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The May filtration on THH and faithfully flat descent

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A B S T R A C T

In this article, we study descent properties of topological Hochschild homology and topological cyclic homology. In particular, we verify that both of these invariants satisfy faithfully flat descent and 1-connective descent for connective $\mathbf{E}_2$-ring spectra. This generalizes a result of Bhatt–Morrow–Scholze from [6] and a result of Dundas–Rognes from [11], respectively. Along the way, we develop some basic theory for cobar constructions and give an alternative presentation of the May filtration on topological Hochschild homology, originally due to Angelini-Knoll–Salch [3].

Contents

1. Introduction

Since their inception in the 1980s, topological Hochschild homology (THH) and topological cyclic homology (TC) have served as very useful tools for studying algebraic K-theory; see, for instance, foundational work of Hesselholt–Madsen on the K-theory of local fields [17] or the very recent work of Hahn–Wilson on Quillen–Lichtenbaum phenomena in the K-theory of ring spectra [18]. Each of these invariants are *localizing* in the sense that they carry exact sequences of small idempotent complete stable ∞-categories to cofiber sequences.[1] One very useful consequence of this property, originally observed by Thomason in [30],

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[1] It is worth remarking that THH and K are additionally finitary, in that they preserve filtered colimits. On the other hand, TC notably lacks this property, as it is constructed via limits rather than colimits.

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is that such invariants satisfy a form of excision for (spectral) schemes called *Nisnevich descent*. In practice, this allows one to decompose calculations with these invariants into more manageable pieces; e.g., for X a scheme, the value of $\mathrm{K}(X)$, $\mathrm{THH}(X)$, or $\mathrm{TC}(X)$ may be determined by gluing together the values of these invariants on an affine chart. However, algebraic K-theory rather famously fails to satisfy *étale descent*, which is strictly stronger than Nisnevich descent. The étale (hyper)sheafification of algebraic K-theory is an extremely rich object, studied by Thomason in [29], and revisted by Clausen–Mathew in [8]. By contrast to algebraic K-theory, THH actually satisfies étale base-change, meaning that if $A \to B$ is an étale extension of commutative rings, then there is a natural equivalence $B \otimes_A \mathrm{THH}(A) \simeq \mathrm{THH}(B)$; note that étale base change implies étale descent by quasicoherent descent. As TC is constructed from THH via a limiting procedure, it is automatic that TC also satisfies étale descent. This was originally observed by Geisser–Hesselholt in [13] and later extended to $\mathbf{E}_\infty$ and $\mathbf{E}_2$-ring spectra by Mathew [22] and Clausen–Mathew [8], respectively. The algebraic precursor of this property for Hochschild homology was originally studied and established by Geller–Weibel [31].

Over the past few years, THH and TC have seen fantastic application to other areas of mathematics, such as p-adic Hodge theory. In their breakthrough work [6], Bhatt–Morrow–Scholze used THH, TC, and variants thereof to introduce a new mixed characteristic cohomology theory called *prismatic cohomology*, which recovers de Rham cohomology, étale cohomology, and crystalline cohomology in special cases. One of the key technical results of [6] is the fact that THH and all of its variants satisfy faithfully flat descent for commutative rings. This is used in an essential way to construct "motivic" filtrations on THH, TC^-, TP, and TC, which are in turn used to define the (Nygaard completed) prismatic cohomology and the syntomic cohomology of rings which satisfy mild torsion and smoothness hypotheses.

Given the above results and their fruitful applications, it is natural to ask whether these invariants satisfy faithfully flat descent for suitably commutative ring spectra. In this article, we will primarily work with $\mathbf{E}_2$-ring spectra, as they have the property that their homotopy groups behave somewhat like ordinary commutative rings and modules. To state our main results, we first recall a few notions from stable homotopy theory and spectral algebraic geometry.

Definition 1.1. A map of connective $\mathbf{E}_2$-ring spectra $f : A \to B$ is said to be faithfully flat, provided that

1. The induced map of commutative rings $\pi_0 f : \pi_0 A \to \pi_0 B$ is faithfully flat.
2. The induced map $\pi_0 B \otimes_{\pi_0 A} \pi_k A \to \pi_k B$ is an isomorphism of $\pi_0 B$-modules for all $k \geq 0$.

As we shall see later in Section 3, given a map of $\mathbf{E}_2$-ring spectra $f : A \to B$ one can form the coaugmented cobar construction of f

$$\mathrm{CB}^\bullet(f)\colon\ A \longrightarrow B \rightrightarrows B \otimes_A B \,\substack{\to\\ \to\\ \to}\, \cdots$$

which is a diagram of $\mathbf{E}_1$-ring spectra.

Definition 1.2. Let $\mathcal{C}$ be a presentable ∞-category. Given a functor $E : \mathrm{Alg}_{\mathbf{E}_1}(\mathrm{Sp}) \to \mathcal{C}$, we say that E satisfies faithfully flat descent for connective $\mathbf{E}_2$-rings provided that the natural map

$$E(A) \to \lim_{\Delta} E(\mathrm{CB}^\bullet(f)),$$

in $\mathcal{C}$ is an equivalence.

We are now able to state our main theorem.

Theorem A. *Topological Hochschild homology, viewed as a functor* $\mathrm{THH} : \mathrm{Alg}_{\mathbf{E}_1}(\mathrm{Sp}) \to \mathrm{CycSp}$*, satisfies faithfully flat descent for connective* $\mathbf{E}_2$*-rings. Consequently, so does* TC.

The rough idea behind the proof of Theorem A is a reduction to the discrete case, by using the *May filtration* on THH, denoted by $\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(A)$, which was originally introduced and studied by Angelini-Knoll–Salch in [3]. Combining an analysis of the May filtration together with our second main result below, we can reduce Theorem A to the aforementioned result of Bhatt–Morrow–Scholze [6, Corollary 3.3].

Theorem B. *Let $f : A \to B$ be a 1-connective map of connective $\mathbf{E}_2$-ring spectra, i.e.* $\mathrm{fib}(f)$ *is 1-connective. Then, for $E = \mathrm{THH}$ or TC the induced map*

$$E(A) \to \lim_{\Delta} E(\mathrm{CB}^\bullet(f))$$

is an equivalence.

Remark 1.3. Theorem B can be viewed as a sort of generalization of [11, Theorem 1.2], due to Dundas–Rognes. Strictly speaking, however, the precise statement of Theorem B does not literally generalize [11, Theorem 1.2]. However, our work in Section 3 does provide an exact generalization. For further details, we refer to the reader to Remark 3.26.

Outline. We now provide a brief outline of the article. In Section 2 we review and carefully prove several folklore results we will need to establish the basic properties of the May filtration. In Section 3, we review the cobar construction and its basic properties and use these to establish Theorem B. In Section 4 we establish Theorem A in the case of generalized Eilenberg–MacLane spectra. In Section 5 we review filtered THH and some of its basic properties and give an ∞-categorical treatment of the May filtration on THH. Section 6 is devoted to proving Theorem A and collecting some consequences of the proof.

Notation. Throughout this article, we freely use the theory of ∞-categories, incarnated via quasicategories, as developed in [19] and [21]. For consistency, wherever possible, we also follow the notation therein. We also handle all set-theoretic issues as is done in [19, 1.2.15], assuming the existence of a Grothendieck universe of κ-small objects, and whenever necessary choosing a larger Grothendieck universe of κ'-small objects, in which the previous universe now resides. Throughout, we refer to the ∞-category of spaces by $\mathcal{S}$ and the ∞-category of spectra by Sp.

A spectrum X is called n-connective provided that $\pi_m X \cong 0$ for $m < n$, and a map $f : X \to Y$ of spectra is n-connective provided that the fiber of f is n-connective, i.e. $\pi_m f$ is an isomorphism in degrees $m < n$ and a surjection when $n = m$. The ∞-category of n-connective spectra will be denoted by $\mathrm{Sp}_{\geq n}$ where $n \in \mathbb{Z}$. We will make frequent use of the ∞-categorical treatment of topological Hochschild homology, topological cyclic homology, topological restriction homology, and cyclotomic spectra found in work of Nikolaus–Scholze [27, III.2.3, II.1.8, II.1.6], Antieau–Nikolaus [2], and McCandless [23].

In the intervening time between the original version of this article and the published version, the theories of filtered objects and filtered cyclotomic spectra have become more well-documented in the literature; we direct the interested reader to [28] and [1] for example. However, for completeness, we have opted to include our own account which originally appeared in a previous version of this article.

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2. Filtered spectra

Our main technical tool is the May filtration on THH, as defined in [3], which we import into the ∞-categorical context. To do so, we need to establish some basic facts regarding $\mathbb{Z}$-filtered and $\mathbb{Z}_{\geq 0}$-filtered spectra. Unless otherwise specified, we view $\mathbb{Z}^{op}$ and $\mathbb{Z}^{op}_{\geq 0}$ as categories via the partial order, $\leq$, and as symmetric monoidal categories via addition. By abuse, we use the same notation for the associated symmetric monoidal ∞-categories.

Definition 2.1. The ∞-category of filtered spectra is $\mathrm{Fil}(\mathrm{Sp}) = \mathrm{Fun}(\mathbb{Z}^{op}, \mathrm{Sp})$, and the ∞-category of $\mathbb{N}$-filtered spectra is $\mathrm{fil}(\mathrm{Sp}) = \mathrm{Fun}(\mathbb{Z}^{op}_{\geq 0}, \mathrm{Sp})$.

It is readily checked that both of these ∞-categories are presentably symmetric monoidal and stable where the symmetric monoidal product in both cases is Day convolution; see [19, 5.5.3.6], [21, 1.1.3.1], [15, 2.13]. Throughout, we will let $\circledast$ denote the Day convolution product and allow context to dictate exactly which product we mean. The unit object in $\mathrm{Fil}(\mathrm{Sp})$, denoted by $\mathbb{1}_{\mathrm{Fil}}$, is the filtered spectrum

$$\cdots \to 0 \to 0 \to \mathbb{S} \to \mathbb{S} \to \cdots$$

which is $\mathbb{S}$ in degrees $n \leq 0$ with identity maps in negative degree. The unit object of $\mathrm{fil}(\mathrm{Sp})$, denoted by $\mathbb{1}_{\mathrm{fil}}$, is the $\mathbb{Z}_{\geq 0}$-filtered spectrum which is $\mathbb{S}$ in degree 0, and 0 otherwise, diagrammatically given by

$$\cdots \to 0 \to 0 \to \mathbb{S}.$$

Note for instance that the restriction of $\mathbb{1}_{\mathrm{Fil}}$ to $\mathbb{Z}^{op}_{\geq 0}$ is $\mathbb{1}_{\mathrm{fil}}$. Because $\mathrm{Fil}(\mathrm{Sp})$ and $\mathrm{fil}(\mathrm{Sp})$ are symmetric monoidal, we may consider the ∞-categories of $\mathcal{O}$-algebras, where $\mathcal{O}$ is an ∞-operad, and we use notation $\mathrm{Alg}^{\mathrm{Fil}}_{\mathcal{O}}$ and $\mathrm{Alg}^{\mathrm{fil}}_{\mathcal{O}}$, leaving Sp implicit. These categories will appear later when we discuss variants of THH. Here are the folklore claims we establish in this section.

1. Restriction along the inclusion $\mathbb{Z}^{op}_{\geq 0} \subseteq \mathbb{Z}^{op}$ exhibits $\mathrm{fil}(\mathrm{Sp})$ as a symmetric monoidal subcategory of $\mathrm{Fil}(\mathrm{Sp})$.
2. The functors $\mathrm{colim}_{\mathbb{Z}^{op}}$ and ev_0 are colimit-preserving symmetric monoidal functors $\mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$ and these agree upon restriction to $\mathrm{fil}(\mathrm{Sp})$.
3. The associated graded functor $\mathrm{gr}^* : \mathrm{fil}(\mathrm{Sp}) \to \mathrm{Sp}$ is colimit-preserving and symmetric monoidal functor.
4. The Whitehead tower functor $\tau_{\geq *} : \mathrm{Sp} \to \mathrm{Fil}(\mathrm{Sp})$ is lax symmetric monoidal.

Remark 2.2. Claims (1)-(3) hold for any stable presentably symmetric monoidal ∞-category $\mathcal{C}$ by the following observation; if $F : \mathcal{C} \to \mathcal{D}$ is a colimit-preserving (lax) symmetric monoidal functor, then for any stable presentably symmetric monoidal ∞-category $\mathcal{E}$, the induced functor $F \otimes \mathrm{id}_{\mathcal{E}} : \mathcal{C} \otimes \mathcal{E} \to \mathcal{D} \otimes \mathcal{E}$ is (lax) symmetric monoidal. A variant of claim (4) will hold if $\mathcal{C}$ is additionally equipped with a nice enough t-structure compatible with the symmetric monoidal structure on $\mathcal{C}$, in the sense of [21, Example 2.2.1.3]. However, we will not need such a result.

2.1. Properties of Fil(Sp), fil(Sp)*, and Day convolution*

Definition 2.3. For $\mathcal{C}$ a stable ∞-category, an object X is said to generate $\mathcal{C}$ provided that

$$\pi_0 \,\mathrm{Map}_{\mathcal{C}}(X, Y) = 0 \quad \text{implies} \quad Y \simeq 0.$$

Similarly, given a collection of objects, $\{X_i\}_{i \in I}$, this collection is said to jointly generate $\mathcal{C}$, provided that $\pi_0 \,\mathrm{Map}(X_i, Y) \simeq 0$ for all $i \in I$ implies $Y \simeq 0$.

Remark 2.4. Note that X being a generator for $\mathcal{C}$ is equivalent to the functor $\mathrm{map}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathrm{Sp}$ being conservative; here, $\mathrm{map}_{\mathcal{C}}$ denotes the mapping spectrum in $\mathcal{C}$. A similar statement is true for a jointly generating collection, replacing Sp with $\prod_{i\in I}\mathrm{Sp}$ and $\mathrm{map}_{\mathcal{C}}(X,-)$ with $\prod_{i\in I}\mathrm{map}_{\mathcal{C}}(X_i,-)$. In the case where $\mathcal{C}$ admits infinite coproducts, having a jointly generating collection of objects is equivalent to $\bigoplus_{i\in I} X_i$ being a generator.

