero since it can not be divisible by two. Thus, $\eta A[57]M_1$ must be a boundary. In Lemma 7.4.7(c) we shall see that $\eta A[52,2] = \eta C[20]A[32,2]$, and thus $\eta A[52,2]M_1\bar{M}_2 = C[20](\eta A[32,2]M_1\bar{M}_2)$ transgresses. If $d^8(\sigma C[44]M_1^2\bar{M}_2) = \eta A[57]M_1$ then $\beta_2 M_1^{14}\bar{M}_3$ transgresses to a nonzero element B which is indecomposable by Lemma 1.3.10. However, in the proof of Lemma 7.4.7(e) we shall see that $2B = \eta A[59,2]$ and $B = C[20]^3$, a contradiction. Thus, $d^{40}(\beta_2 M_1^{14}\bar{M}_3) = \eta A[57]M_1$. Since $\sigma C[44]M_1^2\bar{M}_2$ transgresses,

$$\sigma^2 C[44] = 0. \qquad [7.30]$$

Since $A[52,1] \in \langle\nu,\eta,\eta^2 D[45]\rangle$, $\sigma A[52,1] \in \sigma\langle\nu,\eta,\eta^2 D[45]\rangle$ $= \langle\sigma,\nu,\eta\rangle\eta^2 D[45] = 0$. Since $A[52,2] \in \langle B[47],\nu,\eta\rangle$, $\sigma A[52,2] \in \langle\sigma,B[47],\nu\rangle\eta$ $\subset \eta\cdot \mathrm{Cok}J_{58} = 0$. Thus,

$$\sigma A[52,1] = \sigma A[52,2] = 0. \qquad [7.31]$$

We have thus proved the following theorem.

THEOREM 7.4.6 $\pi_{59}^S = Z_2 A[59,1] \oplus Z_2 A[59,2] \oplus Z_8 \beta_7$

The computations in Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
*19	63	$4\beta_2 M_1^{19}\bar{M}_2$	51	61	$\sigma C[44]M_1^2\bar{M}_2$
22	62	$\nu A[19]M_1^7 M_2^2\langle M_3\rangle$	*52	64	$A[52,1]M_1^3 M_2$
32	62	$A[32,1]M_1^8\bar{M}_3$	53	61	$\eta A[52,2]M_1\bar{M}_2$
38	62	$B[38]M_1^2\bar{M}_2\bar{M}_3$	56	62	$\nu^2 A[50,2]M_1^3$
45	61	$2D[45]M_1\langle M_3\rangle$	59	61	$A[59,1]M_1$, $A[59,2]M_1$

FIGURE 7.4.4: Leaders from Rows 1 to 59 of Degree at Least 61

There are five leaders of degree 61 and four leaders of degree 62. Since $A[59,1]M_1$ and $A[59,2]M_1$ can only bound from below the 54 row, $\nu^2 A[50,2]M_1^3$ must transgress. Recall that in the derivation of π_{59}^S we showed that $2D[45]M_1\langle M_3\rangle$ must bound from below the 34 row. Moreover, $2D[45]M_2\langle M_3\rangle$ equals $2M_2$ times the boundary $D[45]\langle M_3\rangle$ and must bound from below the 34 row. The lowest row of such an element of degree 66 is the 24 row. Therefore, $d^{14}(A[32,1]M_1^8\overline{M}_3)$ $= 2D[45]M_1\langle M_3\rangle$.

Let $\xi = d^{10}(\sigma C[44]M_1^2\overline{M}_2)$. Then $\sigma C[44]M_1^2\overline{M}_2$ has a representative $\mathcal{R} = (C[44] \wedge \mu_2 \wedge \sigma \wedge \overline{\mu}_{01}) \cup (C[44] \wedge \mu_2 \wedge B_{\sigma\nu} \wedge \mu_1) \cup (C[44] \wedge B_{\nu\sigma} \wedge \overline{\mu}_{01})$ $\cup (C[44] \wedge \mu_2 \wedge B_{\langle\sigma,\nu,\eta\rangle}) \cup (B_{C[44]\langle\nu,\sigma,\nu\rangle} \wedge \mu_1)$ with $\partial\,\mathcal{R} = C[44] \wedge [(B_{\nu\sigma} \wedge B_{\nu\eta}) \cup (\nu \wedge B_{\langle\sigma,\nu,\eta\rangle})] \cup [B_{C[44]\langle\nu,\sigma,\nu\rangle} \wedge \eta]$. Observe that $\partial\,[(B_{\nu\sigma} \wedge B_{\nu\eta}) \cup (\nu \wedge B_{\langle\sigma,\nu,\eta\rangle})] = \langle\nu,\sigma,\nu\rangle \wedge \eta$. Thus, $\partial\,\mathcal{R}$ represents an element of $\langle C[44],\langle\nu,\sigma,\nu\rangle,\eta\rangle = \langle C[44],\sigma^2,\eta\rangle$. By Theorem 2.4.2, $\xi M_1 = d^{16}(\eta\sigma^2 A[30]M_1^3M_2^2) = 0$, and ξM_1 must bound from between the 45 and 51 rows, i.e. from the 47 row. There is no leader of degree 63 in the 47 row. Thus, $\xi = 0$ and $\sigma C[44]M_1^2\overline{M}_2$ must bound from below the 40 row. Assume that $d^{14}(B[38]M_1^2\overline{M}_2\overline{M}_3) = \sigma C[44]M_1^2\overline{M}_2$. Since $d^{12}(A[40,1]\overline{M}_2\langle M_3\rangle) = \sigma C[44]M_1^4$, r_{Δ_1} shows that a representative of $B[38]M_1^4\overline{M}_3$ is homologous to a representative of $A[40,1]\overline{M}_2\langle M_3\rangle$. Then r_{Δ_3} shows that a representative of $B[38]M_1^4$ is homologous to a representative of $A[40,1]\overline{M}_2$. Therefore, $4D[45] = \sigma B[38] = d^6(A[40,1]\overline{M}_2)$, a contradiction. The only other possibility for $\sigma C[44]M_1^2\overline{M}_2$ to bound from below the 40 row is $d^{30}(\nu A[19]M_1^7M_2^2\langle M_3\rangle) = \sigma C[44]M_1^2\overline{M}_2$.

We show that $\eta A[59,1] = 0$. Assume that $\eta A[59,1] \neq 0$. Since we shall see that there is no possibility for an element of π_{61}^S to have order 4, $\eta^2 A[59,1] = 0$ by Lemma 3.3.14. Thus, $\eta A[59,1]M_1$ must be a boundary. Clearly $d^6(\eta A[54,2]M_1\overline{M}_2) = \eta A[59,2]M_1$. The arguments in the derivation of π_{61}^S that show that $\eta^2\sigma M_1^{21}M_2^2$, $4\beta_2 M_1^{19}\overline{M}_2$ and $\gamma_2 M_1^{20}$ can not bound $B[60]M_1$ also apply to show that these elements do not bound $\eta A[59,1]M_1$. (The argument there shows that if $d^{42}(4\beta_2 M_1^{19}\overline{M}_2) = \eta A[59,1]M_1$ then $d^6(\eta A[54,2]M_1\overline{M}_2) = \eta A[59,1]M_1$. Hence $d^6(A[54,2]\overline{M}_2) = A[59,1]$, a contradiction.) In the derivation of π_{61}^S we will also show that $d^6(\eta A[50,2]M_1^3\overline{M}_2) = \nu^2 A[50,2]M_1^3$ and that $4D[45]M_1^6M_2$, $\sigma A[32,1]M_1^6M_2^2$ are boundaries. Since $\nu A[50,1]M_1^2M_2 = r_{\Delta_1}(\nu A[50,1]M_1^3M_2)$ and there is no possibility for a hidden differential on $\nu A[50,1]M_1^3M_2$, $\nu A[50,1]M_1^2M_2$ must transgress. Note that $d^8(A[8]D[45]M_1^2\bar{M}_2) = D[45]d^8(A[8]M_1^2\bar{M}_2) = D[45]\eta A[14]M_1$. If $d^8(A[8]D[45]M_1^2\bar{M}_2) = \eta A[59,1]M_1$ then $A[59,1] = A[14]D[45]$. Since $A[59,1] = d^8(\eta\sigma C[44]M_1\bar{M}_2)$ and $A[14] = d^{12}(4\nu M_1^3\bar{M}_2)$, it would follow that twice a representative of $2\nu D[45]M_1^3\bar{M}_2$ union $\partial\ [(B_{4\nu D[45]} \wedge \mu_1 \wedge \mu_2 \wedge \bar{\mu}_{01}) \cup (2\nu D[45] \wedge B_{2\eta} \wedge \mu_2 \wedge \bar{\mu}_{01})]$ represents $\eta\sigma C[44]M_1\bar{M}_2$. Thus, $\eta\sigma C[44] \in \langle 2D[45], 2\nu, \nu\rangle \subset \langle D[45], 4\nu, \nu\rangle = \langle D[45], \eta^3, \nu\rangle = \langle \eta^2 D[45], \eta, \nu\rangle$. Therefore, $\eta\sigma C[44]M_1 = d^6(\eta^2 D[45]M_1\bar{M}_2)$, a contradiction. Hence $d^8(A[8]D[45]M_1^2\bar{M}_2) \neq \eta A[59,1]M_1$. Thus, $\eta A[59,1]M_1$ can not be a boundary, a contradiction. Hence $\eta A[59,1] = 0$ and $A[59,1]M_1$ must bound from below the 52 row. The only possibility is $d^{22}(B[38]M_1^2\bar{M}_2\bar{M}_3) = A[59,1]M_1$.

Now $\eta A[59,2]$ and $B[60] = d^8(\eta A[52,2]M_1\bar{M}_2)$ are nonzero. To identify $2B[60]$ as $\eta A[59,2]$, we derive several relations.

LEMMA 7.4.7 (a) $\eta^2 A[45,2] = \eta A[14]A[32,2]$ and

$\eta A[45,2] \equiv A[14]A[32,2]$ modulo $(\eta^2 C[44], \eta A[45,1])$.

(b) $A[54,2] = A[14]C[20]^2$

(c) $\nu A[52,2] = \nu C[20]A[32,2]$ and

$A[52,2] = C[20]A[32,2]$ modulo $(A[52,1], \eta\sigma C[44])$.

(d) $A[59,2] = A[14]A[45,2]$

(e) $B[60] = C[20]^3$ and $2B[60] = \eta A[59,2]$.

(f) $\eta B[47] = A[8]C[20]^2$

(g) $\eta^2 C[20]^3 = \nu A[14]A[45,2]$

PROOF. (a) $d^6(A[8]C[20]M_1\bar{M}_2) = \eta A[32,2]M_1$. Thus, $\eta^2 A[45,2]M_1$
$= d^6(\eta^2 C[20]^2 M_1\bar{M}_2) = d^6(A[8]A[14]C[20]M_1\bar{M}_2) = A[14]d^6(A[8]C[20]M_1\bar{M}_2)$
$= \eta A[14]A[32,2]M_1$.

(b) $A[14]C[20]^2 \in A[14]\langle \eta, \eta A[32,2], \eta, \nu\rangle \subset \langle \eta, \eta A[14]A[32,2], \eta, \nu\rangle$
$= \langle \eta, \eta^2 A[45,2], \eta, \nu\rangle = A[54,2]$.

(c) $\nu A[52,2] = \eta A[54,2] = \eta A[14]C[20]^2 = \nu A[32,2]C[20]$.

(d) $A[59,2] \in \langle \eta, \nu, A[54,2]\rangle = \langle \eta, \nu, A[14]C[20]^2\rangle = A[14]\langle \eta, \nu, C[20]^2\rangle$ $= A[14]A[45,2]$.

(e) $2C[20]^3 = C[20]^2\langle \nu, \eta, \eta A[14]\rangle = \langle C[20]^2, \nu, \eta\rangle \eta A[14] = A[45,2]\eta A[14]$ $= \eta A[59,2] \neq 0$. The only possibility for $C[20]^3$ is $B[60]$.

(f) $\eta B[47] = \nu A[45,2] = \nu\langle \eta, \nu, C[20]^2\rangle = \langle \nu, \eta, \nu\rangle C[20]^2 = A[8]C[20]^2$.

(g) $\nu A[14]A[45,2] = \eta A[14]B[47] = A[14]A[8]C[20]^2 = \eta^2 C[20]^3$. ∎

The above relations were motivated by relations (d) and (e) which were observed by Mark Mahowald from the Adams spectral sequence. We have now proved the following theorem.

THEOREM 7.4.8 $\pi^S_{60} = Z_4 B[60]$ and $2B[60] = \eta A[59,2]$.

The computations of Section 6 show that we have the following leaders. Since this is the final table of leaders of this section, we inlcude the leaders of all degrees.

Row	Degree	Leader	Row	Degree	Leader
9	63	$\eta^2\sigma M_1^{21}M_2^2$	46	66	$\eta^2 C[44]M_1^7\bar{M}_2$
11	65	$\beta_1 M_1^{20}\bar{M}_3$	47	65	$\nu C[44]M_1^3\langle M_2^2\rangle$
18	64	$4C[18]M_1^7\bar{M}_2 M_2^2\bar{M}_3$	51	65	$\sigma C[44]M_1^4\bar{M}_2$
19	63	$4\beta_2 M_1^{19}\bar{M}_2$	51	63	$\eta A[50,2]M_1^3\bar{M}_2$
23	63	$\gamma_2 M_1^{20}$	52	64	$A[52,1]M_1^3 M_2$
32	66	$A[32,1]M_1^2\langle M_4\rangle$	53	63	$\nu A[50,1]M_1^2 M_2$, $A[8]D[45]M_1^2\bar{M}_2$
34	64	$2B[34]M_1^5\bar{M}_2\bar{M}_3$	54	66	$\eta A[8]D[45]M_1^3\bar{M}_2$
39	63	$\sigma A[32,1]M_1^6 M_2^2$	55	63	$\eta A[54,2]M_1\bar{M}_2$
40	66	$\eta A[39,3]M_1^3\bar{M}_2\langle M_3\rangle$	56	62	$\nu^2 A[50,2]M_1^3$
40	64	$\eta\sigma A[32,1]M_1^5\langle M_3\rangle$	56	66	$A[56]M_1^2\bar{M}_2$
44	66	$2C[44]M_1^2 M_2^3$	59	65	$A[59,1]M_2$, $A[59,1]\bar{M}_2$,

45	65	$2D[45]\langle M_1^4\rangle\langle M_2^2\rangle$			$A[59,2]\bar{M}_2$
45	63	$4D[45]M_1^6M_2$	60	62	$Z_4B[60]M_1$

FIGURE 7.4.5: Leaders from Rows 1 to 60 of Degree at Least 62

5. Computation of π_N^S, $61 \le N \le 64$.

We continue the computations of Section 4. In the tables of leaders we include all leaders of degree less than 67. This will suffice to compute through the 64 stem modulo group extension problems which we can not resolve in the 62 and 63 stems. We use Tangora's computation [59] of E_2 of the Adams spectral sequence through degree 70 to see that A[61] is not divisible by σ, to eliminate one group extension in degree 62 and to eliminate several possible differentials in degree 64. Diagrams summarizing his computation are given in Appendix 6.

