An Algebraic Approach to Results Obtained using Topological Modular Forms

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Abstract

Mark Behrens [B] used the theory of topological modular forms, to derive the existence of a family of modular forms for level 1 having coefficients in $\mathbb{Z}$ which satisfied interesting congruence properties mod powers of p. In this paper we limit ourselves to working mod p and use the algebraic theory of modular forms mod p to duplicate and generalize Behrens' results while viewing them in a wider context, which we believe is probably the "correct" way to look at them.

It is noteworthy that the results obtained by our algebraic methods, although inspired by the topological theorems, go farther than the topological ones did in multiple respects, and one hope of the authors is that this may perhaps in turn inspire some broader topological insights *(at least it seems worth asking such questions)*, although since neither of the authors understands the underlying topological theory, we will have to leave it to others to decide how much, if at all, this is worthy of further pursuit.

Notation. *In this version of the paper, if we refer to a Theorem, Lemma, Definition, etc. in a specific section by a number, it denotes the Theorem, Lemma, Definition, etc. by that number in that specific Section. By contrast, if the notation Theorem 3.2, for example, appears in a different Section, it denotes Theorem 2 of Section 3.*

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Section 0: Introduction

Starting with an arbitrary prime $p\geq 5$, Mark Behrens ([B]) used the theory of topological modular forms, to derive the existence of a family of modular forms for level 1 having coefficients in $\mathbb{Z}$ which satisfied interesting congruence properties mod powers of p. In this paper, for simplicity of exposition and in order to best present a cohesive set of optimal results, we limit ourselves to working mod p, although existence results for higher powers of p were found by the second author and are not that much harder to obtain.

Behrens had apparently put some effort and thought into trying to re-derive the existence of such a family of forms from the point of view of algebraic number theory, but was unsuccessful in the end in doing so. The goal of this paper is to do just that, i.e. to use only the algebraic theory of modular forms mod p to prove the existence of Behrens' forms in general, as well as their uniqueness *(up to scalar multiplication)* in the "large" and "small" cases *(among other broad sets of conditions)*, and in the process, to put his results in a much wider context, which we believe is probably the correct way to look at them *(at least from a number theoretic point of view)*. Although the arguments presented here rely on abstract theory, direct computations yielding explicit formulas for forms satisfying these congruences have also been completed by the second author *(with some guidance and suggestions from the first)*.

It is noteworthy that the number theoretic results, although inspired by the topological theorems, go farther than the topological ones did in multiple respects, and one hope of the authors is that perhaps this may in turn inspire some broader topological insights *(at least it seems worth asking such questions)*, although since neither of the authors understands the underlying topological theory, we will have to leave it to others to decide how much, if at all, this is worthy of further pursuit.

For example, Behrens only obtains mod p results in cases of level 1 filtrations[1] that are divisible by p^2-1, whereas we relax this restriction in the current paper *(forcing us to generalize the properties satisfied by the resulting family of forms ever so slightly)*, and it seems at least worth asking if these more general results are also open to topological interpretations, and/or if they yield any new topological insights.

Note in particular that for each congruence class a mod $p-1$, we obtain complete families of modular forms of level one filtrations congruent to $a(p+1)$ mod p^2-1 that satisfy point by point analogous properties to all of the ones that Behrens' family of forms satisfied in the special case where $a=0$. It seems like there should be some sort of a topological explanation for the existence of these additional families of modular forms analogous to the explanation furnished by Behrens for the existence of his family of forms *(even if it is as simple as the additional families being some sort of "topological twists" of Behrens' family of forms)* and it might be worth at least trying to flesh out what this is.

1. For purposes of this paper, the level N filtration of a modular form f defined over $\mathbb{Z}$ equals the lowest weight k with the property that the q-expansion of f is congruent mod p to a modular form for $\Gamma_0(N)$.

On a slightly different note, Behrens has a result *(Theorem 1.4 of [B])* which in the mod p case asserts that any form f satisfying the "main assertion" of his theorem for a prime $\ell = \ell_0$ that is a generator for $\mathbb{Z}_p^*$ *(as well as satisfying the three other properties that all his forms satisfy)*, must satisfy the analogous "main assertion" for all primes $\ell \neq p$. We are able to strengthen this under any one of a variety of sets of general conditions by allowing ℓ_0 to be an arbitrary prime *(without any restriction other than it not equal p)*[2] as long as the filtration k is divisible by p^2-1 *(which under Behrens' hypothesis that k be divisible by $p-1$ combined with our result that k must be divisible by $p+1$ is equivalent to k not being congruent to $\frac{p^2-1}{2}$ mod p^2-1)*. We are also able to remove one of Behrens' other hypotheses from the statement of this result.[3]

Moreover, if we tweak Behrens' "main assertion" to get a more general formulation made explicit below, we are able to extend these strengthened results to *arbitrary filtrations* k. And even without tweaking his "main assertion", one can still get related results for arbitrary k that fit in a broader context than that considered by Behrens *(an overview of all our results that generalize and/or improve on his is contained in Theorem 1 below)*.

Again it seems worth asking whether these findings and/or this broader context have topological interpretations or implications of any significance. Behrens proves the result we are talking about here by exhibiting an embedding of a cohomological space associated with ℓ_0 into the analogous space associated to an arbitrary $\ell \neq p$ in cases where ℓ_0 is a topological generator for $\mathbb{Z}_p^\times$, and it is possible that the results of this paper might imply that such embeddings could in fact yield isomorphisms for the parts of those spaces corresponding to "large" and "small" drops in filtration mod p, with an image of index at most 2 in general, in those cases corresponding to filtrations divisible by p^2-1 *(which if the authors' intuition based on the number theory is correct, might be the only filtrations for which the mod p spaces corresponding to topological generators for $\mathbb{Z}_p^\times$ are nontrivial)*.

Note that Behrens' main result *(Theorem 1.2 of [B])* includes the assertion that his modular forms f have q-expansions at infinity that satisfy certain bounds for the orders of their zeroes, but it is trivial to show that if there exists a form f that satisfies the other assertions of his theorem, one can always modify f if necessary, so that the bound for the order of its zeroes is also satisfied *(for more details, see explanation at the beginning of Section 3)*. Thus, this condition can be ignored from the point of view of the existence of such forms *(although it does allow one to strengthen the uniqueness results)*.

The remainder of Theorem 1.2 of [B] asserts that Behrens' forms f of level 1 and filtrations $k = i(p^2-1)$ satisfy the property that the level ℓ filtrations of $f - f|V_\ell$ fall by at least $j(p-1)$ for arbitrary $\ell \neq p$ when compared to the filtrations k of the original level 1 forms f, where the values of j are made explicit and depend on $r = \mathrm{ord}_p(k) = \mathrm{ord}_p(i)$.

2. We can do this for example in cases of generalized eigenforms or in cases of "large" or "small" falls in filtrations, and even when we can not get the full result, we can still replace Behrens' requirement that ℓ_0 be a generator of $\mathbb{Z}_p^*$ by the significantly weaker condition that ℓ_0 be a quadratic nonresidue with respect to p.

3. Behrens' result includes a condition on the order of the zeroes of f at ∞ which we do not need. It is unclear to the authors how critical this condition was to Behrens' proof of this particular result.

Thus, his results should probably be properly viewed as answering a special case of a more general question asking when and by how much the level ℓ filtration of a linear combination of f and $f\,|\,V_\ell$ can fall down as compared to the original filtration of the level 1 modular form f. We show this phenomenon can happen for some forms of filtration k if and only if k is divisible by $p+1$ *(since Behrens restricted to k divisible by $p-1$ and had the specific linear combination $f-f|V_\ell$ in mind, he ended up with results for k divisible by (p^2-1))*, and we also answer the question of how much the level ℓ filtration can fall - obtaining an answer that is surprising by itself but consistent with properties satisfied by Behrens' forms - as well as the question of how uniform these drops in filtration are for different primes ℓ, again obtaining an answer that is somewhat surprising.

On an intuitive level, it would seem highly unusual that a linear combination of f and $f|V_\ell$ could have a filtration for $\Gamma_0(\ell)$ lower than the filtration of f for $SL_2(\mathbb{Z})$, and in fact this phenomenon seems like one that should never happen unless it is "forced to". The primary example that springs readily to mind involves G_{p+1}, which is congruent to G_2 mod p. Although G_2 is not a modular form of any level, $G_2-\ell G_2\,|\,V_\ell$ is a form of level ℓ and weight 2, which means that even though G_{p+1} has filtration $p+1$ for $SL_2(\mathbb{Z})$ *(also for level ℓ)*, $G_{p+1}-\ell G_{p+1}\,|\,V_\ell$ has filtration 2 for $\Gamma_0(\ell)$.

We show in this paper that the above example is in "some sense" the only one, by which we mean that the only time a linear combination of f and $f|V_\ell$ can have lower filtration for $\Gamma_0(\ell)$ than the filtration of f for $SL_2(\mathbb{Z})$ is when f is in a natural sense "derived" from G_{p+1} in a precisely describable manner made explicit in Section 1 *(in fact we show that all such forms can be "identified" with scalar multiples of G_{p+1} under the Hecke module isomorphisms between "twists" of the quotient spaces $\mathcal{W}_k$ of modular forms mod p first discovered by Tate and Serre and described in the first author's thesis [J1] and in [J3])*. In the case where f is a Hecke eigenform, this follows from the fact that the minimal weight associated to a mod $\mathcal{P}$ system of eigenvalues not related to an Eisenstein series is independent of the level, as originally conjectured by Serre, but we answer the question for arbitrary forms that are not necessarily eigenforms or even generalized eigenforms mod $\mathcal{P}$.

As mentioned above, we prove here that such forms exist if and only if the level 1 filtration k is divisible by $p+1$, and we obtain general results for all such k that are analogous to those obtained by Behrens in the special case where k is divisible by p^2-1. However, what makes filtrations divisible by p^2-1 special is that in those cases, the linear combination can be taken to be $f-f\,|\,V_\ell$, whereas in the case of general $k\equiv a(p+1)$ mod p^2-1, the linear combination must instead be $f-\ell^a f\,|\,V_\ell$ *(which will only equal $f-f\,|\,V_\ell$ for those primes ℓ whose multiplicative orders mod p divide a)*. Note in particular, when $k\equiv\frac{p^2-1}{2}$ mod p^2-1 which still meets Behrens' restriction that k be divisible by $p-1$, these linear combinations are $f-f|V_\ell$ when ℓ is a quadratic residue with respect to p, but $f+f|V_\ell$ for quadratic non-residues ℓ.[4]

Moreover, we are able to show that such a drop in filtration happens for a given form f of such a filtration k for one prime $\ell_0\neq p$ if and only if it happens for all primes $\ell\neq p$, and that in the case where f is a generalized eigenvector for the Hecke operator *(among other alternative general sets of conditions)* the amount of the drop is the same for all ℓ. This

4. This might be the case most worth next considering if one is thinking about the possibility of generalizing Behrens' results. It is also worth pointing out that forms of such filtrations can interact with forms of filtrations divisible by p^2-1 *(because the two filtrations are congruent mod $p-1$, see Section 6 for a more detailed explanation of how such interactions can affect the results we are interested in here)*.

could potentially be a significant result, possibly giving evidence for the existence of some sort of a "super map" allowing one to raise the level of the mod $\mathcal{P}$ Hecke algebra from level 1 to levels ℓ for all primes $\ell \neq p$ at once in some meaningful way. Indeed it implies, that there might be some way of lifting each generalized eigenspace up as a single cohesive unit and "plunking it down" with one single motion into some kind of space comprised of all the different higher levels in such a way that the resulting changes in filtration are uniform in all levels *(which on the surface seems a bit counterintuitive since the V_ℓ operators entering into our maps are different operators living in different universes for different choices of ℓ).*

As for how far the filtration can fall for a given f, we show that for forms f of filtrations k as above, the maximal drop in filtration for even just one prime ℓ_0 is $j(p-1)$, where $j = \begin{cases} p^r + p^{r-1} - 1 & \text{if } r \geq 1 \\ 1 & \text{if } r = 0 \end{cases}$ for $r = \mathrm{ord}_p(k)$, which is the maximal drop guaranteed by Behrens' results in the case where $a = 0$ *(he finds a family of forms that drop in filtration by precisely that amount for all $\ell \neq p$)*, and if the filtration of $f - \ell_0^a f \mid V_{\ell_0}$ drops by this maximal amount or by other amounts that are "large" or "small" for one $\ell_0 \neq p$ *(whether or not f is a generalized eigenvector for the Hecke operators)*, it must also drop by the ***same amount*** for all $\ell \neq p$. Moreover, in the process of proving this, we prove that if one includes Behrens' condition on the order of the zeroes at infinity, his forms are unique up to scalar multiplication in the maximal case *(with the analogous uniqueness statement also true when "maximal" drops are replaced by ones that are "small" or by ones that are "large" but not necessarily maximal).*

Note that it is easy to show forms that drop in filtration must exist in the cases where $r = 0$ *(i.e. where $p \nmid k$)*, and not too difficult to show that the drop in filtration in such cases is precisely $p-1$, and it is also easy to see that the existence of forms f for $r = 0$ with drops in filtration of $p-1$ implies the existence of forms f^{p^r} for general r with drops in filtration of $p^r(p-1)$.

Since $j = p^r$ when $r \leq 1$, this explains the existence of forms for which the drop in filtration is $j(p-1)$ in these cases, but the mystery that stymied the authors for quite a long time was finding a way to understand why this should imply the existence of forms for which there were larger drops in filtration in cases where $r \geq 2$ *(i.e. in cases where j was strictly bigger than p^r).* We had partial results containing most of the main ideas several years back showing that for each prime $\ell \neq p$, we could find a form f that satisfied the desired drop in filtration when one passed to level ℓ, but it took us some time to fill in the gaps allowing us to conclude that the same form f worked for all primes ℓ.

The task of unraveling this mystery was made even more difficult by the stubborn insistence of the first author on only accepting a theoretical argument that she deemed to be essentially non-computational, despite the fact that the second author was able to compute explicit formulas for modular forms that satisfied these extra drops in filtration in cases where $r \geq 2$. We are happy to finally be able to announce that we have solved this problem in such a way that meets the first author's standards for a "non-computational" proof and we present such an existence argument, among other results, in this paper.

In Section 1, we flesh out our "motivating philosophy", namely that the only time the filtration of a linear combination of f and $f|V_\ell$ falls down even by only a minimal amount for all primes $\ell \neq p$ should be when f can be "identified" with a "twist" of G_{p+1} in some naturally explicit way that is in some sense "consistent" with the action of the Hecke operators, and prove a number of related results, *most dealing with minimal falls*, including the one that guarantees the uniqueness of Behrens' forms in the "minimal" case. In Section 2, we use Serre's now proven modularity conjecture to extend our results for minimal falls by loosening the restriction that the filtration fall for all $\ell \neq p$ to only requiring it fall for a single *(arbitrary)* $\ell_0 \neq p$, proving in particular that if the filtration falls at least minimally for one ℓ_0, then it must fall for all ℓ, and in the process showing that drops in filtration can only occur in cases where $k \equiv a(p+1) \bmod p^2-1$ in which case, the linear combinations in question must *(up to scalar multiples)* be of the form $f - \ell^a f|V_\ell$.[5] We prove our general existence results in Section 3, and in the process show that *(at least if we temporarily ignore Behrens' condition on the orders of the zeroes at infinity)* we can choose our forms to be ***generalized eigenforms*** for the Hecke operators for which the drops in filtration equal $j(p-1)$ where j is as defined above, which are precisely the maximal drops in filtration found by Behrens in the special case where $a=0$.[6] In Section 4, we use the existence of that one family of generalized eigenforms found in Section 3 where the drops in filtration for all primes ℓ were ***precisely*** the maximal ones found by Behrens in the special case where $a=0$, to show that the drop for even one prime $\ell_0 \neq p$ and ***any*** form f can never be more that this amount. We extend our arguments in Section 5 to prove that any two forms of filtration k for which the filtration drops by at least $j'(p-1)$ for one prime ℓ_0 in cases of "large" or "small" j' must differ *(up to scalar multiplication)* by a form of filtration $\leq k - j'(p-1)$, and combine that with Behrens' condition on the order of the zeroes at ∞ to deduce the uniqueness of his forms in cases of "large" and "small" j'. In the same Section, we also upgrade the results obtained from Serre's conjecture in Section 2 *(asserting if the filtration falls for one prime $\ell_0 \neq p$, it falls for all $\ell \neq p$ - without requiring any additional restriction on ℓ_0)* to results that take into account the size of the falls, thus improving on Theorem 1.4 of [B] in the case of "large" and "small" falls in the way that we described above, show that if the drop in filtration is "sufficiently large" the size of the fall must be independent of ℓ, and talk about the implications of these results in cases of arbitrary filtrations not necessarily divisible by p^2-1 if one is only interested in the linear combination $f - f|V_\ell$ rather than the more general $f - \ell^a f|V_\ell$. Finally, in Section 6, we address the question of uniformity in the drops in filtration for varying ℓ's head on, showing that this drop will be uniform for all ℓ "under most circumstances" *(including the generalized eigenform case that we mentioned above)*, provide a simple counterexample showing this is not always so, give precise descriptions of how and why things can "go wrong" due to the interactions of 2 generalized eigenvectors for systems of eigenvalues that are quadratic twists of each other, and improve on Theorem 1.4 of [B] in the cases not covered by our previous results.

Since this project was based on and motivated by trying to reproduce results of Behrens, it seems worth incorporating those of our results that generalize, improve on, and/or help to explain his into one single statement, which we do in the following Theorem, whose proof follows easily by collecting the results in this paper. Note that Behrens' main result *(Theorem 1.2 of [B])* in the mod p context is implied by the special case $a=0$ of the very beginning part of this Theorem.

Theorem 1. *Let $p \geq 5$ and $k \equiv a(p+1)$ mod p^2-1 be any positive integer divisible by $p+1$. Let* $j = \begin{cases} p^r + p^{r-1} - 1 & \text{if } r \geq 1 \\ 1 & \text{if } r = 0 \end{cases}$ *for $r = ord_p(k)$.*

5. If we were only considering forms where the filtrations were known to begin with to drop for *all* primes ℓ, we could have proven these last two facts back in Section 1 without waiting until Section 2 to appeal to Serre's conjecture *(the contribution of his conjecture being that it allows us to prove that it is enough for the filtration to drop for a single prime ℓ_0)*.

6. Behrens finds a family of forms in the $a=0$ case where the drop in filtration is at least that much, and his methods imply that no form can drop by more than that amount for all $\ell \neq p$ in this case.

Then for each $1 \leq j' \leq j$, there exists a mod $\mathcal{P}$ form $f_{k,j'}$ for the full modular group of filtration k with coefficients in $\mathbb{F}_p$ (hence the reduction mod p of a form with coefficients in $\mathbb{Z}$) satisfying the following properties.

i. $w(f_{k,j'}) = k$ *where w denotes the mod p filtration of level 1*[7]

ii. $ord_{f_{k,j'}}(q) > \frac{k - j'(p-1)}{12}$ *or* $ord_{f_{k,j'}}(q) = \frac{k - j'(p-1) - 2}{12}$ *where $ord_{f_{k,j'}}(q)$ denotes the order of the zero of $f_{k,j'}$ at infinity.*

iii. $w_\ell(f_{k,j'} - \ell^a f_{k,j'} \,|\, V_\ell) \leq k - j'(p-1)$ *for all primes $\ell \neq p$, where w_ℓ denotes the level ℓ mod p filtration as defined above.*

Moreover, the forms $f_{k,j'}$ are unique (up to scalar multiplication) in cases where j' is "large" ($j' \geq j - \frac{p-5}{2}$) or "small" ($j' \leq \frac{p+1}{2}$), although not unique for all choices of j'.[8]

Furthermore, any level 1 form f of filtration $k \equiv a(p+1)$ mod $p^2 - 1$ satisfying condition iii for ***any*** *prime $\ell_0 \neq p$ in cases where j' is "large" or "small" as defined above must also satisfy the analogous condition for* ***all*** *$\ell \neq p$, with the same assertion being true for arbitrary j' when f is a generalized eigenvector for the Hecke operators $T_{\ell,\ell \neq p}$, and this is still true in all of these cases if we replace the "$\leq k - j'(p-1)$" by "$= k - j'(p-1)$" except that one has to "scale back" the definition of "small" to exclude $j' = \frac{p+1}{2}$. In other words, under any of these circumstances (with the scaled back definition of "small"), the drop in filtration must be uniform in ℓ.*[9]

Note that if the filtration k of a level 1 form f is not of the form $a(p+1)$, no linear combination of f and $f|V_\ell$ can ever have lower filtration than f for even one prime $\ell \neq p$, whereas if the filtration k of f satisfies $k = a(p+1)$, then the only linear combination of f and $f|V_{\ell_0}$ that can have lower filtration than f for even one $\ell_0 \neq p$ is (up to scalar multiplication) the linear combination $f - \ell_0^a f \,|\, V_{\ell_0}$, and when this occurs, the filtrations of $f - \ell^a f \,|\, V_\ell$ must drop for all ℓ, plus ***the maximal drop in filtration for even one $\ell \neq p$ is $j(p-1)$ for j as above****, so that in particular, the weak inequality can be replaced by equality for any form f satisfying iii above in the case $j' = j$.*

Moreover, for any $\ell_0 \neq p$, a drop in filtration of some linear combination of f and $f|V_{\ell_0}$ will occur if and only if f is an eigenvector for all Hecke operators $T_{\ell,\ell \neq p}$ with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ modulo forms of lower filtration, or equivalently if and only if the image of f in the quotient space $\mathcal{W}_k =_{def.} \widetilde{M}_k / \widetilde{M_{k-(p-1)}}$ can be identified (up to scalar multiple) with that of $\theta^{a-1}(G_{p+1})$ in $\mathcal{W}_{a(p+1)}$ under the map (first introduced by Tate and Serre ([J1] ad [J3])) identifying $\mathcal{W}_k$ with $\mathcal{W}_{a(p+1)}$ as Hecke modules. In particular, if f is itself a generalized eigenvector for all $T_{\ell,\ell \neq p}$, the filtration of some linear combination of f and $f|V_{\ell_0}$ will fall if and only if f corresponds to the above system of eigenvalues for some choice of a.

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7. This a fortiori implies that $f_{k,j}$ is not congruent to 0 mod p, which was one of Behrens' conclusions.

8. See the Remark following the Example in Section 6 that shows that the $f_{k,j'}$ are not always unique.

9. This is no longer always true in the general case, although it is still true if one limits oneself to primes ℓ in the same quadratic residue category with respect to p, so that for any given f, there are at most two different amounts by which the filtration can fall for different ℓ, where the fall in the case of quadratic nonresidues is always strictly less than or equal to that in the quadratic residue case, *(which implies in particular that if the filtration of $f - \ell_0^a f|V_{\ell_0}$ falls by at least $j'(p-1)$ for some $j' \geq 1$ and one quadratic* **nonresidue** *ℓ_0, then the filtration of $f - \ell^a f|V_\ell$ must fall by* **at least** *$j'(p-1)$ for all ℓ)*. Moreover, the case where the size of the fall varies depending on the quadratic residue properties of ℓ can only occur under explicitly describable circumstances caused by the interactions of 2 generalized eigenforms for systems of eigenvalues that are quadratic twists of each other *(See Section 6 for details)*.

Proof. The existence of the $f_{k,j'}$ for $j'=j$ follows from Theorem 3.4, after one realizes that condition *ii* can easily be finessed as explained at the start of Section 3, and this existence for $j'=j$ is immediately seen to imply the existence for all cases of $j'\leq j$. The uniqueness *(up to scalar multiplication)* when one takes into account the order of the zeroes at infinity follows from Corollary 1.4 and the Remark following its statement in the case of $j'=1$ and from Proposition 5.1 in the case of "large" and "small" j'.

