

Since  $1_{M(q,4)}$  has order  $q$ , every element in  $[M(q,4), L_k^h(\pi)]$  must have order dividing  $q$ . This proves 2.4 if  $q = \text{odd}$ .

To prove 2.4 if  $q = 2^n$  it suffices to show  $\exists$  a premap  $pf: L_k^h(\pi) \rightarrow Y$ , where  $Y$  is a product of Eilenberg-Mac Lane spaces and  $pf$  induces an isomorphism (mod odd torsion) on the homotopy groups. To get  $pf$  apply the universal coefficient theorem for ordinary homology to the composition of homomorphisms

$$H_*(L_k^h(\pi), \Omega_0) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)} \subset \Omega_*(L_k^h(\pi) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}) \xrightarrow{\sigma} L_{*+k}^h(\pi) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}.$$

Since 2.4 is true for  $q = \text{odd}$  or  $q = 2^n$ , it is true for all  $q$ .

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## KERVAIRE'S INVARIANT FOR FRAMED MANIFOLDS\*

JOHN JONES AND ELMER REES

1. Pontryagin (see [19]) set up a correspondence between homotopy classes of maps  $S^{n+k} \rightarrow S^n$  and bordism classes of framed  $k$ -dimensional manifolds  $M^k$  embedded in  $\mathbb{R}^{n+k}$ . When  $k = 2$  and  $n \geq 2$  he constructed a map  $q: H_1(M^2) \rightarrow \mathbb{Z}/2$  (all our homology groups will have  $\mathbb{Z}/2$  coefficients) which is quadratic with respect to the intersection pairing. Such a quadratic form has an Arf invariant; see [1], [4] or [21].  $\text{Arf}(q)$  depends only on the framed bordism class of  $M$  and defines an isomorphism  $\pi_{n+2} S^n \rightarrow \mathbb{Z}/2$ . This procedure can be used in practice to show that certain maps are essential—for example this is done in [13].

In [10], M. Kervaire defined an Arf invariant for  $(2l-2)$ -connected,  $(4l-2)$ -dimensional closed manifolds which are almost parallelizable and smooth in the complement of a point. The manifolds  $S^1 \times S^1$ ,  $S^3 \times S^3$  and  $S^7 \times S^7$  may be framed in different ways to have Kervaire invariant one or zero. The Kervaire invariant of any framed  $(4l-2)$ -manifold  $M$  was defined by first performing surgery to make  $M$   $(2l-2)$ -connected. In [10], Kervaire showed that his invariant vanished for closed, smooth 10-dimensional manifolds and constructed a manifold which was smooth in the complement of a point and had Kervaire invariant one. This gave his famous example of a nonsmoothable manifold.

W. Browder [3] extended the definition of the Kervaire invariant and also extended Kervaire's result by showing that a framed, smooth  $M^{4l-2}$  has Kervaire invariant zero if  $l$  is not a power of 2. His definition has since been extended and simplified by E. H. Brown [5]. Browder gave necessary and sufficient conditions in terms of the Adams spectral sequence for the existence of a framed  $M^{2n+1-2}$  with Kervaire invariant one. These conditions have been verified for  $n = 4$  [17] and

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\*These notes are based on a talk given by Elmer Rees.

$n = 5$  (see [15] for an account of other attempts). An explicit  $M^{30}$  was constructed in [7] and directly shown to have a framing of Kervaire invariant one; a sketch of its properties is given in §4 of these notes.

2. It is clearly desirable to be able to define the Kervaire invariant of a framed manifold directly, without having to do surgery first. It is also desirable to understand exactly how the invariant depends on the framing. The Browder-Brown approach enables one to define the quadratic form directly on the middle homology of a framed  $M^{2k}$  and to study the precise relationship between quadratic forms and framings.

