



Ext and the motivic Steenrod algebra over $\mathbb{R}$

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ABSTRACT

We present a descent style, Bockstein spectral sequence computing Ext over the motivic Steenrod algebra over $\mathbb{R}$ and related sub-Hopf algebras. We demonstrate the workings of this spectral sequence in several examples, providing motivic analogues to the classical computations related to $BP\langle n \rangle$ and ko .

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1. Introduction

Motivic stable homotopy theory intertwines classical algebraic geometry and stable algebraic topology. Rather than considering just the stable category of spaces, one considers a category of spaces built out of smooth schemes over a chosen ground field k . This framework, the foundations of which can be found in the work of Morel and Voevodsky [16], allows one to make use of many of the classical tools and techniques from algebraic topology to solve problems like the Milnor conjecture. A nice introduction for topologists is found in the work of Dugger and Isaksen [4] and Borghesi [2].

One of the key features in motivic stable homotopy is that the spheres are bigraded. There is the simplicial sphere $S^{1,0}$ (which is related to the suspension tied to the triangulated structure), and there is a geometric sphere $S^{1,1} = \mathbb{G}_m$. Taking smash products of these spheres and formally desuspending them produces spheres $S^{p,q}$, $p, q \in \mathbb{Z}$, and this yields bigraded homotopy groups:

$$\pi_{p,q}(X) = [S^{p,q}, X].$$

Much of the information known about these groups is due to Morel. He showed that $\pi_{p,q}(S^{0,0}) = 0$ if $p < q$ [15]. Moreover, he showed that $\pi_{n,n}(S^{0,0})$ is the Milnor–Witt K -theory of the ground field [14].

Morel also established the foundations of the motivic Adams spectral sequence [13]. This is a trigraded spectral sequence of the form

$$E_2 = \text{Ext}_{\mathcal{A}^*}^{s,t,u}(H^{*,*}(X), H^{*,*}(\text{Spec}(k))) \Rightarrow \pi_{t-s,u}(\widehat{X}_2),$$

where $\mathcal{A}^*$ is the mod-2 motivic Steenrod algebra, where $H^{*,*}(-)$ is mod-2 motivic cohomology, and where $\widehat{X}_2$ is the nilpotent completion of X with respect to the motivic Eilenberg–MacLane spectrum $H\mathbb{F}_2$. The motivic Steenrod algebra and the motivic cohomology of a point were worked out by Voevodsky, and we will review the salient points below. Working over the ground field $\mathbb{C}$, Dugger and Isaksen have a research program to compute the relevant Ext groups and have run the spectral sequence in low dimensions [5].

This paper serves as a preamble to joint work with Dugger and Isaksen in which we will compute motivic stable homotopy groups using the Adams spectral sequence over $\mathbb{R}$ through a range. The primary tool we will use is a descent style spectral sequence which converts motivic Ext over $\mathbb{C}$ to motivic Ext over $\mathbb{R}$. In this paper, we compute several examples which demonstrate the subtleties of the full form. These examples have the added advantage of being related to motivic BP and

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$BP\langle n \rangle$ and also being conjecturally related v_1 -periodic phenomena. In particular, this computation should detect a large collection of v_1 -periodic homotopy classes and describe their basic relations. We will return to these applications in the last section.

2. Background

2.1. Motivic cohomology and the motivic Steenrod algebra

For completeness and for reference, we summarize the basic results needed about

$$\mathbb{M}_2 = H^{*,*}(\mathrm{Spec}(\mathbb{R}))$$

and of $\mathcal{A}^*$, the mod 2 motivic Steenrod algebra. In this section and in all that follows, we will almost exclusively work over $\mathrm{Spec}(\mathbb{R})$. In a few situations, we will also work over $\mathrm{Spec}(\mathbb{C})$, and in those cases, we will use a superscript $\mathbb{C}$ to distinguish it from the real case.

Theorem 2.1 ([22]). *As an algebra,*

$$\mathbb{M}_2 = \mathbb{F}_2[\tau, \rho],$$

where $|\tau| = (0, 1)$ and $|\rho| = (1, 1)$.

We will, in our description of the dual Steenrod algebra and homology, use the standard convention that $H^{*,*} = H_{-*, -*}$. We will continue to use the names τ and ρ . The class ρ is in the image of the Hurevicz map, and the corresponding element in $\pi_{-1, -1}(S^{0,0})$ is represented in Morel's Milnor–Witt K -theory description by $[-1] \in K_1^M(\mathbb{R})$.

Voevodsky also computed the motivic Steenrod algebra over $\mathbb{R}$ [23,21]. It is the associative algebra over $\mathbb{M}_2$ generated by classes Sq^{2i} and Sq^{2i+1} , in bidegrees $(2i, i)$ and $(2i+1, i)$ respectively, subject to a substantially more complicated form of the Adem relations (which will not be needed directly). The ground ring $\mathbb{M}_2$ is not central, since we have

$$Sq^1 \tau = \rho.$$

This means that the dual Steenrod algebra $\mathcal{A}$ is a Hopf algebroid over $\mathbb{M}_2$, the structure of which was also computed by Voevodsky.

Theorem 2.2 ([23,21]).

$$\mathcal{A} = (\mathbb{M}_2, \mathbb{M}_2[\xi_1, \dots][\tau_0, \dots]/(\tau_i^2 - \rho\tau_{i+1} - \rho\tau_0\xi_{i+1} - \tau\xi_{i+1})).$$

The left unit is the canonical inclusion, while the right unit is given by $\eta_R(\rho) = \rho$ and $\eta_R(\tau) = \tau + \tau_0\rho$. The coproducts on the generators ξ_i and τ_i are the classical coproducts. The bidegrees of the elements are given by $|\xi_i| = (2^{i+1} - 2, 2^i - 1)$ and $|\tau_i| = (2^{i+1} - 1, 2^i - 1)$.

The monomial basis (as an $\mathbb{M}_2$ -algebra) gives rise to a dual basis in $\mathcal{A}^*$ which is the motivic analogue of the Milnor basis. The element dual to $\xi_i^{2^i}$ is $Sq^{2^{i+1}}$, and the element dual to τ_i is the exterior Milnor class Q_i . We remark that these are in general not primitive, since $\rho\tau_i$ occurs in the product τ_{i-1}^2 .

Just as in the classical case, we also consider finitely generated subalgebras of the Steenrod algebra. In the dual case, these give Hopf algebroid quotients of $\mathcal{A}$ which are finitely generated over $\mathbb{M}_2$.

