

The homology of BPO

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1. Introduction

The goal of this note is to calculate $H_*BPO = H_*(BPO, \mathbb{Z}/2)$ where $BPO = B\mathbb{P}\mathbb{R}^{\mathbb{Z}/2}$. Here, $B\mathbb{P}\mathbb{R}$ is the 2-local $\mathbb{Z}/2$ -equivariant irreducible summand of Landweber's Real cobordism spectrum $M\mathbb{R}$ (see [9],[3]). The spectrum $M\mathbb{R}$ was investigated in [6], where we also give a calculation of its coefficients, and hence also of π_*BPO . Our main interest in these spectra is motivated by the possibility of investigating an Adams-type spectral sequence based on them. This will be pursued elsewhere. However, the calculation of H^*BPO contains some surprising complications, and is of independent interest. This will be our main focus here.

To describe H^*BPO , recall the cofibration

$$(1.1) \quad (B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2} \rightarrow BPO \rightarrow \Phi^{\mathbb{Z}/2}B\mathbb{P}\mathbb{R}$$

where $\Phi^{\mathbb{Z}/2}$ denotes geometric fixed points ([10]). This gives a long exact sequence of the form

$$(1.2) \quad \begin{array}{ccccc} H_*((B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) & \longrightarrow & H_*BPO & \longrightarrow & H_*(\Phi^{\mathbb{Z}/2}BPO) \\ & & & & \downarrow \partial \\ & & & & H_{*-1}((B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}). \end{array}$$

THEOREM 1.3. $H_*BPO \cong \text{Ker}(\partial) \oplus \text{Coker}(\partial)$ as comodules over the dual A_* of the Steenrod algebra. Moreover,

$$\text{Ker}(\partial) \cong \mathbb{Z}/2,$$

with generator 1.

To interpret this result, we must describe the map ∂ explicitly. We have

$$H_*(\Phi^{\mathbb{Z}/2}B\mathbb{P}\mathbb{R}) \cong H_*H \cong A_* = \mathbb{Z}/2[\zeta_1, \zeta_2, \dots].$$

We will recall in the next section that we may write

$$(1.4) \quad \Sigma H_*((B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) \cong \mathbb{Z}/2[b_1, b_2, \dots][r^{-1}]\{r^{-1}\}$$

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where $\dim(r^{-1}) = 1$, $\dim(b_i) = 2(2^i - 1)$. By our convention, square brackets denote polynomial generators, while braces denote an additive generator, typically used for comparison with other structures, and dimensional shift.

THEOREM 1.5. *The map ∂ can be characterized by the formula*

$$\partial(\zeta_i) = \sum_{j=0}^i b_{i-j}^{2^j} r^{2^i - 2^{j+1} + 1}$$

where the formula is to be interpreted as multiplicative on the generators, while in the result, all non-negative powers of r are set to 0.

Remark: In particular, we actually have $\partial(\zeta_i) = r^{1-2^i}$.

The result can be extended to calculate also the homology of

$$H_*((\Sigma^{n\alpha} B\mathbb{P}\mathbb{R})^{\mathbb{Z}/2})$$

as an A_* -comodule, for an arbitrary $n \in \mathbb{Z}$, where α is the real sign representation of $\mathbb{Z}/2$. In this case, the long exact sequence (1.2) takes on the form

$$\begin{array}{ccc} H_*((\Sigma^{n\alpha} B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) & \longrightarrow & H_*((\Sigma^{n\alpha} B\mathbb{P}\mathbb{R})^{\mathbb{Z}/2}) \\ & & \downarrow \\ & & H_*(\Phi^{\mathbb{Z}/2} BPO) \\ & & \downarrow \partial_n \\ & & H_{*-1}((\Sigma^{n\alpha} B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) \end{array}$$

since $\Phi^{\mathbb{Z}/2} E = \Phi^{\mathbb{Z}/2} \Sigma^\alpha E$. It turns out that

$$\Sigma H_*((\Sigma^{n\alpha} B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) = \mathbb{Z}/2[b_1, b_2, \dots][r^{-1}]\{r^{-n-1}\},$$

so ∂_n is given by the same formula as ∂ , where in the result, however, all powers of r greater or equal than $-n$ are set to 0.

THEOREM 1.6. *Ker(∂_n) is 0 if $n < 0$, and is spanned by monomials in the conjugates $\overline{\zeta_1}, \overline{\zeta_2}, \dots$ of degree $\leq n$ (where each $\overline{\zeta_i}$ is counted as having degree 1). Moreover, $H_*((\Sigma^{n\alpha} B\mathbb{P}\mathbb{R})^{\mathbb{Z}/2}) \cong \text{Ker}(\partial_n) \oplus \text{Coker}(\partial_n)$ as A_* -comodules.*

To make use of the statements about A_* -comodules structure, we need to understand the A_* -comodule structure on $H_*((B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2})$. When done directly, this is surprisingly complicated. First of all, we have

$$\Sigma H_* \mathbb{R}P^\infty = \mathbb{Z}/2[r^{-1}]\{r^{-1}\} \subset H_*((B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}).$$