The fact that any form f of filtration k satisfying condition *iii* for ***any*** prime $\ell_0\neq p$ must also satisfy the analogous condition for ***all*** $\ell\neq p$ is the main thrust of Theorem 2.1 in the case $j'=1$ and of Corollary 5.2 in the case of large" or "small" j', and follows from Proposition 6.3 when f is a generalized eigenform, with the subsequent assertion that the drop in filtration must be completely independent of ℓ under all of these circumstances *(after one scales back" the definition of "small" as in the statement of this Theorem)* following from the Remark after Corollary 6.2 *(whose proof is implicit in Theorem 6.1).*

The fact that if k is not of the form $a(p+1)$, no linear combination of f and $f|V_\ell$ can ever have lower filtration than f for any prime $\ell\neq p$, is the main assertion of Proposition 2.5, whereas the facts that when $k=a(p+1)$, the only linear combinations of f and $f|V_\ell$ that can have lower filtration than f for any prime $\ell\neq p$ are scalar multiples of the linear combination $f-\ell^a f\,|\,V_\ell$, and that in such a case, the maximal drop in filtration is $j(p-1)$ are contained in Theorem 2.1 and Proposition 4.2 respectively.

Finally, the last paragraph of this Theorem follows easily by combining Theorems 1.1 and 2.1 with the Remark following Theorem 1.1. ☐

Since Behrens' results concern linear combinations of the form $f-f|V_\ell$, it might be worth briefly considering the implications of the above to linear combinations of this specific form, which we do in the following Corollary in the case of "large" and "small" falls, deferring the cases of more general j' to the Remark following the proof of this Corollary.

Corollary 2. *Let f have filtration k. Then for any $\ell_0\neq p$, the filtration of $f-f|V_{\ell_0}$ can possibly fall only if $k\equiv a(p+1) \bmod p^2-1$ for some a divisible by the multiplicative order of ℓ_0 in $(\mathbb{Z}/p\mathbb{Z})^*$, and in such a case, if the filtration of $f-f|V_{\ell_0}$ falls by at least $j'(p-1)$ for some j' that is "large" or "small" as defined in the above Theorem, then the filtration of $f-f|V_\ell$ will fall by at least $j'(p-1)$ for all primes $\ell\neq p$ of orders dividing a, whereas it will not fall at all for primes ℓ not satisfying this property (although the filtration of some linear combination of f and $f|V_\ell$ will fall by at least $j'(p-1)$ for all $\ell\neq p$).*

It follows that when k is divisible by p^2-1, if the filtration of $f-f|V_{\ell_0}$ falls by at least $j'(p-1)$ for one $\ell_0\neq p$ (without any restrictions on ℓ_0), then the filtration of $f-f|V_\ell$ will fall by at least $j'(p-1)$ for all primes $\ell\neq p$. It also follows that if ℓ_0 is a generator for $\mathbb{Z}_p^*$ then the filtration of $f-f|V_{\ell_0}$ can fall for some form f of filtration k only if k is divisible by p^2-1 (although the filtration of some linear combination of f and $f|V_{\ell_0}$ will fall for some forms f of filtration k for all k divisible by $p+1$).

Moreover, the entire Corollary remains true if one replaces all occurrences of the words "at least" by the word "precisely" for all such j' except for $j'=\frac{p+1}{2}$.

Proof. This follows immediately from the fourth paragraph of the above Theorem *(see also Corollary 5.4, which basically just restates these facts, as well as the comments preceding that Corollary).* ☐

Remark. One can also obtain results along similar lines for j' that do not satisfy the above restrictions under suitable conditions implicit in Section 6 *(from which it can be deduced, among other things, that the drop in filtration is uniform for all ℓ under any one of a variety of sets of general conditions, including that of being a generalized eigenform).*

Section 1: Motivating Philosophy and Connection with Hecke Operators

In this Section, we include some basic results that encapsulate much of the "motivating philosophy" underlying this project alluded to in the Introduction, namely that the only time the filtration of a linear combination of g and $g|V_\ell$ can fall down even by only a minimal amount for even one *(hence by the results of the next section for all)* $\ell \neq p$ should be when g is "derived" from G_{p+1} in some natural way that respects the action of the Hecke operators.

More precisely, the fall in filtration associated with G_{p+1} that was pointed out in the Introduction is easily seen to imply an "analogous" fall in filtration for all $g = \theta^{a-1}(G_{p+1})$ $(1 \leq a \leq p)$, which in turn can be shown *(see Lemma 2 below)* to imply a fall in filtration for any g which can be "identified" with one of the $\theta^{a-1}(G_{p+1})$ under the Hecke module isomorphism *(originally discovered by Tate and Serre and written up in [J1] and in [J3])* between the quotient spaces $\mathcal{W}_k =_{def} \widetilde{M}_k \,/\, \widetilde{M_{k-(p-1)}}$ of higher weights and the analogous quotient space $\mathcal{W}_{a(p+1)}$ for $1 \leq a \leq p$ containing *(the image of)* $\theta^{a-1}(G_{p+1})$.

The underlying idea motivating this project was our belief that the converse of this statement should also be true, i.e. the only time that the filtration of a linear combination of g and $g|V_\ell$ could fall down for even one ℓ should be when g can be "identified" *(up to scalar multiple)* with one of the $\theta^{a-1}(G_{p+1})$ under one of the above mentioned Tate-Serre isomorphisms. Since $\theta^{a-1}(G_{p+1})$ is an eigenvector for the Hecke operators with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ and since the generalized eigenspaces for these system of eigenvalues in $\mathcal{W}_{a(p+1)}$ are easily seen to all be one dimensional,[1] this is equivalent to the statement that the only time the filtration of a linear combination of g and $g|V_\ell$ could fall down for even one ℓ should be when the image of g in $\mathcal{W}_k$ is an eigenvector for the Hecke operators with this system of eigenvalues, or equivalently when g is an eigenvector for this system of eigenvalues modulo forms of lower filtration *(i.e. modulo forms of filtration* $\leq k-(p-1)$ *).*

Since we will see in Section 2 that if the filtration of some linear combination of g and $g|V_\ell$ falls down for even one prime $\ell \neq p$, the filtration of g must be congruent to $a(p+1)$ mod p^2-1 for some a, in which case that linear combination must equal *(a scalar multiple of)* $g - \ell^a g \,|\, V_\ell$ and the filtration must fall at least minimally for all $\ell \neq p$, these results *(in both directions)* are the main thrust of the following Theorem, which summarizes the main results of the current Section.

Theorem 1. *Let g be a form of level 1 and filtration $k \equiv a(p+1)$ mod p^2-1, with $1 \leq a \leq p-1$ Then the following are equivalent.*

i. The filtration of $g - \ell^a g|V_\ell$ falls at least minimally for all primes $\ell \neq p$

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1. Since it follows from the Tate-Serre methods alluded to above that $\mathcal{W}_{a(p+1)}$ is isomorphic as a Hecke module to the space $\mathcal{W}_{p+1}$ "twisted" $a-1$ times, it suffices to argue that the corresponding generalized eigenspace in $\mathcal{W}_{p+1}$ is one dimensional, but that is immediate since if this were not the case, there would exist two normalized eigenvectors of weight $p+1$ in characteristic 0 with the same system of eigenvalues for $\ell \neq p$ as G_{p+1} mod $\mathcal{P}$, which since nothing of weight $p+1$ can be in the image of V mod $\mathcal{P}$ could only happen if there was another normalized eigenvector of weight $p+1$ not equal to G_{p+1} but congruent to G_{p+1} mod $\mathcal{P}$. However, since the constant term of G_{p+1} is nonzero mod $\mathcal{P}$, such a normalized eigenvector could not be a cusp form, and it could also not be an Eisenstein series since the space of Eisenstein series of weight $p+1$ for level 1 is one dimensional.

ii. *The image of g in $\mathcal{W}_k$ can be identified (up to a scalar multiple) with that of $\theta^{a-1}(G_{p+1})$ in $\mathcal{W}_{a(p+1)}$ under the Tate-Serre map identifying $\mathcal{W}_k$ with $\mathcal{W}_{a(p+1)}$ as Hecke modules.*

iii. *The image of g in $\mathcal{W}_k$ can be identified (up to a scalar multiple) with that of G_{p+1} in $\mathcal{W}_{p+1}$ under the Tate-Serre map identifying $\mathcal{W}_k$ with $\mathcal{W}_{p+1}[a-1]$ as Hecke modules where $\mathcal{W}_{p+1}[\,a-1\,]$ represents the Hecke module gotten from $\mathcal{W}_{p+1}$ by keeping the underlying vector space structure but replacing the action of T_ℓ by the action of $\ell^{a-1}T_\ell$ for all $\ell \neq p$.*

iv. *The image of g is an eigenvector for the Hecke operators T_ℓ, $\ell \neq p$ with system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq p}$ in the quotient space $\mathcal{W}_k$.*

v. *The image of g is a generalized eigenvector for the Hecke operators T_ℓ, $\ell \neq p$ with system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq p}$ in the quotient space $\mathcal{W}_k$.*

vi. *The modular form g is an eigenvector for the Hecke operators T_ℓ, $\ell \neq p$ with system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq p}$ modulo forms of lower filtration.*

vii. *The modular form g is a generalized eigenvector for the Hecke operators T_ℓ, $\ell \neq p$ with system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq p}$ modulo forms of lower filtration.*

Note in particular that if g itself is a generalized eigenvector for the above system of eigenvalues, the filtration of $g-\ell^a g|V_\ell$ must fall for all primes $\ell \neq p$.

Remark. The hypothesis that $k \equiv a(p+1) \bmod p^2-1$ can be shown to follow directly from each of Conditions *ii* through *vii* using Theorem 6.3 of [J3], which starting with a specific twist of a system of mod $\mathcal{P}$ eigenvalues of weight $\leq p+1$, gives formulas for determining exactly which $\mathcal{W}_k$ contain non-trivial eigenspaces for the twisted system of eigenvalues. One could also modify the proof of Lemma 3 below to deduce that this congruence is implied by Condition *i*, and then use that to conclude that $i \Longrightarrow vii$. However, since assuming this condition to begin with simplifies the computations entering into the proof that $i \Longrightarrow vii$, and since the fact that this condition is implied by *i* is a side result contained in Theorem 2.1 in any case, we will wait until then to deduce this implication.

Proof. The equivalence of *ii* and *iii* follows immediately from the fact that the Tate Serre isomorphism relating $\mathcal{W}_{a(p+1)}$ to $\mathcal{W}_{p+1}[a-1]$ is gotten by an application of the map $\overline{\theta^{a-1}}$ induced by θ^{a-1} on the vector space $\mathcal{W}_{p+1}$. The equivalence of either hence both of these with *iv* and *v* follows from the one dimensionality mentioned above of the associated generalized eigenspaces in the quotient spaces $\mathcal{W}_k$. The equivalence of *iv* and *v* with *vi* and *vii* follows from the definition of the quotient spaces $\mathcal{W}_k$. The fact that $ii \Longrightarrow i$ follows immediately from Lemma 2 combined with the fact that the filtration of $G_{p+1}-\ell G_{p+1}|V_\ell$ is 2 which is lower than the filtration $p+1$ of G_{p+1}, and the fact that $i \Longrightarrow vi$ is just the case $j'=1$ of the first paragraph of Lemma 3. □

Remark. We will see in the proof of Lemma 3 below, that assertion *i* above for a particular prime ℓ_0 implies assertion *vi* for that prime ℓ_0, by which we mean that if the filtration of $g-\ell_0^a g|V_{\ell_0}$ falls at least minimally for one prime $\ell_0 \neq p$, then g must be an eigenvector for the Hecke operator T_{ℓ_0} with eigenvalue $\ell_0^a+\ell_0^{a-1}$ modulo forms of lower filtration.

However, it is worth pointing out that one needs ***all*** $\ell \neq p$ for the converse direction, meaning it is ***not*** true that if g is an eigenvector for T_{ℓ_0} modulo forms of filtration strictly less then k with eigenvalue $\ell_0^a+\ell_0^{a-1}$ for one particular $\ell_0 \neq p$ that the filtration of $g-\ell_0^a g|V_{\ell_0}$ must fall for that ℓ_0.

In fact, one can show that pretty much the opposite is the case, since for example, ***all*** eigenforms of filtration $k=a(p+1)$ are known to have eigenvalues congruent to $\ell_0^a+\ell_0^{a-1}$ mod $\mathcal{P}$ for a set of primes ℓ_0 of positive density,[2] but we show in Proposition 2.2 that the only eigenforms f for all T_ℓ including $\ell=p$ for which the filtration of some linear combination of f and $f|V_{\ell_0}$ falls down for even one $\ell_0\neq p$, are those that are congruent to $\theta^{a-1}(G_{p+1})$ for some a.

We now fill in the details of the proof of the above Theorem and obtain other related results in the process. Our first lemma shows that the isomorphisms between the Hecke modules $\mathcal{W}_k$ and twists of $\mathcal{W}_{k'}$ of lower weights first noticed by Tate and Serre respect the property of having the filtration fall when the level is raised, by which we mean that if a modular form g has an image in $\mathcal{W}_k$ that corresponds to the image of the form g' in $\mathcal{W}_{k'}$ under one of the Tate-Serre isomorphisms, then some linear combination of g and $g|V_\ell$ will have a filtration in level ℓ that is strictly smaller than k, if and only if the "analogous" linear combination of g' and $g'|V_\ell$ also has a drop in filtration.

In addition to allowing us to conclude that $ii\Longrightarrow i$ in the above Theorem, this will play a major role in Section 2 in extending our results for minimal falls by loosening the requirement that the filtration fall for all $\ell\neq p$ to only requiring it fall for a single *(arbitrary)* $\ell_0\neq p$.

Lemma 2. *Let $k,k'>p+1$, suppose g,g' are level 1 forms of filtrations k, k' whose images in $\mathcal{W}_k$ and $\mathcal{W}_{k'}[d]$ correspond to each other under one of the Tate-Serre isomorphisms, where $\mathcal{W}_{k'}[d]$ (for any positive integer d) represents the Hecke module gotten from $\mathcal{W}_{k'}$ by keeping the underlying vector space structure, but "twisting" the action of the Hecke operators d times in the manner made explicit in assertion iii of Theorem 1 above. Then the filtration of a linear combination $g'-\ell^a g'|V_\ell$ will be lower than k' for some prime $\ell\neq p$ if and only if the filtration of $g-\ell^{a+d}g|V_\ell$ will be lower than k.*

Proof. Tate and Serre showed that if $k\not\equiv 1$ mod p and $k\geq 2p+2$, the map $\bar{\theta}$ induced by θ maps $\mathcal{W}_{k'}$ isomorphically *(as a vector space)* onto $\mathcal{W}_k$ where $k'=k-(p+1)$ *(introducing a single twist when these vector spaces are considered as Hecke modules)*, whereas if $k\equiv 1$ mod p and $k>p+1$, the operator U induces a Hecke module isomorphism $\overline{U}$ between $\mathcal{W}_k$ and $\mathcal{W}_{k'}$ where $k'=\frac{k-1}{p}+p<k$. Since the general Tate Serre isomorphisms can be obtained by successive applications of these two maps *(or their inverses)*, it suffices to prove the result separately in each of these two cases.

Case 1: $k\not\equiv 1$ mod p and $k'=k-(p+1)$

In this case, the Tate-Serre isomorphism is induced by the action of θ, so the hypotheses of the lemma are equivalent to the assertion that $\theta(g')=g+h$ where h has filtration strictly smaller than k.

This implies that any linear combination $g+A_\ell g|V_\ell$ can be rewritten as $\theta(g'+\ell^{-1}A_\ell g'|V_\ell)$ modulo forms of filtration strictly smaller than k *(where ℓ^{-1} represents an integer since we are working mod $\mathcal{P}$)*.

2. This follows since as pointed out by Serre, the existence of Deligne's representations attached to eigenforms mod $\mathcal{P}$, together with the Chebotarev Density Theorem, implies that there must be a set of primes ℓ of positive density all congruent to 1 mod p for which the eigenvalue of T_ℓ is congruent to 2 *(hence to $\ell^a+\ell^{a-1}$)* mod $\mathcal{P}$, and also another set of primes ℓ of positive density all congruent to -1 mod p for which the eigenvalue of T_ℓ is congruent to 0 *(hence again to $\ell^a+\ell^{a-1}$)* mod $\mathcal{P}$ for all eigenforms of any given weight and level.

But since $p\nmid k'$, the filtration of $\theta(g'+\ell^{-1}A_\ell g'|V_\ell)$ for any ℓ

$$\begin{cases} =k'+(p+1)=k & \text{if the filtration of } g'+\ell^{-1}A_\ell g'|V_\ell \text{ stays at } k' \\ <k'+(p+1)=\ k & \text{if the filtration of } g'+\ell^{-1}A_\ell g'|V_\ell \text{ is strictly smaller than } k' \end{cases}.$$

Since $g+A_\ell g|V_\ell$ equals $\theta(g'+\ell^{-1}A_\ell g'|V_\ell)$ modulo forms of filtration strictly smaller than k, this immediately implies that the filtration of $g+A_\ell g|V_\ell$ falls if and only if the filtration of $g'+\ell^{-1}A_\ell g'|V_\ell$ does.

Case 2: $k\equiv 1$ mod p and $k'=\frac{k-1}{p}+p$

In this case, the Tate-Serre isomorphism is induced by the action of U, and the hypotheses of the lemma are equivalent to the assertion that $g|U=g'$ modulo forms of filtration strictly smaller than k', which immediately implies that if one applies the U operator to any linear combination $g+A_\ell g|V_\ell$, one obtains the linear combination $g'+A_\ell g'|V_\ell$ modulo forms of filtration strictly smaller than k'.

But since $k\equiv 1$ mod p, the map U induces an embedding of the quotient space $\mathcal{W}_k(\ell)=_{def}\widetilde{M_k(\Gamma_0(\ell))}/\widetilde{M_{k-(p-1)}(\Gamma_0(\ell))}$ into $\mathcal{W}_{k'}(\ell)$ for any level ℓ,[3] hence it follows that the filtration of $g+A_\ell g|V_\ell$ will be smaller than k, if and only if the filtration of $(g+A_\ell g|V_\ell)|U$ is smaller than k', which by the previous paragraph will be true if and only if the filtration of $g'+A_\ell g'|V_\ell$ will be smaller than k' *(one could also see this by looking directly at the filtrations without appealing to the quotient spaces $\mathcal{W}_k(\ell)$ and $\mathcal{W}_{k'}(\ell)$).* □

Remark. An alternate approach to the isomorphisms given by Tate and Serre is given by Robert in [Ro], and it is also easy *(arguably even easier)* to prove the analogue of Lemma 2 using his isomorphisms.

The case $j'=1$ of our next lemma played a key role in the above proof of Theorem 1, and we will also see that one of its corollaries in this case implies the uniqueness of Behrens' forms in the "minimal" case, with a related corollary playing an important role throughout our paper. However, the Lemma goes further, in that it also includes results for "non-minimal" drops in filtration which will be used in Section 3 in the proof of our *(weak)* existence result.[4] The proof of the Lemma involves a simple application of the trace map from modular forms of level ℓ to ones of level 1.

Definition. *We define the Trace map taking us from modular forms of level ℓ to modular forms of level 1 in the usual way, so that if $\{\gamma_i\}_{i=1}^{\ell+1}$ constitute a complete set of right coset representatives for $\Gamma_0(\ell)$ in $SL_2(\mathbb{Z})$ and g is a modular form of weight k for $SL_2(\mathbb{Z})$, then $Tr(g)=\sum_{i=1}^{\ell+1} g\,|_k\,\gamma_i$.*

3. This is a basic step in the Tate/Serre argument that follows easily for example from Lemma 1.9 of [J3] which says that $w(h|U)\le\frac{w(h)-1}{p}+p$ with equality if and only if $w(h)\equiv 1$ mod p. We actually have an isomorphism in this case, since the dimensions of the two spaces can be shown to be the same, even in the case of arbitrary level.

4. We use this lemma in only one place in the proof of that result, to argue that the generalized eigenvectors we are looking at in the quotient spaces $\mathcal{W}_{k^*}(\ell_1)$ are actual eigenvectors *(which is of use to us since it is easy to prove that the associated eigenspaces are one dimensional).*

We could thus have avoided the need for including the cases $j'>1$ of this lemma *(although they are interesting in their own right)* if we could have shown directly that the entire ***generalized*** eigenspaces in the $\mathcal{W}_{k^*}(\ell_1)$ for these systems of eigenvalues are one dimensional *(which the authors are convinced they are)*, but the natural way to do that would be to use results contained in Section 4 of [J3] paper that may not have been generalized to higher levels, even though these generalized results are undoubtably true *(the result of Theorem 4.1 of that paper was generalized by Ash and Stevens in [A-S] but their proof only deals with systems of eigenvalues, and does not on the surface say anything about the dimensions of the underlying generalized eigenspaces, although it is possible that such results might be implied by their proof).*

It is easy to check that the trace of a modular form of weight k for $\Gamma_0(\ell)$ is a modular form for $SL_2(\mathbb{Z})$ of the same weight that does not depend on the choice of the coset representatives.

Lemma 3. *If a modular form g of filtration $k \equiv a(p+1) \bmod p^2-1$ satisfies the property that the filtration of $g - \ell^a g|V_\ell$ falls down by at least $j'(p-1)$ for some $j' \geq 1$ and all $\ell \neq p$, then g must be an eigenvector for the T_ℓ with $\ell \neq p$ modulo forms of filtration $\leq k - j'(p-1)$ with mod $\mathcal{P}$ system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$.*

Moreover, in the case where g is a generalized eigenvector and j' is not divisible by $p+1$, one can replace the weak inequality at the end of the last paragraph by strict inequality, which implies in particular, that (for any $\ell_1 \neq p$) the image of $g - \ell_1^a g|V_{\ell_1}$ must be an actual eigenvector in the quotient space $\mathcal{W}_{k-j'(p-1)}(\ell_1)$ of level ℓ_1 for the T_ℓ, $\ell \neq p$ and U_{ℓ_1} with system of eigenvalues $\{\lambda_\ell\} = \begin{cases} \{\ell^a + \ell^{a-1}\} & \text{if } \ell \neq p, \ell_1 \\ \ell_1^{a-1} & \text{if } \ell = \ell_1 \end{cases}$.

Remark. As summarized in Theorem 1 above, the converse of the first paragraph of the Lemma is also true in the case where $j' = 1$.

Note, however, that this converse is ***not*** true for $j' > 1$, since we will see in Section 3, for example, that there exist generalized eigenvectors g of filtrations $k \equiv a(p+1) \bmod p^2-1$ for the system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ for k not divisible by p, for which the filtrations of $g - \ell^a g|V_\ell$ fall down by ***precisely*** $(p-1)$ for all ℓ, but it follows from the proof of the second paragraph of this lemma that such g would be eigenforms for this system of eigenvalues modulo forms of filtration $\leq k - j'(p-1)$ for j' that are much larger than 1. It is probable that some sort of results in the converse direction are true, but they would be more complicated than the straightforward converse of the first paragraph above.

Proof. Assume that the filtration of g equals k and also that the filtration of $g - \ell^a g \mid V_\ell$ equals $k' \leq k - j'(p-1)$ for some $\ell \neq p$.

Then $g - \ell^a g|V_\ell \equiv h$ for some form h for $\Gamma_0(\ell)$ of weight k'. This implies that $\mathrm{Tr}(g - \ell^a g \mid V_\ell) \equiv \mathrm{Tr}(h) \bmod \mathcal{P}$,[5] and hence $\mathrm{Tr}(g - \ell^a g \mid V_\ell)$ is congruent mod $\mathcal{P}$ to a form for $SL_2(\mathbb{Z})$ of weight k'.

But since g is itself a weight k modular form for $SL_2(\mathbb{Z})$, it follows directly from the definition of the trace map that $\mathrm{Tr}(g) = (\ell+1)g$. On the other hand, it is also easy to see *(again using the fact that g is a weight k modular form for $SL_2(\mathbb{Z})$)* that $\mathrm{Tr}(g \mid V_\ell) = \ell^{1-k} g \mid T_\ell \equiv \ell^{1-2a} g|T_\ell \bmod \mathcal{P}$ for the Hecke operator T_ℓ.[6] Thus $\mathrm{Tr}(g - \ell^a g \mid V_\ell) = (\ell+1)g - \ell^{1-a} g \mid T_\ell$.

Then by combining the above two paragraphs, one concludes that $(\ell+1)g - \ell^{1-a} g \mid T_\ell$ has a weight $k' \leq k - j'(p-1)$, or equivalently that g is a mod $\mathcal{P}$ eigenvector for the Hecke operator T_ℓ with eigenvalue $\ell^a + \ell^{a-1}$ modulo forms of filtration $\leq k - j'(p-1)$. Since the same argument works for all $\ell \neq p$, we have succeeded in proving the first paragraph of the Lemma.

