THE BROWN-PETERSON HOMOLOGY OF ELEMENTARY p-GROUPS

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1. Introduction. For twenty years, oriented bordism and complex bordism have been under active investigation as generalized homology theories ([At], [CF]). Despite deep applications (e.g. [MRW]) and the many beautiful papers on the structure of bordism, researchers have computed the bordism groups for few spaces: for some artifically constructed examples, for complexes with few cells, for spaces for which the Atiyah-Hirzebruch spectral sequence collapses. And for little else. The n-fold product of $B\mathbf{Z}/p$'s—the classifying space for the elementary p-group of rank n—played a central role in Conner and Floyd's work [CF]. Ever since, its bordism has stood out as a desirable candidate for computation. Here, we compute these bordism groups of the n-fold products of $B\mathbf{Z}/p$'s, producing the first example of a sequence of standard spaces with increasingly complicated, but known, bordism groups.

Conner and Floyd computed the first two cases [CF]. In his thesis, Landweber gave an elegant treatment of the bordism of $B\mathbf{Z}/p \times B\mathbf{Z}/p$ [L_3]. Stong [unpublished] computed the $n = 3$ case. These computations are equivalent to computing the reduced Brown-Peterson homology of the n-fold smash product of $B\mathbf{Z}/p$ with itself, $BP_* \wedge^n B\mathbf{Z}/p$ ([A_3], [BP], [Q]).

Let p be the prime associated with BP. Let L_k be the free BP_*-module on generators of degree $2i$, $0 < i < p^k$. For BP_*-modules M and N, denote $M \otimes_{BP_*} N$ by MN and $M \otimes_{BP_*} \cdots \otimes_{BP_*} M$ (k factors) by M^k or by $\otimes^k M$. Adopt the convention that M^0 is the free BP_*-module on one generator in degree zero. Our main theorem is:

Theorem 5.1. *There is a BP_*-module filtration on the reduced Brown-Peterson homology of the n-fold smash product of the classifying space of $\mathbf{Z}/p$ with itself, $BP_* \wedge^n B\mathbf{Z}/p$, such that the associated graded BP_*-module is*:

$$\bigoplus_{i_1+\cdots+i_k=n-k} L_1^{i_1} \cdots L_k^{i_k} \otimes^k BP_* B\mathbf{Z}/p. \qquad \square$$

The proof is by induction on n using Landweber's short exact sequence:

$$0 \to BP_*B\mathbf{Z}/p \otimes_{BP_*} BP_*X \to BP_*(B\mathbf{Z}/p \wedge X) \to \Sigma\, \mathrm{Tor}_1^{BP_*}(BP_*B\mathbf{Z}/p, BP_*X) \to 0. \tag{2.19}$$

The surprisingly nice answer in Theorem 5.1 derives from the following result which conflicts with Tor^{BP_*}'s reputation of being indescribable and intractable.

THEOREM 4.1. *Denote $BP_*B\mathbf{Z}/p$ by N. Then there is an isomorphism of BP_*-modules*

$$\Sigma\, \mathrm{Tor}_1^{BP_*}(N, N^n) \simeq L_n N^n. \qquad \square$$

The filtration of Theorem 5.1 comes from the iterated use of (2.19). If the sequence (2.19) were split—and it may well be—then Theorem 5.1 would follow from Theorem 4.1 and (2.19) inductively. However, there are serious difficulties in forcing (2.19) to have a split-like behavior.

To complete our description, we must study $\otimes^n BP_*B\mathbf{Z}/p$. As a BP_*-module, $BP_*B\mathbf{Z}/p$ is generated by elements $z_m \in BP_{2m-1}B\mathbf{Z}/p$, $m > 0$. For $I = (i_1, \ldots, i_n)$, define $z_I \in \otimes^n BP_*B\mathbf{Z}/p$ to be $z_{i_1} \otimes \cdots \otimes z_{i_n}$. Recall that $BP_* \equiv \mathbf{Z}_{(p)}[v_1, v_2, \cdots]$ and $I_n = (p, v_1, \cdots, v_{n-1})$ (with $I_1 = (p)$).

THEOREM 3.2. *There is a BP_*-module filtration on $\otimes^n BP_*B\mathbf{Z}/p$ with associated graded object free over BP_*/I_n on classes represented by the z_I.* $\square$

For odd primes, Theorem 5.1 determines the oriented bordism of $\times^n B\mathbf{Z}/p \simeq B(\mathbf{Z}/p)^n$, $MSO_* \times^n B\mathbf{Z}/p$. ($(\mathbf{Z}/p)^n = \mathbf{Z}/p \times \cdots \times \mathbf{Z}/p$, n times.) This is the group of bordism classes of free $(\mathbf{Z}/p)^n$-actions on oriented manifolds [CF]. From the time of the publication of Conner and Floyd's monograph [CF] to now, the toral element $[S^1 \times \cdots \times S^1 \to \times^n B\mathbf{Z}/p] \in MSO_n \times^n B\mathbf{Z}/p$ has been of paramount interest. There is a corresponding fundamental class $\gamma_n \in BP_n \wedge^n B\mathbf{Z}/p$. Conner and Floyd's book ends with a conjecture settled by the following.

THEOREM 1.1. (Conner-Floyd Conjecture [CF], Ravenel-Wilson [RW]).

$$I_n = \{x \in BP_* : x\gamma_n = 0\}. \qquad \square$$

We give two separate computations of $BP_* \wedge^n B\mathbf{Z}/p$. The first—the one we have described above—builds on Theorems 3.2 and 4.1. Both 3.2 and 4.1 rely heavily on the Conner-Floyd Conjecture. Our second computation, described below, uses the Adams spectral sequence and gives a new proof of Theorem 1.1 making this paper formally independent of [RW]. The original Ravenel-Wilson proof of the Conner-Floyd Conjecture [RW] was circuitous. Steve Mitchell [Mit] has a direct proof which reduces the Ravenel-Wilson theorem to four pages of elegance. But our new proof is conceptually simple: give a quick computation of $BP_* \wedge^n B\mathbf{Z}/p$ and then read off the annihilator ideal of the fundamental class.

Historically, the two computations of $BP_* \wedge^n B\mathbf{Z}/p$ were thoroughly intertwined. Originally, we used the Conner-Floyd Conjecture to show the collapse of the Adams spectral sequence for $\pi_*(BP \wedge^n B\mathbf{Z}/p)$. Our work with that spectral sequence led us to conjecture the otherwise unexpected Theorem 4.1.

The second computation of $BP_* \wedge^n B\mathbf{Z}/p$ proceeds in a rather curious fashion. We compute the E_2 terms for the Adams spectral sequences for both $BP_* \wedge^n B\mathbf{Z}/p$ and $BP^* \wedge^n B\mathbf{Z}/p$. The spectral sequence for the latter collapses because its E_2 term is concentrated in even degrees. To show that the Adams spectral sequence for the first collapses, we use the duality spectral sequence

$$(2.35) \qquad E_2^{**} = \mathrm{Ext}^*_{BP_*}(BP_*X, BP^*) \Rightarrow BP^*X.$$

Applied to the E_2 term of the Adams spectral sequence for $BP_* \wedge^n B\mathbf{Z}/p$, we get the correct E_2 term for the spectral sequence for $BP^* \wedge^n B\mathbf{Z}/p$. This forces both spectral sequences—the remaining Adams and the duality—to collapse.

We get several by-products of our computations. Following [JW], it is easy to show that for $k \geq n$, the long exact sequences relating the $BP\langle k\rangle_*(-)$ all become short exact:

$$(1.2) \qquad 0 \longrightarrow BP\langle k\rangle_* \wedge^n B\mathbf{Z}/p \xrightarrow{v_k} BP\langle k\rangle_* \wedge^n B\mathbf{Z}/p \longrightarrow BP\langle k-1\rangle_* \wedge^n B\mathbf{Z}/p \longrightarrow 0.$$

This implies that the projective dimension of $BP_* \wedge^n B\mathbf{Z}/p$ as a BP_*-module is n. Perhaps of more significance is the mod p Brown-Peterson homology of $\wedge^n B\mathbf{Z}/p$. This fits into a short exact sequence

$$(1.3)\qquad 0 \longrightarrow \Sigma BP_*(\wedge^{n-1} B\mathbf{Z}/p, \mathbf{Z}/p) \longrightarrow BP_* \wedge^n B\mathbf{Z}/p \xrightarrow{\partial_1} \Sigma^2 BP_* \wedge^n B\mathbf{Z}/p \longrightarrow 0.$$

See the paragraph following Remark 6.2. Also, we obtain an explicit description of $BP^* \wedge^n B\mathbf{Z}/p$ which nicely complements Landweber's elegant computation of this group [L_2].

