European Talbot 2025: Higher Algebra and Chromatic Homotopy Theory

Notes Taken by Mattie Ji

June 30th - July 4th, 2025

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Introduction

European Talbot 2025: Higher Algebra and Chromatic Homotopy Theory is the seventh European Talbot workshop that took place in Kolding, Denmark from June 30th to July 4th, 2025. The workshop is aimed to bring together a group of 30+ graduate students and post-docs to introduce and study selected topics in higher algebra and chromatic homotopy theory, under the guidance of two senior mentors working in the area. This workshop is organized by Daniel Bermudez, Marie-Camille Delarue, Joao Fernandes, Hyeonhee Jin, Filippos Sytilidis and Maxime Wybouw, with the two senior mentors being Gijs Heuts and Ishan Levy.

This workshop was funded by the Hausdorff Center for Mathematics (HCM) and Deutsche Forschungsgemeinschaft (DFG - the German Research Foundation) via the Junior Research Retreats program and the European Mathematical Society (EMS) committee for European Solidarity. For more information, please visit the workshop website here.

The workshop itself is composed of 3-4 main talks per day from Monday (June 30th) to Friday (July 4th), with additional follow-up activities in the evening. Although not explicitly stated, it appears that a working knowledge of higher algebra before the stabilization and stable homotopy theory would be helpful pre-requisites before reading the notes. Below is an outline of the schedule.

Monday (June 30th): Fundamentals of Higher Algebra

- Lecture 1 (Section 1), by Ishan Levy, is an overview of the workshop and its two core topics - higher algebra and chromatic homotopy theory.
- Lecture 2 (Section 2), by Yuqin Kewang, is about the stable ∞-category of spectra, stabilization, and the connections between group-like $\mathbb{E}_\infty$-spaces and spectra.
- Lecture 3 (Section 3), by Julie Bannwart, is about presentable stable ∞-categories, symmetric monoidal ∞-categories, (commutative) algebras over them, and modules over algebras.
- Lecture 4 (Section 4), by Markus Zetto, is about ∞-operads via symmetric sequences, algebras over operads, the Bar-Cobar duality and Koszul duality.
- Lecture 4.5 (Section 5), by Gijs Heuts and Ishan Levy, was an informal lecture carried out in the evening to cover the classic story of complex orientations, formal group laws, and Quillen's theorem of MU carrying the universal formal group law.

Tuesday (July 1st): The Chromatic Perspective on Stable Homotopy Theory

- Lecture 5 (Section 6), by Henry Rice, is about the nilpotence and periodicity theorem in chromatic homotopy theory.
- Lecture 6 (Section 7), by Catherine Li, is about the construction of Morava E-theory with an “unique" $\mathbb{E}_\infty$-ring structure, the construction of Brown-Peterson spectrum as an $\mathbb{E}_2$-ring, and the construction of Morava K-theory $K(n)$ as an $\mathbb{E}_1$-ring.
- Lecture 7 (Section 8), by Maite Carli, is about descendability, the smash product theorem, and the chromatic convergence theorem.
- Lecture 8 (Section 9), by Florian Riedel, introduces $T(n)$ as the telescope of a v_n-self map on a type n complex, the Bousfield-Kuhn functor, and discusses semi-additivity.
- The First Q and Q session (Section 10) was held by Gijs Heuts and Ishan Levy in the evening.

Wednesday (July 2nd): Power Operations

- Lecture 9 (Section 11), by Azélie Picot, gave the definition and some examples of power operations.
- Lecture 10 (Section 12), by Jordan Levin, was about the Dyer-Lashof operations, at least at prime n, and for low n.

- Lecture 10.5 (Section 13), by Ryan Quinn, explains the construction of an $\mathbb{E}_3$-MU-algebra structure on the truncated Brown-Peterson spectra $\mathrm{BP}\langle n\rangle$.
- There was a Gong Show carried out in the evening where participants are encouraged to eat a relatively spicy pepper and explain some math topic for 5 minutes on the board.

Thursday (July 3rd): Filtrations

- Lecture 11 (Section 14), by Lucy Grossman, explained t-structures, filtered objects, and spectral sequences.
- Lecture 12 (Section 15), by Jonathan Pederson, introduced synthetic spectra as a representation of the Adams spectral sequence.
- Lecture 13 (Section 16), by Mattie Ji, explained an application of synthetic spectra to chromatic homotopy theory that shows there is an equivalence between the homotopy category of the ∞-category of E-local spectra and the homotopy category ∞-category of differential E_*E-comodules, for E a p-local Landweber exact homology theory of height n and $p > n^2 + n + 1$.
- Lecture 14 (Section 17), by Emma Brink, explains an application of synthetic spectra to giving multiplicative structures on Moore spectra.
- The Second Q and A session (Section 18) was held by Gijs Heuts and Ishan Levy in the evening.

Friday (July 4th): $T(n)$-local $\mathbb{E}_\infty$ Rings

- Lecture 15 (Section 19), by Preston Cranford, was about McClure's theorem.
- Lecture 16 (Section 20), by Max Blans, was about the Chromatic Nullstellensatz.
- Lecture 17 (Section 21), by Vignesh Subramanian, was about applications of the Chromatic Nullstellensatz.
- Lecture 18 (Section 22), by Gijs Heuts, concludes the seminar by giving some outlooks on open problems in the area.
- There were no formal mathematical activities arranged on Friday evening, but there was a karaoke night that lasted until about 5 am the next day. Nevertheless, the notetaker heard many interesting math conversations happening at the party.

These notes were originally live-texed by Mattie Ji, with additional editing works done by Mattie Ji after the conference concluded. We are thankful to the organizers and the mentors for running this conference; we are thankful to each of the speakers for contributing the talks in the conference and for follow-ing up after the talks; and we are thankful to the participants for coming to the conference. The notetaker, as the presenter of Lecture 13, would also like to thank Mark Behrens, Pengkun Huang, Ishan Levy, and Jinghui Yang for helpful conversations during the writing and preparations of this lecture.

Throughout the notes, we use Spc and $\mathcal{S}$ interchangeably to denote the ∞-category of spaces. Unless it is a typo or stated otherwise, the symbol "$\mathbb{E}_k$" denotes the little k-cubes operad, the symbol E_n denotes the n-th Morava E-theory with respect to some prime p, and if two superscripts appear above as $E_n^{\bullet,\bullet}$, the symbol denotes a certain item on the n-th page of some spectral sequence.

If you find any typos or errors in these notes, please feel free to contact the note taker (Mattie Ji).

1 Lecture 1: Overview by the mentors (by Ishan Levy)

This workshop is about **higher algebra and chromatic homotopy theory**! What are those about?

We will begin first with what higher algebra is about! Let us first look at what we mean by "usual algebra". In usual algebra, we usually study algebraic structures such as $\mathrm{Set}, \mathrm{Ab},$ or Rings. In higher algebra, roughly speaking, we would like to replace these categories with the "∞-categoriy of homotopy types" (or what some people would like to call Anima!).

Question 1.1. Why would we want to do this with higher algebras?

The reason why is because homotopy types show up in many places, even if you are not interested in homotopy theory. They show up in:

1. Geometric topology (ie. when you try to classify manifolds, surgery theory).
2. Algebraic or arithmetic geometry, when you are trying to study algebraic invariants (ex. varieties or arithmetic objects).
3. (**Perhaps a bit more philosophical**): Homotopy types are sort of a "natural replacement" of Sets from a categorical viewpoint.

1.1 What is Higher Algebra?

Let us look at linear algebra for example.

- In usual **linear algebra**, we would like to work with **abelian categories.**
- In **higher algebra**, the fundamental place where we would like to do linear/homological algebra is called the **stable ∞-category**.

The following gives a comparision between the two.

Linear Algebra	Higher Algebra
Abelian Categories	Stable ∞-categories
Exact seqeunces	Fiber sequences
Universal abelian category that acts on others - Ab!	Universal stable ∞-category - spectra!
Symmetric Monoidal by $\otimes$	Symmetric Monoidal by $\wedge$
The unit $\mathbb{Z}$	The unit $\mathbb{S}$

There are some interesting things happening in higher algebra that is not captured by linear algebra though!

Linear Algebra	Higher Algebra
$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$	$\mathrm{Hom}(\mathbb{S}, \mathbb{S}) = \mathrm{colim}_n\, \Omega^n S^n$
Commutativity is strict	Commutativity is parametrized by operads $\mathbb{E}_1, \mathbb{E}_2, ..., \mathbb{E}_\infty$

Encoding algebras and algebraic maps encodes a lot of data! How would we do this in practice, that we will learn about this week:

1. **Constructions** (Thom spectra, algebraic K-theory, ...)
2. **Deformation theory** (Trying to build non-linear objects (ie. $\mathbb{E}_1$-algebras) using linear data).
3. Closely tied to deformation theory is **Obstruction Theory**. Ex. If our goal is to understand $\mathbb{E}_n$-algebras (ie. figuring out what their homotopy groups are), the key thing to first understand the free $\mathbb{E}_n$-algebras, which by the Yoneda lemma is equivalent to understand what are called **Power Operations**. Precisely, we have that $\mathrm{Map}(\mathrm{Free}\, S^0, R) = \Omega^\infty R$. Algebras in general are built under colimits from free ones, so in some sense the free algebra functor with some extra structure entirely encodes what it means to be an $\mathbb{E}_n$-algebra. One key method in this is the **Koszul Duality**.

Example 1.2. Where are the obstructions? Suppose we have the free algebra on $\{x\}$ as $\text{Free}\{x\}$ and consider the pushout:

$$\begin{tikzcd}
\text{Free}\{x\} \arrow[r] \arrow[d] & R \arrow[d] \\
\mathbb{1} \arrow[r] & R'
\end{tikzcd}$$

Consider a map $R \to S$, one can ask if we can lift this to a map $R' \to S$. It turns out you can lift this if and only if some obstructions related to the homotopical information of $\text{Free}\{x\}$ vanishes (more precisely, x comes to be nullhomotopic in S).

4. **Descent**: Let $f : R \to S$ be **faithfully flat**. It was Grothendieck who observed that we can recover $\text{Mod}(R)$ as

$$\text{Mod}(R) \xrightarrow{\sim} \lim_{[n]\in\Delta} \text{Mod}(S^{\otimes n}).$$

In usual algebra we needed faithful flatness for this, but in higher algebra descent is effective way beyond this setting.

Example 1.3. Considering descent for the unit map $\mathbb{S} \to \mathbb{F}_p$ gives the **Adams spectral sequence**. Considering descent for the unit map $\mathbb{S} \to \text{MU}$ gives the **Adams Novikov spectral sequence**.

5. Often in higher algebra, we want to study some object X by finding a **filtration** on X.

Definition 1.4. Let $X \in \mathcal{C}$ (for $\mathcal{C}$ some category), a **filtration on** X is a functor $F : \mathbb{Z} \to \mathcal{C}$ (where $\mathbb{Z}$ is interpreted as the poset category). Write $X_i = F(i)$, we want $\text{colim}\, X_i = X$.

Often times we are also interested in the associated graded gr X_i/X_{i+1} and see the difference between the associated graded and X.

Principle: All spectral sequences come from filtered objects!

This is good because we can apply the techniques of higher algebra to filtered objects.

1.2 What is Chromatic Homotopy Theory?

If spectra is the analog of abelian groups, one natural question we can ask is - what do these spectra look like? In classical algebra, we have the structure theorem for abelian groups, etc., chromatic homotopy theory asks what a spectrum looks like.

Let us look at the abelian group case first. Consider the natural inclusion map $\mathbb{Z} \to \mathbb{Q}$. The information that this map is losing is exactly the cokernel, which fits into

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}.$$

On the other hand, we have that

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_p \text{colim}_k\, \mathbb{Z}/p^k\mathbb{Z}.$$

Thus, understanding $\mathbb{Q}$ can be thought of as understanding $\mathbb{Z}$ and each prime powers!
Similarly, for the sphere spectrum, we have a map $\mathbb{S} \to \mathbb{S}[p^{-1}, p \text{ prime}] \cong \mathbb{Q}$. The cokernel in this case gives a sequence

$$\mathbb{S} \to \mathbb{S}[p^{-1}, p \text{ prime}] \cong \mathbb{Q} \to \mathbb{Q}/\mathbb{S} \cong \bigoplus_p \text{colim}_k\, \mathbb{S}/p^k.$$

Question 1.5. Can we decompose this further?

Well in this case, one can check that $\pi_*\mathbb{S}/p^k$ has an element $v_1^{k_1}$ that generates $\pi_*/$ nilpotence. In general - to understand some object, we can pick an element in it and ask - what happens when we mod out this element or invert this element?

In this case, we can consider another map

$$\Sigma\mathbb{S}/p^{k_1} \xrightarrow{v_1^{k_1}} \mathbb{S}/p^{k_0} \to \mathbb{S}/(p^{k_0}, v_1^{k_1}).$$

In this case $(\pi_*\mathbb{S}/p^{k_1})[v_1^{-1}]$ are completely understood (in the sense that it is a local ring of Krull dimension 0)! On the other hand, we can ask - what about the term $\mathbb{S}/(p^{k_0}, v_1^{k_1})$?

It turns out in this case, we can find an element $v_2^{k_2}$, so we try doing this again, which will give an element $v_3^{k_3}$, and so on ... Putting this together we define

$$X_n = \mathbb{S}/(p^{k_0}, v_1^{k_1}, ..., v_{n-1}^{k_{n-1}}),$$

and the **telescope object of height** n as

$$T(n) = X_n[v_n^{k_n-1}].$$

There is a more canonical localization functor we can define given by

$$\mathrm{Sp} \to \mathrm{Sp}_{T(n)}, X \to \lim_{k_0,k_1,...,k_{n-1}}(X \otimes \mathbb{S}/(p^{k_1}, ..., v_{n-1}^{k_{n-1}})[v_n^{k_n-1}].$$

Definition 1.6. When do these v_i's show up?

Remark 1.7. If we consider the map

$$\mathbb{S} \to \mathrm{MU} \to \mathrm{MU} \otimes \mathrm{MU}\,.$$

There is an object called the moduli stack M_{fg} of formal groups with a sheaf of invariant differentials ω on it. Doing descent on this map, we have a spectral sequence

$$H^s(M_{fg}; \omega^{\otimes t}) \implies \pi_{2t-s}\mathbb{S}.$$

It turns out that $v_n \in H^0(M_{fg}; \omega^{\otimes(p^n-1)}/(v_0, ..., v_{n-1}))$. This set-up is sometimes called the **Adams-Novikov filtration.**

Let $K(n)$ be the Morava K-theory. If we look at the $K(n)$-localization of X, this is tied to

$$L_{K(n)}X = \lim_{\Delta} L_{T(n)}(X \otimes \mathrm{MU}^{\otimes+1})$$

which is sometimes more computable tan $T(n)$-localization. To give some idea on why this might be more computable, there is an alternative description of the terms in the limit, indeed

$$L_{T(n)}(X \otimes \mathrm{MU}^{\otimes+1}) \cong (X\widehat{\otimes}E_n)^{h\mathbb{G}_n}$$

where $\otimes^\wedge$ denotes a "completed tensor product" and we have a lot of understanding of E_n, the Morava E-theory. In fact, we have that

$$\pi_*E_n = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][\beta^{\pm1}], |u_i| = 0, |\beta| = 2$$

Here $W(\mathbb{F}_{p^n})$ is the ring of Witt vectors.

Write L_n to be the **Bousfield localization** of $K(n) \oplus K(n-1) \oplus ... \oplus K(0)$ (or with respect to E_n). L_n^f is a finite version of L_n. These L_n^f glue together information of $L_{K(i)}$ and $L_{T(i)}$ for $i \leq n$.

Question 1.8. Do these L_n's capture all the information of (finite) spectra?

In principle, the answer is yes, due to the chromatic convergence theorem.

Theorem 1.9 (Chromatic Convergence Theorem - One Version)**.** $\mathbb{S}_{(p)} = \lim_{\leftarrow n} L_n\mathbb{S}$.

These **chromatic localizations have a lot of intetresting properties** in $L_{T(n)}$ Sp and $L_{K(n)}$ Sp.

1. **Ambidetextrity** (which makes the set-up feel like working in characteristic zero). Consider a non-trivial vector space V with G actions on the left and right. In this case there is an eequivalent $V_G \simeq V^G$ (from orbits to fixed points) given by
$$x \mapsto \sum_{g \in G} gx.$$
In $L_{T(n)}$ Sp or $L_{K(n)}$ Sp, there is always some canonical map $X_{hG} \xrightarrow{\sim} X^{hG}$ (from the orbit to the fixed points) where G is what is called a π-finite group.

2. $K(n)$-local $\mathbb{E}_\infty$-rings are very nice! For instance, free $K(n)$-local $\mathbb{E}_\infty - E_n$-algebra at the level of $\pi_\bullet$ is a completed polynomial algebra over $\pi_\bullet E_n$. (Here E_n means Morava E-theory).

3. **Bousfield-Kuhn functor**: There is a Bousfield Kuhn functor $\Phi_n : \mathrm{Spc} \to \mathrm{Sp}_{T(n)}$ that allows you to recover a spectrum from the underlying space after $T(n)$-localization. In other words, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Sp} \arrow[rr, "L_{T(n)}"] \arrow[dr, "\Omega^\infty"'] & & \mathrm{Sp}_{T(n)} \\
& \mathrm{Spc} \arrow[ur, "\Phi_n"'] &
\end{tikzcd}$$

Remark 1.10. For MU-modules (ie. complex-oriented), it is true that $K(n)$ and $T(n)$ localizations are the same.

2 Lecture 2: Spectra and stabilization (by Yuqin Kewang)

2.1 Stable ∞-categories

One motivation for introducing stable ∞-categories is that they provide a better model for **triangulated categories**.

Definition 2.1. Let $\mathcal{C}$ be an ∞-category, we say that $\mathcal{C}$ is pointed if $\mathcal{C}$ has a zero object 0, which is both final and initial.

Definition 2.2. Let $\mathcal{C}$ be a pointed ∞-category, a **triangle** in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

A **triangle** is a **fiber sequence** (resp. **cofiber sequence**) if this diagram is a **pullback square** (resp. **pushout square**).

Example 2.3. Take $\mathcal{C} = S_*$ to be the ∞-category of pointed spaces. Here the zero object is the point. For each $X \in S_*$, we can define its **loop space** ΩX and **suspension space** ΣX

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] & & X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X & & 0 \arrow[r] & \Sigma X \end{tikzcd}$$

where the left square is a pullback and the right square is a pushout.