Definition 1.

(1) Let $\mathcal{A}(n)$ denote the quotient Hopf algebroid obtained by reducing modulo the ideal $(\xi_1^{2^n}, \xi_2^{2^{n-1}}, \dots, \xi_n^2, \xi_{n+1}, \dots) + (\tau_{n+1}, \dots)$:

$$\mathcal{A}(n) = (\mathbb{M}_2, \mathbb{M}_2[\xi_1, \dots, \xi_n][\tau_0, \dots, \tau_n]/(\xi_i^{2^{n-i+1}}, \tau_i^2 - \rho\tau_{i+1} - \rho\tau_0\xi_{i+1} - \tau\xi_{i+1})).$$

(2) For $0 \leq n \leq \infty$, let $E(n)$ denote the reduction modulo $(\xi_1, \dots) + (\tau_{n+1}, \dots)$:

$$E(n) = (\mathbb{M}_2, \mathbb{M}_2[\tau_0, \dots, \tau_n]/(\tau_i^2 - \rho\tau_{i+1}, \tau_n^2)).$$

The Hopf algebroid $\mathcal{A}(n)$ is dual to the sub-Hopf algebra of the motivic Steenrod algebra generated by Sq^{2^i} for $i \leq n$, while $E(n)$ is dual to the sub-Hopf algebra generated by Q_i for $i \leq n$.

2.2. Bockstein spectral sequence

The Hopf algebroid $\mathcal{A}$ has a distinguished invariant ideal: (ρ) , and if we reduce modulo ρ , then we recover the complex motivic dual Steenrod algebra $\mathcal{A}^{\mathbb{C}}$. We therefore have a Miller–Novikov style Bockstein spectral sequence for undoing this reduction [18]. This is simply the spectral sequence associated to the filtration of $\mathcal{A}$ by powers of ρ .

Proposition 2.3. *There is a quadruply graded spectral sequence of algebras of the form*

$$E_1 = \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho] \implies \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2).$$

Since the motivic Steenrod algebra is bigraded, the Ext groups are trigraded. We will use the following Adams convention: $|x| = (t - s, s, u)$, where t is the internal topological degree of any class representing x in the cobar complex, s is the cohomological degree, and u is the motivic weight. By definition, the only algebra generator of non-zero Bockstein degree is ρ , so we will only describe the aforementioned tridegrees, even in the Bockstein spectral sequence.

There are some immediate permanent cycles.

Proposition 2.4. *If $x \in \text{Ext}_{\mathcal{A}^c}(\mathbb{M}_2, \mathbb{M}_2)$ can be represented in the cobar complex using only polynomials in the classes τ_i and ξ_i , then x survives the Bockstein spectral sequence.*

This is because the ρ -Bockstein spectral sequence measures the deviation of a cobar differential from being zero. If our classes are classical, then their bar differential is zero regardless of the presence or absence of ρ . We will use this to see that certain classes are permanent cycles.

Before continuing, we present a general lemma on the creation of new permanent cycles in Bockstein spectral sequences. This will help us streamline many of the computations.

Lemma 2.5. *In a Bockstein spectral sequence associated to the filtration of an associative, homotopy-commutative differential graded algebra A by a central element b , if $d_r(x) = b^r \cdot a$, then every element in $a \cdot E_{r+1}$ is a permanent cycle.*

Proof. The reason for this lies in the construction of Bockstein spectral sequences. A d_r -differential $d_r(x) = ab^r$ means that in $H^*(A)$, $ab^r = 0$ while $ab^k \neq 0$ for $k < r$. By associativity and commutativity of the multiplication in $H^*(A)$, we conclude that for all classes $c \in E_{r+1}$, $a \cdot c$ is annihilated by b^r . This means it can neither be the target of a d_s -differential (since it is killed by a smaller power of b) nor be the source of a d_s -differential (the target of which must be b -torsion free). $\square$

Remark. The associativity condition is essential. The homotopy-commutativity can be greatly weakened, but the statement of the lemma becomes much more complicated, since we must consider any product of elements on E_{r+1} , at least one of which is a . For our purposes, A can be taken to be the cobar complex.

Computing $\text{Ext}_{\mathcal{A}^c}(\mathbb{M}_2^c, \mathbb{M}_2^c)$ is at least as difficult as computing the classical Ext groups. We therefore will look at two much simpler cases, illustrating the ρ -Bockstein spectral sequence by using it to compute $\text{Ext}_{E(n)}(\mathbb{M}_2, \mathbb{M}_2)$ for all n and $\text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathbb{M}_2)$.

The starting inputs are the cohomology of $E(n)^c$ and $\mathcal{A}(1)^c$.

Proposition 2.6.

(1) For $0 \leq n \leq \infty$,

$$\text{Ext}_{E(n)^c}(\mathbb{M}_2^c, \mathbb{M}_2^c) \cong \mathbb{M}_2^c[v_0, \dots, v_n],$$

where $|v_i| = (2^{i+1} - 2, 1, 2^i - 1)$.

(2) As an algebra,

$$\text{Ext}_{\mathcal{A}(1)^c}(\mathbb{M}_2^c, \mathbb{M}_2^c) \cong \mathbb{M}_2^c[v_0, \eta, x, v_1^4]/(v_0\eta, \tau\eta^3, \eta x, x^2 - v_0^2v_1^4),$$

where $|\eta| = (1, 1, 1)$, $|x| = (4, 3, 2)$, and the classes v_0 and v_1^4 have the same tridegrees as above.

Proof. For $E(n)$, this is an immediate consequence of a change-of-rings theorem: the Hopf algebra $E(n)^c$ is the classical Hopf algebra $E(n)$ base changed to $\mathbb{M}_2$. For $\mathcal{A}(1)^c$, this computation can be done in many ways, and a resolution-based approach will appear in the thesis of Shkembli [19]. $\square$

In the cohomology of $E(n)$, the classes v_i are all classical. For $\mathcal{A}(1)$, the classes v_0, η, x , and v_1^4 are classical: they have cobar representatives with no reference to ρ or τ (although we must be careful to distinguish between ξ_1 and τ_0^2 when lifting the usual representatives). Proposition 2.4 ensures that these are all permanent cycles.

3. Ext over $E(n)$

3.1. Bockstein spectral sequence

Throughout this section, let n be a positive integer or infinity. In this section, we will prove the following theorem.