Lemma 2.5. *Let K be a simplicial set and let $\mathrm{ev}_k : \mathrm{Fun}(K, \mathrm{Sp}) \to \mathrm{Sp}$ denote evaluation at $k \in K$. For all objects $k \in K$, the functors ev_k admit left adjoints, $L_k : \mathrm{Sp} \to \mathrm{Fun}(K, \mathrm{Sp})$ given on vertices by $L_k X : k' \mapsto X^{\otimes \mathrm{Map}_K(k,k')}$. Additionally, the objects $\mathbb{1}_K(k) = L_k\mathbb{S}$ have the following properties:*

1. *$\mathbb{1}_K(k)$ is compact for all $k \in K$;*
2. *the $\mathbb{1}_K(k)$'s jointly generate $\mathrm{Fun}(K, \mathrm{Sp})$; and*
3. *the collection $\{\Sigma^n \mathbb{1}_K(k)\}_{n\in\mathbb{Z},k\in K}$ generates $\mathrm{Fun}(K, \mathrm{Sp})$ under small colimits.*

Proof. Since ev_k preserves small limits and colimits, it admits a left adjoint $L_k : \mathrm{Sp} \to \mathrm{Fun}(K, \mathrm{Sp})$. The compactness of $\mathbb{1}_K(k)$ follows from the fact that $\mathbb{S}$ is compact in Sp and ev_k preserves colimits. To explicitly identify L_k, we use the following chain of natural equivalences obtained from the end formula, which appears for example, in [14, 5.2]

$$\begin{aligned}\mathrm{Map}_{\mathrm{Fun}(K,\mathrm{Sp})}(X^{\otimes \mathrm{Map}_K(k,-)}, E_\bullet) &\simeq \varprojlim_{i\to j\in \mathrm{Tw}(K)} \mathrm{Map}_{\mathrm{Sp}}\Big(\varinjlim_{\mathrm{Map}_K(k,i)} X, E_j\Big) \\ &\simeq \varprojlim_{i\to j\in \mathrm{Tw}(K)} \varprojlim_{\mathrm{Map}_K(k,i)} \mathrm{Map}_{\mathrm{Sp}}(X, E_j) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}}(X, \varprojlim_{i\to j\in \mathrm{Tw}(K)} E_j^{\mathrm{Map}_K(k,i)})\end{aligned}$$

As the exponential object $E_j^{\mathrm{Map}_K(k,i)}$ is equivalent to the mapping spectrum $\mathrm{map}_{\mathrm{Sp}}(\Sigma^\infty_+ \mathrm{Map}_K(k,i), E_j)$, an application of the end formula and the spectral co-Yoneda lemma yield:

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}}(X, \varprojlim_{i\to j\in \mathrm{Tw}(K)} E_j^{\mathrm{Map}_K(k,i)}) &\simeq \mathrm{Map}_{\mathrm{Sp}}\Big(X, \mathrm{map}_{\mathrm{Fun}(K,\mathrm{Sp})}(\Sigma^\infty_+ \mathrm{Map}_K(k,-), E_\bullet)\Big) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}}(X, E_k).\end{aligned}$$

This proves that L_k is given on vertices as claimed.

To prove the final assertions, note that as $\mathrm{Map}_{\mathrm{Fun}(K,\mathrm{Sp})}(\mathbb{1}_K(k), E_\bullet) \simeq E_k$, the collection $\{\mathbb{1}_K(k)\}_{k\in K}$ jointly generates the stable ∞-category $\mathrm{Fun}(K, \mathrm{Sp})$. This allows us to mimic the proof of [21, 1.4.4.2] to show that the objects $\Sigma^n\mathbb{1}_K(k)$ generate $\mathrm{Fun}(K, \mathrm{Sp})$ under small colimits. □

Note that in the case of Fil(Sp) (resp. fil(Sp)) the objects $\mathbb{1}_K(k)$ are shifts of $\mathbb{1}_{\mathrm{Fil}}$ into higher filtration degrees, and we denote these objects by $\mathbb{1}_{\mathrm{Fil}}(k)$.

Example 2.6. The object $\mathbb{1}_{\mathrm{Fil}}(k)$ is given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{S} \xrightarrow{\mathrm{id}} \mathbb{S} \xrightarrow{\mathrm{id}} \cdots$$

where the leftmost copy of $\mathbb{S}$ is in degree k. Similarly, if $K = \mathbb{Z}^{op}_{\geq 0}$, the object $\mathbb{1}_{\mathrm{Fil}}(k)$ is given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{S} \xrightarrow{\mathrm{id}} \mathbb{S} \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \mathbb{S}$$

where the leftmost copy of $\mathbb{S}$ is in degree k.

Lemma 2.7. *In* Fil(Sp), *there are natural equivalences*

$$\mathbb{1}_{\mathrm{Fil}}(n) \circledast \mathbb{1}_{\mathrm{Fil}}(m) \xrightarrow{\sim} \mathbb{1}_{\mathrm{Fil}}(n+m).$$

Proof. By the construction of $\mathbb{1}_{\mathrm{Fil}}(n) = L_n\mathbb{S}$, it follows that $\mathbb{1}_{\mathrm{Fil}}(n)$ is given by

$$\mathbb{S}^{\otimes \operatorname{Map}_{\mathbb{Z}}(-,n)} \simeq \Sigma^\infty_+ \operatorname{Map}_{\mathbb{Z}}(-,n) : \mathbb{Z}^{op} \to \mathrm{Sp}$$

However, by the universal property of Day convolution, see [15, Section 3] and [26, Corollary 3.7] the functor

$$\mathbb{Z} \to \operatorname{Fun}(\mathbb{Z}^{op}, \mathcal{S}) \xrightarrow{\Sigma^\infty_+} \operatorname{Fun}(\mathbb{Z}^{op}, \mathrm{Sp})$$

is symmetric monoidal, so that we have the desired natural equivalence

$$\mathbb{1}_{\mathrm{Fil}}(n) \circledast \mathbb{1}_{\mathrm{Fil}}(m) \simeq \Sigma^\infty_+ \operatorname{Map}_{\mathbb{Z}}(-,n) \circledast \Sigma^\infty_+ \operatorname{Map}_{\mathbb{Z}}(-,m) \xrightarrow{\sim} \Sigma^\infty_+ \operatorname{Map}_{\mathbb{Z}}(-,n+m) \simeq \mathbb{1}_{\mathrm{Fil}}(n+m). \qquad \square$$

Remark 2.8. More generally, if I is a small symmetric monoidal ∞-category, then $\operatorname{Fun}(I, \mathrm{Sp})$ is presentably symmetric monoidal under the Day convolution product and the functor

$$I \xrightarrow{y} \operatorname{Fun}(I^{op}, \mathcal{S}) \xrightarrow{\Sigma^\infty_+} \operatorname{Fun}(I^{op}, \mathrm{Sp})$$

is symmetric monoidal, where y denotes the Yoneda embedding. In other words, we have a natural equivalence

$$\Sigma^\infty_+ \operatorname{Map}_I(-,i) \circledast \Sigma^\infty_+ \operatorname{Map}_I(-,j) \simeq \Sigma^\infty_+ \operatorname{Map}_I(-,i \otimes j).$$

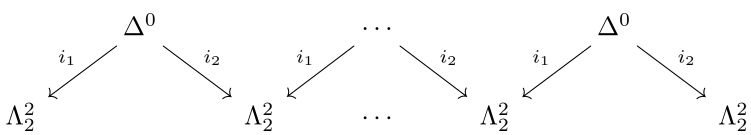
Remark 2.9. One can alternatively prove Lemma 2.7 by using the lax transformation

$$\mathbb{1}_{\mathrm{Fil}}(n) \circledast \mathbb{1}_{\mathrm{Fil}}(m) \to \mathbb{1}_{\mathrm{Fil}}(n+m)$$

and directly calculating the Day convolution product. This calculation may be carried out by appealing to the following lemma.

Lemma 2.10. *Let* $\oplus : \mathbb{Z}^{op} \times \mathbb{Z}^{op} \to \mathbb{Z}^{op}$ *denote the monoidal product and let* $A_k \subseteq B_k = (\mathbb{Z}^{op} \times \mathbb{Z}^{op}) \times_{\mathbb{Z}^{op}} \mathbb{Z}^{op}_{/k}$ *denote the full subcategory of those pairs* (n,m) *such that* $k+1 \geq n+m \geq k$. *Then the inclusion* $A_k \subseteq B_k$ *is cofinal.*

Proof. By Joyal's ∞-categorical version of Quillen's Theorem A (see [19, 4.1.3.1]), it will suffice to verify that for all $(r,s) \in B_k$, $A_k \times_{B_k} (B_k)_{(r,s)/}$ is weakly contractible. By definition, this category is given by the collection of tuples (i,j) in $\mathbb{Z}^{op} \times \mathbb{Z}^{op}$ such that $r \geq i$, $s \geq j$, and $k+1 \geq i+j \geq k$; this is because there must be a path in $\mathbb{Z}^{op} \times \mathbb{Z}^{op}$ from (r,s) to (i,j). As a simplicial set, the category $A_k \times_{B_k} (B_k)_{(r,s)/}$ is isomorphic to the colimit of the diagram



where i_1 denotes the inclusion of the vertex 1 in Λ^2_2 and i_2 denotes the inclusion of the vertex 2 in Λ^2_2. Since anodyne morphisms of simplicial sets are closed under pushouts, the morphism

$$\Lambda^2_2 \amalg_{\Delta^0} \cdots \amalg_{\Delta^0} \Lambda^2_2 \to \Delta^2 \amalg_{\Delta^0} \cdots \amalg_{\Delta^0} \Delta^2$$

is anodyne, hence a weak homotopy equivalence. Now, observe that $\Delta^2 \amalg_{\Delta^0} \cdots \amalg_{\Delta^0} \Delta^2$ is weakly contractible, whence the claim. □

Remark 2.11. An analogue of Lemma 2.10 holds for $\mathbb{Z}^{op}_{\geq 0}$ as well.

Proposition 2.12. *Let $\mathcal{E} \subseteq \mathrm{Fil}(\mathrm{Sp})$ denote the full subcategory of those filtered spectra with the property that the maps $X_n \to X_{n-1}$ are equivalences for $n \leq 0$. Then, $\mathcal{E}$ is a symmetric monoidal subcategory of $\mathrm{Fil}(\mathrm{Sp})$, and the restriction map $\mathrm{Fil}(\mathrm{Sp}) \to \mathrm{fil}(\mathrm{Sp})$ induced by the inclusion $i : \mathbb{Z}^{op}_{\geq 0} \subseteq \mathbb{Z}^{op}$ is a symmetric monoidal equivalence $\mathcal{E} \to \mathrm{fil}(\mathrm{Sp})$.*

Proof. By [19, 4.3.2.15], there is an equivalence of ∞-categories $i^*|_{\mathcal{E}} : \mathcal{E} \to \mathrm{fil}(\mathrm{Sp})$ given by precomposition with $i : \mathbb{Z}^{op}_{\geq 0} \subseteq \mathbb{Z}^{op}$. It will suffice to show that $\mathcal{E}$ is a symmetric monoidal subcategory and that $i^* : \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{fil}(\mathrm{Sp})$ is symmetric monoidal upon restriction to $\mathcal{E}$.

By Lemma 2.5, the objects $\mathbb{1}_{\mathrm{fil}}(n)$ generate $\mathrm{fil}(\mathrm{Sp})$ under desuspensions and small colimits and since the functor $i^*|_{\mathcal{E}}$ is an equivalence, we see that the collection $\{\mathbb{1}_{\mathrm{Fil}}(n)\}_{n\geq 0}$ generates $\mathcal{E}$ under desuspensions and small colimits as well. It is clear that $\mathcal{E}$ is stable under the formation of desuspensions and colimits in $\mathrm{Fil}(\mathrm{Sp})$, so we can reduce to checking that for $n, m \geq 0$, the object $\mathbb{1}_{\mathrm{Fil}}(n) \circledast \mathbb{1}_{\mathrm{Fil}}(m) \in \mathcal{E}$, but this follows immediately from Lemma 2.7.

To check i^* is symmetric monoidal upon restriction to $\mathcal{E}$, we first note that by [26, Corollary 3.8], i^* is a lax symmetric monoidal functor. Therefore, it remains to prove that for all $X, Y \in \mathcal{E}$, the canonical map

$$i^*(X) \circledast i^*(Y) \to i^*(X \circledast Y)$$

is an equivalence. Since i^* preserves all small limits and colimits, it is stable under desuspensions and small colimits separately in each variable, so we may reduce to the case where $X = \mathbb{1}_{\mathrm{Fil}}(n)$ and $Y = \mathbb{1}_{\mathrm{Fil}}(m)$ for $n, m \geq 0$, in which case the result follows from Lemma 2.7. □

2.2. Colimit and evaluation functors

Proposition 2.13. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Then, the functor*

$$\operatorname*{colim}_{\mathbb{Z}^{op}} : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$$

is symmetric monoidal for the Day convolution product on $\mathrm{Fil}(\mathcal{C})$. Additionally, the functor

$$\operatorname*{colim}_{\mathbb{Z}^{op}_{\geq 0}} : \mathrm{fil}(\mathcal{C}) \to \mathcal{C}$$

is symmetric monoidal for the Day convolution product on $\mathrm{fil}(\mathcal{C})$, and $\mathrm{colim}_{\mathbb{Z}^{op}_{\geq 0}} \simeq \mathrm{ev}_0$, where ev_0 denotes precomposition by $0 : \Delta^0 \to \mathbb{Z}^{op}_{\geq 0}$.

Proof. It suffices to prove the claim for $\mathrm{Fil}(\mathcal{C})$ as the case of $\mathrm{fil}(\mathcal{C})$ is identical. Let $p : \mathbb{Z}^{op} \to \Delta^0$ be the unique map to the point, and observe that it is symmetric monoidal, where Δ^0 carries the trivial symmetric monoidal structure; alternatively p can be obtained by observing that $\mathrm{CAlg}(\mathrm{Cat}_\infty)$ admits a terminal object, given by Δ^0. Now, by our presentability assumption, we can apply [26, Corollary 3.8] to conclude that precomposition by p induces a symmetric monoidal operadic left adjoint

$$p_!^{\otimes} : \mathrm{Fun}(\mathbb{Z}^{op}, \mathcal{C})^{\otimes} \xrightarrow{p_!^{\otimes}} \mathcal{C}^{\otimes},$$

such that $(p_!^{\otimes})_{\langle 1\rangle}$ is the left adjoint of p^*, which shows that colim $= p_!$ is indeed symmetric monoidal.

The only claim which remains is to verify that $\mathrm{colim}_{\mathbb{Z}_{\geq 0}^{op}} \simeq \mathrm{ev}_0$. However, this follows from the fact that $\mathbb{Z}_{\geq 0}^{op}$ has a terminal object given by $0 \in \mathbb{Z}_{\geq 0}^{op}$. $\square$

Remark 2.14. By [26, Corollary 3.8], the functor $\mathrm{ev}_0 : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ is lax symmetric monoidal, but will in general fail to be symmetric monoidal as the following example illustrates. Consider the natural map induced by the lax symmetric monoidal structure on ev_0;

$$\mathrm{ev}_0(\mathbb{1}_{\mathrm{Fil}(\mathcal{C})}(1)) \otimes \mathrm{ev}_0(\mathbb{1}_{\mathrm{Fil}(\mathcal{C})}(-1)) \to \mathrm{ev}_0(\mathbb{1}_{\mathrm{Fil}(\mathcal{C})}(1) \circledast \mathbb{1}_{\mathrm{Fil}(\mathcal{C})}(-1)) \simeq \mathbb{1}_{\mathcal{C}}.$$

However, $\mathrm{ev}_0(\mathbb{1}_{\mathrm{Fil}}(-1)) \simeq 0$, so the map above cannot be an equivalence.

2.3. The associated graded functor

Before commencing, we recall some generalities on graded spectra. Let $\mathbb{Z}^{\mathrm{ds}}$ denote the integers viewed as a discrete space, and note that $\mathbb{Z}^{\mathrm{ds}}$ (and thus $(\mathbb{Z}^{\mathrm{ds}})^{op}$) is a symmetric monoidal category with addition as the monoidal product. Similarly, we have the nonnegative variant as well, $\mathbb{Z}^{\mathrm{ds}}_{\geq 0}$. Note that these symmetric monoidal categories are canonically equivalent to their opposites, and upon occasion we may leave this identification implicit.