We hope this paper inspires many future computations of bordism groups for spaces of interest thus increasing the usefulness of BP to homotopy theory problems. The second author has already used these techniques and results to compute BP_*X and BP^*X for $X = BO_n$ and MO_n ($p = 2$). At present, these computations rely heavily on the Adams spectral sequence computations of this paper's Section 6. See [W_2].

A number of open problems remain. We believe that Theorem 5.1's filtration is unnecessary, but we have been unable to eliminate it. We have not attempted the analogous computations for finite dimensional lens spaces or arbitrary abelian p-groups. Possibly, our techniques may go through with only modest modifications. The n-fold tensor product could be described in much greater detail than what we give. What is the action of the symmetric group on n letters on $BP_* \wedge^n B\mathbf{Z}/p$? Our results are always as BP_*-modules. *Never* as BP_*BP-comodules. Steve Mitchell has fresh insights to many of the interesting questions here. We are publishing at this point to encourage Mitchell and others to build on our computations.

Section 2 reviews the basic facts that we need in the rest of the paper. Theorems 3.2, 4.1, and 5.1 are proven in Sections 3, 4, and 5, respectively. Section 6 contains the Adams spectral sequence approach. A short appendix shows how the work of Ravenel and Wilson [RW] goes through for $p = 2$.

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2. Preliminaries. The basic references for Brown-Peterson homology are [A$_3$], [BP], and [Q]. We review the facts we need. The coefficient ring is

(2.1) $$BP^{-*} \simeq BP_* \simeq \mathbf{Z}_{(p)}[v_1, v_2, \cdots], \qquad |v_n| = 2p^n - 2.$$

(2.2) $$BP^*\mathbf{C}P^\infty \simeq BP^*[[x]], \qquad \text{where} \quad x \in BP^2\mathbf{C}P^\infty$$

is the first Conner-Floyd Chern class. The induced map, p^*, from the fibration,

(2.3) $$B\mathbf{Z}/p \xrightarrow{\pi} \mathbf{C}P^\infty \xrightarrow{p} \mathbf{C}P^\infty,$$

defines

(2.4) $$p^*(x) = [p](x) \equiv \sum_i a_i x^{i+1}, \qquad a_i \in BP_{2i}.$$

Applying BP^* to (2.3), we have a short exact sequence with

(2.5) $$BP^*B\mathbf{Z}/p \simeq BP^*[[x]]/[p](x).$$

This is easy to see from the Atiyah-Hirzebruch spectral sequence. It is convenient to let

(2.6) $$x \quad \text{stand for} \quad \pi^*x.$$

For dimensional reasons,

(2.7) $$a_i \in I_n \equiv (p, v_1, \ldots, v_{n-1}), \qquad i < p^n - 1.$$

We also need

(2.8) $$v_n \equiv a_{p^n-1} \qquad \text{modulo } I_n \text{([Ar], [CF], [Mo], [R]).}$$

The reduced Brown-Peterson homology of $B\mathbf{Z}/p$, $BP_*B\mathbf{Z}/p$, is generated by

(2.9) $$z_m \in BP_{2m-1}B\mathbf{Z}/p, \qquad m > 0,$$

subject to the relations

(2.10) $$\sum_{0\le i<m} a_i z_{m-i} = 0.$$

This is proven using the Gysin sequence, which happens to be short exact here:

(2.11) $$0 \longrightarrow BP_{2m+2}\mathbf{C}P^\infty \xrightarrow{[p](x)\cap -} BP_{2m}(\mathbf{C}P^\infty_+) \xrightarrow{t} BP_{2m+1}B\mathbf{Z}/p \longrightarrow 0.$$

The map t has a realization as a stable transfer,

(2.12) $$\Sigma\mathbf{C}P^\infty_+ \to B\mathbf{Z}/p.$$

The inclusion $\iota_m : \mathbf{C}P^m \hookrightarrow \mathbf{C}P^\infty$ gives the bordism class $[\iota_m] \in BP_{2m}\mathbf{C}P^\infty$; $z_{m+1} = t([\iota_m])$.

(2.13) $$x \cap [\iota_m] = [\iota_{m-1}] \quad \text{and} \quad \pi^*(x) \cap z_{m+1} = z_m.$$

We define an algebraic Smith homomorphism, [CF], $\partial : BP_{2m+1}B\mathbf{Z}/p \to BP_{2m-1}B\mathbf{Z}/p$, by

(2.14) $$\partial(z_{m+1}) = z_m, \qquad \partial(z_1) = 0.$$

This is a BP_*-homomorphism, because ∂ preserves the relations (2.10):

(2.15) $$\partial\Big(\sum_{0\le i<m} a_i z_{m-i}\Big) = \sum_{0\le i<m} a_i \partial(z_{m-i}) = \sum_{0\le i<m-1} a_i z_{m-1-i} = 0.$$

Observe that

(2.16) $$\sum_{0\le i} a_i \partial^i \equiv 0.$$

When the functional equation (2.16) is applied to z_m, we get (2.10). Mark Mahowald pointed out to us that this Smith homomorphism occurs geometrically. Mahowald's observation was essential to our original approach to $BP_* \wedge^n B\mathbf{Z}/p$.

Definition 2.17. A *Landweber filtration* of a BP_* (or BP^*) module M is a filtration—possibly infinite—of M by BP_*-submodules

$$0 = M_{-1} \subset M_0 \subset \cdots \subset M_{i-1} \subset M_i \subset \cdots \subset M$$

with $M_i/M_{i-1} \simeq \Sigma^{k_i} BP_*/I_{n_i}$. We call the associated graded BP_*-module a *Landweber presentation.* In [L_1], Landweber showed that BP_*X always has a finite Landweber presentation when X is a finite complex.

We compute using the Künneth-type spectral sequence of [CS],

$$E^2_{s,t} \simeq \mathrm{Tor}^{BP_*}_{s,t}(BP_*X, BP_*Y) \Rightarrow BP_*(X \wedge Y). \tag{2.18}$$

$BP_*B\mathbf{Z}/p$ has BP_*-projective dimension 1 (see (2.11)). Thus for $X = B\mathbf{Z}/p$, (2.18) collapses to the short exact sequence of [L_3]

$$\begin{aligned} 0 \to BP_*B\mathbf{Z}/p \otimes_{BP_*} BP_*Y &\to BP_*(B\mathbf{Z}/p \wedge Y) \\ &\to \Sigma\, \mathrm{Tor}^{BP_*}_1(BP_*B\mathbf{Z}/p, BP_*Y) \to 0. \end{aligned} \tag{2.19}$$

We only use the form (2.19) and, indeed, we reconstruct an explicit proof of (2.19) in Section 5.

Let R be a ring; R could be a sub-algebra of the mod p Steenrod algebra or R could be BP_*. Let M and N be R-modules. Often we must deal with M knowing no more than the associated graded object (e.g. a Landweber presentation) of a specific filtration of M. In such a situation, our approach to $\mathrm{Tor}^R(N, M)$ or to $\mathrm{Ext}_R(N, M)$ must be indirect. Filter m:

$$0 = M_{-1} \subset M_0 \subset \cdots \subset M_{i-1} \subset M_i \subset \cdots \subset M. \tag{2.20}$$

Apply $\mathrm{Tor}^R_*(N, -)$ to the short exact sequences

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0 \tag{2.21}$$

to obtain a homology spectral sequence:

$$E^r_{s,t} \Rightarrow \mathrm{Tor}^R_{s+t}(N, M) \tag{2.22}$$

$$E^1_{s,t} \simeq \mathrm{Tor}^R_{s,t}(N, M_s/M_{s-1})$$

$$d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}.$$

Note that differentials lower homological degree by one (just enough to affect (2.19)). Let $\bar{M}$ be the graded module associated to (2.21) with $\bar{M}_i = M_i/M_{i-1}$. We have analogous cohomological spectral sequences; their differentials raise cohomological degree by one.

$$E_1^{**} \simeq \mathrm{Ext}_R^*(\bar{M}, N) \Rightarrow \mathrm{Ext}_R^*(M, N) \tag{2.23}$$

$$E_1^{**} \simeq \mathrm{Ext}_R^*(N, \bar{M}) \Rightarrow \mathrm{Ext}_R^*(N, M). \tag{2.24}$$

This ends the review of material prerequisite to the proof of the main theorem. The rest of this section prepares for Section 6's Adams spectral sequence computations.