Similarly, we can define Σ and Ω in general.

Remark 2.4. There is an adjunction $\Sigma \dashv \Omega$.

Definition 2.5. We say an ∞-category $\mathcal{C}$ is **stable** if

1. $\mathcal{C}$ is pointed.
2. Every morphism in $\mathcal{C}$ has a fiber and a cofiber.
3. Every triangle in $\mathcal{C}$ is a pullback if and only if it is a pushout.

Proposition 2.6. If $\mathcal{C}$ is stable, then the adjunction $\Sigma \dashv \Omega$ is an equivalence of categories.

Proof. By the definition of stability, we know that for a diagram

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

Since this is a pullback, it is also a pushout, so $\Sigma\Omega X$ is X. Doing the other diagram gives us $X \cong \Omega\Sigma X$. This proves the equivalence. ■

Proposition 2.7. If $\mathcal{C}$ is a stable ∞-category, then its underlying homotopy category $h\mathcal{C}$ has the structure of a **triangulated category**.

Remark 2.8. Being triangulated is a property of a stable ∞-category, whereas for 1-categories it is an extra structure.

Now that we have defined stable ∞-categories, we would like to ask.

Question 2.9. Do we have examples of stable ∞-categories?

Yes! In fact, there is a general process called **stabilization** which gives you stable ∞-categories.

Definition 2.10. Let $\mathcal{C}$ be an ∞-category that admits finite limits, then we consider the ∞-category of pointed objects in $\mathcal{C}$ (call this $\mathcal{C}_*$). Then $\mathcal{C}_*$ also admits finite limits and is pointed. Thus, ΩX is always well-defined.

The **stabilization of** $\mathcal{C}$ is $\mathrm{Sp}(\mathcal{C}) \coloneqq \lim(\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*) \in \widehat{\mathrm{Cat}_\infty}$.

Notation: Cat_∞ is the ∞-category of small ∞-categories, and $\widehat{\mathrm{Cat}}_\infty$ is the ∞-category of large ∞-categories. In general, Spc is not small, but finite spaces are small.

Proposition 2.11. $\mathrm{Sp}(\mathcal{C})$ is a stable ∞-category.

Remark 2.12. If $\mathcal{C}$ has finite limits and colimits, we have that

$$\mathrm{Sp}(\mathcal{C}) = \lim(\ldots \to \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*) \in P_r^R$$

$$= \operatorname{colim}(\mathcal{C}_* \xrightarrow{\Sigma} \mathcal{C}_* \xrightarrow{\Sigma} \ldots) \in P_r^L$$

There is an embedding P_r^R to $\widehat{\mathrm{Cat}_\infty}$ that recovers the original construction. What are these notations? We will see them in Lecture 3!

Definition 2.13. Take $\mathcal{C} = \mathrm{Spc}$ the ∞-category of spaces, then we define Sp as the stabilization of Spc. This is called the ∞-category of spectra.

Concretely, what is Sp?

- The objects are $X \coloneqq \{(X_n)_{n\geq 0}, \delta_n : X_n \xrightarrow{\sim} \Omega X_{n+1}\}$.
- What are the mapping spaces? Well,

$$\mathrm{Map}_{\mathrm{Sp}}(X,Y) = \lim(\ldots \mathrm{Map}_{\mathrm{S}_*}(X,Y_n) \xrightarrow{\Omega} \mathrm{Map}_{\mathrm{S}_*}(X_n,Y_n) \to \ldots)$$

  Concretely, a map $f : X \to Y$ is a sequence of maps $f_n : X_n \to Y_n$ equipped with a homotopy of the following two maps.

$$\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d, "\delta_n"'] & Y_n \arrow[d, "\delta_n^*"] \\
\Omega X_{n+1} \arrow[r, "\Omega f_{n+1}"'] & \Omega Y_{n+1}
\end{tikzcd}$$

Example 2.14. For $A \in \mathrm{Ab}$, we have the Eilenberg Maclane spectrum HA given by

$$\{(K(A,i), K(A,i) \xrightarrow{\sim} \Omega K(A,i+1))\}_{i\geq 0}\}.$$

Proposition 2.15. Sp has all limits and colimits (ie. Sp is a presentable ∞-category). Limits and filtered colimits in Sp can be computed levelwise.

Let us also look at what ΩX for a spectrum X is. Indeed,

$$(\Omega X)_n \cong \Omega X_n \cong X_{n-1},$$

this implies to $\Omega X = X[-1]$ (ie. X shifted down). Similarly, we will also have that $X = X[1]$.

There is a natural functor $\Omega^\infty : \mathrm{Sp} \to S_*$ by $X \mapsto X_0$. Since limits can be computed levelwise, this functor Ω^∞ commutes with all small limits! This admits a left adjoint $\Sigma^\infty : \mathrm{Sp} \to \mathrm{Sp}$ given by

$$X \mapsto ((\mathrm{colim}_i\, \Omega^i \Sigma^{i+n} X)_n, \mathrm{colim}\, \Omega^i \Sigma^{i+n} X \simeq \Omega\, \mathrm{colim}\, \Omega^i \Sigma^{i+n+1} X)$$

Note that since loop functor commutes with colimits, we have that

$$\Omega\, \mathrm{colim}\, \Omega^i \Sigma^{i+n+1} X = \mathrm{colim}\, \Omega^{i+1} \Sigma^{i+n+1} X,$$

which gives the desired equivalence.

Definition 2.16. The image of Σ^∞ are called **suspension spectra**.

Example 2.17. $\Sigma^\infty S^0$ is the sphere spectrum $\mathbb{S}$.

Now we would like to derive some good properties of Sp.

Proposition 2.18 (Property 1)**.** Sp is compactly generated, more precisely $\mathrm{Sp} = \mathrm{Ind}(\mathrm{Sp}^{fin})$, the ∞-category of finite spectra.

To prove this proposition, we first need to define to notion of finite spectra.

Definition 2.19. The category of finite spectrum Sp^{fin} is the colimit

$$\mathrm{colim}(S_*^{fin} \xrightarrow{\Sigma} S_*^{fin} \to \ldots) \in \mathrm{Cat}_\infty$$

Here S_*^{fin} refers to the category of pointed finite spaces. We can view every finite spectrum as $\Sigma^{-n}\Sigma^\infty K$ where K is some fintie space!

Proof Idea of Property 1. Recall that for $\mathcal{C}$ a small ∞-category, $\text{Ind}(\mathcal{C}) = P^{\text{filtered}}(\mathcal{C}) \subseteq P(\mathcal{C}) = \text{Fun}(\mathcal{C}^{op}, \text{Spc})$. The idea is that $P(\mathcal{C})$ is the ∞-category generated by $\mathcal{C}$ freely under colimits.

Concretely, $\text{obj}(\text{Ind}(\mathcal{C}))$ are exactly of the form $\text{colim}_I F_i$ where $F : I \to \mathcal{C}$ and I is filtered. Furthermore, we have that

$$\text{Map}_{\text{Ind}(\mathcal{C})}(\text{colim}_I F_i, \text{colim}_J G_j) \cong \lim_I \text{colim}_J \text{Map}_{\mathcal{C}}(F_i, G_j).$$

Now the proof for the case of Sp comes down the following observations:

1. $\text{Sp}^{\text{fin}} \subseteq \text{Sp}$ is compact (ie. $\text{Map}_{Sp}(X, -)$ commutes with filtered colimits).
2. In general for $\mathcal{C}_* \hookrightarrow \mathcal{C}$ compact, then $\text{Ind}(\mathcal{C}_*) \to \mathcal{C}$ is fully faithful. Hence the induced map $\text{Ind}(\text{Sp}^{fin}) \to \text{Sp}$ is fully-faithful.
3. Essential Surjecitvity: For $X \in \text{Sp}$, we want to show that X can be written as the colimit of finite spectrum. By Yoneda lemma, we can indeed write $X = \text{colim}\, \Sigma^{-n}\Sigma^{\infty} X_n$.

■

2.2 The Recognition Principle

The next property we want to estbalish the the recognition principle - the slogan is that connective spectra are exactly group-like $\mathbb{E}_\infty$-spaces!

Write Fin_* as the 1-category of pointed finite spaces. The objects are $\langle n \rangle = \{*, 1, ..., n\}$ for $n \geq 0$. The maps between $\langle n \rangle$ and $\langle m \rangle$ are set functions f such that $f(*) = *$.

Definition 2.20. A **commutative monoid in spaces** (ie. also called $\mathbb{E}_\infty$-space) is a functor $\underline{M} : \text{Fin}_* \to \text{Spc}$ satisfying the **Segal condition**, ie. the map

$$(\chi_{i,*})_i : \underline{M}(\langle n \rangle) \to \prod_{i=1}^{n} \underline{M}(\langle 1 \rangle)$$

is an equivalence. Each map $\chi_* : \langle n \rangle \to \langle 1 \rangle$ is the unique map that sends $i \to 1$ and everything else to $*$.

Definition 2.21. Let $\underline{M} : \text{Fin}_* \to \text{Spc}$ be an $\mathbb{E}_\infty$ space. Write $M = \underline{M}(\langle 1 \rangle) \in S$ to be the **underlying space** of $\underline{M}$. Now we obtain a multiplicative structure on M given by

$$M \times M \cong M(\langle 2 \rangle) \xrightarrow{m_*} M(\langle 1 \rangle) = M,$$

where $m : \langle 2 \rangle \to \langle 1 \rangle$ is given by sending 1 and 2 both to 1. One can check this is a monoid structure that is **unitial, associative, and commutativie up to higher coherence**.

Definition 2.22. An $\mathbb{E}_\infty$-space $\underline{M}$ is **group-like** if $\pi_0(M)$ is a group. We denote the subcategory as $\text{CMon}^{gp}(\text{Spc})$.

Fact: $\text{CMon}(\text{Spc})$ has 0 objects (ie. the trivial group), and has limits that can be computed level wise.

Proposition 2.23 (Property 2, Recognition Proinciple)**.** There is an equivalence of categories between $\text{CMon}^{gp}(S)$ and connective spectra $\text{Sp}_{\geq 0}$ given by the adjunction $(B^\infty, \Omega^\infty)$.

Here $B^\infty : \mathrm{CMon}^\infty(S) \to \mathrm{Sp}_{\geq 0}$ is constructed as follows.
We define

$$\Omega\underline{M} : \mathrm{Fin}_* \to \mathrm{Spc}, \langle n\rangle \mapsto \Omega\underline{M}(\langle n\rangle).$$

Note that $\pi_i(\Omega\underline{M})_{\cong\pi_{i+1}(\underline{M})}$.CThis suggests that Ω from connected group-like $\mathbb{E}_\infty$-spectrum gives an equivalence to group-like $\mathbb{E}_\infty$-spaces. The inverse of this is then B where

$$B\underline{M} \cong \mathrm{colim}_{\Delta^{op}}(\mathrm{Fin}_* \times \Delta^{op} \to \mathrm{Fin}_* \xrightarrow{\underline{M}} S) \in \mathrm{CMond}(Spc)$$

$$\cong \mathrm{colim}_{\Delta^{op}}(\ldots\underline{M} \times \underline{M} \rightrightarrows \underline{M} \to 0)$$

From here, we define

$$B^\infty\underline{M} = \{(M, BM, B^2M, \ldots), B^nM \simeq \Omega B^{n+1}M\}$$

Here, we have the equivalence $B^nM \simeq \Omega B^{n+1}M$ **precisely because** B is defined to be the inverse of Ω.

Proposition 2.24. $B^\infty\underline{M}$ is connective, ie. $\pi_{-n}(B^\infty\underline{M}) = \pi_0(B^nM) = 0$.

The map $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathrm{CMon}^{gp}(S)$ is given by

$$E \mapsto (\underline{\Omega^\infty E} : \langle n\rangle \to \Omega^\infty(\Sigma^\infty\langle n\rangle \otimes E).$$

Proposition 2.25. $\underline{\Omega^\infty E}$ is grop-like ie. $\pi_*(\Omega^\infty E) = \pi_0(E)$ is a group.

Finally, we give two examples of spectra.

Example 2.26. The connective complex K-theory ku. Consider a functor $\underline{\mathrm{Vect}} : \mathrm{Fin}_* \to \mathrm{Spc}$ given by

$$\langle n\rangle \mapsto \{(v_1, \ldots, v_n) \in \mathrm{G}_r^n \mid v_i \perp v_j, i \neq j\}$$

$$f : \langle m\rangle \to \langle n\rangle \mapsto \mathrm{Vect}\langle m\rangle \to \mathrm{Vect}\langle n\rangle, (v_1, \ldots, v_m) \mapsto (\bigoplus_{i\in f^{-1}(j)} v_i)_{1\leq j\leq n}.$$

One can check this **satisfies the Segal condition**. Consider another map $\prod x_i : \underline{Vect}(\langle n\rangle) \to \prod_{i=1}^n \underline{Vect}(\langle 1\rangle)$ given by

$$(v_1, \ldots, v_n) \mapsto (v_1, \ldots, v_n).$$

This is a **homotopy equivalent** because of the Gram-Schmidt procedure. One can check then that

$$\pi_*(\underline{\mathrm{Vect}}) = \mathbb{N}$$

which then implies that $B^\infty\underline{\mathrm{Vect}}^{\mathrm{gp}} = ku$.

Note that $\Omega^\infty ku = \Omega^\infty B^\infty \underline{Vect}^{gp}$, which is the underlying space of $\underline{Vect}^{gp}$, which is $BU \times \mathbb{Z}$.

Example 2.27. **The Thom spectrum**: For each n consider $BO_n \to S_* \xrightarrow{\Sigma^{-n}\Sigma^{-\infty}}$ Sp given by

$$\mathrm{pt} \mapsto S^n(\text{equipped with } O_n \text{ action}) \to \mathbb{S}.$$

In this case we can lift the maps

$$\begin{tikzcd}
BO_n \arrow[r] \arrow[d] & \mathrm{Sp} \\
BO_{n+1} \arrow[ur] &
\end{tikzcd}$$

To build a map $\tau : \mathrm{BO} \to \mathrm{Sp}$.

From here, we define $\mathrm{MO} \coloneqq \mathrm{colim}(\tau : BO \to Sp)$ and $MU = \mathrm{colim}(BU \to BO \xrightarrow{\tau} \mathrm{Sp})$.

3 Lecture 3: Presentable stable ∞-categories and (symmetric) monoidal structures (by Julie Bannwart)

Presentability will be a kind of smallness condition that are well-behaved, and we want to consider them to work with!

3.1 Symmetric Monoidal ∞-Categories and Their Algebraic Objects

In the classical case, a **symmetric monoidal 1-category** is the data $(\mathcal{C}, \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \mathbb{1}_{\mathcal{C}} \in \mathcal{C})$ with isomorphisms witnessing associativity, commutativity, and unitality, in addition to nice compatibility axioms.

For ∞-categories, the challenge is to **neatly encode** them. Last lecture, we saw how to give multiplicative structures for connected E_∞-spaces. Here we want to generalize:

Definition 3.1. Recall Fin_* is the category of pointed finite sets composing of objects of the form $\langle m \rangle = \{*, 1, ..., m\}$ and morphisms being base-point preserving set functions.

A (small) **symmetric monoidal ∞-category** is a functor $\underline{\mathcal{C}} : \mathrm{Fin}_* \to \mathrm{Cat}_\infty$ that satisfies again the **Segal condition**, ie. for maps $\rho_i : \langle m \rangle \to \langle 1 \rangle$ that sends i to 1 and the rest to $*$,

$$\underline{\mathcal{C}}(\langle m \rangle) \xrightarrow{\{\rho_i\}} \prod \underline{\mathcal{C}}(\langle 1 \rangle)$$

is an equivalence.

Equivalently, by the **straightening and unstraightening theorems**, it turns out this condition is equivalent to specifying a **co-Cartesian fibration** $\mathcal{C}^\otimes \to \mathrm{Fin}_*$ with the Segal conditions. Here, by a co-Cartesian fibration, we mean for each $\langle m \rangle \to \langle n \rangle$, we have an induced fiber functor $\mathcal{C}^\otimes_{\langle m \rangle} \to \mathcal{C}^\otimes_{\langle n \rangle}$ satisfying some specified conditions (ie. certain diagram lifting - “coCarteisan lifts").

Idea:

- The underlying ∞-category $\mathcal{C}$ is denoted by $\underline{\mathcal{C}}\langle 1 \rangle$ or $\mathcal{C}^\otimes_{\langle 1 \rangle}$.
- The unit $\mathbb{1}_{\mathcal{C}}$ is given by $\underline{\mathcal{C}}(\langle 0 \rangle) \simeq * \to \underline{\mathcal{C}}(\langle 1 \rangle) \simeq \mathcal{C}$.
- The tensor product is given by $\mathcal{C} \times \mathcal{C} \simeq \underline{\mathcal{C}}(\langle 2 \rangle) \xrightarrow{f} \underline{\mathcal{C}}(\langle 1 \rangle) \simeq \mathcal{C}$, where the map f is induced by
$$f' : \langle 2 \rangle \mapsto \langle 1 \rangle, f'(1) = f'(2) = 1.$$
- There are some coherence conditions enforced by Fin_*.

Proposition 3.2 (2.1.2.20-21 of [Lur17]). Let $\mathcal{C}$ be a symmetric monoidal 1-category, there exists $\underline{N\mathcal{C}}$ with $\underline{N\mathcal{C}}\langle 1 \rangle \cong N\mathcal{C}$.

The analogous description is - If $\mathcal{C}$ is a symmetric monoidal ∞-category, then $h\mathcal{C}\langle 1 \rangle$ is a symmetric monoidal 1-category.

Proposition 3.3 (2.4.1 of [Lur17]). Let $\mathcal{C}$ be an ∞-category with finite products (resp. coproduct), then $\mathcal{C}$ has a symmetric monoidal structure with tensor products being the categorical product $\times$ (resp. categorical coproduct $\bigsqcup$). (In this case we write $\mathcal{C}^\otimes$ to denote either $\mathcal{C}^\times$ or $\mathcal{C}^\sqcup$)

Classically, a **lax symmetric monoidal functor** $F : \mathcal{C} \to$ is equipped with maps $\lambda_{c,c'} Fc \otimes Fc' \to F(c \otimes c')$ (structure) satisfying some coherence. A **strong symmetric monoidal functor** requires the $\lambda_{c,c'}$ are equivalences (property).