Definition 2.15. The ∞-category of $\mathbb{Z}$-graded spectra is given by

$$\mathrm{Gr}(\mathrm{Sp}) = \mathrm{Fun}((\mathbb{Z}^{\mathrm{ds}})^{op}, \mathrm{Sp}),$$

and the ∞-category of $\mathbb{Z}_{\geq 0}$-graded spectra is given by

$$\mathrm{gr}(\mathrm{Sp}) = \mathrm{Fun}((\mathbb{Z}^{\mathrm{ds}}_{\geq 0})^{op}, \mathrm{Sp}).$$

Both of these ∞-categories are stable and presentably symmetric monoidal, with monoidal product given by Day convolution, and by abuse of notation, we also let $\circledast$ denote the Day convolution product, allowing context to dictate whether we are Day convolving filtered or graded spectra. The unit object in Gr(Sp), denoted by $\mathbb{1}_{\mathrm{Gr}}$, is the graded spectrum given by

$$(\mathbb{1}_{\mathrm{Gr}})_n = \begin{cases} \mathbb{S}, & \text{if } \ n \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, the unit object in gr(Sp) is the nonnegatively graded spectrum given by

$$(\mathbb{1}_{\mathrm{gr}})_n = \begin{cases} \mathbb{S}, & \text{if } \ n = 0 \\ 0, & \text{otherwise} \end{cases}$$

For each $i \in \mathbb{Z}$, we have a functor $\mathrm{gr}^i : \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$ which sends a filtered spectrum X_* to $\mathrm{cofib}\,(X_{i+1} \to X_i)$, furnishing a functor

$$\mathrm{gr} : \mathrm{Fil}(\mathrm{Sp}) \to \prod_{i \in \mathbb{Z}} \mathrm{Sp} \simeq \mathrm{Gr}(\mathrm{Sp});$$

the functor gr clearly preserves small colimits and is additionally symmetric monoidal by [20, 3.2.1]. Every graded spectrum has an associated "underlying spectrum" which is given by the functor

$$\mathrm{und} = \operatorname*{colim}_{\mathbb{Z}^{\mathrm{ds}}} : \mathrm{Gr}(\mathrm{Sp}) \to \mathrm{Sp},$$

or, in the nonnegatively graded case,

$$\mathrm{und} = \operatorname*{colim}_{\mathbb{Z}^{\mathrm{ds}}_{\geq 0}} : \mathrm{gr}(\mathrm{Sp}) \to \mathrm{Sp}\,.$$

Explicitly, these functors are given by $\mathrm{und}(X_*) = \bigoplus_{i \in \mathbb{Z}} X_i$ and $\mathrm{und}(X_*) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} X_i$, respectively. Given a filtered or nonnegatively filtered object X_* we will sometimes abuse notation and refer to both $\mathrm{gr}^*(X_*)$ and $\mathrm{und}(\mathrm{gr}^*(X_*))$ as the "associated graded" of the filtration X_*.

Proposition 2.16. *The functors*

$$\mathrm{und} \circ \mathrm{gr}^* : \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$$

and

$$\mathrm{und} \circ \mathrm{gr}^* : \mathrm{fil}(\mathrm{Sp}) \to \mathrm{Sp}$$

are colimit-preserving and symmetric monoidal. The same statement is true if we replace Sp *by any stable presentably symmetric monoidal* ∞*-category* $\mathcal{C}$*.*

Proof. By Proposition 2.12, the inclusion $\mathrm{fil}(\mathrm{Sp}) \to \mathrm{Fil}(\mathrm{Sp})$ is colimit-preserving and symmetric monoidal, so it suffices to prove the claim for $\mathrm{gr}_* : \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$. The fact that $\mathrm{und} \circ \mathrm{gr}^*$ is colimit-preserving is clear, as both gr and und preserve small colimits. By [20, 3.2.1], it will suffice to show that und is symmetric monoidal. However, as und can be expressed as a left Kan extension, this holds by an identical proof to that of Proposition 2.13 □

2.4. The Whitehead tower

Recall, that for each $n \in \mathbb{Z}$, we have an n-connective cover functor $\tau_{\geq n} : \mathrm{Sp} \to \mathrm{Sp}_{\geq n}$, which is right adjoint to the inclusion $i_n : \mathrm{Sp}_{\geq n} \subseteq \mathrm{Sp}$; for more details, see [21, 1.2.1.7] for instance. Moreover, for each $n \in \mathbb{Z}$, we let $i_{n,n-1} : \mathrm{Sp}_{\geq n} \to \mathrm{Sp}_{\geq n-1}$ denote the inclusion of n-connective spectra into $(n-1)$-connective spectra, and observe there is a canonical equivalence $i_{n-1} \circ i_{n,n-1} \simeq i_n$. As both i_n and i_{n-1} admit right adjoints, given by $\tau_{\geq n}$ and $\tau_{\geq n-1}$, respectively, there is an induced Beck–Chevallay transformation

$$i_{n,n-1} \circ \tau_{\geq n} \to \tau_{\geq n-1},$$

which, after post-composing by the functor i_{n-1}, gives a natural transformation

$$i_n \circ \tau_{\geq n} \to i_{n-1} \circ \tau_{\geq n-1}.$$

Stitching these transformations together induces a functor $\mathbb{Z}^{op} \to \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$,[2] which can be displayed as

[2] This is possible via the following fact; the map $\cdots \Delta^1 \coprod_{\Delta^0} \Delta^1 \coprod_{\Delta^0} \Delta^1 \cdots \to \mathbb{Z}^{op}$, induced by the maps $\Delta^1 \to \mathbb{Z}^{op}$ which pick out the unique arrow $n \to n-1$, is an equivalence of ∞-categories.

$$\cdots \to i_n \circ \tau_{\geq n} \to i_{n-1} \circ \tau_{\geq n-1} \to \cdots \to i_0 \circ \tau_{\geq 0} \to i_{-1} \circ \tau_{\geq -1} \to \cdots$$

Equivalently, this gives a functor $\tau_{\geq *} : \mathrm{Sp} \to \mathrm{Fil(Sp)}$, which takes a spectrum X to its Whitehead tower.

For our purposes, there is a more convenient construction of $\tau_{\geq *}$. Let $\mathcal{W} \subseteq \mathrm{Fil(Sp)}$ denote the full subcategory of filtered spectra spanned by X_* with the property that for all $n \in \mathbb{Z}$, the spectrum X_n is n-connective.

Lemma 2.17. *The ∞-category $\mathcal{W}$ is presentable and the inclusion $\mathcal{W} \to \mathrm{Fil(Sp)}$, preserves small colimits, thereby admitting a right adjoint $R : \mathrm{Fil(Sp)} \to \mathcal{W}$. Furthermore, $\mathcal{W}$ is a symmetric monoidal subcategory of* Fil(Sp) *and R is lax symmetric monoidal.*

Proof. To see presentability, note that $\mathcal{W}$ can be expressed as the pullback of ∞-categories

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathrm{Fil(Sp)} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_n)_{n\in\mathbb{Z}}} \\ \prod_{n\in\mathbb{Z}} \mathrm{Sp}_{\geq n} & \xrightarrow[(i_n)_{n\in\mathbb{Z}}]{} & \prod_{n\in\mathbb{Z}} \mathrm{Sp} \end{array}$$

and each of the categories involved is presentable, and the functors $(i_n)_{n\in\mathbb{Z}}$ and $(\mathrm{ev}_n)_{n\in\mathbb{Z}}$ preserve small colimits. Therefore, by [19, 5.5.3.12], $\mathcal{W}$ is presentable. The fact that $\mathcal{W}$ is closed under small colimits in Fil(Sp) is immediate since colimits in Fil(Sp) are calculated pointwise and each ∞-category $\mathrm{Sp}_{\geq n}$ is closed under the formation of small colimits in Sp. Therefore, by the adjoint functor theorem, we can deduce the existence of R.

To prove the remaining claims, it will suffice, by [21, 2.2.1.1] and [21, 2.2.1.2], to demonstrate that $\mathcal{W}$ contains the unit of Fil(Sp) and is closed under the formation of Day convolution. Certainly, $\mathbb{1}_{\mathrm{Fil}} \in \mathcal{W}$, and if $X_*, Y_* \in \mathcal{W}$, we have that

$$(X_* \circledast Y_*)_n = \operatorname*{colim}_{p+q\geq n} X_p \otimes Y_q \in \mathrm{Sp}_{\geq n}$$

since $X_p \otimes Y_q \in \mathrm{Sp}_{\geq p+q} \subseteq \mathrm{Sp}_{\geq n}$, and since $\mathrm{Sp}_{\geq n}$ is closed under small colimits in Sp. $\square$

Remark 2.18. Unraveling the construction of R in Lemma 2.17, we see that R can be described as

$$R(X_*) = \cdots \to \tau_{\geq n+1} X_{n+1} \to \tau_{\geq n} X_n \to \tau_{\geq n-1} X_{n-1} \to \cdots .$$

Proposition 2.19. *The Whitehead tower functor $\tau_{\geq *} : \mathrm{Sp} \to \mathrm{Fil(Sp)}$ is canonically lax symmetric monoidal.*

Proof. By Lemma 2.17, the composition $\mathrm{Fil(Sp)} \xrightarrow{R} \mathcal{W} \xrightarrow{i} \mathrm{Fil(Sp)}$ is lax symmetric monoidal. Additionally, as we have seen above, the constant diagram functor $\delta : \mathrm{Sp} \to \mathrm{Fil(Sp)}$ is also lax symmetric monoidal. As the functor $\tau_{\geq *}$ is given by the composite $i \circ R \circ \delta$, we are done. $\square$

Remark 2.20. Since $\tau_{\geq *}$ is lax symmetric monoidal, we have induced functors $\mathrm{Alg}_{\mathbf{E}_1} \to \mathrm{Alg}^{\mathrm{Fil}}_{\mathbf{E}_1}$ and $\mathrm{CAlg} \to \mathrm{CAlg}^{\mathrm{Fil}}$. In particular, for A an $\mathbf{E}_1$-ring, the associated Whitehead tower

$$\cdots \to \tau_{\geq n} A \to \tau_{\geq n-1} A \to \cdots \to \tau_{\geq 0} A \to \tau_{\geq -1} A \to \cdots$$

is an $\mathbf{E}_1$-algebra in Fil(Sp). As a consequence, by Proposition 2.16, the associated graded

$$\mathrm{gr}^* \, \tau_{\geq *} A \simeq \bigoplus_{k\in\mathbb{Z}} \Sigma^k H\pi_k A,$$

is an $\mathbf{E}_1$-ring spectrum.

3. The cobar construction

In this section, we review the cobar construction in the setting of ∞-categories, and establish some of its basic features such as functoriality with respect to lax monoidal functors, base-change, and monoidal refinements.

3.1. $\mathcal{O}$-monoidal envelopes

In this subsection, we review the theory of $\mathcal{O}$-monoidal envelopes as developed in [21, 2.2.4], which we will use to define the cobar, and later on, the cyclic bar construction. Recall that if $\mathcal{O}^{\otimes}$ is an ∞-operad, an $\mathcal{O}$-monoidal ∞-category $\mathcal{C}$ is a morphism of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ which is additionally a coCartesian fibration. Informally, such a gadget is an ∞-category equipped with a coherent collection of ways to tensor together objects, indexed by the multimorphisms in $\mathcal{O}$. A lax $\mathcal{O}$-monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is a commutative diagram of ∞-operads

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{F^{\otimes}} & \mathcal{D}^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{O}^{\otimes} & \end{array}$$

and we say that F is $\mathcal{O}$-monoidal if $F^{\otimes}$ additionally carries p-coCartesian arrows to q-coCartesian arrows.

Definition 3.1. [21, 2.2.4.1] Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category. The $\mathcal{O}$-monoidal envelope of $\mathcal{C}^{\otimes}$, is defined as the fiber product

$$\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})} \mathrm{Act}(\mathcal{O}^{\otimes})$$

where $\mathrm{Act}(\mathcal{O}^{\otimes}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ is the full subcategory spanned by the active morphisms, and $\mathrm{Act}(\mathcal{O}^{\otimes}) \to \mathcal{O}^{\otimes}$ is the projection onto the first factor.

By [21, 2.2.4.4], evaluation at $\{1\} \subseteq \Delta^1$ induces a coCartesian fibration of ∞-operads $p' : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$. Additionally, the pullback of the diagonal embedding $\mathcal{O}^{\otimes} \to \mathrm{Act}(\mathcal{O}^{\otimes})$ along $\mathcal{C}^{\otimes}$ induces a lax $\mathcal{O}$-monoidal inclusion

$$i_{\mathcal{C}} : \mathcal{C}^{\otimes} \to \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes};$$

i.e. $i_{\mathcal{C}}$ is a map of ∞-operads such that $p' \circ i_{\mathcal{C}} = p$. The following proposition characterizes $\mathcal{O}$-monoidal functors out of $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$.

Proposition 3.2. [21, 2.2.4.9] *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be $\mathcal{O}$-monoidal ∞-categories. The inclusion $i_{\mathcal{C}} : \mathcal{C}^{\otimes} \subseteq \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{C}}(\mathcal{D}).$$

Here, $\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{C}), \mathcal{D})$ denotes the ∞-category of $\mathcal{O}$-monoidal functors and $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D})$ is the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the maps of ∞-operads.

In the proof of [21, 2.2.4.9], it is shown that restriction along $i_{\mathcal{C}}$ is a trivial fibration, with inverse given by q-left Kan extension along the map $i_{\mathcal{C}}$. We now fix some notation for the lemma below. Given $F^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a lax $\mathcal{O}$-monoidal functor, we write $\mathrm{Env}_{\mathcal{O}}(F)^{\otimes} : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathrm{Env}_{\mathcal{O}}(\mathcal{D})^{\otimes}$ for the q'-left Kan extension of $i_{\mathcal{D}} \circ F^{\otimes}$ along the map $i_{\mathcal{C}}$, where $q' : \mathrm{Env}_{\mathcal{O}}(\mathcal{D})^{\otimes} \to \mathcal{O}^{\otimes}$. In the case where $F^{\otimes}$ is already $\mathcal{O}$-monoidal, $\mathrm{Env}_{\mathcal{O}}(F)^{\otimes}$ is given by $F^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Act}(\mathcal{O}^{\otimes})$. Because the identity functor $\mathrm{id} : \mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}$ is a lax $\mathcal{O}$-monoidal functor, we can form the p-left Kan extension of id along $i_{\mathcal{C}}$, and we denote this $\mathcal{O}$-monoidal functor by $\otimes_{\mathcal{C}} : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$.

Lemma 3.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a lax $\mathcal{O}$-monoidal functor. Then, there is an essentially unique lax $\mathcal{O}$-monoidal natural transformation*

$$\otimes_{\mathcal{D}} \circ \mathrm{Env}_{\mathcal{O}}(F)^{\otimes} \to F^{\otimes} \circ \otimes_{\mathcal{C}},$$

which is an equivalence when F is $\mathcal{O}$-monoidal.