5. This follows directly from the definition of the trace together with the fact that if h and h' are both modular forms for $\Gamma_0(\ell)$ with $\mathcal{P}$-integral coefficients such that $h \equiv h' \bmod \mathcal{P}$, then $h \mid \gamma \equiv h' \mid \gamma \bmod \mathcal{P}$ for all $\gamma \in SL_2(\mathbb{Z})$ which can in turn be deduced from Katz's q-expansion principle ([K]).

6. For a reference see the bottom of page 224 of [S1], but note that the p there equals our ℓ and that when g is a modular form of weight k for $SL_2(\mathbb{Z})$, then $g \mid W = \ell^{\frac{k}{2}} g \mid V_\ell$

It remains only to show that in the case of a generalized eigenvector where j' is not divisible by $p+1$, we can replace the weak inequality appearing in this first paragraph of the Lemma by strict inequality.

But that follows easily since the fact that g is a generalized eigenvector for the above system of eigenvalues that is an actual eigenvector mod forms of weight $\leq k-j'(p-1)$ implies that $g|T_\ell = (\ell^a+\ell^{a-1})g+G$ where G is a form of filtration $\leq k-j'(p-1)$ that must also be a generalized eigenvector for the same system of eigenvalues.

But the Remark following the statement of Theorem 1 above implies that the only filtrations for which there can exist generalized eigenvectors in Level 1 for that system of eigenvalues are ones congruent to k mod p^2-1. Since by hypothesis, j' is not divisible by $p+1$, this implies G can not have filtration exactly $k-j'(p-1)$ and hence must have an even lower filtration. $\square$

The following follows as an easy corollary.

Corollary 4. *If g and g' are two forms of filtration $k \equiv a(p+1)$ mod p^2-1, satisfying the properties that the filtrations of $g-\ell^a g|V_\ell$ and $g'-\ell^a g'|V_\ell$ both fall down at least minimally for all $\ell \neq p$, then (up to scalar multiples) g and g' must differ by a form of filtration strictly less than k.*

Remark. Because of the condition on the order of the zeroes at ∞ given in Behrens' main result *(Theorem 1.2 of [B])*, this immediately gives the uniqueness *(up to scalar multiplication)* of his forms $f_{k,j'}$ as defined in Theorem 0.1 in those cases where $j'=1$ *(one sees this easily by noting that all forms of weight k but filtration strictly less than k in level one must have zeroes at ∞ of orders strictly smaller than those in Behrens' bounds).*

Proof. It follows from the above lemma that g and g' must both be eigenvectors for the $T_{\ell,\ell\neq p}$ with mod $\mathcal{P}$ systems of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq p}$ modulo forms of filtration strictly less than k. This means that their images in $\mathcal{W}_k$ must both be eigenvectors with that system of eigenvalues. But as mentioned above, the eigenspaces *(also the generalized eigenspaces)* for this system of eigenvalues in the quotient spaces $\mathcal{W}_k$ are well known to be one dimensional, so g and g' must have images in $\mathcal{W}_k$ that are scalar multiples of each other, which means that *(up to scalar multiples)* they must differ by a form of filtration less than k. $\square$

Since we will show in the next Section that if the filtration of some linear combination of g and $g|V_{\ell_0}$ falls for one $\ell_0 \neq p$, then we must have $k \equiv a(p+1)$ mod p^2-1 for some a in which case the filtration of $g-\ell^a g|V_\ell$ falls for all primes $\ell \neq p$, we can upgrade the previous Corollary by only requiring that the filtrations fall for one $\ell_0 \neq p$ *(or for one ℓ_0 for each form)* yielding:

Corollary 5. *If g and g' are two forms of the same filtration k satisfying the properties that the filtrations of $g-\ell_0^a g|V_{\ell_0}$ and $g'-\ell_0^a g'|V_{\ell_0}$ both fall down at least minimally for at least one $\ell_0 \neq p$ and some a, then (up to scalar multiples) g and g' must differ by a form of filtration strictly less than k.*

The same conclusion would be true if one started with the hypothesis that the filtration of $g-\ell_0^a g|V_{\ell_0}$ fell down for one prime ℓ_0 and the filtration of $g'-(\ell_0')^a g|V_{\ell_0'}$ fell down for a possibly different prime ℓ_0'.

Section 2: Applications of Serre's Conjecture to minimal falls

The main contribution of this Section is that it allows us to pass from the hypothesis that the filtration of some linear combination of g and $g|V_\ell$ falls *(at least minimally)* for one prime $\ell \neq p$ to the assertion that the filtration of some linear combination falls for all such primes ℓ. A secondary contribution is that we show that drops in filtration can only occur in cases where $k \equiv a(p+1) \bmod p^2-1$ for some a and that in such cases the linear combinations in question must be scalar multiples of $g-\ell^a g|V_\ell$. These last two facts could have been proven in Section 1 in cases where the filtration was known to fall for ***all*** primes $\ell \neq p$ without needing to invoke Serre's conjecture, but we held off on doing so, since we knew they would fall out in this section "for free" without any additional work on our part.

We summarize our main results in Theorem 1 below, which follows easily from Propositions 4 and 5 of this Section. Although we deduce one line in this Theorem from Lemma 1.3, we could have easily incorporated a proof of this instead into the inductive argument contained in the proof of Proposition 4 in this Section.

The basic idea is to make use of Serre's *(now proven)* conjecture [S2][1] that among other things implies that any system of mod $\mathcal{P}$ eigenvalues *(even if one leaves out a finite number of them or omits eigenvalues for a set of primes of density 0)* determines the minimal weight and level of any corresponding mod $\mathcal{P}$ eigenform *(which implies in particular that the minimal weight for any mod $\mathcal{P}$ system of eigenvalues always occurs in the minimal level)* **as long as we are in the case where the associated mod $\mathcal{P}$ representation $\tilde{\rho}$ is irreducible,** which can be easily seen to only fail to be true when the associated eigenform is congruent mod $\mathcal{P}$ to some Eisenstein series of level N, or to a power of θ applied to some Eisenstein series of level N. [2]

We use this to get our result in the case of Hecke eigenforms, and then extend it it to the case of generalized eigenforms, from which it is easy to extend it further to arbitrary modular forms mod $\mathcal{P}$.

We actually only need to appeal to Serre's conjecture when the filtrations are divisible by $p+1$, since work contained in the first author's thesis suffices to show that these are the only filtrations for which the filtration of a linear combination of f and $f|V_\ell$ can fall *(even for a single prime ℓ)*, but since we still need to use Serre's conjecture *(at least his epsilon conjecture, see Footnote 1)* when the filtration is divisible by $p+1$, it is slightly more economical to use it in all cases, so we take that approach in what follows.

Theorem 1. *Let g be a modular form mod $\mathcal{P}$ of level 1 and filtration k. If the filtration of some linear combination of g and $g|V_{\ell_0}$ falls down for one $\ell_0 \neq p$, then we must have $k \equiv a(p+1) \bmod p^2-1$ for some a, in which case (up to scalar multiple) the above linear combination must equal $g-\ell_0^a g|V_{\ell_0}$ and the filtration of $g-\ell^a g|V_\ell$ must fall down for all primes $\ell \neq p$.*

Moreover, if this is the case, the above form g must be an actual eigenform modulo forms of lower filtration for all T_ℓ with $\ell \neq p$ with mod $\mathcal{P}$ system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell \neq p}$.

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1. In fact, for the results we can't prove by other means *(those where the filtration is divisible by $p+1$)*, we are pretty sure that one can get by with Serre's epsilon conjecture *(proven by Ribet in [Ri] and used to show that Weil-Taniyama implies Fermat's Last Theorem)* which is stated in terms of lowering the level in the case of weight 2.

2. This follows easily by just directly writing down the possible reducible representations, and applying formulas for coefficients of Eisenstein series.

Since generalized eigenvectors for distinct systems of eigenvalues must be linearly independent (hence also linearly independent modulo forms of lower filtrations), this implies that if g is itself a generalized eigenform, it must have system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$.[3]

Note in particular that as a side result of the first paragraph, we are able to conclude that if we know the filtration of $g - \ell^a g|V_\ell$ falls down for some fixed a and ***all*** *ℓ, then we must have $k \equiv a(p+1)$ mod p^2-1 for that a.*

Proof. Proposition 5 below implies that the filtration of some linear combination of g and $g|V_{\ell_0}$ can fall down for one $\ell_0 \neq p$ only if k is divisible by $p+1$, so that $k \equiv a(p+1)$ mod p^2-1 for some a, and hence $k = a^*(p+1)$ for some $a^* \equiv a$ mod $p-1$, and then Proposition 4 implies that *(up to scalar multiple)* the above linear combination must equal $g - \ell_0^{a^*} g|V_{\ell_0}$ *(or equivalently $g - \ell_0^a g|V_{\ell_0}$)* and the filtration of $g - \ell^{a^*} g|V_\ell$ *(or equivalently of $g - \ell^a g|V_\ell$)* must fall for all $\ell \neq p$.

But Lemma 1.3 says this can happen only if g is an actual eigenvector modulo forms of lower filtration for all T_ℓ with $\ell \neq p$ with mod $\mathcal{P}$ system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$.[4] $\square$

Convention. *From here on* ***in this Section only****, when we say generalized eigenvector or actual eigenvector of level 1 (resp. level ℓ_0), we mean for all Hecke operators T_ℓ (resp. all T_ℓ for $\ell \neq \ell_0$)* ***including $\boldsymbol{\ell = p}$****. Also, from now on, when we say the filtration falls in this Section, we mean it falls by at least the minimal amount $(p-1)$, as nothing in this Section has anything to do with the extent or size of the fall. Note finally that sometimes when we say "is congruent to" in this Section, we should probably say, "equals" since all modular forms here (as in the rest of the paper) are assumed to be modular forms mod $\mathcal{P}$.*

We now fill in the details of the proof of Theorem 1. In Proposition 2, we prove the result in the case $g = f$ is an actual *(normalized)* eigenvector and in Corollary 3, we extend this to the case where g is a generalized eigenvector for which there exists an actual *(normalized)* eigenvector f of the ***same filtration*** as g for the same system of eigenvalues *(in which case, we end up showing that g must be a scalar multiple of f)*. Proposition 4 than deals with general forms of filtration divisible by $p+1$, and Proposition 5 shows that if the filtration of g is <u>not</u> divisible by $p+1$, the phenomenon we are interested in will never occur. Lemma 6, is just a slightly technical *(but not too messy)* Lemma that is needed for one case of the proof of Proposition 5.

Proposition 2. *If f is a* ***normalized eigenform*** *of level 1 for which the filtration of some linear combination of f and $f|V_{\ell_0}$ falls down for some prime $\ell_0 \neq p$, then f must be congruent to $\theta^{a-1}(G_{p+1})$ for some $1 \leq a \leq p$ and (up to scalar multiple) the above linear combination must equal $f - \ell_0^a f|V_{\ell_0}$, in which case, the filtration of $f - \ell^a f|V_\ell$ falls down for all $\ell \neq p$. Note that in this case, the filtration of f equals $a(p+1)$.*

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3. The reader is cautioned that there are generalized eigenforms g for other systems of eigenvalues for which the filtration of some linear combination of g and $g|V_{\ell_0}$ falls down for one $\ell_0 \neq p$, but they have other filtrations. If one fixed the congruence class of the filtration mod $p-1$, but did not fix the filtration itself, there would be two possible systems of eigenvalues, with this property, namely $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ and $\{\ell^{a'} + \ell^{a'-1}\}_{\ell \neq p}$ for $a' = a \pm \frac{p-1}{2}$ mod $p-1$, where the filtration would be congruent to $a(p+1)$ mod p^2-1 *(respectively $a'(p+1)$ mod p^2-1)* in the first *(respectively second)* case.

4. Although we could also have easily built a direct proof of this result into the inductive argument contained in the proof of Proposition 4 of this Section, we didn't have to do so since Lemma 1.3 which was also used for other purposes involved a stronger result than the one we could obtain here *(in that it took the size of the drop in filtration into account)*.

Proof. We first claim that this can only happen if the mod $\mathcal{P}$ representation $\tilde{\rho_f}$ associated to the eigenform f is reducible.

This is because according to Serre's *(now proven)* conjecture [S2], as long as the associated representation $\tilde{\rho_f}$ is irreducible, the eigenvalues *(minus any finite subset or those for any set of primes of density 0)* determine the minimal weight and level of any mod $\mathcal{P}$ modular representation, which means in particular that the minimal weight for any mod $\mathcal{P}$ system of eigenvalues always occurs in the minimal level, which in this case is level 1 *(so the filtration can not fall when the level rises)*.

We have to be a little bit careful, since for technical reasons, we need to treat the case where $f = \theta^{p-1}(f')$ for some eigenform f' that is ***not*** in the kernel of $U = T_p$ separately from cases of other eigenforms, because according to the definitions, the same mod $\mathcal{P}$ representation is associated with both f' and f *(i.e.* $\tilde{\rho}_f = \tilde{\rho}_{f'}$), although the filtrations of the two eigenforms will not be the same *(so that the filtration of the normalized eigenvector f in this case will not equal the minimal weight of the associated representation)*.

But it is easy to see that in this case too, the filtration of some linear combination of f and $f|V_{\ell_0}$ can fall down for some prime $\ell_0 \neq p$, only if the mod $\mathcal{P}$ representation $\tilde{\rho}_f = \tilde{\rho}_{f'}$ associated to the eigenform f is reducible. This is because the result for f can be easily deduced from the analogous result for f' in the irreducible case.

More precisely, if we let k' denote the filtration of f', then $f = \theta^{p-1}(f')$ is easily seen to have filtration $k'p$ *(just use the fact that* $\theta^{p-1}(f') = f' - f'|U|V$ *where* $f'|U$ *by hypothesis is a nonzero scalar multiple of f)*.

But any linear combination $f + \lambda f|V_\ell$ will equal $\theta^{p-1}(f' + \lambda f'|V_\ell)$ where $f' + \lambda f'|V_\ell$ like f' will be an eigenvector with nonzero mod $\mathcal{P}$ eigenvalue for $U = T_p$.

It follows that in the irreducible case, $f + \lambda f|V_\ell$ will have filtration $k'p$ *(for the same reasons that* $f = \theta^{p-1}(f')$ *had this filtration, since we have already seen that* $f' + \lambda f'|V_\ell$ *will have filtration equal to the filtration* k' *of* f' *in this case)* and hence it will have a filtration equal to that of f.

This means we are done in the irreducible case.

Moreover, as we mentioned above, the only way the representation $\tilde{\rho}$ will fail to be irreducible in the case of level 1 is if the associated normalized modular form is congruent to $f = \theta^{a-1}(G_{k^\star})$ for some $4 \leq k^\star \leq p-3$ or $k^\star = p+1$ and some $1 \leq a \leq p$, where $G_{k^\star}$ denotes the normalized Eisenstein series of weight $k^\star$.

We are done since the easily verifiable fact that the filtration of any such $G_{k^\star}$ in level 1 will be the same as that of any linear combination of $G_{k^\star}$ and $G_{k^\star}|V_{\ell_0}$ in level ℓ_0 unless $k^\star = p+1$ and *(up to scalar multiple)* the above linear combination equals $G_{p+1} - \ell_0 G_{p+1}|V_{\ell_0}$ *(in which case, the filtration of* $G_{p+1} - \ell G_{p+1}|V_\ell$ *falls down for all* ℓ*)* is easily seen to imply that the filtration of $f = \theta^{a-1}(G_{k^\star})$ in level 1 for all $1 \leq a \leq p$ will be the same as that of any linear combination of f and $f|V_{\ell_0}$ in level ℓ_0, unless $k^\star = p+1$ and *(up to scalar multiple)* the above linear combination equals $f - \ell_0^a f|V_{\ell_0}$, *(in which case, the filtration of* $f - \ell^a f|V_\ell$ *falls down for all* ℓ *and the filtration of f equals* $a(p+1)$*)*. $\square$

Remark. We omitted the Eisenstein series of weight $k=p-1$ from the above since G_{p-1} does not have p-integral coefficients *(whereas E_{p-1}, which does have p-integral coefficients, has filtration 0 not $p-1$, and does not satisfy our definition of being a normalized eigenform since it is congruent to a constant mod p)*. However, in order to get the complete picture, one must also take G_{p-1} into account, since after one "raises the level in the right way" G_{p-1} and $G_2=G_{p+1}$ share the same θ-cycle *(or more precisely $G_{p-1}-G_{p-1}|V_\ell$ shares the same θ-cycle with $G_2-\ell G_2|V_\ell$)*, and it was certainly not an accident that this fact played a key role in obtaining the second author's original computational results.[5]

Corollary 3. *Suppose g is a generalized eigenvector of level 1 for which there exists an actual (normalized) eigenvector f of level 1 of the* ***same filtration*** *k for the same system of eigenvalues such that the filtration of some linear combination of g and $g|V_{\ell_0}$ falls down for one $\ell_0\neq p$. Then f must equal $\theta^{a-1}(G_{p+1})$ for some $1\le a\le p$, g must equal a scalar multiple of f, k must equal $a(p+1)$, and (up to scalar multiple) the above linear combination must equal $g-\ell_0^a g|V_{\ell_0}$, in which case, the filtration of $g-\ell^a g|V_\ell$ falls down for all $\ell\neq p$.*

Proof. The fact that f is a *normalized* eigenvector of filtration k for the system of eigenvalues *(including the eigenvalue corresponding to $U=T_p$)* associated to the generalized eigenvector g of level 1 trivially implies that the minimal weight for this system of eigenvalues in level 1 must be k.[6]

By hypothesis, some linear combination of g and $g|V_{\ell_0}$ has filtration strictly smaller than k in level ℓ_0. Since such a linear combination would be a generalized eigenvector for the operators T_ℓ for $\ell\neq\ell_0$ of level ℓ_0 for the same system of eigenvalues as g *(hence as f)*, this can only happen if there is an actual *(normalized)* eigenvector for the T_ℓ for $\ell\neq\ell_0$ *(and for U_{ℓ_0})* in level ℓ_0 of filtration strictly smaller than k corresponding to the above mentioned system of eigenvalues for $\ell\neq\ell_0$.

As in the proof of Proposition 2, it is easy to show this contradicts the fact that the minimal weight associated to this system of eigenvalues in level 1 is k, unless f is congruent to $\theta^{a-1}(G_{p+1})$ for some integer $1\le a\le p$,[7] in which case, $k=a(p+1)$ and the associated mod $\mathcal{P}$ level 1 system of eigenvalues equals $\{\ell^a+\ell^{a-1}\}$.

We are done since as we pointed out in the previous Section, the generalized eigenspace in $\mathcal{W}_k$ for $k=a(p+1)$ corresponding to the above system of eigenvalues is easily seen to be one dimensional, which since there can't be level 1 generalized eigenvectors for this system of eigenvalues of filtration smaller than the filtration of f, implies that g must actually equal a scalar multiple of f. The rest of the assertions of this Corollary then follow immediately from Proposition 2. □

5. Thus the Eisenstein series of weights 2 and $p-1$ are in some sense "companion forms" and one can think of the fact that one needs to add on a non-p-integral constant term in order to make the normalized Eisenstein series of weight $p-1$ in level 1 become a modular form as being some kind of "weird mod p translation" of the fact that one needs to add on a non-holomorphic correction term to make the normalized Eisenstein series of weight 2 in level 1 become an automorphic function, *(with both add-ons "disappearing" when one "raises the level in the right way")*.

6. Recall that there can be only one normalized eigenvector for any system of eigenvalues $\{\lambda_\ell\}$ in level 1 as the eigenvalues λ_ℓ for all ℓ *(including the one for $\ell=p$)* determine the congruence class of the weight mod $p-1$, as well as all the coefficients of the associated normalized eigenform.

7. As in the proof of Proposition 2, one concludes that $\tilde{\rho}_f$ must be reducible and hence f must equal a power of θ applied to some Eisenstein series $G_{k^\star}$, after which it remains only to verify that $k^\star$ must equal $p+1$ *(which one can do among other ways by applying Atkin-Lehner's annihilator K_{ℓ_0} operator ([A-L]) to kill off those coefficients of index divisible by ℓ_0* (thus erasing any differences between the normalized eigenform in level ℓ_0 and the original form f in level 1 at the expense of raising the levels to ℓ_0^2)*, and then using the fact that the only time $G_{k^\star}|K_{\ell_0}$ can have filtration in level ℓ_0^2 lower than the filtration of $G_{k^\star}$ in level 1 is when $k^\star=p+1$ to deduce the analogous fact for powers of θ applied to $G_{k^\star}$)*.

Remark. We make direct use of Corollary 3 only when $p+1 \leq k \leq 2p$, where it is easier to prove than in the general case. In fact, by substituting results of the first author's thesis, we could get by using it only when $k = p+1$, where parts of the proof become rather trivial *(e.g. the conclusion that $k^\star$ must equal $p+1$ in Footnote 7 becomes immediate, making the bulk of that Footnote no longer needed).*

We now prove our main result for arbitrary modular forms mod $\mathcal{P}$ in the case of filtration divisible by $p+1$.

Proposition 4. *Suppose g is a mod $\mathcal{P}$ form in level 1 of filtration $k = a(p+1)$ for some positive integer a, such that the the filtration of some linear combination of g and $g|V_{\ell_0}$ falls down for one $\ell_0 \neq p$. Then (up to scalar multiple) the above linear combination must equal $g - \ell_0^a g|V_{\ell_0}$, and the filtration of $g - \ell^a g|V_\ell$ falls down for all $\ell \neq p$.*

Proof. We prove the result by induction on k.

To prove the result for $k = p+1$, we first note that it suffices to prove it in the case where g is a generalized eigenvector. That is because we can write g as a sum of generalized eigenvectors for ***distinct*** systems of eigenvalues for the T_ℓ all of filtration $p+1$, and since generalized eigenvectors for distinct systems of eigenvalues must always be linearly independent *(hence also linearly independent modulo forms of lower weight)* and these systems of eigenvalues will still be distinct after we omit the prime ℓ_0,[8] it is easy to see that the only way the filtration of some linear combination of g and $g|V_{\ell_0}$ can fall down is if the analogous statement is true for all the generalized eigenvectors that enter into the above decomposition.

We are then done by Corollary 3 because if g is a generalized eigenvector of filtration $p+1$, the fact that there are no modular forms of weight 2 for level 1, implies there must exist an actual *(normalized)* eigenvector f of the ***same filtration*** $p+1$ for the same system of eigenvalues.

The case of $k = a(p+1)$ for $a \geq 2$ then follows easily by induction after appealing to Lemma 1.2, since $\mathcal{W}_{a(p+1)} \cong \mathcal{W}_{p+1}[a-1]$ as a Hecke module under one of the Tate Serre isomorphisms referred to in that Lemma. □

In the next Proposition, we employ a similar argument as that used in the above proof to show that when k is not divisible by $p+1$, the filtration of ***no*** linear combination of g and $g|V_{\ell_0}$ will ever fall down for any ℓ_0. In particular, the argument that we give depends on Corollary 3, hence on Serre's conjecture, although as we mentioned above, it is possible to avoid Serre's conjecture and rely solely on results of the first author's thesis in cases where k is ***not*** divisible by $p+1$.

The idea of this proof is to argue that any $\mathcal{W}_k$ must be isomorphic to a twist of $\mathcal{W}_{k'}$ for $k' \leq 2p$ under one of the Tate-Serre isomorphisms included in Lemma 1.2, thus allowing us to once again use that Lemma to reduce to the cases of "low filtrations", where we hope to be able to obtain our result with the aid of Corollary 3 above.

8. One sees this for example by considering the mod $\mathcal{P}$ Galois representation associated with such systems of eigenvalues.

The only complication occurs in the beginning stages of our induction argument since this time one could have a generalized eigenvector of "low" filtration k for which the corresponding actual eigenvector had lower filtration still *(hence Corollary 3 would not apply)*. But it is easy to see that this can only happen in a case where $\lambda_p = 0$ and $\{\lambda_\ell\}$ is also a system of eigenvalues of weight $4 \leq k' \leq p-1$. We treat that case separately in Lemma 6, although we assume the result of that Lemma in the proof we give below.

Proposition 5. *Suppose g is a form of nonzero filtration k that is **not** divisible by $p+1$. Then the filtration of no linear combination of g and $g|V_{\ell_0}$ will ever fall down for any ℓ_0.*

Proof. We again argue by induction on k, and as we explained above, use the fact that $\mathcal{W}_k$ must be isomorphic to a twist of $\mathcal{W}_{k'}$ for $k' \leq 2p$ to reduce to the case $k \leq 2p$ *(with the aid of Lemma 1.2)*.