From Figure 7.4.5, we see that there are three leaders of degree 62 and nine leaders of degree 63. Let $\mathfrak{C}$ denote the mapping cone of η^2 with $\rho: S \longrightarrow \mathfrak{C}$ the canonical map. If $d^6(\eta A[50,2]M_1^3\bar{M}_2) = \nu^2A[50,2]M_1^3$ then in the Atiyah-Hirzebruch spectral sequence for $\mathfrak{C}_*BP$, $d^6(\rho(\eta)M_1^3\bar{M}_2) = \rho(\nu)^2M_1^3$. Thus, $d^4(\rho(\nu)^2M_1^3) = \rho(\eta A[8])M_1$ is zero, and $\eta A[8]$ is divisible by η^2, a contradiction. If $d^{10}(\eta A[50,2]M_1^3\bar{M}_2) = B[60]M_1$ then $A[50,2]M_1^3\bar{M}_2$ shows that $\eta A[50,2]M_1^2\bar{M}_2$ is homologous to $\eta A[52,2]M_1\bar{M}_2$ since $B[60] = d^8(\eta A[52,2]M_1\bar{M}_2)$. Then $r_{(1,1)}$ shows that $A[50,2]M_1^2$ has a representative with boundary $\eta A[52,2]$ which contradicts that $\nu A[50,2]$ is nonzero and not divisible by η.

We show that $\nu A[57] = 0$. Assume that $\nu A[57] = \eta A[59,2]$. Since $d^{14}(4C[44]M_1^7\bar{M}_2) = A[57]\bar{M}_2$ and $d^8(2C[44]M_1^7\bar{M}_2) = 2\sigma C[44]M_1^3\bar{M}_2$, there must be a hidden differential $d^r(4C[44]M_1^7\bar{M}_2) = X$ with X in either the 53 or 55 row. There are three possibilities for X: $\nu A[50,1]M_1^2M_2$, $A[8]D[45]M_1^2\bar{M}_2$ and

$\eta A[54,2]M_1\bar{M}_2$. Now $\nu A[50,1]M_1^2M_2$ can only bound from below the 36 row, and thus $X \neq \nu A[50,1]M_1^2M_2$. A representative of X will be homologous to a representative of $A[57]\bar{M}_2$. Thus, a representative of X will have boundary which represents $\eta A[59,2]M_1$. We know that this true if $X = \eta A[54,2]M_1\bar{M}_2$ because $A[54,2]\bar{M}_2$ has a representative with boundary A[59,2]. Moreover, we know that this is false if $X = A[8]D[45]M_1^2\bar{M}_2$: $d^8(A[8]M_1^2\bar{M}_2) = A[14]M_1$, $A[14]D[45] \neq A[59,1]$ (see the derivation of π_{60}^S), $A[59,2] = A[14]A[45,2]$ and we can thus choose a representative of D[45] such that $A[14]D[45] = 0$. Therefore, $d^{12}(4C[44]M_1^7M_2) = \eta A[54,2]M_1\bar{M}_2$. Let $\mathfrak{C}$ denote the mapping cone of $\eta A[54,2]$ with $\rho: S \longrightarrow \mathfrak{C}$ the canonical map. In the Atiyah-Hirzebruch spectral sequence for $\mathfrak{C}_*BP$, $d^{14}(\rho(4C[44])M_1^7M_2) = \rho(A[57])\bar{M}_2$ and $d^4(\rho(A[57])\bar{M}_2) = \rho(\eta A[59,2])M_1$. Thus, $\rho(\eta A[59,2]) = 0$ and $\eta A[59,2] \in \eta A[54,2]\cdot\pi_5^S = 0$, a contradiction. Therefore, $\nu A[57] \neq \eta A[59,2]$ and

$$\nu A[57] = 0. \qquad [7.32]$$

Since $B[60] = d^8(\eta A[52,2]M_1\bar{M}_2)$, we see from Theorem 2.4.5(a) that

$$B[60] \in \langle\eta, \eta A[52,2], \eta, \nu\rangle. \qquad [7.33]$$

Thus, $2B[60] \in 2\langle\eta,\eta A[52,2],\eta,\nu\rangle \subset \langle\langle 2,\eta,\eta A[52,2]\rangle,\eta,\nu\rangle = \langle\eta A[54,2],\eta,\nu\rangle$ because $\nu \nmid 2B[60]$, $\pi_{55}^S = Z_2\eta A[54,2] \oplus Z_2\gamma_6$ and $\langle 2\gamma_6,\eta,\nu\rangle = \gamma_6\langle 2,\eta,\nu\rangle = 0$. Thus, $\eta A[54,2]M_2$ has a representative with boundary 2B[60] and $d^6(\eta A[54,2]M_1\bar{M}_2) = 2B[60]M_1$. Note that $\eta A[54,2]M_1\bar{M}_2$ must be nonzero in E^6. Thus, $d^4(A[52,1]M_1^3M_2)$ can not equal $\eta A[54,2]M_1\bar{M}_2$. Therefore, $\nu A[52,1]$ equals 0 and not $\eta A[54,2]$. Since $d^{12}(B[34]M_1^6) = 2D[45]$, $d^{12}(2B[34]M_1^5\bar{M}_2\bar{M}_3)$ $= 4D[45](M_1^6M_2+M_1^2M_3)$. We will show in the derivation of π_{62}^S that $\sigma A[32,1]M_1^6M_2^2$ $= d^{22}(4C[18]M_1^7\bar{M}_2M_2^2\bar{M}_3)$. Since $B[60]M_1$ could only bound from below the 53 row, $\nu A[50,1]M_1^2M_2$, $A[8]D[45]M_1^2\bar{M}_2$ and $\eta A[54,2]M_1\bar{M}_2$ transgress. Since $\gamma_2M_1^{20} = r_{\Delta_1}(\gamma_2M_1^{21})$ and there is no possibility for a hidden differential on $\gamma_2M_1^{21}$, $d^{38}(\gamma_2M_1^{20})$ can not equal $B[60]M_1$. Thus, $\gamma_2M_1^{20}$ transgresses. We shall see in

Section 8.3 that θ_5 exists and that the only possibility is for $\eta^2\sigma M_1^{21}M_2^2$ to transgress to θ_5. Assume that $d^{42}(4\beta_2M_1^{19}\bar{M}_2) = B[60]M_1$. Recall that $A[54,1] = d^{36}(\beta_2M_1^{18})$. By Theorem 2.2.7(b), $\langle A[54,1],4,\eta,\nu\rangle$ is defined. By Theorem 2.4.6 (d), $B[60] \in \langle A[54,1],4,\eta,\nu\rangle \supset \langle 2A[54,1],2,\eta,\nu\rangle$ $= \lambda\langle \eta A[8]D[45],2,\eta,\nu\rangle \supset \lambda\eta D[45]\langle A[8],2,\eta,\nu\rangle = 0$. By Theorem 2.3.1(b), $B[60] \in \text{Indet}\ \langle A[54,1],4,\eta,\nu\rangle = \langle \eta A[54,2],\eta,\nu\rangle + \langle A[54,1],\eta^2,\nu\rangle =$ $d^6(\eta A[54,2]M_2)$ modulo (η,ν). Thus $B[60] = 0$ in E^8, a contradiction. Therefore $4\beta_2M_1^{19}\bar{M}_2$ transgresses. Let $A[61] = d^6(\nu^2A[50,2]M_1^3)$. By Theorem 2.4.4(c),

$$A[61] \in \langle \eta,\nu,\nu^2A[50,2]\rangle. \qquad [7.34]$$

Then $2A[61] \in 2\langle \eta,\nu,\nu^2A[50,2]\rangle = \langle 2,\eta,\nu\rangle\nu^2A[50,2] = 0$. Note that $\eta\sigma A[54,1] = 0$ and $\eta\sigma A[54,2] = \sigma(\nu A[52,2]) = 0$. We shall see that $\eta^2B[60]$ is nonzero. Moreover, $A[61]$ projects to $h_0(A+A')$ in the Adams spectral sequence, an element of filtration degree 9 which can not be divisible by σ. Therefore,

$$\sigma A[54,1] = \sigma A[54,2] = 0. \qquad [7.35]$$

We have thus proved the following theorem.

THEOREM 7.5.1 $\pi_{61}^S = Z_2A[61] \oplus Z_2\eta B[60]$

The computations of Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
9	63	$\eta^2\sigma M_1^{21}M_2^2$	46	66	$\eta^2C[44]M_1^7\bar{M}_2$
11	65	$\beta_1M_1^{20}\bar{M}_3$	47	65	$\nu C[44]M_1^3\langle M_2^2\rangle$
18	64	$4C[18]M_1^7\bar{M}_2M_2^2\bar{M}_3$	51	63	$\eta A[50,2]M_1^3\bar{M}_2$
19	63	$4\beta_2M_1^{19}\bar{M}_2$	51	65	$\sigma C[44]M_1^4\bar{M}_2$
23	63	$\gamma_2M_1^{20}$	52	64	$A[52,1]M_1^3M_2$
32	66	$A[32,1]M_1^2\langle M_4\rangle$	53	63	$\nu A[50,1]M_1^2M_2$, $A[8]D[45]M_1^2\bar{M}_2$
34	64	$2B[34]M_1^5\bar{M}_2\bar{M}_3$	54	66	$\eta A[8]D[45]M_1^3\bar{M}_2$
39	63	$\sigma A[32,1]M_1^6M_2^2$	56	66	$A[56]M_1^2\bar{M}_2$, $\nu^2A[50,2]M_1^2M_2$

40	66	$\eta A[39,3]M_1^3\bar{M}_2\langle M_3\rangle$	59	65	$A[59,1]M_2$, $A[59,1]\bar{M}_2$,
40	64	$\eta\sigma A[32,1]M_1^5\langle M_3\rangle$			$A[59,2]\bar{M}_2$
44	66	$2C[44]M_1^2M_2^3$	60	66	$B[60]\bar{M}_2$
45	65	$2D[45]\langle M_1^4\rangle\langle M_2^2\rangle$	61	63	$A[61]M_1$, $\eta B[60]M_1$
45	63	$4D[45]M_1^6M_2$			

FIGURE 7.5.1: Leaders from Rows 1 to 61 of Degree at Least 63

From Figure 7.5.1, we see that there are eleven leaders of degree 63 and four leaders of degree 64. We observed in the derivation of π_{61}^S that $d^{12}(2B[34]M_1^5\bar{M}_2\bar{M}_3) = 4D[45](M_1^6M_2+M_1^2M_3)$. If $4C[18]M_1^7\bar{M}_2M_2^2\bar{M}_3$ survives to E^{28} then $d^{28}(4C[18]M_1^7\bar{M}_2M_2^2\bar{M}_3) = 2D[45]M_1^2M_3$ since $d^{28}(4C[18]M_1^{11}M_2) = D[45]$ and $2D[45](M_1^6M_2+M_2^3)$ is a d^{12}-boundary. However, $d^4(2D[45]M_1^2M_3) = 2\nu D[45]M_3 \neq 0$. Thus, $4C[18]M_1^7\bar{M}_2M_2^2\bar{M}_3$ must hit an element below the 45 row. The only possibility is $d^{22}(4C[18]M_1^7\bar{M}_2M_2^2\bar{M}_3) = \sigma A[32,1]M_1^6M_2^2$.

We show that $\eta\sigma A[32,1]M_1^5\langle M_3\rangle$ transgresses and that $d^{10}(A[52,1]M_1^3M_2) = A[61]M_1$. Since $\nu\cdot\pi_{55}^S = 0$, $A[8]A[50,2] \in \langle\eta,\nu,2\nu\rangle A[50,2] = \eta\langle\nu,2\nu,A[50,2]\rangle$ $\subset \langle\nu,2\nu,\eta A[50,2]\rangle = \langle\nu,2\nu,\nu^2A[45,1]\rangle = \langle\nu,2\nu,\nu^2\rangle A[45,1] = 0$. Thus, $A[8]A[50,2] = 0$. Observe that $\eta A[61] \in \eta\langle\nu^2A[50,2],\nu,\eta\rangle \subset \eta\langle A[50,2],\nu^3,\eta\rangle$ $= \eta\langle A[50,2],\eta A[8],\eta\rangle \supset \eta\langle A[50,2],A[8],\eta^2\rangle \subset \langle A[50,2],A[8],\eta^3\rangle$ $= \langle A[50,2],A[8],4\nu\rangle \supset 2\nu\langle A[50,2],A[8],2\rangle \subset 2\nu\cdot\pi_{59}^S = 0$. Thus, $\eta A[61] \in (\eta^2,\eta A[50,2])$, $\eta A[61] = 0$ and $A[61]M_1$ must be a boundary. Since $\eta\sigma A[32,1]M_1^5\langle M_3\rangle = r_{\Delta_1}(\eta A[39,3]M_1^3\bar{M}_2\langle M_3\rangle)$ (see the derivation of π_{64}^S) and there is no possibility for a hidden differential on $\eta A[39,3]M_1^3\bar{M}_2\langle M_3\rangle$, $\eta\sigma A[32,1]M_1^5\langle M_3\rangle$ can bound neither $A[8]D[45]M_1^2\bar{M}_2$, $\eta A[50,2]M_1^3\bar{M}_2$, $A[61]M_1$ nor $\eta B[60]M_1$. Thus, $\eta\sigma A[32,1]M_1^5\langle M_3\rangle$ transgresses. The only remaining possibility is $d^{10}(A[52,1]M_1^3M_2) = A[61]M_1$.

Now $B[62] = d^{40}(\gamma_2M_1^{20})$, $A[62,1] = d^{44}(\eta^2\sigma M_1^{21}M_2^2)$, $A[62,2] = d^{44}(4\beta_2M_1^{19}\bar{M}_2)$, $A[62,3] = d^{10}(\nu A[50,1]M_1^2M_2)$, $A[62,4] = d^{12}(\eta A[50,2]M_1^3\bar{M}_2)$,

$A'[62] = d^{10}(A[8]D[45]M_1^2\overline{M}_2)$ and $\eta^2B[60]$ are nonzero. We shall see in Section 8.3 that $A[62,1] = \theta_5$ and that $2\theta_5 = 0$. By Theorem 2.4.5(b),

$$A[62,3] \in \langle\eta,\nu,\nu A[50,1],\nu\rangle. \qquad [7.36]$$

Thus, $2A[62,3] \in 2\langle\eta,\nu,\nu A[50,1],\nu\rangle \subset \langle\langle 2,\eta,\nu\rangle,\nu A[50,1],\nu\rangle = \langle 0,\nu A[50,1],\nu\rangle = \nu\cdot\pi_{59}^S$. Now $\nu A[59,2] = \eta^2B[60]$ and $A[59,1]M_1^2$ bounds. Thus, $\nu A[59,1] \in (\eta) = Z_2\eta^2B[60]$, and there is a choice of $A[59,1]$ such that

$$\nu A[59,1] = 0. \qquad [7.37]$$

As observed by Mark Mahowald, $A[62,3]$ is represented by by h_5n in the Adams spectral sequence from which it follows that

$$A[62,3] \in \langle A[30],2,A[31]\rangle. \qquad [7.38]$$

Then $2A[62,3] \in 2\langle A[30],2,A[31]\rangle = \langle 2,A[30],2\rangle A[31] = \eta A[30]A[31] = 0$. Note that $\nu\langle 2,\eta,\eta A[14]\rangle = 2\langle\eta,\eta A[14],\nu\rangle = 2(2C[20]) = \nu(\nu A[14])$ and $\eta\langle 2,\eta,\eta A[14]\rangle = \langle\eta,2,\eta\rangle\eta A[14] = 2\nu(\eta A[14]) = 0$. Thus,

$$\langle 2,\eta,\eta A[14]\rangle = \nu A[14]. \qquad [7.39]$$

Since $d^8(A[8]D[45]M_1^2\overline{M}_2) = D[45]d^8(A[8]M_1^2\overline{M}_2) = D[45](\eta A[14]M_1)$, $A'[62] \in \langle\eta,\eta A[14],D[45]\rangle$. Thus, $2A'[62] \in 2\langle\eta,\eta A[14],D[45]\rangle = \langle 2,\eta,\eta A[14]\rangle D[45] = \nu A[14]D[45] = 0$. We shall see in the derivation of π_{63}^S that $2B[62] = A'[62]$. Thus,

$$2B[62] \in \langle\eta,\eta A[14],D[45]\rangle. \qquad [7.40]$$

Since $A[62,2] = d^{44}(4\beta_2M_1^{19}\overline{M}_2)$ and $A[56] = d^{38}(2\beta_2M_1^{16}\overline{M}_2)$, Theorem 2.4.6(d) implies that

$$A[62,2] \in \langle\nu,\eta,2,A[56]\rangle. \qquad [7.41]$$

Now $2A[62,2] \in 2\langle A[56],2,\eta,\nu\rangle \subset \langle\langle 2,A[56],2\rangle,\eta,\nu\rangle \subset \langle\eta A[56],\eta,\nu\rangle = \langle\nu A[54,1],\eta,\nu\rangle \supset A[54,1]\langle\nu,\eta,\nu\rangle = A[8]A[54,1] = \langle\eta,\nu,2\nu\rangle A[54,1] = \eta\langle\nu,2\nu,A[54,1]\rangle$. Thus, $2A[62,2] \in (\eta,\nu) = Z_2\eta^2B[60]$. Assume that $2A[62,2] = \eta^2B[60]$. Let $\mathfrak{C}$ denote the mapping cone of $A[56]$ with $\rho: S \longrightarrow \mathfrak{C}$ the canonical map and $\partial:\pi_*\mathfrak{C} \longrightarrow \pi_{*-57}^S$ the connecting homomorphism in the long exact sequence induced by multiplication by $A[56]$. In the Atiyah-Hirzebruch spectral sequence for $\mathfrak{C}_*BP$, $\rho(A[62,2])M_1$ is nonzero in E^6 because