Definition 3.4. Given symmetric monoidal ∞-categories $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ and $\mathcal{D}^{\otimes} \to \mathrm{Fin}_*$, a lax (resp. strong) symmetric monoidal functor is a commutative diagram:

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D}^{\otimes} \arrow[dl] \\
& \mathrm{Fin}_* &
\end{tikzcd}$$

such that F carries coCartesian lifts of inert edges (defined as $F : \langle m \rangle \to \langle m \rangle$ such that $f^{-1}(i)$ is a singleton for all $1 \leq i \leq m$) (resp. all edges) to coCartesian edges.

Note that symmetric monoidal functors between $\underline{\mathcal{C}}, \underline{\mathcal{D}} : Fin_* \to \mathrm{Cat}_{\infty}$ are exactly natural transformations.

Example 3.5. Symmetric monoidal functors between (co)Cartesian symmetric monoidal structures (that is, $\mathcal{C}^{\times} \to \mathcal{D}^{\times}$ or $\mathcal{C}^{\sqcup} \to \mathcal{D}^{\sqcup}$ are exactly those that preserves finite (co)products.

The (partial) idea is that tenosring in $\mathcal{C}$ are coCartesian lifts $m : \langle 2 \rangle \to \langle 1 \rangle$ of the form $[c, c'] \to c \otimes c'$. For a symmetric monoidal functor F, we would have a coCartesian lift $[Fc, Fc'] \to F(c \otimes c')$. The uniqueness of coCarteisna lifts implies that $F(c \otimes c') \simeq Fc \otimes Fc'$.

Classically, for symmetric monoidal 1-category, we can use them to define associative and commutative algebra objects over them.

Definition 3.6. An **algebra object** is $E \in (\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$ (for 1-categories) is equipped with maps $\mathbb{1}_{\mathcal{C}} \to E$ and $E \otimes E \to E$, satisfying suitable axioms.

Let us try to generalize this definition.

Definition 3.7. A **commutative algebra** in $\rho : \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ is a section of ρ that sends inert edges to co-Cartesian ones (lax symmetric monoidal functor $\mathrm{Fin}_* \to \mathcal{C}^{\otimes}$).

To be able to obtain the **associative algebras** for $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$, we would like to consider the functor/diagram

$$\begin{tikzcd}
\Delta^{op} \arrow[rr] \arrow[dr] & & \mathcal{C}^{\otimes} \arrow[dl] \\
& \mathrm{Fin}_* &
\end{tikzcd}$$

mapping **convex edges** to co-Cartesian ones. Here by convex edges, we mean **injective + image is an interval (consecutive integers)**.

Remark 3.8. Why should Δ^{op} serve as the model for associative algebras? Well, the intuitive idea is that since the maps are order-preserving in Δ, there is no way to “swap". Thus, we lose the commutativity part.

Example 3.9. Commutative algebras in $\mathrm{Spc}^{\times}$ is exactly an $\mathbb{E}_{\infty}$-space in Lecture 2! Here $\mathrm{Spc}^{\times}$ is the symmetric monoidal structure on Spc given by product of spaces.

We can now view

$$\mathrm{CAlg}(\mathcal{C}^{\otimes}) \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C}^{\otimes}) \text{ and } \mathrm{Alg}(\mathcal{C}^{\otimes}) \subseteq \mathrm{Fun}(\Delta^{op}, \mathcal{C}^{\otimes})$$

Remark 3.10. In 1-categories, we always have that $\mathrm{CAlg}(\mathcal{C}) \subseteq \mathrm{Alg}(\mathcal{C})$, and being commutative is just extra property. This is in general false for ∞-categories! There is only really a forgetful map.

Example 3.11. $\mathrm{CAlg}(\mathrm{Cat}_\infty^\times)$ can be identified with the collection of symmetric monoidal small ∞-categories. Again, $\mathrm{Cat}_\infty^\times$ denotes Cat_∞ with a symmetric monoidal structure given by its product $\times$.

Proposition 3.12. Let F be a lax symmetric monoidal functor $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$, then it induces a map $\mathrm{Alg}(\mathcal{C}^\otimes) \to \mathrm{Alg}(\mathcal{D}^\otimes)$ (and similarly for CAlg).

3.2 Spectra and Presentable Categories

What is a presentable category?

Definition 3.13. An ∞-category $\mathcal{C}$ is **presentable** if:

1. $\mathcal{C}$ admits small colimits.
2. $\mathcal{C}$ is accessible in the following sense - it is generated under κ-filtered colimits by a small sub-category, where κ is a regular ordinal.

By a **theorem of Simpson**, this is equivalent to $\mathcal{C}$ be an “accessible localization of a presheaf category".

Fun Fact: If $\mathcal{C}$ is presentable and $\mathcal{C}^{op}$ is presentable, then $\mathcal{C}$ is a POSET.

Definition 3.14. We define $\mathrm{Pr}^L \subseteq \widehat{\mathrm{Cat}_\infty}$ is the subcategory of **presentable categories** whose morphisms are **left adjoint functors**.

We use $\mathrm{Pr}^L_{st} \subseteq \mathrm{Pr}^L$ to denote the full sub-category of stable presentable categories.

Proposition 3.15 (4.8.15 of [Lur17]). There exists a symmetric monoidal structure on Pr^L such that:

1. $\mathrm{Pr}^L \hookrightarrow \widehat{\mathrm{Cat}_\infty}$ is **lax symmetric monoidal**.
2. The map $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_{\mathrm{Pr}^L} \mathcal{D}$ in Cat_∞ is initial amongst functors preserving colimits in both variables. Note that we can identify $\mathcal{C} \otimes_{\mathrm{Pr}^L} \mathcal{D}$ as the category of right adjoint functors from $\mathcal{C}^{op} \to \mathcal{D}$ (ie. $\mathrm{RFun}(\mathcal{C}^{op}, \mathcal{D}) \simeq \mathrm{RFun}(\mathcal{C}^{op}, \mathcal{D})_*$).
3. $- \otimes_{Pr^L} -$ preserveing colimits in both
4. Pr^L_{st} has an induced symmetric monoidal structure.

Fact: (from 4.8.2 of [Lur17])

1. $\mathrm{Spc} \in \mathrm{Pr}^L$ is the unit of the symmetric monoidal structure.
2. $\mathcal{C} \otimes \mathrm{Spc}_* \simeq \mathcal{C}_*$.
3. The stabilization $\mathrm{Sp}(\mathcal{C})$ is $\mathcal{C} \otimes \mathrm{Sp} \simeq \mathrm{RFun}(\mathcal{C}^{op}, \mathrm{Sp}) = \lim \mathrm{RFun}(\mathcal{C}^{op}, \mathrm{Spc}_*) = \lim \mathcal{C}_* \simeq \mathrm{Sp}(\mathcal{C})$. Sp is the unit in Pr^L_{st}

4. $\mathrm{CAlg}(\mathrm{Pr}^{L,\otimes})$ are precisely the presentably symmetric monoidal categories

Theorem 3.16 (4.8.2.19 of [Lur17])**.** There exists a unique symmetric monoidal structure on Sp such that the unit is $\mathbb{S}$ and $- \otimes -$ preserves small colimits in both variables.

Proof Idea. For **uniqueness**, let $E \in \mathrm{Sp}$, recall we could write $E = \mathrm{colim}_m \Sigma^m \Sigma^\infty_+ X_m$ for spaces X_m from Lecture 2. Let $F \in \mathrm{Sp}$, then

$$\begin{aligned} E \otimes F &= (\mathrm{colim}_m \Sigma^m \Sigma^\infty_+ X_m) \otimes F \\ &= \mathrm{colim}_m \Sigma^m \,\mathrm{colim}_{X_m} ((\Sigma^\infty_+ *) \otimes F) \\ &= \mathrm{colim}_m \Sigma^m \,\mathrm{colim}_{X_m} (\mathbb{S} \otimes F) && \Sigma^\infty_+ * \simeq \mathbb{S} \\ &= \mathrm{colim}_m \Sigma^m \,\mathrm{colim}_{X_m} (F) \end{aligned}$$

For **existence**, this is because Sp is the unit in Pr^L_{st}, which implies that $\mathrm{Sp} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$. Alternatively, this follows from Sp "being idempotent" in Pr^L, that is - $\mathrm{Sp} \simeq \mathrm{Sp} \otimes \mathrm{Spc} \xrightarrow{id \otimes \Sigma^\infty_+} \mathrm{Sp} \otimes \mathrm{Sp}$ is an equivalence. ■

3.3 Ring Spectra and Examples

From now on, we always assume Sp is equipped with the aforementioned symmetric monoidal structure.

Definition 3.17. $\mathrm{CAlg}(\mathrm{Sp})$ is called $\mathbb{E}_\infty$-rings. $\mathrm{Alg}(\mathrm{Sp})$ is called $\mathbb{E}_1$-rings.

Our goal now is to look at interpolations between the two. Let us look at ku for Lecture 2 for example. Now before, this we should first discuss how what we talked about last lecture fits within the context:

1. The group-like $\mathbb{E}_\infty$-spaces $\mathrm{CMon}(\mathrm{Spc})^{gp}$ are exactly $\mathrm{CAlg}(\mathrm{Spc}^\times)^{gp}$.
2. $\underline{\mathrm{Vect}}_{\mathbb{C}}^{gp}$ is given as an element in $\mathrm{CMon}(\mathrm{Spc})^{gp}$.
3. Because tensor product of vector bundles exist, this actually makes $\underline{\mathrm{Vect}}_{\mathbb{C}}^{gp}$ into a commutative algebra:
$$\underline{\mathrm{Vect}}_{\mathbb{C}}^{gp} \in \mathrm{CAlg}(\mathrm{CMon}(\mathrm{Spc})^{gp,\otimes})$$
4. The functor $B^\infty : \mathrm{CMon}(\mathrm{Spc})^{gp,\otimes} \to \mathrm{Sp}^{\otimes}_{\geq 0}$ is **symmetric monoidal**!

Now let us see how ku fits into this:

Example 3.18. Recall last lecture, we constructed ku as $B^\infty \underline{\mathrm{Vect}}_{\mathbb{C}}^{gp} \in \mathrm{CAlg}(\mathrm{Sp}^{\otimes}_{\geq 0})$. In other words, ku is a commutative algebra over connective spectra - in particular this means that

$$B^\infty \underline{Vect}_{\mathbb{C}}^{gp} \in \mathrm{CAlg}(\mathrm{Sp}^{\otimes})$$

so ku is an $\mathbb{E}_\infty$-ring spectrum.

We can also construct a non-connective version of ku as follows.

Definition 3.19. The **bott element** is the map $\beta : \mathbb{S}^2 \to ku$ corresponding to $\beta : S^2 \to \Omega^\infty ku = \mathbb{Z} \times BU = (\bigsqcup_m BU_m)^{gp}$ given by the map $1 - \theta(1)$ (here 1 is the trivial bundle and $\theta(1)$ is the tautological line bundle).

From here we define $KU := ku[\beta^{-1}] \in \mathrm{CAlg}(\mathrm{Sp})$. Note that in this case

$$\Sigma^2 KU \simeq \mathbb{S}^2 \otimes KU \xrightarrow{\beta \otimes id} KU \otimes KU \xrightarrow{\mu} KU$$

is an equivalence. This is called **Bott periodicity**.

In the last lecture, we also saw $\underline{MU}$ as the colimit in Spectra $\operatorname{colim}^{\mathrm{Sp}}(BU \to BO \xrightarrow{j} \mathrm{Sp})$ (here we use $j : BO \to \mathrm{Sp}$ to denote this being the J-homomorphism).

Theorem 3.20 (Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, Charles Rezk [ARB$^+$13])**.** Let Pic be ∞-groupoid of invertible objects in Sp. There is a symmetric monoidal functor (called the **Thom spectrum functor**) $M : \mathrm{Spc}\,/\,\mathrm{Pic} \to \mathrm{Sp}$

$$(X \xrightarrow{F} \mathrm{Pic}) \mapsto \operatorname{colim}_X^{\mathrm{Sp}} F$$

left-Kan-extending the inclusion $*/\,\mathrm{Pic} \simeq \mathrm{Pic} \hookrightarrow \mathrm{Sp}$ wit respect to the **Day convolution** on $\mathrm{Spc}\,/\,\mathrm{Pic} \simeq \mathcal{P}(\mathrm{Pic})$. Here by Day convolution, we mean

$$(X \to \mathrm{Pic}) \otimes (Y \to \mathrm{Pic}) \simeq (X \times Y \to \mathrm{Pic} \times \mathrm{Pic} \xrightarrow{\mu} \mathrm{Pic}).$$

Remark 3.21. In general, there is a Day convolution that can be considered as follows:

- There is a symmetric monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for $\mathcal{C}, \mathcal{D}$ both symmetric monoidal. In this case, commutative algebras over lax symmetric monoidal functors.
- Now $F \otimes_{Day} \mathcal{G}$ is the left Kan extension of $\mathcal{C} \times \mathcal{C} \xrightarrow{F \times G} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes_{\mathcal{D}}} \mathcal{D}$ along $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes_{\mathcal{C}}} \mathcal{C}$.

Example 3.22. We can realize MU as $M(BU \to BO \xrightarrow{j} \mathrm{Pic})$. Here the j-homomorphism is given by the diagram

$$\begin{tikzcd}
BO \arrow[r, "j"] \arrow[d, hook] & \mathrm{Pic} \\
\mathbb{Z} \times BO \simeq (\coprod_m BO_m)^{gp} \arrow[ur, dashed] & \\
\coprod_m BO_m \arrow[u] \arrow[uur] &
\end{tikzcd}$$

where the map $\coprod_m BO_m \to \mathrm{Pic}$ is induced by sending a real vector space to the Σ^∞ of the one-point-compactification of its Thom space.

Note that the j-homomorphism is an $\mathbb{E}_\infty$-map, so this yields an $\mathbb{E}_\infty$-algebra in $\mathrm{Spc}\,/\,\mathrm{Pic}$. Thus, MU is an $\mathbb{E}_\infty$-ring.

3.4 Modules

Recall how modules are defined for 1-categories:

Definition 3.23. Let $E \in \mathrm{Alg}(\mathcal{C}, \otimes)$, a module F over E is a the data of a map $E \otimes F \to F$ satisfying relevant unitality and associativity.

Let us try to look at the set-up for ∞-categories:

Definition 3.24. Let $A : \Delta^{op} \to \mathcal{C}^{\otimes}$ be an (associative) algebra in $\mathcal{C}$. A **left A-module** is $F : \Delta^{op} \times [1] \to \mathcal{C}^{\otimes}$ such that:

- Post-composition with $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ is a map $\Delta^{op} \times [1] \to \mathrm{Fin}_*$ such that

$$([m], 0) \mapsto \langle m+1 \rangle, ([m], 1) \mapsto \langle m \rangle.$$

- Restriction to $\Delta^{op} \times \{1\}$ is the functor A.
- $F(id_{[m]}, 0 \to 1)$ and $F(\alpha^{op}, id_0)$ are coCartesian in $\mathcal{C}^{\otimes}$ for all inert $\alpha : [m] \to [m]$ with $\alpha(m) = m$.

Note that there is a similar notion of right A-module, and if $A \in \mathrm{CAlg}(\mathcal{C})$, the two notions agree.

Remark 3.25. We could recover the underlying module $M := F([0], 0) \in \mathcal{C}^{\otimes}_{\langle 1 \rangle}$. $F([n], 0)$ has to be $(A, ..., A, M) \in \mathcal{C}^{\otimes}_{\langle m+1 \rangle} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^{m+1}$ (where A spans m times). Any coCartesian lift $\langle 2 \rangle \to \langle 1 \rangle$ with source $(A, M) = F([1], 0)$ has to have target $F([0], 0) = M$. This gives the required map $A \otimes M \to M$.

For our purposes, we mostly work over Sp with $\mathbb{E}_1$ or $\mathbb{E}_\infty$-rings R.

Proposition 3.26 (7.1.2.7 of [Lur17])**.** The ∞-category of left R-modules - $\mathrm{LMod}_R(\mathrm{Sp})$ - is a presentable, symmetric monoidal category that is functorial with respect to $R \in \mathrm{CAlg}(\mathrm{Sp})$. There is a forgetful functor $\mathrm{LMod}_R(\mathrm{Sp}) \to \mathrm{Sp}$ with left-adjoint given underlying by $R \otimes -$.

4 Lecture 4: Operads, algebras, and Koszul duality (by Markus Zetto)

4.1 Operads via Symmetric Sequences

Let us fix $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$, ie. $\mathcal{V}$ is a presentable, stable, symmetric monoidal ∞-category. In this case, we have a sequence of forgetful functors

$$\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}^{\mathrm{colim}})) \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{colim}}) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}) \to \widehat{\mathrm{Cat}}.$$

This sequence admits a sequence of left adjoints back as

$$\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}^{\mathrm{colim}})) \xleftarrow{-\otimes\mathcal{V}} \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{colim}}) \xleftarrow{\mathcal{P}^{\mathrm{colim}}} \mathrm{CAlg}(\widehat{\mathrm{Cat}}) \xleftarrow{\mathrm{Sym}} \widehat{\mathrm{Cat}}.$$

By composing the left-adjoints and restricting to Cat, one obtains a functor

$$\mathrm{Cat} \to \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^L)), \quad \mathcal{C} \to \mathcal{P}(\mathrm{Sym}\,\mathcal{C}) \otimes \mathcal{V},$$

where one should think of $\mathcal{P}$ as the presheaves valued in spaces. Thematically, one can interpret the right term as

$$\mathcal{P}(\mathrm{Sym}\,\mathcal{C}) \otimes \mathcal{V} \cong \mathrm{Fun}(\bigsqcup_{n\geq 0}(\mathcal{C}^{\times n}_{h\Sigma_n})^{op}, \mathcal{V}).$$

Definition 4.1. A **$\mathcal{V}$-enriched operad with colors** $X \in \mathcal{S}$ is a monad on

$$\mathcal{P}(\mathrm{Sym}\,X) \otimes \mathcal{V} \in \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^L))$$

Explicitly, a $\mathcal{V}$-enriched operad with colors X, as a monad, is an algebra in the endomorphism category $\mathrm{End}^{L,\otimes}_{\mathcal{V}}(\mathrm{PSym}\,X \otimes \mathcal{V}) = \mathrm{Fun}(X, \mathrm{PSym}\,X \otimes \mathcal{V})$. Note that

$$\mathrm{Fun}(X, \mathrm{PSym}\,X \otimes \mathcal{V}) = \mathrm{Fun}(X \times \bigsqcup_{n\geq 0} X^{\times n}_{h\Sigma_n}, \mathcal{V}) =: \mathrm{SSeq}_X(\mathcal{V}).$$

Here, $\mathrm{SSeq}_X(\mathcal{V})$ denotes the ∞-category of symmetric sequences valued in $\mathcal{V}$ with colors X, and is defined as above.