Moreover, if $k \leq p-1$, the result is trivial since no nonconstant modular forms of ***weight*** k for such k for any level can ever have filtration smaller than k.

We can thus assume $p+3 \leq k \leq 2p$ *(k can not equal $p+1$ under the hypothesis of this Proposition, that case having already been handled in Proposition 4 above)*, where we can reduce to the subcase where g is a generalized eigenform, using an argument similar to the one used above in the case $k = p+1$ *(except that this time when we write our form as a sum of generalized eigenvectors, we could have some generalized eigenvectors of lower filtration, but these will not effect whether or not the filtration of a linear combination of the original form g and $g|V_\ell$ falls)*.

We thus let g be a generalized eigenvector of filtration $p+3 \leq k \leq 2p$ corresponding to a system of eigenvalues $\{\lambda_\ell\}$.

Then if $\{\lambda_\ell\}$ is *not* also a system of eigenvalues of weight $k' = k-(p-1)$, we are done by Corollary 3 as in the proof of Proposition 4, since in such cases, there must exist an actual eigenform f of the ***same filtration*** for the same system of eigenvalues.

The case where $\{\lambda_\ell\}$ is also a system of eigenvalues of weight $k' = k-(p-1)$ is handled in Lemma 6. $\square$

It remains only to fill in the gap not explicitly handled above, where g is a generalized eigenvector of filtration k but the corresponding eigenvector f has lower filtration, and we do that in the Lemma below.

Lemma 6. *Let g be a generalized eigenvector of filtration $p+3 \leq k \leq 2p$ in level 1 for which the associated system of eigenvalues $\{\lambda_\ell\}$ is also a system of eigenvalues of weight $k' = k-(p-1)$. Then no linear combination of g and $g|V_{\ell_0}$ will ever have a filtration that falls down lower than k for any ℓ_0.*

Proof. Note first that since k is a filtration strictly larger than $p+1$, the operator $U = T_p$ must contract the filtration, which is easily seen to only happen in the U-nilpotent case, which means that the mod $\mathcal{P}$ eigenvalue λ_p must be 0 *(see for example Lemma 3.2 of [J2] keeping in mind that the ℓ in that paper equals our p)*.

Then since no system of eigenvalues of filtration 0 or $p+1$ will ever satisfy $\lambda_p = 0$ *(U acts bijectively on the constants and on $\widetilde{M_{p+1}}$ in level 1)* and since $\{\lambda_\ell\}$ by hypothesis is also a system of eigenvalues of weight $k' = k-(p-1)$, we must have $\{\lambda_\ell\}$ is a system of eigenvalues of filtration $4 \leq k' \leq p-1$.

This means there exists an eigenform for $\{\lambda_\ell\}$ of filtration $4 \leq k' \leq p-1$ that is in the kernel of U, and in such cases, one immediately sees by looking at the associated θ-cycle that the "partner system" $\{\ell^{p+1-k'}\lambda_\ell\}$ will always have filtration $p+3-k'$, which is also a filtration between 4 and $p-1$.

Now since g is a generalized eigenvector with system of eigenvalues $\{\lambda_\ell\}$ of filtration $k = p + k' - 1$ *(for* $4 \leq k' \leq p-1$*)*, then $\theta^{p+1-k'}(g)$ will be a generalized eigenvector of filtration $p^2 + 3p - k'p$ *(since each application of* θ *up to and including the* $(p+1-k')^{th}$ *application adds exactly* $p+1$ *to the filtration)* with system of eigenvalues $\{\ell^{p+1-k'}\lambda_\ell\}$.

But it is known that the map V maps the generalized eigenspace of filtration $p+3-k'$ with system of eigenvalues $\{\ell^{p+1-k'}\lambda_\ell\}$ ***isomorphically onto*** the generalized eigenspace of filtration $p^2+3p-k'p$ with the same system of eigenvalues[9] *(the map is clearly injective, so surjectivity can be deduced from the fact that the two generalized eigenspaces are seen to have the same dimensions - note that this argument would not work if we were not in the case where* λ_p *equaled 0 since when* λ_p *is nonzero mod* $\mathcal{P}$*, the dimensions are not always the same.)*[10]

Combining the above two paragraphs, we conclude that $\theta^{p+1-k'}(g)$ equals $h|V$ for some generalized eigenvector h of filtration $p+3-k'$ ***modulo the space of forms of lower filtrations***, which says that

$$\theta^{p+1-k'}(g) = h|V + h^* \tag{1}$$

for some h^* of filtration strictly smaller than $p^2+3p-k'p$.

But Equation (1) implies that for any linear combination $g + A_\ell g|V_\ell$, we have

$$\theta^{p+1-k'}(g + A_\ell g|V_\ell) = (h + \ell^{p+1-k'} A_\ell h|V_\ell)|V + (h^* + \ell^{p+1-k'} A_\ell h^*|V_\ell) \tag{2}$$

and we claim this equation can be used to deduce that the filtration of any linear combination of g and $g|V_\ell$ in level ℓ will always equal that of g.

More precisely, Equation (2) easily implies that the filtration of $\theta^{p+1-k'}(g + A_\ell g|V_\ell)$ in level ℓ must be the same as the filtration of $\theta^{p+1-k'}(g)$ in level 1 *(i.e. it must equal* $p^2+3p-k'p$*)* since the filtration of $(h + \ell^{p+1-k'} A_\ell h|V_\ell)|V$ would still be $p^2+3p-k'p$, as the filtration of $h + \ell^{p+1-k'} A_\ell h|V_\ell$ would still be $p+3-k'$ *(it could not fall down since* $4 \leq p+3-k' \leq p-1$ *which means there is literally no room to fall)*, and the filtration of $h^* + \ell^{p+1-k'} A_\ell h^*|V_\ell$ is still strictly smaller than $p^2+3p-k'p$.

9. By a generalized eigenspace of a certain filtration j, we mean the vector space formed by taking a generalized eigenspace of ***weight*** j and modding out by the subspace spanned by the generalized eigenforms of smaller filtration. We can identify this with the corresponding generalized eigenspace in the quotient space $\mathcal{W}_j$.

10. The proof that the dimensions are the same when $\lambda_p = 0$ is not that hard. *(Keeping in mind that the letters* p *and* ℓ *are switched in that paper)* it boils down to combining Tate and Serre's argument showing any generalized eigenspace of filtration $p+3 \leq j \leq 2p$ in level one has a semisimplification isomorphic to that of a "modified direct sum" of twists of two generalized eigenspaces of lower weights, with the result summarized in Part (1) of Lemma 6.2 in [J3] stipulating exactly when two systems of eigenvalues both of "low weights" can equal twists of each other in this $\lambda_p = 0$ case *(the proof of which follows from the shapes of the* $\theta-$ *cycles for the associated normalized eigenforms).*

But this implies the filtration of $g+A_\ell g|V_\ell$ must be the same as that of g for if it fell down to $k'=k-(p-1)$, a direct application of θ-cycles for forms of filtration $< p+1$ *(see Examples 7.6 and 7.8 of [J2] with the k there being replaced by k')* would imply that the filtration of $\theta^{p+1-k'}(g+A_\ell g|V_\ell)$ would equal either $p+3-k'$ or $2p+2-k'$, neither of which can equal $p^2+3p-k'p$.[11] $\square$

This completes the proof of Theorem 1.

11. The level in [J2] was assumed to be 1, but the exact same θ-cycles work for these weights in cases of higher level. Note also that we don't know which of these two examples applies, since we don't know if g and hence if $g+A_\ell g|V_\ell$ is in the kernel of U or not.

Section 3: Existence Result

In this Section, we provide a proof of the existence of Behrens' forms $f_{k,j'}$ as defined in Theorem 0.1. Note that Behrens' results contain a condition on the order of the zeroes at infinity of his modular forms, but it is easy to show that if there exists a form f that satisfies the other assertions of his theorem, one can always modify f if necessary, so that the bound for the order of its zero is also satisfied *(one sees this for example by using the fact that the space of modular forms of weight k for level 1 has a basis consisting of elements that are products of Eisenstein series of decreasing weights times increasing powers of* Δ *and noting that one can always raise the order of the zero at infinity of a modular form f to that satisfying Behrens' bounds by subtracting off a linear combination of those elements of this basis that have filtration* $\leq k - j'(p-1)$*, hence not changing the status of whether the condition that the filtration of* $f - f|V_\ell$ *be* $\leq k - j'(p-1)$ *is satisfied or not).*

This implies, in particular, that if one of Behrens' $f_{k,j'}$ exists for some k and j', then so does $f_{k,j'}$ for all smaller j'. Since the numbers j' as set forth in Theorem 0.1 all satisfy $1 \leq j' \leq j$ for j depending on k as defined in that Theorem *(also defined below)*, it suffices to prove the existence of his $f_{k,j'}$ in the maximal cases where $j' = j$.

Also, although Behrens just stipulates that the drop in filtration of $f_{k,j} - f_{k,j}|V_\ell$ be at least $j(p-1)$, his methods imply that the drop in filtration is precisely this amount for all ℓ, and we explicitly show the analogous precision in the drops in filtration is satisfied for each of our family of forms. Moreover, we will use this precision for the set of forms we construct here *(combined with the fact that our forms equal generalized eigenvectors for the Hecke operators)* to prove in the next Section that the drop in filtration for any modular form f of filtration k can never be more than $j(p-1)$ for any prime ℓ and such a j.[1]

We start with a Lemma that makes it easier for us to prove there exist generalized eigenvectors for which the filtration drops by a specified amount *(and no further)* upon applying linear combinations involving the V_ℓ operators. More precisely, it says that it suffices to find forms *(not necessarily generalized eigenforms)*, satisfying this specified drop in filtration that act in one crucial way the way generalized eigenvectors satisfying those same drops in filtration would have to act *(as illustrated for example by the second paragraph of Lemma 1.3)*. We only need to apply this in cases where p does not divide the filtration k, as we are able to deduce our results for general k from these.

Lemma 1. *Let* $\lambda \geq 1$*, and let f be a level 1 modular form of filtration* $k \equiv a(p+1) \bmod p^2 - 1$ *with the property that for all primes* ℓ_1 *in a set* $\mathcal{S}$ *of one or more primes* $\ell_1 \neq p$*,* $f - \ell_1^a f \,|\, V_{\ell_1}$ *has filtration equal to* $k - \lambda(p-1)$ *and has an image in* $\mathcal{W}_{k-\lambda(p-1)}(\ell_1)$ *that is a generalized eigenvector for* $T_{\ell, \ell \neq \ell_1, p}$ *with system of eigenvalues* $\{\ell^a + \ell^{a-1}\}_{\ell \neq \ell_1, p}$ *.*

Then there exists a generalized eigenform f^* *for all* T_ℓ *with* $\ell \neq p$ *in level 1 also of filtration k corresponding to the mod* $\mathcal{P}$ *system of eigenvalues* $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$*, satisfying the property that* $f^* - \ell_1^a f^* \,|\, V_{\ell_1}$ *equals* $f - \ell_1^a f \,|\, V_{\ell_1}$ *for all* $\ell_1 \in \mathcal{S}$ *modulo forms of filtration* ***strictly less*** *than* $k - \lambda(p-1)$*. In particular,* $f^* - \ell_1^a f^* \,|\, V_{\ell_1}$ *must also have precise filtration* $k - \lambda(p-1)$ *for all* $\ell_1 \in \mathcal{S}$*.*

1. In the case of $a = 0$, Behrens' methods imply that no form can drop by more than that amount for all $\ell \neq p$, although they do not bound the size of the drop for a single $\ell \neq p$.

The reader is cautioned that the fact that the drop in filtration for our forms is precisely $j(p-1)$ as well as the fact that our forms are generalized eigenforms for the Hecke operators could change in principle after one adjusts the orders of the zeroes at ∞ to meet the conditions stipulated by Behrens, although the precision in the size of the drops even after adjusting for the orders of the zeroes is guaranteed in this case by Proposition 4.2.

Proof. We first note that it follows from Theorem 2.1 that f must be an eigenvector for $T_{\ell,\ell\neq p}$ modulo forms of lower filtration with system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq p}$. But since generalized eigenvectors for distinct systems of eigenvalues must be linearly independent, hence linearly independent mod lower filtrations, this implies that when we write f as a sum of generalized eigenvectors for distinct systems of eigenvalues for the T_ℓ with $\ell\neq p$, the only one of these generalized eigenvectors having filtration k must be the one corresponding to the above system of eigenvalues *(which we denote by f^*)*.

Note next that whenever g is a generalized eigenvector for a particular system of eigenvalues, the same must be true for $g-\ell_1^a g\,|\,V_{\ell_1}$ *(not counting the Hecke operator T_{ℓ_1})*, and keep in mind that *(by hypothesis)* $f-\ell_1^a f\,|\,V_{\ell_1}$ has a nonzero image in $\mathcal{W}_{k-j(p-1)}(\ell_1)$ that is a generalized eigenvector for $T_{\ell,\ell\neq\ell_1,p}$ with system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq\ell_1,p}$.

But then since f^* is the only generalized eigenvector in the above mentioned decomposition for f corresponding to the system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq p}$, since any two mod $\mathcal{P}$ systems of eigenvalues for the Hecke operators $T_{\ell,\ell\neq p}$ in level one that agree except possibly for a finite number of primes $\ell\neq p$ must coincide[2], and since generalized eigenvectors for distinct systems of eigenvalues must be linearly independent mod lower filtrations, we conclude that $f^*-\ell_1^a f^*\,|\,V_{\ell_1}$ must have an image in $\mathcal{W}_{k-j(p-1)}(\ell_1)$ equaling that of $f-\ell_1^a f\,|\,V_{\ell_1}$. $\square$

We next present a Lemma that while true for all r will only be used in the cases where $r\leq 1$ which, among other things, allows us to prove in Lemma 3 that the filtration of a modular form f can never fall more than $p^r(p-1)$ in those cases. We will in turn make use of that result in Theorem 4 to prove the existence of ***generalized eigenforms*** for arbitrary r that satisfy a drop in filtration at least as large as the predicted maximal drop, and also in Theorem 5 to prove that the drop in filtration for these forms is ***precisely*** the predicted maximal amount. The proof of Theorem 4 also makes use of Lemma 2 directly to start off our inductive argument *(i.e. Lemma 2 gives the result in the cases r=0,1)*.

Lemma 2. *For every $k\equiv a(p+1)$ mod p^2-1 such that $ord_p(k)=r$, there exists a generalized eigenform f^* of filtration k in level 1 with system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq p}$ satisfying the property that $f^*|(identity-\ell_1^a V_{\ell_1})$ and also $f^*|(identity-\ell_1^a V_{\ell_1})(identity-\ell_2^a V_{\ell_2})$ have filtration precisely $k-p^r(p-1)$ for all primes $\ell_1,\ell_2\neq p$.*

Proof. Note first of all that it suffices to find a form $f=f_k$ *(not necessarily a generalized eigenform)* of filtration k satisfying the property about drops in filtration that we want f^* to satisfy and also satisfying the property that $f-\ell_1^a f|V_{\ell_1}$ has an image in $\mathcal{W}_{k-p^r(p-1)}(\ell_1)$ that is an eigenvector for T_ℓ, $\ell\neq\ell_1,p$ with system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq\ell_1,p}$ for all primes $\ell_1\neq p$, because then Lemma 1 above allows us to replace the form f by a generalized eigenform f^* such that $f^*|(\text{identity}-\ell_1 V_{\ell_1})$ equals $f|(\text{identity}-\ell_1 V_{\ell_1})$ for all primes ℓ_1 modulo forms of filtration strictly less than $k-p^r(p-1)$, which in turn implies that $f|(\text{identity}-\ell_1^a V_{\ell_1})(\text{identity}-\ell_2^a V_{\ell_2})$ equals $f^*|(\text{identity}-\ell_1^a V_{\ell_1})(\text{identity}-\ell_2^a V_{\ell_2})$ for all primes $\ell_1,\ell_2\neq p$ modulo forms of filtration strictly less than $k-p^r(p-1)$ *(so that in particular f^* satisfies the same drops in filtration as f)*.

2. As pointed out in a Footnote in the previous Section, one sees this for example by considering the mod $\mathcal{P}$ Galois representation associated with such systems of eigenvalues.

Note furthermore that it clearly suffices to prove this result in the case $p \nmid k$, since the result will then follow easily for general k by setting $f_k = f_{k/p^r} \,|\, V^r$ or equivalently *(since all the coefficients of the forms f in this lemma are in $\mathbb{F}_p$)* $f_k = (f_{k/p^r})^{p^r}$.

But then I claim the form $f = G_{p+1}^a$ gives us everything that we want.

To see this, note first that f has the right filtration by Lemma 3 of [C-K] *(the proof in the published version of their paper, having been refereed by the first author of this paper, is correct)*, and one can show that the filtrations of $f|(\text{identity}-\ell_1^a V_{\ell_1})$ and also of $f|(\text{identity}-\ell_1^a V_{\ell_1})(\text{identity}-\ell_2^a V_{\ell_2})$ are precisely $k-(p-1)$ for all primes $\ell_1,\ell_2 \neq p$ by combining that lemma in [C-K] with a repeated application of Lemma 7.2 in [T]

So it remains only to check that $f - \ell_1^a f \,|\, V_{\ell_1}$ is a (nonzero) eigenvector in $\mathcal{W}_{k-(p-1)}(\ell_1)$ for all primes $\ell_1 \neq p$ with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq \ell_1, p}$. But this follows easily by repeated applications of Lemma 7.2 and Fact 12 in [T]. □

Remark. It is not an accident that the forms f which we found in the above lemma could be viewed as "the" ones that correspond to G_{p+1} under the alternate approach to the "Tate-Serre" isomorphisms mentioned in Section 1 that were given by Robert in [Ro]. One probably could do something similar using the Tate-Serre maps themselves, but its justification *(while arguably in some sense simpler)* might seem more convoluted, at least at first glance.

We now use the existence of the generalized eigenforms f^* found in the previous lemma to prove that in the case where $r = \text{ord}_p(a) = 0, 1$, no modular form of filtration k can ever fall more than the "predicted maximal amount". We limit ourselves to the case where h is a generalized eigenform for the Hecke operators with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ for some a, because that is the case which we will use in this Section, but the analogous result for general modular forms *(at least in the case of $h|(identity-\ell_1^a V_{\ell_1})$)* is contained in Proposition 4.2.

Lemma 3. *Let h be a mod $\mathcal{P}$ form of filtration k in level 1 where $r = ord_p(k) = 0, 1$ that is a generalized eigenform for the Hecke operators with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ for some a. Then the filtration of $h|(identity-\ell_1^a V_{\ell_1})$ as well as the filtration of $h|(identity-\ell_1^a V_{\ell_1})(identity-\ell_2^a V_{\ell_2})$ can not fall by more than $p^r(p-1)$ for any $\ell_1, \ell_2 \neq p$. (This is no longer true for $r \geq 2$, as in those cases, we produce generalized eigenforms in Theorem 4 for which the size of the fall is greater than $p^r(p-1)$.)*

Proof. Note first that it suffices to prove the result for the filtration of $h|(identity-\ell_1^a V_{\ell_1})(identity-\ell_2^a V_{\ell_2})$, since that certainly implies the result for the filtration of $h|(identity-\ell_1^a V_{\ell 1})$.

Note also that in order for there to exist a generalized eigenvector h of filtration k for the above system of eigenvalues, we must have $k \equiv a(p+1) \bmod p^2-1$ *(this follows for example from the Remark following the statement of Theorem 1.1).*

Then let f represent the form that was denoted by f^* in Lemma 2 for such a k, so that f is a generalized eigenform of filtration $k \equiv a(p+1) \bmod p^2-1$ with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ satisfying the property that $f|(identity-\ell_1^a V_{\ell_1})(identity-\ell_2^a V_{\ell_2})$ has filtration ***precisely*** $k - p^r(p-1)$ for all primes $\ell_1, \ell_2 \neq p$.

Suppose there existed another generalized eigenform h for the T_ℓ $(\ell \neq p)$ for the same system of eigenvalues and same filtration k, with the filtration of $h|(identity - \ell_1^a V_{\ell_1})(identity - \ell_2^a V_{\ell_2})$ falling by more than $p^r(p-1)$ for some primes ℓ_1, ℓ_2.

Since the mod $\mathcal{P}$ generalized eigenspace in $\mathcal{W}_k$ for the system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ is known to be one dimensional *(as we explained, for example, in Section 1)*, it follows that if we change one of f or h by a scalar multiple if necessary, $g =_{\rm def} f - h$ must have filtration strictly smaller than k.

But then

$$g|(identity - \ell_1^a V_{\ell_1})(identity - \ell_2^a V_{\ell_2}) =$$

$$f|(identity - \ell_1^a V_{\ell_1})(identity - \ell_2^a V_{\ell_2}) - h|(identity - \ell_1^a V_{\ell_1})(identity - \ell_2^a V_{\ell_2})$$

where the filtration of $f|(identity - \ell_1^a V_{\ell_1})(identity - \ell_2^a V_{\ell_2})$ equals $k - p^r(p-1)$ and the filtration of $h|(identity - \ell_1^a V_{\ell_1})(identity - \ell_2^a V_{\ell_2})$ by hypothesis is even smaller, so the filtration of $g|(identity - \ell_1^a V_{\ell_1})(identity - \ell_2^a V_{\ell_2})$ must be precisely $k - p^r(p-1)$. In particular, this implies the filtration of g can not be less than $k - p^r(p-1)$ and hence by the previous paragraph, it must equal k' for some $k - p^r(p-1) \leq k' \leq k - (p-1)$.

But since f and h are both generalized eigenforms with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$, the same must be true of g, which implies in particular that the filtration of g must be congruent to k mod $p^2 - 1$ *(again by the Remark following the statement of Theorem 1.1, for example)*.

The Lemma is therefore proven since the fact that $r = 0$ or 1, implies that no k' between $k - p^r(p-1)$ and $k - (p-1)$ can possibly satisfy the above congruence condition mod $p^2 - 1$. $\square$

We are now ready to prove our main existence result.

As pointed out at the start of this Section, in order to prove the existence of the $f_{k,j'}$ *(using the notation introduced in Theorem 0.1)*, it suffices to find one form h of filtration k for which the filtration of $h - \ell^a h \,|\, V_\ell$ falls down by at least $j'(p-1)$ for all $\ell \neq p$ since the condition on the order of the zeroes at infinity can then easily be negotiated, and it therefore suffices to prove the existence of such h when j' equals the maximal value j set forth in the statement of Theorem 0.1.

We do in the following Theorem *(the existence of Behrens' forms follows from the case $a = 0$ since his forms all have filtrations divisible by $p^2 - 1$)*.

Theorem 4. *Let $k \equiv a(p+1)$ mod $p^2 - 1$ for $1 \leq a \leq p-1$ be a positive integer with $r = ord_p(k)$ and let $j = \begin{cases} p^r + p^{r-1} - 1 & \text{if } r \geq 1 \\ 1 & \text{if } r = 0 \end{cases}$. Then there exists a* ***generalized eigenform*** *h in level 1 for the mod $\mathcal{P}$ system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ of filtration k such that $h - \ell^a h \,|\, V_\ell$ has filtration $\leq k - j(p-1)$ for all $\ell \neq p$.*

Remark. It turns out to be surprisingly easy to upgrade the $\leq$ to ***precise equality***, which we will do in Theorem 5 below. As mentioned above, we use this equality to prove in the next section that the drop in filtration for any modular form f can never be more than $j(p-1)$ for any prime ℓ and such a j.

Proof. We prove the result by induction on $r = \mathrm{ord}_p(k)$. When $r \leq 1$, we have $j = p^r$, so the result is contained in Lemma 2 above.

We therefore assume $r = \mathrm{ord}_p(k) \geq 2$.

Then $j = p^r + p^{r-1} - 1$, and our goal is to prove the existence of a generalized eigenform h of filtration k for the system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ for which the filtration of $h - \ell^a h \mid V_\ell$ falls by at least $j(p-1) = (p^r + p^{r-1} - 1)(p-1)$ for all ℓ, bringing us to filtration $\leq k - (p^r + p^{r-1} - 1)(p-1)$.