Proof. We first establish the existence of this lax $\mathcal{O}$-monoidal natural transformation. To do so, note that there is a commutative diagram of the form

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{F^{\otimes}} & \mathcal{D}^{\otimes} \\ {\scriptstyle i_{\mathcal{C}}}\downarrow & \nearrow{\scriptstyle F^{\otimes} \circ \otimes_{\mathcal{C}}} & \downarrow{\scriptstyle q} \\ \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes} \end{array}$$

Therefore, by the universal property of q-left Kan extensions, it will suffice to show that $\otimes_{\mathcal{D}} \circ \mathrm{Env}_{\mathcal{O}}(F)^{\otimes}$ is the q-left Kan extension of $F^{\otimes}$ along $i_{\mathcal{C}}$. However, this follows from [21, 2.2.4.9], using the equivalences

$$\otimes_{\mathcal{D}} \circ \mathrm{Env}_{\mathcal{O}}(F)^{\otimes} \circ i_{\mathcal{C}} \simeq \otimes_{\mathcal{D}} \circ i_{\mathcal{D}} \circ F^{\otimes} \simeq F^{\otimes}.$$

To see this unique natural transformation is an equivalence when $F^{\otimes}$ is $\mathcal{O}$-monoidal, first note that $F^{\otimes} \circ \otimes_{\mathcal{C}}$ is also $\mathcal{O}$-monoidal. The desired claim now follows from [21, 2.2.4.9] combined with the observation that $F^{\otimes} \circ \otimes_{\mathcal{C}} \circ i_{\mathcal{C}} \simeq F^{\otimes}$. $\square$

We will now restrict our attention to the cases where $\mathcal{O}^{\otimes} = \mathrm{Fin}_*$ or $\mathbf{E}_1^{\otimes}$, and offer a brief summary of [21, 2.2.4.3, 2.2.4.6]. In the case where $\mathcal{O}^{\otimes} = \mathrm{Fin}_*$, we write $\mathrm{Env}(\mathcal{C})^{\otimes}$ in place of $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$, and we call this the symmetric monoidal envelope. Additionally, we can identify $\mathrm{Env}(\mathcal{C})^{\otimes}_{\langle 1\rangle}$ with the subcategory of $\mathcal{C}^{\otimes}_{\mathrm{act}} \subseteq \mathcal{C}^{\otimes}$ spanned by the active morphisms, and this ∞-category carries a canonical symmetric monoidal structure. Objects in this ∞-category can be identified with finite tuples of objects in $\mathcal{C}$, and the symmetric monoidal structure can be informally described as concatenation of tuples. From this, we can also see that the underlying functor of $\otimes_{\mathcal{C}} : \mathrm{Env}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ is informally given by sending $(X_1, \ldots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n$.

As we will see below, similar results are true in the case when $\mathcal{O}^{\otimes} = \mathbf{E}_1^{\otimes}$. Recall that by [21, 5.1.0.7], we have an equivalence of ∞-operads $\mathbf{E}_1^{\otimes} \simeq \mathrm{Assoc}^{\otimes}$, which means there is no loss of generality in identifying $\mathrm{Env}_{\mathbf{E}_1}(\mathcal{C})^{\otimes} \simeq \mathrm{Env}_{\mathrm{Assoc}}(\mathcal{C})^{\otimes}$. Moreover, we have an Assoc-monoidal functor $\mathrm{Env}_{\mathrm{Assoc}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ whose underlying functor is given by sending $(X_1, \ldots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n$. We will conclude this subsection be describing $\mathrm{Env}_{\mathrm{Assoc}}(\mathrm{Assoc})$ which we will need to form the cobar construction. Now, consider

$$\mathrm{Act}(\mathrm{Assoc}^{\otimes}) \times_{\mathrm{Assoc}^{\otimes}} \{\langle 1\rangle\}$$

whose objects are active maps $\alpha : \langle n\rangle \to \langle 1\rangle$ in $\mathrm{Assoc}^{\otimes}$, and whose morphisms are commutative diagrams

$$\begin{array}{ccc} \langle n\rangle & \xrightarrow{\quad f \quad} & \langle m\rangle \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & \langle 1\rangle & \end{array}$$

in $\mathrm{Assoc}^{\otimes}$; note that as both α and β are active, this implies that f is active as well. Because α and β are active, we have total orderings on $\langle n\rangle^{\circ}$ and $\langle m\rangle^{\circ}$, and f must restrict to an order preserving map $\langle n\rangle^{\circ} \to \langle m\rangle^{\circ}$.

3.2. The cobar construction of $\mathbf{E}_1$-algebras

To give a rigorous definition of the cobar construction in the language of higher categories, we produce the "universal example" of the cobar construction in an $\mathbf{E}_1$-monoidal category by appealing to the theory of monoidal envelopes as found in [21, 2.2.4]. This construction essentially appears as [24, Construction 2.7], where it is phrased for symmetric monoidal ∞-categories.

Construction 3.4. Recall that we have an equivalence of ∞-operads $\mathbf{E}_1^{\otimes} \simeq \mathrm{Assoc}^{\otimes}$ by [21, 5.1.0.7]. By [21, Construction 2.2.4.1], we have that $\mathrm{Env}_{\mathbf{E}_1}(\mathbf{E}_1) = \mathrm{Env}_{\mathbf{E}_1}(\mathbf{E}_1)^{\otimes}_{\langle 1\rangle}$ can be described as the full subcategory of $\mathrm{Act}(\mathbf{E}_1^{\otimes})$ spanned by the active morphisms $\langle n\rangle \to \langle 1\rangle$ in $\mathbf{E}_1^{\otimes}$. By the equivalence of ∞-operads $\mathbf{E}_1^{\otimes} \simeq \mathrm{Assoc}^{\otimes}$ from [21, 5.1.0.7], we can identify $\mathrm{Env}_{\mathbf{E}_1}(\mathbf{E}_1) \simeq \mathrm{Env}_{\mathrm{Assoc}}(\mathrm{Assoc})$. Now, let Δ_+ denote the augmented simplex category, with initial object $[-1] = \varnothing$. In order to define the desired functor

$$\mathrm{cb} : \Delta_+ \to \mathrm{Env}_{\mathrm{Assoc}}(\mathrm{Assoc}),$$

it will suffice to specify a 1-categorical functor since both the source and target are 1-categories. On objects, this functor is given by sending $[n]$ to $\langle n+1\rangle \cong \{0 < 1 < \cdots < n-1\} \coprod \{*\} \to [0] \coprod \{*\} \cong \langle 1\rangle$, where $[n] \to [0]$ is the unique map to the terminal object. Each map of finite ordered sets $f : [n] \to [m]$ is sent to

$$\langle n+1\rangle \cong \{0 < 1 < \cdots < n-1\} \coprod \{*\} \xrightarrow{f \coprod \{*\}} \{0 < 1 < \cdots < m-1\} \coprod \{*\} \cong \langle m+1\rangle,$$

and it is clear from the definition that cb preserves composition and the identity. Because f is a map of finite ordered sets $\mathrm{cb}(f)$ is an active map in $\mathrm{Assoc}^{\otimes}$, which provides the desired functor

$$\mathrm{cb} : \Delta_+ \to \mathrm{Env}_{\mathrm{Assoc}}(\mathrm{Assoc}).$$

Definition 3.5. Let $\mathcal{C}$ be an ∞-category and let T be a monad in $\mathcal{C}$, i.e. $T \in \mathrm{Alg}_{\mathbf{E}_1}(\mathrm{Fun}(\mathcal{C}, \mathcal{C}))$. Then, the (coaugmented) T-cobar construction of an object $X \in \mathcal{C}$ is given by the following composition

$$\Delta_+ \xrightarrow{\mathrm{cb}} \mathrm{Env}_{\mathbf{E}_1}(\mathbf{E}_1) \xrightarrow{\mathrm{Env}_{\mathbf{E}_1}(T)} \mathrm{Env}_{\mathbf{E}_1}(\mathrm{Fun}(\mathcal{C}, \mathcal{C})) \xrightarrow{\circ} \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \times \{X\} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \times \mathcal{C} \xrightarrow{\mathrm{ev}} \mathcal{C}.$$

We denote this coaugmented cosimplicial object as $\mathrm{CB}^{\bullet}(T; X)$ and note that it can be displayed diagrammatically as follows:

$$X \longrightarrow TX \rightrightarrows T^2X \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

Example 3.6. If $(L \dashv R) : \mathcal{C} \to \mathcal{D}$ is an adjunction of ∞-categories, then $T = RL$ defines a monad on $\mathcal{C}$, and any object $X \in \mathcal{C}$ admits a monadic resolution $\mathrm{CB}^{\bullet}(T; X)$.

While the definition we have offered is quite general, in the sequel, we only deal with examples of the following form.

Example 3.7. Let $f : A \to B$ be a morphism of $\mathbf{E}_1$-algebras in a presentably symmetric monoidal ∞-category $\mathcal{C}$ The extension and restriction of scalars adjunction $f_! \dashv f^*$ determines a monad $f^*f_! : \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C})$. In fact, the adjunction above is monadic and consequently we have an equivalence

$$\mathrm{LMod}_T(\mathrm{LMod}_A(\mathcal{C})) \simeq \mathrm{LMod}_B(\mathcal{C}).$$

Now, we can form the $f^*f_!$ - cobar construction of any object $M \in \mathrm{LMod}_A(\mathcal{C})$, and in the case where $A \in \mathrm{LMod}_A(\mathcal{C})$ we recover a coaugmented cosimplicial object which serves as the analogue of the cobar construction from commutative algebra;

$$A \longrightarrow B \rightrightarrows B \otimes_A B \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \cdots$$

We call this the cobar construction associated to f, and we denote it by $\mathrm{CB}^\bullet(f)$. In the case when we apply the cobar construction to a left A-module M, we shall use the notation $\mathrm{CB}^\bullet(f; M)$.

Given a morphism of $\mathbf{E}_1$-algebras $f : A \to B$, there is an alternative way to produce the object $\mathrm{CB}^\bullet(f)$ using the theory of bimodules. In fact, this construction will produce $\mathrm{CB}^\bullet(f)$ as a coaugmented cosimplicial object in ${}_A\mathrm{BMod}_A$. Recall that by [21, 3.4.1.7], the morphism of $\mathbf{E}_1$-algebras $f : A \to B$ in a presentably $\mathbf{E}_1$-monoidal ∞-category $\mathcal{C}$ determines an $\mathbf{E}_1$-algebra in ${}_A\mathrm{BMod}_A$ under the relative tensor product $\otimes_A$; i.e. B is an $\mathbf{E}_1$-algebra in (A, A)-bimodules, with left and right action given by multiplying on the left or the right via the morphism f.

$$\Delta_+ \xrightarrow{\mathrm{Env}_{\mathbf{E}_1}(f)\,\circ\,\mathrm{cb}} \mathrm{Env}_{\mathbf{E}_1}({}_A\mathrm{BMod}_A(\mathcal{C})) \xrightarrow{\otimes} {}_A\mathrm{BMod}_A(\mathcal{C}) \xrightarrow{(-)\otimes_A M} \mathrm{LMod}_A(\mathcal{C}) \tag{3.8}$$

Now, recall that by [4, Theorem 4.2], there is an equivalence of $\mathbf{E}_1$-monoidal ∞-categories

$${}_A\mathrm{BMod}_A \to \mathrm{Fun}^L(\mathrm{LMod}_A, \mathrm{LMod}_A),$$

which carries M to the functor $M \otimes_A (-)$, and whose inverse is given by evaluation at A. With suitable modifications to the proof of [4, Theorem 4.2], this result is true $\mathcal{C}$-linearly, for $\mathcal{C}$ a presentably symmetric monoidal stable ∞-category. That is, there is an equivalence of $\mathbf{E}_1$-monoidal ∞-categories

$${}_A\mathrm{BMod}_A(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}^L_{\mathcal{C}}(\mathrm{LMod}_A(\mathcal{C}), \mathrm{LMod}_A(\mathcal{C})),$$

where $\mathrm{Fun}^L_{\mathcal{C}}$ denotes colimit-preserving $\mathcal{C}$-linear functors, and the target is equipped with the composition monoidal structure. Furthermore, we note that the forgetful functor

$$\mathrm{Fun}^L_{\mathcal{C}}(\mathrm{LMod}_A(\mathcal{C}), \mathrm{LMod}_A(\mathcal{C})) \to \mathrm{Fun}(\mathrm{LMod}_A(\mathcal{C}), \mathrm{LMod}_A(\mathcal{C}))$$

is one of $\mathbf{E}_1$-monoidal ∞-categories. Consequently, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Env}_{\mathbf{E}_1}({}_A\mathrm{BMod}_A(\mathcal{C})) & \xrightarrow[\otimes]{} & {}_A\mathrm{BMod}_A(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Env}_{\mathbf{E}_1}(\mathrm{Fun}(\mathrm{LMod}_A(\mathcal{C}), \mathrm{LMod}_A(\mathcal{C}))) & \xrightarrow[\circ]{} & \mathrm{Fun}(\mathrm{LMod}_A(\mathcal{C}), \mathrm{LMod}_A(\mathcal{C})) \end{array}$$

The next lemma is now immediate from the preceding discussion.

Lemma 3.9. *Let $f : A \to B$ be a morphism of $\mathbf{E}_1$-algebras in a presentably symmetric monoidal stable ∞-category $\mathcal{C}$. Then, the $f^*f_!$-cobar construction of an object $M \in \mathrm{LMod}_A(\mathcal{C})$ is naturally equivalent to Diagram (3.8).*

We now establish two basic lemmas about the $f^* f_!$-cobar construction using the bimodule perspective above. The first lemma is an "Ur-base change" statement and the second deals with multiplicative refinements of the cobar construction.

Lemma 3.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbf{E}_1$-monoidal categories whose tensor products preserve geometric realizations separately in each variable, and let $F : \mathcal{C} \to \mathcal{D}$ be a lax $\mathbf{E}_1$-monoidal functor. Let $f : A \to B$ be a morphism of $\mathbf{E}_1$-algebras in $\mathcal{C}$ and let $g = F(f) : F(A) \to F(B)$ be the induced map of $\mathbf{E}_1$-algebras in $\mathcal{D}$. Then, for any $M \in \mathrm{LMod}_A(\mathcal{C})$, there is a canonical natural transformation*

$$\mathrm{CB}^\bullet(g; F(M)) \to F\left(\mathrm{CB}^\bullet(f; M)\right) \tag{3.11}$$

of coaugmented cosimplicial objects in LMod_A, which is an equivalence when F is $\mathbf{E}_1$-monoidal and preserves geometric realizations.