Recall that the A_* -comodule structure on $H_* \mathbb{R}P^\infty$ is given by

$$(1.7) \quad r^{-k} \mapsto \left(\sum_{\ell \geq 0} r^{2^\ell} \otimes \zeta_\ell \right)^{-k}$$

with non-negative powers of r set to 0. As it turns out, the formula for b_i is

$$(1.8) \quad b_i \mapsto \sum_{j=0}^i b_j^{2^{i-j}} \left(\sum_{k=0}^{i-j} (r^{2^i - 2^{i-j+1} + 2^k} \otimes \zeta_k) \cdot \sum_{\ell \geq 0} (r^{2^\ell} \otimes \zeta_\ell) + \left(\sum_{k < i-j} r^{2^i - 2^{i-j+1} + 2^k} \otimes \zeta_k^2 \right) \cdot \left(\sum_{\ell \geq 0} r^{2^\ell} \otimes \zeta_\ell \right)^{-2^i} \right).$$

This is, again, understood as a multiplicative formula together with (1.7), with non-negative powers of r set to 0. The formula is analogous for the case of suspension by $\Sigma^{n\alpha}$, with the difference that we set to 0 all powers of r greater or equal to $-n$.

These formulas, however, seem almost too complicated to be useful. We will show in the next section how they can be understood in a different light by using the equivariant language, and considering all the suspensions $\Sigma^{n\alpha}$, $n \in \mathbb{Z}$, at the same time.

2. The equivariant language

The Adams isomorphism gives

$$(2.1) \quad (BPR \wedge EZ/2_+)^{\mathbb{Z}/2} = BPR \wedge_{\mathbb{Z}/2} EZ/2_+.$$

Thus,

$$(2.2) \quad \begin{aligned} H_*((BPR \wedge EZ/2_+)^{\mathbb{Z}/2}) &= (H \wedge (BPR \wedge_{\mathbb{Z}/2} EZ/2_+))_* \\ &= ((H_{\mathbb{Z}/2} \wedge BPR) \wedge_{\mathbb{Z}/2} EZ/2_+)_* = (H_{\mathbb{Z}/2} \wedge BPR)_*^f. \end{aligned}$$

On the right hand side, $H_{\mathbb{Z}/2}$ denotes $\mathbb{Z}/2$ -equivariant Mackey cohomology with constant coefficients, but could alternately denote for example the Borel cohomology theory.

The right hand side notation refers to the *Tate cofibration sequence* [5]

$$(2.3) \quad E^f \rightarrow E^c \rightarrow E^t$$

for any $\mathbb{Z}/2$ -equivariant $RO(\mathbb{Z}/2)$ -graded spectrum E . The first map (also known as the *norm map*) is the map

$$E^f = E \wedge EZ/2_+ \simeq F(EZ/2_+, E) \wedge EZ/2_+ \rightarrow F(EZ/2_+, E) = E^c$$

induced by the collapse $EZ/2_+ \rightarrow S^0$. The Tate spectrum E^t is then defined simply as the cofiber of this map.

Now for $E = H_{\mathbb{Z}/2} \wedge BPR$, the long exact sequence associated with the cofibration sequence (2.3) has been completely calculated in [6]. Recall our coefficient convention: for a $\mathbb{Z}/2$ -equivariant spectrum E , $E_* = (E^{\mathbb{Z}/2})_*$, while E_* denotes the complete system of coefficients

$$E_{k+\ell\alpha} = [S^{k+\ell\alpha}, E].$$

Then, for $E = H_{\mathbb{Z}/2} \wedge BPR$, (2.3) gives a short exact sequence

$$(2.4) \quad 0 \rightarrow (H_{\mathbb{Z}/2} \wedge BPR)_*^c \rightarrow (H_{\mathbb{Z}/2} \wedge BPR)_*^t \rightarrow \Sigma(H_{\mathbb{Z}/2} \wedge BPR)_*^f \rightarrow 0$$

which has the form

$$(2.5) \quad 0 \rightarrow \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]_a^\wedge \rightarrow \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]_a^\wedge[a^{-1}] \rightarrow a^{-1}\mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a^{-1}] \rightarrow 0.$$

Here the element ρ is of dimension $\alpha - 1$, the element a is of dimension $-\alpha$ and the element ξ_i is of dimension $(2^i - 1)(1 + \alpha)$. To compute the right A_* -comodule structure, we further compare (2.5) with the analogous sequence for $H_{\mathbb{Z}/2}$ using the Thom reduction

$$(2.6) \quad BPR \rightarrow H_{\mathbb{Z}/2}.$$

Thus, we obtain a diagram of the form

$$(2.7) \quad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]_a^\wedge & \longrightarrow & \mathbb{Z}/2[\zeta_i, \sigma, \sigma^{-1}, a]_a^\wedge = (H \wedge H)_*^c \\ \downarrow & & \downarrow \\ \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]_a^\wedge[a^{-1}] & \xrightarrow{f} & \mathbb{Z}/2[\zeta_i, \sigma, \sigma^{-1}, a]_a^\wedge[a^{-1}] \\ \downarrow & & \downarrow \\ a^{-1}\mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a^{-1}] & \xrightarrow{f_1} & a^{-1}\mathbb{Z}/2[\zeta_i, \sigma, \sigma^{-1}, a^{-1}] \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Here the A_* -comodule structure on the bottom row is determined by the (a) -complete $\mathbb{Z}/2[a]$ -Hopf algebroid structure on $(H \wedge H)_*^c$. This is described in [6]. We have

$$(2.8) \quad \psi(\zeta_i) = \sum_{j=0}^i \zeta_j^{2^{i-j}} \otimes \zeta_{i-j}$$

and

$$(2.9) \quad \eta_R(\sigma) = \sum_{i \geq 0} \zeta_i \sigma^{2^i} a^{2^i - 1}.$$

Similarly, to determine the vertical maps (2.7), it suffices to consider the middle map f .