Let H^*X be the mod p cohomology of X. Let $\mathbf{Z}_p$ be the p-adic integers and A be the mod p Steenrod algebra. By a powerful change of rings, the Adams spectral sequence [A_2]

$$E_2^{**} \simeq \mathrm{Ext}_A^{**}(H^*X, H^*Y) \Rightarrow \{Y, X\}_* \otimes \mathbf{Z}_p \tag{2.25}$$

can be adapted to compute

$$BP^*Y \simeq \{Y, BP\}_{-*} \quad \text{and} \quad BP_*X \simeq \{S^0, BP \wedge X\}_*. \tag{2.26}$$

Let

$$E_s \equiv E[Q_s, Q_{s+1}, \ldots] \tag{2.27}$$

be the exterior algebra on the indicated Milnor primitives [Mi]. $E \equiv E_0$ is a normal subalgebra of A and we have

$$E \hookrightarrow A \to A//E \simeq H^*BP. \tag{2.28}$$

By the Cartan-Eilenberg change of rings spectral sequence we can replace:

$$\text{(2.29)} \qquad \mathrm{Ext}_A^{**}(H^*(BP \wedge X), H^*Y) \quad \text{with} \quad \mathrm{Ext}_E^{**}(H^*X, H^*Y).$$

(See [CE] or, for this particular case, [M].) So the forms of the Adams spectral sequence we use are

$$\text{(2.30)} \qquad \mathrm{Ext}_E^{**}(H^*X, \mathbf{Z}/p) \Rightarrow BP_*X$$

and

$$\text{(2.31)} \qquad \mathrm{Ext}_E^{**}(\mathbf{Z}/p, H^*Y) \Rightarrow BP^{-*}Y.$$

In our Adams spectral sequence computations, we need two more forms of the Cartan-Eilenberg change of rings spectral sequence [CE]. From the sequence,

$$\text{(2.32)} \qquad E[Q_s] \hookrightarrow E_s \to E_{s+1} \simeq E_s//E[Q_s],$$

we have

$$\text{(2.33)} \qquad \mathrm{Ext}^*_{E_{s+1}}(\mathrm{Tor}_*^{E[Q_s]}(M, \mathbf{Z}/p), \mathbf{Z}/p) \Rightarrow \mathrm{Ext}^*_{E_s}(M, \mathbf{Z}/p)$$

and

$$\text{(2.34)} \qquad \mathrm{Ext}^*_{E_{s+1}}(\mathbf{Z}/p, \mathrm{Ext}^*_{E[Q_s]}(\mathbf{Z}/p, M)) \Rightarrow \mathrm{Ext}^*{}_{E_s}(\mathbf{Z}/p, M).$$

Finally, we need the duality spectral sequence of [A_1], [CS], and [JW, 5.17]

$$\text{(2.35)} \qquad E_2^{s,t} \simeq \mathrm{Ext}_{BP_*}(BP_*X, BP^*) \Rightarrow BP^*X.$$

3. Tensor products. Our subject is $\otimes^n BP_*B\mathbf{Z}/p$, the n-fold BP_*-tensor product of the reduced Brown-Peterson homology of $B\mathbf{Z}/p$. $B\mathbf{Z}/p$ is the classifying space for $\mathbf{Z}/p$ where p is the prime associated with BP. Our goal is to give a Landweber presentation (2.17) of this, our subject. Recall the BP_*-generators $z_m \in BP_{2m-1}B\mathbf{Z}/p$, $m > 0$ (2.9). For each n-tuple of positive integers $I = (i_1, \ldots, i_n)$, let

(3.1) $$z_I \equiv z_{i_1} \otimes \cdots \otimes z_{i_n} \in \otimes^n BP_* B\mathbf{Z}/p.$$

THEOREM 3.2. *The tensor product $\otimes^n BP_* B\mathbf{Z}/p$ has a Landweber presentation which is free over BP_*/I_n on classes represented by the z_I.*

Hence the BP_*/I_n-basis of the Landweber presentation of $\otimes^n BP_* B\mathbf{Z}/p$ is indexed by the n-tuples of positive integers. We delay the proof of Theorem 3.2 until after that of its corollary.

COROLLARY 3.3. *The iterated Künneth homomorphism,*

$$\chi : \otimes^n BP_* B\mathbf{Z}/p \to BP_* \wedge^n B\mathbf{Z}/p,$$

is injective.

Proof of 3.3. Suppose z is in the kernel of χ. By Theorem 3.2, we can write $z = \Sigma c_I z_I$, $c_I \in BP_*$. We may assume either $c_I = 0$ or $c_I \not\equiv 0$ modulo I_n. Choose an n-tuple $J = (j_1 + 1, \ldots, j_n + 1)$, $j_k \geq 0$, such that the degree of z_J is maximal for those z_I in z with nonzero c_I. Let $x_i = 1 \otimes \cdots \otimes \pi^*(x) \otimes \cdots \otimes 1$ with the $\pi^*(x)$ in the i^{th} factor; $x_i \in \otimes^n BP^* B\mathbf{Z}/p$. Then

$$0 = x_1^{j_1} \cdots x_n^{j_n} \cap \chi(z) = \chi(x_1^{j_1} \otimes \cdots \otimes x_n^{j_n} \cap z)$$

$$= c_J \chi(z_{(1,\ldots,1)}).$$

By the solution to the Conner-Floyd Conjecture (1.1), $c_J \equiv 0$ modulo I_n. By our choices of the c_I and of J, this means z must be zero. □

Remark 3.4. Theorem 3.2 has a different phrasing. Every element z of $\otimes^n BP_* B\mathbf{Z}/p$ has an expression of the form $z = \Sigma c_{I,L} v^L z_I$, $v^L = v_n^{\ell_n} v_{n+1}^{\ell_{n+1}} \cdots$ for $L = (\ell_n, \ell_{n+1}, \ldots, \ell_{n+k}, 0, 0, \ldots)$. Here $c_{I,L}$ is one of the integers $0, 1, \ldots, p-1$. *And this expression is unique.* Back in the 1960's, Bob Stong knew that z has such an expression. But the uniqueness of the expression—the BP_*/I_n-free statement of (3.2)—depends on the solution of the Conner-Floyd Conjecture. Corollary 3.3 appears in [CF] as 44.4, but the earlier 44.4's argument has a flaw first noticed by Larry Smith. The lack of a Corollary 3.3 was a major obstacle in the computation of various interpretations of $BP_* \wedge^n B\mathbf{Z}/p$. Steve Mitchell suggested the present derivation of Corollary 3.3 from Theorem 3.2; his argument improves on our original one.

Proof of Theorem 3.2. Order n-tuples of positive integers lexicographically: $I < J$ if $i_1 = j_1, \ldots, i_{k=1} = j_{k-1}, i_k < j_k$. Define $F_J \subset \otimes^n BP_*B\mathbf{Z}/p$ to be the BP_*-submodule generated by all z_I, $I \leq J$. This gives a filtration of $\otimes^n BP_*B\mathbf{Z}/p$ which is indexed by the n-tuples J. Let $E_{0,n} = E_0(\otimes^n BP_*B\mathbf{Z}/p)$ be the graded BP_*-module associated to the filtration. We must show that $E_{0,n}$ is a BP_*/I_n-module, i.e. that $I_n E_{0,n} = 0$. The proof that $E_{0,n}$ is BP_*/I_n-free is the same as Corollary 3.3's proof.

We prove that $I_{n+1}E_{0,n+1} = 0$ using the inductive assumption that $I_n E_{0,n} = 0$. (Induction begins with the $n = 0$ vacuous case.) We must show that $I_{n+1}(z_I \otimes z_i) = 0$ in $E_{0,n+1}$ where I is an n-tuple. By (2.7) and (2.8), $I_{n+1} = I_n + (a_{p^n-1})$. By our induction, $(I_n z_I) \otimes z_i = 0$ in $E_{0,n+1}$. It remains to consider $a_{p^n-1}(z_I \otimes z_i)$. By (2.10),

$$-a_{p^n-1} z_I \otimes z_i = \sum_{0<j} a_{p^n-1-j} z_I \otimes z_{i+j} + \sum_{0\leq j} a_{p^n-1+j} z_I \otimes z_{i-j}.$$

By (2.7), $a_{p^n-1-j} \in I_n$ and $a_{p^n-1-j} z_I \otimes z_{i+j} = 0$ in $E_{0,n+1}$. But $z_I \otimes z_{i-j}$ is of lower filtration than $z_I \otimes z_i$. Thus in $E_{0,n+1}$, $-a_{p^n-1} z_I \otimes z_i = 0$ as desired. □

This proof of Theorem 3.2 has, as a corollary, the following result of Conner and Floyd.