Our inductive hypothesis implies that there exists a generalized eigenform f° for the above system of eigenvalues of filtration $\frac{k}{p}$ where the filtration of $f^\circ - \ell^a f^\circ \mid V_\ell$ falls down by at least $(p^{r-1} + p^{r-2} - 1)(p-1)$ for all $\ell \neq p$ bringing us to filtration $\leq \frac{k}{p} - (p^{r-1} + p^{r-2} - 1)(p-1)$.

Then if we let $f = f^\circ | V$ *(where $V = V_p$ multiplies the filtration by p)*, we get a generalized eigenform for the same system of eigenvalues of filtration exactly k such that the filtration of $f - \ell^a f \mid V_\ell$ falls down by at least $p(p^{r-1} + p^{r-2} - 1)(p-1) = (p^r + p^{r-1} - p)(p-1)$ for all ℓ bringing us to filtration $\leq k - (p^r + p^{r-1} - p)(p-1)$, and our goal is to show that we can lower this bound by an extra $(p-1)(p-1)$.

But Lemma 2 in the case where p does not divide the filtration implies that we also have another generalized eigenform g for the same system of eigenvalues whose filtration $k - (p^r + p^{r-1} - p)(p-1) + (p-1) = k - (p^r + p^{r-1} - p - 1)(p-1) = k - ((p+1)p^{r-1} + 1)$ is not divisible by p and is congruent to k mod $p^2 - 1$, such that the filtration of $g - \ell^a g \mid V_\ell$ falls by at least $p-1$ and hence *(by Lemma 3 above)* by exactly $p-1$, for all ℓ, bringing us again to filtration $k - (p^r + p^{r-1} - p)(p-1)$ *(this time with known equality)*.

Moreover, since f and g are both generalized eigenvectors of higher filtrations for the system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ whose filtrations fall to $k - (p^r + p^{r-1} - p)(p-1)$ for all $\ell_1 \neq p$ after applying the operator (*identity*$-V_{\ell_1}$), it follows from the second paragraph of Lemma 1.3[3] that for any prime $\ell_1 \neq p$, $f - \ell_1^a f \mid V_{\ell_1}$ and $g - \ell_1^a g \mid V_{\ell_1}$ both have images in the quotient space $\mathcal{W}_{k-(p^r+p^{r-1}-p)(p-1)}(\ell_1)$ that are eigenvectors for the Hecke operators T_ℓ for $\ell \neq \ell_1, p$ and for U_{ℓ_1} with systems of eigenvalues $\{\lambda_\ell\} = \begin{cases} \ell^a + \ell^{a-1} & \text{if } \ell \neq p, \ell_1 \\ \ell_1^{a-1} & \text{if } \ell = \ell_1 \end{cases}$, where *(by Lemma 3 above again)* at least $g - \ell_1^a g \mid V_{\ell_1}$ has a nonzero image in this quotient space.

3. As explained in a footnote in Section 1, we could avoid the need of appealing to Lemma 1.3 here if we could show directly that the entire ***generalized*** eigenspaces in the $\mathcal{W}_{k^*}(\ell_1)$ for this system of eigenvalues are one dimensional, but the natural way to do that would be to use dimension results embedded in Section 4 of [J3] that may not have been generalized to arbitrary levels, even though these generalized results are undoubtably true *(the result of Theorem 4.1 of that paper was generalized by Ash and Stevens to arbitrary levels, but their proof, at least on the surface, seems to only deal with systems of eigenvalues, not with the dimensions of the underlying generalized eigenspaces)*.

Since it is easy to show that this eigenspace is one dimensional,[4] and since the image of $g - \ell_1^a g \mid V_{\ell_1}$ is nonzero in this quotient space, we conclude that if we change g by a scalar multiple if necessary, the images of $f - \ell_1^a f \mid V_{\ell_1}$ and $g - \ell_1^a g \mid V_{\ell_1}$ must coincide in $\mathcal{W}_{k-(p^r+p^{r-1}-p)(p-1)}(\ell_1)$.

This means if we let $h = f - g$ *(after changing g by a scalar multiple if necessary)*, the form $h - \ell_1^a h|V_{\ell_1}$ will have filtration strictly smaller than $k - (p^r + p^{r-1} - p)(p-1)$.

Note that h *(like f and g)* is a generalized eigenvector in level 1 for the Hecke operators with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$, from which it immediately follows that $h - \ell_1^a h \mid V_{\ell_1}$ is a generalized eigenvector in level ℓ_1 for the $T_{\ell, \ell \neq \ell_1, p}$ and for U_{ℓ_1} with the system of eigenvalues $\{\lambda_\ell\}$ defined up above.

However, it is show in [J3] that only a very limited class of filtrations can contain nontrivial generalized eigenvectors for any given system of eigenvalues $\{\lambda_\ell\}$. In particular, *(an easily proven slight generalization of)* Theorem 6.3 of that paper fleshed out in the appendix[5] implies that any generalized eigenvector in level ℓ_1 for the system of eigenvalues $\{\lambda_\ell\}$ must have filtration congruent to $k \pm (p-1)$ mod $p^2 - 1$ *(or in cases where $\ell_1 \equiv 1$ mod p one would have to also add filtrations congruent to k mod $p^2 - 1$)*[6].

Since we know that the filtration of $h - \ell_1^a h|V_{\ell_1}$ is strictly smaller than $k - (p^r + p^{r-1} - p)(p-1) = k - (p^{r-1} - 1)(p^2 - 1) - (p-1)$, this can only happen if this filtration is less than or equal to $k - (p^{r-1})(p^2 - 1) + (p-1) = k - (p^r + p^{r-1} - 1)(p-1)$.

4. The proof is epsilon more complicated than that of the analogous fact in the case of Level 1, since the quotient spaces $\mathcal{W}_{k^*}(\ell_1)$ we are considering here are isomorphic under the Tate-Serre maps to twists of the eigenspace $\mathcal{W}_{p+3}(\ell_1)$ whose Hecke structure is related to that of weight 2 and weight $p-1$ *(whereas in the case of Level 1, the quotient spaces we were considering were all twists of the quotient space $\mathcal{W}_{p+1}$ which was isomorphic to the space $\widetilde{M_{p+1}}$ since there are no forms of weight 2 in Level 1)*. Nevertheless, it is still easy to show that the eigenspaces in question are all one dimensional.

More precisely, under the previously mentioned Tate-Serre maps, the assertion becomes equivalent to the claim that the actual eigenspace in $\mathcal{W}_{p+3}(\ell_1)$ associated with the specified twist of $\{\lambda_\ell\}$ is one dimensional, which in turn is equivalent to the assertion that the corresponding eigenspace in $\widetilde{M_{p+3}(\Gamma_0(\ell_1))}$ is one dimensional *(as it is easy to see by a direct application of θ-cycles, that no eigenform of filtration 4 can ever belong to such a twist of $\{\lambda_\ell\}$)*. Moreover, this eigenspace in $\widetilde{M_{p+3}(\Gamma_0(\ell_1))}$ is clearly one dimensional since nothing of weight $p+3$ can be in the image of V.

5. That theorem is stated for the full modular group, but as noted in that paper, generalizes to other levels *(modulo some simple modifications that must be taken into account in the current case since when $\ell \equiv 1$ mod p there exists a generalized eigenform of filtration $p+1$ for which the corresponding actual eigenform has filtration 2, as we explain in the following footnote)*.

Note the numbers for level ℓ are different than the analogous ones for level 1 since we have non-constant eigenforms of filtration $\leq p-1$ in level ℓ whose system of eigenvalues are twists of $\{\lambda_\ell\}$. This fact is responsible for making all our arguments work.

6. The extra addition in the case $\ell_1 \equiv 1$ mod p is due to the fact that whereas in general, $G_{p+1} - \ell_1 G_{p+1}|V_{\ell_1}$ is an eigenvector for U_{ℓ_1} with eigenvalue 1 of filtration 2 and $G_{p+1} - G_{p+1}|V_{\ell_1}$ is an eigenvector for U_{ℓ_1} with eigenvalue ℓ_1 *(which in the case $\ell_1 \not\equiv 1$ mod p has filtration $p+1$)*, when $\ell_1 \equiv 1$ mod p, these two forms are congruent mod p, and by subtracting them from each other and dividing by $p^{\mathrm{ord}_p(\ell_1 - 1)}$, one obtains a generalized eigenvector with the same system of eigenvalues of filtration $p+1$, for which the actual eigenvector has filtration 2.

Another way of saying this is that in the case where $\ell_1 \equiv 1$ mod p, the system of eigenvalues $\{\lambda_\ell\} = \begin{cases} \{1+\ell^{-1}\} & \text{if } \ell \neq p, \ell_1 \\ \ell_1^{-1} & \text{if } \ell = \ell_1 \end{cases}$ of filtration 2 and the system of eigenvalues $\{\lambda'_\ell\} = \begin{cases} \{1+\ell^{-1}\} & \text{if } \ell \neq p, \ell_1 \\ 1 & \text{if } \ell = \ell_1 \end{cases}$ of filtration $p+1$ are the same, where when we say that a system of eigenvalues $\{\mu_\ell\}$ has filtration k^*, we mean that $\mathcal{W}_{k^*}(\ell_1)$ has a nonzero eigenspace corresponding to $\{\mu_\ell\}$, or equivalently that $\widetilde{M_{k^*}(\Gamma_0(\ell_1))}$ has a ***generalized*** eigenform corresponding to $\{\mu_\ell\}$ having filtration k^*. Note that this doesn't necessarily mean that $\widetilde{M_{k^*}(\Gamma_0(\ell_1))}$ has an actual eigenform corresponding to $\{\mu_\ell\}$ of filtration k^*, as the actual eigenform corresponding to $\{\mu_\ell\}$ could have lower filtration, which is what happens in the case we are considering here.

We can do this for any prime $\ell_1 \neq p$, although in theory the scalar multiple we used to change the form g up above could have depended on the choice of the prime ℓ_1, which means that the form h could have depended on ℓ_1 as well.

To complete our proof, we must show that the same form h works for all primes ℓ.

However, if this were not the case, we would have two linear combinations $f + A_1 g$ and $f + A_2 g$ where $A_1 \neq A_2$ that fall to filtration $k - (p^r + p^{r-1} - 1)(p-1)$ for primes ℓ_1 and ℓ_2 respectively. But then both of them must fall to at least this filtration when one applies the composition (identity$-\ell_1^a V_{\ell_1}$)(identity$-\ell_2^a V_{\ell_2}$).

Subtracting one gets that $(A_1 - A_2)g$ must fall to at least this filtration when one applies the above mentioned composition of operators, but since g has filtration not divisible by p, this contradicts Lemma 3 above. $\square$

We end this section, by proving that the drop in filtration for the forms found in the previous theorem is precisely equal to the bound found in that theorem.

Theorem 5. *The result of Theorem 4 can be upgraded to say that $h - \ell^a h \,|\, V_\ell$ has* ***precise*** *filtration $k - j(p-1)$ for all $\ell \neq p$.*

Proof. Since Theorem 4 says that $h - \ell^a h \,|\, V_\ell$ has filtration $\leq k - j(p-1)$ for all $\ell \neq p$, it suffices to prove the reverse inequality, in other words it suffices to show that the filtration of $h - \ell^a h \,|\, V_\ell$ must be greater than or equal to $k - j(p-1)$.

This follows immediately from Lemma 3 and the definition of j in the cases where $r = 0$ or 1, so we assume that $r \geq 2$.

Since $h = f - g = f^\circ|V - g$ *(keeping the notation we used in the previous proof)*, it follows that $\theta(h) = -\theta(g)$. Moreover, since g has filtration $w(g)$ which was not divisible by p, this implies that the filtration of $\theta(h)$ equals $w(g) + p + 1$.

Let us call this filtration k', and note that a trivial computation shows that $r' =_{def.} \mathrm{ord}_p(k') = 1$.

Then since $\theta(h)$ is a generalized eigenform with system of eigenvalues $\{\ell^{a+1} + \ell^a\}_{\ell \neq p}$, it follows immediately from Lemma 3 in the case $r' = 1$, that $\theta(h) - \ell^{a+1}\theta(h)|V_\ell$ has filtration $\geq k' - p(p-1)$ for all ℓ.

But $\theta(h) - \ell^{a+1}\theta(h)|V_\ell = \theta(h - \ell^a h|V_\ell)$, which means $\theta(h - \ell^a h|V_\ell)$ has filtration $\geq k' - p(p-1)$.

Since $k' - p(p-1)$ is easily seen to equal $k - j(p-1) + (p+1)$, and since an application of θ can increase the filtration by at most $p+1$, this implies that the filtration of $h - \ell^a h|V_\ell$ must be greater than or equal to $k - j(p-1)$ as desired. $\square$

Section 4: Bound for Size of Maximal Drop in Filtration

In this Section, we show that the drop in filtration of $j(p-1)$ found for the forms appearing in Theorem 3.5 equals the maximal amount by which the filtration of any linear combination of h and $h|V_\ell$ can fall for any level one form h of filtration k ***and any prime $\ell \neq p$***.

We first prove a simple lemma that could be viewed as a variant of Lemma 3.1, although we limit it here to the case of a single prime ℓ_0.

Lemma 1. *Let $\lambda \geq 0$. Let g^* be a level 1 modular form of filtration k^* with the property that there exists a prime ℓ_0 for which $g^* - \ell_0^a g^*|V_{\ell_0}$ has filtration equal to $k^* - \lambda(p-1)$ and has an image in $\mathcal{W}_{k^*-\lambda(p-1)}(\ell_0)$ that is a generalized eigenvector for $T_{\ell,\ell\neq\ell_0,p}$ with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell\neq\ell_0,p}$.*

Then λ is strictly greater than 0 and there exists a level 1 generalized eigenform g for all T_ℓ with $\ell \neq p$ corresponding to the mod $\mathcal{P}$ system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell\neq p}$ of filtration $k' \equiv a(p+1)$ mod p^2-1 such that $k^* - (\lambda-1)(p-1) \leq k' \leq k^*$, satisfying the property that $g - \ell_0^a g|V_{\ell_0}$ equals $g^* - \ell_0^a g^*|V_{\ell_0}$ modulo forms of filtration ***strictly less*** *than $k^* - \lambda(p-1)$ (so that in particular, $g - \ell_0^a g|V_{\ell_0}$ drops to the same filtration as $g^* - \ell_0^a g^*|V_{\ell_0}$).*

Remark. The proof is essentially the same as that of Lemma 3.1 except that since we do not know the congruence class of k^* mod p^2-1, the linear combination for which the filtration falls for one ℓ_0 does not determine the one for which the filtration falls for all ℓ, and therefore the argument used to conclude that g^* has the same filtration as g in the proof of Lemma 3.1 does not work here *(for an example where g^* and g have different filtrations, see the Example given in Section 6 although to make it work, you will have to choose "a" in the Lemma to equal "$a'=3$" in the Example, "g^*" in the Lemma to equal "g" in the Example–so that "k^*" in the Lemma equals 30, "g" in the Lemma to equal "g_2" in the Example–which means "k'" in the Lemma equals 18, and apply it to a prime ℓ_0 that is a quadratic residue with respect to $p=5$).*

Proof. We can write g^* as a sum of generalized eigenvectors for distinct systems of eigenvalues for the T_ℓ with $\ell \neq p$. When we pass from g^* to $g^* - \ell_0^a g^*|V_{\ell_0}$, we end up with a form of precise filtration $k^* - \lambda(p-1)$ that is a simultaneous eigenvector for $T_{\ell,\ell\neq\ell_0,p}$ modulo forms of lower filtration with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell\neq\ell_0,p}$. This implies in particular that there must be one generalized eigenvector g in the above mentioned decomposition of g^* of filtration k' between k^* and $k^* - \lambda(p-1)$ corresponding to the system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell\neq\ell_0,p}$ satisfying the property that $g - \ell_0^a g|V_{\ell_0}$ has an image in $\mathcal{W}_{k^*-\lambda(p-1)}(\ell_0)$ equaling that of $g^* - \ell_0^a g^*|V_{\ell_0}$.

Moreover, since any generalized eigenvector in level 1 for the system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell\neq\ell_0,p}$ must also be a generalized eigenvector for T_{ℓ_0} with the eigenvalue $\ell_0^a + \ell_0^{a-1}$,[1] it follows that g must be a generalized eigenvector for all the Hecke operators T_ℓ, $\ell \neq p$ for the system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell\neq p}$.

1. This follows immediately from the fact that if two mod $\mathcal{P}$ systems of eigenvalues for the Hecke operators T_ℓ, $\ell \neq p$ in level one agree except possibly for a finite number of primes $\ell \neq p$, they must coincide, which (as pointed out in footnotes in Sections 2 and 3) one sees for example by considering the mod $\mathcal{P}$ Galois representations associated to such systems of eigenvalues.

Finally note that it follows from the Remark following Theorem 1.1 that if $k'=w(g)$, then $k'\equiv a(p+1)$ mod p^2-1, and also it follows from Theorem 1.1, that k' *(hence also* k^**)* must be strictly bigger than $k-\lambda(p-1)$ so that in particular $\lambda\neq 0$. $\square$

We are now in a position to show that the maximal amount that the filtration of a linear combination of h and $h|V_\ell$ can fall for any form h of filtration k and any prime ℓ is $j(p-1)$ for j as defined in the previous section. We limit ourselves to linear combinations of the form $h-\ell^a h|V_\ell$ for $k\equiv a(p+1)$ mod p^2-1 as we have shown in Section 2 that these are the only filtrations k and *(up to scalar multiple)* the only linear combinations of h and $h|V_\ell$ for which the filtration can slip down even for a single prime ℓ.

Proposition 2. *If h is a level 1 form of filtration $k\equiv a(p+1)$ mod p^2-1 with $r=ord_p(k)$, then the filtration of $h-\ell^a h\,|\,V_\ell$ can never be less than $k-j(p-1)$ for any prime $\ell\neq p$, where $j=\begin{cases} p^r+p^{r-1}-1 & \text{if } r\geq 1\\ 1 & \text{if } r=0\end{cases}$ as in Section 3 (so that $j(p-1)$ is the maximal fall for any ℓ).*

Proof. We start with the generalized eigenform f for system of eigenvalues $\{\ell^a+\ell^{a-1}\}_{\ell\neq p}$ of filtration k found in Theorems 3.4 and 3.5 such that $f-\ell^a f\,|\,V_\ell$ has filtration ***precisely*** $k-j(p-1)$ for all $\ell\neq p$.

Suppose there existed another form h of filtration k with the filtration of $h-\ell_0^a h|V_{\ell_0}$ strictly smaller than $k-j(p-1)$ for some particular ℓ_0. Since f and h have the same filtration k with the filtrations of $f-\ell_0^a f\,|\,V_{\ell_0}$ and $h-\ell_0^a h\,|\,V_{\ell_0}$ both lower than k, it follows from Corollary 1.5 that if we change one of f or h by a scalar multiple if necessary, $g^*=_{\text{def}} f-h$ must have filtration strictly smaller than k.

Then $g^*-\ell_0^a g^*\,|\,V_{\ell_0}=(f-\ell_0^a f\,|\,V_{\ell_0})-(h-\ell_0^a h|V_{\ell_0})$ where the filtration of $f-\ell_0^a f\,|\,V_{\ell_0}$ equals $k-j(p-1)$ precisely and the filtration of $h-\ell_0^a h|V_{\ell_0}$ by hypothesis is even smaller, and so $g^*-\ell_0^a g^*\,|\,V_{\ell_0}$ must itself have precise filtration $k-j(p-1)$. In particular, this implies that the filtration of g^* can not be less than $k-j(p-1)$, and hence by the above paragraph, it must equal k^* for some $k-j(p-1)\leq k^*\leq k-(p-1)$.

Moreover, since $(f-\ell_0^a f\,|\,V_{\ell_0})-(g^*-\ell_0^a g^*|V_{\ell_0})=h-\ell_0^a h|V_{\ell_0}$, which was assumed to have filtration strictly smaller than $k-j(p-1)$, then the images of $f-\ell_0^a f\,|\,V_{\ell_0}$ and $g^*-\ell_0^a g^*\,|\,V_{\ell_0})$ must coincide in $\mathcal{W}_{k-j(p-1)}(\ell_0)$. But since f was chosen to be a generalized eigenform for the T_ℓ for the above system of eigenvalues, it follows immediately that $f-\ell_0^a f\,|\,V_{\ell_0}$ *(and hence also* $g^*-\ell_0^a g^*|V_{\ell_0}$*)* has an image in $\mathcal{W}_{k-j(p-1)}(\ell_0)$ that is a generalized eigenvector for this system of eigenvalues *(omitting the eigenvalue for the prime* ℓ_0*)*.

Lemma 1 then implies that there exists a generalized eigenvector g for the above system of eigenvalues of filtration $k'\equiv a(p+1)$ mod p^2-1 with $k-j(p-1)\leq k'\leq k-(p-1)$ such that $g-\ell_0 g|V_{\ell_0}$ has the same filtration as $g^*-\ell_0 g^*|V_{\ell_0}$.[2]

From now on, the only things that we need to know about g are that it is a form for $SL_2(\mathbb{Z})$ of filtration $k'\equiv a(p+1)\equiv k$ mod p^2-1 with $k-j(p-1)\leq k'\leq k-(p-1)$ such that $g-\ell_0 g|V_{\ell_0}$ has filtration $k-j(p-1)$ for one particular choice of ℓ_0.

2. Actually, Lemma 1 implies the stronger statement that $k-(j-1)(p-1)\leq k'\leq k-(p-1)$ but we don't gain anything by pointing this out in the text of our proof, since the possibility that $k'=k-j(p-1)$ is eliminated anyway along with other filtrations in what follows, and eliminating this possibility at this point would force us to treat the case where $r=1$ separately from the case where $r=0$.

It remains to show that this leads to a contradiction.

Note in particular that when $r=0$ or 1, no k' between $k-j(p-1)$ and $k-(p-1)$ can possibly satisfy the above congruence condition mod p^2-1, so the Theorem is proven in those cases.[3]

We therefore assume that $r\geq 2$. Even though the above congruence condition mod p^2-1 no longer yields an immediate contradiction, an inductive argument *(first noticed by the second author)* allows us to get the contradiction we want in these cases. Although the numbers get a tiny bit messy, it is easy to explain the main idea, namely we show $r'=\mathrm{ord}_p k'$ is strictly smaller than $r=\mathrm{ord}_p k$, so our inductive hypothesis bounds the amount that the filtration of $g-\ell g|V_\ell$ can fall for any ℓ, and we get a contradiction by demonstrating that this bound is not large enough to allow $g-\ell_0 g|V_{\ell_0}$ to reach its stipulated filtration *(for that one particular ℓ_0)* of $k-j(p-1)$.

More precisely, since the two extremes of the inequality $k-j(p-1)\leq k'=w(g)\leq k-(p-1)$ do not satisfy the congruence restriction $k'\equiv k \bmod p^2-1$ mentioned above, we will never have k' equaling either extreme, and in fact we can refine this inequality taking this congruence relation into account to get $k-(j-p)(p-1)\leq k'=w(g)\leq k-(p^2-1)$.

This says that $1\leq\frac{k-k'}{p^2-1}\leq\frac{(j-p)(p-1)}{p^2-1}=\frac{(p^r+p^{r-1}-p-1)(p-1)}{p^2-1}=\frac{(p^{r-1}-1)(p^2-1)}{p^2-1}=p^{r-1}-1$, which implies $0\leq\mathrm{ord}_p(k')\leq\mathrm{ord}_p(k)-2=r-2$.

But then, since $r'=_{def.}\mathrm{ord}_p(k')<r$, we know by induction that the most the filtration of $g-\ell^a g|V_\ell$ could possibly fall for any ℓ (including ℓ_0), starting from $w(g)=k'$, is $j'(p-1)$ *(where j' is defined in a manner analogous to j using r' instead of r, i.e. $j'=\begin{cases} p^{r'}+p^{r'-1}-1 & \text{if } r'\geq 1\\ 1 & \text{if } r'=0\end{cases}$).*

Let us now let b denote $\frac{k-k'}{p^2-1}$, so that $w(g)=k'=k-b(p^2-1)$ where by above, we have

$$1\leq b\leq p^{r-1}-1 \tag{1}$$

Since the previous paragraph implies that the most the filtration of $g-\ell^a g|V_\ell$ can fall is $j'(p-1)$, this says that

$$w_\ell(g-\ell^a g|V_\ell)\geq k'-j'(p-1)=k-b(p^2-1)-j'(p-1)=k-[b(p+1)+j'](p-1) \tag{2}$$

for all ℓ.