Referring to [6], we have

$$f(\rho) = \eta_R(\sigma),$$

and the images of the ξ_i 's are given by the recursion

$$(2.10) \quad \begin{aligned} f(\xi_0) &= 1 \\ f(\xi_i) &= \frac{1}{a^{2^i}} \left(\frac{\zeta_{i-1}^2}{\eta_R(\sigma)} + \zeta_i a + \frac{f(\xi_{i-1})}{\sigma^{2^i - 1}} \right), \quad i \geq 1. \end{aligned}$$

Remark: If we are only interested in the coefficients in dimensions $k + 0\alpha$, this already gives the formulas from the previous section: Put

$$(2.11) \quad \begin{aligned} b_i &= \xi_i \rho^{1-2^i}, \\ r &= \rho a. \end{aligned}$$

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or $H_{\mathbb{Z}/2}$ using the

This gives (1.4). To compute the Steenrod coaction, we shall use the map f , and the fact that

$$\psi(\eta_R(\sigma)) = 1 \otimes \eta_R(\sigma),$$

$$\psi(f(\xi_i)) = \sum_{j=0}^i f(\xi_{i-j})^{2^j} \otimes f(\xi_j)$$

(see [6]). Along with (2.11), by expanding the formula (2.10) to explicit form in the $f(\xi_i)$'s, we obtain the formula (1.8).

It will also be useful for us to make formula (2.10) explicit in the ζ_i 's.

LEMMA 2.12. *Identifying $f(\xi_i)$ with ξ_i and ρ with $\eta_R(\sigma)$, we have*

$$\zeta_i = \sum_{j=0}^i \xi_j^{2^{i-j}} \rho^{-2^{i-j}+1} a^{2^i-2^{i-j}+1} + \sigma^{-2^{i-1}} \sum_{j=0}^{i-1} \xi_j^{2^{i-1-j}} \rho^{-2^{i-1-j}+1} a^{1-2^{i-j}}.$$

Proof: Induction. For $i = 0$, the statement is obviously correct. Processing the second formula of (2.10) gives

$$\zeta_i = a^{2^{i-1}} \xi_i + \xi_{i-1} \sigma^{-2^{i-1}} a^{-1} + \zeta_{i-1} \rho^{-1} a^{-1}.$$

By the induction hypothesis, the right hand side is

$$a^{2^{i-1}} \xi_i + \xi_i \sigma^{-2^{i-1}} a^{-1} + \left(\sum_{j=0}^{i-1} \xi_j^{2^{i-j}} \rho^{-2^{i-j}+2} a^{2^i-2^{i-j}+2} \right) \rho^{-1} a^{-1} + \left(\sigma^{-2^{i-1}} \sum_{j=0}^{i-1} \xi_j^{2^{i-1-j}} \rho^{-2^{i-1-j}+2} a^{2-2^{i-j}} \right) \rho^{-1} a^{-1}.$$

This is the induction step. $\square$

Now (2.10) completely determines the map f . We shall use this to completely determine the connecting map ∂ of (1.2). As a first step, we will find the following statement useful:

LEMMA 2.13. *Let E be a spectrum (non-equivariant) and let X be a G -spectrum indexed over the G -complete universe. Let i be the inclusion from trivial to complete G -universe. Then we have a natural equivalence*

$$(i_* E_{\text{fixed}} \wedge X)^G \simeq E \wedge (X^G),$$

where E_{fixed} denotes E thought of as a fixed G -equivariant spectrum indexed over the trivial universe.

Proof: We will construct a map

$$\phi: E \wedge (X^G) \rightarrow (i_* E_{\text{fixed}} \wedge X)^G.$$

By adjunction, this is equivalent to having a map

$$i_*(E \wedge (X^G))_{\text{fixed}} \rightarrow i_* E_{\text{fixed}} \wedge X.$$

We have

$$i_*((E \wedge (X^G))_{\text{fixed}}) = i_*(E_{\text{fixed}}) \wedge i_*(X^G)_{\text{fixed}},$$

so it suffices to have a map

$$i_*((X^G)_{\text{fixed}}) \rightarrow X.$$

We choose this to be the adjoint to the identity $X^G \rightarrow X^G$. To show that ϕ is an equivalence, it suffices to consider the case $E = S^0$. In this case the procedure we

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described in [6].

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ms $k + 0\alpha$, this

described starts with the identity $X^G \rightarrow X^G$ and adjoins it forth and back, to get back the identity. $\square$

By Lemma 2.13, our goal is equivalent to computing

$$(2.14) \quad (i_* H\mathbb{Z}/2 \wedge BPR)_*.$$

Remark: In [6], it was incorrectly claimed that $i_* H\mathbb{Z}/2$ is Real-oriented (which would trivialize this problem). However, the statement about $i_* H\mathbb{Z}/2$ is given only as a curiosity in [6], while the homology theory used in the applications is $H_{\mathbb{Z}/2}$. Consequently, other results of [6] are unaffected by the mistake.