This yields an equivalent definition for operads.

Definition 4.2. We call $\mathrm{Op}_X(\mathcal{V})$ as the $\mathrm{Alg}(\mathrm{SSeq}_X(\mathcal{V})$ (ie. we call them the operads over $\mathcal{V}$.)

We also give a similar definition for co-operads.

Definition 4.3. We call $\mathrm{coOp}_X(\mathcal{V})$ as the $\mathrm{coAlg}(\mathrm{SSeq}_X(\mathcal{V}))$. When $X = *$, we may drop the symbol X.

Remark 4.4. Observe that an operad $\mathcal{O} \in \mathrm{Op}_X(\mathcal{V})$ specifies a functor

$$\mathrm{Mul}_{\mathcal{O}} : X \times \bigsqcup_{n\geq 0} X^{\times n}_{h\Sigma n} \to \mathcal{V},$$

with the identifies and compositions on $\mathrm{Mul}_{\mathcal{O}}$ specified by the algebra structure on $\mathcal{O}$.

Let us specialize to the case where $X = *$ is a point. In this case one recovers the set-up where

$$\mathrm{SSeq}(\mathcal{V}) = \mathrm{Fun}(\bigsqcup_{n\geq 0} B\Sigma_n, \mathcal{V}).$$

An operad $\mathcal{O}$ here can be thought of as a collection $\mathcal{O}(n), n \geq 0$ equipped with Σ_n-action on $\mathcal{O}(n)$.

There is an operation on operads $\mathcal{O}, \mathcal{P}$ given by

$$(\mathcal{O} \odot \mathcal{P})(n) = \bigsqcup_{r \geq 0} (\mathcal{O}(r) \otimes \mathcal{P}^{\otimes r}(n))_{h\Sigma n}$$

Here the operation in the middle is seen as a **Day-Convolution**, and $\mathcal{P}^{\otimes r}(n) := \operatorname{colim}_{n+1+...+n_r=r} \mathcal{O}(n_i)$. There is a multiplicative unit given by

$$\mathbb{1}(n) = \begin{cases} \emptyset, n \geq 1 \\ 1_{\mathcal{V}}, n = 1 \end{cases} .$$

Definition 4.5. Let $\mathcal{O}$ be an operad over $\operatorname{SSeq}(\mathcal{V})$. An $\mathcal{O}$-algebra is the data of a map $1_{\mathcal{V}} \to \mathcal{O}(1)$ (called the unit map) and a map

$$(\mathcal{O} \odot \mathcal{O})(n) \to \mathcal{O}(n)$$

for each n, satisfying some compatibility conditions.

Here we give some examples of operads.

Example 4.6.

1. The unit operad $\mathbb{1}$ given above.
2. The commutative operad $\mathbb{E}_\infty$ given by $\mathbb{E}_\infty(n) := 1_{\mathcal{V}}$ for all n.
3. When $\mathcal{V} = \operatorname{Spc}$, the $\mathbb{E}_k$-operads are given by configuration spaces $\mathbb{E}_k(n) := \operatorname{Emb}(\{1, ..., n\}, \mathbb{R}^k)$.
4. When $\mathcal{V} = \operatorname{Vect}_k$ with k having characteristic zero, the Lie operad Lie is given by the operad freely generated by $[\bullet, \bullet] \in \operatorname{Lie}(2)$ with respect to the relations of anti-symmetry and the Jacobi identity.

Remark 4.7. If $P \in \operatorname{SSeq}(\mathcal{V})$ is concentrated in only degree 0, then so is $\mathcal{O} \odot P$ This gives an action of $\operatorname{SSeq}(\mathcal{V})$ on $\mathcal{V}$, or equivalently, it specifies a map

$$\operatorname{SSeq}(\mathcal{V}) \xrightarrow{\otimes} \operatorname{End}(\mathcal{V}); P \mapsto (\operatorname{Sym}_P : v \mapsto P \odot v[0]),$$

where $P \odot v[0]$ is in degree 0 and hence stays in $\mathcal{V}$.

Definition 4.8. We define $\operatorname{Alg}_{\mathcal{O}}(\mathcal{V})$ as $\operatorname{LMod}_{\operatorname{Sym}\mathcal{O}}(\mathcal{V})$ for $\mathcal{O} \in \operatorname{Op}(\mathcal{V})$ with respect to the action specified in the action above. Similarly, we also define $\operatorname{coAlg}_{\mathcal{Q}}^{nil,d.p.}(\mathcal{V})$ as $\operatorname{coLMod}_{\operatorname{Sym}\mathcal{Q}}(\mathcal{V})$ for $\mathcal{Q} \in \operatorname{coOp}(\mathcal{V})$.

Remark 4.9. An actual co-algebra should have a map $A \to \prod(\mathcal{Q}(n) \otimes A^{\otimes n})^{h\Sigma_n}$.

Explicitly, the algebra structure specified above gives a map

$$(\mathcal{O} \odot v[0])(0) = \bigsqcup_{r \geq 0} (\mathcal{O}(r) \otimes v^{\otimes r})_{h\Sigma_r} \to v.$$

Remark 4.10. The presentation of operads here is different from that of Lurie's in [Lur17]. The notetaker believe they are equal over spaces when X is one point. References for this presentation of operads in the notes here can be found in [Bra17, BCN24, Hau22].

4.2 Bar-Cobar Duality

Definition 4.11. For $\mathcal{C}$ a monoidal ∞-category and $A \in \mathrm{Alg}^{aug}(\mathcal{C}) := \mathrm{Alg}(\mathcal{C})/1_{\mathcal{C}}$, we say $\mathrm{Bar}(\mathcal{C})$ is the **bar construction** of A if for all $c \in \mathcal{C}$, we have that

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{Bar}\,A, c) \simeq \mathrm{Map}_{{}_A\,\mathrm{BiMod}_A(\mathcal{C})}(A, \rho(c))$$

where $\rho : \mathcal{C} \to_A \mathrm{BiMod}_A(\mathcal{C})$ restrocts scalars along the augmentation.

Proposition 4.12. If $\mathcal{C}$ admits a geometric realization and $\otimes$ is compatible with them (in fact this is not necessary), then $\mathrm{Bar}(A) := 1 \otimes_A 1 \simeq 1 \otimes_A A \otimes_A 1 = \mathrm{colim}_{[n]\in\Delta^{op}}(1 \otimes A^{\otimes n} \otimes 1)$.

Observe that $\mathrm{Bar}(A)$ is in fact a co-algebra. Indeed, we have a clear sequence of maps

$$\mathrm{Bar}(A) = 1 \otimes_A 1 \xrightarrow{\sim} 1 \otimes_A A \otimes_A 1 \to 1 \otimes_A (1 \otimes_A 1) \otimes_A 1 \xrightarrow{\sim} (1 \otimes_A 1) \otimes_A (1 \otimes_A 1) = \mathrm{Bar}(A) \otimes_A \mathrm{Bar}(A).$$

This now defines a functor

$$\mathrm{Bar} : \mathrm{Alg}^{aug}(\mathcal{C}) \to \mathrm{coAlg}^{aug}(\mathcal{C}).$$

Similarly, if $\mathcal{C}$ admits totalization, we can get a similar construction called the **CoBar**. Now we have a pair of functors

$$\mathrm{Bar} : \mathrm{Alg}^{aug}(\mathcal{C}) \leftrightarrow \mathrm{coAlg}^{aug}(\mathcal{C}) : \mathrm{coBar}\,.$$

Proposition 4.13. The pair of functors above form an adjunction.

Proof Idea. we first set-up a few preliminary definitions.

Definition 4.14. A **pairing** of ∞-categories $\mathcal{C}, \mathcal{D}$ is a right fibration $\mathcal{M} \to \mathcal{C} \times \mathcal{D}$ (in other words, it is a functor $\mathcal{C}^{op} \times \mathcal{D}^{op} \to \mathcal{S}$. The **pairing** is **left representable** if for all $c \in \mathcal{C}$, $\mathcal{M} \times_{\mathcal{C}} \{c\}$ has a final object (ie. there is a factorization $\mathbb{D} : \mathcal{C}^{op} \to \mathcal{D} \subseteq \mathcal{P}(\mathcal{D})$). The pairing is **right representable** if for all $d \in \mathcal{D}$, $\mathcal{M} \times_{\mathcal{D}} \{d\}$ has a final object (ie. there is a factorization $\mathbb{D}' : \mathcal{D}^{op} \to \mathcal{C} \subseteq \mathcal{P}(\mathcal{C})$).

It is a general fact that if the pairing is both left and right representable, then there is an adjunction $(\mathbb{D}')^{op} \dashv \mathbb{D}$.

Example 4.15. Lurie's straightening map $\mathrm{Map}_{\mathcal{C}} : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{S}$ is both left and right representable, and it corresponds to the adjunction $id : \mathcal{C} \leftrightarrow \mathcal{C} : id$. This corresponds to a map $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C}$.

This definition can now be applied to conclude the proposition. ■

Let us now consider the setting with $\mathbb{E}_k$-monoidal ∞-categories.

Definition 4.16. A pairing of $\mathbb{E}_k$-monoidal categories is an $\mathbb{E}_k$-monoidal functor $\mathcal{M} \to \mathcal{C} \times \mathcal{D}$ whose underlying functor is still a right fibration.

Observe in this case that the induced map below is still a pairing:

$$\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{M}) \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \times \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{D}).$$

Theorem 4.17. Let $\mu : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ be a pairing of $\mathbb{E}_k$-monoidal ∞-categories such that:

- $\mathcal{M} \times_{\mathcal{D}} \{1_{\mathcal{D}}\} \simeq \mathcal{C}$.
- $\mathcal{C}$ is left representable.
- $\mathcal{D}$ has totalizations.

Then $\mathrm{Alg}_{\mathbb{E}_k}(\mu)$ is a left presentable pairing.

Example 4.18. Suppose $1_{\mathcal{C}}$ is final and $\mathcal{C}$ has totalization in Example 4.15 (yes, here we are applying the theorem above with $\mathcal{D} = \mathcal{C}$ and $\mathcal{C} = \mathcal{C}^{op}$), then the theorem above tells us that $\mathrm{Alg}_{\mathbb{E}_k}(\lambda)$ is a left representable pairing. This defines a functor

$$\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})^{op} \xrightarrow{\mathrm{Bat}^{(k)}} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}^{op}).$$

Here is another example:

Example 4.19. Consider the case where $\mathcal{C} = \mathcal{D}$. If $\mathcal{D}$ has geometric realizations and totalizations and $1_{\mathcal{D}}$ is the zero object, then we have an adjunction

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \arrow[r, shift left, "\mathrm{Bar}^{(k)}"] & \mathrm{coAlg}_{\mathbb{E}_k}(\mathcal{C}) \arrow[l, shift left, "\mathrm{CoBar}^{(k)}"] \end{tikzcd}$$

5 Lecture 4.5: Complex Orientations and Quillen's Theorem (by Gijs Heuts and Ishan Levy)

This is a very improvised last minute addition to the workshop. As a result, the two mentos are figuring out what to cover live as this is being written. The content division is roughly:

1. Complex orientations
2. The role of MU and Quillen's theorem.
3. M_{fg} and height.

Let us begin with complex orientations!

Definition 5.1. We say a ring spectrum E is **complex orientable** if the inclusion map $\mathbb{C}P^1 \to \mathbb{C}P^\infty$ induces a map $\tilde{E}^2(\mathbb{C}P^\infty) \to \tilde{E}^2(\mathbb{C}P^1) \cong \tilde{E}^0(S^0)$ is surjective. A choice of an element $t \in \tilde{E}^2(\mathbb{C}P^\infty)$ that is mapped to $1 \in \tilde{E}^0(S^0)$.

Complex orientations give, in essence, a good method to compute analogs of **Chern classes**. Here are some interesting facts about them:

1. If E is complex orientable, then the **Atiyah-Hirzeburch spectral sequence** implies
$$E^*(\mathbb{C}P^\infty) \cong E^*[[t]] \text{ and } E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x,y]]$$
This t should be thought of as the first Chern class of the universal line bundle on $\mathbb{C}P^\infty$, adopted for the setting of E specifically.
2. $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ may not be true for a general complex orientable E! To be precise, consider the classifying map of tensor product of line bundles:
$$m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$
This induces a map
$$m^* : E^*[[t]] \to E^*[[x,y]].$$
Write $f(x,y) = m^*(t) \in E^*[[x,y]]$.
3. The up-shot is that this power series f is an example of what is called a **formal group law** over E^*, that is:
    - (Symmetric): $f(x,y) = f(y,x)$.
    - (Unital): $f(x,0) = x$ and $f(0,y) = y$.
    - (Associativity): $f(x, f(y,z)) = f(f(x,y),z)$.

Here are some examples of complex-oriented cohomology theories.

Example 5.2. $H\mathbb{Z}$ is complex orientable. KU is complex orientable. MU is also complex orientable.

In fact, we have the following:

Proposition 5.3. Let E be such that $\pi_*(E)$ is concentrated in even degrees, then E is complex orientable.

Proof Sketch. The proof comes from obstruction theory. Consider a lift

$$\begin{tikzcd}
\mathbb{C}P^\infty \arrow[dr, dashed] & \\
\mathbb{C}P^1 \arrow[u] \arrow[r] & \Omega^{\infty-2}E
\end{tikzcd}$$

The obstructions for these lifts existing all exist in odd homotopy groups. ■

Consider the functor FGL : CRings → Sets that assigns any ring R to the collection of formal group laws over R.

Proposition 5.4. It turns out this functor FGL is **corepresentable** by some ring L.

Proof. We can construct the Lazard ring L as $\mathbb{Z}[a_{ij}]_{i,j=0}^\infty / \sim$ where $\sim$ the minimal set of relations such that $F(x,y) = \sum_{i,j}^\infty a_{ij}x^iy^j$ is a formal group law. This is the desired ring. ■

Theorem 5.5 (Lazard's Theorem). The Lazard ring L is in fact isomorphic to a polynomial ring as $\mathbb{Z}[a_1, a_2, ...]$ with $|a_i| = 2i$.

Now MU_* carries a formal group law, so there is a classifying map from $L \to MU_*$.

Theorem 5.6 (Quillen's Theorem, First Part). The map $L \to MU_*$ is an isomorphism.

A much easier result is as follows: for a complex orientable cohomology theory E, the data of a complex orientation of E is exactly a map $MU \to E$, so MU can be thought of as a **topological lift of** L. What we can do is we can write

$$MU = \operatorname{colim}_n \Sigma^{-n}\Sigma^\infty Th(\gamma_n \to BU(n)).$$

One can compute that $E^*(BU(n)) \cong E^*[[c_1, ..., c_n]]$ using the Atiyah-Hirzeburch spectral sequence, and this will give the desired topological lift.

Theorem 5.7 (Quillen's Theorem, Second Part). If E is complex oriented, then $E_*(MU) \cong E_*[b_1, b_2, ...]$. Now observe that $E_*MU = \pi_*(E \otimes MU)$, now we have two formal group laws $E \otimes MU[[t_E]] = (E \otimes MU)^{\mathbb{C}P^\infty} = E \otimes MU[[t_M]]$ given by E and MU respectively. Since they are isomorphic to the same underlying ring, they must differ by some **power series**, say

$$g(t) = t + b_1t^2 + b_2t^3 + ...$$

Claim: These b_i's are exactly the ones appearing in $E_*[b_1, b_2, ...]$.

As a result - When we take $E = MU$, we learn that $MU_*(MU)$ is exactly **the ring parametrizing a universal graded formal group law AND a (strict) automorphism of the fgl**.

Now consider the diagram

$$\begin{tikzcd}
MU \arrow[r, shift left=2, "i_R"] \arrow[r, shift right=2, "i_L"'] & MU \otimes MU \arrow[l, "m" description]
\end{tikzcd}$$

where i_L, i_R are inclusions to the left and right factors and m is multiplication. Taking homotopy groups on both sides, we have that

$$\begin{tikzcd}
\text{fgls over R} \arrow[r, "\text{identity}" description] & \text{isomorphisms of fgls} \arrow[l, shift right=3, "\text{target}"'] \arrow[l, shift left=3, "\text{source}"]
\end{tikzcd}$$

This whole fact relates to the Adams-Novikov spectral sequence. Now recall we considered the following co-simplicial object $MU^{\otimes +1}$ as

$$\begin{tikzcd}
\mathbb{S} \arrow[r] & \mathrm{MU} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \mathrm{MU} \otimes \mathrm{MU} \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & \ldots
\end{tikzcd}$$

Now consider $\lim \Delta_{\leq n} MU^{\bullet +1}$. Via some Dold-Kan correspondence, there is apparently a way to extract a spectral sequence out of this business.

Now MU_* defines a map from $\mathrm{CRings} \to \mathrm{Sets}$. Because of the simplicial object given in $MU^{\otimes +1}$, this really gives a map $\mathrm{CRings} \to \mathrm{sSets}$, from then we can extend into

$$\mathrm{CRings} \to \mathrm{sSet} \to \mathrm{Spc}\,.$$

In this case, it turns out we have a commutative diagram of the following, where M_{fg} sends a ring R to the groupoid of formal groups over R, whose morphisms are isomorphisms of the formal groups themselves. The inclusion $\mathrm{Grpd} \to \mathrm{Spc}$ is just the inclusion of 1-groupoids to ∞-groupoids.

$$\begin{tikzcd}
 & & \mathrm{Grpd} \arrow[d, hook] \\
\mathrm{CRing} \arrow[r] \arrow[urr, "M^*_{fg}"] & \mathrm{sSet} \arrow[r] \arrow[ur] & \mathrm{Spc}
\end{tikzcd}$$

> **Question 5.8.** What can we say about classifying formal groups?

Let us try to classify formal groups over algebraic closed fields, up to isomorphisms.