COROLLARY 3.5. [CF, 46.4]. *$I_n z_{(1,\ldots,1)} = 0$ in $\otimes^n BP_*B\mathbf{Z}/p$.* □

4. Torsion products. We compute $\mathrm{Tor}_1^{BP_*}(BP_*B\mathbf{Z}/p, \otimes^n BP_*B\mathbf{Z}/p)$ where the n-fold tensor product is as in the beginning of Section 3. Let M be any BP_*-module. We use the following simplified notation in this section.

$$N \equiv BP_*B\mathbf{Z}/p$$

$$N^n \equiv N \otimes \cdots \otimes N, \text{ the } n\text{-fold } BP_*\text{-tensor product}$$

$$\Sigma M \equiv \text{the suspension of } M \; (\simeq BP_*(S^1) \otimes_{BP_*} M)$$

$$MN \equiv M \otimes_{BP_*} N$$

$$N*M = \mathrm{Tor}_1^{BP_*}(N, M)$$

Let L_n be the free BP_*-module on generators y_i in degrees $2i$, $0 < i < p^n$. In this notation, our goal is to compute $\Sigma N*N^n$ and our result is surprisingly elegant.

THEOREM 4.1. $\Sigma N*N^n \simeq L_n N^n$.

Bob Stong proved the $n = 1$, $p = 2$, case of Theorem 4.1 during someone's talk at the 1970 Madison conference.

Following 2.11, we construct a free BP_*-resolution of ΣN.

$$0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} \Sigma N \longrightarrow 0. \tag{4.2}$$

The element z_m is now in $(\Sigma N)_{2m}$. Both F_1 and F_0 are formally isomorphic to $BP_*\mathbf{C}P^\infty$. We have the BP_*-basis elements:

$$y_m \in (F_1)_{2m} \simeq (F_0)_{2m}, \qquad m > 0. \tag{4.3}$$

Then $f_1(y_m) = \Sigma_i a_i y_{m-i}$ and $f_0(y_m) = z_m$. $\mathrm{Tor}_*^{BP_*}(-, M)$ applied to (4.2) gives us the four term exact sequence for any M (e.g. $M = N^n$):

$$0 \longrightarrow \Sigma N*M \longrightarrow F_1 M \xrightarrow{f_1 \otimes 1} F_0 M \xrightarrow{f_0 \otimes 1} \Sigma NM \longrightarrow 0. \tag{4.4}$$

Our approach to Theorem 4.1 is to construct a map $\phi : L_n N^n \to F_1 N^n$ whose image is in $f_1 \otimes 1$'s kernel. This gives a map

$$\tilde{\phi} : L_n N^n \to \Sigma N*N^n. \tag{4.5}$$

Our construction will guarantee injectivity; a counting argument will then give surjectivity.

Recall the Smith homomorphism $\partial : N \to N$ of (2.14). Let $\partial_i : N^n \to N^n$ be $\partial_i = 1 \otimes \cdots \otimes 1 \otimes \partial \otimes 1 \cdots$, the Smith homomorphism in the i-th factor of N^n. In the obvious way, extend ∂_i to a homomorphism $\partial_i : MN^n \to MN^n$ $(= M \otimes_{BP_*} N^n)$.

LEMMA 4.6. *Let E be a free BP_*-module. To define a BP_*-homomorphism*

$$\phi : L_n N^n \to EN^n,$$

it suffices to define $\phi_0(y_k \otimes z_I)$ for all k and I satisfying $\partial_i \phi_0 = \phi_0 \partial_i$,

$i = 1, \ldots, n$. (Note that k ranges over integers, $0 < k < p^n$. The symbol I ranges over all n-tuples of positive integers.)

Proof. We have ϕ_0 defined on generators; we extend ϕ_0's definition linearly, but we must check that the resulting extension ϕ is well-defined and multilinear. We must check that ϕ preserves the relations (2.10). By symmetry, it is enough to check linearity in the n-th coordinate. Write $z_I = z_J \otimes z_j$ (i.e. $(i_1, \ldots, i_n) = (j_1, \ldots, j_{n-1}, j)$). We compute:

$$\begin{aligned}\phi(y_k \otimes z_J \otimes \Sigma_i a_i z_{j-i}) &= \Sigma_i a_i \phi_0(y_k \otimes z_J \otimes z_{j-i})\\ &= \Sigma_i a_i \phi_0 (\partial_n)^i (y_k \otimes z_J \otimes z_j)\\ &= \Sigma_i a_i (\partial_n)^i \phi_0 (y_k \otimes z_J \otimes z_j)\\ &= 0 \quad \text{by} \quad (2.16). \qquad \square\end{aligned}$$

LEMMA 4.7. *There is a BP_*-homomorphism*

$$\phi : L_n N^n \to F_1 N^n$$

which satisfies for all k, $0 < k < p^n$:

(i) $\phi(y_k \otimes z_J) = y_k \otimes z_J + \Sigma_{p^n \le i}\, y_i \otimes w_i$, $w_i \in N^n$;
(ii) $\partial_i \phi = \phi \partial_i$, $i = 1, \ldots, n$; *and*
(iii) $(f_1 \otimes 1)\phi \equiv 0$.

Proof of Theorem 4.1 (assuming 4.7). From 4.7(iii), we have a map (4.5). From 4.7(i), $\tilde{\phi}$ is injective. All groups in (4.4) for $M = N^n$ are finite in each degree. Since $F_1 N^n$ and $F_0 N^n$ are formally isomorphic, the kernel of $f_1 \otimes 1$, $\Sigma N{*}N^n$, and the cokernel, ΣN^{n+1}, must each have the same order in each degree. From 3.2, we know the orders of $L_n N^n$ and ΣN^{n+1}. If they coincide, our injection is a surjection. Let B be the $\mathbf{Z}/p$-vector space generated by the z_m, $m > 0$, ΣN^{n+1} and $\Sigma B^{n+1} \otimes_{\mathbf{Z}/p} BP_*/I_{n+1}$ have the same orders. The order of $L_n N^n$ is the same as the order of $L_n \otimes_{BP_*} B^n \otimes_{\mathbf{Z}/p} BP_*/I_n$. As free right $\mathbf{Z}/p[v_{n+1}, v_{n+2}, \ldots]$-modules: $\Sigma B \otimes_{\mathbf{Z}/p} BP_*/I_{n+1}$ has a basis given by the z_m, $m > 0$; $L_n \otimes_{BP_*} BP_*/I_n$ has a basis given by $y_i \otimes v_n^j$, $0 < i < p^n$, $0 \le j$. Each basis has a single element in each even positive degree. If we decorate these two free right $\mathbf{Z}/p[v_{n+1}, v_{n+2}, \ldots]$-modules with $B^n \otimes_{\mathbf{Z}/p} -$, their orders remain equal. We have a sequence of four graded groups with equal orders:

ΣN^{n+1}, $\Sigma B^{n+1} \otimes_{\mathbf{Z}/p} BP_*/I_{n+1}$, $L_n \otimes_{BP_*} B^n \otimes_{\mathbf{Z}/p} BP_*/I_n$, $L_n N^n$. □

Proof of Lemma 4.7. By 4.6, it is enough to define $\phi_0(y_k \otimes z_I)$ such that (i), (ii), and (iii) hold for ϕ_0. We define $\phi_0(y_k \otimes z_I)$ by induction on the degree of z_I. For $I = (1, \ldots, 1)$, define $\phi_0(y_k \otimes z_I) = y_k \otimes z_I \in F_1 N^n$. Parts (i) and (ii) are trivial. To get (iii), we check that

$$(f_1 \otimes 1)(y_k \otimes z_I) = \Sigma_i a_i y_{k-i} \otimes z_I = 0$$

because $a_i \in I_n$ (recall $k < p^n$, (2.7)) and by Corollary 3.5.