To calculate (2.14), we will consider the diagram

$$\begin{array}{ccc} \Phi\mathbb{Z}/2(i_* H\mathbb{Z}/2 \wedge BPR) & \xrightarrow{\partial} & \Sigma(i_* H\mathbb{Z}/2 \wedge BPR \wedge E\mathbb{Z}/2_+) \\ \downarrow g & & \downarrow = \\ i_* H\mathbb{Z}/2 \wedge BPR & \longrightarrow & \Sigma(i_* H\mathbb{Z} \wedge BPR \wedge E\mathbb{Z}/2_+). \end{array}$$

On coefficients, this is

$$\begin{array}{ccc} \mathbb{Z}/2[\zeta_i, a, a^{-1}] & \xrightarrow{\partial} & \Sigma a^{-1} \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a^{-1}] \\ \downarrow g & & \downarrow = \\ \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]_a^{\wedge} [a^{-1}] & \longrightarrow & \Sigma a^{-1} \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a^{-1}]. \end{array}$$

Theorems (1.3), (1.5) can now be restated as follows:

THEOREM 2.15. *The map g (and hence the map ∂) is given by*

$$g(\zeta_i) = \sum_{j=0}^i \xi_{i-j}^{2^j} \rho^{-2^j+1} a^{2^i-2^{j+1}+1}.$$

Moreover, in twist 0, the kernel of the map ∂ is $\mathbb{Z}/2\{1\}$. (The maps f, g are maps of rings.)

The proof of the Theorem hinges on the following result:

$$\text{LEMMA 2.16. } gf(\zeta_i) = \sum_{j=0}^i (\sigma a)^{-2^i+2^j} \zeta_j.$$

Proof that Lemma 2.16 implies Theorem 2.15: Similarly as in the proof of Lemma 2.12, we suppress, for the moment, f from the notation (this is OK since f is injective). From the statement of Lemma 2.16, we now notice that

$$g(\zeta_i) = (\sigma a)^{-2^{i-1}} g(\zeta_{i-1}) + \xi_i.$$

Thus, by induction, to prove the formula of Theorem 2.15, it suffices to prove that

$$\sum_{j=0}^i \xi_{i-j}^{2^j} \rho^{-2^j+1} a^{2^i-2^{j+1}+1} = (\sigma a)^{2^{i-1}} \sum_{j=0}^{i-1} \xi_{i-j-1}^{2^{j-1}} \rho^{-2^j+1} a^{2^{i-1}-2^{j+1}+1} + \xi_i.$$

This is Lemma 2.12.

and back, to get $\square$

To prove the injectivity of ∂ , let $r = \sigma a$, $u_i = gf(\zeta_i)$. Thus, we have

$$u_i = gf(\zeta_i) = \sum_{j=0}^i r^{-2^i+2^j} \zeta_i.$$

-oriented (which $\mathbb{Z}/2$ is given only cations is $H_{\mathbb{Z}/2}$.

Notice that on the right hand side, all summands involve a negative power of r , except the summand ζ_i . Our statement is now equivalent to the following result. $\square$

CLAIM 2.17. *Suppose $p(u_1, u_2, \dots) = q(\zeta_1, \zeta_2, \dots)$ where p, q are polynomials with coefficients in $\mathbb{Z}/2$. Then $p = q = 1$ or 0 .*

Proof of Claim: Let $x = r^{-1}$. We have

$$u_n = \zeta_n + u_{n-1}x^{2^{n-1}}.$$

Thus, the equation of the Claim reads

$$(2.18) \quad p(u_1, \dots, u_n) = q(u_1 + x, u_2 + u_1x^2, \dots, u_n + u_{n-1}x^{2^{n-1}}),$$

since p, q are polynomials with coefficients in $\mathbb{Z}/2$. Separating the terms free of x on the right, we see that

$$p = q \in \mathbb{Z}/2[x_1, \dots, x_n].$$

Now grade $\mathbb{Z}/2[x_1, \dots, x_n]$ by giving x_i degree 2^{i-1} . A monomial is given the sum of degrees of its factors. Let p_{top} be the sum of monomials of p of top degree. We see that the sum of monomials of

$$p(u_1 + x, u_2 + u_1x^2, \dots, u_n + u_{n-1}x^{2^{n-1}})$$

carrying the highest power of x is

$$p_{top}(x, u_1x^2, \dots, u_{n-1}x^{2^{n-1}}).$$

Thus, (2.18) implies that

$$p_{top}(x, u_1x^2, \dots, u_{n-1}x^{2^{n-1}}) = 0,$$

while we know that

$$p_{top} \neq 0.$$

This is impossible, since $x, u_1x^2, \dots, u_{n-1}x^{2^{n-1}}$ are algebraically independent. $\square$

3. Real Morava K -theories and the image of ζ_i

The purpose of this section is to prove Lemma 2.16. Let $r = \sigma a$. Note that for any $\mathbb{Z}/2$ -equivariant spectrum E , we have an obvious natural map

$$i_*H \wedge F(E\mathbb{Z}/2_+, E) \rightarrow F(E\mathbb{Z}/2_+, i_*H \wedge E)$$