Proof. Our assumptions about $\mathcal{C}$ and $\mathcal{D}$ guarantee that both ${}_A\mathrm{BMod}_A(\mathcal{C})$ and ${}_{F(A)}\mathrm{BMod}_{F(A)}(\mathcal{D})$ are $\mathbf{E}_1$-monoidal ∞-categories, and that the induced functor

$$\mathrm{BMod}(F) : {}_A\mathrm{BMod}_A(\mathcal{C}) \to {}_{F(A)}\mathrm{BMod}_{F(A)}(\mathcal{D})$$

is lax $\mathbf{E}_1$-monoidal; see [21, 4.4.3.12]. Applying Lemma 3.3 to the lax $\mathbf{E}_1$-functor ${}_A\mathrm{BMod}_A(\mathcal{C}) \to {}_{F(A)}\mathrm{BMod}_{F(A)}(\mathcal{D})$, we obtain a natural transformation

$$\alpha : \otimes \circ \mathrm{Env}_{\mathbf{E}_1}(\mathrm{BMod}(F)) \to \mathrm{BMod}(F) \circ \otimes.$$

Now, let M be a left A-module, and note that as F is lax $\mathbf{E}_1$-monoidal we have a natural transformation

$$\beta : (-) \otimes_{F(A)} F(M) \circ \mathrm{BMod}(F) \to \mathrm{LMod}(F) \circ (-) \otimes_A M.$$

Both of these fit into the diagram below, furnishing the desired natural transformation

$$\begin{array}{ccccccc}
\Delta_+ \xrightarrow{\mathrm{Env}_{\mathbf{E}_1}(f)\,\circ\,\mathrm{cb}} & \mathrm{Env}_{\mathbf{E}_1}({}_A\mathrm{BMod}_A(\mathcal{C})) & \longrightarrow & {}_A\mathrm{BMod}_A(\mathcal{C}) & \longrightarrow & \mathrm{LMod}_A(\mathcal{C}) \\
 & \downarrow & \overset{\alpha}{\Nearrow} & \downarrow & \overset{\beta}{\Nearrow} & \downarrow \\
 & \mathrm{Env}_{\mathbf{E}_1}({}_{F(A)}\mathrm{BMod}_{F(A)}(\mathcal{D})) & \longrightarrow & {}_{F(A)}\mathrm{BMod}_{F(A)}(\mathcal{D}) & \longrightarrow & \mathrm{LMod}_{F(A)}(\mathcal{D})
\end{array}$$

The assumption that F is $\mathbf{E}_1$-monoidal and preserves geometric realizations implies that $\mathrm{BMod}(F)$ is an $\mathbf{E}_1$-monoidal functor. This implies that α and β are invertible transformations, whence the claim. □

Lemma 3.12. *For $1 \le n \le \infty$, we let $\mathcal{C}$ be an $\mathbf{E}_{n+1}$-monoidal category which admits geometric realizations and whose tensor product preserves geometric realizations separately in each variable. If $f : A \to B$ is a map of $\mathbf{E}_{n+1}$-algebras in $\mathcal{C}$ then the (coaugmented) cobar construction for f refines to a coaugmented cosimplicial object in ${}_A\mathrm{BMod}_A(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C}))$. Moreover, if we have a lax $\mathbf{E}_{n+1}$-monoidal functor $F : \mathcal{C} \to \mathcal{D}$, and $g = F(f)$, the transformation*

$$\mathrm{CB}^\bullet(g) \to F(\mathrm{CB}^\bullet(f))$$

is a map of $(F(A), F(A))$-bimodules in $\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{D})$, which is an equivalence when F is $\mathbf{E}_{n+1}$-monoidal and preserves geometric realizations.

Proof. By Dunn additivity [12] and [21, 3.4.1.7], we have a chain of equivalences

$$\left(\mathrm{Alg}_{\mathbf{E}_{n+1}}(\mathcal{C})\right)_{A/} \simeq \left(\mathrm{Alg}_{\mathbf{E}_1}(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C}))\right)_{A/} \simeq \mathrm{Alg}_{\mathbf{E}_1}\left({}_A\mathrm{BMod}_A(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C}))\right),$$

so we may view $f : A \to B$ as an $\mathbf{E}_1$-algebra in ${}_A\mathrm{BMod}_A(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C}))$ and form the f-cobar construction, producing the desired coaugmented cosimplicial object in ${}_A\mathrm{BMod}_A(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C}))$. To see this the desired refinement we apply Lemma 3.10 to the forgetful functor $\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C}) \to \mathcal{C}$, which is $\mathbf{E}_n$-monoidal; this implies that the cobar construction formed in ${}_A\mathrm{BMod}_A(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C}))$ agrees with the one formed in ${}_A\mathrm{BMod}_A(\mathcal{C})$. That this transformation is an equivalence follows similarly to the proof of Lemma 3.10. □

We now specialize the results above to a more restrictive use-case. Throughout, fix $\mathcal{C}$ stable and presentably symmetric monoidal. Recall that by [21, 4.8.5.16], there is a symmetric monoidal functor

$$\mathrm{LMod} : \mathrm{Alg}_{\mathbf{E}_1}(\mathcal{C}) \to \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{St}}).$$

Consequently, for A an $\mathbf{E}_{n+1}$-algebra in $\mathcal{C}$, the ∞-category $\mathrm{LMod}_A(\mathcal{C})$ is presentably $\mathbf{E}_n$-monoidal. This product admits an explicit description as the composite

$$\mathrm{LMod}_A(\mathcal{C}) \times \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{LMod}_{A\otimes A}(\mathcal{C}) \xrightarrow{m_!} \mathrm{LMod}_A(\mathcal{C}),$$

where $m_!$ is the extension of scalars functor associated to the $\mathbf{E}_n$-algebra map $m : A \otimes A \to A$. Similarly, any map of $\mathbf{E}_{n+1}$-algebras $f : A \to B$ determines an $\mathbf{E}_n$-monoidal functor $f_! : \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{C})$.

Now, for any $\mathbf{E}_2$-algebra A, the forgetful functor ${}_A\mathrm{BMod}_A(\mathrm{LMod}_A(\mathcal{C})) \xrightarrow{\sim} \mathrm{LMod}_A(\mathcal{C})$ induces an equivalence of $\mathbf{E}_1$-monoidal categories, and the composition

$$\mathrm{LMod}_A(\mathcal{C}) \simeq {}_A\mathrm{BMod}_A(\mathrm{LMod}_A(\mathcal{C})) \to {}_A\mathrm{BMod}_A(\mathcal{C})$$

is an $\mathbf{E}_1$-monoidal functor since the monoidal product in $\mathrm{LMod}_A(\mathcal{C})$ can be calculated as the relative tensor product. Therefore, given an $\mathbf{E}_1$-A-algebra B, we can calculate the cobar construction in either $\mathrm{LMod}_A(\mathcal{C})$ or ${}_A\mathrm{BMod}_A(\mathcal{C})$; in particular, we may do this for a map of $\mathbf{E}_2$-algebras $A \to B$.

Corollary 3.13. *Let $f : A \to B$ be a map of $\mathbf{E}_2$-algebras in $\mathcal{C}$, and let $C \in \mathrm{Alg}_{\mathbf{E}_1}(\mathrm{LMod}_A(\mathcal{C}))$, with unit map given by $g : A \to C$. Then there is a canonical equivalence*

$$B \otimes_A \mathrm{CB}^\bullet(g) \xrightarrow{\sim} \mathrm{CB}^\bullet(B \otimes_A g).$$

For all $n \geq 2$, if $f : A \to B$ is a map of $\mathbf{E}_{n+1}$-algebras and C is an $\mathbf{E}_m$-algebra for $2 \leq m \leq n$, this equivalence refines to an equivalence of $\mathbf{E}_{m-1}$-algebras.

Proof. By the discussion preceding the statement of the Corollary, we have an $\mathbf{E}_1$-monoidal functor $B \otimes_A (-) : \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{C})$. Therefore, given $C \in \mathrm{Alg}_{\mathbf{E}_1}(\mathrm{LMod}_A(\mathcal{C}))$, we can apply the same argument appearing in Lemma 3.12 to conclude the natural map

$$B \otimes_A \mathrm{CB}^\bullet(g) \xrightarrow{\sim} \mathrm{CB}^\bullet(B \otimes_A g)$$

is an equivalence of coaugmented cosimplicial left B-modules. The fact that this refines to a multiplicative equivalence follows similarly as in the proof of Lemma 3.12. □

3.3. Applications to THH

Before applying the results from the previous section to THH, we recall the following well-known lemma regarding 1-connective descent, of which Adams–Novikov descent is a consequence.

Lemma 3.14. *Let $f : A \to B$ be a map of connective $\mathbf{E}_1$-rings which is 1-connective. Then, for any left A-module M which is bounded below, the natural map*

$$M \to \lim_{\Delta} \mathrm{CB}^{\bullet}(f; M)$$

is an equivalence.

Proof. By suspending, we can reduce to the case where M is connective. Using the ∞-categorical Dold–Kan correspondence, it suffices to check that the natural map

$$M \to \lim_{n} \mathrm{Tot}_n(\mathrm{CB}^{\bullet}(f; M))$$

is an equivalence. However, by [24, 2.11] and the proof of [24, 2.14], the fiber of this map can be identified with the limit of a tower whose n-th term is

$$\mathrm{fib}\Big(M \to \mathrm{Tot}_n(\mathrm{CB}^{\bullet}(f; M))\Big) \simeq I^{\otimes_A (n+1)} \otimes_A M,$$

where $I = \mathrm{fib}(f) \in \mathrm{Sp}_{\geq 1}$, by assumption. Observe that $I^{\otimes_A (n+1)} \otimes_A M \in \mathrm{Sp}_{\geq n+1}$ by the connectivity assumption on M, and connectivity tends to ∞ linearly in n. The t-structure on Sp is left separated, so the limit over the objects $I^{\otimes_A (n+1)} \otimes_A M$ is zero, whence the claim. □

Lemma 3.15. *Let $f : A \to B$ be a map of $\mathbf{E}_2$-rings. Then, the natural map*

$$\mathrm{CB}^{\bullet}(\mathrm{THH}(f)) \to \mathrm{THH}(\mathrm{CB}^{\bullet}(f))$$

in ${}_{\mathrm{THH}(A)}\mathrm{BMod}_{\mathrm{THH}(A)}(\mathrm{CycSp})$ is an equivalence. Moreover, if $f : A \to B$ is a map of $\mathbf{E}_{n+1}$-rings, for $1 \leq n \leq \infty$, then this is an equivalence of $\mathbf{E}_{n-1}$-algebras in CycSp.

Proof. By [27, IV.2], the functor $\mathrm{THH} : \mathrm{Alg}_{\mathbf{E}_1}(\mathrm{Sp}) \to \mathrm{CycSp}$ is symmetric monoidal. By Lemmas 3.10 and 3.12, we have the desired claim. □

In [11] the authors verify that $E = \mathrm{K}$, THH, TC, and their variants all satisfy 1-connective descent. In particular, for A a connective $\mathbf{E}_\infty$-ring and B a connective $\mathbf{E}_1$-A-algebra with 1-connective unit map $\eta : A \to B$, Dundas and Rognes show (see [11, Theorem 3.14]) that we have an equivalence

$$E(A) \xrightarrow{\sim} \lim_{\Delta} E(\mathrm{CB}^{\bullet}(f)).$$

To prove our flat descent result, we require a generalization of [11, Theorem 3.14], removing the $\mathbf{E}_\infty$ assumption on A, and whose proof is similar to that of [11]. Before commencing, we recall the following standard lemma, whose proof is essentially identical to [2, Lemma 2.1]. We would like to thank the anonymous referee for suggestions which have helped to improve the clarity of the following argument.

Lemma 3.16. *Let I be a small ∞-category, and let G be a finite group or $\mathbf{T}$. Assume there exists some natural number $d \geq 0$ such that $\lim_I$ carries objects of $(\mathrm{Sp}^{\geq 0})^I$ to $\mathrm{Sp}^{\geq -d}$, then:*

1. *Both* $(-)_{hG}$ *and* $(-)^{tG}$ *preserve limits of uniformly bounded below* I*-shaped diagrams in* Sp.
2. *The forgetful functor* $\mathrm{CycSp} \to \mathrm{Sp}^{B\mathbf{T}}$ *preserves and reflects limits of uniformly bounded below* I*-shaped diagrams of cyclotomic spectra.*

Proof. For (1), using the norm cofiber sequence (and desuspending if $G = \mathbf{T}$) it is enough to prove the claim for $(-)_{hG}$. We do so by producing a convenient formula for the homotopy orbits of a connective spectrum with G-action. Write $BG \simeq \mathrm{colim}_n BG^{(n)}$ where $BG^{(n)}$ is an n-skeleton of BG, and hence a finite CW complex; we will also write $BG = BG^{(\infty)}$ for convenience below. Recall that by the convergence of Postnikov towers of spectra, we have an equivalence

$$\mathrm{Sp}_{\geq 0} \xrightarrow{\sim} \lim_m \tau_{\leq m}(\mathrm{Sp}_{\geq 0}),$$

where $\tau_{\leq m}(\mathrm{Sp}_{\geq 0})$ denotes the full subcategory of m-truncated connective spectra in the sense of [19, 5.5.6]; by [21, 1.2.1.9], this is equivalent to $\mathrm{Sp}_{\geq 0} \cap \mathrm{Sp}_{\leq m}$. Moreover, this equivalence is given by sending a spectrum X to $(\tau_{\leq m} X)_{m \geq 0}$ with inverse $(X_m)_{m \geq 0} \mapsto \lim_m X_m$. Observe that for any pair of positive integers m, n with $0 \leq m \leq n \leq \infty$, we have a commuting square of functors of the form $\mathrm{Sp}^{BG}_{\geq 0} \to \tau_{\leq m}(\mathrm{Sp}_{\geq 0})$:

$$\begin{array}{ccc} \mathrm{colim}_{BG^{(n)}} \tau_{\leq m} & \xrightarrow{\sim} & \tau_{\leq m} \mathrm{colim}_{BG^{(n)}} \\ \downarrow \sim & & \downarrow \sim \\ \mathrm{colim}_{BG^{(m)}} \tau_{\leq m} & \xrightarrow[\sim]{} & \tau_{\leq m} \mathrm{colim}_{BG^{(m)}} \end{array}$$

The horizontal maps are equivalences by uniqueness of left adjoints. The vertical maps are equivalences because any functor $BG^{(n)} \to \tau_{\leq m}(\mathrm{Sp}_{\geq 0})$ is uniquely determined by its restriction $BG^{(m)}$; this implies the colimit indexed by $BG^{(n)}$ is the same as the colimit indexed by $BG^{(m)}$. Therefore, as a functor $\mathrm{Sp}^{BG}_{\geq 0} \to \mathrm{Sp}$, we have

$$(-)_{hG} \simeq \lim_m \tau_{\leq m}(-)_{hG} \simeq \lim_n \lim_m \tau_{\leq m}\left(\mathrm{colim}_{BG^{(n)}}(-)\right) \simeq \lim_n \mathrm{colim}_{BG^{(n)}}(-).$$

With our formula in hand, let X be an I-indexed diagram of uniformly bounded below spectra with G-action. By shifting, we may assume that X has the form $I \to \mathrm{Sp}^{BG}_{\geq d}$, and we set $\overline{X} = \lim_i X_i \in \mathrm{Sp}^{BG}_{\geq 0}$. As each $BG^{(n)}$ is a finite CW complex and Sp is stable, $\mathrm{colim}_{BG^{(n)}}$ commutes with small limits. Because the formula for $(-)_{hG}$ is valid for each X_i and for $\overline{X}$, the claim follows.

For the second claim, by [27, II.1.5(v)], it suffices to show that $(-)^{tC_p}$ preserves limits of this form, which follows immediately from (1). □

Example 3.17. We enumerate a few examples of ∞-categories which satisfy the hypothesis of Lemma 3.16.

1. The “punctured n-cube” given by removing the initial vertex of $(\Delta^1)^{\times n}$ has this property for $d = n$. This can be proved by induction together with the following standard technique from Goodwillie calculus. If $\mathcal{X}$ is an n-cube, it can be expressed as a natural transformation of two $(n-1)$-cubes $\mathcal{X}_0 \to \mathcal{X}_1$. Moreover, we have a fiber sequence of the form

$$\mathrm{tfib}(\mathcal{X}) \simeq \mathrm{fib}\left(\mathrm{tfib}(\mathcal{X}_0) \to \mathrm{tfib}(\mathcal{X}_1)\right).$$

2. The diagrams $I = \mathbb{Z}^{op}$ and $\mathbb{Z}^{op}_{\geq 0}$ have this property for $d = 1$. This can be deduced from the following fact: given a I as above and given a diagram $X : I \to \mathrm{Sp}$, we have a fiber sequence of the form

$$\lim_I(X) \to \prod_{i \in I} X(i) \to \prod_{i \in I} X(i).$$

3. The diagram $I = \mathbf{T} = B\mathbb{Z}$ has this property when $d = 1$. This follows by observation that $\lim_{B\mathbb{Z}}$ can be identifed with taking the homotopy fixed points with respect to the endomorphism determined by a diagram $B\mathbb{Z} \to \mathrm{Sp}$.