$\nu\partial^{-1}(\eta^2) = \rho(A[62,2])$ would imply $A[62,2] \in \langle\nu, A[56], \eta^2\rangle$ and $2A[62,2] \in 2\langle\eta^2, A[56], \nu\rangle = \langle 2, \eta^2, A[56]\rangle\nu$, $A[59,2] \in \langle 2, \eta^2, A[56]\rangle$ and $\eta A[59,2] \in \eta\langle 2, \eta^2, A[56]\rangle = \langle\eta, 2, \eta^2\rangle A[56] = 0$, a contradiction. By 7.41, $d^6(\partial^{-1}(2)M_1M_2) = \rho(A[62,2])M_1$ and $\rho(A[62,2]) \in \langle\nu, \eta, \partial^{-1}(2)\rangle$. Then $2\rho(A[62,2]) \in 2\langle\nu, \eta, \partial^{-1}(2)\rangle \subset \langle 2\nu, \eta, \partial^{-1}(2)\rangle \supset \nu\langle 2, \eta, \partial^{-1}(2)\rangle$. Thus, $\rho(A[59,2])$ is an element of $\langle 2, \eta, \partial^{-1}(2)\rangle$. Then $\rho(\eta A[59,2]) \in \eta\langle 2, \eta, \partial^{-1}(2)\rangle$ $= \langle\eta, 2, \eta\rangle\partial^{-1}(2) = 2\nu\partial^{-1}(2)$ and $\rho(B[60])$ is divisible by ν. Thus, $\rho(\eta B[60]) = 0$ and $\eta B[60]$ is divisible by $A[56]$, a contradiction. Therefore, $2A[62,2] \neq \eta^2 B[60]$ and $2A[62,2] = 0$. We shall see that $\eta A[62,4] \neq 0$. Therefore, $A[62,4]$ is not divisible by 2. We have thus proved the following theorem.

THEOREM 7.5.2 π^S_{62} has a composition series

$$Z_2A[62,4],\ Z_2A[62,1] \oplus Z_2A[62,2] \oplus Z_2A[62,3] \oplus Z_4B[62] \oplus Z_2\eta^2B[60]$$

where $2A[62,4] \in Z_2\eta^2B[60]$.

The computations of Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
11	65	$\beta_1 M_1^{20}\bar{M}_3$	52	66	$A[52,1]M_1^7$
23	65	$2\gamma_2 M_1^{18}\bar{M}_2$	54	66	$\eta A[8]D[45]M_1^3\bar{M}_2$
32	66	$A[32,1]M_1^2\langle M_4\rangle$	56	66	$A[56]M_1^2\bar{M}_2$, $\nu^2A[50,2]M_1^2M_2$
40	64	$\eta\sigma A[32,1]M_1^5\langle M_3\rangle$	59	65	$A[59,1]M_2$, $A[59,1]\bar{M}_2$,
40	66	$\eta A[39,3]M_1^3\bar{M}_2\langle M_3\rangle$			$A[59,2]\bar{M}_2$
44	66	$2C[44]M_1^2M_2^3$	60	66	$B[60]\bar{M}_2$
45	65	$2D[45]\langle M_1^4\rangle\langle M_2^2\rangle$	62	64	$A[62,1]M_1$, $A[62,2]M_1$,
46	66	$\eta^2C[44]M_1^7\bar{M}_2$			$A[62,4]M_1$, $2B[62]M_1$, $\eta^2B[60]M_1$
47	65	$\nu C[44]M_1^3\langle M_2^2\rangle$	62	66	$A[62,3]M_1^2$
51	65	$\sigma C[44]M_1^4\bar{M}_2]$			

FIGURE 7.5.2: Leaders from Rows 1 to 62 of Degree at Least 64

From Figure 7.5.2 we see that there are six leaders of degree 64 and eight leaders of degree 65. Clearly $A[59,1]M_2$ transgresses. Since $\nu A[59,1] = 0$, $A[59,1]\bar{M}_2$ transgresses. Since $\nu A[59,2] = \eta^2 B[60]$, $d^4(A[59,2]\bar{M}_2) = \eta^2 B[60]M_1$. Since $2D[45]\langle M_1^4\rangle\langle M_2^2\rangle$ is twice a boundary, it must be a boundary. Observe that a representative of $8M_1^{26}M_2^2+M_1^{32}$ has boundary $2\eta^2\sigma M_1^{21}\bar{M}_2^2$ union $\beta_1 M_1^{20}M_2^2$ modulo $(2\beta_1)$ and elements of filtration degree 50. Since $r_{\Delta_1}(\beta_1 M_1^{20}\bar{M}_3)$ $= \beta_1 M_1^{20}M_2^2$, $\beta_1 M_1^{20}\bar{M}_3$ has a representative $\mathcal{R}$ with boundary $(A[62,1]2 \wedge \mu_1) \cup (A[62,1] \wedge B_{2\eta})$. Then $\mathcal{R} \cup (B_{A[62,1]2} \wedge \mu_1)$ represents $\beta_1 M_1^{20}\bar{M}_3$ and has boundary $(B_{A[62,1]2} \wedge \eta) \cup (A[62,1] \wedge B_{2\eta})$. Thus, $\beta_1 M_1^{20}\bar{M}_3$ transgresses to an element $B[64,1]$ of $\langle A[62,1],2,\eta\rangle$ and $2B[64,1] \in 2\langle\eta,2,A[62,1]\rangle = \langle 2,\eta,2\rangle A[62,1] = \eta^2 A[62,1]$.

We show that $\eta A'[62] = 0$ and that $A'[62] = 2B[62]$. By 7.40, $A'[62] \in \langle\eta^2,A[14],D[45]\rangle$ and $\eta A'[62] \in \eta\langle\eta^2,A[14],D[45]\rangle \subset \langle\eta^3,A[14],D[45]\rangle =$ $\langle 4\nu,A[14],D[45]\rangle \subset \langle 2\nu,0,D[45]\rangle = 2\nu\cdot\pi_{60}^S + D[45]\cdot\pi_{18}^S = \{C[18]D[45],\eta\alpha_2 D[45]\}$. Now $C[18]D[45] \in D[45]\langle\sigma,2\sigma,\nu\rangle = \langle D[45],\sigma,2\sigma\rangle\nu \subset \nu\cdot\pi_{60}^S = 0$. Since $\eta\cdot\pi_{53}^S =$ $Z_2\eta A[8]D[45]$, $\eta\alpha_2 D[45] \in \eta D[45]\langle\sigma,16,\alpha_1\rangle = \eta\alpha_1\langle D[45],\sigma,16\rangle = k\eta\alpha_1 A[8]D[45]$ $= k\alpha_1\nu^3 D[45] = 0$. Thus, $\eta A'[62] = 0$ and $A'[62]M_1$ must be a boundary. Assume that $d^{12}(\sigma C[44]M_1^4\bar{M}_2) = A'[62]M_1$. Then a representative of $A[40,1]M_1^2\bar{M}_2\langle M_3\rangle$ would show that $\sigma C[44]M_1^6$ is homologous to $A[8]D[45]M_1^2\bar{M}_2$. It would follow that $d^{14}(\eta A[40,1]M_1^3\bar{M}_2\langle M_3\rangle) = \eta A[8]D[45]M_1^3\bar{M}_2$. However, this would contradict the fact that $\eta A[40,1]M_1^3\bar{M}_2\langle M_3\rangle$ is a d^8-boundary. Thus, $\sigma C[44]M_1^4\bar{M}_2$ transgresses. Similarly, if $d^{16}(\nu C[44]M_1^3\langle M_2^2\rangle) = A'[62]M_1$ then a representative of $A[40,1]M_1^6\langle M_2^2\rangle$ would show that $\nu C[44]M_1^2\langle M_2^2\rangle$ is homologous to $A[8]D[45]M_1^2\bar{M}_2$. Then $d^{14}(\eta A[40,1]M_1^7\langle M_2^2\rangle) = \eta A[8]D[45]M_1^3\bar{M}_2$ which contradicts the fact that $\eta A[40,1]M_1^7\langle M_2^2\rangle$ is a d^8-boundary. The only remaining possibility is $d^{40}(2\gamma_2 M_1^{21}) = A'[62]M_1$. Since $2\gamma_2 M_1^{21} = (2M_1)(\gamma_2 M_1^{20})$, $d^{40}(2\gamma_2 M_1^{21}) = 2B[62]M_1$ and $A'[62] = 2B[62]$.

We show that $A[62,4]M_1$ can not be a boundary. Assume that $d^{16}(\nu C[44]M_1^3\langle M_2^2\rangle) = A[62,4]M_1$. Let $\mathfrak{C}$ denote the mapping cone of $\eta C[44]$ with $\rho: S \longrightarrow \mathfrak{C}$ the canonical map. Let $\partial: \pi_*^S \longrightarrow \pi_{*-46}^S$ denote the connecting homomorphism in the long exact sequence induced by multiplication by $\eta C[44]$. The following observations show that $\rho(A[62,4])M_1$ is nonzero in E^{16} of the Atiyah-Hirzebruch spectral sequence for $\mathfrak{C}_*BP$.

1) η does not divide $A[62,4]$.

2) $d^2((\partial^{-1}(\gamma_1)M_1) = \partial^{-1}(\eta\gamma_1)$ and $\partial^{-1}(\eta A[14])M_1 = d^8(\partial^{-1}(A[8])M_1^2\bar{M}_2)$.

3) The only elements from $Z_8\partial^{-1}(\beta_1) \otimes H_6BP$ which surivive to E^6 are d^{12}-boundaries.

4) The only elments from $[Z_2\partial^{-1}(\eta A[8]) \oplus Z_2\partial^{-1}(\eta^2\sigma) \oplus Z_2\partial^{-1}(\alpha_1)] \otimes H_8BP$ which survive to E^8 are d^{10}-boundaries.

5) The only element from $Z_8\partial^{-1}(2\sigma) \otimes H_{10}BP$ which survives to E^{10} is $\partial^{-1}(2\sigma)M_1^5$, and $d^{10}(\partial^{-1}(2\sigma)M_1^5) = \partial^{-1}(A[16])$.

6) The only elements from $Z_8\partial^{-1}(\nu) \otimes H_{16}BP$ which survive to E^{14} are $Z_2\partial^{-1}(\nu)\{M_1^8, M_1\langle M_3\rangle\}$. Assume that one of these elements $\partial^{-1}(\nu)X$ bounds $\rho(A[62,4])M_1$. Let $\mathfrak{C}'$ denote the mapping cone of η with $\rho': S \longrightarrow \mathfrak{C}'$ the canonical map and ∂' the connecting homomorphism in the long exact sequence induced by multiplication by η. In the Atiyah-Hirzebruch spectral sequence for $\mathfrak{C}'_*BP$, $\rho'(A[62,4])M_1 = d^{12}(C[44]\partial'^{-1}(\nu)X) = C[44]d^{12}(\partial'^{-1}(\nu)X)$. Thus, $\rho'(A[62,4])$ is divisible by $C[44]$. The only possibility is $\rho'(A[62,4]) \in \{C[44]\rho'(C[18]), C[44]\rho'(\eta\alpha_2)\}$ and $A[62,4] \in \{C[44]C[18], C[44]\eta\alpha_2\} + Z_2\eta^2B[60]$. Now $C[44]C[18] = d^{12}(2\sigma C[44]M_1^6) = 0$ in E^{12} and $A[62,4]$ is nonzero in E^{12}. Thus, $A[62,4]$ is divisible by η, a contradiction. Therefore, neither element of $Z_2\partial^{-1}(\nu)\{M_1^8, M_1\langle M_3\rangle\}$ can bound $\rho(A[62,4])M_1$.

Now $d^{16}(\rho(\nu C[44])M_1^3\langle M_2^2\rangle) = \rho(A[62,4])M_1$. Applying r_{Δ_1}, we see that $\rho(C[44])M_1M_2\langle M_2^2\rangle$ shows that $\rho(\nu C[44])M_1^2\langle M_2^2\rangle$ is homologous to $\rho(\sigma C[44])M_1^3M_2$ not

$\rho(\eta A[50,2])M_1^3\bar{M}_2$, a contradiction. Thus, $d^{16}(\nu C[44]M_1^3<M_2^2>)$ is not $A[62,4]M_1$. There is no other way for $A[62,4]M_1$ to bound from below the 51 row, and $A[62,4]M_1$ is not a boundary.

Now $A[62,1]M_1$, $A[62,2]M_1$ and $\eta\sigma A[32,1]M_1^5<M_3>$ can only bound from below the 40 row. There are no such leaders of degree 65 remaining which could bound any of these elements. Thus, $A[63] = d^{24}(\eta\sigma A[32,1]M_1^5<M_3>)$, $\eta A[62,1]$, $\eta A[62,2]$ and $\eta A[62,4]$ are nonzero. Since $\eta^2 A[62,1]$ and $\eta^2 A[62,4]$ will be seen to be nonzero, $2A[63] \in Z_2\eta A[62,2]$. We have thus proved the following theorem.

THEOREM 7.5.3. π_{63}^S has a composition series:

$$Z_2A[63],\ Z_2\eta A[62,1] \oplus Z_2\eta A[62,2] \oplus Z_2\eta A[62,4] \oplus Z_{128}\gamma_7$$

where $2A[63] \subset Z_2\eta A[62,2]$.

The computations of Section 6 show that we have the following leaders.