(indeed, this is adjoint to the map

$$E\mathbb{Z}/2_+ \wedge i_*H \wedge E = E\mathbb{Z}/2_+ \wedge i_*H \wedge F(E\mathbb{Z}/2_+, E) \rightarrow i_*H \wedge E$$

given by the collapse $E\mathbb{Z}/2_+ \rightarrow S^0$). Thus, in the right hand column of (2.7), $(H \wedge H)^c$ can be replaced by $i_* H \wedge H^c$. Passing to twist 0, the bottom square of (2.7) can be replaced by

$$(3.1) \quad \begin{array}{ccc} H_* \Phi^{\mathbb{Z}/2} B\mathbb{P}\mathbb{R} & \xrightarrow{\partial} & \Sigma H_*(E\mathbb{Z}/2_+ \wedge B\mathbb{P}\mathbb{R}^{\mathbb{Z}/2}) \\ \downarrow \bar{f} & & \downarrow \bar{f}_1 \\ H_* \Phi^{\mathbb{Z}/2}(H^c) & \xrightarrow{\bar{\partial}} & \Sigma H_*((E\mathbb{Z}/2_+ \wedge H^c)^{\mathbb{Z}/2}) \end{array}$$

which on coefficients (in twist 0) is

$$(3.2) \quad \begin{array}{ccc} \mathbb{Z}/2[\zeta_i] & \xrightarrow{\partial} & (a^{-1}\mathbb{Z}/2[\zeta_i, \rho, \rho^{-1}, a^{-1}])_{*+0\alpha} \\ \downarrow \bar{f} & & \downarrow \\ \mathbb{Z}/2[\zeta_i, r, r^{-1}] & \xrightarrow{\bar{\partial}} & r^{-1}\mathbb{Z}/2[\zeta_i, r^{-1}]. \end{array}$$

Since (3.1) is obtained by applying H_* to a diagram of (non-equivariant) spectra, it is a diagram of right A_* -comodules. On the other hand, notice that the bottom row of (3.2) consists of extended right A_* -comodules, that the elements r, r^{-1} are primitive (since they come from $\pi_* \Phi^{\mathbb{Z}/2} H^c$) and that, in fact, the augmentation is induced by the map

$$i_* H \wedge H^c \longrightarrow H^c \wedge H^c \xrightarrow{\phi} H^c.$$

The augmentation takes the form

$$(3.3) \quad \begin{array}{ccc} \mathbb{Z}/2[\zeta_i, r, r^{-1}] & \xrightarrow{\bar{\partial}} & r^{-1}\mathbb{Z}/2[\zeta_i, r^{-1}] \\ \epsilon \downarrow & & \downarrow \epsilon_1 \\ \mathbb{Z}/2[r, r^{-1}] & \xrightarrow{proj} & r^{-1}\mathbb{Z}/2[r^{-1}] \end{array}$$

where $\epsilon(\zeta_i) = 0$, $\epsilon(\zeta_i) = 0$ for $i > 0$. It is then a general fact about extended comodules that

$$\bar{f}(\zeta_i) = \phi(\epsilon \otimes 1)\psi(\zeta_i) = \sum_{j=0}^i \epsilon \bar{f}(\zeta_i)^{2^{i-j}} \otimes \zeta_{i-j}.$$

Thus, we need to show that

$$\epsilon \bar{f}(\zeta_i) = r^{-2^i+1}.$$

Since, however, projection in (3.3) is an isomorphism in positive dimensions, we have reduced the proof of Lemma 2.16 to the following statement.

LEMMA 3.4. $\epsilon_1 \bar{\partial} \bar{f}(\zeta_i) \neq 0$ for $i > 0$.

To prove Lemma 3.4, we will recall certain constructions from [6]. Specifically, we have constructed a 'quotient spectrum' $B\mathbb{P}\mathbb{R}/(x_1, \dots, x_n)$ if $x_1, \dots, x_n \in B\mathbb{P}_*$ (non-equivariant!) is a regular sequence. Let

$$\begin{aligned} k\mathbb{R}(n) &= B\mathbb{P}\mathbb{R}/(v_i, i \neq n), \\ k\mathbb{R}(n)/v_n^2 &= B\mathbb{P}\mathbb{R}/(v_i, i \neq n, v_n^2). \end{aligned}$$

column of (2.7),
bottom square of

We constructed a cofibration sequence of the form

$$\begin{array}{c}
 \Sigma^{(2^n-1)(1+\alpha)} H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+ \\
 \downarrow t \\
 k\mathbb{R}(n)/v_n^2 \wedge E\mathbb{Z}/2_+ \\
 \downarrow \\
 H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+ \\
 \downarrow Q_n \\
 \Sigma^{(2^n-1)(1+\alpha)+1} H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+.
 \end{array}$$

($H\mathbb{Z}/2$ is Mackey cohomology with constant coefficients, but could be $i_*H\mathbb{Z}/2$ or $H\mathbb{Z}/2^c$.)