Assume that we have defined $\phi_0(y_k \otimes z_J)$ satisfying (i), (ii), and (iii) for all z_J's with degree less than z_I. Think of the Smith homomorphisms, ∂_i, as partial derivatives $\partial/\partial x_i$. We want to solve the "exact partial differential equations":

$$d\phi_0(y_k \otimes z_I) = \sum_{1 \le i \le n} \partial_i \phi_0(y_k \otimes z_I) dx_i = \sum_{1 \le i \le n} \phi_0 \partial_i (y_k \otimes z_I) dx_i.$$

Because $\partial_i \partial_j = \partial_j \partial_i$, this can be solved up to an "integration constant":

$$\phi_0(y_k \otimes z_I) = \phi_1(y_k \otimes z_I) + c \otimes z_{(1,\ldots,1)}.$$

Here $\phi_1(y_k \otimes z_I)$ satisfies (i) and (ii) and has $z_{(1,\ldots,1)}$-coefficient zero. Part (ii) holds for $\phi_0(y_k \otimes z_I)$ no matter the choice of c. We must choose c so that both (i) and (iii) hold. For (iii), we want $(f_1 \otimes 1)\phi_0(y_k \otimes z_I) = 0$. Well, $x \equiv (f_1 \otimes 1)\phi_1(y_k \otimes z_I)$ has

$$\partial_i (f_1 \otimes 1)\phi_1(y_k \otimes z_I) = (f_1 \otimes 1)\phi_1 \partial_i (y_k \otimes z_I) = 0$$

by induction on the degree of z_I. Since the $\partial_i x = 0$, $x = w \otimes z_{(1,\ldots,1)}$, $w \in F_1$. By the exactness of (4.4) ($M = N^n$), $0 = (f_0 \otimes 1)(x) = (f_0 \otimes 1)(f_1 \otimes 1)\phi_1(y_k \otimes z_I)$. By Lemma 4.8 (following), we have

$$w \otimes z_{(1,\ldots,1)} = (f_1 \otimes 1)\Big(\sum_{p^n \le i} c_i y_i \otimes z_{(1,\ldots,1)}\Big).$$

To complete the proof, we define our integration constant c to be $-\Sigma_{p^n \le i} c_i y_i$. □

LEMMA 4.8. *Let* $z = z_{(1,\ldots,1)}$. *If* $(f_0 \otimes 1)(w \otimes z) = 0$, *then*

$$w \otimes z = (f_1 \otimes 1)\Big(\sum_{p^n \leq i} c_i y_i \otimes z\Big), \qquad c_i \in BP_*.$$

Proof. Write $w \otimes z$ in the form

$$\sum_{1 \leq i \leq m} d_i v_n y_i \otimes z + \sum_i e_i y_i \otimes z, \qquad d_i, e_i \in BP_*.$$

By Theorem 3.2, we can do this with the non-zero e_i not in I_{n+1} and the non-zero d_i not in I_n. If $m = 1$,

$$0 = (f_0 \otimes 1)(w \otimes z) = d_1 v_n z_1 \otimes z + \sum_i e_i z_i \otimes z.$$

By Corollary 3.5, the first term is zero, but by Theorem 3.2 (and our assumptions about the e_i), all the e_i must be zero. By Corollary 3.5, by (2.7) and (2.8),

$$\begin{aligned} w \otimes z &= d_1 v_n y_1 \otimes z = d_1 a_{p^n - 1} y_1 \otimes z \\ &= (f_1 \otimes 1)(d_1 y_{p^n} \otimes z). \end{aligned}$$

Now suppose $m > 1$.

$$\begin{aligned} (f_1 \otimes 1)(d_m y_{m+p^n-1} \otimes z) = &\sum_{0 \leq i < p^n - 1} d_m a_i y_{m+p^n-1-i} \otimes z \\ &+ d_m a_{p^n-1} y_m \otimes z \\ &+ \sum_{p^n \leq i} d_m a_i y_{m+p^n-1-i} \otimes z. \end{aligned}$$

The first terms are zero because $a_i z = 0$ ((2.7) and Corollary 3.5). The second term is equal to $d_m v_n y_m \otimes z$. For $i \geq p^n$, write $d_m a_i$ as before:

$$d_m a_i = d'_{m+p^n-1-i} v_n + e'_{m+p^n-1-i}.$$

So

$$\begin{aligned} (w - f_1(d_m y_{m+p^n-1})) \otimes z = &\sum_{0 \leq i < m} (d_i - d_i') v_n y_i \otimes z \\ &+ \sum_i (e_i - e_i') y_i \otimes z. \end{aligned}$$

This element satisfies the lemma's hypothesis. By induction on m, we are done. $\square$

Once upon a time, an "algebraic Conner-Floyd conjecture" had a certain popularity. This conjecture asked that the annihilator ideal of $z_1 \otimes \cdots \otimes z_1$ in the n-fold tensor product $\otimes^n BP_*B\mathbf{Z}/p$ be I_n. Well, it is. The dream was to prove this weaker, algebraic analog of the Conner-Floyd Conjecture and to then prove a Corollary 3.3. The two would solve the Conner-Floyd Conjecture (Theorem 1.1). History did not happen that way. Observe that Theorem 3.2 and Theorem 4.1 only depend on the "algebraic Conner-Floyd conjecture." (And note that Corollary 3.3 is no mere corollary.) It would interest us to see a simple, direct approach to Theorems 3.2 and 4.1 which would circumvent the full strength of Theorem 1.1.

5. The main theorem. We complete the computation of the reduced Brown-Peterson homology of the n-fold smash product of $B\mathbf{Z}/p$ with itself, $BP_* \wedge^n B\mathbf{Z}/p$. Recall that L_k is the free BP_*-module on generators of degree $2i$, $0 < i < p^k$.

THEOREM 5.1. *There is a BP_*-module filtration of $BP_* \wedge^n B\mathbf{Z}/p$ such that the associated graded BP_*-module is isomorphic to*

$$\oplus J_n \otimes_{BP_*} \cdots \otimes_{BP_*} J_1 .$$

The direct sum is over all such tensor products where each J_i is either L_k or $BP_*B\mathbf{Z}/p \simeq N$. Here k is the number of J_j, $j < i$, which are N.

This is an equivalent rephrasing of the introduction's version of Theorem 5.1. Some examples:

Module	Associated Graded Module
$BP_*B\mathbf{Z}/p$	N
$BP_* \wedge^2 B\mathbf{Z}/p$	$(N \otimes N) \oplus (L_1 \otimes N)$
$BP_* \wedge^3 B\mathbf{Z}/p$	$(N \otimes N \otimes N) \oplus (N \otimes L_1 \otimes N) \oplus (L_2 \otimes N \otimes N)$ $\oplus (L_1 \otimes L_1 \otimes N)$

Here, as throughout this section, $N \otimes N$ means $N \otimes_{BP_*} N$, etc. Note that when $n \leq 2p - 2$ (e.g. $n = 2$), $BP_* \wedge^n B\mathbf{Z}/p$ is isomorphic to its associated graded module [L3]. By Theorem 4.1, the above associated graded module for $BP_* \wedge^3 B\mathbf{Z}/p$ is isomorphic to:

$$(N \otimes BP_* \wedge^2 B\mathbf{Z}/p) \oplus (\Sigma \operatorname{Tor}_1^{BP_*}(N, BP_* \wedge^2 B\mathbf{Z}/p)).$$

The two summands are the two end terms in Landweber's short exact sequence (2.19). This example suggests the proof of our main theorem. Complications arise since we do not identify $BP_* \wedge^n B\mathbf{Z}/p$ with its associated graded module. Theorem 5.1's proof occupies the rest of this section.

We need a geometric realization of the resolution (4.2) and an explicit proof of the short exact sequence (2.19). Let C be the stable cofibre of the Bockstein π (in the cofibration)

(5.2) $$B\mathbf{Z}/p \xrightarrow{\pi} \mathbf{C}P^\infty \to C.$$

Using the long exact sequence for integral and mod p homology, we see that $H_*(C, \mathbf{Z})$ is free abelian on one generator in each positive even dimension. The homomorphism $BP_*(\pi)$ is zero; so we get the free resolution (4.2).

(5.3)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & BP_*\mathbf{C}P^\infty & \longrightarrow & BP_*C & \longrightarrow & BP_*\Sigma B\mathbf{Z}/p & \longrightarrow & 0 \\ & & \simeq & & \simeq & & \simeq & & \\ 0 & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \longrightarrow & \Sigma N & \longrightarrow & 0. \end{array}$$

Smash Y with the stable cofibration (5.2) and apply $BP_*(-)$ to obtain:

(5.4)
$$\begin{array}{ccc} BP_*\mathbf{C}P^\infty \otimes BP_*Y & \xrightarrow{f_1 \otimes 1} & BP_*C \otimes BP_*Y \\ \simeq & & \simeq \\ BP_*(\mathbf{C}P^\infty \wedge Y) & \longrightarrow & BP_*(C \wedge Y) \\ & {\scriptstyle (\pi\wedge 1)_*} \nwarrow \quad \swarrow {\scriptstyle \delta_*} & \\ & BP_*(B\mathbf{Z}/p \wedge Y) & \end{array}$$

From (5.4), we get the short exact sequence

(5.5) $$0 \to \text{Image } (f_1 \otimes 1) \to BP_*(B\mathbf{Z}/p \wedge Y)$$
$$\to \text{Kernel } (f_1 \otimes 1) \to 0.$$

By (5.3) and the four-term exact sequence (4.4) ($M = BP_*Y$), (5.5) becomes (2.19):

$$(5.6)\qquad 0 \to BP_*B\mathbf{Z}/p \otimes BP_*Y \to BP_*(B\mathbf{Z}/p \wedge Y) \to \Sigma \mathrm{Tor}_1^{BP_*}(BP_*B\mathbf{Z}/p, BP_*Y) \to 0.$$

We have now proven:

LEMMA 5.7. *The image of* $(\pi \wedge 1)_*$ *is* $\mathrm{Tor}_1^{BP_*}(BP_*B\mathbf{Z}/p, BP_*Y)$. □

We use (5.6) to filter $BP_* \wedge^n B\mathbf{Z}/p$ inductively with 2^{n-1} BP_*-modules. The first half (2^{n-2} many) of these terms come from the tensor product, the second half from the torsion product.