Theorem 3.1. *As an algebra,*

$$\text{Ext}_{E(n)}(\mathbb{M}_2, \mathbb{M}_2) = \mathbb{F}_2[\rho, \tau^{2^{n+1}}, v_i(j) | 0 \leq i \leq n, 0 \leq j] / \rho^{2^{i+1}-1} v_i(j),$$

and subject to the additional relations:

- (1) If $i \geq k$, then $v_i(j) \cdot v_k(\ell) = v_i(j + 2^{k-i}\ell) \cdot v_k(0)$,
- (2) and if $j \geq 2^{n-i}$, then $v_i(j) = \tau^{2^{n+1}} v_i(j - 2^{n-i})$.

Moreover, the class $v_i(j)$ is represented on E_1 by $\tau^{2^{i+1}j} v_i$.

While the statement of the theorem is complicated, the method of computation is less so. The E_1 -term of the Bockstein spectral sequence is

$$E_1 = \text{Ext}_{E(n)\mathbb{C}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho] = \mathbb{F}_2[\rho, \tau, v_0, \dots, v_n].$$

Proposition 2.4 guarantees that the classes v_0 through v_n are all permanent cycles (and in fact, the cobar representatives τ_0 through τ_n are visibly primitive in $E(n)$), and by construction, the class ρ is as well. We therefore need only understand the differentials on τ .

Theorem 3.2.

(1) For $i \leq n$, $E_{2i} = \dots = E_{2^{i+1}-1}$ and

$$E_{2^{i+1}-1} = \mathbb{F}_2[\rho, \tau^{2^i}, v_j(k), v_i, \dots, v_n | 0 \leq j \leq i-1, 0 \leq k | \rho^{2^{i+1}-1} v_j(k),$$

subject to two additional families of relations:

$$v_m(r) \cdot v_j(\ell) = v_m(r + 2^{j-m}\ell) \cdot v_j(0),$$

if $m \leq j$, and if $r \geq 2^{i-m}$,

$$v_m(r) = \tau^{2^{i+1}} v_m(r - 2^{i-m}).$$

(2) The classes $v_m(r)$ are represented on E_1 by $\tau^{2^{m+1}r} v_m$ and are permanent cycles.

(3) The $d_{2^{i+1}-1}$ -differential is determined by

$$d_{2^{i+1}-1}(\tau^{2^i}) = \rho^{2^{i+1}-1} v_i.$$

Proof. We prove this by induction on i . The base case of $i = 0$ is the earlier analysis of the Bockstein E_1 -term, and we already determined that τ is the only generator that is not a permanent cycle. We begin by showing the third part of the theorem. The first and second parts will follow easily from this and [Lemma 2.5](#).

The d_k -differential on τ^j is computed by reducing $\eta_L(\tau^j) - \eta_R(\tau^j)$ modulo ρ^{k+1} . The units are ring homomorphisms, so

$$\eta_R(\tau^{2^i}) = (\eta_R(\tau))^{2^i} = \tau^{2^i} + (\rho\tau_0)^{2^i}.$$

Since in $E(n)$, $\tau_i^2 = \rho\tau_{i+1}$ for $i < n$ and $\tau_n^2 = 0$, we conclude immediately that

$$\eta_R(\tau^{2^i}) = \begin{cases} \tau^{2^i} + \rho^{2^{i+1}-1} \tau_i & i \leq n, \\ \tau^{2^i} & \text{otherwise.} \end{cases}$$

The difference between this and the left unit gives the desired differential:

$$d_j(\tau^{2^i}) = \begin{cases} \rho^{2^{i+1}-1} v_i & i \leq n, \quad j = 2^{i+1} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can now complete our inductive argument. We assume that $E_{2^{i+1}-1}$ is of the stated form. By the above argument, there is a differential $d_{2^{i+1}-1}(\tau^{2^i}) = \rho^{2^{i+1}-1} v_i$.

By the induction hypothesis, the classes annihilated by $\rho^{2^{i+1}-1}$ are in the ideal generated by the infinite families of permanent cycles $v_0(j), \dots, v_{i-1}(j)$. The collection of generators for this ideal is closed under τ^{2^i} -multiplication, so the only new $d_{2^{i+1}-1}$ -cycle is $\tau^{2^{i+1}}$. This explains the reason for including the redundant generators in our list of permanent cycles: the bookkeeping at this point is substantially simplified.

Let $v_i(j)$ denote the class $\tau^{2^{i+1}j} v_i$. Since v_i was truncated by $d_{2^{i+1}-1}$, [Lemma 2.5](#) shows that for every $j \geq 0$, $v_i(j)$ is a permanent cycle. We therefore conclude that

$$E_{2^{i+1}} = \mathbb{F}_2[\rho, \tau^{2^{i+1}}, v_j(k), v_{i+1}, \dots, v_n | 0 \leq j \leq i, 0 \leq k | \rho^{2^{i+1}-1} v_j(k),$$

subject additionally to the relations

$$v_m(r) \cdot v_j(\ell) = v_m(r + 2^{j-m}\ell) \cdot v_j(0),$$

if $m \leq j$, and if $r \geq 2^{i-m}$,

$$v_m(r) = \tau^{2^{i+1}} v_m(r - 2^{i-m}).$$

Since all of the generators with the exception of $\tau^{2^{i+1}}$ are permanent cycles, and since we have directly determined the differentials on $\tau^{2^{i+1}}$, we conclude that

$$E_{2^{i+1}} = \dots = E_{2^{i+2}-1},$$

completing the proof of the first part. $\square$

We remark that the “obvious” relations for the classes $v_i(j)$ are actually inherited from E_1 : here we can just multiply as we normally do. We can therefore conclude these relations through E_∞ .

Corollary 3.3. *The E_∞ -page of the Bockstein spectral sequence for $\text{Ext}_{E(n)}(\mathbb{M}_2, \mathbb{M}_2)$ is*

$$E_\infty = \mathbb{F}_2[\rho, \tau^{2^{n+1}}, v_i(j) | 0 \leq i \leq n, 0 \leq j] / \rho^{2^{i+1}-1} v_i(j),$$

and subject to

$$v_i(j) \cdot v_k(\ell) = v_i(j + 2^{k-i}\ell) \cdot v_k(0)$$

whenever $i \leq k$, and

$$v_i(j) = \tau^{2^{n+1}} v_i(j - 2^{n-i})$$

whenever $j \geq 2^{n-i}$.

The tridegree of the element $v_i(j)$ is $(2^{i+1} - 2, 1, 2^i - 1 - 2^{i+1}j)$.