Theorem 3.18. *Let $f : A \to B$ be a 1-connective map of connective $\mathbf{E}_2$-ring spectra. Then, for any bounded below left $\mathrm{THH}(A)$-module M, the induced map of left $\mathrm{THH}(A)$-modules in CycSp*

$$M \to \lim_{\Delta} \left(\mathrm{THH}(\mathrm{CB}^\bullet(f)) \otimes_{\mathrm{THH}(A)} M\right) \simeq \lim_{\Delta} \mathrm{CB}^\bullet(\mathrm{THH}(f); M)$$

is an equivalence. Additionally, for F any of the functors TC, TR, $(-)_{hG}$, $(-)^{hG}$, or $(-)^{tG}$ (where G is either $\mathbf{T}$ or a finite subgroup thereof), the induced map

$$F(M) \to \lim_{\Delta} F\big(\mathrm{THH}(\mathrm{CB}^\bullet(f)) \otimes_{\mathrm{THH}(A)} M\big)$$

is an equivalence as well.

Proof. The fact that the natural map above is an equivalence of spectra with $\mathbf{T}$-action follows immediately from Lemmas 3.14 and 3.15 together with the observation that for f as above, the induced map $\mathrm{THH}(A) \to \mathrm{THH}(B)$ is 1-connective. To show this is an equivalence of cyclotomic spectra, by [27, II.1.5(v)], we only need to verify that the C_p-Tate construction preserves the limit $\lim_\Delta \mathrm{CB}^\bullet(\mathrm{THH}(f); M)$. In fact, we will prove something stronger, namely that $(-)_{hG}$, $(-)^{hG}$, and $(-)^{tG}$ preserve this limit, for G any subgroup of $\mathbf{T}$. The claim for $(-)^{hG}$ is immediate, and by the norm cofiber sequence the claim for $(-)^{tG}$ follows from the claim for $(-)_{hG}$. Writing $I = \mathrm{fib}(\mathrm{THH}(f))$, we have an equivalence of spectra with $\mathbf{T}$-action

$$\lim_{\Delta} \mathrm{CB}^\bullet(\mathrm{THH}(f); M) \simeq \lim_{n} \mathrm{cofib}\left(I^{\otimes_{\mathrm{THH}(A)} n} \otimes_{\mathrm{THH}(A)} M \to M\right),$$

just as in the proof of Lemma 3.14. Suspending M if necessary, our assumptions imply that

$$\mathrm{cofib}\left(I^{\otimes_{\mathrm{THH}(A)} n} \otimes_{\mathrm{THH}(A)} M \to M\right)$$

is connective for all n, so that the claim follows by Example 3.17 and Lemma 3.16.

The claims for TC and TR follow immediately since both are limit-preserving functors with codomain CycSp. □

As an immediate consequence, we are able to deduce an analogous result for algebraic K-theory.

Corollary 3.19. *Let $f : A \to B$ be a 1-connective map of connective $\mathbf{E}_2$-ring spectra. Then, for any connective $\mathbf{E}_1$-A-algebra C, the induced map*

$$\mathrm{K}(A) \to \lim_{\Delta} \mathrm{K}(\mathrm{CB}^\bullet(f); C),$$

is an equivalence.

Proof. This is a direct consequence of the Dundas–Goodwillie–McCarthy theorem [10] combined with Theorem 3.18. Indeed, it will suffice to prove the analogous claim for, $\mathrm{K}^{\mathrm{inv}}$, the fiber of the cyclotomic trace. However, by the Dundas–Goodwillie–McCarthy theorem, we have an equivalence $\mathrm{K}^{\mathrm{inv}}(R) \simeq \mathrm{K}^{\mathrm{inv}}(\pi_0 R)$ for R a connective $\mathbf{E}_1$-ring. Consequently, as f is 1-connective, the diagram $\mathrm{K}^{\mathrm{inv}}(\mathrm{CB}^\bullet(f; C))$ is constant, whence the claim. □

To conclude this subsection, we will examine the ways in which TC and TR effect the connectivity of maps between cyclotomic spectra. This will allow us to deduce some decent results of a similar flavor to Theorem 3.18. For this, we need to understand to what extent the functors TR and TC preserve connectivity. In the case where X is bounded below and p-complete, this has been documented in [9, Lemma 2.5 or Remark 2.14], which we restate for the convenience of the reader. We claim no originality for this observation. As we were unable to find the desired statement for integral TR in the literature, we provide the argument below.

Lemma 3.20. *Let X be a cyclotomic spectrum.*

1. *If $X \in \mathrm{CycSp}_{\geq n}$, then $\mathrm{TR}(X) \in \mathrm{Sp}_{\geq n}$.*
2. *[9, Lemma 2.5 or Remark 2.14]: If $X \in \mathrm{CycSp}_{\geq n}$ is p-complete, then $\mathrm{TC}(X) \in \mathrm{Sp}_{\geq (n-1)}$.*

Proof. By shifting, we may assume X is connective. Recall that by [27, II.3.8], [23, 3.3.10], and [23, 3.3.12], we have natural equivalences

$$\mathrm{TR}(X) \simeq \mathrm{TR}^{\mathrm{gen}}(X) \simeq \lim_n X^{C_{n!}},$$

where we view X as a genuine cyclotomic spectrum, and the limit is taken over the maps

$$X^{C_{n!}} \simeq (X^{C_n})^{C_{(n-1)!}} \to (\Phi^{C_n} X)^{C_{(n-1)!}} \simeq X^{C_{(n-1)!}}.$$

The isotropy separation sequence guarantees a cofiber sequence of the form

$$(X_{hC_n})^{C_{(n-1)!}} \to X^{C_{n!}} \simeq (X^{C_n})^{C_{(n-1)!}} \to (\Phi^{C_n} X)^{C_{(n-1)!}} \simeq X^{C_{(n-1)!}}.$$

Using the fact that X_{hC_n} preserves connectivity, induction combined with a Milnor sequence argument establishes the desired claim. We note that an identical proof works for p-typical topological restriction homology. The second part of the lemma is [9, Lemma 2.5 or Remark 2.14]. □

Remark 3.21. Lemma 3.20 is very much false for TC in the integral setting. For example, we have an equivalence $\mathrm{TC}(H\mathbf{Q}^{\mathrm{triv}}) \simeq \mathrm{map}(\mathbf{CP}^\infty, H\mathbf{Q})$, which by the Atiyah–Hirzebruch spectral sequence, is a power series ring on a generator of degree -2.

Proposition 3.22. *Let $n \in \mathbb{Z}$ and let $f : X \to Y$ be an n-connective map of bounded below cyclotomic spectra.*

1. *The induced map $\mathrm{TR}(X) \to \mathrm{TR}(Y)$ is n-connective.*
2. *If X and Y are both p-complete, then the induced map $\mathrm{TC}(X) \to \mathrm{TC}(Y)$ is $(n-1)$-connective.*
3. *Let $n \geq 1$. If f is of the form $\mathrm{THH}(A) \to \mathrm{THH}(B)$, induced by an n-connective map of connective $\mathbf{E}_1$-rings $A \to B$, then $\mathrm{TC}(A) \to \mathrm{TC}(B)$ is $(n+1)$-connective.*[3]

Proof. Both (1) and (2) follow immediately from Lemma 3.20 since a map is n-connective if and only if the fiber is n-connective. To complete the argument, assume f is of the form $\mathrm{THH}(A) \to \mathrm{THH}(B)$ where $A \to B$ is an n-connective map of connective ring spectra. By the Dundas–Goodwillie–McCarthy theorem [10], we have an equivalence

$$\mathrm{fib}\,(\mathrm{K}(A) \to \mathrm{K}(B)) \simeq \mathrm{fib}\,(\mathrm{TC}(A) \to \mathrm{TC}(B))\,,$$

[3] We are grateful to Georg Tamme for making us aware of this fact.

where K denotes the connective algebraic K-theory of these ring spectra. Therefore, it will suffice to show that $\mathrm{K}(f)$ is an $(n+1)$-connective map, which is a classical fact, originally proved in [5, Proposition 10.9]. □

Remark 3.23. The discrepancy between (2) and (3) is somewhat surprising to the author. We believe it would be very interesting to identify other maps of cyclotomic spectra for which TC either preserves or increases connectivity.

Proposition 3.24. *Let $n \geq 1$, let $f : R \to S$ be an n-connective map of connective $\mathbf{E}_1$-algebras in* CycSp*, and let $M \in \mathrm{LMod}_A(\mathrm{CycSp})$ be bounded below.*

1. *The natural map*

$$\mathrm{TR}(M) \to \lim_{\Delta} \mathrm{CB}^{\bullet}(\mathrm{TR}(f); \mathrm{TR}(M)),$$

is an equivalence.

2. *If $n \geq 2$ and both R and S are p-complete, then the natural map*

$$\mathrm{TC}(M) \to \lim_{\Delta} \mathrm{CB}^{\bullet}(\mathrm{TC}(f); \mathrm{TC}(M)),$$

is an equivalence.

3. *If f is of the form $\mathrm{THH}(A) \to \mathrm{THH}(B)$ for $A \to B$ an n-connective map of connective $\mathbf{E}_2$-rings, then the natural map*

$$\mathrm{TC}(M) \to \lim_{\Delta} \mathrm{CB}^{\bullet}(\mathrm{TC}(f); \mathrm{TC}(M))$$

is an equivalence.

Proof. Each of these results will follow immediately from Lemma 3.14 once we verify that the maps $\mathrm{TC}(R) \to \mathrm{TC}(S)$ and $\mathrm{TR}(R) \to \mathrm{TR}(B)$ are at least 1-connective. This claim follows immediately from Proposition 3.22. □

Remark 3.25. Interestingly, while TC (and TR) will fail to be symmetric monoidal, Theorem 3.18 and Proposition 3.24 imply that for $A \to B$ a 1-connective map of connective $\mathbf{E}_2$-rings the natural map

$$\lim_{\Delta} \mathrm{TC}(B)^{\otimes_{\mathrm{TC}(A)} \bullet + 1} \to \lim_{\Delta} \mathrm{TC}(B^{\otimes_A \bullet + 1})$$

is an equivalence.

Remark 3.26. We now carefully explain how our work in this subsection directly generalizes [11, Theorem 1.2]. Following the notation of [11], Dundas–Rognes show that if $R \to B$ a 1-connective map of connective $\mathbf{E}_\infty$-rings and A is a connective $\mathbf{E}_1$-R-algebra, the induced map

$$F(A) \to \lim_{\Delta} F\left(A \otimes_R \mathrm{CB}^{\bullet}(f)\right)$$

is an equivalence, for $F = \mathrm{THH}, \mathrm{TC}$, or K. Clearly, Theorem 3.18 implies this result when F is either THH or TC, and Corollary 3.19 does so for $F = \mathrm{K}$.

4. A special case of the descent result

In this section, we prove a special case of our main theorem by combining Theorem 3.18 together with the fact that THH satisfies flat descent for discrete rings as shown by Bhatt–Morrow–Scholze in [6, Corollary 3.4, Remark 3.5].

Theorem 4.1. *Let $f : A \to B$ be a faithfully flat map of connective $\mathbf{E}_2$-rings. Then, the induced map*

$$\mathrm{THH}(H\pi_*A) \to \lim_{\Delta} \mathrm{THH}(\mathrm{CB}^\bullet(H\pi_* f))$$

is an equivalence of cyclotomic spectra.

Proof. By Lemma 3.12, the coaugmented cobar construction associated to the map of $\mathbf{E}_\infty$-rings $\pi_0 f : H\pi_0 A \to H\pi_0 B$, refines to a coaugmented cosimplicial object in $\mathrm{CAlg}(\mathrm{LMod}_{H\pi_0 A})$; similarly, the coaugmented cobar construction associated to the Postnikov truncation $p : H\pi_* A \to H\pi_0 A$ refines to a coaugmented cosimplicial object in $\mathrm{Alg}_{\mathbf{E}_1}(\mathrm{LMod}_{H\pi_* A})$, which, using the $\mathbf{E}_2$-map $H\pi_0 A \to H\pi_* A$, gives a diagram $\Delta_+ \to \mathrm{Alg}_{\mathbf{E}_1}(\mathrm{LMod}_{H\pi_0 A})$. Forming the relative tensor product, and applying THH we obtain the following diagram

$$\mathrm{THH}\left(\mathrm{CB}^\bullet(\pi_0 f) \otimes_{H\pi_0 A} \mathrm{CB}^\bullet(p)\right) : \Delta_+ \times \Delta_+ \to \mathrm{Alg}_{\mathbf{E}_1}(\mathrm{LMod}_{H\pi_0 A}) \xrightarrow{\mathrm{THH}} \mathrm{Sp}\,.$$

By Lemma 3.10, this is equivalent to the diagram

$$\mathrm{THH}(\mathrm{CB}^\bullet(\pi_0 f)) \otimes_{\mathrm{THH}(H\pi_0 A)} \mathrm{THH}(\mathrm{CB}^\bullet(p)).$$

To proceed, we will make a few observations about this augmented bicosimplicial object.

1. Our flatness assumption guarantees the natural map $H\pi_0 B \otimes_{H\pi_0 A} H\pi_* A \to H\pi_* B$ is an equivalence of connective $\mathbf{E}_2$-rings. Combining this with Corollary 3.13 we have an equivalence of coaugmented cosimplicial objects
$$\mathrm{THH}(\mathrm{CB}^\bullet(\pi_0 f)) \otimes_{\mathrm{THH}(H\pi_0 A)} \mathrm{THH}(\mathrm{CB}^{-1}(p)) \simeq \mathrm{THH}(\mathrm{CB}^\bullet(H\pi_* f)).$$
2. We claim that the coaugmented cosimplicial object
$$\mathrm{THH}(\mathrm{CB}^m(\pi_0 f)) \otimes_{\mathrm{THH}(H\pi_0 A)} \mathrm{THH}(\mathrm{CB}^\bullet(p))$$
is a limit diagram for all $[m] \in \Delta_+$. This, however, follows immediately from Theorem 3.18.
3. We now claim that for all $m \geq 0$ coaugmented cosimplicial object
$$\mathrm{THH}(\mathrm{CB}^\bullet(\pi_0 f)) \otimes_{\mathrm{THH}(H\pi_0 A)} \mathrm{THH}(\mathrm{CB}^m(p))$$
is also a limit diagram, and we argue by induction. The case $m = 0$ follows from the [6, Corollary 3.4], and the inductive step follows from the fact that for each $m \geq 1$, $\mathrm{THH}(\mathrm{CB}^m(p)) \simeq \mathrm{THH}((H\pi_0 A)^{\otimes_{H\pi_* A} m})$ is a retract of $\mathrm{THH}(\mathrm{CB}^{m-1}(p))$.