Claim: There is one formal group up to isomorphism in characteristic 0 (which is the additive formal group). In characteristic p, there are infinitely many formals groups up to isomorphism, and they are classified by an invariant called their **height**, which is valued in $\mathbb{N} \cup \{+\infty\}$.

How do we define this height? Well, write $f(x, y) = x +_F y$ notationally, we define

$$[p]_F(t) = t +_F t +_F \ldots +_F t$$

where $+_F$ is given $p - 1$ times. Now we **claim that** there is an g and a unique h such that

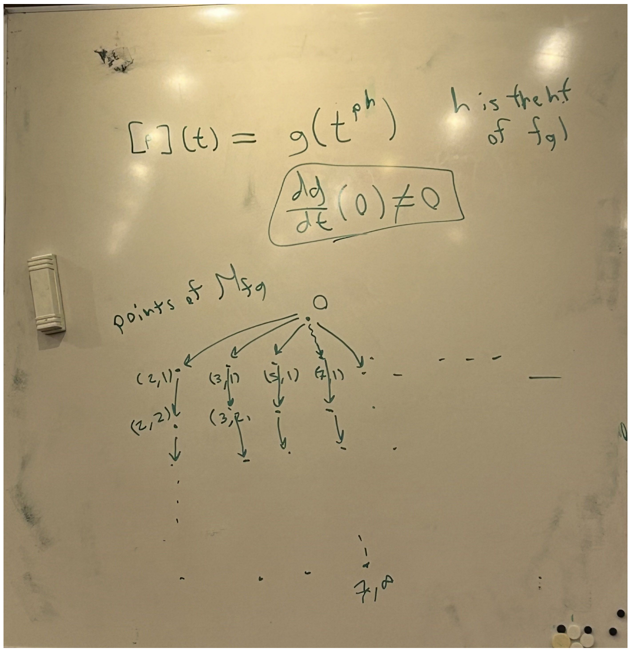
$$[p]_F(t) = g(t^{p^h})$$

where g is power series and $\frac{dg}{dt}(0) \neq 0$. In this case, h is called the **height** of F at prime p.

What this height help us is as follows. So far, we have M_{fg} as a geometric object, and we would like to ask: what are the points of M_{fg}? Well, not being super rigorous, but the claim we had above tells us the points roughly look like:

1. There is one point corresponding to 0 (in characteristic 0) at the top.
2. The first row consists of $(2, 1), (3, 1), (5, 1), (7, 1)\ldots$ going over the primes in the first coordinate.
3. The second row consists of $(2, 2), (3, 2), (5, 2), (7, 2), \ldots$ and so on.
4. The specialization of each point goes downward, forming a "stratification" given by height.

Pictorially, we have that:



6 Lecture 5: Nilpotence and periodicity (by Henry Rice)

Last night, we reviewed formal group laws and heights, so we will omit this during the lecture. The next key concept we want to consider is what we call **self maps**!

Broadly speaking, the idea of self maps appear in the following scenarios:

1. Let X be a spectrum, a self map is a map $\Sigma^d X \to X$.
2. Let $f : F \to X$, we can consider its smash product.
3. Let R be a ring spectrum and take $r \in \pi_0(R)$, we can product a map $R \to R$ that mimicks multiplication by r between homotopy groups.

6.1 Nilpotence Theorem

The idea of **nilpotence theorem** is that the possible nilpotence for all of these self-maps can be detected by some homology theory.

Theorem 6.1 (Nilpotence Theorem)**.** There are three statements about nilpotence:

1. Let R be a ring spectrum, then the kernel of the map $\pi_*(R) \to \mathrm{MU}_*(R)$ (induced by smashing R with the unit map $\mathbb{S} \to MU$) consists of (individually) nilpotent elements.
2. Let F be a finite spectrum and consider a map $f : F \to X$. Suppose the map $MU \otimes f : MU \otimes F \to MU \otimes X$ is null-homotopic, then f is smash nilpotent in the sense that $f^{\otimes k} \simeq *$ for $k >> 0$.
3. Consider a sequence of maps
$$... \to X_n \xrightarrow{f_n} X_{n+1} \to ...$$
Suppose X_n is c_n-connected with $c_n \geq mn+b$ for $m, b \in \mathbb{Z}$. If $MU_*(f_n) : MU_*(X_n) \to MU_*(X_{n+1})$ are all zero maps, then $\lim_{\to} X_n \simeq *$.

6.2 Morava K-Theory

Nilpotence is closely tied to the concept of **Morava K-theory** $K(n)$, which are ring spectrums defined for each height n and each prime p. Rather than giving an explicit construction here, let us first describe what we want $K(n)$'s to look at before constructing this. Roughly speaking, we want these objects to satisfy the following:

1. $K(n)_* = \pi_*(K(n)) \cong \mathbb{F}_p[v_n^{\pm 1}]$ where $|v_n| = 2(p^n - 1)$.

Now, how are we going to construct $K(n)$? Well, recall that $MU_* \cong \mathbb{Z}[t_1, t_2, ...]$ with $|t_i| = 2i$ and let F be the universal formal group law over MU_*. To construct $K(n)$'s, we first localize MU to $MU_{(p)}$. In this case, the homotopy groups of $MU_{(p)}$ becomes:

$$\pi_*(MU_{(p)}) = \mathbb{Z}_{(p)}[t_i \mid i \neq p^n - 1],$$

with the understanding that $t_{p^i - 1} = v_i$.

Definition 6.2. We construct the Morava K-theory $K(n)$ as:

$$K(n) = v_n^{-1} MU_{(p)} / (t_i, v_j | i \neq p^n - 1, j \neq n)$$

The existence, uniqueness, and (some notion of) algebra spectrum structure, of such $K(n)$ is made rigorous by the following theorem.

Theorem 6.3 (Strickland's Theorem, 1999). Let R be a commutative ring spectrum such that $\pi_*(R)$ only has even components. Suppose A_* is a localized regular quotient of R and 2 is invertible in A_*, then there exists a unique homotopy commutative R-ring spectrum A such that $\pi_*(A) \cong A$. This is sometimes called a **strong realization**.

Note that this theorem only holds for $p \neq 2$. For $p = 2$, existence still holds, but there are actually two choices for $K(n)$.

By convention, we write $K(\infty) = H\mathbb{F}_p$.

Remark 6.4. Some words can be said of the **obstruction theory** to the commutativity statement above. Let R be a ring spectrum and $x \in \pi_*(R)$

Thus, from the construction of $K(n)$, they in fact satisfy many nice properties:

1. $K(n)_*$ is a graded field.
2. All modules over $K(n)_*$ are free.
3. $K(n) \otimes X$ is a free $K(n)$-module, and hence $K(n) \otimes X \cong \bigoplus_i \Sigma^{a_i} K(n)$ where the a_i's depend on X.
4. (Kunneth Formula): $K(n)(X \otimes Y) \cong K(n)_*(X) \otimes K(n)_*(Y)$.
5. $K(n)_*(X) = 0$ implies $K(n-1)_*(X) = 0$. (This is apparently a consequence of the **Landweber filtration theorem**).

The most relevant property to the previous statement about nilpotence is that - $K(n)$ detects nilpotence!

Theorem 6.5 ($K(n)$ detects nilpotence). There are a few ways for how $K(n)$ detects nilpotence:

1. Let R be a p-local pring spectrum and $\alpha \in \pi_*(R)$, then α is **nilpotent** if and onlt if $K(n)_*(\alpha)$ is nilpotent for all n (here we view α as the self-map that induces multiplication by α interchangeably).
2. Let $f : \Sigma^k F \to F$ be a map where F is finite and pointed spectrum. f is **nilpotent** if and only if $K(n)_*(f)$ is nilpotent for all n.
3. Let F be a finite spectrum and $f : F \to X$, then f is **smash nilpotent** if and only if $K(n)_*(f)$ is nilpotent for all n.

Proof Sketch. The theorem follows from an application of the nilpotence theorem. The idea is to show that MU detects nilpotence if and only if all the $K(n)$'s can, for which one can discern from the explicit construction of the $K(n)$'s. ■

6.3 Periodicity Theorem

The periodicity theorem uses the thick subcategory theorem in its proof. In this lecture, however, we will be doing this backwards.

Definition 6.6. Let X be a p-local finite spectrum, we say X is **type n** if n is the least integer such that $K(n)_*(X) \neq 0$.

Question 6.7. Let X be a p-local finite spectrum of type n, does X have a v_n-self map. By this, we mean a map $f : \Sigma^d X \to X$ such that $K(n)_*(f)$ is an isomorphism and $K(m)_*(f)$ is nilpotent for $m \neq n$.

We remark that a v_n-self map from a p-local finite spectrum that does not have type n is not that meaningful of a question.

- If X has type $m > n$, then the zero map from $X \to X$ is a desired v_n-self map!
- If X has type $m < n$, it is not too terrible to show that there are never any v_n-self maps.

The periodicity theorem tries to answer this question of when there exists a v_n-self map for X of type n.

Definition 6.8. Let V_n denote the subcategory of p-local finite spectrum admitting v_n self maps, and let F_n denote the p-local finite spectrum of type n.

Theorem 6.9 (Periodicity Theorem)**.** $V_n = F_n$.

Proof Sketch. One can see that $F_{n+1} \subset V_n \subset F_n$, and check that V_n is a **thick subcategory** of the p-local finite spectra. By the **thick subcategory theorem** (which we have not stated yet), this shows that either V_n is F_{n+1} or V_n is F_n. Then one just need to construct an explicit v_n-self map of a p-local finite spectra of type n to conclude the proof. The explicit construction is actually quite tricky, but can be done. ■

6.4 Thick Subcategory Theorem

Let us actually look at what the **thick subcategory theorem** at the end. Let $F(x, y)$ be a formal group law over L and let Γ be the group of power series of the form $x + b_1x^2 + b_2x^3 + ...$

There is an action of Γ on formal group laws over L given by

$$F \mapsto g(F(g^{-1}(x), g^{-1}(y))).$$

Generalizing, we can let $C\Gamma$ denote the category of finitely presented $\pi_*(MU) = L$-modules M with an action of Γ such that for $\alpha \in L, x \in M, \gamma \in \Gamma$, we have that

$$\gamma(\alpha \cdot x) = \gamma(\alpha) \cdot \gamma(x).$$

Remark 6.10. Based on the discussion above and the last lecture, MU_* is a functor from the category of finite spectra to $C\Gamma$.

Theorem 6.11 (Landweber Filtration Theorem)**.** Let $M \in C\Gamma$, then M has a finite filtration $F_1M \subset F_2M \subset ... \subset F_kM = M$ such that the successive quotients $F_iM/F_{i-1}M$ is isomorphic to some suspension of MU_*/I_{n_i} for some n_i. Furthermore, I_{n_i} has the structure of the ideal $(\langle p, v_1, ..., v_{n-1})$ for some p, n depending on n_i.

Definition 6.12. Let $\mathcal{A}$ be an abelian category, a sub-category $\mathcal{C}$ of $\mathcal{A}$ is **thick** if for any short exact sequence of the form

$$0 \to A \to B \to C \to 0,$$

$B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$.

Similarly, in our case for ∞-categories, we can define an analogous definition.

Definition 6.13. Let F be the category of p-local finite spectra, a subcategory $\mathcal{C}$ is **thick** if it is closed under fibers and cofibers and $X \oplus Y \in \mathcal{C}$ implies $X, Y \in \mathcal{C}$.

Finally, we will state the thick subcategory theorem, which also follows from an application of the nilpotence theorem.

Theorem 6.14 (Thick Subcategory Theorem)**.** The thick subcategories of p-local finite spectra are exactly:

1. The whole category.
2. The trivial subcategory (consisting only of the suspension spectrum of a point).
3. The categories F_n's.

7 Lecture 6: Examples of ring spectra in chromatic homotopy theory (by Catherine Li)

The goal of this lecture is to introduce some of the main characters for chromatic homotopy theory. Our goal is to introduce:

- The Morava E-theory E_n.
- The Brown-Peterson spectrum BP.

We also want to introduce some compatible multiplicative structures:

- E_n as $\mathbb{E}_\infty$-rings.
- BP as $\mathbb{E}_2$-rings (which is not the highest we could go, more on this in Lecture 10.5)
- $K(n)$ as an $\mathbb{E}_1$-ring (recall the last lecture only constructed this to be homotopy commutative).

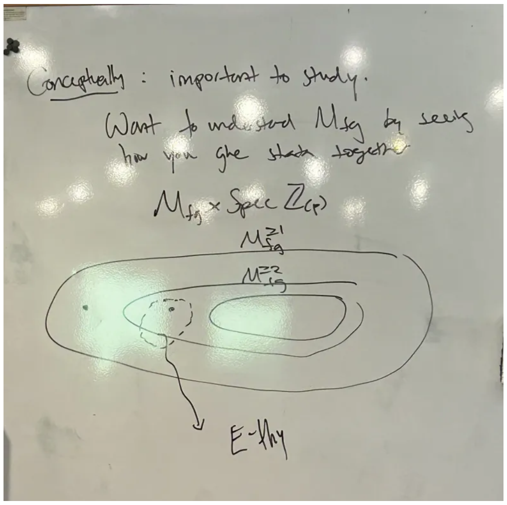
7.1 Morava E-theory

Motivation: Recall from Lecture 4.5 that the moduli stack $M_{fg}^{=n}$ of formal groups of height exactly n has a **unique geometric point**. In more concrete terms, this means that over **an algebraic closed field**, there are exactly **one formal group of height** n **up to isomorphism**. This corresponds to our object $K(n)$ discussed in Lecture 5.

Observation: The inclusion of the geometric point is not flat!!!

In particular, this means that $K(n)$ itself is not Landweber exact. The intuition behind Morava E-theory is as follows - if we can construct a formal neighborhood around this geometric point, maybe the inclusion of the neighborhood will be **flat**? This will be what ends up being what we want to consider as **Morava E-theories**.

Conceptually, we want to understand M_{fg} by seeing how you can glue together the data of $M_{fg} \times \operatorname{Spec} \mathbb{Z}_{(p)}$ based on a stratification given by $M_{fg}^{\geq n}$ for $n \geq 1$. Pictorially, we have a picture:



To achieve this, we need to consider some tools in algebra.

Definition 7.1. An **infinitesimal thickening** of a field k is a local ring $(A, \mathfrak{m})$ with a surjection $\rho : A \to k$ such that $\mathfrak{m} = \ker(p)$, such that

1. $\mathfrak{m}^i = 0$, for $i >> 0$.

2. $\mathfrak{m}^i/\mathfrak{m}^{(i+1)}$ is a finite dimensional k-vector space for all i.

Example 7.2. $A = k[\xi]/(\xi^2)$ is an example of a thickening of k.

Definition 7.3. Let $\mathbb{G}_0 \to \operatorname{Spec} k$ be a formal group of height n over k. A **deformation** of $\mathbb{G}_0$ to A is a formal group $\mathbb{G} \to \operatorname{Spec} A$, such that we have the following pullback:

$$\begin{tikzcd}
\mathbb{G} \times_A \operatorname{Spec} k \simeq \mathbb{G}_0 \arrow[r] \arrow[d] & \mathbb{G} \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

In coordinates, what this means is we have a formal group law $F_0 \in k[[x, y]]$ corresponding to $\mathbb{G}_0$, a formal group law F corresponding to $\mathbb{G}$, such that

$$F \mod \mathfrak{m} \equiv F_0.$$

Example 7.4. Consider a formal group law F over $k[\xi]/(\xi^2)$ such that

$$[p]_F(x) = \xi x^p + x^{p^2} + \ldots$$

In this case, F_0 is the reduction of F mod (ξ), and hence we have that

$$[p]_{F_0} = x^{p^2} + \ldots$$

which has exactly **height 2**! In this case, F is a **deformation of** F_0 **to** A.

Notation: From now on, we will write $\operatorname{Def}_{\mathbb{G}_0}(A)$ denote the **groupoid of defomrations** of $\mathbb{G}_0$ to A where A is some infinitesimal thickening (groupoid here means we include the information of isomorphisms).

Theorem 7.5. Let k be a perfect field of characteristic p and $\mathbb{G}_0 \to \operatorname{Spec} k$ be a formal group of height $n < \infty$. There exists a formal group $\mathbb{G} \to \operatorname{Spf}(E_0)$ where

$$E_0 = W(k)[[u_1, \ldots, u_{n-1}]]$$

such that this formal group is the "universal deformation" of $\mathbb{G}_0$, ie. it induces for any infinitesimal thickening A

$$\operatorname{Spf}(E_0)(A) \cong \operatorname{Def}_{\mathbb{G}}(A).$$

Remark 7.6. **Again, in coordinates**, the universal deformation can be interpreted as: We have a correspondence between $F_0 \sim \mathbb{G}_0$. Here F is a "universal deformation" of F_0 if

$$v_i(F) = u_i, 1 \leq i \leq n-1 \text{ and } F \text{ reduces to } F_0 \text{ under } E_0 \to k.$$

In this case $[p]_F = px + u_1 x^p + \ldots + u_{n-1} x^{p^{n-1}} +$ term, where the additioanl term looks something like $[p]_{F_0}$.

Definition 7.7. Fix $k, \mathbb{G}$, by **Landweber exactness**, there is an even periodic homotopy commutative, homotopy associative spectrum, denoted $_n$ (or $E(k, \mathbb{G})$), called Morava E-theory, whose homology theory is given

$$(E_n)_*(X) = \mathrm{MU}_*(X) \otimes_{MU_*} E_0[[u^{\pm 1}]], |u| = 2.$$

It turns out that phantom maps between Landweber exact spectra are null-homotopic, so there is no need to worry about the uniqueness issue here.

7.2 Separability and $\mathbb{E}_\infty$-Structure

Classically, an $\mathbb{E}_\infty$-structure on E_n was constructed using Goerss-Hopkins obstruction theory. Here we cover a relatively new way to construct this using the method of separability outlined in [Ram23]. To construct the $\mathbb{E}_\infty$-structure on Morava E-theory, we need to introduce the notion of **separability** for $\mathbb{E}_\infty$-structures. Separable algebras are “nice" in the sense that it allows a lift of “homotopy things" to “∞-category properties". Roughly speaking, you might expect for A spearable in $\mathrm{Alg}(\mathcal{C})$, then A being homotopy commutative in $\mathrm{CAlg}(\mathrm{h}(\mathcal{C}))$ extends to A being commutative in $\mathrm{CAlg}(\mathcal{C})$.