We need an inductive computation of

$$(5.8)\qquad \mathrm{Tor}_*^{BP_*}(BP_*B\mathbf{Z}/p, BP_* \wedge^{n-1} B\mathbf{Z}/p) = \{\mathrm{Tor}_0^{BP_*}(BP_*B\mathbf{Z}/p, BP_* \wedge^{n-1} B\mathbf{Z}/p), \mathrm{Tor}_1^{BP_*}(BP_*B\mathbf{Z}/p, BP_* \wedge^{n-1} B\mathbf{Z}/p), 0, 0, \cdots\}.$$

By induction, we know the associated graded module of $BP_* \wedge^{n-1} B\mathbf{Z}/p$ as in Theorem 5.1. To compute (5.8), we must use the spectral sequence (2.22). By Theorem 4.1, the E^1 term of the spectral sequence is isomorphic to

$$\oplus J_n \otimes \cdots \otimes J_1$$

where each J_i, $1 \leq i \leq n$, is either L_k or N. And k is the number of J_j, $j < i$, which are N. The $J_n = N$ terms give Tor_0; the $J_n = L_k$ ones give Tor_1. The differentials of the spectral sequence (2.22) can go only from Tor_1 to Tor_0. So it suffices to show that no element of Tor_0 is ever hit by a differential in (2.22).

For $i = 1, \ldots, n$, define

$$(5.9)\qquad X_i = (\wedge^{n-i} B\mathbf{Z}/p) \wedge \mathbf{C}P^\infty \wedge (\wedge^{i-1} B\mathbf{Z}/p),$$

and maps

$$(5.10)\qquad \pi_i = 1 \wedge \cdots \wedge 1 \wedge \pi \wedge 1 \wedge \cdots : \wedge^n B\mathbf{Z}/p \to X_i$$

induced by the Bockstein π (5.2) on the i-th coordinate from the right. We define a BP_*-filtration on BP_*X_i, using 2^{n-1} BP_*-modules in almost the same way as with $BP_* \wedge^n B\mathbf{Z}/p$. However at the stage

$$(5.11) \qquad BP_*(\mathbf{C}P^\infty \wedge^{i-1} B\mathbf{Z}/p) \simeq BP_*\mathbf{C}P^\infty \otimes BP_* \wedge^{i-1} B\mathbf{Z}/p$$

where no Tor_1 term appears, we use zero for the first 2^{i-2} modules and then index the 2^{i-2} modules coming from (5.11) between $2^{i-2}+1$ and 2^{i-1}. The maps $\pi_{i*}: BP_* \wedge^n B\mathbf{Z}/p \to BP_*X_i$ now preserve filtration and we can use 5.7.

Assume the following for $BP_* \wedge^{n-1} B\mathbf{Z}/p$: (i) that Theorem 5.1 holds and (ii) that the spectral sequence (2.22) collapses. Since $BP_*\mathbf{C}P^\infty$ is BP_*-free, the spectral sequences for BP_*X_i corresponding to (2.22) collapse, $1 \le i \le n$. For $BP_* \wedge^n B\mathbf{Z}/p$, $2^{n-2}-1$ of the terms of Tor_0—all but one—are of the form

$$(5.12) \qquad N \otimes \cdots \otimes N \otimes L_k \otimes J_{i-1} \otimes \cdots \otimes J_1$$

with an L_k in the i-th factor and N's to its left. The filtration-preserving homomorphism $(\pi_i)_*: BP_* \wedge^n B\mathbf{Z}/p \to BP_*X_i$ carries (5.12) injectively into the corresponding factor

$$(5.13) \qquad N \otimes \cdots \otimes N \otimes BP_*\mathbf{C}P^\infty \otimes J_{i-1} \otimes \cdots \otimes J_1$$

of BP_*X_i. This is by (5.7) and the proof of Theorem 4.1. This injection of the term (5.12) into the term (5.13) of the collapsed spectral sequence for BP_*X_i means that (5.12) contains no target of a differential.

One term of Tor_0 remains: $N \otimes \cdots \otimes N$. By Corollary 3.3, it injects into $BP_* \wedge^n B\mathbf{Z}/p$ and thus contains no target of a differential. Once again, we need the solution of the Conner-Floyd Conjecture (in its strongest form here) at a critical point in a proof. □

6. The Adams spectral sequence approach. We compute two Adams spectral sequences: for $BP_* \wedge^n B\mathbf{Z}/p$ and for $BP^* \wedge^n B\mathbf{Z}/p$. Both collapse. The second one collapses for the simple reason that its E_2 term is concentrated in even degrees. A surprising argument shows how the second's collapse implies the collapse of the Adams spectral sequence of central interest: that for $BP_* \wedge^n B\mathbf{Z}/p$. Our original proof used the Conner-Floyd Conjecture and an inductive comparison with the spaces X_i as in

Section 5. This present proof has the great advantage of solving the Conner-Floyd Conjecture as a corollary. Thus our paper becomes self-contained and formally does not depend on [RW]. (We first started thinking about this problem a decade ago; we should have been able to write this paper then.)

We use reduced homology and cohomology with coefficients in $\mathbf{F}_p$, the field with p elements (whose additive group is $\mathbf{Z}/p$). All tensor products are over $\mathbf{F}_p$. For any graded module N_*, let N^* be its vector space dual and N^{-*} be the dual with negative grading. Let $M_* = H_*B\mathbf{Z}/p$ and let B_* be the odd-degree part of M_*. Define $L_{s*} \subset M_*$ to be the even-degree part in degrees less than $2p^s$. Let $M_s^* = M^*/L_s^*$ and $C_s^* = M^*/(B^* \oplus L_s^*)$. To summarize, these graded vector spaces have single basis elements in the following dimensions:

M^*	$1, 2, 3, 4, 5, 6, 7, \ldots$
B^*	$1, 3, 5, 7, \ldots$
L_s^*	$2, 4, \ldots, 2p^s - 2$
M_s^*	$1, 3, \ldots, 2p^s - 1, 2p^s, 2p^s + 1, 2p^s + 2, \ldots$
C_s^*	$2p^s, 2p^s + 2, 2p^s + 4, \ldots.$

Note in the following that we compute $BP^{-*} \wedge^n B\mathbf{Z}/p$ and then have to change the grading to get $BP^* \wedge^n B\mathbf{Z}/p$. Both Adams spectral sequences which we compute are modules over $\mathbf{F}_p[v_0, v_1, \ldots] \simeq \mathrm{Ext}_E(\mathbf{F}_p, \mathbf{F}_p)$, $v_n \in \mathrm{Ext}_E^{1,2p^n-1}(\mathbf{F}_p, \mathbf{F}_p)$.

THEOREM 6.1. *The Adams spectral sequence converging to* $\pi_*(BP \wedge^n B\mathbf{Z}/p) \simeq BP_* \wedge^n B\mathbf{Z}/p$ ($\{\wedge^n B\mathbf{Z}/p; BP\}_* \equiv BP^{-*} \wedge^n B\mathbf{Z}/p$) *collapses. There is a filtration on the* E_2*-term with associated graded object given by*

$$\oplus J_1 \otimes \cdots \otimes J_n \otimes \mathbf{F}_p[v_m, v_{m+1}, \ldots].$$

Here J_i *is either* L_{k*} (L_k^{-*}) *or* B_* (C_k^{-*}) *and* k *is the number of* J_j, $j < i$, *which are* B_* (C_s^{-*}, *for various* s) *and* m *is the number of* J_j, $j \leq n$, *which are* B_* (C_s^{-*}, *for various* s).

Remark 6.2. This confirms the Conner-Floyd Conjecture (1.1), because it is now clear that $(B_*)^n \otimes BP_*/I_n$ injects.