Row	Degree	Leader	Row	Degree	Leader
11	65	$\beta_1 M_1^{20}\bar{M}_3$	54	66	$\eta A[8]D[45]M_1^3\bar{M}_2$
32	66	$A[32,1]M_1^2<M_4>$	56	66	$A[56]M_1^2\bar{M}_2$, $\nu^2 A[50,2]M_1^2M_2$
40	66	$\eta A[39,3]M_1^3\bar{M}_2<M_3>$	59	65	$A[59,1]\bar{M}_2$, $A[59,1]M_2$
44	66	$2C[44]M_1^2M_2^3$	60	66	$B[60]\bar{M}_2$
45	65	$2D[45]<M_1^4><M_2^2>$	62	66	$A[62,1]M_1^2$, $A[62,2]M_1^2$, $A[62,3]M_1^2$,
46	66	$\eta^2 C[44]M_1^7\bar{M}_2$			$2B[62]M_1^2$, $A[62,4]M_1^2$
47	65	$\nu C[44]M_1^3<M_2^2>$	63	65	$A[63]M_1$, $\eta A[62,1]M_1$,
51	65	$\sigma C[44]M_1^4\bar{M}_2$			$\eta A[62,2]M_1$, $\eta A[62,4]M_1$
52	66	$A[52,1]M_1^7$			

FIGURE 7.5.3: Leaders from Rows 1 to 63 of Degree at Least 65

There are ten leaders of degree 65 and fourteen leaders of degree 66. In the derivation of π^S_{63} we showed that $2D[45]\langle M_1^4\rangle\langle M_2^2\rangle$ must bound. Since $2D[45]\langle M_1^4\rangle\langle M_2^2\rangle$ can only bound from below the 34 row, the only possibility is $d^{14}(A[32,1]M_1^2\langle M_4\rangle) = 2D[45]\langle M_1^4\rangle\langle M_2^2\rangle$. Clearly $A[62,1]M_1^2$, $A[62,2]M_1^2$, $A[62,3]M_1^2$, $2B[62]M_1^2$ and $A[62,4]M_1^2$ transgress. Since $\nu B[60] = \nu C[20]^3 = 0$, $B[60]\bar{M}_2$ transgresses. Note that $d^{38}(2\beta_2 M_1^{20}\bar{M}_2) = A[56]M_1^4$, $r_{\Delta_1}(2\beta_2 M_1^{20}\bar{M}_2) = 2\beta_2 M_1^{22}$ and $d^{44}(2\beta_2 M_1^{22}) = A[62,2]$. Thus, $A[56]M_1^4$ is homologous to $A[62,2]M_1$, $d^8(A[56]M_1^2\bar{M}_2) = \eta A[62,2]M_1$ and

$$\sigma A[56] = \eta A[62,2] \qquad [7.42]$$

In addition, $\nu^2A[50,2]M_1^2M_2$ can not hit $\eta A[62,1]M_1$, $\eta A[62,4]M_1$ or $A[63]M_1$ because $\nu^2A[50,2]M_1^2M_2$ is in Image r_{Δ_1} and there is no possibility for a hidden differential on $\nu^2A[50,2]M_1^3M_2$. Thus, $\nu^2A[50,2]M_1^2M_2$ also transgresses. Clearly $\eta A[8]D[45]M_1^3\bar{M}_2$ survives to E^{10} and $d^{10}(\eta A[8]D[45]M_1^3\bar{M}_2) = \eta d^{10}(A[8]D[45]M_1^2\bar{M}_2)M_1 = \eta(2B[62])M_1 = 0$. Therefore, $\eta A[8]D[45]M_1^3\bar{M}_2$ must transgress. Since $\nu A[37] = \eta\sigma A[32,1] + \eta A[39,3]$, $\eta A[39,3]M_1^5\langle M_3\rangle = \eta\sigma A[32,1]M_1^5\langle M_3\rangle$ in E^6. Applying r_{Δ_1}, we see that $d^{24}(\eta A[39,3]M_1^3\bar{M}_2\langle M_3\rangle) = A[63]M_1$. Since $\beta_1 M_1^{20}\bar{M}_3$, $\nu C[44]M_1^3\langle M_2^2\rangle$, $\sigma C[44]M_1^4\bar{M}_2$ and $A[59,1]M_2$ can only bound from below the 40 row, none of these elements can be a boundary.

The remaining leaders of degree 66 are $\eta^2C[44]M_1^7\bar{M}_2$, $2C[44]M_1^4M_3$ and $A[52,1]M_1^7$. Note that $A[59,1]M_2$, $A[59,1]\bar{M}_2$ can only bound from below the 38 row, 52 row, respectively. If $d^{14}(\eta^2C[44]M_1^7\bar{M}_2) = A[59,1]\bar{M}_2$ then $\eta^2C[44]M_1^7$ has a representative with boundary $A[59,1]$ and $d^{14}(\eta^2C[44]M_1^5\bar{M}_2) = A[59,1]M_1$ which contradicts that $\eta^2C[44]M_1^5\bar{M}_2$ is a d^8-boundary and that $A[59,1]M_1$ bounds from the 38 row. Thus, the only possible differentials are $d^{18}(\eta^2C[44]M_1^7\bar{M}_2) = \eta A[62,4]M_1$, $d^{16}(2C[44]M_1^2M_2^3) = A[59,1]\bar{M}_2$, $d^{20}(2C[44]M_1^2M_2^3) = \eta A[62,4]M_1$ or $d^{12}(A[52,1]M_1^7) = \eta A[62,4]M_1$. However, we see from the Adams spectral sequence that the order of π^S_{64} is 2^8. Thus, $\eta^2C[44]M_1^7\bar{M}_2$, $2C[44]M_1^2M_2^3$ and $A[52,1]M_1^7$ must transgress.

Let $A[64,1] = d^{14}(\sigma C[44]M_1^4\bar{M}_2)$, $A[64,2] = d^6(A[59,1]M_1^3)$, $A[64,3] = d^6(A[59,1]M_2)$, $B[64,1] = d^{54}(\beta_1 M_1^{20}\bar{M}_3)$ and $B[64,2] = d^{18}(\nu C[44]M_1^3\langle M_2^2\rangle)$. Then $A[64,1]$, $A[64,2]$, $A[64,3]$, $B[64,1]$, $B[64,2]$, $\eta^2 A[62,1]$ and $\eta^2 A[62,4]$ must be nonzero. By Theorem 2.4.6(c),

$$A[64,1] \in \langle \eta, \nu, \sigma C[44], \sigma \rangle. \qquad [7.43]$$

Thus modulo $2\cdot\text{Indet}\ \langle \eta, \nu, \sigma C[44], \sigma \rangle$, $2A[64,1] \in 2\langle \eta, \nu, \sigma C[44], \sigma \rangle$ $= 2\langle \eta, \nu, \sigma, \sigma C[44] \rangle = \langle\langle 2, \eta, \nu\rangle, \sigma, \sigma C[44]\rangle = \langle 0, \sigma, \sigma C[44]\rangle = \sigma C[44]\cdot\pi_{13}^S = 0$. Then $2A[64,1] \in 2\cdot\text{Indet}\ \langle \eta, \nu, \sigma C[44], \sigma\rangle = 2\langle \eta, \nu, X\rangle \cup 2\langle \eta, Y, \sigma\rangle \cup 2\langle Z, \sigma C[44], \sigma\rangle$ where $X \in \pi_{59}^S$ with $\nu X = 0$, $Y \in \pi_{55}^S$ with $\eta Y = \sigma Y = 0$ and $Z \in \pi_5^S = 0$. Thus, $2A[64,1] \in \langle 2, \eta, \nu\rangle X \cup \langle 2, \eta, \eta A[54,2]\rangle\sigma \cup 2\sigma\cdot\pi_{57}^S = \sigma\langle 2, \eta, \eta A[54,2]\rangle$. Now $\eta\langle 2, \eta, \eta A[54,2]\rangle = \langle \eta, 2, \eta\rangle A[54,2] = 2\nu A[54,2] = 0$. Thus, $2A[64,1] = \sigma(\eta A[56])$ $= 0$. $\nu A[61] \in \nu\langle \eta, \nu, \nu^2 A[50,2]\rangle = \langle \nu, \eta, \nu\rangle\nu^2 A[50,2] = A[8]\nu^2 A[50,2] = 0$, and

$$\nu A[61] = 0. \qquad [7.44]$$

By Theorem 2.4.4(a),

$$A[64,2] \in \langle \eta, A[59,1], \nu\rangle. \qquad [7.45]$$

Thus, $2A[64,2] \in 2\langle \eta, A[59,1], \nu\rangle = \langle 2, \eta, A[59,1]\rangle\nu \subset \nu\cdot\pi_{61}^S = 0$. By Theorem 2.4.4(b),

$$A[64,3] \in \langle A[59,1], \eta, \nu\rangle. \qquad [7.46]$$

Thus, $2A[64,3] \in 2\langle A[59,1], \eta, \nu\rangle = \langle 2, A[59,1], \eta\rangle\nu \subset \nu\cdot\pi_{61}^S = 0$. Recall that in the derivation of π_{63}^S we showed that $2B[64,1] = \eta^2 A[62,1]$ and

$$B[64,1] \in \langle \eta, 2, A[62,1]\rangle. \qquad [7.47]$$

If $2A[62,4] = 0$ then, by Lemma 3.3.14, $\eta^2 A[62,4]$ must be divisble by 2. If $2A[62,4] = \eta^2 B[60]$ then $\eta^2 A[62,4] \in \langle 2, \eta, 2\rangle A[62,4] \subset \langle 2, \eta, 2A[62,4]\rangle$ $= \langle 2, \eta, \eta^2 B[60]\rangle = \langle 2, \eta, \nu A[59,2]\rangle \equiv \langle 2, \eta, \nu\rangle A[59,2] = 0$ modulo (2). Thus, in both cases $\eta^2 A[62,4]$ is divisible by 2. The only possiblity is $2B[64,2] = \eta^2 A[62,4]$. We have thus proved the following theorem.

THEOREM 7.5.4 $\pi_{64}^S = Z_2 A[64,1] \oplus Z_2 A[64,2] \oplus Z_2 A[64,3] \oplus Z_4 B[64,1] \oplus Z_4 B[64,2] \oplus Z_2 \eta\gamma_7$

where $2B[64,1] = \eta^2 A[62,1]$ and $2B[64,2] = \eta^2 A[62,4]$.

In the Adams spectral sequence gg_2 = C[20]C[44] and h_3Q_2 = σA[57] are nonzero infinite cylces distinct from B[64,1], 2B[64,1], B[64,2] and 2B[64,2]. Since both C[20]C[44] and σA[57] are zero in E^{14}, {C[20]C[44],σA[57]} = {A[64,2],A[64,3]}. Note that $\sigma A[57]M_1^2 = d^8(A[57]\langle M_2^2\rangle) = 0$ in E^8 because $A[57]\langle M_2^2\rangle$ is a d^{14}-boundary. Since $A[64,2]M_1^2$ is a d^6-boundary and $A[64,3]M_1^2$ is nonzero in E^8,

$$\sigma A[57] = A[64,2] \text{ and } C[20]C[44] = A[64,3]. \qquad [7.48]$$

6. Tentative Differentials

In this section we give the tentative differentials determined by the differentials on leaders of degree greater than or equal to 47 which were determined in this chapter. We continue the computations made in Chapter 6, Section 4 and use the same format and notation. These computations are complete through degree 68.

<u>DEGREE 9:</u> $\eta^2\sigma$ and α_1

The only differential in degrees less than 70 is $d^{30}(\eta^2\sigma M_1^{21}M_2^2) = A[62,1]$.

<u>DEGREE 11:</u> β_1

The leading differential $d^{40}(4\beta_1(M_1^7\overline{M}_2^3\overline{M}_3+M_1^{10}\overline{M}_2^2\overline{M}_3+M_1^{14}\overline{M}_2^3)) = A[50,1]M_2$ determines tentative differentials whose kernel in degrees less than 70 is given below. These differentials are computed by making the following assignments to monomials in $Z_8\beta_1 \otimes H_{46}BP$: $M_1^7M_2^3M_3$ is assigned 1 and all other monomials are assigned 0.

DEGREE	GROUP	GENERATOR	DEGREE	GROUP		GENERATOR
(54,11)	Z_2	20 0 1 0	(58,11)	Z_2	4/	7 0 1 1
					4/	13 3 1 0
					6/	4 6 1 0
					6/	10 4 1 0
					1/	20 3 0 0
					5/	22 0 1 0
					1/	26 1 0 0

The leading differential $d^{52}(\beta_1 M_1^{20}\bar{M}_3) = B[64,1]$ determines tentative differentials by making the following assignments to monomials in $Z_8\beta_1 \otimes H_{54}BP$: $M_1\bar{M}_2^4\bar{M}_3^2$, $M_1^3\bar{M}_2\bar{M}_3^3$, $M_1^3\bar{M}_2^3\bar{M}_4$, $M_1^5\bar{M}_3\bar{M}_4$, $M_1^9\bar{M}_2^6$, $M_1^{13}\bar{M}_3^2$, $M_1^{17}\bar{M}_2\bar{M}_3$, $M_1^{21}\bar{M}_2^2$ are assigned 1 and all other monomials are assigned 0. We have the tentative differential $d^{52}(\beta_1 M_1^{22}\bar{M}_3) = B[64,1]M_1^2$. There are no remaining elements in degrees less than 70.

DEGREE 17: $\eta^2\gamma_1$ and α_2

The leading differential $d^{32}(\eta^2\gamma_1 M_1^{19}) = \eta A[47]M_1^3$ is computed by assigning 1 to the monomials $\eta^2\gamma_1 M_1\bar{M}_2^6$, $\eta^2\gamma_1 M_1^7\bar{M}_2^4$, $\eta^2\gamma_1 M_1^{13}\bar{M}_2^2$, $\eta^2\gamma_1 M_1^{19}$ and assigning 0 to all the other monomials of degree 55. The tentative differentials have kernel in degrees less than 70 given by the table below, and the new leader is $\alpha_1 M_1^{14}\bar{M}_2$.

DEGREE	BASIS	DEGREE	BASIS	DEGREE	BASIS
(34,17)	14 1 0 0	(36,17)	15 1 0 0	(42,17)	14 0 1 0
	15 2 0 0	(44,17)	15 0 1 0	(46,17)	13 1 1 0
					23 0 0 0
	14 3 0 0	(48,17)	14 1 1 0		15 3 0 0
(50,17)	12 2 1 0		15 1 1 0	(52,17)	13 2 1 0
					19 0 1 0

The leading differential $d^{32}(\alpha_2 M_1^{14}\bar{M}_2) = A[50,1]$ is computed by assigning 1 to the monomials $\eta^2\gamma_1 M_1^5\bar{M}_2^4$, $\eta^2\gamma_1 M_1^{11}\bar{M}_2^2$, $\eta^2\gamma_1 M_1^{17}$, $\alpha_2 M_1^2\bar{M}_2^5$, $\alpha_2 M_1^8\bar{M}_2^3$, $\alpha_2 M_1^{14}\bar{M}_2$ and 0 to all other monomials of degree 51. These tentative differentials have the following kernel in degrees less than 70:

DEGREE	BASIS	DEGREE	BASIS
(42,17)	15 2 0 0	(46,17)	13 1 1 0
			23 0 0 0

The leading differential $d^{38}(\eta^2\gamma_1 M_1^{15}\bar{M}_2^2) = A[52,1]M_1^3$ determines tentative differentials which are a monomorphism on the remaining elements in degrees less than 70. Thus, there are no remaining elements.

DEGREE 18: C[18]

The differential $d^{22}(4C[18]M_1^7\bar{M}_2 M_2^2\bar{M}_3) = \sigma A[32,1]M_1^6M_2^2$ leaves no remaining elements in degrees less than 69.

DEGREE 19: β_2

The leading differential $d^{36}(\beta_2 M_1^{18}) = A[54,1]$ determines tentative differentials whose kernel in degrees less than 70 is given below. These differentials are computed by making the following assignments to monomials of $Z_8\beta_2 \otimes H_{36}BP$: M_1^{18} is assigned 1, and all other monomials are assigned 0. The new β_2-leader is $2\beta_2 M_1^{16}\bar{M}_2$.

DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR
(38,19)	Z_2	2/ 16 1 0 0	(42,19)	Z_2	14 0 1 0		Z_2	2/ 18 1 0 0
(44,19)	Z_4	2/ 19 1 0 0	(46,19)	Z_2	1/ 14 3 0 0		Z_2	2/ 20 1 0 0
		6/ 22 0 0 0			6/ 20 1 0 0			
(48,19)	Z_4	14 1 1 0	(50,19)	Z_2	1/ 15 1 1 0		Z_2	2/ 16 3 0 0
					7/ 18 0 1 0			
					1/ 22 1 0 0		Z_2	2/ 22 1 0 0

The leading differentials $d^{38}(2\beta_2 M_1^{16}\bar{M}_2) = A[56]$ determines tentative differentials by making the following assignments to monomials of $Z_8\beta_2 \otimes H_{38}BP$: M_1^{19} is assigned 1, $M_1^{13}M_2^2$ is assigned 6 and all other monomials are assigned 0. The kernel of these differentials in degrees less than 70 is given by the table below and the new β_2-leader is $\beta_2 M_1^{14}\bar{M}_3$.

DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR
(42,19)	Z_2	1/ 14 0 1 0 6/ 18 1 0 0	(44,19)	Z_2	4/ 19 1 0 0	(46,19)	Z_2	1/ 14 3 0 0 6/ 20 1 0 0
(48,19)	Z_4	14 1 1 0	(50,19)	Z_2	1/ 15 1 1 0 7/ 18 0 1 0 7/ 22 1 0 0			

The leading differential $d^{40}(\beta_2 M_1^{14}\bar{M}_3) = \eta A[57]M_1$ determines tentative differentials which are computed by assigning 1 to $\beta_2 M_1^{15}M_2^2$ and 0 to all other monomials of $Z_8\beta_2 \otimes H_{42}BP$. The kernel of these differentials in degrees less than 70 is given by the table below.

DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR
(44,19)	Z_2	4/ 19 1 0 0	(48,19)	Z_2	2/ 14 1 1 0

The leading differential $d^{44}(4\beta_2 M_1^{19}\bar{M}_2) = A[62,2]$ determines tentative differentials which are computed by making the following assignments to monomials of $Z_8\beta_2 \otimes H_{44}BP$: $M_1^{19}M_2$ is assigned 1, $M_1^{16}M_2^2$ and M_1^{22} are assigned 2, $M_1^{13}M_2^3$ is assigned 6, $M_1^{15}M_3$ and $M_1^{12}M_2M_3$ are assigned 4 while all other monomials are assigned 0. There are no tentative differentials in degrees less than 70. The only remaining element is $2\beta_2 M_1^{14}\bar{M}_2\bar{M}_3$.

DEGREE 21: $\nu C[18]$

The leading differential $d^{24}(\nu C[18]M_1^6\bar{M}_2\bar{M}_3) = (\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2$ determines tentative differentials which are a monomorphism on the remaining elements of $Z_2(\nu C[18]) \otimes H_*BP$ in degrees less than 68.

DEGREE 22: $\nu A[19]$

The leading differential $d^{30}(\nu A[19]M_1^7M_2^2M_3) = \sigma C[44]M_1^2\bar{M}_2$ leaves no remaining elements in degrees less than 69.

DEGREE 23: γ_2

The leading differential $d^{40}(\gamma_2 M_1^{20}) = B[62]$ determines tentative differentials by assigning 1 to $\gamma_2 M_1^{20}$ and 0 to all other monomials of $Z_{16}\gamma_2 \otimes H_{40}BP$. The kernel in degrees less than 70 is given by the table below, and the new γ_2-leader is $2\gamma_2 M_1^{18}\overline{M}_2$.

DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR
(42,23)	Z_2	2/ 18 1 0 0	(44,23)	Z_4	2/ 22 0 0 0	(46,23)	Z_2	2/ 23 0 0 0 6/ 20 1 0 0
	Z_8	2/ 20 1 0 0						

The leading differential $d^{40}(2\gamma_2 M_1^{18}\overline{M}_2) = 2B[62]M_1$ determines tentative differentials by assigning 1 to $\gamma_2 M_1^{21}$ and 0 to all other monomials of $Z_{16}\gamma_2 \otimes H_{42}BP$. The kernel in degrees less than 70 is given by the table below, and the new γ_2-leader is $2\gamma_2 M_1^{22}$.

DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR
(44,23)	Z_4	2/ 22 0 0 0	(46,23)	Z_4	4/ 20 1 0 0

DEGREE 24: $\eta A[23]$

The leading differential $d^{22}(\eta A[23]M_1^{15}\overline{M}_2) = 2D[45]M_1^4 M_2$ determines tentative differentials which are a monomorphism on the remaining elements of degree less than 69.

DEGREE 30: A[30]

The leading differential $d^{16}(A[30]\langle M_4\rangle) = \sigma A[32,1]M_1^4 M_2^2$ determines tentative differentials which are a monomorphism on $Z_2A[30]\langle M_4\rangle \otimes Z_2[\langle M_1^4\rangle^2, \langle M_2^2\rangle^2, \langle M_3\rangle^2, \langle M_4\rangle^2, \{M_5\}, \dots, \{M_n\}, \dots]$. There are no remaining elements.

<u>DEGREE 32:</u> A[32,1]

The leading differential $d^8(A[32,1](M_1^2M_3+M_2^3) = A[39,1]M_1^2M_2$ determines tentative differentials with kernel in degrees less than 69 given by $Z_2A[32,1]\{M_1^8\overline{M}_3, M_1^2\overline{M}_4+M_1^2\overline{M}_2^5+M_1^8\overline{M}_2^3+M_1^{14}\overline{M}_2\}$. The new A[32,1]-leader is $A[32,1]M_1^8\overline{M}_3$. The leading differential $d^{14}(A[32,1]M_1^8\overline{M}_3) = 2D[45]M_1\langle M_3\rangle$ determines no tentative differentials by assigning 1 to $A[32,1]M_1^8M_3$ and 0 to all other monomials of $Z_2A[32,1] \otimes H_{30}BP$. The only remaining element in degree less than 69 is $A[32,1](M_1^2\overline{M}_4+M_1^2\overline{M}_2^5+M_1^8\overline{M}_2^3+M_1^{14}\overline{M}_2)$. Moreover, $d^{32}(A[32,1](M_1^2\overline{M}_4+M_1^2\overline{M}_2^5+M_1^8\overline{M}_2^3+M_1^{14}\overline{M}_2)) = 2D[45]\langle M_1^4\rangle\langle M_2^2\rangle$.

<u>DEGREE 34:</u> B[34]

The leading differential $d^8(B[34]M_1^4M_2) = \eta A[40,1]M_1^3$ determines tentative differentials by making the following assignments to monomials of $Z_2\otimes [Z_4B[34] \otimes H_{14}\ BP]$: $M_1^4M_2$ and M_1^7 are assigned 1 and all other monomials are assigned 0. The kernel of these tentative differentials in degrees less than 69 is given by the table below.

DEGREE	GENERATOR	DEGREE	GENERATOR	DEGREE	GENERATOR
(12,34)	6 0 0 0	(14,34)	0 0 1 0 4 1 0 0	(18,34)	2 0 1 0 6 1 0 0
(20,34)	4 2 0 0	(22,34)	4 0 1 0 8 1 0 0	(24,34)	6 2 0 0
(26,34)	0 2 1 0 4 3 0 0		6 0 1 0 10 1 0 0	(28,34)	14 0 0 0
(30,34)	8 0 1 0 12 1 0 0		2 2 1 0 6 3 0 0	(34,34)	4 2 1 0 8 3 0 0 10 0 1 0 14 1 0 0

The leading differential $d^6(B[34]M_1^6) = 2D[45]$ determines tentative differentials by assigning 1 to $B[34]M_1^6$ and 0 to all other monomoials of

$Z_2 \otimes (Z_4B[34] \otimes H_{12}BP)$. These tentative differentials are a monomorphism on the elements of the table above. There are no elements of $Z_2 \otimes [Z_4B[34] \otimes H_{14} BP]$ remaining in degrees less than 69.

The leading differential $d^{12}(2B[34]M_1^5\overline{M}_2\overline{M}_3) = 4D[45]M_1^6M_2$ determines the tentative differential $d^{12}(2B[34]M_1\overline{M}_2^3\overline{M}_3) = 2D[45]M_1^5\langle M_2^2\rangle$. There are no remaining elements in degrees less than 69.

<u>DEGREE 36:</u> A[36]

The leading differential $d^{10}(A[36]M_1^2\overline{M}_3) = 4D[45]M_1M_2$ determines tentative differentials which have kernel in degrees less than 69 equal to $Z_2(A[36]M_1^6\overline{M}_2^2)$. Moreover, $d^{16}(A[36]M_1^6\overline{M}_2^2) = \nu A[50,1]M_1^3$.

<u>DEGREE 38:</u> $\eta\sigma A[30] = \eta A[37]$

The leading differential $d^{10}(\eta\sigma A[30]M_1^3\overline{M}_2) = \eta^2D[45]M_1$ determines tentative differentials which are a monomorphsim on $Z_2\eta\sigma A[30]M_1^3\overline{M}_2 \otimes B\langle 4\rangle$. There are no remaining elements.

<u>DEGREE 38:</u> B[38]

The relation $\sigma B[38] = 4D[45]$ determines tentative d^8-differentials in degrees less than 69 with kernel $Z_2(B[38]M_1^2\overline{M}_2\overline{M}_3)$. Moreover, $d^{22}(B[38]M_1^2\overline{M}_2\overline{M}_3) = A[59,1]M_1$.

<u>DEGREE 39:</u> $\eta B[38]$

The leading differential $d^{12}(\eta B[38]M_1^3\overline{M}_2) = \eta A[47]M_1$ determines tentative differentials which are a monomorphism on $Z_2\eta B[38]M_1^3\overline{M}_2 \otimes B\langle 4\rangle$. There are no remaining elements.

<u>DEGREE 39:</u> $A[39,1] = \sigma A[32,3]$

The leading differential $d^8(A[39,1]M_1^5) = \eta^2 C[44]M_1$ determines tentative differentials which in degrees less than 70 have kernel
$Z_2A[39,1]\{M_1^2M_2, M_1^3M_2, M_1^6M_2+M_1^3M_2^2, M_1M_2^3, M_1M_2M_3, M_1^2M_2M_3, M_1^{10}M_2, M_1^3M_2M_3+M_1^4M_2^3, M_1^{11}M_2, M_1^6\bar{M}_2^3\}$
Thus, the new $A[39,1]$-leader is $A[39,1]M_1^2M_2$.

The leading differential $d^8(A[32,1](M_1^2M_3+M_2^3)) = A[39,1]M_1^2M_2$ determines tentative differentials which in degrees less than 70 have cokernel
$Z_2A[39,1]\{M_1^3M_2, M_1M_2^3, M_1M_2M_3, M_1^2M_2M_3, M_1^3M_2M_3+M_1^4M_2^3, M_1^{11}M_2, M_1^6\bar{M}_2^3\}$. Thus, the new $A[39,1]$-leader is $A[39,1]M_1^3M_2$.

The leading differential $d^{10}(A[39,1]M_1^3M_2) = A[50,2]$ determines tentative differentials which are a monomorphism on $Z_2(A[39,1]M_1^3M_2) \otimes B\langle 4\rangle$. The only remaining elements in degrees less than 70 are $A[39,1]\{M_1M_2M_3, M_1^6\bar{M}_2^3\}$.
Moreover, $d^{14}(A[39,1]M_1M_2M_3) = A[52,1]M_1M_2$.

<u>DEGREE 39:</u> $A[39,3]$

The leading differential $d^8(A[39,3]M_1^2\bar{M}_2) = \eta A[45,1]M_1$ determines tentative differentials which are a monomorphism on $Z_2(A[39,3]M_1^2\bar{M}_2) \otimes B\langle 4\rangle$. There are no remaining elements.

<u>DEGREE 39:</u> $\sigma A[32,1]$

The leading differential $d^{10}(\sigma A[32,1]M_1^4\bar{M}_2) = \nu A[45,1]M_1^2$ determines tentative differentials with kernel in degrees less than 68 equal to
$Z_2\sigma A[32,1]\{M_1^4M_2^2, M_1^6M_2^2\}$. Moreover, $d^{16}(A[30]\langle M_4\rangle) = \sigma A[32,1]M_1^4M_2^2$ and
$d^{22}(4C[18]M_1^7\bar{M}_2M_2^2\bar{M}_3) = \sigma A[32,1]M_1^6M_2^2$.

DEGREE 40: $C[20]^2$

The leading differential $d^8(2C[20]^2M_1\bar{M}_2) = B[47]$ determines tentative differentials which are a monomorphism on $Z_2(2C[20]^2)\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B<4>$. There are no remaining elements.

DEGREE 40: $A[40,1]$

The leading differential $d^8(A[40,1]M_1^6) = \nu C[44]M_1^2$ determines tentative differentials with kernel in degrees less than 69 equal to $Z_2A[40,1]\{\bar{M}_2\bar{M}_3, M_1^2\bar{M}_2\bar{M}_3\}$. Thus, the new $A[40,1]$-leader is $A[40,1](\bar{M}_2\bar{M}_3+M_1^2M_2^2)$.

The leading differential $d^{12}(A[40,1]\bar{M}_2\bar{M}_3) = \sigma C[44]M_1^4$ determines the tentative differential $d^{12}(A[40,1]M_1^2\bar{M}_2\bar{M}_3) = \sigma C[44]M_1^6$. There are no remaining elements.

DEGREE 40: $A[40,2]$

The leading differential $d^8(A[40,2]M_1^6\bar{M}_2) = A[47]M_1^2\bar{M}_2$ determines only the tentative differential $d^8(A[40,2]M_1^6\bar{M}_3) = A[47]M_1^6\bar{M}_2$ in degrees less than 69. There are no remaining elements.

DEGREE 40: $\eta A[39,3]$

The leading differential $d^{20}(\nu C[18]M_1^6\bar{M}_2\bar{M}_3) = (\eta A[39,3]+\eta\sigma A[32,1])M_1^3\bar{M}_2$ determines tentative differentials with image $Z_2(\eta A[39,3]M_1^3\bar{M}_2+\eta\sigma A[32,1]M_1^3\bar{M}_2)\{1,<M_1^4>,<M_2^2>,<M_1^4>^2\}$ in degrees less than 69. The only remaining element is $\eta A[39,3]M_1^3\bar{M}_2<M_3>$.

DEGREE 40: $\eta\sigma A[32,1]$

The leading differential $d^{16}(\eta\sigma A[32,1]M_1^5<M_3>) = A[63]$ determines the tentative differential $d^{16}(\eta\sigma A[32,1]M_1\langle M_2^2\rangle\langle M_3\rangle) = A[63]M_1^2$. There are no remaining elements in degrees less than 69.

DEGREE 41: $\eta A[40,1]$

The leading differential $d^8(B[34]M_1^4M_2) = \eta A[40,1]M_1^3$ determines tentative differentials which are a monomorphsim on the remaining elements in degrees less than 68.

DEGREE 42: $C[42]$

The leading differential $d^6(2C[42]M_1^3) = A[47]$ determines tentative differentials which are a monomorphism on $Z_2(2C[42])\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$. There are no remaining elements in degrees less than 69.

DEGREE 42: $\eta^2C[20]^2$

The leading differential $d^6(\eta^2C[20]^2M_1\bar{M}_2) = \eta^2A[45,2]M_1$ determines tentative differentials which are a monomorphism on $Z_2\eta^2C[20]^2\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 44: $C[44]$

The leading differential $d^4(C[44]M_1^2) = \nu C[44]$ determines tentative differentials which are a monomorphism on $Z_2 \otimes (Z_8C[44]\{M_1^2, \bar{M}_2, M_1^2\bar{M}_2\} \otimes B\langle 4\rangle)$. The remaining elements from $Z_2 \otimes (Z_8C[44] \otimes H_*BP)$ in degrees less than 69 are $Z_2C[44]\{\langle M_1^4\rangle, \langle M_2^2\rangle, \langle M_3\rangle, \langle M_1^4\rangle\langle M_2^2\rangle, \langle M_1^4\rangle\langle M_3\rangle, \langle M_1^4\rangle^3\}$. Thus, the new $C[44]$-leader is $C[44]\langle M_1^4\rangle$.