Warning: Morava E-theory E_n is **not separable**, it is only what is called “**homotopy ind-separable**". Fortunately, things are okay because of nice properties for $K(n)$-local spectra, which we will see now.

Definition 7.8. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category. An allgebraic $A \in \mathrm{Alg}(\mathcal{C})$ is called **separable** if the multiplication map

$$m : A \otimes A \to A$$

admits a section $s : A \to A \otimes A$ (as a map of $A \otimes A^{op}$-modules).

Definition 7.9. Let $\mathcal{C}$ be a compactly generated, presentably stable, symmetric monoidal ∞-category, then $A \in \mathrm{CAlg}(\mathcal{C})$ is called **ind-separable** if there is a subset

$$S \subseteq \pi_* \operatorname{map}(\mathbb{1}, A \otimes A)$$

such that the multiplication $m : A \otimes A \to A$ realizes $(A \otimes A)[S^{-1}]$:

$$A \otimes A \xrightarrow{m} A \simeq (A \otimes A)[S^{-1}].$$

Question 7.10. Why would we expect this to relate to Morava E-theories?

Here is a motivating example of these objects.

Example 7.11. Let $R \to A$ be a G-Galois extension for G a finite group, then A is **separable** as an R-algebra.

Morava E-theory E_n is the **(pro) Galois extension** of $L_{K(n)}\mathbb{S}$. This example sort of suggests that Morava E-theory is related to something that is ind-separable, rather than separable.

Theorem 7.12. Let $\mathcal{C}$ satisfy the following assumptions:

1. $\mathcal{C}$ is compactly generated, stable symmetric monoidal ∞-category in which $\otimes$ commutes with colimits in each variable. The compact objects $\mathcal{C}^\omega$ are closed under non-empty tensor products.

2. If $X, Y \in \mathcal{C}$ such that there exists $f : \pi_* \operatorname{Map}(-, X) \xrightarrow{\cong} \pi_* \operatorname{Map}(-, Y)$ (ie. an isomorphism on the cohomology theories they represent), then there exists $\tilde{f} : X \to Y$ that is an equivalence.

Let $A \in \operatorname{CAlg}(ho(\mathcal{C}))$ be **homotopy commutative, homotopy associative, homotopy algebra, homotopy ind-separable** in $\mathcal{C}$, which receives **no phantom maps** from any tensor powers of A, then for all $1 \leq d \leq \infty$, then:

$$\text{The moduli space } \operatorname{Alg}_{\mathbb{E}_d}(\mathcal{C})^{\simeq} \times_{\operatorname{Alg}_{ho(\mathcal{C})}} \{A\} \text{ is contractible}$$

In other words, there exists a unique lift of the structure on A to a $\mathbb{E}_d$-algebra.

Proof Theme. The theme of this approach is that:

- Separability is akin to projectivity.
- Separable A implies it is projective as an $A \otimes A^{op}$-module, and any A-module looks like a retract of $A \otimes M$ for some M.

This implies the mapping spaces are discrete (at least simply connected), because of this discreteness it should be believable hopefully that lifting to $\mathbb{E}_d$ is controlled by the homotopy category. The sort of **analogy** to have in mind is the term $\operatorname{Ext}_{A \otimes A^{op}}(A, A)$. ■

Proposition 7.13. $\operatorname{Sp}_{K(n)}$ satisfies the hypothesis of Theorem 7.12.

Proof Sketch. The idea for why $\operatorname{Sp}_{K(n)}$ satisfies this is as follows:

1. They compactly generate $L_{K(n)}$
2. For the second condition it suffices to show that $\operatorname{ho}(\mathcal{C}^{\omega})$ is contractible in this case.

■

Our goal now is to explain why Morava E-theory satisfies the assumption of this theorem.

Proposition 7.14. Morava E-theory is homotopy ind-separable and receives no phantom maps from tensor powers.

Proof Sketch. The lack of phantom maps $(E_n)^{\otimes k} \to E_n$ is similar to the earlier discussion on Landwever exactness. There is some subtlety for doing this in the $K(n)$-local category, but it ends up being okay. ■

Putting this all together, we have shown the following:

Theorem 7.15. E_n has a unique $\mathbb{E}_d$-structure for $1 \leq d \leq \infty$.

7.3 Brown-Peterson Spectrum

Now we will construct an $\mathbb{E}_2$-structure on BP, the approach here is adapted from [CM15]. Now recall that $\pi_*(MU_{(p)}) = \mathbb{Z}_{(p)}[v_1 \mid |v_i| = 2p^i - 2] \otimes \mathbb{Z}_{(p)}[b_m \mid m \neq p^k - 1]$.

Definition 7.16. We define BP as

$$\mathrm{BP} = MU_{(p)}/\langle b_m \mid m \neq p^k - 1 \rangle.$$

Note that $\mathrm{MU}_{(p)} = \bigoplus \Sigma^{2m} BP$.

Remark 7.17. Quillen has an alternative construction of BP. There is a map $\epsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ (called the Quillen idempotent), and we can write

$$\mathrm{BP} = \mathrm{colim}(\mathrm{MU}_{(p)} \xrightarrow{\epsilon} \mathrm{MU}_{(p)} \xrightarrow{\epsilon} ...).$$

BP is landweber exact and hence homotopy commutative and homotopy associativty.

Theorem 7.18. The Quillen idempotent $\epsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ is represented by $\mathbb{E}_2$-maps of ring spectra.

Proof Sketch. Observe that a homotopy ring map $\mathrm{MU} \to \mathrm{MU}$ is equivalent to specifying a complex orientation on MU. Since a complex orientation is alternatively defined by $\mathbb{C}P^\infty$, this is equivalent to maps of pointed spectra from $MU(1)$ (after taking the suspension spectrum) to MU. By Thom isomorphism, one has a sequence of equivalence

$$\mathrm{Map}_{\mathrm{Alg}(ho(Sp))}(MU, MU) \cong \mathrm{Map}_{\mathrm{Alg}(Sp_*)}(MU(1), MU) \cong \mathrm{Map}_{\mathcal{S}_*}(\Sigma^{-2}\mathbb{C}P^\infty, \mathrm{GL}_1(MU)).$$

From here, one can use show that the obstructions to lifting ϵ to an $\mathbb{E}_2$-map vanishes. Here, by $MU(1)$, we mean $\Sigma^{-2}\,\mathrm{Th}(\gamma_1)$, where γ_1 is the universal line bundle on $\mathbb{C}P^\infty$. ■

Corollary 7.19. BP is $\mathbb{E}_2$.

Proof. Since the Quillen idempotent map $\epsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ is $\mathbb{E}_2$, it follows that the colimit

$$\mathrm{BP} = \mathrm{colim}(\mathrm{MU}_{(p)} \xrightarrow{\epsilon} \mathrm{MU}_{(p)} \xrightarrow{\epsilon} ...),$$

is also $\mathbb{E}_2$. ■

Theorem 7.20. The natural map $MU \to BP$ is represented by a map of $\mathbb{E}_2$-ring spectra.

Proof Sketch. Assuming we already have an $\mathbb{E}_2$-map $MU \to R$ where R is even and $\mathbb{E}_2$, it turns out in general that we have an equivalence

$$\mathrm{Map}_{Alg_{\mathbb{E}_2}(Sp)}(MU, R) \cong \mathrm{Map}_{\mathcal{S}}(BSU, B^2\,\mathrm{GL}_1(R)).$$

From which one can check that there is a surjection on π_0 from $\mathrm{Map}_{\mathcal{S}}(BSU, B^2\,\mathrm{GL}_1(R)) \to \mathrm{Map}_{\mathrm{Alg}(ho(Sp))}(MU(1), R)$. Thus, we can lift to a map of $\mathbb{E}_2$-ring spectra. ■

Remark 7.21. There is a simpler explanation of the proof in the case where $R = MU$. Indeed, this is because that we have a sequence of equivalences:

$$\begin{aligned}
\mathrm{Map}_{Alg_{\mathbb{E}_2}(Sp)}(MU, MU) &\cong \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(Sp)}(\Sigma^\infty_+ BU, MU) \\
&\cong \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})}(BU, \Omega^\infty MU) && \text{Suspension-Loop Adjunction} \\
&\cong \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})}(BU, \mathrm{GL}_1(MU)) && \text{Since } BU \text{ lands in one component} \\
&\cong \mathrm{Map}_{\mathcal{S}}(B^2 BU, B^2\,\mathrm{GL}_1(MU)) \\
&\cong \mathrm{Map}_{\mathcal{S}}(BSU, B^2\,\mathrm{GL}_1(MU)).
\end{aligned}$$

Here the last isomorphism following from an adjunction between the group of units functor (right) and inclusion of group-like $\mathbb{E}_n$-algebras to $\mathbb{E}_n$-spaces (left). The isomorphism here is in fact canonical because we have an identity map $id_{MU} : MU \to MU$.

7.4 $\mathbb{E}_1$-Structure on $K(n)$

By **Angeltieit** [Ang08], there is an obstruction theory as follows.

Theorem 7.22. Let R be an even $\mathbb{E}_\infty$-ring and I the ideal generated by some regular sequence in $\pi_*(R)$. If $A = R/I$ is homotopy associative, then this extends to an $\mathbb{E}_1$-structure on A.

In the case of Morava K-theory, we apply to MU with $I = (p, v_1, ..., v_{n-1}, v_{n+1}, b_m \mid m \neq p^k - 1)$. This gives an $\mathbb{E}_1$-structure on $k(n)$ (here $k(n)$ is not capitalized intentionally). From here, we get $K(n) = v_n^{-1}k(n)$. It turns out that inverting by v_n preserves the $\mathbb{E}_1$-structure, which is quite non-trivial of a fact (and has something to do with Bousfield localization accordingly).

Remark 7.23. The $\mathbb{E}_\infty$-structure on E_n is very canonical, but the structures for $K(n)$ is somewhat ad-hoc and may not have good universal properties.

Remark 7.24. Morava K-theory is not $\mathbb{E}_2$.

8 Lecture 7: Descendability, the smash product theorem, and chromatic convergence (by Maite Carli)

The topic of today's talk is the **smash product theorem**.

Theorem 8.1 (Smash Product Theorem)**.** L_{E_n} (notationally we write this as L_n) is a **smashing localization** for all $X \in \mathrm{Sp}$. In other words,

$$L_n X \simeq X \otimes L_n \mathbb{S}.$$

Note that if we have a map $X \otimes L_n\mathbb{S} \to L_n X$ that is an E_n-equivalence, then we have that $E_n \otimes (X \otimes L_n\mathbb{S}) \to E_n \otimes L_n\mathbb{S}$ is an equivalence.

If $X \otimes L_n\mathbb{S}$ is already E_n-local, then the E_n-equivalence follows.
The outline of this talk is as follows:

1. We will first introduce “descent and nilpotence".
2. Explain the proof of the smash product theorem from this.
3. Give a proof of the chromatic convergence theorem.

For the first two parts, we mainly follow a paper of Akhil Mathew.

8.1 Descent and Nilpotence

Let us recall some definitions from this morning.

Definition 8.2. A subcategory is called **thick** if it is closed under fibers and cofibers, contains 0, and is idempotent complete.

Let $\mathcal{C}$ be some symmetric monoidal stable idempotent complete ∞-category, and $A \in \mathrm{Alg}(\mathcal{C})$.

Definition 8.3. An object $X \in \mathcal{C}$ is **A-nilpotent** if it belongs to $\mathrm{Thick}^{\otimes}(A) = \mathrm{Thick}(\{A \otimes Y \mid Y \in \mathcal{C}\})$. We also refer to such collection to be Nil_A.

Example 8.4. If $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ is the derived ∞-category for $\mathbb{Z}$ and $A = \mathbb{Z}/p\mathbb{Z}$, then $X \in \mathrm{Nil}_A$ if and only if there exists $n \geq 0$ such that $p^n : X \to X \simeq 0$.

Question 8.5. How can we approximate $X \in \mathcal{C}$ by elements in Nil_A?

The idea is to introduce the **cobar construction**.

Definition 8.6. The **augmented cobar construction** $\mathrm{CB}^{aug}(A)$ is the augmented cosimplicial diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, shift left=2] \arrow[r, shift right=2] & A \otimes A \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & A \otimes A \otimes A \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots
\end{tikzcd}$$

The cobar construction is $CB^{\bullet}(A) := \mathrm{CB}^{aug}(A)|_{\Delta}$.

Remark 8.7. $\mathrm{CB}^{aug}(A)$ admits an extra degeneracy. After tensoring with A, for any functor $F : \mathcal{C} \to \mathcal{D}$, we have that

$$F(A) \xrightarrow{\cong} \mathrm{Tor}(F(CB^{aug}(A) \otimes A)).$$

Proposition 8.8. Let $X \in \mathrm{Nil}_A$. For all stable ∞-categor $\mathcal{D}$ and exact functor $F : \mathcal{C} \to \mathcal{D}$, we have an equivalence

$$F(X) \xrightarrow{\cong} \mathrm{Tot}(F(X \otimes A)) \rightrightarrows F(X \otimes A \otimes A)...$$

Proof. We can check that the class for which equivalence holds is thick. If $X = A \otimes Y$, then we are done by the previous remark. ■

Definition 8.9. A tower in $\mathcal{C}$ is a functor $\mathbb{Z}_{\geq 0}^{op} \to \mathcal{C}$. Towers form a stable ∞-category called $\mathrm{Tow}(\mathcal{C})$.

To any cosimplical object $X^\bullet : \Delta \to \mathcal{C}$, we can associated a tower

$$... \to \mathrm{Tot}_{\leq n}(X^\bullet) \to \mathrm{Tot}_{\leq n-1}(X^\bullet) \to ...$$

Here, $\mathrm{Tot}_{\leq n}(X^\bullet)$ refers to $\lim_{[i] \in \Delta, i \leq n} X^i$.
Fact: $\mathrm{Fun}(\Delta, \mathcal{C})$ is equivalent to $\mathrm{Tow}(\mathcal{C})$.

Definition 8.10. A tower $\{X_i\}_{i \geq 0}$ is **nilpotent** if there exists $N > 0$ such that for all $i \geq 0$, the map $X_{i+N} \to X_i$ is equivalent to zero. A tower $\{X_i\}_{i \geq 0}$ is **quickly converging** if it is in the thick subcategory generated by the constant and nilpotent towers. A cosimplicial object $X^\bullet$ is **quickly converging** if the associated tower is.

Let us examine some properties of co-simplicial towers.

Proposition 8.11. Suppose $X^\bullet \in \mathrm{Fun}(\Delta, \mathcal{C})$ is quickly converging, then we have:

1. For all $F : \mathcal{C} \to \mathcal{D}$ exact, $F(X^\bullet)$ is quickly converging.
2. Let $\mathcal{D}$ be stable, idempotent complete, and suppose F is exact, then we have that

$$F(\mathrm{Tot}(X^\bullet)) \xrightarrow{\sim} \mathrm{Tot}(F(X^\bullet)).$$

3. Suppose $\mathrm{Tot}(X^\bullet) \in \mathrm{Thick}(X^i \mid i \in \mathbb{N})$, $\lim(\mathrm{Tot}_{\leq i} X^\bullet)$ is a retract of some $\mathrm{Tot}_{\leq i} X^\bullet$.

Proposition 8.12. $X \in \mathrm{Nil}_A$ if and only if $\mathrm{CB}^\bullet(A) \otimes X$ is quickly convergent with limit X.

Proof. For the proof, we need the following two facts.

1. If $X \in \mathrm{Nil}_A$, then there exists some $n > 0$ such that $I^{\otimes n} \to \mathbb{1}$ is null-homotopic after tensoring with X, (where $I \to \mathbb{1} \to A$).
2. We have a cofiber sequence

$$(X \otimes I^{\otimes \bullet})_n \to \mathrm{Const}(X) \to \mathrm{Tot}_{\leq n}(\mathrm{CB}^\bullet(A) \otimes X).$$

If X is nilpotent, then by ■

Definition 8.13. If $\mathbb{1} \in \mathrm{Nil}_A$, ie. $\mathcal{C} = \mathrm{Thick}^{\otimes}(A)$, then A is called **descendable**.

8.2 Smash Product Theorem

We wish to show that “E_n is descendable in L_n-local category".

Proposition 8.14. Let E be an $\mathbb{E}_1$-ring spectrum. If $CB^\bullet(E)$ is quickly converging, then

1. $\mathrm{Tot}(CB^\bullet(E)) = L_E\mathbb{S}$.
2. $L_E\mathbb{S}$ is E-nilpotent.
3. $L_E : \mathrm{Sp} \to \mathrm{Sp}$ is a smashing localization.

Proof. For (1), by quick convergence, we get that $E \otimes \mathrm{Tot}(CB^\bullet(E)) \cong \mathrm{Tot}(CB^\bullet(E) \otimes E) \cong E$. This shows that $\mathrm{Tot}(CB^\bullet(E)) \cong L_E\mathbb{S}$ with an E-equivalence and both sides are E-local.

For (2), this goes to the previous proposition.

For (3), it suffices to show that for all $X \in \mathrm{Sp}$, $L_E\mathbb{S} \otimes X$ is E-local. We know that $L_E\mathbb{S} \in \mathrm{Thick}^{\otimes}(E)$, so $X \otimes L_E\mathbb{S} \in \mathrm{Thick}^{\otimes}(E)$, which is E-local. ■

Theorem 8.15. $\mathrm{CB}^\bullet(E_n)$ is quickly converging.

Proof. In general, we can associate a spectral sequence called the **Bousfield-Kan spectral sequence**, to the tower associated to $\mathrm{CB}^\bullet(A) \otimes A$, with signature given by

$$E_2^{s,t} := H^s(\pi_t(\mathrm{CB}^\bullet(A) \otimes X)) \implies \pi_{t-s}(\mathrm{CB}^\bullet(A) \otimes X).$$

Proposition 8.16. If $\mathrm{CB}^\bullet(A)$ is quickly convergent, then the associated BK spectral sequence admits a horizontal vanishing line, ie. there exists $N \geq 2$ and $h \geq 0$ such that $E_N^{s,t} = 0$ for all $s \geq h$.

Furthermore, we have the following.

Theorem 8.17. Let $X^\bullet$ be a co-simplicial spectrum. If there exists $s \geq 1$ such that for any finite spectrum F, the BKSS associated to $X^\bullet \otimes F$ vanishes at $E_s^{p,q}$ for all $p \geq s$, then $X^\bullet$ is **quickly convergent**.