LEMMA 3.5. Let δ be the composition

$$BP\mathbb{Z} \wedge S^{\infty\alpha} \xrightarrow{\bar{\delta}} \Sigma BP\mathbb{R} \wedge E\mathbb{Z}/2_+ \longrightarrow \Sigma H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+.$$

Then there exists a lift

$$\begin{array}{ccc}
 & \Sigma^{-(2^n-1)(1+\alpha)} H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+ & \\
 \delta_0 \nearrow & \downarrow Q_n & \\
 BP\mathbb{R} \wedge S^{\infty\alpha} & \xrightarrow{\delta} & \Sigma H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+.
 \end{array}$$

Furthermore, if $1 : S^0 \rightarrow BPO \wedge S^{\infty\alpha}$ is the unit, then $\delta_0(1) \neq 0$.

Proof: We consider the diagram

(3.6)

$$\begin{array}{ccccc}
 C & \xrightarrow{\iota} & C \wedge S^{\infty\alpha} & \xrightarrow{\bar{\delta}_1} & \Sigma^{-(2^n-1)(1+\alpha)} H\mathbb{Z}/2_f \\
 \bar{\pi} \downarrow & & \downarrow \pi & & \downarrow Q_n \\
 BP\mathbb{R} & \xrightarrow{\bar{\iota}} & BP\mathbb{R} \wedge S^{\infty\alpha} & \xrightarrow{\delta} & \Sigma H\mathbb{Z}/2_f \\
 \downarrow v_n & & \downarrow v_n \wedge S^{\infty\alpha} & & \downarrow t \\
 \Sigma^{-(2^n-1)(1+\alpha)} BP\mathbb{R} & \longrightarrow & \Sigma^{-(2^n-1)(1+\alpha)} BP\mathbb{R} \wedge S^{\infty\alpha} & \xrightarrow{\delta_1} & \Sigma^{1-(2^n-1)(1+\alpha)} (k\mathbb{R}(n)/v_n^2)_f.
 \end{array}$$

The vertical lines of this diagram are cofibration sequences, which defines C . The map δ_1 is constructed as follows: By general module-theoretical considerations, we have a diagram

$$\begin{array}{ccc}
 \Sigma^{(2^n-1)(1+\alpha)} k\mathbb{R}(n)_f & \longrightarrow & \Sigma^{(2^n-1)(1+\alpha)} (k\mathbb{R}(n)/v_n)_f \\
 \downarrow v_n & & \downarrow t \\
 k\mathbb{R}(n)_f & \xrightarrow{\theta} & (k\mathbb{R}(n)/v_n^2)_f.
 \end{array}$$

variant) spectra,
that the bottom
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augmentation is

about extended

dimensions, we

[6]. Specifically,
 $x_n \in BP_*$ (non-

The map δ_1 is the composition

$$B\mathbb{P}\mathbb{R} \wedge S^{\infty\alpha} \xrightarrow{\bar{\delta}} \Sigma B\mathbb{P}\mathbb{R}_f \longrightarrow \Sigma k\mathbb{R}(n)_f \xrightarrow{\theta} \Sigma(k\mathbb{R}(n)/v_n^2)_f$$

where the second map is the $M\mathbb{R}$ -module theoretic quotient map.

We now turn our attention to the diagram (3.6). In [6], it was proved that

$$(3.7) \quad v_n \wedge S^{\infty\alpha} = 0.$$

Thus, $t\delta = 0$, which immediately implies the existence of the map δ_0 .

Now consider the unit $1 \in \pi_0 B\mathbb{P}\mathbb{R}$. Let $\bar{1} = 1 \wedge S^{\infty\alpha} \in \pi_0 B\mathbb{P}\mathbb{R} \wedge S^{\infty\alpha}$. Note that since $v_n \wedge S^{\infty\alpha} = 0$, we have

$$C \wedge S^{\infty\alpha} = (B\mathbb{P}\mathbb{R} \wedge S^{\infty\alpha}) \vee (\Sigma^{-(2^n-1)(1+\alpha)-1} B\mathbb{P}\mathbb{R} \wedge S^{\infty\alpha}).$$

Thus, $\bar{1}$ lifts to $1' \in \pi_0 C \wedge S^{\infty\alpha}$. Note that we may define $\bar{\delta}_1 = \delta_0\pi$. Thus, the non-vanishing statement is equivalent to

$$(\bar{\delta}_1)_* 1' \neq 0.$$

This is equivalent to saying that $1' \notin \text{Im}(\iota_*) : \pi_0 C \rightarrow \pi_0 C \wedge S^{\infty\alpha}$. Assume this was not true, i.e.

$$(3.8) \quad 1' = \iota_* 1'', \quad 1'' \in \pi_0 C.$$

Recall that $\bar{\iota} : \pi_0 B\mathbb{P}\mathbb{R} \rightarrow \pi_0 B\mathbb{P}\mathbb{R} \wedge S^{\infty\alpha}$ is a mod 2 isomorphism, so we know that

$$\bar{\pi}_*(1'') \equiv 1 \pmod{2}.$$

But note that this is not possible, since by the calculation of $\pi_* B\mathbb{P}\mathbb{R}$ ([6]),

$$v_n \cdot e \neq 0$$

for any class $e \in \pi_0 B\mathbb{P}\mathbb{R}$ congruent to 1 mod 2. Thus, (3.8) leads to a contradiction and Lemma 3.5 is proved. $\square$