The leading differential $d^8(C[44]\langle M_1^4\rangle) = \sigma C[44]$ determines tentative differentials which are a monomorphism on $Z_2C[44]\{\langle M_1^4\rangle, \langle M_2^2\rangle, \langle M_3\rangle, \langle M_1^4\rangle\langle M_2^2\rangle, \langle M_1^4\rangle\langle M_3\rangle, \langle M_1^4\rangle^3\}$. There are no remaining elements from $Z_2 \otimes (Z_8C[44] \otimes H_*BP)$ in degrees less than 69.

The leading differential $d^{12}(2C[44]M_1^6) = 2\sigma C[44]M_1^2$ determines tentative differentials with kernel on $Z_2 \otimes [Z_4(2C[44]) \otimes H_*BP]$ in degrees less than 69 equal to $Z_2(2C[44])\{M_1^2M_2^3, M_1^5M_3+M_1^6M_2^2+M_1^3M_2^3\}$.

The leading differential $d^{14}(4C[44]M_1^7) = A[57]$ determines tentative differentials which are a monomorphism on $Z_2(4C[44])\{M_1^7, M_1^6\bar{M}_2, M_1^7\bar{M}_2, M_1^4\bar{M}_3\}$. The only remaing element in degrees less than 69 is $4C[44]M_1^3M_2^3$.

<u>DEGREE 45:</u> D[45]

The leading differential $d^{12}(B[34]M_1^6) = 2D[45]$ determines tentative differentials with cokernel in $Z_2 \otimes (Z_8(2D[45]) \otimes H_*BP)$ in degrees less than 68 given by the table below. The 2D[45]-leader is $2D[45]M_1^2$.

DEGREE	GENERATOR	DEGREE	GENERATOR	DEGREE	GENERATOR
(4,45)	2 0 0 0	(6,45)	3 0 0 0	(8,45)	1 1 0 0
(10,45)	2 1 0 0	(12,45)	3 1 0 0		6 0 0 0
(14,45)	1 2 0 0		7 0 0 0	(16,45)	1 0 1 0
	2 2 0 0		5 1 0 0	(18,45)	2 0 1 0
	3 2 0 0		6 1 0 0	(20,45)	3 0 1 0
	0 1 1 0		1 3 0 0		4 2 0 0
	7 1 0 0		10 0 0 0	(22,45)	1 1 1 0
	4 0 1 0		2 3 0 0		5 2 0 0
	11 0 0 0				

The d^8-differentials determined by the relation $\sigma B[38] = 4D[45]$ have cokernel on the elements from $Z_4(4D[45]) \otimes H_*BP$ in degrees less than 68 equal to

$(Z_2(4D[45])\{M_1M_2, M_1^3M_2\} \otimes B<4>) \oplus Z_2 4D[45]M_1^6\bar{M}_2$. Thus, the new 4D[45]-leader is $4D[45]M_1M_2$, and the new 8D[45]-leader is 0.

The leading differential $d^4(D[45]M_1^2) = \nu D[45]$ determines leading differentials which are a monomorphism on $[Z_4 \otimes (Z_{16}D[45]\{M_1^2, \bar{M}_2, M_1^2\bar{M}_2\} \otimes B<4>)]$ $\oplus [Z_2 \otimes ((Z_8(2D[45])\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B<4>)]$. The remaining elements from $Z_4 \otimes (Z_{16}D[45] \otimes H_*BP)$ in degrees less than 68 are $Z_2(2D[45])\{M_1<M_2^2>, M_1<M_3>, M_2<M_3>, <M_1^4><M_2^2>, <M_1^4>M_3, M_1^5<M_2^2>\}$. Thus, the new D[45]-leader is 0, and the new 2D[45]-leader is $2D[45]M_2<M_1^4>$.

The leading differential $d^{22}(\eta A[23]M_1^{15}\bar{M}_2) = 2D[45]M_1^4M_2$ determines tentative differentials with image $Z_2(2D[45])\{M_1^4M_2, M_2<M_3>, M_3<M_1^4>\}$ in degrees less than 68. The remaining elements from $Z_2 \otimes [Z_8(2D[45]) \otimes H_*BP]$ are $Z_2(2D[45])\{M_1<M_3>, <M_1^4><M_2^2>, M_1^5<M_2^2>\}$, and the new 2D[45]-leader is $2D[45]M_1<M_3>$.

The leading differentials $d^{14}(A[32,1]M_1^8\bar{M}_3) = 2D[45]M_1<M_3>$ and $d^{14}(A[32,1]M_1^2<M_4>) = 2D[45]<M_1^4><M_2^2>$ determine no tentative differentials in degrees less than 68. The only remaining element from $Z_2 \otimes [Z_8(2D[45]) \otimes H_*BP]$ is $2D[45]M_1^5<M_2^2>$.

The leading differential $d^{10}(A[36]M_1^2\bar{M}_3) = 4D[45]M_1M_2$ determines tentative differentials with cokernel on the remaining elements from $Z_4(4D[45]) \otimes H_*BP$ in degrees less than 68 equal to $Z_2(4D[45])\{M_1^3M_2, M_1^6\bar{M}_2, M_1^3M_2<M_1^4>\}$. Thus, the new 4D[45]-leader is $4D[45]M_1^3M_2$.

The leading differential $d^{12}(4D[45]M_1^3M_2) = A[52,1]M_1^2$ determines tentative differentials which are a monomorphism on $Z_2(4D[45])\{M_1^3M_2, M_1^3M_2<M_1^4>\}$. Thus, the only remaining element from $Z_4(4D[45]) \otimes H_*BP$ in degrees less than 68 is $4D[45]M_1^6M_2$.

The leading differential $d^{12}(2B[34]M_1^5\bar{M}_2\bar{M}_3) = 4D[45]M_1^6M_2$ determines the tentative differential $d^{12}(2B[34]M_1\overline{M}_2^3\overline{M}_3) = 2D[45]M_1^5\langle M_2^2\rangle$. There are no remaining elements from $Z_8C[44] \otimes H_*BP$ in degrees less than 69.

DEGREE 45: A[45,1]

The leading differential $d^4(A[45,1]M_1^2) = \nu A[45,1]$ determines tentative differentials which are a monomorphism on $Z_2A[45,1]\{M_1^2, \overline{M}_2, M_1^2\overline{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 45: A[45,2]

The leading differential $d^4(A[45,2]\overline{M}_2) = \eta B[47]M_1$ determines tentative differentials which are a monomorphism on $Z_2A[45,2]\{\overline{M}_2, M_1^2\overline{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 46: $\eta D[45]$ and $\eta A[45,2]$

Both $\eta^2D[45]$ and $\eta^2A[45,2]$ are nonzero. Thus, the only element of E^4 with a representative in $(Z_2\eta D[45] \oplus Z_2\eta A[45,2]) \otimes H_*BP$ is 0.

DEGREE 46: $\eta^2C[44]$

The leading differential $d^2(\eta C[44]M_1) = \eta^2C[44]$ determines tentative differentials with image $Z_2(\eta^2C[44]) \otimes B\langle 2\rangle$ and cokernel $Z_2(\eta^2C[44]M_1) \otimes B\langle 2\rangle$. The $\eta^2C[44]$-leader is $\eta^2C[44]M_1$.

The leading differential $d^8(A[39,1]M_1^5) = \eta^2C[44]M_1$ determines tentative differentials with cokernel in degrees less than 69 equal to $Z_2\eta^2C[44]\{M_1^7\overline{M}_2, M_1^5\overline{M}_2^2\}$. Moreover, $d^{14}(\eta^2C[44]M_1^7\overline{M}_2) = A[59,1]\overline{M}_2$.

DEGREE 46: $\eta A[45,1]$

The leading differential $d^2(A[45,1]M_1) = \eta A[45,1]$ determines tentative differentials with image $Z_2(\eta A[45,1]) \otimes B<2>$ and cokernel $Z_2(\eta A[45,1]M_1) \otimes B<2>$. The $\eta A[45,1]$-leader is $\eta A[45,1]M_1$.

The leading differential $d^8(A[39,3]M_1^2\bar{M}_2) = \eta A[45,1]M_1$ determines tentative differentials with image $Z_2\eta A[45,1]M_1 \otimes B<4>$. The remaining elements are $Z_2\eta A[45,1]\{M_1^3, M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B<4>$, and the new $\eta A[45,1]$-leader is $\eta A[45,1]M_1^3$.

The leading differential $d^6(\eta A[45,1]M_1^3) = 2\sigma C[44]$ determines tentative differentials which are a monomorphism on $Z_2\eta A[45,1]\{M_1^3, M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B<4>$. There are no remaining elements.

DEGREE 47: $\eta^2 A[45,2] = 2B[47] = \eta A[14]A[32,2]$

The leading differential $d^2(\eta A[45,2]M_1) = \eta^2 A[45,2]$ determines tentative differentials with image $Z_2(\eta^2 A[45,2]) \otimes B<2>$ and cokernel $Z_2(\eta^2 A[45,2]M_1) \otimes B<2>$. The $\eta^2 A[45,2]$-leader is $\eta^2 A[45,2]M_1$.

The leading differential $d^6(\eta^2 C[20]^2 M_1\bar{M}_2) = \eta^2 A[45,2]M_1$ determines tentative differentials with image $Z_2\eta^2 A[45,2]\{M_1, M_1^3\} \otimes B<4>$. The remaining elements are $Z_2\eta^2 A[45,2]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B<4>$, and the new $\eta^2 A[45,2]$-leader is $\eta^2 A[45,2]M_1\bar{M}_2$.

The leading differential $d^8(\eta^2 A[45,2]M_1M_2) = A[54,2]$ determines tentative differentials which are a monomorphism on $Z_2\eta^2 A[45,2]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B<4>$. Thus, there are no remaining elements.

DEGREE 47: $\eta^2 D[45]$

The leading differential $d^2(\eta D[45]M_1) = \eta^2 D[45]$ determines tentative differentials with image $Z_2(\eta^2 D[45]) \otimes B\langle 2\rangle$ and cokernel $Z_2(\eta^2 D[45]M_1) \otimes B\langle 2\rangle$. The $\eta^2 D[45]$-leader is $\eta^2 D[45]M_1$.

The tentative differentials determined by the leading differential $d^{10}(\eta\sigma A[30]M_1^3\bar{M}_2) = \eta^2 D[45]M_1$ have image $Z_2\eta^2 D[45]M_1 \otimes B\langle 4\rangle$. The remaining elements are $Z_2(\eta^2 D[45])\{M_1^3, M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$, and the new $\eta^2 D[45]$-leader is $\eta^2 D[45]M_1^3$.

The leading differential $d^6(\eta^2 D[45]M_1^3) = A[52,1]$ determines tentative differentials which are a monomorphism on $Z_2(\eta^2 D[45])\{M_1^3, M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

<u>DEGREE 47:</u> $\nu C[44]$

The leading differential $d^4(C[44]M_1^2) = \nu C[44]$ determines tentative differentials with image $Z_2\nu C[44]\{1, M_1, M_2\} \otimes B\langle 4\rangle$ and cokernel $Z_2\nu C[44]\{M_1^2, M_1^3, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$. The $\nu C[44]$-leader is $\nu C[44]M_1^2$.

The leading differential $d^8(A[40,1]M_1^6) = \nu C[44]M_1^2$ determines tentative differentials whose image in degrees less than 68 is all of $Z_2\nu C[44]\{M_1^2, M_1^3, M_1^2M_2\} \otimes B\langle 4\rangle$ except for $Z_2\nu C[44]M_1^3\langle M_2^2\rangle$. The remaining elements in degrees less than 68 are $Z_2\nu C[44]M_1^3\langle M_2^2\rangle \oplus Z_2\nu C[44]\{M_1M_2, M_1^3M_2\} \otimes B\langle 4\rangle$, and the new $\nu C[44]$-leader is $\nu C[44]M_1M_2$.

The leading differential $d^8(\nu C[44]M_1M_2) = \eta\sigma C[44]M_1$ determines tentative differentials which are a monomorphism on $Z_2\nu C[44]\{M_1M_2, M_1^3M_2\} \otimes B\langle 4\rangle$. The only remaining element in degrees less than 68 is $\nu C[44]M_1^3\langle M_2^2\rangle$. Moreover, $d^{18}(\nu C[44]M_1^3\langle M_2^2\rangle) = B[64,2]$.

<u>DEGREE 47:</u> $A[47] = \sigma A[40,2]$

The leading differential $d^6(2C[42]M_1^3) = A[47]$ determines leading differentials with image $Z_2A[47]\{1,M_1^2,\bar{M}_2\} \otimes B\langle 4\rangle$. Since $\eta A[47] \neq 0$, the remaining elements are $Z_2A[47]M_1^2\bar{M}_2 \otimes B\langle 4\rangle$, and the new A[47]-leader is $A[47]M_1^2\bar{M}_2$.

The leading differential $d^8(A[40,2]M_1^6\bar{M}_2) = A[47]M_1^2\bar{M}_2$ determines tentative differentials with image in degrees less than 68 equal to $Z_2A[47]M_1^2\bar{M}_2 \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 47: B[47]

The leading differential $d^6(2C[20]^2M_1\bar{M}_2) = B[47]$ determines tentative differentials with image $Z_2B[47]\{1,M_1^2\} \otimes B\langle 4\rangle$. Since $\eta B[47] \neq 0$, the remaining elements are $Z_2 \otimes (Z_4B[47]\{\bar{M}_2,M_1^2\bar{M}_2\} \otimes B\langle 4\rangle)$, and the new B[47]-leader is $B[47]\bar{M}_2$.

The leading differential $d^6(B[47]\bar{M}_2) = A[52,2]$ determines tentative differentials which are a monomorphism on $Z_2 \otimes (Z_4B[47]\{\bar{M}_2,M_1^2\bar{M}_2\} \otimes B\langle 4\rangle)$. There are no remaining elements.

DEGREE 48: $\eta A[47]$

The leading differential $d^2(A[47]M_1) = \eta A[47]$ determines tentative differentials with image $Z_2(\eta A[47]) \otimes B\langle 2\rangle$ and cokernel $Z_2(\eta A[47]M_1) \otimes B\langle 2\rangle$. The $\eta A[47]$-leader is $\eta A[47]M_1$.

The leading differential $d^4(\eta B[38]M_1^3\bar{M}_2) = \eta A[47]M_1$ determines tentative differentials with image $Z_2\eta A[47]M_1 \otimes B\langle 4\rangle$. The remaining elements are $Z_2\eta A[47]\{M_1^3,M_1\bar{M}_2,M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$, and the new $\eta A[47]$-leader is $\eta A[47]M_1^3$.

The leading differential $d^{32}(\eta^2\gamma_1M_1^{19}) = \eta A[47]M_1^3$ determines tentative differentials which are surjective in degrees less than 69. Thus, there are no remaining elements.

<u>DEGREE 48:</u> $\nu A[45,1]$

The leading differential $d^4(A[45,1]M_1^2) = \nu A[45,1]$ determines tentative differentials with cokernel $Z_2\nu A[45,1]\{M_1^2, M_1^3, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B<4>$. The new $\nu A[45,1]$-leader is $\nu A[45,1]M_1^2$.

The leading differential $d^{10}(\sigma A[32,1]M_1^4\bar{M}_2) = \nu A[45,1]M_1^2$ determines tentative differentials with image in degrees less than 69 equal to $Z_2\nu A[45,1]\{M_1^2, M_1M_2\} \otimes B<4>$. The remaining elements in degrees less than 69 are $Z_2\nu A[45,1]\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B<4>$, and the new $\nu A[45,1]$-leader is $\nu A[45,1]M_1^3$.