On the other hand, we also know from above that $w_{\ell_0}(g-\ell_0^a g|V_{\ell_0})=k-j(p-1)$ for one particular ℓ_0. We will get a contradiction by showing

$$b(p+1)+j'<j \tag{3}$$

which when combined with (2) says the amount that the filtration of $g-\ell^a g|V_\ell$ could possibly fall would never be large enough to reach the known filtration of $k-j(p-1)$ for $g-\ell_0^a g|V_{\ell_0}$ for that one ℓ_0.

It thus remains only to prove inequality (3).

3. This is the same argument used in the proof of Lemma 3.3, and since g' is a generalized eigenvector, we could have just quoted that Lemma here.

To do this, we first note that we can improve on inequality (1) in the general case. More precisely, inequality (1) is equivalent to the fact that

$$k-(p^{r-1}-1)(p^2-1)\leq k'\leq k-(p^2-1)$$

but *(except in the case* $\mathrm{ord}_p(k')=r'=0$*)* neither of the ends of this inequality can possibly equal k' since they are not divisible by the right power of p. Then taking into account that $\mathrm{ord}_p(k')=r'\leq r-2$ and that $k'\equiv k \bmod p^2-1$, it is easy to see that we can refine this inequality in the general case *(where this refinement changes nothing in the case* $r'=0$*)* to get

$$k-(\,p^{r-1}-p^{r'})(p^2-1)\leq k'\leq k-p^{r'}(p^2-1),$$

or equivalently

$$p^{r'}\leq b=\frac{k-k'}{p^2-1}\leq p^{r-1}-p^{r'},$$

and since $r\geq 2$ hence also ≥ 1 means that $j+1=p^{r-1}(p+1)$, we can rewrite the right-hand inequality if we want as

$$b\leq\frac{j+1}{p+1}-p^{r'}.$$

That means we have

$$b(p+1)+j'\leq j+1-p^{r'}(p+1)+j'=j+[j'+1]-[p^{r'+1}+p^{r'}],$$

hence to prove inequality (3) and complete the proof it suffices to check that

$$p^{r'+1}+p^{r'}>j'+1.$$

But this is immediate since $j'+1=\begin{cases}p^{r'}+p^{r'-1} & \text{if } r'\geq 1\\ 2 & \text{if } r'=0\end{cases}$, so we are done. □

As an immediate Corollary we get the following result, which deals with the case where $p\nmid k$.

Corollary 3. *Let h be a form of filtration k, where $r=ord_p(k)=0$ or equivalently where $j=1$, such that the filtration of $h-\ell_0^a h\,|\,V_{\ell_0}$ falls by at least $p-1$ for one $\ell_0\neq p$, then the filtration of $h-\ell^a h\,|\,V_\ell$ must fall by exactly $p-1$ for all ℓ.*

Proof. Theorem 2.1, says if the filtration of $h-\ell_0^a h\,|\,V_{\ell_0}$ falls by at least $p-1$ for one $\ell_0\neq p$, then the filtration of $h-\ell^a h\,|\,V_\ell$ must fall by at least $p-1$ for all ℓ. But the above Proposition says the fall can never be more than $p-1$ for such k. □

The proof of the above Proposition actually gives us more than the Proposition itself, in that the argument used to eliminate the possibility of the existence of a form g of filtration k' satisfying $k'\equiv k \bmod p^2-1$ with $k-j(p-1)\leq k'\leq k-(p-1)$ such that $g-\ell_0 g|V_{\ell_0}$ has filtration $k-j(p-1)$ for one ℓ_0 actually shows that in cases where such numbers k' exist, the filtration of $g-\ell_0 g|V_{\ell_0}$ must have filtration $\geq k-(j-(p-1))(p-1)$. Moreover, if we loosen the congruence equating k' to $k \bmod p^2-1$ to the analogous congruence mod $\frac{p^2-1}{2}$, and go through the step by step analogous argument, we can still obtain the following result, which will become useful to us in Sections 5 and 6.

Lemma 4. *Let $k\equiv a(p+1) \bmod p^2-1$ with $r=\mathrm{ord}_p(k)$, let j be as above, and let $k'\equiv k \bmod \frac{p^2-1}{2}$ such that $k-j(p-1)\leq k'\leq k-(p-1)$. Then if $r=0$ or equivalently if $j=1$, no such k' can ever exist, whereas if $r\geq 1$, the farthest any level one form g of filtration k' can possibly fall is to filtration $\geq k-\left(j-\frac{p-3}{2}\right)(p-1)$.*

Section 5: Uniqueness of Behrens' Forms Plus Improving Theorem 1.4 of [B]

Recall that in Section 2 we used Serre's now proven modularity conjecture to prove that if the filtration of some linear combination of h and $h|V_{\ell_0}$ falls down *(at least minimally)* for one $\ell_0 \neq p$, the filtration of an "analogous" linear combination must fall down *(at least minimally)* for all ℓ. The main purpose of this section is to extend this result to one that takes into account the size of the falls, while at the same time improving on Behrens' Theorem 1.4 ([B]) that we discussed in the Introduction to this paper.

In the process of doing this, we prove that any two forms of filtration k for which the filtration drops by at least $j'(p-1)$ for one prime ℓ_0 in cases of "large" or "small" j' must differ *(up to scalar multiplication)* by a form of filtration $\leq k - j'(p-1)$ and combine that with Behrens' condition on the order of the zeroes at ∞ to deduce the uniqueness of his forms in cases of such j'.[1] Although we could have proven this uniqueness without using our Serre's conjecture results *(since we know that Behrens' forms have a filtration that falls for **all** $\ell \neq p$)*, we would not have been able to prove the stronger results alluded to in the previous paragraph.

More precisely, we prove in Corollary 2 below that if the filtration of some linear combination of h and $h|V_{\ell_0}$ falls down by at least $j'(p-1)$ for one $\ell_0 \neq p$ in cases of "large" or "small" j', the filtration of the "analogous" linear combination of h and $h|V_\ell$ must fall down by at least that amount for all primes ℓ *(where the definition of "analogous" linear combination depends on the congruence properties of the filtration of h mod p^2-1)*. Moreover, this immediately implies that the size of the fall is uniform for all ℓ in the case of "large" falls, with an analogous result in the case "small" falls proven in Section 6.

Among other things, this gives a significant strengthening of Theorem 1.4 of [B] in cases of "large" and "small" j' *(with the comparable strengthening for $j'=1$ having been already obtained in Theorem 2.1 and a slightly weaker strengthening in the case of arbitrary j' contained in Corollary 6.2)*, in which Behrens obtains a somewhat similar result but puts restrictions on the choice of the prime ℓ_0 *(he insists it be a topological generator of the p-adic units, which if you limit his results to mod p instead of mod powers of p is equivalent to it being a generator for $\mathbb{Z}_p^*$)* and also does not conclude that the sizes in the falls in filtration must be uniform for all ℓ.

We start with our uniqueness results.

Proposition 1. *Let $k \equiv a(p+1)$ mod p^2-1 and, continue, as in Sections 3 and 4, to let $j = \begin{cases} p^r + p^{r-1} - 1 & \text{if } r \geq 1 \\ 1 & \text{if } r = 0 \end{cases}$ for $r = ord_p(k)$. Then any two forms h and h' of filtration k for which the filtration of $h - \ell_0^a h \,|\, V_{\ell_0}$ and $h' - \ell_0^a h' \,|\, V_{\ell_0}$ both drop by at least $j'(p-1)$ for **at least one** $\ell_0 \neq p$ where $1 \leq j' \leq \frac{p+1}{2}$ or $j' \geq j - \frac{p-5}{2} > 1$ must (after changing one of them by a scalar multiple if necessary) differ by a form of filtration $\leq k - j'(p-1)$.*

When combined with the condition on the zeroes at infinity, this implies that the forms $f_{k,j'}$ appearing in Theorem 0.1 are unique (up to scalar multiples) for such j' (as one immediately sees by noting that all forms of filtration $\leq k - j'(p-1)$ must have zeroes of orders strictly smaller than those in Behrens' bounds).

1. This uniqueness result can ***not*** be extended to arbitrary j' *(as illustrated by the Example following Proposition 6.3 and the subsequent Remark)*.

Remark. *1)* Note in particular that if $r \le 1$ the above conditions on j' must always be satisfied for all j' as in Theorem 0.1, except in the case where $r = 1$ and $j' = \frac{p+3}{2}$, so this implies that $f_{k,j'}$ must be unique for such k except when $j' = \frac{p+3}{2}$.

2) Although the first paragraph of this Proposition can be significantly strengthened in cases when h and h' are both generalized eigenforms of filtration k, it is noteworthy that the result is false in the case of arbitrary j' even if one restricts to generalized eigenforms. This makes it all the more intriguing that Corollary 3 which is proven using this result remains true for arbitrary j' in the case of generalized eigenforms *(with a weaker variant remaining true if one omits the generalized eigenform hypothesis)*. For relevant proofs and counterexamples, see Section 6.

Proof. Recall Corollary 1.5 says that whenever the filtrations of two level one forms of the same filtration k both drop *(even minimally)* for at least one $\ell_0 \neq p$ they must differ *(up to a scalar multiples)* by a form of filtration ***strictly*** less than k.

Therefore *(after changing one of h or h' by a scalar multiple if necessary)*, $g =_{\text{def.}} h - h'$ must have filtration k' for some $k' < k$. Moreover, $g - \ell_0^a g \,|\, V_{\ell_1} = (h - \ell_0^a h \,|\, V_{\ell_0}) - (h' - \ell_0^a h' | V_{\ell_0})$ must have filtration $\le\ k - j'(p-1)$ *(since both of the forms inside parentheses on the right-hand side of this equation have this property)*. To prove the Proposition we must show that g itself has filtration $\le\ k - j'(p-1)$.

We thus assume this is not the case, which means we assume that $k - (j'-1)(p-1) \le k' \le k - (p-1)$, where k' as above equals the filtration of g, which implies in particular that $g - \ell_0^a g \,|\, V_{\ell_0}$ has a filtration lower than the filtration of g.

Then Theorem 2.1 implies that k' must be divisible by $p+1$ and hence $k' \equiv k \bmod p+1$, but since g by definition equals the difference of two forms of filtration k, we know that k' must also be congruent to k mod $p-1$, and the only way that both these conditions can be satisfied is if $k' \equiv k \bmod \frac{p^2-1}{2}$.

The fact that k' is strictly smaller than k then implies that $k' \le\ k - \frac{p^2-1}{2}$, which contradicts our assumption that k' is strictly bigger than $k - j'(p-1)$ in the case where $1 \le j' \le\ \frac{p+1}{2}$.

That leaves us with the case where $j' \ge j - \frac{p-5}{2} > 1$, but this case can also be easily ruled out using the above mentioned fact that $g - \ell_0^a g \,|\, V_{\ell_1}$ must have filtration $\le\ k - j'(p-1)$, since Proposition 4.2 implies that $j' \le j$, which when combined with the above constraints on k' gives $k - (j-1)(p-1) \le k' \le k - (p-1)$ and $k' \equiv k \bmod \frac{p^2-1}{2}$, so that Lemma 4.4 asserts that the farthest the filtration of $g - \ell_0^a g \,|\, V_{\ell_0}$ can fall is to $k - \Big(j - \frac{p-3}{2}\Big)(p-1)$. $\square$

We are now able to present a Corollary that extends the main result contained in Section 2 to one that takes the size of the drops into account. Note that among other things, as mentioned at the start of this Section, this would seem to ***significantly improve*** Theorem 1.4 of [B], in the cases of "large" or "small" j' *(with arbitrary j' addressed in the following Section)*.

2

Corollary 2. ***(Improvement of Theorem 1.4 of [B] in cases of "large" and "small" j')*** *Let h be a form of filtration $k \equiv a(p+1)$ mod p^2-1 with $r = ord_p(k)$ and let j be as defined above. If the filtration of $h - \ell_0^a h \mid V_{\ell_0}$ slips down by at least $j'(p-1)$ for one $\ell_0 \neq p$ where $1 \leq j' \leq \frac{p+1}{2}$ or $j' \geq j - \frac{p-5}{2} > 1$, then the filtration of $h - \ell^a h \mid V_\ell$ will slip down by at least $j'(p-1)$ for all $\ell \neq p$.*

Proof. Suppose there exists a form h of filtration k for which the filtration of $h - \ell_0^a h \mid V_{\ell_0}$ slips down by at least $j'(p-1)$ for one ℓ_0.

We know there exists a form f such that the filtration of $f - \ell^a f \mid V_\ell$ slips down by at least $j'(p-1)$ for all $\ell \neq p$ *(for example take the form found in Theorem 3.5 where the filtration falls down by the maximal amount $j(p-1)$ for all ℓ)*[2]. We are clearly done if $f = h$, so assume this is not the case.

Then Proposition 1 implies that *(after replacing one of them by a scalar multiple if necessary)* f and h differ by a form $g =_{\text{def.}} f - h$ of filtration $\leq k - j'(p-1)$. This implies in particular, that the filtration of $g - \ell^a g | V_\ell$ for arbitrary ℓ must also be $\leq k - j'(p-1)$.

But then $h - \ell^a h \mid V_\ell = (f - \ell^a f \mid V_\ell) - (g - \ell^a g | V_\ell)$ where the filtrations of both forms inside parentheses on the right-hand side are $\leq k - j'(p-1)$ for all $\ell \neq p$, which implies that the filtration of $h - \ell^a h \mid V_\ell$ must also have this property. □

Moreover the above Corollary when applied to varying j' allows us to immediately conclude that if the filtration of $f - \ell_0^a f | V_{\ell_0}$ falls by a "large enough" amount for one prime ℓ_0, the filtration of $f - \ell^a f | V_\ell$ must fall by ***precisely*** the same amount for all primes ℓ.

The analogous result for "small enough" falls will be obtained in the next Section together with results about how much uniformity there is for different ℓ in the general case. Note that this result was already proven in Corollary 4.3 in the special case where $p \nmid k$ *(or equivalently where $j = 1$)*.

Thus we have

Corollary 3. *Keep the notation of the previous Corollary. Then if the filtration of $h - \ell_0^a h \mid V_{\ell_0}$ slips down by precisely $j'(p-1)$ for $j' = j = 1$ or $j' \geq j - \frac{p-5}{2} > 1$ for one $\ell_0 \neq p$, the filtration of $h - \ell^a h \mid V_\ell$ must slip down by precisely $j'(p-1)$ for all $\ell \neq p$. In particular this must be true in all cases where $j' = j$ (i.e. in all cases of maximal drops).*

Some Comments Relating this to Theorem 1.4 of [B]: It seems that to completely understand the import of Behrens' Theorem 1.4, one should view it within the more general context set up in this Section.

Note that Behrens' results were constrained by the fact that he was only considering linear combinations $h - h \mid V_\ell$, instead of considering more general linear combinations of the form $h - \ell^a h | V_\ell$ for filtrations $k \equiv a(p+1)$ mod $p^2 - 1$ as we did above.

2. Although the existence of such a form was proven in Theorem 3.5, the fact that this drop was the maximal amount possible was not proven until Proposition 4.2.

Although we have proven in Theorem 2.1 that the filtration of a linear combination of h and $h|V_{\ell_1}$ can only fall in cases where $k \equiv a(p+1) \bmod p^2-1$ in which case the linear combination must equal $h - \ell_0^a h \,|\, V_{\ell_0}$, the reader is cautioned that this it is not the same as saying that the only time the filtration of $h - \ell_0^a h \,|\, V_{\ell_0}$ falls is in cases where $k \equiv a(p+1) \bmod p^2-1$ *(as ℓ_0^a for a single prime ℓ_0 may not uniquely determine the congruence class a mod $p-1$).*

Thus, if one only considers the linear combination $h - h \,|\, V_\ell$, the filtration could fall for arbitrary primes $\ell \neq p$ only in cases where $a \equiv 0 \bmod p-1$ *(in which case k would be divisible by p^2-1)*, but one could have the filtration of $h - h \,|\, V_{\ell_0}$ fall for some ℓ_0 for other choices of a without it falling for arbitrary ℓ as long as $\ell_0^a \equiv 1 \bmod p$, for that particular choice of ℓ_0 and a, *(where in such a case, the filtration of $h - h \,|\, V_\ell$ would also have to fall for all ℓ satisfying $\ell^a \equiv 1$ mod p).*

This is probably related to why Behrens proved his mod p result in the above mentioned Theorem under the limiting hypothesis that ℓ_0 generate $(\mathbb{Z}/p\mathbb{Z})^*$ because those are the cases where ℓ_0^a does uniquely determine the congruence class of a mod $p-1$, so one could not have the filtration of $h - h \,|\, V_{\ell_0}$ fall for such ℓ_0 unless a was 0 mod $p-1$ *(or equivalently, the filtration k was divisible by p^2-1)*, in which case, the filtration would fall for all $\ell \neq p$. But if one knew in advance that the filtration of k was divisible by p^2-1, there would be no need to put any restrictions on the choice of ℓ_0, whereas if one knew in advance that k was ***not*** divisible by p^2-1, then it would never be the case that the filtration of $h - h \,|\, V_{\ell_0}$ could fall for primes ℓ_0 satisfying his restriction.

Note that since Behrens limited himself to forms of filtration divisible by $p-1$ and since we have proven that if the filtration k of f is not of the form $a(p+1)$, no linear combination of f and $f|V_{\ell_0}$ can ever have lower filtration than f for even one prime $\ell_0 \neq p$, there are only two types of filtrations that he has to consider, namely those where $k = i(p^2-1)$ and those where $k = \frac{p^2-1}{2} + i(p^2-1)$.[3]

In the first case, there will exist forms f for which the filtration of $f - f|V_\ell$ falls for all primes $\ell \neq p$, and in the second case, there will exist forms f for which the filtration of $f - f \,|\, V_\ell$ falls for all quadratic residues ℓ with respect to p while the filtration of $f + f|V_\ell$ falls for all quadratic non-residues ℓ. In either case, it suffices to know the filtration falls for one $\ell_0 \neq p$ *without any restriction on the prime ℓ_0* in order to conclude that it falls for all primes $\ell \neq p$, except one has to have the right $\pm$ sign in the linear combinations in order for the phenomenon to occur at all.

If we want to restrict to considering the linear combination $f - f|V_\ell$ and do not impose any preconditions on the filtration k, it seems like the most general result we would have is the following. We state this in terms of drops $j'(p-1)$ where j' is as in Corollary 2, although using results of Section 6, one can prove similar results for arbitrary j' in cases of generalized eigenforms or under any one of a number of alternate sets of general conditions.

Corollary 4. *Let k be the filtration of a level one modular form f, and restrict to j' appearing in Corollary 2. Then*

i. *If k is not divisible by $p+1$, the filtration of $f - f|V_\ell$ will never fall for any prime $\ell \neq p$.*

3. As pointed out in one of the footnotes in Section 0, forms of filtrations of this second type might be the case most worth next considering if one is thinking about the possibility of generalizing results that Behrens has already obtained.

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ii. *For any* $\ell_0 \neq p$*, if* $k \equiv a(p+1)$ *mod* p^2-1 *for some* $1 \leq a \leq p-1$*, the filtration of* $f - f|V_{\ell_0}$ *can possibly fall only if* ℓ_0 *has multiplicative order in* $(\mathbb{Z}/p\mathbb{Z})^*$ *dividing* a*. Moreover, in such a case, if the filtration of* $f - f|V_{\ell_0}$ *falls by at least* $j'(p-1)$ *, then the filtration of* $f - f|V_\ell$ *will fall by at least* $j'(p-1)$ *for all primes* $\ell \neq p$ *of order dividing* a*, whereas it will not fall at all for primes* ℓ *not satisfying this property (although the filtration of some linear combination of* f *and* $f|V_\ell$ *will fall by at least* $j'(p-1)$ *for all* $\ell \neq p$*).*

iii. *In the special case where* $a = p-1$ *so that* k *is divisibly by* p^2-1*, if the filtration of* $f - f|V_{\ell_0}$ *falls by at least* $j'(p-1)$ *for one* $\ell_0 \neq p$ *(without any restrictions on* ℓ_0*), then the filtration of* $f - f|V_\ell$ *will fall by at least* $j'(p-1)$ *for all primes* $\ell \neq p$*.*

iv. *If* ℓ_0 *is a generator for* $\mathbb{Z}_p^*$ *then the filtration of* $f - f|V_{\ell_0}$ *can fall for some form of filtration* k *only if* k *is divisible by* p^2-1 *(although the filtration of some linear combination of* f *and* $f|V_{\ell_0}$ *will fall for some forms* f *of filtration* k *for all* k *divisible by* $p+1$*).*

Moreover, one can replace the words "at least" by the word "precisely" in both the hypotheses and conclusions of all of the above for all such j' *except for* $j' = \frac{p+1}{2}$*.*

Proof. Part *i* follows immediately from Proposition 2.5, and Parts *ii* and *iii* follow easily by combining Proposition 2.4 with Corollary 2 of this Section. Part *iv* follows automatically from Parts *i* and *ii*. Furthermore, the last sentence follows from Corollary 3 of this Section if $j=1$ or if $j' \geq j - \frac{p-5}{2} \geq 1$ and from the Remark following Corollary 6.2 in cases where $1 \leq j' \leq \frac{p-1}{2}$. $\square$

The authors do not know if statements such as these have any nontrivial topological implications along the lines of those that entered into Behrens' proof of Theorem 1.4 in [B], but thought it was at least worthwhile asking such questions.

Section 6: When and how can the drop in filtration vary with ℓ

Assume temporarily that $w(f) \equiv a(p+1) \bmod p^2-1$. Then we saw in the last Section that if the filtration of $f - \ell_0^a f|V_{\ell_0}$ falls by a "large enough" amount for one prime ℓ_0 , the filtration of $f - \ell^a f|V_\ell$ must fall by precisely the same amount for all primes ℓ. In this Section, we investigate further how much uniformity there is in the size of the drop in filtration for different ℓ in general, and show that this drop will be uniform for all ℓ "under most circumstances", and that even when this is not so, there will still be uniformity within each quadratic residue category with respect to p *(so that for any given f, there are at most two different amounts by which the filtration can fall for different ℓ)* where the fall in the case of quadratic nonresidues is always less than that in the quadratic residue case.

This implies in particular that if the filtration of $f - \ell_0^a f|V_{\ell_0}$ falls by at least $j'(p-1)$ for some $j' \geq 1$ and one quadratic ***nonresidue*** ℓ_0, then the filtration of $f - \ell^a f|V_\ell$ must fall by ***at least*** $j'(p-1)$ for all ℓ, so that we are able to improve on Theorem 1.4 of [B] in the general case. In other words, while we can not remove all restrictions on the prime ℓ_0 for general modular forms as we could in Corollary 5.2 in the case of "large" and "small" j' *(although we can if the form is a generalized eigenform and under other sets of general conditions)*, we can always replace his requirement that ℓ_0 be a generator of $\mathbb{Z}_p^*$ in the mod p case by the significantly weaker condition that ℓ_0 be a quadratic nonresidue with respect to p.

Moreover, as we have used again and again throughout this paper *(proven back in Section 2)* if f is a form of filtration k for which some linear combination of f and $f|V_{\ell_0}$ falls in filtration for even one prime ℓ_0, then we must have $k \equiv a(p+1) \bmod p^2-1$ for some a, in which case, that linear combination must be *(a scalar multiple of)* $f - \ell_0^a f \mid V_{\ell_0}$. Thus the question about uniformity in the size of the falls can be reworded as follows.

Question. If f is a modular form of filtration $k \equiv a(p+1) \bmod p^2-1$ for which the filtration of $f - \ell_0^a f \mid V_{\ell_0}$ falls down by precisely $j'(p-1)$ for some integer $j' \geq 1$ and some $\ell_0 \neq p$, must the filtration of $f - \ell^a f \mid V_\ell$ fall down by precisely $j'(p-1)$ for every $\ell \neq p$?