First, consider quadratic forms  $q$  defined on a mod 2 vector space  $V$  relative to a nonsingular pairing  $\langle \cdot, \cdot \rangle$ , i.e.,  $q(x + y) = q(x) + q(y) + \langle x, y \rangle$ . Note that the existence of such a form implies  $\langle x, x \rangle = 0$  for all  $x \in V$ . If  $q_1$  and  $q_2$  are two such forms then their difference  $q_1 - q_2$  is linear and so by the nonsingularity of the pairing there is an element  $v$  such that  $q_1(x) - q_2(x) = \langle x, v \rangle$ . This shows that if there is one quadratic form on  $V$  then there is a 1-1 correspondence between the set of quadratic forms and  $V$  itself. It is easy to show that  $\text{Arf}(q_1) + \text{Arf}(q_2) = q_1(v) = q_2(v)$ .

Browder and Brown defined the Kervaire invariant of a framed manifold  $M^{2k} \subset R^{2k+N}$  as follows. It is the Arf invariant of a quadratic form defined on  $H^k(M) \cong H_k(M)$ . Let  $\nu$  be the normal bundle of  $M$ ,  $T(\nu)$  its Thom complex, and let  $t$  be the given framing. Then the Pontryagin-Thom construction gives a map  $S^{2k+N} \rightarrow T(\nu)$  and  $t$  gives a homeomorphism  $T(\nu) \rightarrow \Sigma^n(M_+)$  where  $M_+$  denotes  $M$  together with a disjoint base point. Given  $a \in H^k(M) = [M_+, K_k]$ , where  $K_k$  denotes an Eilenberg-Mac Lane space  $K(\mathbb{Z}/2, k)$ , we may form the composite  $S^{2k+N} \rightarrow \Sigma^n M_+ \rightarrow \Sigma^{Na} \Sigma^N K_k$ , which is an element of the group  $\pi_{2k+N}(\Sigma^N K_k)$ . A calculation shows that this group is  $\mathbb{Z}/2$ . So, from our framed manifold  $(M^{2k}, t)$ , we have constructed a function  $q_t: H^k(M) \rightarrow \mathbb{Z}/2$ . One may check that  $q_t$  is quadratic with respect to the intersection pairing, and that its Arf invariant depends only on the framed bordism class of  $(M, t)$ . We will denote this invariant by  $K(M, t)$ ; it equals Kervaire's invariant when that is defined. The following theorem, due to E. H. Brown [5], shows the relationship between this quadratic form and those considered by Pontryagin and Kervaire.

**THEOREM.** *If the Poincaré dual of  $a \in H^k(M)$  is represented by an embedded  $S^k \subset M^{2k}$  then  $q_t(a) = \varepsilon(a) + h(a)$  where*

$$\varepsilon(a) = \begin{cases} 0 & \text{if the normal bundle of } S^k \subset M^{2k} \text{ is trivial,} \\ 1 & \text{otherwise;} \end{cases}$$

$$h(a) = \begin{cases} 0 & \text{if } \varepsilon(a) = 1, \\ \text{the Hopf invariant of the framed embedding} \\ S^k \subset M^{2k} \subset R^{2k+N} & \text{if } \varepsilon(a) = 0. \end{cases}$$

The quadratic form clearly depends at most on the framing restricted to the  $k$ -skeleton of  $M$ ; it is important to understand precisely what it depends on. Browder, in [3], completely analysed this. Browder's work depends on the notion of a Wu-orientation. Note that a framing corresponds to a lift in the diagram

$$\begin{array}{ccc} & O & \\ & \downarrow & \\ M & \nearrow & EO \\ & \downarrow & \\ & BO & \end{array}$$

The different choices of framing correspond to maps from  $M$  to the fibre,  $O$ , of this principal bundle. One wants structures, analogous to framings, which give quadratic forms. Since quadratic forms on  $H^k(M)$  are in 1-1 correspondence with  $H^k(M)$ , one would like the structures to be in 1-1 correspondence with  $H^k(M)$ . Therefore one needs a principal fibration over  $BO$  with fibre  $K_k$ . Such fibrations correspond to  $H^{k+1}(BO)$ . The element chosen to classify this fibration is the Wu-class  $v_{k+1} \in H^{k+1}(BO)$ . The total space of this fibration is denoted by  $BO\langle v_{k+1} \rangle$  and a lifting to  $BO\langle v_{k+1} \rangle$  of the classifying map for the normal bundle of  $M$  is called a Wu-orientation of  $M$ . One good reason for the choice of  $v_{k+1}$  is that it is the only  $(k+1)$ -dimensional characteristic class that is zero on all  $2k$ -manifolds. Hence every  $2k$ -manifold admits a Wu-orientation. A framing gives a Wu-orientation as is easily seen from the commutative diagram

$$\begin{array}{ccc} EO & \longrightarrow & BO\langle v_{k+1} \rangle \\ & \searrow & \nearrow \\ & BO & \end{array}$$