The leading differential $d^4(\nu A[45,1]M_1^3) = \nu^2A[45,1]M_1 = \eta A[50,2]M_1$ determines tentative differentials which are a monomorphism on $Z_2\nu A[45,1]\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B<4>$. There are no remaining elements in degrees less than 69.

<u>DEGREE 48:</u> $\nu D[45]$

The leading differential $d^4(D[45]M_1^2) = \nu D[45]$ determines tentative differentials with image $[Z_4(\nu D[45])\{1, M_1, M_2\} \otimes B<4>] \oplus [Z_2(2\nu D[45])\{M_1^2, M_1M_2\} \otimes B<4>]$. The remaining elements are $[Z_4(\nu D[45])\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B<4>] \oplus [Z_2(\nu D[45])\{M_1^2, M_1M_2\} \otimes B<4>]$. Thus, the $\nu D[45]$-leader is $\nu D[45]M_1^2$.

The leading differential $d^4(\nu D[45]M_1^2) = \nu^2D[45]$ determines tentative differentials which are a monomorphism on $Z_2 \otimes (Z_4\nu D[45]\{M_1^2, M_1^3, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B<4>)$. The remaining elements are $Z_2(2\nu D[45])\{M_1^3, M_1^2M_2, M_1^3M_2\}$, and the new $\nu D[45]$-leader is $2\nu D[45]M_1^3$.

The leading differential $d^6(2\nu D[45]M_1^3) = A[8]D[45]$ determines tentative differentials which are a monomorphism on $Z_2(2\nu D[45])\{M_1^3, M_1^2M_2, M_1^3M_2\}$. Thus, there are no remaining elements.

DEGREE 48: $\eta B[47]$

The leading differential $d^2(B[47]M_1) = \eta B[47]$ determines tentative differentials with image $Z_2(\eta B[47]) \otimes B<2>$ and cokernel $Z_2(\eta B[47]M_1) \otimes B<2>$. The $\eta B[47]$-leader is $\eta B[47]M_1$.

The leading differential $d^4(A[45,2]\bar{M}_2) = \eta B[47]M_1$ determines tentative differentials with image $Z_2\eta B[47]\{M_1, M_1^3\} \otimes B<4>$. The remaining elements are $Z_2\eta B[47]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B<4>$, and the new $\eta B[47]$-leader is $\eta B[47]M_1\bar{M}_2$. The leading differential $d^6(\eta B[47]M_1\bar{M}_2) = \eta A[52,2]M_1$ determines tentative differentials which are a monomorphism on $Z_2\eta B[47]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B<4>$. Thus, there are no remaining elements.

DEGREE 50: $A[50,1]$

The leading differential $d^{34}(\eta^2\gamma_1 M_1^{17}) = A[50,1]$ determines tentative differentials with image in degrees less than 69 equal to $Z_2A[50,1]\{1, M_1\} \otimes B<4>$. Thus, the remaining elements are $Z_2A[50,1]\{M_1^2, M_1^3, M_2, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B<4>$, and the new $A[50,1]$-leader is $A[50,1]M_1^2$.

The leading differential $d^4(A[50,1]M_1^2) = \nu A[50,1]$ determines tentative differentials which are a monomorphism on $Z_2A[50,1]\{M_1^2, M_1^3, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B<4>$. The remaining elements in degrees less than 69 are $Z_2A[50,1]M_2 \otimes B<4>$, and the new $A[50,1]$-leader is $A[50,1]M_2$.

The leading differential $d^{40}(4\beta_1(M_1^7\bar{M}_2^3\bar{M}_3 + M_1^{10}\bar{M}_2^2\bar{M}_3 + M_1^{14}\bar{M}_2^3)) = A[50,1]M_2$ determines tentative differentials with image $Z_2A[50,1]M_2 \otimes B<4>$ in degrees less than 69. Thus, there are no remaining elements.

DEGREE 50: $A[50,2]$

The leading differential $d^{12}(A[39,1]M_1^3M_2) = A[50,2]$ determines tentative differentials with image equal to $Z_2A[50,2] \otimes B<4>$. Since $\eta A[50,2]$ is nonzero, the remaining elements are $Z_2A[50,2]\{M_1^2, \bar{M}_2, M_1^2\bar{M}_2\} \otimes B<4>$, and the $A[50,2]$-leader is $A[50,2]M_1^2$.

The leading differential $d^4(A[50,2]M_1^2) = \nu A[50,2]$ determines tentative differentials which are a monomorphism on $Z_2A[50,2]\{M_1^2, \bar{M}_2, M_1^2\bar{M}_2\} \otimes B<4>$. Thus, there are no remaining elements.

<u>DEGREE 51:</u> $\sigma C[44]$

The leading differential $d^8(C[44]<M_1^4>) = \sigma C[44]$ determines tenatiative differentials with image $Z_2(\sigma C[44])\{1, M_1^2, \bar{M}_2, M_2^2, \bar{M}_3, M_1^8\}$ in degrees less than 68. Since $\eta\sigma C[44] \neq 0$, the remaining elements are $Z_2(\sigma C[44])\{M_1^4, M_1^2\bar{M}_2, M_1^6, M_1^4\bar{M}_2, M_1^2M_2^2\}$. Thus, the $\sigma C[44]$-leader is $\sigma C[44]M_1^4$.

The leading differential $d^{12}(A[40,1]\bar{M}_2<M_3>) = \sigma C[44]M_1^4$ determines the tentative differential $d^{12}(A[40,1]M_1^2\bar{M}_2<M_3>) = \sigma C[44]M_1^6$. The remaining elements in degrees less than 68 are $Z_2(\sigma C[44])\{M_1^2\bar{M}_2, M_1^4\bar{M}_2, M_1^2M_2^2\}$, and the new $\sigma C[44]$-leader is $\sigma C[44]M_1^2\bar{M}_2$.

The leading differential $d^{30}(\nu A[19]M_1^7M_2^2<M_3>) = \sigma C[44]M_1^2\bar{M}_2$ determines no tentative differentials in degrees less than 68. The remaining elements in degrees less than 68 are $Z_2(\sigma C[44])\{M_1^4\bar{M}_2, M_1^2M_2^2\}$ and the new $\sigma C[44]$-leader is $\sigma C[44]M_1^4\bar{M}_2$. Moreover, $d^{14}(\sigma C[44]M_1^4\bar{M}_2) = A[64,1]$.

<u>DEGREE 51:</u> $2\sigma C[44]$

The leading differential $d^6(\eta A[45,1]M_1^3) = 2\sigma C[44]$ determines tentative differentials with image $Z_2(2\sigma C[44])\{1, M_1, M_2\} \otimes B<4>$. The remaining elements are $Z_2(2\sigma C[44])\{M_1^2, M_1^3, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B<4>$, and the $2\sigma C[44]$-leader is $2\sigma C[44]M_1^2$.

The leading differential $d^8(2C[44]<M_2^2>) = 2\sigma C[44]M_1^2$ determines tentative differentials with image in degrees less than 68 equal to all the remaining elements listed above.

DEGREE 51: $\nu^2 D[45]$

The leading differential $d^4(\nu D[45]M_1^2) = \nu^2 D[45]$ determines tentative differentials with image $Z_2\nu^2 D[45]\{1, M_1, M_1^2, M_2, M_1M_2\} \otimes B<4>$. The remaining elements are $Z_2\nu^2 D[45]\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B<4>$, and the $\nu^2 D[45]$-leader is $\nu^2 D[45]M_1^3$.

The leading differential $d^4(\nu^2 D[45]M_1^3) = \eta A[8]D[45]M_1$ determines tentative differentials which are a monomorphism on $Z_2\nu^2 D[45]\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B<4>$. There are no remaining elements.

DEGREE 51: $\eta A[50,2]$

The leading differential $d^2(A[50,2]M_1) = \eta A[50,2]$ determines tentative differentials with image $Z_2\eta A[50,2] \otimes B<2>$ and cokernel $Z_2\eta A[50,2]M_1 \otimes B<2>$. Thus, the $\eta A[50,2]$-leader is $\eta A[50,2]M_1$.

The leading differential $d^4(\nu A[45,1]M_1^3) = \eta A[50,2]M_1$ determines tentative differentials with image $Z_2\eta A[50,2]\{M_1, M_1^3, M_1\bar{M}_2\} \otimes B<4>$. The remaining elements are $Z_2\eta A[50,2]M_1^3\bar{M}_2 \otimes B<4>$, and the new $\eta A[50,2]$-leader is $\eta A[50,2]M_1^3\bar{M}_2$. The leading differential $d^6(\eta A[50,2]M_1^3\bar{M}_2) = A[62,4]$ determines tentative differentials which are a monomorphism on $Z_2\eta A[50,2]M_1^3\bar{M}_2 \otimes B<4>$. There are no remaining elements.

DEGREE 52: $A[52,1]$

The leading differential $d^6(\eta^2 D[45]M_1^3) = A[52,1]$ determines tentative differentials with image $Z_2A[52,1]\{1,M_1,M_2\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2A[52,1]\{M_1^2,M_1^3,M_1M_2,M_1^2M_2,M_1^3M_2\} \otimes B\langle 4\rangle$, and the A[52,1]-leader is $A[52,1]M_1^2$.

The leading differential $d^8(4D[45]M_1^3M_2) = A[52,1]M_1^2$ determines tentative differentials with image $Z_2A[52,1]M_1^2 \otimes B\langle 4\rangle$. The remaining elements are $Z_2A[52,1]\{M_1^3,M_1M_2,M_1^2M_2,M_1^3M_2\} \otimes B\langle 4\rangle$, and the A[52,1]-leader is $A[52,1]M_1^3$.

The leading differential $d^{36}(\eta^2\gamma_1M_1^{15}\overline{M}_2^2) = A[52,1]M_1^3$ determines tentative differentials with image in degrees less than 69 equal to $Z_2A[52,1]\{M_1^3,M_1^2M_2\}$. The remaining elements in degrees less than 69 are $Z_2A[52,1]\{M_1M_2,M_1^3M_2,M_1^7,M_1^5M_2\}$, and the new A[52,1]-leader is $A[52,1]M_1M_2$.

The leading differential $d^{14}(A[39,1]M_1M_2M_3) = A[52,1]M_1M_2$ determines no tentative differentials in degrees less than 69. The remaining elements in degrees less than 69 are $Z_2A[52,1]\{M_1^3M_2,M_1^7,M_1^5M_2\}$, and the new A[52,1]-leader is $A[52,1]M_1^3M_2$.

The leading differential $d^{10}(A[52,1]M_1^3M_2) = A[61]M_1$ determines tentative differentials which are a monomorphism on $Z_2A[52,1]M_1^3M_2 \otimes B\langle 4\rangle$. The only remianing elements in degrees less than 69 are $Z_2A[52,1]\{M_1^7,M_1^5M_2\}$.

<u>DEGREE 52:</u> A[52,2]

The leading differential $d^6(B[47]\overline{M}_2) = A[52,2]$ determines tentative differentials with image $Z_2A[52,2]\{1,M_1^2\} \otimes B\langle 4\rangle$. Since $\eta A[52,2]$ is nonzero, the remaining elements are $Z_2A[52,2]\{\overline{M}_2,M_1^2\overline{M}_2\} \otimes B\langle 4\rangle$, and the A[52,2]-leader is $A[52,2]\overline{M}_2$.

The leading differential $d^4(A[52,2]\overline{M}_2) = \eta A[54,2]M_1$ determines tentative differentials which are a monomorphism on $Z_2A[52,2]\{\overline{M}_2,M_1^2\overline{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 52: $\eta\sigma C[44]$

The leading differential $d^2(\sigma C[44]M_1) = \eta\sigma C[44]$ determines tentative differentials with cokernel $Z_2\eta\sigma C[44]M_1 \otimes B\langle 2\rangle$, and the $\eta\sigma C[44]$-leader is $\eta\sigma C[44]M_1$.

The leading differential $d^6(\nu C[44]M_1M_2) = \eta\sigma C[44]M_1$ determines tentative differentials with image $Z_2\eta\sigma C[44]\{M_1, M_1^3\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2\eta\sigma C[44]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$, and the new $\eta\sigma C[44]$-leader is $\eta\sigma C[44]M_1\bar{M}_2$.

The leading differential $d^8(\eta\sigma C[44]M_1\bar{M}_2) = A[59,1]$ determines tentative differentials which are a monomorphism on $Z_2(\eta\sigma C[44])\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 53: $\eta A[52,2]$

The leading differential $d^2(A[52,2]M_1) = \eta A[52,2]$ determines tentative differentials with image $Z_2\eta A[52,2] \otimes B\langle 2\rangle$ and cokernel $Z_2\eta A[52,2]M_1 \otimes B\langle 2\rangle$. The $\eta A[52,2]$-leader is $\eta A[52,2]M_1$.

The leading differential $d^6(\eta B[47]M_1\bar{M}_2) = \eta A[52,2]M_1$ determines tentative differentials with image $Z_2\eta A[52,2]\{M_1, M_1^3\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2\eta A[52,2]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$, and the new $\eta A[52,2]$-leader is $\eta A[52,2]M_1\bar{M}_2$.

The leading differential $d^8(\eta A[52,2]M_1\bar{M}_2) = B[60]$ determines tentative differentials which are a monomorphsim on $Z_2\eta A[52,2]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 53: $A[8]D[45]$

The leading differential $d^6(2\nu D[45]M_1^3) = A[8]D[45]$ determines tentative differentials with image $Z_2A[8]D[45]\{1, M_1^2, \bar{M}_2\} \otimes B\langle 4\rangle$. Since $\eta A[8]D[45] \neq 0$, the remaining elements are $Z_2A[8]D[45]M_1^2\bar{M}_2 \otimes B<4>$, and the $A[8]D[45]$-leader is $A[8]D[45]M_1^2\bar{M}_2$.

The leading differential $d^8(A[8]D[45]M_1^2\bar{M}_2) = 2B[62]$ determines tentative differentials which are a monomorphism on $Z_2A[8]D[45]M_1^2\bar{M}_2 \otimes B<4>$. There are no remaining elements.

DEGREE 53: $\nu A[50,1]$

The leading differential $d^4(A[50,1]M_1^2) = \nu A[50,1]$ determines tentative differentials with image $Z_2\nu A[50,1]\{1, M_1, M_1^2, M_2, M_1M_2\} \otimes B<4>$. The remaining elements are $Z_2\nu A[50,1]\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B<4>$, and the $\nu A[50,1]$-leader is $\nu A[50,1]M_1^3$.

The leading differential $d^{18}(A[36]M_1^6\bar{M}_2^2) = \nu A[50,1]M_1^3$ determines no tentative differentials in degrees less than 68. The remaining elements are $(Z_2\nu A[50,1]\{M_1^2M_2, M_1^3M_2\} \otimes B<4>) \oplus Z_2\nu A[50,1]M_1^7$, and the new $\nu A[50,1]$-leader is $\nu A[50,1]M_1^2M_2$.

The tentative differential $d^{10}(\nu A[50,1]M_1^2M_2) = A[62,3]$ determines tentative differentials which are a monomorphism on $Z_2\nu A[50,1]\{M_1^2M_2, M_1^3M_2\} \otimes B<4>$. The only remaining element in degrees less than 68 is $\nu A[50,1]M_1^7$.

DEGREE 53: $\nu A[50,2]$

The leading differential $d^4(A[50,2]M_1^2) = \nu A[50,2]$ determines tentative differentials with image $Z_2\nu A[50,2]\{1, M_1, M_2\} \otimes B<4>$. The remaining elements are $Z_2\nu A[50,2]\{M_1^2, M_1^3, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B<4>$, and the $\nu A[50,2]$-leader is $\nu A[50,2]M_1^2$.

