

4. THE RATIONAL CALCULATION

The calculation of the rational homotopy is an immediate consequence of the following result.

Theorem 6.

a) There are elements $\zeta \in H^1(\mathbb{G}_2, (E_2)_0)$ and $\epsilon_3 \in H^3(\mathbb{G}_2^1, (E_2)_0)$ such that

$$H^s(\mathbb{G}_2, (E_2)_0) \cong \Lambda(e_3, \zeta_2) \otimes \mathbb{Z}_3[\beta^2 \Delta^{-1}]/(3\beta^2 \Delta^{-1})$$

b) Let $t = 4 \cdot 3^k m$ with $m \not\equiv 0 \pmod{3}$. Then $3^{k+1} H^*(\mathbb{G}_2, (E_2)_t) = 0$.

c) In all other cases $H^s(\mathbb{G}_2, (E_2)_t) = 0$.

Part (b) and (c) are easy consequences of the product decomposition $\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_3$. Part (a) follows from Theorem 5 and a straightforward Bockstein spectral sequence argument. The crucial point is that Theorem 5 gives $E_2^{s,t,0} = 0$ if $s = 1, 2$.

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Recent Computational work on EO_n

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(joint work with Michael J. Hopkins and Douglas C. Ravenel)

The Hopkins-Miller theorem ensures that there is an action of the Morava stabilizer group S_n on the Lubin-Tate spectrum E_n by E_∞ ring maps. If n is divisible by $p - 1$, then S_n has infinite cohomological dimension arising from p -torsion elements in the group. Since the $K(n)$ -local sphere, $L_{K(n)}S^0$, is the homotopy fixed points of S_n acting on E_n [1], it is hoped that by restricting attention to finite subgroups (which carry the bulk of the higher cohomology), we can understand computationally the homotopy of $L_{K(n)}S^0$. Indeed, this was successfully done by Adams-Baird and Ravenel for $n = 1$ [2] and by Goerss-Henn-Mahowald-Rezk, and Behrens for $n = 2$, $p = 3$ [3, 4]. The algebraic approximation to $\pi_* L_{K(n)}S^0$ by a finite subgroup G can be made rigid by considering the homotopy fixed point spectra $EO_n(G) = E_n^{hG}$ of Hopkins and Miller.

The computations of the homotopy of EO_{p-1} by Hopkins and Miller allowed Nave to demonstrate quite strong results about the non-existence of Smith-Toda complexes [5]. This computation relied on an understanding of the homotopy of the Lubin-Tate spectrum E_{p-1} as an algebra over $\mathbb{Z}/p$ and was facilitated by a reinterpretation of this algebra using judiciously chosen invariant elements in the

mod p homotopy. This talk focused on generalizations of this computation to higher heights divisible by $p - 1$.

Using formal group machinery, Devinatz and Hopkins computed the action of S_n on the homotopy groups of E_n [6]. While complete, their description was difficult to apply to computations. Hopkins conjectured that there is a more natural collection of generators of E_{n*} for which the action of finite subgroups is especially simple. We begin by recalling that S_n is the group of units in the maximal order $\mathcal{O}_n$ of the division algebra D_n over $\mathbb{Q}_p$ of Hasse invariant $\frac{1}{n}$. The natural left action of S_n on $\mathcal{O}_n$ commutes with the right action of $\mathbb{Z}_{p^n}$, the Witt vectors for $\mathbb{F}_{p^n}$, and this makes $\mathcal{O}_n$ into a $\mathbb{Z}_{p^n}[S_n]$ -module, the Dieudonné module M_n .

Conjecture 1 (Hopkins). *If $G \subset S_n$ is a finite subgroup, then there is a G -equivariant isomorphism*

$$E_{n*} \cong S_{\mathbb{Z}_{p^n}}(M_n)[\Delta^{-1}]_I^\wedge,$$

where S denotes the symmetric algebra functor, M_n is placed in degree -2 , Δ is a trivial representation corresponding to the multiplicative norm over the group on M_n , and I is an ideal in degree 0.

The conjecture is most important when $p - 1$ divides n and p divides the order of G , as here an obstruction theory argument reduces the proof of this conjecture to verifying it for $\mathbb{Z}/p \subset G$. By using the theory of formal A -modules, Hopkins, Ravenel, and I have made significant headway in proving this conjecture.

Let $A = \mathbb{Z}_p[\zeta]$, where ζ is a p^{th} root of unity. There is an inclusion of A into $\mathcal{O}_n$, and this induces a formal A -module structure on F_n , the Honda formal group of height n . If we write $n = (p - 1)f$, then as a formal A -module, F_n has height f , and there is a Lubin-Tate deformation theory of formal A -modules similar to that of formal groups, corepresented by a ring E_{f*}^A . Since A has a p^{th} root of unity, we can easily describe the action of $\mathbb{Z}/p$ on E_{f*}^A . Moreover, if we forget down to formal groups, then we get a natural $\mathbb{Z}/p$ -equivariant map of corepresenting rings:

$$E_{(p-1)f*} \rightarrow E_{f*}^A.$$

This map is surjective, and it therefore produces a spectral sequence computing the cohomology of $\mathbb{Z}/p$ with coefficients in $E_{(p-1)f*}$ from the cohomology of $\mathbb{Z}/p$ with coefficients in E_{f*}^A . This spectral sequence has the advantage of having a well understood algebraic model, and by mirroring Devinatz and Hopkins original arguments, we have been able to show that through a large range, these spectral sequences coincide.