Using these results, we can show that $\mathrm{CB}^\bullet(E_n)$ is quickly converging as follows. Observe that it is quickly converging if and only if $\mathrm{CB}^\bullet(E_n) \otimes L_n\mathbb{S}$ is quickly converging, which holds if and only if there exists a finite type 0 spectrum X such that $\mathrm{CB}^\bullet(E_n) \otimes X$ is (here we implicitly use the thick subcategory theory is).

The hope is to find an appropriate X such that the spectral sequence associated to $\mathrm{CB}^\bullet(E_n) \otimes X$ on the E_2-page already has the horizontal vanishing line. In this case, we consider the map

$$(E_n)_*(X) \to (E_n \otimes E_n)_*(X) \to (E_n \otimes E_n \otimes E_n)_*(X) \to \ldots \quad (\dagger)$$

It turns out that $\underline{\text{there is an equivalence between } ((E_n)_*, (E_n)_*(E_n))\text{-comodules and quasicoherent sheaves on } \mathrm{M}_{fg}^{\leq n}}$. Therefore, we can interpret (†) above as a free-resolution of certain quasicoherent sheaf. In other words, the E_2-page of the spectral sequence is equal to the cohomology $H^s(M_{fg}^{\leq n}, F_{\Sigma^k X})$.

Thus the vanishing line result inductively reduces to showing that $H^0(M_{fg}^{\leq n}, \mathcal{F}) = 0$ for $s >> 0$ and $\mathcal{G} \in \mathrm{QCoh}(M_{fg}^{\leq n})$. ∎

8.3 Chromatic Convergence Theorem

Theorem 8.18 (Chromatic Convergence Theorem)**.** If X is a p-local finite spectrum, then

$$X \simeq \lim(... \to L_2X \to L_1X \to L_0X).$$

Proof with Ishan Levy's Trick. All spectrum satisfying the conclusion of this theorem is a thick subcategory, so it suffices to show this for $\mathbb{S}_{(p)}$ by the **thick subcategory theorem**. Now for $\mathbb{S}_{(p)}$ itself, the **Adams Novikov filtration** tells us that

$$\mathbb{S}_{(p)} = \mathrm{Tot}(\mathrm{MU}_{(p)} \rightrightarrows \mathrm{MU}_{(p)} \otimes \mathrm{MU}_{(p)} ...)$$

Claim 1: Chromatic convergence holds for free $MU_{(p)}$-modules.
Claim 2: $\mathrm{CB}^\bullet(L_n\, \mathrm{MU}_{(p)})$ is quickly convergent.

Proof of Claim 2: Since E_n is complex oriented and is E_n-local, we have a ring map $L_n\,\mathrm{MU} \to E_n$, so E_n is an L_nMU-module. This implies that $E_n \in \mathrm{Thick}^{\otimes}(X^{\otimes}L_nMU)$, which implies that $L_n\mathbb{S} \in \mathrm{Thick}^{\otimes}(L_nMU)$.

By Claim 2 and the Adams-Novikov filtration on $\mathbb{S}_{(p)}$, we have that

$$L_n\mathbb{S} \otimes \mathrm{Tot}(CB^\bullet(\mathrm{MU}_{(p)})) \cong L_n\mathbb{S} \otimes \mathbb{S}_{(p)} \cong L_n\mathbb{S} \cong \mathrm{Tot}(CB^\bullet(L_nMU)) \cong \mathrm{Tot}(L_n(B^\bullet(MU))).$$

Thus, we have that

$$\begin{aligned} \mathbb{S}_{(p)} &= \mathrm{Tot}(\lim_n L_nCB^\bullet(MU)) \\ &= \lim_n \mathrm{Tot}(L_nCB^\bullet(MU)) \\ &= \lim_n L_n\mathbb{S}. \end{aligned}$$

∎

9 Lecture 8: Special features of monochromatic homotopy theory (by Florian Riedel)

9.1 $T(n)$ and Bousfield-Kuhn Functors

Really, our goal in life is to understand $\mathbb{S}$. In chromatic homotopy theory, we want to understand the unit map

$$\mathbb{S}_{(p)} \to \mathrm{MU}_{(p)}$$

which **"detects nilpotence"** (this is the so-called nilpotence theorem). A consequence of the nilpotence theorem is the Nishida nilpotence theorem. Consider the map

$$\varphi : \pi_*\mathbb{S}_{(p)} \to \pi_* MU_{(p)} \cong \mathbb{Z}_{(p)}[t_0, t_1, ...]$$

Theorem 9.1 (Nishida Nilpotence Theorem)**.** We learn from this that $\pi_*\mathbb{S}_{(p)}$ is **all nilpotent** for $* > 0$.

Another goal of chromatic homotopy theory is to try to **interpolate** between

$$\mathrm{Sp} \text{ and } (MU_* \to MU_*(MU) \rightrightarrows ...) \approx M_{fg}.$$

M_{fg} itself comes with a **stratification by height**, so we in particular have a decomposition

$$M_{fg}^{=n} \subseteq M_{fg}^{\leq n} \subseteq M_{fg}.$$

Recall: We have the Morava K-theories given by $MU_{(p)}/(t_0, t_1, ...)[v_n^{-1}]$ for which we have

$$\pi_*(K(n)) = \mathbb{F}_p[v_n^{\pm 1}] \text{ and } K(0) = H\mathbb{Q}.$$

We also have **Bousfield localizations** $L_{K(n)} : \mathrm{Sp} \to \mathrm{Sp}_{K(n)}$, and $L_n = L_{E_n} = L_{K(0)\oplus ... \oplus K(n)}$.

Here is a picture of what is going on:

$$\begin{array}{ccccc} \mathrm{Sp}_{\mathbb{Q}} & \mathrm{Sp}_{K(1)} & ... & \mathrm{Sp}_{K(n)} & \\ & & & & \\ 0 & 1 & ... & n & \text{height} \end{array}$$

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

It turns out that these $K(n)$-local spectra sits inside a category called $\mathrm{Sp}_{T(n)}$, which is what we mean by **monochromatic categories** here.

Definition 9.2. Let V be a **type** n **complex**, ie. V is a compact object and $K(m)_*(V) = 0$ for $m < n$ and $K(n)_*(V) \neq 0$. Recall by the **periodicity theorem**, we have a v_n-self-map $v : \Sigma^t V \to V$ that is an isomorphism on $K(n)_*$.

From here $T(n) = \Sigma^\infty V[v^{-1}]$, where we inverted the self map v. Note that this required a choice of a v_n-self map, and one can show they are all Bousfield equivalent. Alternatively, we can take $T(n)$ to be $\mathbb{S}/(p^{m_0}, v_1^{m_1}, ..., v_{n-1}^{m_{n-1}})[v_n^{-1}]$.

So now we have a commtutative diagram of this form:

$$\begin{tikzcd}
\mathrm{Sp} \arrow[r, "L_{T(n)}"] \arrow[dr, "L_{K(n)}"'] & \mathrm{Sp}_{T(n)} \\
& \mathrm{Sp}_{K(n)} \arrow[u]
\end{tikzcd}$$

Our goal of this talk is to understand $\mathrm{Sp}_{T(n)}$.

Remark 9.3. When $n = 0, 1$, $Sp_{K(n)}$ and $Sp_{T(n)}$ are the same. When $n > 1$, their equality is called the **telescope conjecture**, which is very recently proven to be false!

Theorem 9.4 (Bousfield-Kuhn)**.** There exists a factorization for $L_{T(n)} : \mathrm{Sp} \to \mathrm{Sp}_{T(n)}$ of the form:

$$\begin{tikzcd}
\mathrm{Sp} \arrow[r, "L_{T(n)}"] \arrow[d, "\Omega^\infty"'] & \mathrm{Sp}_{T(n)} \\
S_* \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

where Φ is called the Bousfield-Kuhn functor.

Remark 9.5. The surprising part is not necessarily about the connectivity of spectrum in this theorem because $T(n)$-localization introduces periodicity. The surprising part is that it forgets about the $\mathbb{E}_\infty$-structure entirely.

Let us look at how this might work. Let (V, v) be a type n space and $v : \Sigma^t V \to V$ be a v_n-self map. Let $A \in S_*$ and suppose $\Omega^t A \simeq A$, then A deloops to a **periodic spectrum** $\{A_i\}_{i\in\mathbb{Z}}$ where:

$$A_0 = A, A_1 = \Omega^{t-1}A, ..., A_{t-1} = \Omega A, A_t = A, ...$$

Now we will use A interchangeably as the space and its periodic spectrum.

Definition 9.6. The Bousfield-Kuhn functor associated to (V, v) is

$$\Phi_V(A) := \mathrm{colim}_{\to}(\mathrm{Map}_*(V, A) \xrightarrow{v} \mathrm{Map}_*(\Sigma^t V, A) \xrightarrow{v} ...)$$

Exercise 9.7. Check that $\Omega^t \Phi_v(A) \simeq \Phi(A)$ implies $\Phi_v(A) \in \mathrm{Sp}$.

Lemma 9.8. Let $X \in \mathrm{Sp}$, $\Phi_v(\Omega^\infty X) \simeq L_{T(n)} X^V \simeq L_{T(n)} X \otimes D(\Sigma^\infty V)$.

Proof. Let $A \in S_*$, then we **claim** that $\Phi_v(A)$ is $T(n)$-local, which follows from the following:

Let $W \in S_*^\omega$ (ie. a finite space). Then there is a sequence of equivalences

$$\begin{aligned}
\Phi_V(A)^W &\simeq \mathrm{colim}_{\to}(\mathrm{Map}_*(V, A)^W \xrightarrow{v} \mathrm{Map}_*(\Sigma^t V, A)^W \to ...) \\
&\simeq \mathrm{colim}_{\to}(\mathrm{Map}_*(V \wedge W, A) \xrightarrow{W\wedge v} \mathrm{Map}_*(\Sigma^t V \wedge W, A) \to ...) \\
&\simeq \Phi_{V\wedge W}(A).
\end{aligned}$$

Moreover we consider the map $\Sigma^s W \xrightarrow{w} W$ and have that $\Phi_V(A)^{\Sigma^\infty W[w^{-1}]} \simeq \lim(\Phi_{V\wedge W}(A) \xleftarrow{V\wedge w} \Sigma^s \Phi_{V\wedge W}(A) \leftarrow ...)$. Then use the fact that self maps are nilpotent on complexes of the wrong type to conclude that

$$\Phi_V(\Omega^\infty X) \simeq \Phi_V(\Omega^\infty L_{T(n)} X).$$

Claim implies that $\Phi_v(\Omega^\infty X) \approx \Phi_v(\Omega^\infty L_{T(n)} X)$, then we have that

$$\Phi_v(\Omega^\infty X) = \mathrm{colim}_{\to}(\mathrm{Map}_*(V, \Omega^\infty X) \xrightarrow{v} ...)$$

$$= \mathrm{colim}_{\to}(\mathrm{Map}_*(\Sigma^\infty V, X) \xrightarrow{\sim} \mathrm{Map}_*(\Sigma^\infty V, X) \xrightarrow{cofib} \mathrm{Map}_*(\Sigma^\infty V/\sim, X)...)$$

Eventually, the cofiber becomes 0, and since X is T(n)-local, $\Sigma^\infty/\sim$ is type $n+1$. ■

Now **contemplate** the functoriality of $(V, v) \xrightarrow{\Phi} \Phi_V$, which gives us:

$$\begin{tikzcd}
\Phi_{(-)}: & \mathrm{Sp}^{fin}_{\geq n} \arrow[r] \arrow[d, hook] & \mathrm{Fun}(S_*, Sp) \\
& \mathrm{Sp}^{fin} \arrow[ur] &
\end{tikzcd}$$

Definition 9.9. We construct the Bousfield-Kun functor Φ as the right Kan extension of $\Phi_{(-)}(\mathbb{S})$ ...

9.2 Higher Semi-Additivity

Now, **consequences** - **higher semi-additivity**.

Definition 9.10. Call a space $A \in \mathrm{Spc}$ to be m-finite for $m \geq -2$ if

1. When $m = -2$, A is a point.
2. When $m = -1$, $A \in \{*, \emptyset\}$.
3. When $m \geq 0$, $\pi_*(A)$ is finite for all $*$ and 0 for $* > n$.

Example 9.11. Let G be a finite group, then BG is 1-finite.

Definition 9.12. Let $\mathcal{C}$ be a presentable category, we call $\mathcal{C}$ m-**semi-additive** if it is $(m-1)$-semi-additive and we have a natural equivalence between:

For any $X \in \mathcal{C}^A := \mathrm{Fun}(A, \mathcal{C})$, $\mathrm{Num}_A : \mathrm{colim}_A X \xrightarrow{\sim} \lim_A X$.

Remark 9.13.

1. Any $\mathcal{C}$ is -2-semi-additive.
2. If $\mathcal{C}$ is $(m-1)$-semi-additive, then there is a canonical map $\mathrm{Nm}_A : \mathrm{colim}_A X \to \lim_A X$ for all $A \in S^{m-finite}$. This means that m-semi-additivty is a property of $\mathcal{C}$, rather than an extra structure.
3. If $m = -1$, then $\mathrm{colim}_\emptyset \simeq \emptyset_{\mathcal{C}} \simeq *_{\mathcal{C}} = \lim_\emptyset$. This is the same as requiring $\mathcal{C}$ to be pointed!
4. If $m = 0$, then consider $A \in S^{m-finite}$. In this case, for a finite set and $X \in \mathcal{C}^A$ if and only if $\{X_a\}_{a \in A}$.

We have an equivalence in this case

$$\mathrm{Nm}_A : \mathrm{colim}_A X = \coprod_{a \in A} X_a \xrightarrow{\sim} \prod_{a \in A} X_a = \mathrm{colim}_{a \in A} X_a.$$

This map is really the identity matrix, ... classically this just means $\mathcal{C}$ is **semi-additive** in the classical sense.

5. If $m = 1$, and take $A = BG$ and let $X \in \mathcal{C}^{BG}$ and G acting on X, the map $\mathrm{Nm}_{BG} : X_{hG} \to X^{hG}$ is literally the norm map!
6. Check $\mathcal{C} = \mathrm{Sp}_{\mathbb{Q}}$ is 1-semiadditive. $\mathcal{C} = \mathrm{Mod}_{\mathbb{F}_2}\,\mathrm{Sp}$ is NOT 1-semi-addittive. Pr^L is ∞-semi-additive.

Theorem 9.14 (Hopkins-Lurie, Carmeli-Schlank-Yanosk)**.** $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ are ∞-semi-additive.

The 1-semi-additivty case has a “fun proof" using the Bousfield-Kuhn functor.

10 The First Q and A Session (by Gijs Heuts and Ishan Levy)

For Fun: Every time the mentor says the word “local", you are supposed to take a sip from alcohol.

Question 10.1. What is localization?

The idea is given $E \in \mathrm{Sp}$ a spectrum, there is a category $L_E\,\mathrm{Sp}$ (or Sp_E) that “formally inverts E-equivalences". To be more precise:

Definition 10.2. We say $X \in \mathrm{Sp}$ is E-acyclic if $E \otimes X \simeq 0$. We say that $Y \in \mathrm{Sp}$ is E-local if $\mathrm{Map}(X, Y) \cong *$ whenever X is E-acyclic.

What Bousfield proved is the following:

Theorem 10.3. Any spectrum $Z \in \mathrm{Sp}$ admits a E-localization $Z \mapsto L_E Z$ satisfying the relevant universal properties. If Z is already E-local, then $L_E Z = Z$.

Definition 10.4. The full subcategory of E-local spectrum is $L_E\,\mathrm{Sp}$.

Exercise 10.5. $L_E : \mathrm{Sp} \to L_E\,\mathrm{Sp}$ is what is called a **reflective localization**, meaning L_E admits a left adjoint.

Remark 10.6. Not every localization in life is an example of a reflective localization. In general for a category $\mathcal{C}$ with weak equivalences W, it is possible to produce a category that inverts the weak equivalences, ie. a map $\mathcal{C} \to \mathcal{C}[W^{-1}]$. This localization map may not be reflective in general.

Question 10.7. Is there a good formula to compute $L_{K(n)}X$ in practice?

Let X be a finite spectrum, it turns out we have that [DH04]

$$L_{K(n)}X = (X \otimes E_n)^{h\mathbb{G}_n}$$

where $\mathbb{G}_n$ is called the **generalized Morava stabilizer group**, which is the automorphism group on the formal group associated to E_n. This is accompanied by a spectral sequence called the **homotopy fixed points spectral sequence**.

Question 10.8. How do you prove something is a smashing localization in practice?

There are only really two ways. The first is how we did it in the lecture with L_n and the smash product theorem. The second is how we do with $L_n^f := L_{T(n)\oplus \ldots \oplus T(0)}$, which is not done in the lecture.

Question 10.9. Can you give us an example of how to perform smashing localization for MU modules?

We give an example for the, L_1 localiztion of X, an MU-module. This is the pullback:

$$\begin{tikzcd}
L_1 X \arrow[r] \arrow[d] & X[v_1^{-1}] \arrow[d] \\
X[p^{-1}] \arrow[r] & X[p^{-1}, v_1^{-1}]
\end{tikzcd}$$

Question 10.10 (Asked Jokingly)**.** What is the easiest example you can use to tell apart $K(n)$ and $T(n)$ in terms of localizations?

This is the **telescope conjecture**. This joke was very funny when asked in the Q and A session, it made the crowd laugh for about a good minute or so.

Question 10.11. What is the easiest way to see $K(n)$ is not smashing for $n > 0$?

$K(n)$ involves p-localization. Consider the sequence

$$\mathbb{S}/p \to \mathbb{S}/p^2 ... \to \mathbb{Q}/\mathbb{S}_p$$

Let us $K(1)$-localizae $\mathbb{Q}/\mathbb{S}_p$, we will get $\Sigma L_{K(1)}\mathbb{S}$. On the other hand the $K(1)$-localization of the intermediate terms are 0, but the limit at the end is evidently not.