The leading differential $d^4(\nu A[50,2]M_1^2) = \nu^2A[50,2]$ determines tentative differentials which are a monomorphism on

$Z_2\nu A[50,2]\{M_1^2, M_1^3, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 54: A[54,1]

The leading differential $d^{36}(\beta_2 M_1^{18}) = A[54,1]$ determines tentative differentials with image $Z_2A[54,1]\{1, M_1, M_1^2, M_2, M_1M_2\} \otimes B\langle 4\rangle$ in degrees less than 69. The remaining elements in degrees less than 69 are $Z_2A[54,1]\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$, and the A[54,1]-leader is $A[54,1]M_1^3$.

The leading differential $d^4(A[54,1]M_1^3) = \eta A[56]M_1$ determines tentative differentials which are a monomorphism on $Z_2A[54,1]\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$. There are no remaining elements in degrees less than 69.

DEGREE 54: $A[54,2] = A[14]C[20]^2$

The leading differential $d^8(\eta^2A[45,2]M_1\bar{M}_2) = A[54,2]$ determines tentative differentials with image $Z_2A[54,2]\{1, M_1^2\} \otimes B\langle 4\rangle$. Since $\eta A[54,2] \neq 0$, the remaining elements are $Z_2A[54,2]\{\bar{M}_2, M_1^2\bar{M}_2\} \otimes B\langle 4\rangle$, and the A[54,2]-leader is $A[54,2]\bar{M}_2$.

The leading differential $d^6(A[54,2]\bar{M}_2) = A[59,2]$ determines tentative differentials which are a monomorphism on $Z_2A[54,2]\{\bar{M}_2, M_1^2\bar{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 54: $\eta A[8]D[45]$

The leading differential $d^2(A[8]D[45]M_1) = \eta A[8]D[45]$ determines tentative differentials with image $Z_2\eta A[8]D[45] \otimes B\langle 2\rangle$ and cokernel $Z_2\eta A[8]D[45]M_1 \otimes B\langle 2\rangle$. The $\eta A[8]D[45]$-leader is $\eta A[8]D[45]M_1$.

The leading differential $d^4(\nu^2 D[45]M_1^3) = \eta A[8]D[45]M_1$ determines tentative differentials with image equal to $Z_2(\eta A[8]D[45])\{M_1, M_1^3, M_1\bar{M}_2\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2\eta A[8]D[45]M_1^3\bar{M}_2 \otimes B\langle 4\rangle$, and the new $\eta A[8]D[45]$-leader is $\eta A[8]D[45]M_1^3\bar{M}_2$.

DEGREE 55: $\eta A[54,2]$

The leading differential $d^2(A[54,2]M_1) = \eta A[54,2]$ determines tentative differentials with image $Z_2\eta A[54,2] \otimes B\langle 2\rangle$ and cokernel $Z_2\eta A[54,2]M_1 \otimes B\langle 2\rangle$. The $\eta A[54,2]$-leader is $\eta A[54,2]M_1$.

The leading differential $d^4(A[52,2]\bar{M}_2) = \eta A[54,2]M_1$ determines tentative differentials with image $Z_2\eta A[54,2]\{M_1, M_1^3\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2\eta A[54,2]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$.

The leading differential $d^6(\eta A[54,2]M_1\bar{M}_2) = \eta A[59,2]M_1$ determines tentative differentials which are a monomorphism on $Z_2\eta A[54,2]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 56: $A[56]$

The leading differential $d^{38}(2\beta_2 M_1^{16}\bar{M}_2) = A[56]$ determines tentative differentials with image in degrees less than 69 equal to $Z_2A[56]\{1, M_1^2, \bar{M}_2\} \otimes B\langle 4\rangle$. Since $\eta A[56] \neq 0$, the only remaining element is $A[56]M_1^2\bar{M}_2$.

DEGREE 56: $\nu^2 A[50,2]$

The leading differential $d^4(\nu A[50,2]M_1^2) = \nu^2 A[50,2]$ determines tentative differentials with image $Z_2(\nu^2 A[50,2])\{1, M_1, M_1^2, M_2, M_1M_2\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2(\nu^2 A[50,2])\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$, and the $\nu^2 A[50,2]$-leader is $\nu^2 A[50,2]M_1^3$.

The leading differential $d^6(\nu^2 A[50,2]M_1^3) = A[61]$ determines tentative differentials which are a monomorphism on $Z_2(\nu^2 A[50,2])\{M_1^3, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 57: A[57]

The leading differential $d^{14}(4C[44]M_1^7) = A[57]$ determines tentative differentials with image $Z_2A[57]\{1, M_1^2, \bar{M}_2, M_1^4\}$ in degrees less than 68. Since $\eta A[57] \neq 0$, the only remaining element in degrees less than 68 is $A[57]M_1^2\bar{M}_2$.

DEGREE 57: ηA[56]

The leading differential $d^2(A[56]M_1) = \eta A[56]$ determines tentative differentials with image $Z_2\eta A[56] \otimes B\langle 2\rangle$ and cokernel $Z_2\eta A[56]M_1 \otimes B\langle 2\rangle$. The $\eta A[56]$-leader is $\eta A[56]M_1$.

The leading differential $d^4(A[54,1]M_1^3) = \eta A[56,2]M_1$ determines tentative differentials with image $Z_2\eta A[56]\{M_1, M_1^3, M_1\bar{M}_2\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2(\eta A[56]M_1^3\bar{M}_2) \otimes B\langle 4\rangle$, and the new $\eta A[56]$-leader is $\eta A[56]M_1^3\bar{M}_2$.

DEGREE 58: ηA[57]

The leading differential $d^2(A[57]M_1) = \eta A[57]$ determines tentative differentials with image $Z_2\eta A[57] \otimes B\langle 2\rangle$ and cokernel $Z_2\eta A[57]M_1 \otimes B\langle 2\rangle$. The $\eta A[57]$-leader is $\eta A[57]M_1$.

The leading differential $d^8(\beta_2 M_1^{14}\bar{M}_3) = \eta A[57]M_1$ determines tentative differentials with image in degrees less than 69 equal to $Z_2\eta A[57]\{M_1, M_1^3, M_1\bar{M}_2, M_1^5\}$. There are no remaining elements.

DEGREE 59: A[59,1]

The leading differential $d^8(\eta\sigma C[44]M_1\bar{M}_2) = A[59,1]$ determines tentative differentials with image $Z_2A[59,1]\{1, M_1^2\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2A[59,1]\{M_1, M_1^3, M_2, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$, and the A[59,1]-leader is $A[59,1]M_1$.

The leading differential $d^{22}(B[38]M_1^2\bar{M}_2\bar{M}_3) = A[59,1]M_1$ determines no tentative differentials in degrees less than 68. The remaining elements are $Z_2A[59,1]\{M_1^3, M_2, M_1M_2\}$, and the new A[59,1]-leaders are $A[59,1]M_1^3$ and $A[59,1]M_2$.

The leading differentials $d^6(A[59,1]M_1^3) = A[64,2]$ and $d^6(A[59,1]M_2) = A[64,3]$ determine tentative differentials which are a monomorphism on $Z_2A[59,1]\{M_1^3, M_2, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$. (The first leading differential determines tentative differentials by assigning 1 to $A[59,1]M_1^3$ and 0 to $A[59,1]M_2$. The second leading differential determines tentative differentials by assigning 1 to $A[59,1]M_1^3$ and $A[59,1]M_2$.) There are no remaining elements in degrees less than 68.

DEGREE 59: A[59,2] = A[14]A[45,2]

The leading differential $d^6(A[54,2]\bar{M}_2) = A[59,2]$ determines tentative differentials with image $Z_2A[59,2]\{1, M_1^2\} \otimes B\langle 4\rangle$. Since $\eta A[59,2] \neq 0$, the remaining elements are $Z_2A[59,2]\{\bar{M}_2, M_1^2\bar{M}_2\} \otimes B\langle 4\rangle$, and the A[59,2]-leader is $A[59,2]\bar{M}_2$.

The leading differential $d^4(A[59,2]\bar{M}_2) = \eta^2 B[60]M_1$ determines tentative differentials which are a monomorphism on $Z_2A[59,2]\{\bar{M}_2, M_1^2\bar{M}_2\} \otimes B\langle 4\rangle$. There are no remaining elements.

DEGREE 60: $B[60] = C[20]^3$

The leading differential $d^8(\eta A[52,2]M_1\bar{M}_2) = B[60]$ determines tentative differentials with image equal to $Z_2B[60]\{1, M_1^2\} \otimes B\langle 4\rangle$. Since $\eta B[60] \neq 0$, the remaining elements from $Z_2 \otimes (Z_4B[60] \otimes H_*BP)$ are $Z_2B[60]\{\bar{M}_2, M_1^2\bar{M}_2\} \otimes B\langle 4\rangle$, and the B[60]-leader is $B[60]\bar{M}_2$.

DEGREE 60: $2B[60] = \eta A[59,2]$

The leading differential $d^2(A[59,2]M_1) = \eta A[59,2]$ determines differentials with image $Z_2(\eta A[59,2]) \otimes B\langle 2\rangle$ and cokernel $Z_2(\eta A[59,2]M_1) \otimes B\langle 2\rangle$. The $\eta A[59,2]$-leader is $\eta A[59,2]M_1$.

The leading differential $d^6(\eta A[54,2]M_1\bar{M}_2) = \eta A[59,2]M_1$ determines tentative differentials with image $Z_2\eta A[59,2]\{M_1, M_1^3\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2\eta A[59,2]\{M_1\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$, and the new $\eta A[59,2]$-leader is $\eta A[59,2]M_1\bar{M}_2$.

DEGREE 61: A[61]

The leading differential $d^6(\nu^2 A[50,2]M_1^3) = A[61]$ determines tentative differntials with image $Z_2A[61]\{1, M_1^2, \bar{M}_2\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2A[61]\{M_1, M_1^3, M_1\bar{M}_2, M_1^2\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$, and the A[61]-leader is $A[61]M_1$.

The leading differential $d^{10}(A[52,1]M_1^3M_2) = A[61]M_1$ determines tentative differentials with image $Z_2A[61]M_1^3M_2 \otimes B\langle 4\rangle$. The remaining elements are $Z_2A[61]\{M_1^3, M_1\bar{M}_2, M_1^2\bar{M}_2, M_1^3\bar{M}_2\} \otimes B\langle 4\rangle$, and the new A[61]-leader is $A[61]M_1^3$.

DEGREE 61: $\eta B[60]$

Since $\eta^2 B[60] \neq 0$, the only element of E^4 with a representative in $Z_2\eta B[60] \otimes H_*BP$ is zero.

DEGREE 62: $A[62,1] = \theta_5$

The leading differential $d^{54}(\eta^2\sigma M_1^{21}M_2^2) = A[62,1]$ determines no tentative differentials in degrees less than 69. Since $\eta A[62,1] \neq 0$, the remaining elements in degrees less than 69 are $Z_2A[62,1]\{M_1^2,\bar{M}_2\}$, and the A[62,1]-leader is $A[62,1]M_1^2$.

DEGREE 62: A[62,2]

The leading differential $d^{44}(4\beta_2 M_1^{19}\bar{M}_2) = A[62,2]$ determines no tentative differentials in degrees less than 69. Since $\eta A[62,2] \neq 0$ the remaining elements are $Z_2A[62,2]\{M_1^2,\bar{M}_2\}$, and the A[62,2]-leader is $A[62,2]M_1^2$.

DEGREE 62: A[62,3]

The leading differential $d^{10}(\nu A[50,1]M_1^2M_2) = A[62,3]$ determines tentative differentials with image $Z_2A[62,3]\{1,M_1\} \otimes B\langle 4\rangle$. The remaining elements are $Z_2A[62,3]\{M_1^2,M_1^3,M_2,M_1M_2,M_1^2M_2,M_1^3M_2\} \otimes B\langle 4\rangle$, and the A[62,3]-leader is $A[62,3]M_1^2$.

DEGREE 62: A[62,4]

The leading differential $d^{12}(\eta A[50,2]M_1^3\bar{M}_2) = A[62,4]$ determines tentative differentials with image $Z_2A[62,4] \otimes B\langle 4\rangle$. Since $\eta A[62,4] \neq 0$, the remaining elements are $Z_2A[62,4]\{M_1^2,\bar{M}_2,M_1^2\bar{M}_2\} \otimes B\langle 4\rangle$, and the A[62,4]-leader is $A[62,4]M_1^2$.

DEGREE 62: B[62]

The leading differential $d^{40}(\gamma_2 M_1^{20}) = B[62]$ determines tentative differentials whose image in degrees less than 69 equals $Z_2B[62]\{1, M_1, M_1^2, \bar{M}_2\}$. The only remaining element is $B[62]M_1^3$.

DEGREE 62: 2B[62]

The leading differential $d^{10}(A[8]D[45]M_1^2\bar{M}_2) = 2B[62]$ determines tentative differentials with image $Z_2(2B[62]) \otimes B\langle 4\rangle$. The remaining elements are $Z_2(2B[62])\{M_1, M_1^2, M_1^3, M_2, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$, and the 2B[62]-leader is $2B[62]M_1$.

The leading differential $d^{40}(2\gamma_2 M_1^{18}\bar{M}_2) = 2B[62]M_1$ determines a tentative differential with image $2B[62]M_2$ in degrees less than 69. The only remaining elements are $Z_2(2B[62])\{M_1^2, M_1^3\}$.

DEGREE 62: $\eta^2B[60]$

The leading differential $d^2(\eta B[60]M_1) = \eta^2B[60]$ determines tentative differentials with image $Z_2\eta^2B[60] \otimes B\langle 2\rangle$ and cokernel $Z_2\eta^2B[60]M_1 \otimes B\langle 2\rangle$. The $\eta^2B[60]$-leader is $\eta^2B[60]M_1$.

The leading differential $d^4(A[59,2]\bar{M}_2) = \eta^2B[60]M_1$ determines tentative differentials with image $Z_2(\eta^2B[60])\{M_1, M_1^3\} \otimes B\langle 4\rangle$. There are no elements remaining in degrees less than 69.

DEGREE 63: A[63]

The leading differential $d^{24}(\eta\sigma A[32,1]M_1^5\langle M_3\rangle) = A[63]$ determines tentative differentials with image $Z_2A[63]\{1, M_1^2\}$ in degrees less than 68. The only remaining element in degrees less than 68 is $A[63]M_1$ and $d^{24}(A[39,3]M_1^3\bar{M}_2) = A[63]M_1$.

<u>DEGREE 63:</u> $\eta A[62,1]$ and $\eta A[62,4]$

Since $\eta^2A[62,1] \neq 0$ and $\eta^2A[62,4] \neq 0$, the only element of E^4 with a representative in $(Z_2\eta A[62,1] \oplus Z_2\eta A[62,4]) \otimes H_*BP$ is zero.

<u>DEGREE 63:</u> $\eta A[62,2]$

The leading differential $d^2(A[62,2]M_1) = \eta A[62,2]$ determines tentative differentials with image $Z_2\eta A[62,2] \otimes B\langle 2\rangle$. The remaining elements are $Z_2\eta A[62,2]M_1 \otimes B\langle 2\rangle$, and the $\eta A[62,2]$-leader is $\eta A[62,2]M_1$.

The leading differential $d^8(A[56]M_1^2\bar{M}_2) = \eta A[62,2]M_1$ determines tentative differentials with image $Z_2\eta A[62,2]M_1 \otimes B\langle 4\rangle$. The remaining elements are $Z_2\eta A[62,2]\{M_1^2, M_1^3, M_2, M_1M_2, M_1^2M_2, M_1^3M_2\} \otimes B\langle 4\rangle$, and the new $\eta A[62,2]$-leader is $\eta A[62,2]M_1^2$.