Proposition 3 below provides an affirmative answer in the case of generalized eigenforms, which as pointed out in the Introduction could potentially be a significant result, possibly giving evidence for the existence of some sort of a "super map" allowing one to raise the level of the mod $\mathcal{P}$ Hecke algebra from level 1 to levels ℓ for all primes $\ell \neq p$ at once in some meaningful way. Indeed it implies, that there might be some way of lifting each generalized eigenspace up as a single cohesive unit and "plunking it down" with one single motion into some kind of space comprised of all the different higher levels in such a way that the resulting changes in filtration are uniform in all levels *(which on the surface seems a bit counterintuitive since the V_ℓ operators entering into our maps are different operators living in different universes for different choices of ℓ)*

Since one of the themes motivating this project has been that one can build on results about generalized eigenforms to obtain analogous results for general modular forms, it is both noteworthy and somewhat disappointing that the analogous result turns out to be false in the case of ***non***-generalized eigenforms *(although as pointed out above, it is still true if one limits oneself to primes ℓ in the same quadratic residue category with respect to p).*

However, this is entirely understandable once one realizes that each vector space of level 1 mod $\mathcal{P}$ modular forms of a high enough weight $k = a(p+1)$ mod p^2-1 contains two different generalized eigenspaces for two different systems of eigenvalues *(quadratic twists of each other)* for which the filtrations of "appropriate linear combinations" will fall within each generalized eigenspace, and these generalized eigenspaces can and do interact with each other in a way that easily explains why under the right circumstances the amount that the filtration drops will differ depending on the quadratic residue properties of ℓ. For a simple example that illustrates how this works, see the Example following the proof of Proposition 3 below.

We summarize the main results of this Section in the following Theorem.

Theorem 1. *Let f be a level 1 modular form mod p of filtration $k \equiv a(p+1)$ mod p^2-1 for which the filtration of $f - \ell_0^a f | V_{\ell_0}$ drops by precisely $j'(p-1)$ for some integer $1 \leq j' \leq j$ and one prime $\ell_0 \neq p$, where $j(p-1)$ is the maximal value (made explicit in Proposition 4.2) by which the filtration of any level 1 form of filtration k can possibly fall for even one ℓ_0. Then*

a) if f is a generalized eigenvector for the Hecke operators, or if j' is "small" ($j' \leq \frac{p-1}{2}$)[1] *or "large" ($j' \geq j - \frac{p-5}{2}$), the filtration of $f - \ell^a f | V_\ell$ must drop by precisely $j'(p-1)$ for all primes $\ell \neq p$.*

b) if f is not a generalized eigenvector and j' is neither "small" or "large" as defined in Part a, then either

i) the filtration of $f - \ell^a f | V_\ell$ drops by precisely $j'(p-1)$ for all primes $\ell \neq p$,

or

*ii) there exist two positive integers $j_1' < j_2'$ (one of which must equal j') with $j_1' \equiv \frac{p+1}{2}$ mod $p+1$ and $\frac{p-1}{2} < j_1' < j_2' < j - \frac{p-5}{2}$ for which the filtration drops by precisely $j_1'(p-1)$ for all primes ℓ that are quadratic <u>non</u>residues with respect to p, and by precisely $j_2'(p-1)$ for all primes ℓ that are quadratic residues with respect to p. Note in particular that the drop for quadratic <u>non</u>residues is **<u>strictly</u> <u>smaller</u>** than the quadratic residue one in this case.*

ci) Furthermore, if $a' = a \pm \frac{p-1}{2}$ and we write $f = g + h$ where $g = g_1 + g_2$ for g_1 a generalized eigenvector with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$, g_2 a generalized eigenvector for system of eigenvalues $\{\ell^{a'} + \ell^{a'-1}\}_{\ell \neq p}$, and h the sum of generalized eigenforms for other systems of eigenvalues, then the condition described in Part ii of b (that of unequal falls depending on the quadratic residue properties of ℓ) will occur if and only if the filtration of g_2 is strictly greater than both the filtration of h and that of $g_1 - \ell^a g_1 | V_\ell$ (which by Part a will be the same for all ℓ).

1. Note that the definition of "small" used here is slightly different than that used in Corollary 5.2 or in Theorem 0.1 in that it does not include the case $j' = \frac{p+1}{2}$. This is because when $j' = \frac{p+1}{2}$, even though it is still true *(as stated in Corollary 5.2 also in Theorem 0.1 and in Corollary 2 of this Section)* that if the filtration of $f - \ell_0^a f | V_{\ell_0}$ slips down by at least $j'(p-1)$ for one $\ell_0 \neq p$, then the filtration of $f - \ell^a f | V_\ell$ will slip down at least that far for all $\ell \neq p$, it is no longer true that if the filtration of $f - \ell_0^a f | V_{\ell_0}$ slips down by precisely $j'(p-1)$ for such a j', it must slip down by precisely that amount for all ℓ, as illustrated in the Example given below in the case $p = 5$ and $k = 30$.

cii) In particular, if the filtration of $g-\ell^a g|V_\ell$ is the same for all ℓ, this must also be true for the filtration of $f-\ell^a f|V_\ell$, although the converse direction is false in general. This could be interpreted as saying that the addition of the form h (i.e. the addition of generalized eigenvectors for "other" systems of eigenvalues) may or may not "cover up" previously existing differences in the drops in filtration for different ℓ's, but can never introduce such differences on its own.

ciii) Moreover, if the filtration of $h\geq$ that of g_2, then the filtration of $f-\ell^a f|V_\ell$ will always fall by the same amount for all ℓ, whereas if the filtration of h is strictly smaller than the filtration of g_2, then the filtration of $f-\ell^a f|V_\ell$ will fall by the same amount for all ℓ if and only if the analogous statement is true about the filtration of $g-\ell^a g|V_\ell$

civ) Finally, the filtration of f will always fall to the maximum of the filtrations of $g_1-\ell^a g_1|V_\ell$, $g_2-\ell^a g_2|V_\ell$, and h for each ℓ, which for quadratic nonresidues ℓ in the above mentioned case will always equal the filtration of g_2, with a strictly lower filtration for quadratic residues ℓ, (whereas in the case where the size of the fall is independent of ℓ it will always equal the maximum of the filtrations of $g_1-\ell^a g_1|V_\ell$ and h).

Proof. The generalized eigenvector result of Part a follows immediately from Proposition 3. The rest of the Theorem is contained in Lemma 6,[2] *which depends on Lemmas 4 and 5 (although the result in Part a for "large" j' was also proven in Corollary 5.3).* ☐

As an immediate Corollary to the above Theorem, we are able to improve on Behrens' Theorem 1.4 of [B] in the general case. Among other things this gives an independent proof of Corollary 5.2 in the case of "large" and "small" falls.

Corollary 2. ***(Extension of Theorem 1.4 of [B])*** *If f is a form of filtration $k\equiv a(p+1)$ mod p^2-1 for which the filtration of $f-\ell_0^a f|V_{\ell_0}$ falls by at least $j^*(p-1)$ for some $j^*\geq 1$ and one quadratic* ***nonresidue*** *ℓ_0, then the filtration of $f-\ell^a f|V_\ell$ must fall by* ***at least*** *$j^*(p-1)$ for all ℓ.*

Furthermore, one can drop the restriction that ℓ_0 be a quadratic nonresidue if f is a generalized eigenform for the Hecke operators, if j^* is either "small" or "large" as defined in Theorem 1, or if $j^*=\frac{p+1}{2}$ is just "over the line" of being "small" as defined in Theorem 1, so that in these cases, the result is true for arbitrary $\ell_0\neq p$.

Remark. The second paragraph in the above Corollary remains true when the words "by at least $j^*(p-1)$" are replaced by the words "by precisely $j^*(p-1)$" in both the hypothesis and conclusion as long as one excludes the case $j^*=\frac{p+1}{2}$, as does the first paragraph if one adds the extra hypothesis that $j^*\not\equiv\frac{p+1}{2}$ mod $p+1$. In other words, under the hypotheses of the Corollary with these modifications, the drop in filtration must be independent of ℓ.

Note in particular that if $r=\mathrm{ord}_p k=0$, then $j=1$, so the above conclusion must always be true, as must also be so when $r=1$ (or equivalently when $j=p$) except for the case where the drop in filtration is $\frac{p+1}{2}(p-1)$ for all quadratic nonresidues and $\frac{p+3}{2}(p-1)$ for all quadratic residues. Theorem 1 also implicitly contains multiple additional sets of general conditions sufficient to force this conclusion to be true.

2. For example, Lemma 6 explicitly contains Result *bii*, which implies the non-generalized eigenvector assertion of Part a since it says that drops that are non-uniform for all ℓ must have size $j^*(p-1)$ where j^* equals either j_1 or j_2, and in particular, must be either $>\frac{p-1}{2}$ or $<j-\frac{p-5}{2}$

Some interesting details that give valuable insights into how things work in the general case are contained in the statements and proofs of Lemmas 4, 5, and 6. We begin however with the generalized eigenform case.

Proposition 3. *Let g be a* ***generalized eigenform*** *of filtration $k \equiv a(p+1) \bmod p^2-1$ with $r = \mathrm{ord}_p(k)$ for which the filtration of $g - \ell_0^a g \mid V_{\ell_0}$ falls down by precisely $j'(p-1)$ for some integer $j' \geq 1$ and some $\ell_0 \neq p$. Then the filtration of $g - \ell^a g \mid V_\ell$ must fall down by precisely $j'(p-1)$ for every $\ell \neq p$.*

Proof. We fix the number a so that Theorem 2.1 implies that the system of eigenvalues associated to g must be $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$, and argue by induction on k.

Note it follows from Proposition 4.2 that $1 \leq j' \leq j$ with $j = \begin{cases} p^r + p^{r-1} - 1 & \text{if } r \geq 1 \\ 1 & \text{if } r = 0 \end{cases}$, and our desired result is contained in Corollary 5.3 in the case where $j' = j$ *(when the drop in filtration is maximal, as in such cases, among others, the result is true for all modular forms).*

This gives us the result in cases of low filtration since for these cases, the fact that k must be divisible by $p+1$ implies that $r = \mathrm{ord}_p k = 0$, so $1 = j' = j$.[3] More generally, it allows us to reduce to the case where j' is strictly less than j, so we assume that this is the case.

Then recall that as shown in Section 1, the generalized eigenspace corresponding to $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ in the quotient space $\mathcal{W}_k = \widetilde{M}_k / \widetilde{M_{k-(p-1)}}$ is one dimensional. Thus if f is a generalized eigenvector of filtration k for the above system of eigenvalues for which the drop in filtration for $f - \ell^a f \mid V_\ell$ is the maximal value of $j(p-1)$ at all primes ℓ *(such forms exist by Theorems 3.4 and 3.5)*, it follows that $g = \lambda f + g^*$ for some nonzero scalar λ and some generalized eigenform g^* for the same system of eigenvalues of $\underline{\text{lower}}$ $\underline{\text{filtration}}$ $\underline{k^*}$ still congruent to $a(p+1) \bmod p^2 - 1$ *(since as pointed out in the Remark following Theorem 1.1 for example, any generalized eigenvector for the above system of eigenvalues must have a filtration satisfying this congruence – here is where we use the generalized eigenvector hypothesis).*

However, since the filtration for $\lambda f - \ell_0^a \lambda f \mid V_{\ell_0}$ is $k - j(p-1)$, whereas the filtration for $g - \ell_0^a g \mid V_{\ell_0}$ is by hypothesis the larger value $k - j'(p-1)$, this can only happen if the filtration of $g^* - \ell_0^a g^* \mid V_{\ell_0}$ also equals the larger value $k - j'(p-1)$.

But since $k^* < k$, our inductive hypothesis implies that $g^* - \ell^a g^* | V_\ell$ must have the same filtration for ***all*** $\ell \neq p$ *(which by above $= k - j'(p-1)$)*, and this combined with the fact that f was chosen so that $f - \ell^a f \mid V_\ell$ always has a lower filtration, immediately gives us the desired result about the filtration of $g - \ell^a g \mid V_\ell$ for all ℓ. $\square$

Note that the above proof depended on the same a appearing in the mod $p^2 - 1$ congruences satisfied by the filtrations for g as for g^* *(resulting in the same power of ℓ appearing in the linear combinations $g - \ell^a g \mid V_\ell$ and $g^* - \ell^a g^* | V_\ell$ lowering the filtrations in the case of each form)* allowing us to deduce that the drops in filtrations for g were independent of ℓ from the analogous statement *(that followed from our inductive hypothesis)* for g^*, which is why we needed to restrict to the case of generalized eigenforms.

3. One can also see this directly as summarized in Corollary 4.3 for such cases.

Note also that since, as used in the above proof, the generalized eigenspaces in the quotient spaces $\mathcal{W}_k$ for the system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ for k as above are all one dimensional, then one can find a basis for the generalized eigenspace in $\widetilde{\mathcal{M}}_k$ corresponding to this system of eigenvalues consisting of forms f_i of decreasing filtrations k_i, each of which drop a maximal amount for their respective filtration for all primes ℓ, and the filtration to which any g in this generalized eigenspace falls for any ℓ depends only on which of the f_i appear with nonzero coefficients in the representation of g as a linear combination of the elements of this basis *(since one can show that no two f_i can fall to the same filtration)*. Note in particular that the only way one gets a non-maximal fall for a generalized eigenform g of filtration k is when the maximal fall for some lower filtration k_i brings you to a filtration that is higher than the maximal fall brings you to starting with the higher filtration k. Since we have previously seen that the above result is true in the case of maximal drops, this gives an alternate proof of Proposition 3, but it is of course mathematically equivalent to the proof given above.

That brings us to the more general case. Although *(as we already mentioned)* one might conjecture at first that the analogous Proposition should be true in general, it is easy to construct an infinite supply of counterexamples by adding two properly chosen generalized eigenforms of different filtrations whose systems of eigenvalues are twists of each other by the unique quadratic character of conductor p *(equivalently by the mod p operator $\theta^{\frac{p-1}{2}}$)* as we do in the following example.

Example. Let $p=5$ and let $g_1 = \theta^3(\Delta) = \theta^4(G_{p+1})$, which is an eigenform with system of eigenvalues $\{\ell+1\}_{\ell \neq p}$ of filtration $30 \equiv 1(p+1) \bmod p^2-1$ for which the filtration of $g_1 - \ell g_1|V_\ell$ drops by $p(p-1) = 20$ to filtration 10 for all primes ℓ and for which this linear combination of g_1 and $g_1|V_\ell$ is *(up to scalar multiple)* the only one for which the filtration falls for any ℓ. [4]

$$\textit{level 1:}\ 6 = p+1 \longrightarrow [12 \longrightarrow 18 \longrightarrow 24 \longrightarrow 30] \longrightarrow 12\ \textit{not 6}$$

$$\textit{level } \ell\textit{:}\ \begin{array}{l} 2 \longmapsto [8 \longmapsto 14 \longmapsto 20] \longmapsto 10 \\ 4 \longmapsto [10] \longmapsto 8 \end{array}$$

Similarly, let $g_2 = \theta(\Delta) = \theta^2(G_{p+1})$, which has system of eigenvalues $\{\ell^2(\ell+1)\}_{\ell \neq p} = \{\ell^3 + \ell^2\}_{\ell \neq p}$ and filtration $18 \equiv 3(p+1) \bmod p^2-1$ for which the filtration of $g_2 - \ell^3 g_2|V_\ell$ drops by $(p-1) = 4$ to filtration 14 for all ℓ, and for which this linear combination of g_2 and $g_2|V_\ell$ is *(up to scalar multiple)* the only one for which the filtration falls for any ℓ. [5]

Note in particular that this implies the filtration of $g_2 - \ell g_2|V_\ell$ drops to filtration 14 for all quadratic residues ℓ *(since for such ℓ, $\ell \equiv \ell^3$ mod p)*, although for all quadratic nonresidues ℓ, the filtration of $g_2 - \ell g_2|V_\ell$ does not drop at all *(since $\ell \not\equiv \ell^3$ mod p for such ℓ, and as stated above, the linear combination $g_2 - \ell^3 g_2|V_\ell$ is up to scalar multiplication the only one for which the filtration falls)*, which means it stays at filtration 18.

Finally, let $g = g_1 + g_2$ which has filtration 30, and note that since $g - \ell g|V_\ell = (g_1 - \ell g_1|V_\ell) + (g_2 - \ell g_2|V)$, the above two paragraphs immediately imply that although the filtration of $g - \ell g|V_\ell$ falls for all ℓ, it falls to filtration 14 for quadratic residues ℓ but only to filtration 18 for quadratic nonresidues.

4. One can see this, for example, by looking at the θ-cycles for G_{p+1} in level 1 and for $G_{p+1} - \ell G_{p+1}|V_\ell$ in level ℓ and then appealing to Theorem 2.1.

5. One can see this also by using a similar argument to the one mentioned in Footnote 4.

The fact that the filtration of g is 30, follows from the fact that the filtration of the level 1 form g_1 "dominates" over that of the level 1 form g_2. However, after one raises the level, the situation switches since the filtration of $g_1 - \ell g_1|V_\ell$ drops so much that it is overshadowed by the filtration of $g_2 - \ell g_2|V_\ell$ *(which we saw above is 14 in the case of quadratic residues and 18 in the case of quadratic nonresidues).*

Lastly, note that g is an eigenvector mod forms of lower filtration for all T_ℓ as Theorem 2.1 says it must be, but only an actual eigenvector *(or even a generalized eigenvector)* for the quadratic residues ℓ.

Remark. The above example shows that Proposition 5.1 *(uniqueness of Behrens' forms $f_{k,j'}$ in cases where j' is "large" or "small")* ***cannot*** be extended to arbitrary j', as the forms g and g_1 in the above example both satisfy condition *iii* of Theorem 0.1 for $k=30$ and $j'=4$, but do not differ by a form of filtration $\leq k - j'(p-1) = 14$, and hence will still be linearly independent after one adjusts for the size of the zeroes set forth in Condition *ii* of that Theorem using the method described at the beginning of Section 3. Moreover, although this example does not satisfy Behrens' stipulation that k be divisible by p^2-1, it is easy to construct similar counterexamples that do.

In fact, the above example is actually quite typical, and the only examples of forms for which the filtration can fall by differing amounts for different ℓ's are ones that are inherently similar to the one we just described. The keys to understanding this lie in the following three Lemmas which in effect reduce the proof of Theorem 1 to the case where f is the sum of two generalized eigenvectors, one with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$, and the other with system $\{\ell^{a'} + \ell^{a'-1}\}_{\ell \neq p}$ *(where $a' = a \pm \frac{p-1}{2} \bmod p-1$, so that when $p=5$, and $a=1$, then $a'=3$ as in the example given above)* and also tell you how the filtrations work when this is the case.

In order to prove these last three Lemmas, we need to keep in mind the following three basic Facts about systems of eigenvalues and generalized eigenforms and their filtrations, which are similar to facts used throughout this paper.

Fact A: If g is a level 1 generalized eigenvector for the Hecke operators $T_{\ell,\ell\neq p}$ with system of eigenvalues $\{\lambda_\ell\}_{\ell\neq p}$, then any linear combination of g and $g|V_{\ell_0}$ will be a level ℓ_0 generalized eigenvector for the $T_{\ell,\ell\neq\ell_0,p}$ with system of eigenvalues $\{\lambda_\ell\}_{\ell\neq\ell_0,p}$.

Fact B: If two level 1 systems of eigenvalues $\{\lambda_\ell\}_{\ell\neq p}$ and $\{\lambda'_\ell\}_{\ell\neq p}$ are distinct, they must differ at an infinite set of primes ℓ *(indeed at a set of primes of positive density)*[6]. In particular, if the level one systems of eigenvalues $\{\lambda_\ell\}_{\ell\neq p}$ and $\{\lambda'_\ell\}_{\ell\neq p}$ are distinct, the same must be true for the level ℓ_0 systems $\{\lambda_\ell\}_{\ell\neq\ell_0,p}$ and $\{\lambda'_\ell\}_{\ell\neq\ell_0,p}$. By combining this with Fact A, one concludes that if g and g' are level 1 generalized eigenvectors whose systems of eigenvalues are distinct from each other, then any linear combination of g and $g|V_{\ell_0}$ and any linear combination of g' and $g'|V_{\ell_0}$ will be level ℓ_0 generalized eigenvectors *(for the $T_{\ell,\ell\neq\ell_0,p}$)* whose systems of eigenvalues must still be distinct.

6. As pointed out in Footnotes in Sections 2 and 4, one sees this for example by considering the mod $\mathcal{P}$ Galois representations associated to such systems of eigenvalues.

Fact C: Since generalized eigenforms for distinct systems of eigenvalues must be linearly independent, hence linearly independent modulo forms of lower filtration, the filtration of a sum of such generalized eigenforms always equals the maximum of the filtrations of each generalized eigenform appearing in the sum.

We are now in a position to make use of the above Facts to complete the proof of Theorem 1. We do this in the next three Lemmas.

Lemma 4. *Let f be a level 1 modular form of filtration $k \equiv a(p+1) \bmod p^2-1$, and write $f = g + h$ where $g = g_1 + g_2$ for g_1 a generalized eigenvector with system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$, g_2 a generalized eigenvector for system of eigenvalues $\{\ell^{a'} + \ell^{a'-1}\}_{\ell \neq p}$ where $a' = a \pm \frac{p-1}{2}$, and h the sum of generalized eigenforms for other systems of eigenvalues.*

Then the filtration of $f - \ell_0^a f|V_{\ell_0}$ will fall down to a filtration $\leq k - j^*(p-1)$ for some $j^* \geq 1$ if and only if the filtration of $g - \ell_0^a g|V_{\ell_0}$ falls down to at least that low, and the filtration of h was at least that low in level 1 to begin with. In particular, this can only happen if the forms g and g_1 also have filtration k, and h and g_2 have lower filtrations.

Proof. We write f as a linear combination of generalized eigenforms corresponding to distinct systems of eigenvalues and note that any generalized eigenform g' appearing in this decomposition must have filtration k' for some k' congruent to k mod $p-1$. Then if the filtration of some linear combination of g' and $g'|V_{\ell_0}$ falls for even one ℓ_0, it follows from Theorem 2.1 that k' must be divisible by $p+1$, which given the congruence property of k' mod $p-1$ can only happen if either $k' \equiv a(p+1) \equiv k$ mod p^2-1 or $k' \equiv a'(p+1) \equiv k + \frac{p^2-1}{2}$ mod p^2-1.

Moreover, the last paragraph of Theorem 2.1 implies that g' must correspond to the system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ in the first case and to $\{\ell^{a'} + \ell^{a'-1}\}_{\ell \neq p}$ in the second, which means g' must equal g_1 in the first case and g_2 in the second.

Furthermore, none of the generalized eigenforms appearing in the decomposition of h can ever fall in filtration, which by appealing to Facts A, B and C can be easily seen to imply that h itself can not fall in filtration.

The first sentence of the conclusion of the Lemma immediately follows so that in particular, the filtration of h must be strictly less than k, which implies that the filtration of $g = g_1 + g_2$ must equal k, which by Fact C means that the maximum of $w(g_1)$ and $w(g_2)$ must equal k. But since as pointed out above, the fact that g_2 is a generalized eigenvector for system of eigenvalues $\{\ell^{a'} + \ell^{a'-1}\}_{\ell \neq p}$ implies that it must have filtration congruent to $k + \frac{p^2-1}{2}$ mod p^2-1 and hence can not have filtration k, this can only happen if g_1 has filtration k and g_2 has smaller filtration. $\square$

We now temporarily ignore the modular form h and consider what happens to the modular form $g = g_1 + g_2$ under the above hypotheses.

Lemma 5. *Keep the setup and results of the previous Lemma, which means that $k \equiv a(p+1) \bmod p^2-1$, and $g = g_1 + g_2$ with g_1 a generalized eigenvector of filtration k for system of eigenvalues $\{\ell^a + \ell^{a-1}\}_{\ell \neq p}$ and g_2 a generalized eigenvector of filtration $k' < k$ with $k' \equiv k + \frac{p^2-1}{2} \bmod p^2-1$ for system of eigenvalues $\{\ell^{a'} + \ell^{a'-1}\}_{\ell \neq p}$ where $a' = a \pm \frac{p-1}{2}$*

Then the filtration of $g-\ell^a g|V_\ell$ must fall down for all primes $\ell \neq p$, as must the filtration of $g_1-\ell^a g_1|V_\ell$, whereas the filtration of $g_2-\ell^a g_2|V_\ell$ will fall only for the quadratic residues ℓ, with the filtration of $g_2+\ell^a g_2|V_\ell$ falling for the quadratic nonresidues.

Moreover, the falls in filtration for both g_1 and g_2 taken separately must each be the same amount for all primes ℓ except that the "type of linear combinations" producing these falls for g_2 are different than those for g_1 in the case of quadratic nonresidues ℓ (namely $g_2+\ell^a g_2|V_\ell$ versus $g_1-\ell^a g_1|V_\ell$ for such ℓ).