The final issue to tackle is the resolution of hidden extensions. We have a family of multiplicative relations, but these are only *a priori* true modulo ρ . We must show that the last family of relations has no ρ -divisible correction factors. Better said, we must show that

$$v_i(j) \cdot v_k(\ell) - v_i(j + 2^{k-i}\ell) \cdot v_k(0)$$

is zero. While the argument is not difficult (it amounts to elementary 2-adic number theory), it is somewhat involved. Once this point is resolved, however, we have completed the proof of [Theorem 3.1](#).

3.2. Non-existence of hidden extensions

Theorem 3.4. *In $\text{Ext}_{E(n)}(\mathbb{M}_2, \mathbb{M}_2)$,*

$$v_i(j) \cdot v_k(\ell) - v_i(j + 2^{k-i}\ell) \cdot v_k(0) = 0$$

whenever $k \geq i$.

We argue this using tridegree considerations, explicitly determining that there is only a single non-zero class in the tridegree of $v_i(j) \cdot v_k(\ell)$. Since this has cohomological degree 2 and since ρ has cohomological degree 0, we conclude that anything in this tridegree has the form $\rho^m v_r(s) \cdot v_q(t)$. In all that follows, we will assume we have constants m, r, s, q, t, i, j, k , and ℓ such that

$$|\rho^m v_r(s) \cdot v_q(t)| = |v_i(j) \cdot v_k(\ell)| \quad (1)$$

holds true. We may additionally assume without loss of generality that $r \geq q$ and $i \geq k$. We present without proof two small lemmata which allow us to conclude this result. The arguments are elementary, requiring only an analysis of the degrees and weights of the elements in question, and we leave them to the reader.

Lemma 3.5. *If Eq. (1) holds, then $r = i$.*

Lemma 3.6. *If Eq. (1) holds, then $q = k, m = 0$, and*

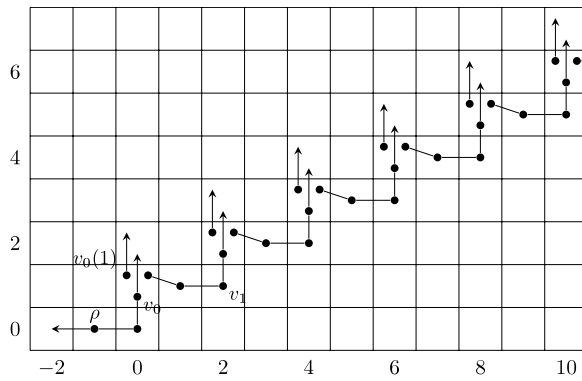
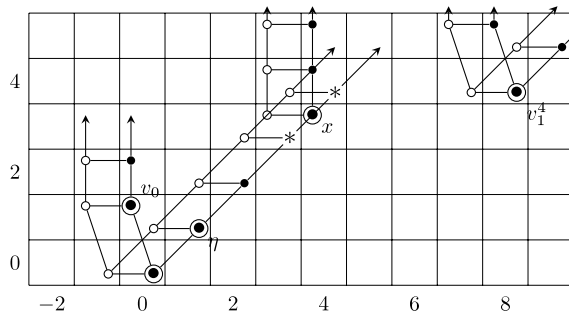
$$2^{r+1}s + 2^{q+1}t = 2^{i+1}j + 2^{k+1}\ell.$$

Proof of Theorem 3.4. Lemmata 3.5 and 3.6 show that there is a unique class in the tridegree under consideration. We therefore conclude that the multiplication in the associated graded is the correct one. $\square$

We close this section with a picture of $\text{Ext}_{E(1)}(\mathbb{M}_2, \mathbb{M}_2)$ (see [Fig. 1](#)). This has the advantage of simultaneously being non-trivial (unlike $\text{Ext}_{E(0)}(\mathbb{M}_2, \mathbb{M}_2)$) and small enough to easily draw (unlike $\text{Ext}_{E(2)}(\mathbb{M}_2, \mathbb{M}_2)$ and beyond). This computation is also used in our subsequent computations. In the picture, the horizontal axis shows $t-s$ while the vertical axis gives s . A black dot represents $\mathbb{F}_2[\tau^4]$. Vertical lines represent multiplication by v_0 , while horizontal lines (read right-to-left) denote multiplication by ρ .

4. Ext over $\mathcal{A}(1)$

Pictures are indispensable in the computation of $\text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathbb{M}_2)$. Since our spectral sequences are quadruply graded, we must choose two gradings for our axes. We will draw our figures like Adams charts: the horizontal axis is the topological degree $t-s$ and the vertical axis is the cohomological degree s . We suppress explicit mention of both the motivic weight and the Bockstein degree.

Fig. 1. $\text{Ext}_{E(1)}(M_2, M_2)$.Fig. 2. The ρ -Bockstein E_1 -page.

By construction, the target of a d_r -differential is divisible by ρ^r . We will use a solid, horizontal line to denote multiplications by ρ , so the only possible targets for a d_r -differentials are classes which are followed on the right by exactly r solid lines (thus the Bockstein degree is implicitly reflected in the picture). Moreover, the Bockstein differentials preserve the internal degree and the weight while increasing the filtration by 1. This means that in our pictures, the Bockstein differentials appear as Adams d_1 -differentials. These two observations about the structure and targets of differentials greatly simplifies the story, allowing a visual check for the existence of differentials.

Given the plethora of elements in all of the figures, we also adopt a convention: any class which is ρ -torsion is not drawn on the following pages. This corresponds to running the ρ -Bockstein spectral sequence with ρ inverted, uncluttering the picture. The algebraic machinery of the spectral sequence keeps track of the ρ -torsion elements for us.

In all of the pictures that follow, an open circle represents a polynomial algebra on ρ (times whatever is linked to the open circle by horizontal lines). Circled classes are algebra generators. For E_1 , black dots represent $\mathbb{F}_2[\tau]$, and stars are $\mathbb{F}_2$ as a graded $\mathbb{F}_2[\tau]$ -module.

4.1. The E_1 -page

We present a picture of the E_1 -page in Fig. 2.

Proposition 4.1. *The d_1 -differential is determined by*

$$d_1(\tau) = v_0\rho.$$

Proof. The d_1 -differential is given by computing the difference of the left and right units on τ and reducing modulo ρ^2 . For degree reasons, and from our earlier observation about permanent cycles, this is the only generator supporting a d_1 . $\square$

This produces two d_1 -cycles: τ^2 and $\eta_0 = \eta\tau$.