Now recall that we can identify

$$(3.9) \quad \pi_*(H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+) = \Sigma^{-1}a^{-1}\mathbb{Z}/2[\sigma, \sigma^{-1}, a^{-1}]$$

via the quotient map

$$\widehat{H\mathbb{Z}/2} \rightarrow \Sigma H\mathbb{Z}/2_f$$

where $\widehat{H\mathbb{Z}/2}_* = \mathbb{Z}/2[\sigma, \sigma^{-1}, a, a^{-1}]$. Using (3.9), Lemma 3.5 implies that

$$(3.10) \quad (\delta_0)_*(1) = \sigma^{-2^n} a^{2-2^{n+1}}.$$

Now consider the map

$$\begin{array}{ccc} (i_* H\mathbb{Z}/2 \wedge \Sigma^{-(2^n-1)(1+\alpha)} H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+)_* & \xrightarrow{(1 \wedge Q_\alpha)_*} & (i_* H\mathbb{Z}/2 \wedge \Sigma H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+)_* \\ & & \downarrow \phi \wedge Id \\ & & (i_* H\mathbb{Z}/2 \wedge \Sigma E\mathbb{Z}/2_+)_* \end{array}$$

In twist 0, this is, by Lemma 2.13,

$$(3.11) \quad \begin{array}{ccc} H\mathbb{Z}/2_*(\Sigma^{-(2^n-1)(1+\alpha)} H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+) & \longrightarrow & H\mathbb{Z}/2_*(\Sigma H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+) \\ & \searrow \lambda_1 & \downarrow \\ & & H\mathbb{Z}/2_* E\mathbb{Z}/2_+ \end{array}$$

On the upper left corner of (3.11), $\pi_*(X) \subset H_*(X)$ (where $X = \Sigma^{-(2^n-1)(1+\alpha)} H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+$) sits as the submodule of primitives, and

$$(3.12) \quad H\mathbb{Z}/2_* X = \pi_* X \otimes A_*.$$

By (3.10) and Lemma 3.5, the statement of Lemma 3.4 in the language of (3.12) would follow from

$$(3.13) \quad \lambda_1(\sigma^{-2^n} a^{2-2^{n+1}} \otimes \zeta_{n+1}) \neq 0.$$

Using the short exact sequence

$$0 \rightarrow \pi_* H\mathbb{Z}/2^c \rightarrow \pi_* \widehat{H\mathbb{Z}/2} \rightarrow \pi_* \Sigma H\mathbb{Z}/2_f \rightarrow 0,$$

we may pull $\sigma^{-2^n} a^{2-2^{n+1}}$ back to $\widehat{H\mathbb{Z}/2}_*$, divide by $a^{2-2^{n+1}}$, and pull back to $(H\mathbb{Z}/2^c)_*$. Considering the composition

$$(3.14) \quad \begin{array}{ccc} (i_* H\mathbb{Z}/2 \wedge \Sigma^{-(2^n-1)(1+\alpha)} H\mathbb{Z}/2^c)_* & \xrightarrow{1 \wedge F(E\mathbb{Z}/2_+, \mathbb{Q}_n)} & (i_* H\mathbb{Z}/2 \wedge \Sigma H\mathbb{Z}/2^c)_* \\ & \searrow \lambda & \downarrow \phi \\ & & (\Sigma H\mathbb{Z}/2^c)_* \end{array}$$

the inequality (3.13) would now follow from

$$(3.15) \quad \lambda(\sigma^{-2^n} \otimes \zeta_{n+1}) \neq 0.$$

However, we have forgetful maps to non-equivariant coefficients, where (3.14) is simply the cap product

$$A_* \xrightarrow{\cap \mathbb{Q}_n} A_* \xrightarrow{\epsilon} \mathbb{Z}/2.$$

Thus, (3.15) follows from the fact that non-equivariantly,

$$(\zeta_{n+1} \cap \mathbb{Q}_n) \neq 0.$$

This concludes the proof of Lemma 3.4.

4. The twisted dimensions

We shall now calculate $H_*(\Sigma^{n\alpha} B\mathbb{P}\mathbb{R})^{\mathbb{Z}/2}$ for $n \in \mathbb{Z}$. Proceeding analogously as in the twist 0 case, we have the map

$$(4.1) \quad \Phi^{\mathbb{Z}/2}(i_* H\mathbb{Z}/2 \wedge B\mathbb{P}\mathbb{R}) \xrightarrow{\partial_n} \Sigma(\Sigma^{n\alpha} i_* H\mathbb{Z}/2 \wedge B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}$$

and an extension of the form

$$(4.2) \quad 0 \rightarrow \text{Ker}(\partial_n) \rightarrow H_*(\Sigma^{n\alpha} B\mathbb{P}\mathbb{R})^{\mathbb{Z}/2} \rightarrow \text{Coker}(\partial_n).$$

Furthermore,

$$\begin{aligned} \Sigma(\Sigma^{n\alpha} i_* H\mathbb{Z}/2 \wedge B\mathbb{P}\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2} &\equiv \\ \Sigma(\Sigma^{n\alpha} B\mathbb{P}\mathbb{R})_f^{\mathbb{Z}/2} &= r^{-(n+1)} \mathbb{Z}/2[\xi_1, \xi_2, \dots][r^{-1}]. \end{aligned}$$

the map ∂_n is determined by the map g of Theorem 2.15. Thus, the question reduces to determining the behaviour of the map ∂_n , and the extension (4.2) in the category of A_* -comodules.