Finally, the filtration of g will fall the same amount for all ℓ if and only if $w(g_2)$ is $\leq$ $w_\ell(g_1-\ell^a g_1|V_\ell)$ for one and hence for all ℓ, and when this is not the case, the size of the fall will still be the same for all ℓ satisfying the same quadratic residue property, with the fall in the quadratic nonresidue case always strictly less than that in the quadratic residue case.

Proof. Theorems 1.1 and 2.1 imply that the filtrations of $g_1-\ell^a g_1|V_\ell$ and $g_2-\ell^{a'} g_2|V_\ell$ must fall for all ℓ, and that these are the only linear combinations of g_1 and $g_1|V_\ell$ *(respectively g_2 and $g_2|V_\ell$)* for which the filtration can fall for even one ℓ.

Since $\ell^{a'} = \begin{cases} \ell^a & \text{if } \left(\frac{\ell}{p}\right)=1 \\ -\ell^a & \text{if } \left(\frac{\ell}{p}\right)=-1 \end{cases}$, this implies in particular that the filtration of $g_2-\ell^a g_2|V_\ell$ will fall only for the quadratic residues ℓ, with the filtration of $g_2+\ell^a g_2|V_\ell$ falling for the quadratic nonresidues.

Moreover, since $k' < k$, the fact that the filtration of $g-\ell^a g|V_\ell$ must fall down for all primes $\ell \neq p$ follows immediately from the analogous fact about the filtration of $g_1-\ell^a g_1|V_\ell$.

Also, the fact that the falls in filtration for $g_1-\ell^a g_1|V_\ell$ and $g_2-\ell^{a'} g_2|V_\ell$ must each be the same size for all ℓ follows from Proposition 3 when combined with the congruence properties of k and k' mod p^2-1.

Thus it remains only to prove the last paragraph, which we do by first noting that

$$w_\ell(g-\ell^a g|V_\ell) = max(w_\ell(g_1-\ell^a g_1|V_\ell), w_\ell(g_2-\ell^a g_2|V_\ell)) \tag{1}$$

Then if $w(g_2)$ is $\leq w_\ell(g_1-\ell^a g_1|V_\ell)$ for one and hence for all ℓ, the fact that $w_\ell(g_2-\ell^a g_2|V_\ell)$ cannot be higher than $w(g_2)$ implies that the right hand side of Equation 1 must equal $w_\ell(g_1-\ell^a g_1|V_\ell)$, and in particular must be independent of ℓ.

On the other hand, if $w(g_2) > w_\ell(g_1-\ell^a g_1|V_\ell)$, the fact that the filtration of $g_2-\ell^a g_2|V_\ell$ falls for all quadratic residues and only for quadratic residues, implies that the right hand side of Equation (1) equals $w(g_2)$ in the case of quadratic nonresidues ℓ *(so that in particular, the size of the fall in the filtration of g will be the same for all such ℓ)*, whereas it will equal a strictly lower amount in the quadratic residue case, where the fact that the size of the fall will be the same for all quadratic residues ℓ follows by combining the previous statements made in this proof. $\square$

Our final Lemma, which encapsulates all of the results included in Theorem 1 above in the ***non*** generalized eigenform case, shows how the above result is affected when one adds back a third form h that is the sum of generalized eigenforms for other systems of eigenvalues.

Lemma 6. *Continue to keep the above notation, except add back the form h to the form g, so that now $f = g + h$ for g and h as above and assume that the filtration of $f - \ell^a f|V_\ell$ falls down for at least one (and hence by Theorem 2.1 for all) ℓ.*

Then the filtration of $f - \ell^a f|V_\ell$ will fall by the same amount for all ℓ, if and only if $w(g_2)$ is $\leq$ either $w(h)$ or $w_\ell(g_1 - \ell^a g_1|V_\ell)$ for some and hence for all ℓ, and when this is ***not*** *the case, the fall in the filtration of $f - \ell^a f|V_\ell$ will still be the same for all ℓ satisfying the same quadratic residue property, with the fall for the quadratic nonresidue category strictly less than the fall for the quadratic residue one.*

In particular, if the filtration of $g - \ell^a g|V_\ell$ is the same for all ℓ, the same must be true for the filtration of $f - \ell^a f|V_\ell$, although the converse direction is false.

Finally note that $w_\ell(f - \ell^a f|V_\ell)$ will equal the maximum of the filtrations $w_\ell(g_1 - \ell^a g_1|V_\ell)$, $w_\ell(g_2 - \ell^a g_2|V_\ell)$, and $w(h)$ for each ℓ, which in the case where the falls are independent of ℓ equals the maximum of the first and last of these, and in the case where the falls vary with ℓ will always equal the filtration of g_2 for quadratic nonresidues ℓ. Moreover, in this second case, if the size of the fall is $j_1'(p-1)$ for quadratic nonresidues ℓ and $j_2'(p-1)$ for quadratic residues ℓ, one must have $j_1' \equiv \frac{p+1}{2} \bmod p+1$ and $\frac{p-1}{2} < j_1' < j_2' < j - \frac{p-5}{2}$ where $j(p-1)$ is the maximal value that any form of filtration k can fall for any ℓ.

Proof. We have

$$f - \ell^a f|V_\ell = (g_1 - \ell^a g_1|V_\ell) + (g_2 - \ell^a g_2|V_\ell) + (h - \ell^a h|V_\ell)$$

where $w_\ell(h - \ell^a h|V) = w(h)$ for all ℓ.

It then follows immediately from Facts A, B, and C that

$$w_\ell(f - \ell^a f|V_\ell) = max(w_\ell(g_1 - \ell^a g_1|V_\ell), w_\ell(g_2 - \ell^a g_2|V_\ell), w(h)) \tag{2}$$

Moreover, if $w(g_2) \leq w_\ell(g_1 - \ell^a g_1|V_\ell)$ or $w(g_2) \leq w(h)$ the same inequality must be satisfied by $w_\ell(g_2 - \ell^a g_2|V_\ell)$, and hence the maximum of the three values on the right hand side of Equation 2 must equal the maximum of $w_\ell(g_1 - \ell^a g_1|V_\ell)$ and $w(h)$, neither of which vary with ℓ, so the size of the drops are uniform for all ℓ.

On the other hand, if $w(g_2) > w_\ell(g_1 - \ell^a g_1|V_\ell)$ and also $w(g_2) > w(h)$, the fact *(pointed out in the proof of the previous lemma)* that the filtration of g_2 falls for all quadratic residues and only for quadratic residues implies that the right hand side of Equation (2) equals $w(g_2)$ in the case of quadratic nonresidues ℓ, whereas it must equal a strictly lower amount in the quadratic residue case. Furthermore, we can deduce that the size of the fall will be the same for all ℓ within a given quadratic residue category from the fact that $w_\ell(g_2 - \ell^a g_2|V_\ell)$, which *(as follows from the proofs of the previous two lemmas)* is the only filtration on the right hand side of Equation 2 to vary with ℓ is uniform within each quadratic residue category *(since the plus or minus sign in the expression $\ell^a = \pm \ell^{a'}$ is the same within any fixed such category).*

The next to the last paragraph of the Lemma *which says "In particular, if the filtration of $g - \ell^a g|V_\ell$ is the same for all ℓ, the same must be true for the filtration of $f - \ell^a f|V_\ell$, although the converse direction is false."* follows immediately since we saw in the previous lemma that $w_\ell(g - \ell^a g|V_\ell)$ will be the same for all ℓ, if and only if $w(g_2) \leq w_\ell(g_1 - \ell^a g_1|V_\ell)$, $\checkmark$ and the precise value for $w_\ell(f - \ell^a f|V_\ell)$ contained in the first sentence of the following paragraph *which says "Finally note that $w_\ell(f - \ell^a f|V_\ell)$ will equal the maximum of the filtrations $w_\ell(g_1 - \ell^a g_1|V_\ell)$, $w_\ell(g_2 - \ell^a g_2|V_\ell)$, and $w(h)$ for each ℓ, which in the case where the falls are independent of ℓ equals the maximum of the first and last of these,* $\checkmark$ *and in the case where the falls vary with ℓ will always equal the filtration of g_2 for quadratic nonresidues ℓ* $\checkmark$" follows easily from Equation 2 combined with some of the assertions made above.

For the last sentence, *which says "Moreover, in this second case,(i.e. non uniform size of falls) if the size of the fall is $j_1'(p-1)$ for quadratic nonresidues ℓ and $j_2'(p-1)$ for quadratic residues ℓ, one must have $j_1' \equiv \frac{p+1}{2} \bmod p+1$ and $\frac{p-1}{2} < j_1' < j_2' < j - \frac{p-5}{2}$ where $j(p-1)$ is the maximal value that any form of filtration k can fall for any ℓ"* first note that $j_1' < j_2'$ and $j_1' \equiv \frac{p+1}{2} \bmod p+1$ since the fall in the quadratic nonresidue case in this situation must be less than that for quadratic residues $\checkmark$ and must take us to $w(g_2) = k' \equiv k + \frac{p^2-1}{2} \bmod p^2 - 1$. $\checkmark$

Moreover, the fact that $j_2' < j - \frac{p-5}{2}$ in this case follows immediately from Corollary 5.3, which states that the filtration of $f - \ell_0^a f \mid V_{\ell_0}$ can slip down by $j'(p-1)$ for $j' \geq j - \frac{p-5}{2}$ for even one ℓ_0, only in the case where the drop in filtration is independent of ℓ. $\square$

Possible Appendix: Generalizing Theorem 6.3 of [J3] to level ℓ in the case where we use it in this paper

In this Section, we prove a result that we used in the proof of our existence theorem *(Theorem 3.4)* about which filtrations in level $\ell_1 \neq p$ contain nonzero generalized eigenvectors belonging to the system of eigenvalues $\{\lambda_\ell\} = \begin{cases} \{\ell^a + \ell^{a-1}\} & \text{if } \ell \neq p, \ell_1 \\ \ell_1^{a-1} & \text{if } \ell = \ell_1 \end{cases}$ for the Hecke operators $T_{\ell, \ell \neq \ell_1, p}$ and for U_{ℓ_1} for a fixed $a \in \mathbb{Z}/(p-1)\mathbb{Z}$.[1]

This extends Theorem 6.3 of the first author's [J3] to modular forms over $\Gamma_0(\ell_1)$ for this system of eigenvalues $\{\lambda_\ell\}$.

Note that $\{\lambda_\ell\} = \{\ell^{a-1}\delta_\ell\}$ where $\{\delta_\ell\} = \begin{cases} \{\ell + 1\} & \text{if } \ell \neq p, \ell_1 \\ 1 & \text{if } \ell = \ell_1 \end{cases}$ is the system of eigenvalues that corresponds to the weight 2 modular form $g = G_2 - \ell_1 G_2 | V_{\ell_1}$, and also $\{\lambda_\ell\} = \{\ell^a \delta'_\ell\}$ where $\{\delta'_\ell\} = \{\ell^{-1}\delta_\ell\} = \begin{cases} \{1 + \ell^{-1}\} & \text{if } \ell \neq p, \ell_1 \\ \ell_1^{-1} & \text{if } \ell = \ell_1 \end{cases}$ is the system of eigenvalues that corresponds to the filtration $p-1$ modular form $g' = G_{p-1} - G_{p-1} | V_{\ell_1}$.

Convention. We say that a system of eigenvalues $\{\mu_\ell\}$ has filtration k_* if $\mathcal{W}_{k_*}(\ell_1)$ has a nonzero eigenspace corresponding to $\{\mu_\ell\}$, or equivalently if $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ has a ***generalized*** eigenform corresponding to $\{\mu_\ell\}$ having filtration k_*. Note that this doesn't necessarily mean that $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ has an actual eigenform corresponding to $\{\mu_\ell\}$ having filtration k_*, as the actual eigenforms corresponding to $\{\mu_\ell\}$ could have lower filtrations.

Theorem. *Fix a prime $\ell_1 \neq p$, and work with modular forms mod $\mathcal{P}$ over $\Gamma_0(\ell_1)$. Let $a \in \mathbb{Z}/(p-1)\mathbb{Z}$, $k_* \geq 2$, and $\{\lambda_\ell\} = \begin{cases} \{\ell^a + \ell^{a-1}\} & \text{if } \ell \neq p, \ell_1 \\ \ell_1^{a-1} & \text{if } \ell = \ell_1 \end{cases}$ be as above. Then the quotient space $\mathcal{W}_{k_*}(\ell_1)$ of level ℓ_1 contains a nonzero eigenspace corresponding to $\{\lambda_\ell\}$ if and only if $k_* \equiv a(p+1) \pm (p-1)$ mod $p^2 - 1$, except that in the case $\ell_1 \equiv 1$ mod p, one has to also add $k_* \equiv a(p+1)$ mod $p^2 - 1$.*

Proof. Theorem 2.4 of [J3] shows that $\mathcal{W}_{k_*}(\ell_1) \cong \mathcal{W}_{k_* - (p+1)}(\ell_1)[1]$ *(where the* [1] *in this notation indicates that the actions of the T_ℓ and U_{ℓ_1} are replaced by those of ℓT_ℓ and $\ell_1 U_{\ell_1}$ respectively)* as long as $k_* \geq 2p + 2$, so it suffices to prove the Theorem for weights $k_* \leq 2p$.[2] That means it suffices to prove that $\mathcal{W}_{k_*}(\ell_1)$ has a nonzero eigenspace corresponding to $\{\lambda_\ell\}$ in cases where $2 \leq k_* \leq 2p$ if and only if $k_* = 2$ and $a = 1$, $k_* = p-1$ and $a = 0$, $k_* = p+3$ and $a = 2$, or $k_* = 2p$ and $a = 1$, except that when $\ell_1 \equiv 1$ mod p, one also has to add $k_* = p+1$ and $a = 1$.

1. The reader is warned that there are still nonzero generalized eigenspaces if we change the eigenvalue for U_{ℓ_1} to $\lambda'_{\ell_1} = \ell_1^a$ *(keeping the eigenvalues for the $T_{\ell, \ell \neq \ell_1, p}$ the same)*, but the filtrations change unless $\ell_1 \equiv 1$ mod p.

2. In fact, it would be completely trivial to reduce this further to $k_* \leq p+1$ if the relationship between systems of eigenvalues of filtrations $p+3 \leq k_* \leq 2p$ with precise twists of systems of eigenvalues of lower weights contained in the proof of Theorem 4.1 of [J3] in cases of low levels was known to carry over to general level $\ell_1 \neq p$.

Ash and Stevens [A-S] generalized the result of that Theorem to arbitrary level, but their proof seems to be based on a weaker assertion than the explicit formulation that we are talking about here. Although it is relatively easy to combine the Ash-Stevens theorem with the theory of θ-cycles and some other basic facts from [J2] and [J3] to spell out exactly which twists of which systems of eigenvalues of filtrations $\leq p+1$ can be related to a given system of eigenvalues of filtration $p+3 \leq k_* \leq 2p$, we thought it best to include a self contained proof of the result we need here.

Moreover, one direction of this if and only if statement is immediate since we have the eigenforms g and g' introduced above, where g has filtration 2 and system of eigenvalues $\{\delta_\ell\}$ *(*$a=1$*)* and g' has filtration $p-1$ with system of eigenvalues $\{\ell^{-1}\delta_\ell\}$ *(*$a=0$*)*, which implies that $\theta(g)$ is a filtration $p+3$ eigenform with system of eigenvalues $\{\ell\delta_\ell\}$ *(*$a=2$*)* and $\theta(g')$ is a filtration $2p$ eigenform with system of eigenvalues $\{\delta_\ell\}$ *(*$a=1$*)*. Furthermore, in cases where $\ell_1\equiv 1 \bmod p$, $G_{p+1}|V_{\ell_1}$ *(hence also* G_{p+1}*)* is a generalized eigenvector *(although not an actual eigenvector)* with system of eigenvalues $\{\delta_\ell\}$ *(*$a=1$*)* of filtration $p+1$.[3]

For the converse direction, we assume that $\mathcal{W}_{k_*}(\ell_1)$ has a nonzero eigenspace corresponding to $\{\lambda_\ell\}$ for some $2\leq k_*\leq 2p$, and divide the proof into the following two cases.

Case 1: $\widetilde{M_{k_*}(\Gamma_0(\ell))}$ ***contains an actual eigenvector*** f ***of filtration*** k_* ***corresponding to the system of eigenvalues*** $\{\lambda_\ell\}$***. We will show that this can only happen if*** $k_*=2$ ***and*** $a=1$, $k_*=p-1$ ***and*** $a=0$, $k_*=p+3$ ***and*** $a=2$, ***or*** $k_*=2p$ ***and*** $a=1$***.***

Note first that since $2\leq k_*\leq 2p$, the only case where such an f could be in the image of V is if $k_*=2p$ and $f=f_1|V$ for some eigenvector f_1 of filtration 2, but in that case, we could easily replace f by $f+f_1$ and obtain another eigenvector of the same filtration and same system of eigenvalues that was not in the image of V.

We can therefore always choose f so that it is not in the image of V, which since it is an eigenvector for the T_ℓ and U_{ℓ_1} implies that its coefficient of index 1 will not equal 0, so we can assume without loss of generality that this coefficient equals 1.

But then *(since the coefficients of indices not divisible by* p *are determined by the eigenvalues for* $T_{\ell,\ell\neq p,\lambda_1}$ *and for* U_{ℓ_1}*)* the fact that $\{\lambda_\ell\}=\{\ell^{a-1}\delta_\ell\}$ immediately implies that $\theta^i(f)=\theta^{i+a-1}(g)$ for all positive integers i, which says that f and g must "share" the same θ-cycle *(also "shared" by the form* g'*)* although f and g might "jump into" that θ cycle at different points with one of them proceeding through the θ-cycle a few steps behind the other.

But then a quick look at the possible shapes of the θ cycles for forms f of filtration $2\leq k_*\leq 2p$ when compared with the known shape of the θ-cycle for g shows that this can only happen in the cases specified above *(See [J2] for the general shapes of* θ*-cycles of forms of low weights, and throw in the cycles starting with forms of weight 2 both in the kernel of U case and not, which weren't included there, since that paper was limited to the case of Level 1).* ⊠

Case 2: $\widetilde{M_{k_*}(\Gamma_0(\ell))}$ ***contains a generalized eigenvector*** h ***of filtration*** k_* ***corresponding to the system of eigenvalues*** $\{\lambda_\ell\}$ ***but all actual eigenvectors corresponding to that system of eigenvalues have strictly lower filtrations. We will show that this case can only happen if*** $\ell_1\equiv 1 \bmod p$, $k_*=p+1$, ***and*** $a=1$***.***

This case can not occur if $2\leq k_*\leq p-3$, because in those cases the weight is so small that $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ contains no elements of lower filtration.

3. As pointed out in a Footnote in Section 3, $G_{p+1}-\ell_1 G_{p+1}|V_{\ell_1}$ is an eigenvector for U_{ℓ_1} with eigenvalue 1 of filtration 2 and $G_{p+1}-G_{p+1}|V_{\ell_1}$ is an eigenvector for U_{ℓ_1} with eigenvalue ℓ_1, which in the case $\ell_1\not\equiv 1 \bmod p$ has filtration $p+1$. However, when $\ell_1\equiv 1 \bmod p$, these two forms are congruent mod p, and by subtracting them from each other and dividing by $p^{\mathrm{ord}_p(\ell_1-1)}$, one obtains a generalized eigenvector with the same system of eigenvalues *(a scalar multiple of* $G_{p+1}|V_{\ell_1}$*)* of filtration $p+1$.

It also can't occur if $k_*=p-1$ because in that case, the only elements of $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ of filtration smaller than k_* are the constants, which will only be eigenvectors with system of eigenvalues $\{\lambda_\ell\}$ if $a=0$ and $\ell_1\equiv 1 \bmod p$ *(where this congruence condition on the prime ℓ_1 is needed so that the eigenvalue for the operator U_{ℓ_1} which equals 1 for the constants coincides with λ_{ℓ_1} which equals ℓ_1^{-1} when $a=0$)*, but in that case, $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ will also contain a separate eigenvector ($g'=G_{p-1}-G_{p-1}|V_\ell$ *defined up above*) for the system of eigenvalues $\{\lambda_\ell\}$ of filtration $k_*=p-1$ contradicting the defining hypothesis for this case.

It can occur if $k_*=p+1$, but only if $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ contained a nonzero eigenvector of filtration 2 corresponding to $\{\lambda_\ell\}$, in which case the results in Case 1 above show that $a=1$, or equivalently $\{\lambda_\ell\}=\{\delta_\ell\}$. Moreover, this can only happen if $\ell_1\equiv 1 \bmod p$ since otherwise the generalized eigenspace of weight $p+1$ for $\Gamma_0(\ell_1)$ corresponding to $\{\delta_\ell\}$ is easily seen to be 1 dimensional[4] *(hence only containing scalar multiples of g which has filtration 2, and in particular containing no generalized eigenvectors h of filtration $p+1$)*.

We are thus reduced to the case $p+3\le k_*\le 2p$, where by hypothesis, $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ would have to contain a nonconstant[5] eigenvector f of lower filtration $k_*'\equiv k_* \bmod p-1$ for the system of eigenvalues $\{\lambda_\ell\}$. By the results of Case 1, this could only happen in the case $k_*'=2$ *(in which case $k_*=2p$)* and $a=1$ or the case $k_*'=p-1$ *(in which case $k_*=2p-2$)* and $a=0$.

But if $k_*=2p$ and $a=1$, then $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ contains an actual eigenvector $\theta^{p-1}(g)$ of filtration $2p$ *(in addition to a separate eigenvector g of filtration 2)* for the system of eigenvalues $\{\lambda_\ell\}$, which means it does not satisfy the hypothesis of Case 2, and indeed it is included in the results described above in Case 1.

It therefore remains only to show that $\mathcal{W}_{k_*}(\ell_1)$ can not contain a nonzero generalized eigenspace corresponding to $\{\lambda_\ell\}$ when $k_*=2p-2$ and $a=0$.

If such a case were to occur, $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ would contain the actual eigenform g' of filtration $p-1$ corresponding to the system of eigenvalues $\{\lambda_\ell\}$ and it would also contain some generalized eigenform of filtration $2p-2$ for the same system of eigenvalues.

But since $2p-2>p+1$, Lemma 3.2 of [J2] *(which follows trivially from the fact that U contracts the filtration for filtrations bigger than $p+1$)* then implies that $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ would have to also contain a normalized eigenvector f' in the kernel of U with the same system of eigenvalues $\{\lambda_\ell\}$ for all $\ell\ne p$ *(in addition to the eigenform g' which is not in the kernel of U)*. But this is impossible since one would then have $g'-f'$ would be a nonconstant form of weight $k_*=2p-2$ in the image of V and no such form can possibly exist *(since $\widetilde{M_{k_*}(\Gamma_0(\ell_1))}$ contains no nonconstant forms of filtration divisible by p for $k_*=2p-2$)*. $\square$

4. Otherwise there would exist two normalized eigenvectors for $\Gamma_0(\ell_1)$ of weight $p+1$ with system of eigenvalues $\{\delta_\ell\}$, which since nothing of weight $p+1$ can be in the image of V mod $\mathcal{P}$ could only happen if in addition to $G_{p+1}-\ell G_{p+1}|V_{\ell_1}$ there was another normalized eigenvector of weight $p+1$ also congruent to g mod $\mathcal{P}$. But since *(in the case $\ell_1\not\equiv 1$ mod p)* the constant term of g is nonzero mod $\mathcal{P}$, such a normalized eigenvector could not be a cusp form, and it could also not be an Eisenstein series since the space of Eisenstein series of weight $p+1$ in characteristic 0 for $\Gamma_0(\ell_1)$ is known to be 2 dimensional spanned by $G_{p+1}-\ell G_{p+1}|V_{\ell_1}$ and $G_{p+1}-G_{p+1}|V_{\ell_1}$ which are easily seen to be linearly independent mod p as long as $\ell_1\not\equiv 1 \bmod p$.

5. When $k_*=2p-2$, one has to give momentary consideration to the constants which have filtration $0\equiv k_* \bmod p-1$, but as we pointed out when we dealt with filtration $p-1$ above, the constants will only be eigenvectors with system of eigenvalues $\{\lambda_\ell\}$ in cases where there is also a nonconstant eigenvector g' for the same system of eigenvalues of filtration $p-1$.

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