Proposition 4.2. *The class η_0 is a permanent cycle.*

Proof. The class η_0 is represented by τ_0^2 in the cobar complex. Since τ_0 is primitive, so is this class, and we conclude that it is a permanent cycle. $\square$

We pause here to identify some extensions that increase the Bockstein filtration by 1. These arise from Massey products built out of τ . From E_2 on, the class τ no longer exists, so multiplication by it is not defined. However, many of the τ multiples of classes persist, and we can represent these as a Massey product:

$$\tau \cdot (-) = \langle \rho, v_0, - \rangle.$$

In particular, we learn that the class η_0 is the bracket $\langle \rho, v_0, \eta \rangle$ (this is also immediate from the form of η_0).

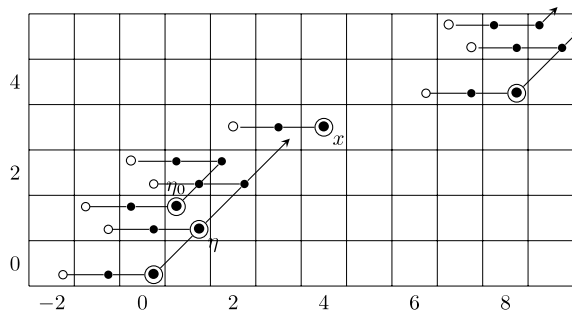


Fig. 3. The ρ -Bockstein E_2 -page.

$$v_0 \eta_0 = \rho \eta \eta_0$$

and

$$\eta\eta_0^2 = \rho x.$$

$$\eta_0 \cdot v_0 = \langle \rho, v_0, \eta \rangle \cdot v_0 = \rho \cdot \langle v_0, \eta, v_0 \rangle = \rho \eta_0 \eta,$$

where the last bracket, the motivic analogue of $\langle 2, \eta, 2 \rangle = \eta^2$, follows from an elementary computation in the cobordism complex.

$$\eta_0^2 \eta = \eta_0 \cdot (\eta_0 \eta) = \langle \rho, v_0, \eta \rangle \cdot \eta_0 \eta = \rho \langle v_0, \eta, \eta_0 \eta \rangle = \rho x. \quad \square$$

These relations will be essential for later differentials and cycles. In particular, we conclude immediately that modulo ρ^2 ,

$$\tau^4 \eta^3 = \rho \tau^2 \chi.$$

4.2. The E_2 -page

$$\mathbb{F}_2[\tau^2, v_0, \eta, \eta_0, x, v_1^4, \rho]/(v_0\rho, v_0\eta, \tau^2\eta^3, \eta x, x^2 - v_0^2v_1^4, \eta_0\eta^2, \eta_0^2 - \tau^2\eta^2, \eta_0x, v_0\eta_0).$$

We present a picture in Fig. 3. Here a dot indicates $\mathbb{F}_2[\tau^2]$.

Proposition 4.4. *The d_2 -differential is determined by*

$$d_2(\tau^2) = \rho^2 \eta_0.$$

This is immediate from the cobar complex.

In addition to τ^4 , we have new d_2 -cycles:

$$v_0(1) = v_0\tau^2, \quad b = x\tau^2, \quad \text{and } c = \eta\eta_0\tau^2,$$

since

$$\eta_0 \cdot (v_0, x, \eta\eta_0) \equiv 0 \pmod{(\rho)}.$$

We remark that [Lemma 2.5](#) does not apply to the class c , since c is not η_0 -divisible on E_3 .

Proposition 4.5. *There is a hidden η -multiplication:*

$$v_0(1)\eta = \rho\eta_0^2.$$

Proof. From E_3 on, τ^2 multiplication is no longer well defined. Instead, we have a bracket formulation:

$$\tau^2 \cdot (-) = \langle \eta_0, \rho^2, - \rangle \text{ or } \langle \rho^2, \eta_0, - \rangle.$$

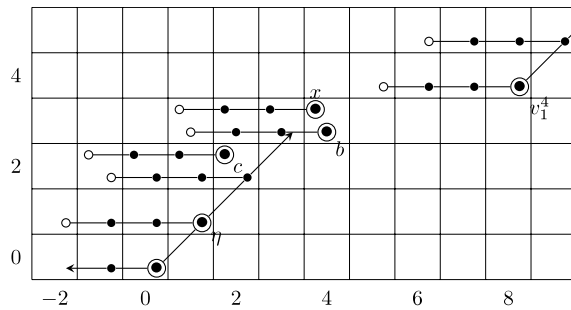
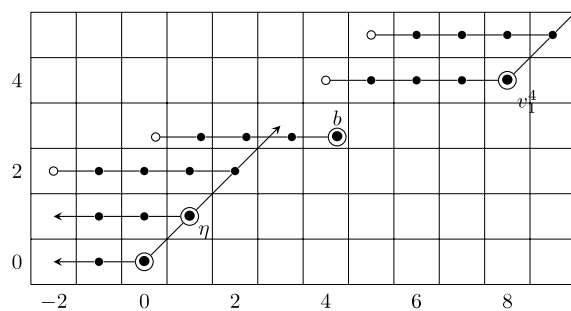
Since v_0 is ρ -torsion, we use the former, rearranging slightly to see that

$$v_0(1) = \langle \eta_0 \rho, \rho, v_0 \rangle.$$

If we multiply by η , then we get

$$v_0(1)\eta = \langle \eta_0\rho, \rho, v_0 \rangle \eta = \eta_0\rho \langle \rho, v_0, \eta \rangle = \rho\eta_0^2,$$

where the last equality follows from the definition of η_0 . $\square$

Fig. 4. The ρ -Bockstein E_3 -page.Fig. 5. The ρ -Bockstein E_4 -page.

We remark that as with most statements, it is also easy to give a justification of this using explicit cycles in the cobar complex. The class $\tau^2 v_0$ is represented by

$$\tau^2 \tau_0 + \tau \rho \tau_0^2 + \rho^2 \tau_0^3,$$

and multiplication by ξ_1 can be easily rearranged to give $\rho \tau_0^2 | \tau_0^2$.

Just as before, we can work out additional hidden extensions:

$$v_0(\tau^2 \eta^2) = \langle \rho^2, \eta_0, \eta^2 \rangle v_0 = \rho^2 \langle \eta_0, \eta^2, v_0 \rangle = \rho^2 x.$$

This is consistent with the previous result:

$$v_0(\eta_0^2) = (\rho \eta \eta_0) \cdot \eta_0 = \rho \cdot (\rho x).$$

These relations actually immediately tell us that we are not finished. The classes v_0 and η_0 are ρ -torsion while x is not.