Let $V(0)$ be the mod p Moore spectrum. The Adams spectral sequence converging to the mod p Brown-Peterson homology of $\wedge^{n-1}B\mathbf{Z}/p$, $\pi_*(BP \wedge V(0) \wedge^{n-1} B\mathbf{Z}/p) \simeq BP_*(\wedge^{n-1}B\mathbf{Z}/p, \mathbf{Z}/p)$, collapses. The graded object associated to the E_2 term has the same description as that of Theorem 6.1 except that the B_* in the J_1-place is just a one-dimensional vector space in degree zero. The proof proceeds exactly as that of Theorem 6.1. The exact sequence (1.3) of the introduction follows.

Since $H^* \wedge^n B\mathbf{Z}/p \simeq \otimes^n M^*$, the E_2 terms of the two Adams spectral sequences are (by (2.30) and (2.31)):

$$\text{(6.3)} \qquad \mathrm{Ext}_E^{**}(\otimes^n M^*, \mathbf{Z}/p) \Rightarrow BP_* \wedge^n B\mathbf{Z}/p$$

and

$$\mathrm{Ext}_E^{**}(\mathbf{Z}/p, \otimes^n M^*) \Rightarrow BP^{-*} \wedge^n B\mathbf{Z}/p.$$

These Ext groups are contained in this more general computation. Recall that $E_s = E[Q_s, Q_{s+1}, \ldots]$ (2.27).

LEMMA 6.4. *There is a filtration on* $\mathrm{Ext}_{E_s}^{**}(\otimes^n M^*, \mathbf{Z}/p)$ $(\mathrm{Ext}_{E_s}^{**}(\mathbf{Z}/p, \otimes^n M^*))$ *such that the associated graded object is given by*

$$\oplus J_1 \otimes \cdots \otimes J_n \otimes \mathbf{F}_p[v_m, v_{m+1}, \ldots]$$

where J_i *is either* L_{k*} (L_k^{-*}) *or* B_* (C_k^{-*}) *and* $k - s$ *is the number of* J_j, $j < i$, *which are* B_* $(C_u^{-*}$, *for various* u). *And* $m - s$ *is the number of* J_j, $j \leq n$, *which are* B_* $(C_u^{-*}$, *for various* u).

We defer the proof of Lemma 6.4. Before confirming the collapse part of Theorem 6.1, we prove two simple lemmas.

LEMMA 6.5. $\mathrm{Ext}_{BP_*}^{**}(BP_*/I_m, BP^*) = \mathrm{Ext}_{BP_*}^{m,*}(BP_*/I_m, BP^*) \simeq \Sigma^S BP^*/I_m$ where $S = \Sigma_{0 \leq i < m} 2p^i - 2$.

Proof. Induct on m. Apply $\mathrm{Ext}_{BP_*}^{**}(-, BP^*)$ to the short exact sequence

$$0 \longrightarrow \Sigma^{2p^m-2}BP_*/I_m \xrightarrow{v_m} BP_*/I_m \longrightarrow BP_*/I_{m+1} \longrightarrow 0.$$

The resulting long exact sequence degenerates to:

$$0 \longrightarrow \operatorname{Ext}^{m,*}_{BP_*}(BP_*/I_m, BP^*) \xrightarrow{v_m^*} \operatorname{Ext}^{m,*}_{BP_*}(\Sigma^{2p^m-2}BP_*/I_m, BP^*)$$

$$\simeq \qquad\qquad \simeq$$

$$0 \longrightarrow \Sigma^S BP^*/I_m \xrightarrow{v_m} \Sigma^{S+2p^m-2}BP^*/I_m$$

$$\xrightarrow{\delta^*} \operatorname{Ext}^{m+1,*}_{BP_*}(BP_*/I_{m+1}, BP^*) \longrightarrow 0$$

$$\simeq$$

$$\longrightarrow \Sigma^{S+2p^m-2}BP^*/I_{m+1} \longrightarrow 0 \quad \square$$

LEMMA 6.6. *Consider* $\operatorname{Ext}^{**}_{BP_*}(J_1 \otimes \cdots \otimes J_n \otimes \mathbf{F}_p[v_m, \ldots], BP^*)$ *where each J_i is either L_{k*} or B_* and $k - s$ is the number of J_j, $j < i$, which are B_*. And m is the number of J_j, $j \leq n$,* which are B_*. *Then this* Ext *module is isomorphic to* $J_1^* \otimes \cdots \otimes J_n^* \otimes \mathbf{F}_p[v_m, \ldots]$. If *$J_i$ is L_{k*}, J_i^* is L_k^*. If J_i is B_*, J_i^* is C_k^*.*

Proof. Observe that a typical element of C_j^* is of the form $Q_j z_{i_j}$ for $z_{i_j} \in B^*$. Define a correspondence

$$C_0^* \otimes \cdots \otimes C_{m-1}^* \otimes \mathbf{F}_p[v_m, \ldots]$$

$$\to \operatorname{Ext}^{**}_{BP_*}(\otimes^m B_* \otimes \mathbf{F}_p[v_m, \ldots], BP^*)$$

$$\simeq \otimes^m B^* \otimes \operatorname{Ext}^{m,*}_{BP_*}(\mathbf{F}_p[v_m, \ldots], BP^*)$$

$$\simeq \otimes^m B^* \otimes \Sigma^S \mathbf{F}_p[v_m, \ldots]$$

by $Q_0 z_{i_1} \otimes \cdots \otimes Q_{m-1} z_{i_m} \otimes 1 \mapsto z_{i_1} \otimes \cdots \otimes z_{i_m} \otimes \Sigma^S 1 = z_I \otimes \Sigma^S 1$. Note that $z_I \otimes \Sigma^S 1$ has bidegree $(m, |z_I| + S)$ and a total degree $|z_I| + S + m$, the same as its preimage. This gives a vector space isomorphism. The L_k^* and L_{k*} correspond directly; so their insertion preserves this isomorphism. $\square$

We introduce a third spectral sequence (2.35)

(6.7) $$E_2^{s,t} \simeq \operatorname{Ext}^{s,t}_{BP_*}(BP_*X, BP^*) \Rightarrow BP^*X$$

of [JW, 5.17]. Let X stand for $\wedge^n B\mathbf{Z}/p$. Consider our three spectral sequences in a triangle

$$(6.8)\qquad \begin{array}{ccc} \operatorname{Ext}_E(\otimes^n M^*, \mathbf{Z}/p) & & \operatorname{Ext}_E(\mathbf{Z}/p, \otimes^n M^*) \\ \swarrow\!\!\!\swarrow & & \searrow\!\!\!\searrow \\ BP_*X & & BP^{-*}X \end{array}$$

$$\operatorname{Ext}^{**}_{BP_*}(BP_*X, BP^*) \Rightarrow BP^*X.$$

To compute BP^*X from data on $\otimes^n M^*$, we have a direct route: $\operatorname{Ext}_E(\mathbf{Z}/p, \otimes^n M^*) \Rightarrow BP^{-*}X$. Lemma 6.4 (to be proved) tells that $\operatorname{Ext}_E(\mathbf{Z}/p, \otimes^n M^*)$ is concentrated in even dimensions and so this Adams spectral sequence collapses. Thus any Landweber presentation (2.17) of $\operatorname{Ext}_E(\mathbf{Z}/p, \otimes^n M^*)$ serves as one for $BP^{-*}X$. Lemma 6.4 also gives a Landweber presentation for $\operatorname{Ext}_E(\otimes^n M^*, \mathbf{Z}/p)$. Lemma 6.6 shows that if we apply $\operatorname{Ext}^{**}_{BP_*}(-, BP^*)$ to the Landweber presentation for $\operatorname{Ext}_E(\otimes^n M^*, \mathbf{Z}/p)$, we get a suitable Landweber presentation for $\operatorname{Ext}_E(\mathbf{Z}/p, \otimes^n M^{-*})$—which must serve as one for BP^*X. We can only conclude that there are no differentials in the two spectral sequences in the indirect route to BP^*X in (6.8) (as well as the spectral sequence of the Landweber presentation (2.23)). All three spectral sequences in (6.8) collapse. In particular $\operatorname{Ext}_E(H^* \wedge^n B\mathbf{Z}/p, \mathbf{Z}/p) \Rightarrow BP_* \wedge^n B\mathbf{Z}/p)$ collapses. Modulo Lemma 6.4, this concludes the proof of Theorem 6.1. After a pause to reminisce, we shall prove Lemma 6.4. □

The philosophy behind this collapse proof goes back to Remark 5.19 of [JW] where we muse that "it may be possible to play one (of $BP^*(X)$ and $BP_*(X)$) against the other." We failed to pursue this tack and promptly forgot it. Recently, Douglas Ravenel suggested this section's approach to proving collapse. We are grateful to Ravenel and we are grateful to have (6.7) so useful in its first application. It completely clarifies the "perverted duality" discussed in [W_1]. Quite possibly, this technique can be repeated in other applications, but don't hope for too much. Look at the three-cell complex (cells in dimensions 1, 2, 4) with non-trivial Sq^1 and $\mathrm{Sq}^2 : X$. The two Adams spectral sequences converging to BP_*X and to $BP^{-*}X$ collapse, but the duality spectral sequence (6.7) does not.