Now, write $X^{\bullet,\bullet} = \mathrm{THH}(\mathrm{CB}^\bullet(\pi_0 f)) \otimes_{\mathrm{THH}(H\pi_0 A)} \mathrm{THH}(\mathrm{CB}^\bullet(p))$. Now, consider the following commutative diagram

$$\begin{array}{ccc} X^{-1,-1} & \longrightarrow & \lim_{[n]\in\Delta} X^{n,-1} \\ \downarrow & & \downarrow \\ \lim_{[m]\in\Delta} X^{-1,m} & \longrightarrow & \lim_{[n]\in\Delta} \lim_{[m]\in\Delta} X^{n,m} \end{array}$$

By observation (1) above, we wish to show the top horizontal map is an equivalence. Therefore, it will be enough to show the other three are equivalences. The left vertical map is an equivalence by observation (2), and the right vertical map can be obtained by taking the limit over the equivalences

$$X^{n,-1} \xrightarrow{\sim} \lim_{[m]\in\Delta} X^{n,m},$$

which are also guaranteed by observation (2). Finally, a similar argument but using observation (3) shows the bottom horizontal map is an equivalence, whence the claim. □

5. Filtered THH and the May filtration

In this section, we review the construction and basic properties of filtered topological Hochschild homology. These notions were originally considered by Brun in [7]. A more modern account also appears in [1]. We will also offer a ∞-categorical construction and account of the May filtration which was originally considered by Angelini-Knoll and Salch in [3].

Before proceeding, we briefly indicate these constructions and introduce some relevant notation. Let $A_* \in \mathrm{Alg}^{\mathrm{Fil}}_{\mathbf{E}_1}$ (or $\mathrm{Alg}^{\mathrm{fil}}_{\mathbf{E}_1}$ for a nonnegatively filtered variant). Then, the filtered topological Hochschild homology of A_*, denoted by $\mathrm{THH}^{\mathrm{Fil}}(A_*)$ is geometric realization of the cyclic bar construction

$$A_* \leftleftarrows A_* \circledast A_* \,\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\, A_* \circledast A_* \circledast A_* \,\overset{\longleftarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longleftarrow}}}\, \cdots$$

formed in the category Fil(Sp). Given an $\mathbf{E}_1$-ring A, the May filtration on THH is obtained by taking filtered THH of the Whitehead tower of A, and we will write

$$\mathrm{Fil}^*_{\mathrm{May}} \mathrm{THH}(A) = \mathrm{THH}^{\mathrm{Fil}}(\tau_{\geq *} A) \quad \text{and} \quad \mathrm{gr}^*_{\mathrm{May}} \mathrm{THH}(A) = \mathrm{gr}^* \mathrm{Fil}^*_{\mathrm{May}} \mathrm{THH}(A)$$

for this filtered object and its associated graded, respectively.

Remark 5.1. The cyclic bar construction can be performed for any associative algebra in a symmetric monoidal ∞-category which admits geometric realizations and whose monoidal product preserves these realizations separately in each variable. Since we are primarily interested in the Whitehead towers of ring spectra and the Day convolution product, we will not work in such general terms, though many of the results in this section will hold in greater generality.

5.1. Recollections on filtered THH

Definition 5.2. Let $A_* \in \mathrm{Alg}^{\mathrm{Fil}}_{\mathbf{E}_1}$, given by a map $A^{\otimes}_* : \mathbf{E}^{\otimes}_1 \to \mathrm{Fil}(\mathrm{Sp})^{\otimes}$. We let $\mathrm{THH}^{\mathrm{Fil}}(A_*)$ denote the filtered spectrum with $\mathbf{T}$-action obtained as the geometric realization of the diagram

$$N(\Lambda^{op}) \xrightarrow{V^o} \mathrm{Env}(\mathbf{E}_1) \xrightarrow{\mathrm{Env}(A^{\otimes}_*)} \mathrm{Env}(\mathrm{Fil}(\mathrm{Sp})) \xrightarrow{\circledast} \mathrm{Fil}(\mathrm{Sp}),$$

where V^o is the map appearing in [27, B.1].

Remark 5.3. This is essentially the definition of THH as given in [27], the only salient difference being that we are using the Day convolution of filtered spectra to form the cyclic bar construction as opposed to the smash product. Additionally, we know that $\mathrm{THH}^{\mathrm{Fil}}$ acquires a circle action by the well-known fact that the geometric realization of any cyclic object admits such an action [27, B.5].

We now establish a few basic properties of filtered THH.

Proposition 5.4. *The functor* $\mathrm{THH}^{\mathrm{Fil}} : \mathrm{Alg}^{\mathrm{Fil}}_{\mathbf{E}_1} \to \mathrm{Fil}(\mathrm{Sp})^{B\mathbf{T}}$ *is symmetric monoidal. Consequently, if* A_* *is an* $\mathbf{E}_{k+1}$*-algebra in* $\mathrm{Fil}(\mathrm{Sp})$*, then* $\mathrm{THH}^{\mathrm{Fil}}(A_*)$ *is an* $\mathbf{E}_k$*-algebra in* $\mathrm{Fil}(\mathrm{Sp})^{B\mathbf{T}}$.

Proof. The first claim essentially follows from the fact that Day convolution preserves geometric realizations separately in each variable and that Δ^{op} is a sifted ∞-category. The second claim is an immediate consequence of the first. $\square$

Next, we establish that filtered THH is compatible with ordinary THH, via the functor $\mathrm{colim}_{\mathbb{Z}^{op}} : \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$.

Proposition 5.5. *There is a natural symmetric monoidal equivalence of functors* $\mathrm{Alg}^{\mathrm{Fil}}_{\mathbf{E}_1} \to \mathrm{Sp}^{B\mathbf{T}}$,

$$\operatorname*{colim}_{\mathbb{Z}^{op}} \circ \mathrm{THH}^{\mathrm{Fil}} \xrightarrow{\sim} \mathrm{THH} \circ \operatorname*{colim}_{\mathbb{Z}^{op}} .$$

Proof. Let A_* be a filtered $\mathbf{E}_1$-ring and let $A = \mathrm{colim}_{\mathbb{Z}^{op}}(A_*)$. By Proposition 2.13 and Lemma 3.10, we have a commutative diagram of symmetric monoidal functors

$$\begin{array}{ccccc}
\mathrm{Env}(\mathbf{E}_1) & \xrightarrow{\mathrm{Env}(A_*^{\otimes})} & \mathrm{Env}(\mathrm{Fil}(\mathrm{Sp})) & \xrightarrow{\circledast} & \mathrm{Fil}(\mathrm{Sp}) \\
& \searrow^{\mathrm{Env}(A^{\otimes})} & \downarrow{\scriptstyle \mathrm{Env}(\mathrm{colim}_{\mathbb{Z}^{op}})} & & \downarrow{\scriptstyle \mathrm{colim}_{\mathbb{Z}^{op}}} \\
& & \mathrm{Env}(\mathrm{Sp}) & \xrightarrow[\otimes]{} & \mathrm{Sp}
\end{array}$$

Precomposing by the functor V^o, and taking the geometric realization yields the result. $\square$

Remark 5.6. Note that we have actually proved a slightly stronger statement, namely that the cyclic bar construction is compatible with $\mathrm{colim}_{\mathbb{Z}^{op}}$ rather than just $\mathrm{THH}^{\mathrm{Fil}}$. Moreover, an identical proof to the one above shows that the cyclic bar construction (in any presentably symmetric monoidal ∞-category $\mathcal{C}$) is compatible with symmetric monoidal functors.

We note the following useful corollary of Proposition 5.5 regarding nonnegatively filtered THH, which follows directly by consideration of Proposition 2.13.

Corollary 5.7. *Let* $i : \mathrm{Alg}^{\mathrm{fil}}_{\mathbf{E}_1} \to \mathrm{Alg}^{\mathrm{Fil}}_{\mathbf{E}_1}$ *denote the symmetric monoidal inclusion of nonnegatively filtered* $\mathbf{E}_1$*-algebras into filtered* $\mathbf{E}_1$*-algebras. Then, there is a natural symmetric monoidal equivalence of functors* $\mathrm{Alg}^{\mathrm{fil}}_{\mathbf{E}_1} \to \mathrm{Fil}(\mathrm{Sp})^{B\mathbf{T}}$

$$i \circ \mathrm{THH}^{\mathrm{fil}} \to \mathrm{THH}^{\mathrm{Fil}} \circ i,$$

consequently, by applying $\mathrm{colim}_{\mathbb{Z}^{op}}$*, we also obtain an equivalence*

$$\mathrm{ev}_0 \circ \mathrm{THH}^{\mathrm{fil}} \xrightarrow{\sim} \mathrm{THH} \circ \mathrm{ev}_0 .$$

There is also a graded variant of THH which we will need in the sequel.

Definition 5.8. Let $A_* \in \mathrm{Alg}^{\mathrm{Gr}}_{\mathbf{E}_1}$, given by a map $A_*^{\otimes} : \mathbf{E}_1^{\otimes} \to \mathrm{Gr}(\mathrm{Sp})^{\otimes}$. We let $\mathrm{THH}^{\mathrm{Gr}}(A_*)$ denote the filtered spectrum with $\mathbf{T}$-action obtained as the geometric realization of the diagram

$$N(\Lambda^{op}) \xrightarrow{V^o} \mathrm{Env}(\mathbf{E}_1) \xrightarrow{\mathrm{Env}(A_*^{\otimes})} \mathrm{Env}(\mathrm{Gr}(\mathrm{Sp})) \xrightarrow{\circledast} \mathrm{Gr}(\mathrm{Sp}).$$

The next proposition shows that filtered and graded THH can be intertwined by the associated graded functor.

Proposition 5.9. *There is a natural symmetric monoidal equivalence of functors* $\mathrm{Alg}^{\mathrm{Fil}}_{\mathbf{E}_1} \to \mathrm{Gr}(\mathrm{Sp})^{B\mathbf{T}}$

$$\mathrm{gr} \circ \mathrm{THH}^{\mathrm{Fil}} \xrightarrow{\sim} \mathrm{THH}^{\mathrm{Gr}} \circ \mathrm{gr}$$

Proof. The proof is identical to that of Proposition 5.5. □

Finally, we show that $\mathrm{THH}^{\mathrm{Gr}}$ and THH may be intertwined via the underlying spectrum functor.

Proposition 5.10. *There is a natural symmetric monoidal equivalence of functors* $\mathrm{Alg}^{\mathrm{Gr}}_{\mathbf{E}_1} \to \mathrm{Sp}^{B\mathbf{T}}$

$$\mathrm{und} \ \circ \mathrm{THH}^{\mathrm{Gr}} \xrightarrow{\sim} \mathrm{THH} \circ \ \mathrm{und}$$

Proof. The proof is identical to that of Proposition 5.5. □

Remark 5.11. The observant reader will notice that the proof of Proposition 5.5 may be easily generalized to establish the following well-known fact: if $F : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor between presentably symmetric monoidal ∞-categories, then for any $A \in \mathrm{Alg}_{\mathbf{E}_1}(\mathcal{C})$, the natural map

$$\mathrm{HH}(F(A)/\mathcal{D}) \to F(\mathrm{HH}(A/\mathcal{C})),$$

is an equivalence; here $\mathrm{HH}(-/\mathcal{C})$ and $\mathrm{HH}(-/\mathcal{D})$ denote the Hochschild homologies computed in $\mathcal{C}$ and $\mathcal{D}$, respectively.

Remark 5.12. All of the folklore results above were also independently established by Antieau–Mathew–Morrow–Nikolaus in [1]. However, the results of [1] are strictly stronger than those we have presented as they also establish compatibility between various kinds of graded and filtered cyclotomic structures.

We conclude this subsection by establishing several basic properties of filtered THH. Before commencing, we recall a bit of notation related to filtered objects. If $X \in \mathrm{Sp}$ is equipped with a descending filtration $\mathrm{Fil}^* X \in \mathrm{Fil}(\mathrm{Sp})$, i.e. if we have a map

$$\operatorname*{colim}_{\mathbb{Z}^{op}} \mathrm{Fil}^* X \to X,$$

we say that $\mathrm{Fil}^* X$ is *exhaustive* provided that $\mathrm{colim}_{\mathbb{Z}^{op}} \mathrm{Fil}^* X \to X$ is an equivalence, and we say that $\mathrm{Fil}^* X$ is *complete* provided that $\lim_{\mathbb{Z}^{op}} \mathrm{Fil}^* X \simeq 0$. We direct the reader to [16] for more details on complete filtered objects.

Proposition 5.13. *Let* $A_* \in \mathrm{Alg}^{\mathrm{Fil}}_{\mathbf{E}_1}$.

1. $\mathrm{THH}^{\mathrm{Fil}}(A_*)$ *defines an exhaustive filtration on* $\mathrm{THH}(A)$.

2. *If the connectivity of A_i tends to ∞ linearly in i, and if each A_i is connective, then* $\mathrm{THH}^{\mathrm{Fil}}(A_*)$ *is complete.*

Proof. The first claim is precisely the content of Proposition 2.13. The second claim will follow if we can show that the connectivity of $\mathrm{THH}^{\mathrm{Fil}}(A_*)_i$ tends to infinity linearly in i. Let $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be the nondecreasing function determined by $f(i) = \mathrm{conn}(A_i)$. Because connective objects are closed under colimits in Sp, the claim will follow if we can verify that the spectrum

$$\mathrm{ev}_n(A_*^{\circledast k}) = \operatorname*{colim}_{n+1 \geq i_1+\cdots+i_i \geq n} A_{i_1} \otimes \cdots \otimes A_{i_k}$$

is n-connective. However, because each A_{i_j} is $f(i_j)$-connective, and f is non-decreasing, we have that

$$n \leq i_1 + \cdots + i_k \leq f(i_1) + \cdots + f(i_k) = \mathrm{conn}\left(A_{i_1} \otimes \cdots \otimes A_{i_k}\right),$$

where the final equality follows from the fact that the smash product of spectra is compatible with the t-structure. □

Remark 5.14. In general, it is extremely useful to have filtrations which are complete. For example, a map $X_* \to Y_*$ of complete filtered objects is an equivalence if and only if the induced map $\mathrm{gr}(X_*) \to \mathrm{gr}(Y_*)$ of graded objects is an equivalence; see [16, Theorem 1.11]. This observation will (essentially) be used to prove our main theorem.

5.2. The May filtration and variants

We now turn to the following ∞-categorical analogue of the May filtration, originally considered in [3].

Definition 5.15. Let A be an $\mathbf{E}_1$-ring spectrum. The May filtration on $\mathrm{THH}(A)$, which we denote by $\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(A)$, is defined to be the filtered spectrum given by $\mathrm{THH}^{\mathrm{Fil}}(\tau_{\geq *}A)$, where $\tau_{\geq *}A$ denotes the Whitehead tower of A. Additionally, we let $\mathrm{gr}^*_{\mathrm{May}}\,\mathrm{THH}(A)$ denote the associated graded object in $\mathrm{Gr}(\mathrm{Sp})^{B\mathbf{T}}$. By abuse of notation, we will sometimes use the notation $\mathrm{gr}^*_{\mathrm{May}}\,\mathrm{THH}(A)$ for the underlying spectrum with $\mathbf{T}$-action determined by this graded object.

Remark 5.16. In the case where A is an $\mathbf{E}_\infty$-ring spectrum, there is an alternative means to define this filtration, following [3]. Recall that $\mathrm{CAlg}^{\mathrm{Fil}} = \mathrm{CAlg}(\mathrm{Fil}(\mathrm{Sp}))$ is a presentable ∞-category, so that it is tensored over the ∞-category of spaces by [19, 4.4.4]. Using this, fact, combined with the main result of [25], we could alternatively define $\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(A)$ via the formula $\tau_{\geq *}A \circledast \mathbf{T}$, where $(-) \circledast \mathbf{T}$ denotes the tensor with the circle; indeed writing $\mathbf{T}$ as a colimit of its simplices, shows that this object is precisely the cyclic bar construction. In fact, one can make an even more general construction, as was done in [3, 3.3.3]. In particular, if $A_* \in \mathrm{CAlg}^{\mathrm{Fil}}$, then given some finite CW complex X, one can form $A_* \circledast X$. In the case where $X = \mathbf{T}^n$ or S^n, then this construction produces a filtration on iterated and higher THH, respectively.