4.3. The E_3 -page

From this point on, giving generators and relations for the pages of the spectral sequence becomes unenlightening. Instead, we rely solely on the pictures of the pages. We present the E_3 page as Fig. 4. Here black dots are $\mathbb{F}_2[\tau^4]$.

Proposition 4.6. *There is a d_3 -differential*

$$d_3(c) = \rho^3 x.$$

Proof. The class c was a d_2 -cycle because $\eta_0^2 \eta$ was zero modulo ρ . However, Proposition 4.3 shows that $\eta_0^2 \eta = \rho x$. Thus the differential on c actually takes the form

$$c \mapsto \rho^2 \eta_0^2 \eta = \rho^3 x. \quad \square$$

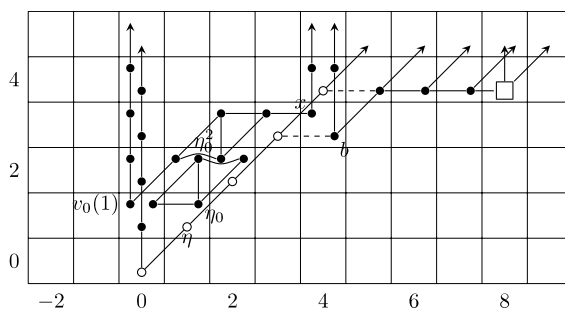
For degree reasons, there are no more d_3 -differentials. Additionally, since all of the classes which annihilate $\rho^3 x$ also annihilate c , we produce no new cycles. This in turn means we do not have new multiplicative relations.

4.4. The E_∞ -page and Ext

The E_4 -page is shown in Fig. 5. The only untruncated cycles at this stage are τ^4 , η , b , and v_1^4 . This leads to tremendous sparseness in the spectral sequence.

The class b is a permanent cycle for a very simple reason: ρb is a permanent cycle, so b must also be a permanent cycle (this is an artifact of the fact that higher Bockstein differentials never further truncate classes).

This class is also present in a hidden extension not readily predicted from shuffling considerations.

Fig. 6. $\text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathbb{M}_2)$.

Proposition 4.7. *There is a hidden η -multiplication of the form*

$$\eta \cdot b = \rho^3 v_1^4.$$

Proof. We will prove this by comparing with $\text{Ext}_{E(1)}(\mathbb{M}_2, \mathbb{M}_2)$. Let $C(\eta)$ denote the subcomodule algebra of $\mathcal{A}(1)$ generated by ξ_1 . A change-of-rings argument shows that

$$\text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, C(\eta)) = \text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathcal{A}(1) \square_{E(1)} \mathbb{F}_2) = \text{Ext}_{E(1)}(\mathbb{M}_2, \mathbb{M}_2),$$

so we can interpret the effect in Ext of the canonical quotient map $\mathcal{A}(1) \rightarrow E(1)$ as the reduction modulo η .

The induced map

$$\text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathbb{M}_2) \rightarrow \text{Ext}_{E(1)}(\mathbb{M}_2, \mathbb{M}_2)$$

takes v_1^4 to v_1^4 . By Theorem 3.1, $\rho^3 v_1^4 = 0$ in $\text{Ext}_{E(1)}(\mathbb{M}_2, \mathbb{M}_2)$, so we conclude that the class $\rho^3 v_1^4$ must be divisible by η in $\text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathbb{M}_2)$. The only possible relation is the given one, for tridegree reasons. $\square$

We remark that from this, we get an amusing formula:

$$\tau^4 \eta^4 = \rho^4 v_1^4.$$

This is a strange cobar analogue of $\tau_0^4 = 0$.

The Bockstein differentials record the ρ -torsion, and during the computations on earlier pages, we recorded hidden ρ -based extensions. This allows us to determine the structure of $\text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathbb{M}_2)$ as a ring. We present the answer in Fig. 6. Some words about the notation here are in order. Solid dots represent $\mathbb{F}_2[\tau^4]$, while open dots represent $\mathbb{F}_2[\rho, \tau^4]$. Solid horizontal lines are ρ -multiplication; solid vertical lines are v_0 -multiplication, and solid lines of slope 1 are η -multiplication. The box in $(8, 4)$ represents the class v_1^4 , and but for the ρ -multiplications drawn in the picture, this represents another copy of the picture. Finally, the reader will note the presence of several dashed horizontal lines. These indicate that ρ on the right-hand class is τ^4 on the left-hand class.

Given the plethora of generators, it is impossible to simply record all multiplicative relations in the picture. The following proposition summarizes those not immediately seen.

Proposition 4.8. *We have*

$$v_0(1)^2 = \tau^4 v_0^2, \quad v_0(1)b = \tau^4 x, \quad x^2 = v_0^2 v_1^4, \quad xb = v_0 v_0(1)v_1^4, \quad b^2 = \tau^4 v_1^4(v_0^2 + \rho^2 \eta^2).$$

Proof. With the exception of the last equality, these are all immediate from the ρ -Bockstein spectral sequence. While there are additional classes in the same bidegree as v_0^2 and x^2 , there are no additional ρ -torsion classes in these bidegrees. The leading term of the last equality is immediate from the spectral sequence. The correction term follows from the two possible ways to associate $\tau^4 \eta^3 b$. $\square$

5. Applications

Hopkins has remarked that the motivic Adams spectral sequence converges to the completion with respect to 2 and η . This has recently been verified by Hu et al. [11, 10], who show that with the obvious connectivity hypotheses, the η -completion is unnecessary. We will apply this and the previous computations to determine the 2-completed motivic homotopy of $B\text{PGL}$ and $B\text{PGL}\langle n \rangle$ over $\mathbb{R}$ and that of a conjectural connective spectrum $ko^{\mathbb{R}}$.

We begin with a general remark about comparison with the complex and classical cases. There is a complexification map from the real Adams spectral sequence to the complex Adams spectral sequence, and the map on the E_2 -terms is reduction modulo ρ . In the cases we consider, the complex Adams spectral sequence collapses with no extensions. This collapse puts large restrictions on the possible Adams differentials and multiplicative extensions. By naturality of the Adams spectral sequence, we conclude that the targets of any Adams differential over $\mathbb{R}$ must be divisible by ρ , as must the correction terms for any multiplicative extensions. In the $ko^{\mathbb{R}}$ case, this greatly simplifies computations.

5.1. The Adams spectral sequence for BPGL

Let MGL be the Thom spectrum of the universal bundle over BGL , and let $BPGL$ be the standard summand of the 2-localization. The work of Borghesi has shown that from the view-point of homology, $BPGL$ behaves like the classical BP [1].