To this end, let $P(n) \subset A_*$ denote the subgroup generated by all monomials of order $\leq n$ in the variables $\bar{\zeta}_1, \bar{\zeta}_2, \dots$ (the conjugates of $\zeta_1, \zeta_2, \dots$) where each of the

variables $\overline{\zeta_i}$ is given the degree 1. Note that $P(n) \subset A_*$ is a right A_* -submodule. Thus, we get a short exact sequence of right A_* -modules

$$(4.3) \quad 0 \rightarrow P(n) \rightarrow A_* \rightarrow Q(n) \rightarrow 0.$$

Proof of Theorem 1.6: We will begin by considering the geometrically defined cofibration sequence

$$(4.4) \quad \Sigma^{-1}MO \xrightarrow{\partial} (\Sigma^{n\alpha}MR)_f^{\mathbb{Z}/2} \xrightarrow{\kappa} (\Sigma^{n\alpha}MR)^{\mathbb{Z}/2} \xrightarrow{\lambda} MO$$

which in homotopy induces a long exact sequence

$$(4.5) \quad \rightarrow \mathbb{Z}/2[a_1, a_2, \dots] \rightarrow \Sigma H_*(\Sigma^{n\alpha}MR)_f^{\mathbb{Z}/2} \rightarrow \Sigma H_*(\Sigma^{n\alpha}MR)^{\mathbb{Z}/2} \rightarrow \dots$$

Recall that the generators a_i of H_*MO are obtained as elements in the image of

$$(4.6) \quad H_*\Sigma^{-1}(BO(1))^{\gamma_1} \rightarrow H_*MO.$$

We have, of course,

$$BO(1)^{\gamma_1} \cong B\mathbb{Z}/2.$$

Now consider the map

$$(4.7) \quad i_n : \Sigma^{-n}BO(n)^{\gamma_n} \rightarrow MO.$$

By a general principle, $\partial i_n = 0$, so i_n lifts to a map

$$(4.8) \quad \iota_n : \Sigma^{-n}BO(n)^{\gamma_n} \rightarrow (\Sigma^{n\alpha}MR)_f^{\mathbb{Z}/2}, \quad i_n = \lambda \iota_n.$$

(Geometrically, ι_n is adjoint to the Thomification of the classifying map of the complexification of γ_n - which is a Real bundle on the fixed space $BO(n)$ - with the inclusion of fixed points.)

Thus, we have proven

LEMMA 4.9. *The kernel of $H_*\partial$ contains $H_*\Sigma^{-n}BO(n)^{\gamma_n}$, which consists of monomials of degree $\leq n$ in the variables $a_1, a_2, \dots$ (where each a_i is given degree 1).*

□

To relate this to the statement of Theorem 1.6, we need to apply the Quillen idempotent to (4.4), (4.5), and determine the images of the a_i 's in terms of the ζ_i 's. This is done as follows:

Recall that the a_i 's are the coefficients of the exponential function of the universal Formal Group Law on MO_* (classifying the FGL's G which satisfy $[2]_G = 0$). In other words, the series

$$(4.10) \quad \sum_{i \geq 0} a_i x^{i+1}$$

is the functional inverse of the series

$$(4.11) \quad \sum_{i \geq 0} m_i x^{i+1}$$

where m_i are the coefficients of the logarithm of the universal FGL on MO_* . Now on the coefficients of the logarithm, the Quillen idempotent

$$(4.12) \quad e : MO \rightarrow MO$$

has the effect

$$(4.13) \quad \begin{aligned} m_{2^k-1} &\mapsto \zeta_k \\ m_i &\mapsto 0, \quad i \neq 2^k - 1. \end{aligned}$$

We conclude that in terms of the ζ_k 's, the a_i 's are the coefficients of the functional inverse of the series

$$(4.14) \quad \sum_{k \geq 0} \zeta_k x^{2^k}.$$

In other words, (4.12) sends

$$(4.15) \quad \begin{aligned} a_{2^k-1} &\mapsto \overline{\zeta_k}, \quad k = 0, 1, 2, \dots \\ a_i &\mapsto 0, \quad i \neq 2^k - 1. \end{aligned}$$

Thus, applying now

$$(4.16) \quad e : M\mathbb{R} \rightarrow M\mathbb{R}$$

to (4.4), (4.5), (4.15) implies that

$$(4.17) \quad P(n) \subseteq \text{Ker}(\partial_n).$$

On the other hand, when examining the image of $\overline{\zeta_k}$ under (4.1), we see from the ζ_k summand of $\overline{\zeta_k}$ that the coefficient of the lowest power of r (=highest power of r^{-1}), which is r^{-1} , contains the summand

$$(4.18) \quad a_{k-1} r^{-1}.$$

Since the elements (4.18) are algebraically independent (over $\mathbb{Z}/2$), we see that the image of any polynomial in the $\overline{\zeta_i}$'s of degree $> n$ will be of degree $> n$ in r^{-1} . Therefore,

$$(4.19) \quad \text{Ker}(\partial_n) \subseteq P(n).$$

Furthermore, (4.8) shows that $P(n)$ is a right A_* -subcomodule of $H_*(\Sigma^{n\alpha} BPR)^{\mathbb{Z}/2}$, which proves the splitting claimed in the statement of Theorem 1.6. $\square$

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