Let  $\bar{\gamma}$  be the bundle over  $BO\langle v_{k+1} \rangle$  obtained by pulling back the universal bundle over  $BO$ . A Wu-orientation gives a bundle map  $w: \nu \rightarrow \bar{\gamma}$ . Given  $a \in H^k(M) = [M_+, K_k]$  we may form the composite  $S^{2k+N} \rightarrow T(\nu) \rightarrow_\Delta M_+ \wedge T(\nu) \rightarrow_{a \wedge T(w)} K_k \wedge T(\bar{\gamma})$  where we have taken  $\nu$  and  $\bar{\gamma}$  to be  $N$ -dimensional, and  $\Delta$  is induced by the diagonal. Hence the Wu-orientation  $w$  gives a function  $Q_w: H^k(M) \rightarrow \pi_{2k+N}(K_k \wedge T(\bar{\gamma}))$ . If the Wu-orientation arises from a framing then  $Q_w$  factors through the map  $i: \pi_{2k+N}(K_k \wedge S^N) \rightarrow \pi_{2k+N}(K_k \wedge T(\bar{\gamma}))$  induced by the inclusion of a fibre of  $\bar{\gamma}$ . E. H. Brown [5] shows that the homomorphism  $i$  is injective (because  $v_{k+1}(\bar{\gamma}) = 0$ ) and that one may choose an epimorphism  $\varepsilon: \pi_{2k+N}(K_k \wedge T(\bar{\gamma})) \rightarrow \mathbb{Z}/4$  so that the composite  $\varepsilon \circ i$  is injective. It is easily shown that the function  $q_w = \varepsilon \circ Q_w$  is quadratic in the sense that  $q_w(x + y) = q_w(x) + q_w(y) + j(x \cdot y)$  where  $j: \mathbb{Z}/2 \rightarrow \mathbb{Z}/4$  is the usual inclusion. The choice of  $\varepsilon$  ensures that if the Wu-orientation came from a framing then this quadratic form takes values in  $\mathbb{Z}/2 \subset \mathbb{Z}/4$  and equals the one previously defined.

The  $\mathbb{Z}/4$  here is needed only to study quadratic phenomena on  $2k$ -manifolds whose  $k$ th Wu-class is nonzero. For in this case there is a class  $x \in H^k(M)$  with  $x \cdot x = 1$ , and as we have already observed the existence of a  $\mathbb{Z}/2$ -valued quadratic form associated to the intersection pairing implies that  $x \cdot x = 0$  for all  $x \in H^k(M)$ . Since our main interest is in framed manifolds we will assume our quadratic forms are  $\mathbb{Z}/2$ -valued.

Suppose now that we have two Wu-orientations, differing by  $v \in H^k(M)$ , giving quadratic forms  $q_1$  and  $q_2$ . Then it can be shown that the quadratic forms also differ by  $v$ , that is  $q_1(x) + q_2(x) = x \cdot v$ . This shows that the relationship between

Wu-orientations and quadratic forms is indeed the simplest possible. So the quadratic form for a framed manifold only depends on the Wu-orientation arising from the framing. We now analyse how a change of framing affects the Wu-orientation, and this then explains the way the quadratic form depends on the framing.