Theorem 5.1. *As an $\mathcal{A}$ -comodule algebra,*

$$H_{*,*}(BPGL; \mathbb{F}_2) = \mathcal{A} \square_{E(\infty)} \mathbb{M}_2.$$

A standard change-of-rings argument then shows us that the E_2 -term of the Adams spectral sequence is a ring we computed in Section 3.

Corollary 5.2. *As an algebra,*

$$E_2 = \text{Ext}_{\mathcal{A}}(\mathbb{M}_2, H_{*,*}(BPGL)) = \text{Ext}_{E(\infty)}(\mathbb{M}_2, \mathbb{M}_2).$$

A combinatorial analysis of the tridegrees shows the following theorem.

Theorem 5.3. *The Adams spectral sequence for BPGL collapses at E_2 with no exotic extensions.*

While there are no filtration jumping extensions, there are group extensions detected by $v_0(0)$.

Lemma 5.4. *In the Adams spectral sequence for BPGL, multiplication by $v_0(0)$ detects multiplication by 2.*

Proof. The primary operation τ_0 detects the attaching map between the 1 and 2-cells in $B\mu_2$: $1 - \epsilon$, where ϵ is the flip map on $S^{0,0} \wedge S^{0,0}$ [23]. This means $v_0(0)$ represents $1 - \epsilon$ multiplication. Morel's description of $\pi_{0,0}S^{0,0}$ shows that $2 - v_0(0) = \rho\eta$ [14]. Our Adams spectral sequence computation shows that $\pi_{1,1}BPGL = 0$, so we conclude that the image of η under the unit map is 0. Thus in $BPGL$, $2 = v_0(0)$. $\square$

To provide intuition for our result about the homotopy of $BPGL$, we give another argument, connecting this to equivariant homotopy and the work of Hu and Kriz [8,9]. There is a realification functor t from the motivic stable homotopy category over $\mathbb{R}$, $\mathcal{H}_{\mathbb{R}}$, to the $\mathbb{Z}/2$ -equivariant stable homotopy category, $\mathcal{H}^{\mathbb{Z}/2}$, and it induces a map of bigraded homotopy rings

$$\pi_{*,*}(X) \rightarrow \pi_{*,*}(t(X)).$$

In the target, the bigraded homotopy groups are actually graded by the real representation ring of $\mathbb{Z}/2$. The specific dictionary is provided by the following: $t(S^{1,0}) = S^1$, the one point compactification of the trivial representation, and $t(\mathbb{G}_m) = S^\sigma$, the one point compactification of the sign representation. Thus $\pi_{p,q}(X) \mapsto \pi_{(p-q)+q\sigma}(t(X))$. It is also immediate that

$$t(\rho) = a_\sigma : S^0 \rightarrow S^\sigma,$$

the inclusion of the origin and the point at infinity into S^σ .

Hu and Kriz showed that $t(BPGL) = BP\mathbb{R}$, the summand of 2-localization of the cobordism spectrum of Real manifolds. They also computed the $RO(\mathbb{Z}/2)$ -graded homotopy groups of $BP\mathbb{R}$. This ring is generated by classes $v_i(j)$ for $0 \leq i$ and for $j \in \mathbb{Z}$ and a_σ , subject to the relations identical to those in Section 3:

$$v_i(j) \cdot v_k(\ell) = v_i(j + 2^{k-i}\ell) \cdot v_k(0)$$

when $i \leq k$, and $a_\sigma^{2^{i+1}-1}v_i(j) = 0$.

The realification functor takes the motivic $v_i(j)$ to the equivariant $v_i(j)$. We therefore conclude that this embeds $\pi_{*,*}BPGL$ into $\pi_{*,*}BP\mathbb{R}$. The image of $\pi_{*,*}BPGL$ is the subring of $\pi_{*,*}BP\mathbb{R}$ generated by all elements with bidegrees of the form (p, q) , $p \geq q$.

This last result can be classically shown using the slice spectral sequence. For spectra like MGL and $BPGL$, realification takes the motivic slice filtration to the equivariant one described by Dugger [3]. In both cases, the slice associated graded spectrum is

$$\bigvee_{p \in I} S^{|p|, |p|/2} \wedge H\mathbb{Z}_2,$$

where the indexing set I runs through all monomials in MU_* [7,8,6]. The associated map of slice E_2 -terms is then exactly the inclusion

$$H_{*,*}^{\text{Mot}}(S^{|p|, |p|/2}) \hookrightarrow H_{*,*}^{\mathbb{Z}/2}(S^{\frac{|p|}{2}(1+\sigma)}).$$

This map is simply a suspension of the ordinary realification map $H_{*,*}^{\text{Mot}} \rightarrow H_{*,*}^{\mathbb{Z}/2}$, the image of which is precisely the homology classes whose degrees satisfy Morel's connectivity result: $x \in H_{p,q}^{\mathbb{Z}/2}$ is in the image iff $p \geq q$.

The class $\tau^2 \in \pi_{0,-2}H\mathbb{Z}_2$ maps to a class $u_{2\sigma} \in \pi_{2,-2\sigma}H\mathbb{Z}_2$, and this class represents a kind of orientation of the representation 2σ . Though not stated in this language, the results of Hu and Kriz show that all of the slice differentials in the equivariant slice spectral sequence are determined by the differentials on the powers of $u_{2\sigma}$ [6]. By naturality, we

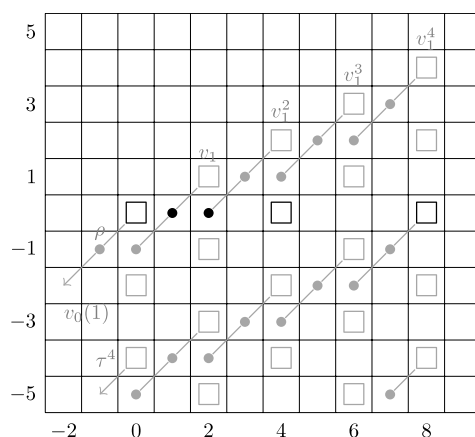


Fig. 7. The motivic homotopy of kgl .

conclude that the map from the motivic slice spectral sequence to the equivariant one is an injection on all pages, and the same is therefore true for the homotopy rings.