Proof of Lemma 6.4. The proof is by induction on n. We compute

$$\operatorname{Ext}^{**}_{E_s}(\otimes^{n+1} M^*, \mathbf{Z}/p)$$

assuming the lemma. Observe that

$$0 \to L_s^* \hookrightarrow M^* \to M_s^* \to 0$$

is split as E_s-modules. So

$$\text{(6.9)} \qquad \operatorname{Ext}_{E_s}^{**}(\otimes^{n+1}M^*, \mathbf{Z}/p) \simeq \operatorname{Ext}_{E_s}^{**}(L_s^* \otimes^n M^*, \mathbf{Z}/p) \oplus \operatorname{Ext}_{E_s}^{**}(M_s^* \otimes^n M^*, \mathbf{Z}/p).$$

$\operatorname{Ext}_{E_s}^{**}(L_s^* \otimes^n M^*, \mathbf{Z}/p)$ is just $L_{s*} \otimes \operatorname{Ext}_{E_s}(\otimes^n M^*, \mathbf{Z}/p)$ which is computed by the induction hypothesis.

Dealing with $\operatorname{Ext}_{E_s}^{**}(M_s^* \otimes^n M^*, \mathbf{Z}/p)$ is harder. Filter M_s^* by

$$F_k = \{x : |x| \text{ odd} \geq k,\ |x| \text{ even} \geq k + 2p^s - 1\}.$$

As an E_s-module, the associated graded module $\bar{M}_s^*$ becomes $E[Q_s] \otimes B^*$, where B^* has trivial E_s-structure. Tensor this with $\otimes^n M^*$ to obtain the spectral sequence (2.23)

$$\text{(6.10)} \qquad \operatorname{Ext}_{E_s}(\bar{M}_s^* \otimes^n M^*, \mathbf{Z}/p) \Rightarrow \operatorname{Ext}_{E_s}^{**}(M_s^* \otimes^n M^*, \mathbf{Z}/p).$$

We use the change of rings spectral sequence (2.33) to compute

$$\text{(6.11)} \qquad \operatorname{Ext}_{E_s}^{**}(\bar{M}_s^* \otimes^n M^*, \mathbf{Z}/p) \Leftarrow \operatorname{Ext}_{E_{s+1}}^{**}(\operatorname{Tor}_{**}^{E[Q_s]}(\bar{M}_s^* \otimes^n M^*, \mathbf{Z}/p), \mathbf{Z}/p).$$

$\bar{M}_s^* \otimes^n M^* \simeq (E[Q_s] \otimes B^*) \otimes^n M^*$ is $E[Q_s]$-free; so the spectral sequence (6.11) collapses. The Tor term is just

$$\text{(6.12)} \qquad \mathbf{Z}/p \otimes_{E[Q_s]} (\bar{M}_s{}^* \otimes^n M^*) \simeq E_{s+1} \otimes_{E_s} (\bar{M}_s{}^* \otimes^n M^*).$$

The use of this spectral sequence costs us v_s-module information. It is easy to see that (6.12) is isomorphic to $B^* \otimes^n M^*$ as E_{s+1}-modules (trivial structure on B^*). Thus (6.11) becomes

$$\text{(6.13)} \qquad \operatorname{Ext}_{E_{s+1}}^{**}(B^* \otimes^n M^*, \mathbf{Z}/p) \simeq B_* \otimes \operatorname{Ext}_{E_{s+1}}^{**}(\otimes^n M^*, \mathbf{Z}/p).$$

We know (6.13) by our inductive hypothesis, but this is just the E_1 term of the spectral sequence (6.10). Over BP_*, (6.13) is generated by

$$B_* \otimes \mathrm{Ext}^0_{E_{s+1}}(\otimes^n M^*, \mathbf{Z}/p). \tag{6.14}$$

The differentials are BP_*-module maps and raise cohomological degree by one. Any non-trivial differential must eliminate one element of (6.14); so if the spectral sequence does not collapse, some E_{s+1}-generator of $B^* \otimes^n M^*$ (B^* trivial) is not an E_s-generator of $M_s^* \otimes^n M^*$. But M_s^* is $E[Q_s]$-free with basis B^*; so this is impossible.

To compute

$$\mathrm{Ext}^{**}_{E_s}(\mathbf{Z}/p, \otimes^{n+1} M^*) \tag{6.15}$$

we use the same basic argument. This becomes

$$\mathrm{Ext}^{**}_{E_s}(\mathbf{Z}/p, L_s^* \otimes^n M^*) \oplus \mathrm{Ext}^{**}_{E_s}(\mathbf{Z}/p, M_s^* \otimes^n M^*). \tag{6.16}$$

The first term of (6.16) is $L_s^{-*} \otimes \mathrm{Ext}^{**}_{E_s}(\mathbf{Z}/p, \otimes^n M^*)$. We use (2.24) and (2.34) to get the second term as

$$\begin{aligned} &\mathrm{Ext}^{**}_{E_{s+1}}(\mathbf{Z}/p, \mathrm{Ext}^{**}_{E[Q_s]}(\mathbf{Z}/p, \bar{M}_s^* \otimes^n M^*)) \\ &\quad \simeq \mathrm{Ext}^{**}_{E_{s+1}}(\mathbf{Z}/p, C_s^* \otimes^n M^*) \simeq C_s^{-*} \otimes \mathrm{Ext}^{**}_{E_{s+1}}(\mathbf{Z}/p, \otimes^n M^*). \end{aligned} \tag{6.17}$$

This time the spectral sequence is concentrated in even degrees. □

Appendix. $K(n)_* \underline{K}_*$, $p = 2$. The paper [RW] solving the Conner-Floyd Conjecture is restricted to odd primes. We sketch how most of the results hold as stated for $p = 2$. In particular, Theorem 1.1 of the present paper is true. These results are due to Urs Würgler. We present our interpretation of them as communicated to us by Douglas Ravenel.

In Remarks 5.7 (p. 474) and 7.3 (p. 479) of [Wü], it is observed that there are several products on the mod 2 Morava K-theories, $K(n)_*(-)$, but they are not necessarily commutative. Pick a multiplication m. The obstruction to commutativity, $m - mT$, is computed explicitly [Wü, 4.12] to be in degree zero (but not equal to 1) of the exterior algebra

$$K(n)^*(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \subset K(n)^*(K(n) \wedge K(n)) \simeq K(n)^*K(n) \otimes K(n)^*K(n),$$

where

$$|a_i| = |b_i| = 2^{i+1} - 1 \quad \text{and} \quad |v_n| = -(2^{n+1} - 2).$$

For dimensional reasons, our obstruction must be

$$m - mT = v_n \sum_{I,J} m(a^I \times b^J)$$

with I and J nonzero. Since the a_i and b_i are odd degree, it follows immediately that if X is a commutative H-space with $K(n)_*X$ concentrated in even degrees, then $K(n)_*X$ is a bicommutative Hopf algebra.

We turn now to the computation of $K(n)_*\underline{K}_*$ in [RW]. Computing $K(n)_*\underline{K}_*$ is easy and the above argument shows it is bicommutative. The same argument gives $\circ$-product commutativity on the generators $a_{(i)} \in K(n)_*\underline{K}_1$. Assuming $K(n)_*\underline{K}_i$ is as described in [RW], the computation of $K(n)_*\underline{K}_{i+1}$ is exactly the same, because the bar spectral sequence is commutative. Since $K(n)_*\underline{K}_{i+1}$ is concentrated in even degrees, there are no non-commutativity extension problems and we are done.

Everything else proceeds as in [RW], including Theorem 1.1 of this paper, with the exception of the "global" description of $K(n)_*\underline{K}_*$ as the free Hopf ring on $K(n)_*\underline{K}_1$. The $\circ$-product commutativity for odd primes, $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$, forced $a_{(i)} \circ a_{(i)}$ to be zero, but not for $p = 2$. A simple inductive proof shows that $a_{(i)} \circ a_{(i)}$ is zero in $K(n)_*\underline{K}_2$: if $a_{(i-1)} \circ a_{(i-1)} = 0$, then $a_{(i)} \circ a_{(i)}$ is a primitive in a degree with no primitives. Thus the free Hopf ring on $K(n)_*\underline{K}_1$ is too big and the relations $a_{(i)} \circ a_{(i)} = 0$ must be imposed.

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