Consider the commutative diagram of principal fibrations

$$\begin{array}{ccc} O & \xrightarrow{x_k} & K_k \\ \downarrow & & \downarrow \\ EO & \longrightarrow & BO\langle v_{k+1} \rangle \\ \downarrow & & \downarrow \\ BO & = & BO \end{array}$$

The fibration  $BO\langle v_{k+1} \rangle \rightarrow BO$  is induced by  $v_{k+1}$  so  $x_k$  is the map  $\Omega v_{k+1}$  obtained by applying the loop functor to  $v_{k+1}: BO \rightarrow K_{k+1}$ . Therefore if two framings of  $M$  differ by the map  $g: M \rightarrow O$  then the induced Wu-orientations differ by  $g^*x_k$ . This leads to the change of framing formula:

**THEOREM.** *Let  $t_1$  and  $t_2$  be two framings of  $M^{2k}$  differing by  $g: M \rightarrow O$ . Then  $q_{t_1}(x) + q_{t_2}(x) = x \cdot g^*x_k$  and  $K(M, t_1) + K(M, t_2) = q_{t_1}(g^*x_k) = q_{t_2}(g^*x_k)$ .*

An easy calculation shows that  $v_{k+1}$  is decomposable unless  $k+1$  is a power of 2. The functor  $\Omega$  annihilates decomposables and so  $x_k = 0$  unless  $k+1$  is a power of 2. Hence one has

**COROLLARY.** *The quadratic form of a framed manifold  $M^{2k}$  is independent of the framing unless  $k+1$  is a power of 2.*

Browder's theorem that the Kervaire invariant of a framed  $M^{4l-2}$  vanishes unless  $l$  is a power of 2 then follows easily from the following consequence of the Kahn-Priddy theorem due to Nigel Ray [20].

**THEOREM.** *If  $\alpha$  is an element of the 2-primary component of  $\pi_{n+N}(S^N)$  for  $N > n+1$ , then there is a manifold  $M^n$  with framings  $t_1$  and  $t_2$  such that  $[M, t_1] = 0$ , and  $[M, t_2] = \alpha \in \pi_{n+N}(S^N)$ .*

The Kervaire invariant of a framed boundary is zero so Browder's theorem follows since we have shown that if  $l$  is not a power of 2 then the quadratic form and so certainly the Kervaire invariant of a framed  $M^{4l-2}$  is independent of the framing.

Conversely one can use this kind of approach to give sufficient conditions for the existence of framed manifolds  $M^{4l-2}$  with Kervaire invariant one when  $l$  is a power of 2. Perhaps the simplest is the following result of [7].

**THEOREM.** *There is a framed manifold  $M^{2k}$ ,  $k = 2^i - 1$ , with Kervaire invariant one if and only if there is an element  $\theta \in \pi_{2k}^s(SO)$  detected by  $Sq^{k+1}$ , i.e., the class  $Sq^{k+1}x_k$  is nonzero in the cofibre of  $\theta$ .*

This theorem is also true with  $SO$  replaced by  $RP^\infty$ . It can be used to yield another proof of Browder's result that there is a framed manifold in dimension  $2^{i+1} - 2$  with Kervaire invariant one if and only if the element  $h_i^2$  in the  $E_2$ -term of the mod 2 Adams spectral sequence survives to  $E_\infty$ . The details are worked out in [7].

3. It is of interest to find criteria under which a manifold will have a framing with Kervaire invariant one. For brevity we will call a manifold "Arf-changeable" if it has framings with different Kervaire invariants. By Ray's result, if there is a framed  $M^{4l-2}$  with Kervaire invariant one, then there is an Arf-changeable manifold in the same dimension. Of course the change of framing formula gives a necessary and sufficient condition for a manifold to be Arf-changeable. There is also the following simple criterion for a manifold to be Arf-changeable. Unfortunately we have only been able to use it constructively in simple cases.

**THEOREM [7].** *Let  $M^{2k}$  be a framed manifold and  $g_1, g_2: M \rightarrow O$  be two maps such that  $g_1^*x_k \cdot g_2^*x_k = 1$ . Then  $M$  is Arf-changeable.*

**PROOF.** Define  $\alpha_i = g_i^*x_k$ ,  $i = 1, 2, 3$ , where  $g_3(x) = g_1(x)g_2(x)$ . Then  $\alpha_3 = \alpha_1 + \alpha_2$  and if  $q$  is a quadratic form coming from some framing then  $q(\alpha_3) = q(\alpha_1) + q(\alpha_2) + 1$ . Therefore it follows by the change of framing formula that one of  $g_1, g_2$  or  $g_3$  changes the Kervaire invariant.