At this point we can further connect to the slice spectral sequence. If we filter by powers of $v_0(0)$ and the usual slice filtration (in both the motivic and the equivariant context), then we have a spectrum whose associated graded is

$$\bigvee_{p \in I'} S^{|p|, |p|/2} \wedge H\mathbb{F}_2,$$

where now I' is the associated graded of MU_* filtered by powers of 2. The equivariant version can again be completely determined using the results of Hu–Kriz, and we learn that the differentials are again completely determined by those on the powers of u_σ (equivariant homology class corresponding to τ):

$$u_\sigma^{2^n} \mapsto \rho^{2^{n+1}-1} v_n.$$

By naturality, these are also the motivic slice differentials. However, these are also exactly the differentials computed in Section 3! The author admits to finding this a surprising connection.

5.2. The Adams spectral sequence for $BPGL\langle n \rangle$

Continuing to mirror the classical constructions, we can form spectra $BPGL\langle n \rangle$. These arise by killing off the homotopy classes $v_{n+1}(0), v_{n+2}(0), \dots$, just as is done classically. The work of Hopkins and Morel shows that $BPGL\langle 1 \rangle$ is a spectrum $kgl_{(2)}$ which localizes to KGL [7], and Ormsby has shown that the homology of $BPGL\langle n \rangle$ is just as in the classical cases [17].

Theorem 5.5. *As an $\mathcal{A}$ -comodule,*

$$H_{*,*}BPGL\langle n \rangle = \mathcal{A} \square_{E(n)} \mathbb{M}_2.$$

The standard change-of-rings arguments then imply that the Adams E_2 -term is what we computed in Section 3.

Corollary 5.6. *The E_2 -term of the Adams–Novikov spectral sequence for $BPGL\langle n \rangle$ is*

$$E_2 = \text{Ext}_{\mathcal{A}}(\mathbb{M}_2, H_{*,*}(BPGL\langle n \rangle)) = \text{Ext}_{E(n)}(\mathbb{M}_2, \mathbb{M}_2).$$

While we make no assumptions about the algebra structure on such a spectrum, essentially all of the arguments for $BPGL$ carry over *mutatis mutandis*. With the single exception of the element $\tau^{2^{n+1}} \in \text{Ext}_{E(n)}^{0,0,-2^{n+1}}(\mathbb{M}_2, \mathbb{M}_2)$, all of the elements in Ext are in the image of the map from the Adams E_2 -term for $BPGL$. This implies they are all permanent cycles. The analysis from Section 3 also shows that $\text{Ext}_{E(n)}^{s,t,*}(\mathbb{M}_2, \mathbb{M}_2) = 0$ for $s > 0$ and $t - s < 0$, so we conclude that all elements in $\text{Ext}_{E(n)}^{0,0,*}(\mathbb{M}_2, \mathbb{M}_2)$ survive.

Corollary 5.7. *The Adams spectral sequence for $BPGL\langle n \rangle$ collapses with no exotic extensions.*

In particular, if we restrict attention to $n = 1$, then we recover Suslin's result connecting the algebraic K -theory of $\mathbb{R}$ with the homotopy of ko [20]. We present a picture of the homotopy in Fig. 7. In this, the horizontal axis is the degree and the vertical axis is the motivic weight. Boxes represent copies of $\mathbb{Z}_2$ while dots are copies of $\mathbb{Z}/2$. To draw attention to the weight zero piece (the groups of which are ko_*), we have grayed out all other weights.

5.3. The Adams spectral sequence for $ko^{\mathbb{R}}$

In this section, we assume the following conjecture.

Conjecture 5.8. *There is a ring spectrum $ko^{\mathbb{R}}$ that fits into a cofiber sequence*

$$\Sigma^{1,1}ko^{\mathbb{R}} \xrightarrow{\eta} ko^{\mathbb{R}} \rightarrow kgl.$$

Working over $\text{Spec}(\mathbb{C})$, Isaksen and Shkembli have produced a spectrum that is appropriately connective and sits in an analogous cofiber sequence [12]. The question becomes one of appropriately passing from $\text{Spec}(\mathbb{C})$ to $\text{Spec}(\mathbb{R})$ and of showing the existence of a structured multiplication.

If $ko^{\mathbb{R}}$ exists, then classical techniques and the results of Hopkins–Morel used in the previous section show that the homology of this spectrum is analogous to the classical one:

$$H_{*,*}ko^{\mathbb{R}} = \mathcal{A} \square_{\mathcal{A}(1)} \mathbb{M}_2.$$

The aforementioned change-of-rings argument tells us that we computed the Adams E_2 -term for $ko^{\mathbb{R}}$ in Section 4.

Corollary 5.9. *As an algebra,*

$$E_2 = \text{Ext}_{\mathcal{A}}(\mathbb{M}_2, H_{*,*}(ko^{\mathbb{R}})) = \text{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathbb{M}_2).$$

To determine if there are Adams differentials, we will again argue by sparseness.

Theorem 5.10. *The Adams spectral sequence for $\pi_{*,*}ko^{\mathbb{R}}$ collapses at E_2 .*

This in turn should have applications to v_1 -periodic elements in $\pi_{*,*}S^{0,0}$. Classically, part of the v_1 -periodic elements can be detected via the ko -Hurewicz map. Our conjectural spectrum $ko^{\mathbb{R}}$, being a ring spectrum, receives a map from $S^{0,0}$, the effect of which on Adams E_2 -terms is the canonical quotient. This allows us to identify a large part of the Hurewicz image and determine that not only does there seem to be many more of these v_1 -periodic elements than in classical homotopy but also their algebra structure is much richer.

Let $P(-)$ denote Adams' periodicity operator. The motivic May spectral sequence shows that the classes $P^n(\eta)$ map to $v_1^{4n}\eta$ (this is immediate in the complex case, and the naturality of the Bockstein spectral sequence gives the rest). The following corollary to our computation has no classical antecedent.

Corollary 5.11. *If $P^n(\eta)$ survives the Adams spectral sequence to give a non-zero homotopy class, then for all i, j , and k , $\rho^i \eta^j (P^n(\eta))^k$ survives and is non-zero.*

Proof. That these classes are permanent cycles is immediate from the survival of $P^n(\eta)$. The element $\rho^i \eta^j (P^n(\eta))^k$ has a non-trivial image in Adams E_2 -term for $ko^{\mathbb{R}}$, and since the target spectral sequence collapses, we conclude that this class cannot be the target of any differentials. $\square$

This is in stark contrast to what happens classically. Classically the classes like $P^n(\eta)$ are nilpotent and multiplication by η^3 is zero on them. In this motivic context, we see that if $P^n(\eta)$ survives, then it spawns a host of new classes.

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