Assuming Hopkins' conjecture, we have also been able to describe the E_2 term of and compute the differentials in the homotopy fixed point spectral sequence of $\pi_*EO_{(p-1)f}(\mathbb{Z}/p)$. The differentials generalize those found by Hopkins and Miller in their original analysis of $\pi_*EO_{p-1}(\mathbb{Z}/p)$, and their construction is very similar.

Proposition 2. *As an algebra, the E_2 term of the homotopy fixed point spectral sequence for $EO_{(p-1)f}(\mathbb{Z}/p)$ is*

$$E(\alpha_1, \dots, \alpha_f) \otimes P(\beta) \otimes P(\delta_1, \dots, \delta_f^{\pm 1}) \oplus \text{Free},$$

where the bidegrees of the elements, written as $(t - s, s)$ are $|\alpha_i| = (-3, 1)$, $|\beta| = (-2, 0)$, and $|\delta_i| = -2p$.

The elements referred to as “Free” arise from free summands of E_{n*} and pair trivially with all elements of higher filtration. They also lie in the image of the transfer map from $EO_{(p-1)f}(\{1\})$, making them permanent cycles. The element β is the periodicity generator of $\mathbb{Z}/p$ cohomology, and the elements α_i , β , and δ_i are related by the power operation

$$\beta \mathcal{P}^0(\alpha_i) = \langle \alpha_i, \dots, \alpha_i \rangle = \beta \delta_i.$$

Using formal group arguments lifted from the analogous story for E_f^A , we can relate the elements α_i to the elements $h_{i,0}$, appropriately translated by powers of δ_f , and the corresponding elements $b_{i,0}$ are similarly related to the classes labeled δ_i and β . These relations allow us to understand differentials that arise on norm classes in the homotopy fixed point spectral sequence.

Proposition 3. *The differentials are algebraically determined by the following properties.*

- (1) *There are differentials $d_{1+2(p^i-1)}(\delta_f^{p^{i-1}}) = \delta_f^{p^{i-1}} h_{i,0} \beta^{p^i-1}$.*
- (2) *There are corresponding $d_{1+2(p-1)(p^i-1)}$ Toda style differentials truncating the β towers on δ_i .*
- (3) *The classes $\Delta_i = \delta_i / \delta_f$ are permanent cycles.*
- (4) *The class $\delta_f^{p^f}$ is a permanent cycle, and these describe all of the differentials.*

With the exception of the final statement, these results are all proved in essentially the same way: there exist universal examples for certain differentials in homotopy fixed point spectral sequences. Let g be a generator of $\mathbb{Z}/p$. If $u: S^k \rightarrow E_n$, then let

$$Nu = u \cdot gu \dots g^{p-1}u: S^{pk} \rightarrow E_n.$$

Since S_n acts on E_n by E_∞ maps, this map is $\mathbb{Z}/p$ -equivariant and therefore descends to homotopy fixed points. The spectrum $(S^{pk})^{h\mathbb{Z}/p}$ is the Spanier-Whitehead dual of a Thom spectrum, and the attaching maps of the top cell determine differentials on the class represented by Nu . The classes $\delta_f^{p^i}$ and the classes Δ_i are of the form Nu for appropriately chosen u , and this general argument produces the described differentials.

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Localising subcategories of the stable module category of a finite group

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(joint work with Dave Benson and Srikanth Iyengar)

Let G be a finite group and k a field of characteristic p . We write $\mathbf{StMod} kG$, respectively $\mathbf{stmod} kG$, for the stable module category whose objects are kG -modules, respectively finitely generated kG -modules, and whose morphisms are module homomorphisms modulo those that factor through a projective module. It is well known that this is a triangulated category. The thick subcategories of $\mathbf{stmod} kG$ were classified in [2]. More accurately, this was achieved when G is a finite p -group; for an arbitrary finite group only the tensor closed thick subcategories were classified. If G is a finite p -group then all thick subcategories are tensor closed.

Hopkins [5] and Neeman [8] classified the thick subcategories of perfect complexes over a commutative Noetherian ring. Neeman, in the same article, classified the localising subcategories of all complexes, thereby proving the algebraic version of the telescope conjecture in this context.

In this report we present the classification of the localising subcategories of the stable module category $\mathbf{StMod} kG$. Our main theorem is as follows.

Theorem. *Let G be a finite group and k a field of characteristic p . Then the tensor closed localising subcategories of $\mathbf{StMod} kG$ are in one to one correspondence with subsets of the set of non-maximal homogeneous prime ideals in $H^*(G, k)$. If S is such a subset then the corresponding localising subcategory is the full subcategory whose objects are the modules having their support contained in S .*

Let us discuss some of the essential ingredients of this work. First of all, we pass from $\mathbf{StMod} kG$ to the closely related category $\mathbf{K}(\mathbf{Inj} kG)$. Here, $\mathbf{K}(\mathbf{Inj} kG)$ denotes the category of complexes of injective kG -modules and homotopy classes of degree preserving morphisms of complexes. Note that injective and projective modules are the same in this context. There is the following useful recollement [7]

$$\mathbf{StMod} kG \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathbf{K}(\mathbf{Inj} kG) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathbf{D}(\mathbf{Mod} kG)$$

which expresses the fact that the stable category $\mathbf{StMod} kG$ can be identified with the localising subcategory of $\mathbf{K}(\mathbf{Inj} kG)$ formed by all acyclic complexes and that the derived category $\mathbf{D}(\mathbf{Mod} kG)$ is equivalent to the corresponding quotient. In