**COROLLARY.** *If  $N_1^k$  and  $N_2^k$  are framed manifolds with  $k = 1, 3$  or  $7$ , then  $N_1 \times N_2$  is Arf-changeable.*

**PROOF.** There are maps  $f_i: N_i \rightarrow SO$  with  $f_i^*x_k \neq 0$ . The maps  $g_1 = f_1 \circ \pi_1$  and  $g_2 = f_2 \circ \pi_2$ , where  $\pi_i: N_1 \times N_2 \rightarrow N_i$  is the projection, have the required property.

We will now use a theorem due to Stong [22] to show that highly connected manifolds are not Arf-changeable.

**THEOREM.** *If  $M^{2k}$ ,  $k = 2^r - 1$ , is a framed manifold and  $s$ -connected where  $\phi(s+1) \geq r+1$ , then  $M$  is not Arf-changeable.*

So for example 8-connected  $M^{30}$ 's and 9-connected  $M^{62}$ 's are not Arf-changeable.

**PROOF.** Stong's theorem says that the map  $q_{s+1}: BO\langle s+1 \rangle \rightarrow BO$  satisfies  $q_{s+1}^*w_{2r} = 0$  if  $\phi(s+1) > r$ . Here  $BO\langle s+1 \rangle$  is the  $(s+1)$ th connected cover of  $BO$ . The theorem follows immediately.

An interesting class of stably parallelizable manifolds are the hypersurfaces, that is compact codimension one submanifolds of Euclidean space. If they are Arf-changeable then one can prove stronger connectivity results. Suppose  $M^{2k} \subset S^{2k+1}$  is a hypersurface. Let  $A, B$  be the closures of the components of the complement of  $M$  in  $S^{2k+1}$ . The Mayer-Vietoris sequences for cohomology and real  $K$ -theory show that  $i_A^* + i_B^*: H^k(A) \oplus H^k(B) \rightarrow H^k(M)$  and  $i_A^* + i_B^*: KO^{-1}(A) \oplus KO^{-1}(B) \rightarrow KO^{-1}(M)$  are isomorphisms, where  $i_A: M \rightarrow A$  and  $i_B: M \rightarrow B$  are the inclusions of the boundaries. So given  $g: M \rightarrow O$  write  $g = g_1 + g_2$  where  $g_1 = i_A^*g_A$  and  $g_2 = i_B^*g_B$  where  $g_A: A \rightarrow O$ ,  $g_B: B \rightarrow O$ . Therefore  $g^*x_k = i_A^*\alpha_A + i_B^*\alpha_B$ , where  $\alpha_A = g_A^*x_k$  and  $\alpha_B = g_B^*x_k$ . The natural framing of  $M^{2k}$  coming from its embedding in  $S^{2k+1}$  extends over both  $A$  and  $B$ . Let  $q$  be the quadratic form coming from this framing; then it follows that  $q$  vanishes on both  $i_A^*H^k(A)$  and  $i_B^*H^k(B)$ . Therefore

$$q(g^*x_k) = q(i_A^*\alpha_A + i_B^*\alpha_B) = i_B^*\alpha_B \cdot i_A^*\alpha_A = g_1^*x_k \cdot g_2^*x_k.$$

It follows that if a hypersurface is Arf-changeable then one may always use the above theorem from [7] to prove it.

PROPOSITION. If  $M^{30}$  is a 7-connected hypersurface then it is not Arf-changeable.

PROOF. Let  $g_1, g_2: M \rightarrow 0$  be maps; then from Stong [22] we know that if  $\alpha_i = g_i^* x_{15}$  then  $\alpha_i = (\text{Sq}^7 + \text{Sq}^4 \text{Sq}^2 \text{Sq}^1) \beta_i$  for some  $\beta_i \in H^8(M)$  and further  $\text{Sq}^2 \beta_i = 0$ . The result is proved by showing  $((\text{Sq}^7 + \text{Sq}^4 \text{Sq}^2 \text{Sq}^1) \beta_1) \cdot ((\text{Sq}^7 + \text{Sq}^4 \text{Sq}^2 \text{Sq}^1) \beta_2) = 0$  using the parallelizability of  $M$  and the Adem relations.