We now establish some of the basic properties of the May filtration.

Proposition 5.17. *Let A be an $\mathbf{E}_1$-ring spectrum, and let $H\pi_*A$ denote* $\mathrm{gr}^*\,\tau_{\geq *}A = \bigoplus_{i\in\mathbb{Z}} \Sigma^i H\pi_i A$.

1. *There is a natural $\mathbf{T}$-equivariant equivalence* $\mathrm{colim}_{\mathbb{Z}^{op}}\,\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(A) \simeq \mathrm{THH}(A)$*, i.e. the May filtration is exhaustive.*
2. *There is a natural $\mathbf{T}$-equivariant equivalence* $\mathrm{und}\left(\mathrm{gr}^*_{\mathrm{May}}\,\mathrm{THH}(A)\right) \simeq \mathrm{THH}(H\pi_*A)$.
3. *If A is additionally connective, then* $\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(A)$ *is a complete filtered object.*

If, in addition, A is an $\mathbf{E}_{k+1}$*-ring spectrum for* $k \geq 1$*, the equivalences of (1) and (2) are equivalences of* $\mathbf{E}_k$*-ring spectra.*

Proof. Parts (1) and (2), as well ask the $\mathbf{E}_{k+1}$-algebra variants, follow immediately from Propositions 5.5, 5.9, and 5.10. Part (3) follows from Proposition 5.13. □

We conclude this section by discussing some variants of the May filtration and their properties. Before making our definitions, recall that if A is an $\mathbf{E}_1$-ring, we write $\mathrm{TC}^-(A) = \mathrm{THH}(A)^{h\mathbf{T}}$ and $\mathrm{TP}(A) = \mathrm{THH}(A)^{t\mathbf{T}}$.

Definition 5.18. Let A be an $\mathbf{E}_1$-ring. For G a subgroup of $\mathbf{T}$, we define the May filtration of $\mathrm{THH}(A)^{hG}$ to be $\mathrm{Fil}^*_{\mathrm{May}} \mathrm{THH}(A)^{hG} = (\mathrm{Fil}^*_{\mathrm{May}} \mathrm{THH}(A))^{hG}$, and similarly for $\mathrm{THH}(A)_{hG}$ and $\mathrm{THH}(A)^{tG}$. In the case where $G = \mathbf{T}$, we use the special notation

$$\mathrm{Fil}^*_{\mathrm{May}} \mathrm{TC}^-(A) = (\mathrm{Fil}^*_{\mathrm{May}} \mathrm{THH}(A))^{hG} \quad \text{and} \quad \mathrm{Fil}^*_{\mathrm{May}} \mathrm{TP}(A) = (\mathrm{Fil}^*_{\mathrm{May}} \mathrm{THH}(A))^{tG}$$

Remark 5.19. We stress that each of these filtrations, except for $\mathrm{Fil}^*_{\mathrm{May}} \mathrm{THH}(A)_{hG}$, may fail to be exhaustive without additional connectivity assumptions on A.

We now establish an analogue of Proposition 5.17 for the May filtration of $\mathrm{THH}(A)_{hG}$.

Proposition 5.20. *Let A be an* $\mathbf{E}_1$*-ring spectrum and let G be a subgroup of* $\mathbf{T}$*. Then* $\mathrm{Fil}^*_{\mathrm{May}} \mathrm{THH}(A)_{hG}$ *is an exhaustive filtration of* $\mathrm{THH}(A)_{hG}$*, and the underlying spectrum of* $\mathrm{gr}^*_{\mathrm{May}} \mathrm{THH}(A)_{hG}$ *is naturally equivalent to* $\mathrm{THH}(H\pi_* A)_{hG}$*. If, additionally, A is assumed to be connective, then* $\mathrm{Fil}^*_{\mathrm{May}} \mathrm{THH}(A)_{hG}$ *is complete.*

Proof. This follows immediately from Proposition 5.17 together with the observations that homotopy orbits commute with all small colimits and preserve connectivity. □

6. Proof of Theorem A and consequences

In this section we prove our main result. This is achieved by combining Theorem 4.1 with some of the basic properties of the May filtration.

Proposition 6.1. *Let* $f : A \to B$ *be a faithfully flat map of connective* $\mathbf{E}_2$*-ring spectra, and let* $H\pi_* f : H\pi_* A \to H\pi_* B$ *denote the induced map. Then, the natural map*

$$\mathrm{CB}^\bullet(H\pi_* f) \xrightarrow{\sim} H\pi_* \mathrm{CB}^\bullet(f)$$

is an equivalence of coaugmented cosimplicial $\mathbf{E}_1$*-ring spectra. Consequently, the induced map of coaugmented cosimplicial objects*

$$\mathrm{THH}(\mathrm{CB}^\bullet(H\pi_* f)) \xrightarrow{\sim} \mathrm{gr}^*_{\mathrm{May}} \mathrm{THH}(\mathrm{CB}^\bullet(f))$$

is an equivalence.

Proof. As the functor $H\pi_* : \mathrm{Sp} \to \mathrm{Sp}$ can be factored into a composite of lax symmetric monoidal functors, it too is lax symmetric monoidal. Therefore, by Lemma 3.10, we obtain the desired multiplicative map

$$\mathrm{CB}^\bullet(H\pi_* f) \to H\pi_* \mathrm{CB}^\bullet(f).$$

It will suffice to show that at each level, the induced map

$$H\pi_*B \otimes_{H\pi_*A} \cdots \otimes_{H\pi_*A} H\pi_*B \to H\pi_*\left(B \otimes_A \cdots \otimes_A B\right),$$

is an equivalence. However, this follows from our flatness assumption by a Tor spectral sequence computation. □

Corollary 6.2. *The functors* $\mathrm{gr}^*_{\mathrm{May}}\,\mathrm{THH}(-)$ *and* $\mathrm{gr}^*_{\mathrm{May}}\,\mathrm{THH}(-)_{hG}$ *satisfy faithfully flat descent, for* $G = \mathbf{T}$ *or any finite subgroup thereof. Consequently, so do the functors* $\mathrm{gr}^i_{\mathrm{May}}\,\mathrm{THH}(-)$ *and* $\mathrm{gr}^i_{\mathrm{May}}\,\mathrm{THH}(-)_{hG}$*.*

Proof. By Theorem 4.1, Proposition 5.17, and Proposition 6.1, we have the following commutative diagram with labeled equivalences

$$\begin{array}{ccc} \mathrm{THH}(H\pi_*A) & \xrightarrow{\sim} & \mathrm{gr}^*_{\mathrm{May}}\,\mathrm{THH}(A) \\ \sim\downarrow & & \downarrow \\ \lim_\Delta \mathrm{THH}(\mathrm{CB}^\bullet(H\pi_*f)) & \xrightarrow[\sim]{} & \lim_\Delta \mathrm{gr}^*_{\mathrm{May}}\,\mathrm{THH}(\mathrm{CB}^\bullet(f)) \end{array}$$

which proves the first claim. The proof for $\mathrm{gr}^*_{\mathrm{May}}\,\mathrm{THH}(-)_{hG}$ follows similarly, but now we use Proposition 5.20. Since each $\mathrm{gr}^i_{\mathrm{May}}$ is naturally a retract of the underlying spectrum, our last claim follows. □

Before commencing with the proof of our main theorem, we recall the following standard lemma regarding cyclotomic spectra.

Theorem 6.3. *Let* $f : A \to B$ *be a faithfully flat map of connective* $\mathbf{E}_2$*-ring spectra. Then, the induced map*

$$\mathrm{THH}(A) \to \lim_\Delta \mathrm{THH}(\mathrm{CB}^\bullet(f))$$

is an equivalence of cyclotomic spectra. Additionally, the same is true for any of the functors TC*,* TR*,* $\mathrm{THH}(-)^{hG}$*,* $\mathrm{THH}(-)_{hG}$*, and* $\mathrm{THH}(-)^{tG}$*, for* $G = \mathbf{T}$ *or any finite subgroup thereof.*

Proof. First, we prove that the map above is an equivalence of spectra with $\mathbf{T}$-action. By Corollary 6.2, we know that the theorem is true for the associated graded terms of the May filtration. Now, consider the following cofiber sequences of filtrations

$$\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(-) \to \mathrm{THH}(-) \to \mathrm{THH}(-)/\,\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(-),$$

where we view $\mathrm{THH}(-)$ as a constant filtered object. Restricting to connective $\mathbf{E}_1$-rings, and taking the limit of these filtrations, we obtain an equivalence

$$\mathrm{THH}(-) \xrightarrow{\sim} \lim_{\mathbb{Z}^{op}} \mathrm{THH}(-)/\,\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(-),$$

in virtue of Proposition 5.17. Therefore, it will suffice to exhibit the claim for each

$$\mathrm{THH}(-)/\,\mathrm{Fil}^i_{\mathrm{May}}\,\mathrm{THH}(-)$$

with $i \geq 1$. For this, observe that for all $i \geq 0$, we have a cofiber sequence of the form

$$\mathrm{gr}^i_{\mathrm{May}}\,\mathrm{THH}(-) \to \mathrm{THH}(-)/\,\mathrm{Fil}^{i+1}_{\mathrm{May}}\,\mathrm{THH}(-) \to \mathrm{THH}(-)/\,\mathrm{Fil}^i_{\mathrm{May}}\,\mathrm{THH}(-),$$

which, by induction combined with Corollary 6.2 proves the claim.

Now, as limits commute with limits, the claim for $\mathrm{THH}(-)^{hG}$ follows immediately for all subgroups of $\mathbf{T}$. By the Norm cofiber sequence for G, the claims for $\mathrm{THH}(-)^{tG}$ and $\mathrm{THH}(-)_{hG}$ are equivalent, and by Lemma 3.16, we can reduce the remainder of the claims to $\mathrm{THH}(-)_{hG}$. However, this essentially follows by the same reasoning as above by applying Proposition 5.20 and Corollary 6.2, whence the claim. □

Remark 6.4. From our work above, it is not hard to show that $\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(-)$, $\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(-)^{hG}$, $\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(-)^{tG}$, and $\mathrm{Fil}^*_{\mathrm{May}}\,\mathrm{THH}(-)_{hG}$, viewed as functors $\mathrm{Alg}^{\mathrm{cn}}_{\mathbf{E}_2} \to \mathrm{Fil}(\mathrm{Sp})$, also satisfy faithfully flat descent.

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Declaration of competing interest

I certify that I have no financial or personal relationships which have influenced the appearing in this article.

References

[1] Benjamin Antieau, Akhil Mathew, Matthew Morrow, Thomas Nikolaus, On the Beilinson fiber square, Duke Math. J. 171 (18) (2022) 3707–3806.
[2] Benjamin Antieau, Thomas Nikolaus, Cartier modules and cyclotomic spectra, J. Am. Math. Soc. 34 (1) (2021) 1–78.
[3] Gabe Angelini-Knoll, Andrew Salch, A May-type spectral sequence for higher topological Hochschild homology, Algebraic Geom. Topol. 18 (5) (2018) 2593–2660.
[4] Jonathan Beardsley, On bialgebras, comodules, descent data and Thom spectra in ∞-categories, Homol. Homotopy Appl. 25 (2) (2023) 219–242.
[5] Marcel Bökstedt, Ib Madsen, Topological cyclic homology of the integers, K-theory, in: K-Theory, Strasbourg, 1992, Astérisque 226 (1994) 57–143.
[6] Bhargav Bhatt, Matthew Morrow, Peter Scholze, Topological Hochschild homology and integral p-adic Hodge theory, Publ. Math. IHÉS 129 (1) (2019) 199–310.
[7] Morten Brun, Filtered topological cyclic homology and relative K-theory of nilpotent ideals, Algebraic Geom. Topol. 1 (1) (2001) 201–230.
[8] Dustin Clausen, Akhil Mathew, Hyperdescent and étale K-theory, Invent. Math. 225 (3) (2021) 981–1076.
[9] Dustin Clausen, Akhil Mathew, Matthew Morrow, K-theory and topological cyclic homology of henselian pairs, J. Am. Math. Soc. 34 (2) (2021) 411–473. MR 4280864.
[10] Bjørn Ian Dundas, Thomas G. Goodwillie, Randy McCarthy, The Local Structure of Algebraic K-Theory, vol. 18, Springer Science & Business Media, 2012.
[11] Bjørn Ian Dundas, John Rognes, Cubical and cosimplicial descent, J. Lond. Math. Soc. 98 (2) (2018) 439–460.
[12] Gerald Dunn, Tensor product of operads and iterated loop spaces, J. Pure Appl. Algebra 50 (3) (1988) 237–258.
[13] Thomas Geisser, Lars Hesselholt, Topological cyclic homology of schemes, in: Algebraic K-Theory, Seattle, WA, 1997, in: Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 41–87. MR 1743237.
[14] David Gepner, Rune Haugseng, Thomas Nikolaus, Lax colimits and free fibrations in ∞-categories, Doc. Math. 22 (2017) 1255–1266.
[15] Saul Glasman, Day convolution for infinity categories, arXiv preprint arXiv:1308.4940, 2013.
[16] Owen Gwilliam, Dmitri Pavlov, Enhancing the filtered derived category, J. Pure Appl. Algebra 222 (11) (2018) 3621–3674.
[17] Lars Hesselholt, Ib Madsen, On the K-theory of local fields, Ann. Math. 158 (1) (2003) 1–113.
[18] Jeremy Hahn, Dylan Wilson, Redshift and multiplication for truncated Brown–Peterson spectra, Ann. Math. 196 (3) (2022) 1277–1351.
[19] Jacob Lurie, Higher Topos Theory, Princeton University Press, 2009.
[20] Jacob Lurie, Rotation invariance in algebraic K-theory, Preprint, available at https://www.math.ias.edu/~lurie, 2015.
[21] Jacob Lurie, Higher algebra, Preprint, available at https://www.math.ias.edu/~lurie, 2017.
[22] Akhil Mathew, THH and base-change for Galois extensions of ring spectra, Algebraic Geom. Topol. 17 (2) (2017) 693–704.
[23] Jonas McCandless, On curves in K-theory and TR, arXiv preprint arXiv:2102.08281, 2021.
[24] Akhil Mathew, Niko Naumann, Justin Noel, Nilpotence and descent in equivariant stable homotopy theory, Adv. Math. 305 (2017) 994–1084.
[25] J. McClure, R. Schwänzl, R. Vogt, THH(R) ≅ R ⊗ S 1 for E∞ ring spectra, J. Pure Appl. Algebra 121 (2) (1997) 137–159.
[26] Nikolaus Thomas, Stable ∞-operads and the multiplicative Yoneda lemma, arXiv preprint, arXiv:1608.02901, 2016.

[27] Nikolaus Thomas, Peter Scholze, On topological cyclic homology, Acta Math. 221 (2) (2018) 203–409.
[28] Arpon Raksit, Hochschild homology and the derived de Rham complex revisited, September 2020.
[29] R.W. Thomason, Algebraic K-theory and etale cohomology, Ann. Sci. Éc. Norm. Supér. 18 (3) (1985) 437–552.
[30] Robert W. Thomason, Thomas Trobaugh, Higher algebraic K-theory of schemes and of derived categories, in: The Grothendieck Festschrift, Springer, 2007, pp. 247–435.
[31] Charles A. Weibel, Susan C. Geller, Étale descent for Hochschild and cyclic homology, Comment. Math. Helv. 66 (1) (1991) 368–388.