4. In [7] the ideas we are describing are used successfully to show that a certain explicitly constructed  $M^{30}$  is Arf-changeable. We now describe this manifold. Consider  $X_5$  the orientable surface of genus 5; it has a smooth free action of the dihedral group  $D_4$  (the symmetries of a square). The quotient space of this action is the non-orientable surface  $Y_1$  of Euler characteristic  $-1$ , the connected sum of the projective plane and the torus. The action is best described by giving a homomorphism  $\phi: \pi_1 Y_1 \rightarrow D_4$ . The group  $\pi_1 Y_1$  has generators  $A_1, A_2, B$  and one relation:  $A_1 A_2 A_1^{-1} A_2^{-1} = B^2$ . The group  $D_4$  permutes the four vertices of the square and in this way can be regarded as a subgroup of  $S_4$ . The homomorphism  $\phi$  is then given by  $\phi(A_1) = (14)(23)$ ,  $\phi(A_2) = (13)(24)$  and  $\phi(B) = (13)$ . In the following one may assume that  $X_5$  is the  $D_4$  covering of  $Y_1$  associated with  $\phi$ ; the fact that  $X_5$  has genus 5 is not necessary.

We now define  $M^{30}$  to be  $X_5 \times_{D_4} (S^7)^4$  where  $D_4$  acts as a permutation group on  $(S^7)^4$ . It is readily checked that  $M^{30}$  is stably parallelizable. Moreover one can construct a map  $g: M^{30} \rightarrow SO$  as follows:

$$M^{30} \xrightarrow{1 \times \omega^4} X_5 \times_{D_4} (SO(8))^4 \xrightarrow{\phi \times 1} ED_4 \times_{D_4} (SO(8))^4 \xrightarrow{D} SO(32).$$

The first map is induced by a map  $\omega: S^7 \rightarrow SO(8)$  such that  $\omega^* x_7$  is nontrivial,  $\tilde{\phi}: X_5 \rightarrow ED_4$  is the equivariant map that covers  $\phi$  and  $D$  is a finite version of the Dyer-Lashof map for  $SO$  and is described explicitly in [14]. Using the results of [12], it is straightforward to calculate  $\alpha = g^* x_{15}$  once one knows enough about  $H^*(M)$ . It turns out that if  $q$  is any quadratic form coming from a framing then  $q(\alpha) \neq 0$ . The change of framing formula shows that this  $M^{30}$  is Arf-changeable. Although the method outlined here only proves that this  $M^{30}$  is Arf-changeable, it is in fact possible with a little care to identify a framing on  $M$  which has Kervaire invariant one.

In the calculation of  $q(\alpha)$  one can use the following lemma which may be of independent interest.

LEMMA [7]. If  $q$  comes from a framing on  $M^{4l-2}$  then  $q(\text{Sq}^1 a) = a \cdot \text{Sq}^2 a$  for any  $a \in H^{4l-2}(M)$ .

PROOF. One considers the map  $\text{Sq}^1_*: \pi_{4l-2}^*(K_{2l-2}) \rightarrow \pi_{4l-2}^*(K_{2l-1})$  and checks that this is an epimorphism. Moreover an element  $\theta$  that maps nontrivially is such that  $\theta^*(c_{2l-2} \cdot \text{Sq}^2 c_{2l-2}) \neq 0$ .

The first-named author has recently proved several further results of this kind.

COROLLARY. If  $M^{4l-2}$  is stably parallelizable and  $a \in H^{2l-2}(M)$  is such that  $\text{Sq}^1 a = 0$  then  $a \cdot \text{Sq}^2 a = 0$ .

This corollary for  $l$  odd can be deduced from Theorem 1.2 of [16] and for  $l = 3$  is given in [18]. It is of interest to note that this corollary is false in general for mani-

folds all of whose Stiefel-Whitney classes vanish; the manifolds  $CP^n$  for  $n+1$  a power of 2 are examples. It is also false for manifolds  $M^{4l}$  as the example  $SU(3)$  shows. It might be interesting to construct examples in other dimensions.

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