For $p > 2$, what are the homotopy groups of $L_{K(1)}\mathbb{S}$? Well, it turns out by Adams operations, we have

$$L_{K(1)}\mathbb{S} = (KU_p^\wedge)^{h\mathbb{Z}_p^\times}.$$

Pick $\ell \in \mathbb{Z}_p^\times$ a generator, we have now a sequence

$$L_{K(1)}\mathbb{S} \to KU_p^\wedge \xrightarrow{1-\psi^\ell} KU_p^\wedge$$

In terms of homotopy groups, $1 - \psi$ maps

$$\mathbb{Z}_p[\beta^{\pm 1}] \to \mathbb{Z}_p[\beta^{\pm 1}], \beta^n \mapsto \beta^n(1 - \ell^n)$$

At $n = 0$, from here we learn that π_0 is the kernel of this map and π_1 is the cokernel of this map. In other words, we have that

$$\pi_0(L_{K(1)}\mathbb{S}) = \mathbb{Z}_p, \pi_{-1}(L_{K(1)}\mathbb{S}) = \mathbb{Z}_p.$$

In general, we can figure out that

$$\pi_n(L_{K(1)}\mathbb{S}) = \begin{cases} \mathbb{Z}_p, n = 0, -1 \\ \mathbb{Z}/p^{k+1}\mathbb{Z}, n+1 = (p-1)p^k m, m \not\equiv 0 \mod p, \\ 0, \text{ otherwise} \end{cases}.$$

Question 10.12. Are the choices of v_n canonical?

The thing that is canonical is the ideal generated by $(p, v_1, ..., v_n)$, but not necessarily the elements themselves. The v_n self-maps themselves are "**asymptotically unique**", in the sense that they are homotopy equivalent after taking a large amount of repeated iterates. Indeed, suppose we have $f : X \to Y$, then the v_n self maps are unique in the sense:

- Let v act on X and v' act on Y, there exists k, ℓ such that the following is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "v^k"'] & Y \arrow[d, "(v')^\ell"] \\ \Sigma^{-?}X \arrow[r] & \Sigma^{-?}Y \end{tikzcd}$$

- If we take f to the identity map on X, this recovers the "asymptotically unique" statement.

Question 10.13. Is it ever important to know how the v_n-self-maps are constructed?

Yes, the existence of such self-maps are quite difficult. To name some explicit examples,

- When $p > 2$ (odd prime), there is a v_1 self map on $\mathbb{S}/p$.
- For $p = 2$, there is a minimal self map v_1^4 self map on $\mathbb{S}/2$.
- When $p > 3$ (a prime at least 5), there is a v_2 self map on $\mathbb{S}/(p, v_1)$.
- Consider $\mathbb{S}/(2, \eta)$, where η generates $\pi_1(\mathbb{S})$ is the Hopf map, then there is a v_1-self map in this case.

These small v_n-self-maps were important in proving the height 1 telescope conjecture.

Question 10.14. Is there an unstable thick subcategory theorem?

Sort of yes. There is the following theorem that says you can oragnize the Bousfield class if you know the suspensions.

Theorem 10.15 (Bousfield). Let X, Y be pointed spaces in S_*^{fin}. Consider the Bousfield class $\langle \bullet \rangle$, then

$$\langle \Sigma X \rangle \leq \langle \Sigma Y \rangle \iff (1) \text{ and } (2).$$

Here the two conditions are:

1. type of Y is $\leq$ the type of X.
2. The connectivity of Y is $\leq$ the connectivity of X.

An interesting conjecture is as follows: what is the lowest connectivity of a finite space of type n? One can check this is at least n, but not quite sharp.

Question 10.16. Would it be computationally useful to have type n spaces with low connectivity?

Not sure.

Question 10.17. What is the role of BP?

BP plays more of a technical role in the sense is that it is a “smaller version" of MU without some of the possible distractions from polynomial generators. Now recall that

$$\pi_* \mathrm{BP} = \mathbb{Z}_{(p)}[v_1, ...]$$

For the Adams-Novikov-Spectral-Sequence, for example, we did this with MU, but we could have also done it with BP instead (which is much smaller) as:

$$\begin{tikzcd} MU_* \arrow[r] & MU_*(MU) \arrow[r] & MU_* \otimes MU_* \otimes MU_* \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{BP}_* \arrow[r] & \mathrm{BP}_*(\mathrm{BP}) \arrow[r] & \mathrm{BP}_* \otimes \mathrm{BP}_* \otimes \mathrm{BP} \end{tikzcd}$$

There is a formal group on BP that comes from the Quillen idempotent, its p-series is given by

$$[p](t) = pt +_F vt^p +_F v_2 t^{p^2} +_F +...$$

Question 10.18. What are the truncated BPs?

The $\mathrm{BP}\langle n \rangle$ comes from cutting the v_i's at the n-th one.

Question 10.19. Can you say some words about Koszul duality?

Let us do a simple case of Kozsul duality. It is very generally true that for a monoidal category $(\mathcal{C}, \otimes)$ with sufficient limits and colimits, there is an adjunction given by:

$$\mathrm{Bar}, \mathrm{CoBar} : \mathrm{Alg}^{g}_{\mathbb{E}_1}(\mathcal{C}) \leftrightarrow \mathrm{coAlg}^{1}_{|Ebb_1}(\mathcal{C}).$$

This adjunction is typically not an equivalence, the (co)unit is typically a (co)completion.

Example 10.20. If $\mathcal{C} = \mathrm{Mod}_k$. Take $R = k[x_1, ..., x_n]$, then $\pi_*(k \otimes_R k) = \mathrm{Tor}^R_*(k, k)$. This Tor group can be classically computed as the exterior powers.

Question 10.21. How does localization of ring spectra work?

Let R be an $\mathbb{E}_\infty$-ring. In higher algebra, it is often difficult to construct various kinds of algebraic structures, but fortunately localization is still okay.

Take $r \in \pi_*(R)$, we can form a localization $R \to R[r^{-1}]$, and the claim is that it is an $\mathbb{E}_\infty$-ring such that $\pi_*(R) \to \pi_*(R[r^{-1}])$ is preclisely localization, with an appropriate universal property.

Here we can define $R[r^{-1}]$ as $\mathrm{colim}(R \xrightarrow{r} \Sigma R \to \Sigma R \to ...)$. The $\mathbb{E}_\infty$-structure can be constructed using a localization from $\mathrm{Mod}(R)$.

11 Lecture 9: Definition and examples of power operations (by Azélie Picot)

Let us start with an introduction on what power operations are and why we care about that? Indeed, gien a spectrum $E \in \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp})$, we can produce some algebraic structures $\pi_*(E)$ out of this. The usage of **power operations** provides **obstructions** to desired properties.

Example 11.1. $K(n)$ can be shown to not be an $\mathbb{E}_\infty$-ring spectrum using these lines of methods.

11.1 Multiplicative Operations

For the set-up today, we work with an operad $\mathcal{O}$ (ex. $\mathcal{O} = \mathbb{E}_n, \mathbb{E}_\infty$) and $E \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ (coefficients, ex. $E = \mathbb{S}, H\mathbb{F}_p$), and we will be working in $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$.

Proposition 11.2. There is a sequence of adjunctions given by

$$\begin{tikzcd}
\mathrm{Sp} \arrow[r, shift right, "\mathrm{Sym}_{\mathcal{O}}(-)"'] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \arrow[l, shift right] \arrow[r, shift right, "E\otimes\mathrm{Sym}_{\mathcal{O}}(-)"'] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E) \arrow[l, shift right]
\end{tikzcd}$$

where the functors going left are all forgetful functors. This is called a **free-forgetful adjunction**.

Here, to be clear,

$$\mathrm{Sym}_{\mathcal{O}}(X) = \bigoplus_{h\geq 0} \mathrm{Sym}^{(h)}_{\mathcal{O}}(X) := \bigoplus_{h\geq 0} (\Sigma^\infty_+ \mathcal{O}(h) \otimes X^{\otimes h})_{h\Sigma_k}.$$

$\mathcal{O}$ acts on homology in a way we want to outline for this lecture.

Proposition 11.3. Let E be an $\mathbb{E}_\infty$-ring spectrum and $\mathcal{O}$ an operad.

1. $E_*(\mathcal{O})$ is an operad in E_*-mod.
2. For $X \in \mathrm{ALg}_{\mathcal{O}}(\mathrm{Mod}_E)$, then $E_*\mathcal{O}$ acts on E_*X (or E_*X is an algebra over $E_*\mathcal{O}$)

Example 11.4. Take $\mathcal{O} = \mathbb{E}_n$ for $n \geq 1$ and $E = \mathbb{S}$. In this case,

- $\mathbb{E}_n(1) = \mathrm{Conf}_2(\mathbb{R}^n) \simeq S^{n-1}$.

Now if $X \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$, then $\pi_0\mathbb{E}_n$ admits an action on $\pi_*(X)$. In practice, if $n = 1$, then $\pi_0\mathbb{E}_n = \mathrm{Associative}$ and if $n \geq 1$, then $\pi_0\mathbb{E}_n = \mathrm{Commutative}$. In particular, this implies that $\pi_*(X)$ is an associative algebra if $n = 1$ and a commutative algebra if $n > 1$.

Here is a second example.

Example 11.5 (The Browder Bracket)**.** Let $\pi_{n-1}(S^{n-1}) = \pi_{n-1}(\mathrm{Conf}_2(\mathbb{R}^n))$ and take a generator λ. Let $\alpha_1 \in \pi_{o_1}(X)$ and $\alpha_2 \in \pi_{o_2}(X)$. The **Browder bracket** is constructed as follows.

The generator λ specifies a map $\lambda : S^{n-1} \to \Sigma^\infty_+ \mathbb{E}_n(2)$. Take $\alpha_1 \in \pi_{a_1}(X)$ and $\alpha_2 \in \pi_{a_2}(X)$, they are given by maps

$$\alpha_1 : S^{a_1} \to X \text{ and } \alpha_2 : S^{a_2} \to X.$$

Combining them together we have a map

$$S^{a_1} \otimes S^{n-1} \otimes S^{a_2} \to X \otimes \Sigma^\infty_+ \mathbb{E}_n(2) \otimes X \to \Sigma^\infty_+ \mathbb{E}_n(2) \otimes X^2 \to X$$

where the last step is given by the operad multiplication. This is called the **Browder bracket**.

Definition 11.6. The resulting class is the **Browder bracket** of α_1 and α_2 is $[\alpha_1, \alpha_2] \in \pi_{a_1+(n-1)+a_2}(X)$.

Proposition 11.7. For every $\mathbb{E}_n$-algebra X, there is a generator $\lambda \in \pi_{n-1}(\mathbb{E}_n(2))$ that defines a bilinear bracket

$$[-,-] : \pi_{\alpha_1}(X) \otimes \pi_{\alpha_2}(X) \to \pi_{\alpha_1+(n-1)+\alpha_2}(X).$$

satisfying the following properties (modulo possible sign mistakes):

1. (Symmetry) $[\alpha, \beta] = -(-1)^{(|\alpha|+n-1)(|\beta|+n-1)}[\beta, \alpha]$. Note $|\alpha| = a_1, |\beta| = a_2$.
2. (Leibiniz) $[\alpha, \beta\gamma] = [\alpha, \beta]\gamma + (-1)^{|\beta|(|\alpha|+n-1)}\alpha[\beta, \gamma]$.
3. (Jacobi) $0 = (-1)^\bullet[\alpha, [\beta, \gamma]] + (-1)^\bullet[\beta, [\gamma, \alpha]] + (-1)^\bullet[\gamma, [\alpha, \beta]]$ and $(-1)^\bullet$ denotes a power of sign omitted in the lecture.

Remark 11.8. When $n = 1$, we have that $[\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha$.

11.2 Power Operations

Definition 11.9. Let $m, n \geq 0$, a **homotopy operation** in $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$ is a natural transformation

$$\pi_m(-) \to \pi_n(-)$$

that induces a **homology operation** $\mathbb{E}_m(-) \to \mathbb{E}_n(-)$. We use $\mathrm{Op}^E_{\mathcal{O}}(m, n)$ to denote the group of these operations.

Our first proposition is a sort of classification statement on $\mathrm{Op}^E_{\mathcal{O}}(m, n)$

Proposition 11.10. In $\mathrm{ho}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E))$ where we have an equivalence between

$$\pi_m(A) \cong [E \otimes \mathrm{Free}_{\mathcal{O}}(S^m); A]_{\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)}$$

Proof Sketch. This follows from the free-forgetful adjunction. ■

Corollary 11.11. Applying the Yoneda lemma, we have that $\mathrm{Op}^G_{\mathcal{O}}(m, n) \cong \pi_n(E \otimes \mathrm{Free}_{\mathcal{O}}(S^m))$.

Question 11.12. Can we consider a multi-input version of this set-up?

Definition 11.13. Let $m, n \geq 0$, a **homotopy operation** in $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$ is a natural transformation

$$\prod_{i=1}^{k} \pi_{m_i} \to \pi_n(-)$$

that induces a **homology operation** $\mathbb{E}_m(-) \to \mathbb{E}_n(-)$. We use $\mathrm{Op}_{\mathcal{O}}^{E}(m_1, ..., m_k, n)$ to denote the group of these operations.

Using a similar line of arguments, we may get that:

Proposition 11.14. $\mathrm{Op}_{\mathcal{O}}^{E}(m_1, ..., m_k, n) \cong \pi_n(E \otimes \mathrm{Free}(S^{m_1} \oplus ... \oplus S^{m_k}))$.

Example 11.15. The Browder bracket defines last section comes from a class $[o_1, o_2] \in \mathrm{Op}_{\mathbb{S}}^{\mathbb{E}_n}(p, q, p + (n - 1) + q)$.

Definition 11.16. The group of power operations of **weight** k on **degree** m in $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$ is $\mathrm{Pow}_{\mathcal{O}}^{E}(m, k) = \bigoplus_{\ell \in \mathbb{Z}} \pi_{m+\ell}(E \otimes (\Sigma_+^{\infty} \mathcal{O}(k) \otimes (S^n)^{\otimes \ell}))$.

In practice, we have that:

- A map $\varphi : S^{m+r} \to E \otimes \mathrm{Sym}_{\mathcal{O}}^{(k)}(S^m)$.
- Let A be an $\mathcal{O}$-algebra and $x : S^m \to E \otimes A$, we consider a map

$$\varphi(x) : S^{m+r} \xrightarrow{\varphi} E \otimes \mathrm{Sym}_{\mathcal{O}}^{(k)}(S^m) \xrightarrow{x} E \otimes A.$$

11.3 Examples of Power Operations

Example 11.17. The **Dyer-Lashof operations** (also known as **Araki-Kudo**) is the special case when $\mathcal{O} = \mathbb{E}_n$ and $E = H\mathbb{F}_p$.

Theorem 11.18. $\mathbb{E}_n$-algebra in $\mathrm{Mod}_{H\mathbb{F}_2}$ has **Dyer-Lashof operations**

$$Q_i : \pi_m \to \pi_{2m+i}, \text{ for } 0 \leq i \leq n-1$$

such that it satisfies:

- (Additivity) $Q_r(x + y) = Q_r(x) + Q_r(y)$ for $r < n - 1$.
- (Square): $Q_0(x) = x^2$.
- (Unit) $Q_j(1) = 0$ for $j > 0$.
- (Cartan Formula) $Q_r(xy) = \sum_{p+q=r} Q_p(x) Q_q(y)$.
- (Adams Relations) $Q_r Q_s(x) = \sum_j \binom{j-s-1}{2j-r-s} Q_{r+2s-2j} Q_j(x)$ for $r > s$.
- (Stability) $\sigma Q_0 = 0$ and $\sigma Q_r = Q_{r-1}$ for $r > 0$.

- (Extension) If an $\mathbb{E}_n$-algebra structure extends to an $\mathbb{E}_{n+1}$-algebra structure, the operations Q_r for $\mathbb{E}_{n+1}$-algebras coincide with the operations Q_r for $\mathbb{E}_n$-algebras.

Compatibility with the Browder bracket $[-,-]$

- $[x, Q_r y] = 0$ for $r < n-1$.
- $Q_{n-1}(x+y) = Q_{n-1}(x) + Q_{n-1}(y) + [x,y]$.
- $Q_{n-1}(xy) = \sum_{p+q=n-1} Q_p(x)Q_q(y) + x[x,y]y$.
- $[x, Q_{n-1}y] = [y,[y,x]]$.
- Can extend an $\mathbb{E}_n$-algebra to $\mathbb{E}_{n+1}$-algebra if the bracket is zero.

Remark 11.19. Dyer-Lashoof operations live in weight 2.

Remark 11.20. If the set-up is already $\mathbb{E}_\infty$, the Browder brackets conditions vanish.

Warning: Different sources may have different ways to index their set-up.

11.4 K(1)-local Power Operations

Let us now work in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}_{K(1)})$.

Theorem 11.21 (McClure). $\pi_0(\mathrm{Free}_{\mathbb{E}_\infty}^{\mathrm{Sp}_{K(n)}}(\{x\})) \cong (\mathrm{Free}_\delta(\{x\}))_p^\wedge$. Here $\{x\}$ should be thought of as S^0.

Here Free_δ denotes the free δ ring!

Definition 11.22. A δ-ring is a ring $(R, S : R \to S)$ such that:

1. $S(x,y) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$
2. $\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$.

(These formulas are forced to make φ a ring homomorphism).

Equivalently, this is a ring “admits lift of **Frobenius**" in the sense that the following diagram commutes

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[d] & R \arrow[d] \\
R/p \arrow[r, "\text{Frobenius}"'] & R/p
\end{tikzcd}$$

with $\varphi(x) = x^p + p\delta(x)$.

12 Lecture 10: Dyer–Lashof operations (by Jordan Levine)

We saw in the last talk that the whole notion of power operations are essentially controlled by homotopy groups of $\mathbb{E}_n$-algebras.

Question 12.1. What can we say about homotopy groups (rings) of $\mathrm{Free}_{\mathbb{E}_n}^{\mathbb{F}_p}(\Sigma^+\mathbb{F}_2)$, for example?

For this talk, we will work with $p = 2$ (the case for $p > 2$ is slightly harder). The central claim of our talk is as follows.

Claim: We can use pure algebra to get these rings, using the following ingredients:

1. Koszul duality. For our purposes, we will use the following form of Koszul duality.

   Theorem 12.2. The bar and cobar construction gives an adjoint equivalence:
   $$\mathrm{Alg}_{\mathbb{E}_k}^{\text{connected,augmented}}(\mathrm{Mod}_{\mathbb{F}_2}^{gr}) \text{ and } \mathrm{coAlg}_{\mathbb{E}_1}^{\text{connected,coaugmented}}(\mathrm{Alg}_{\mathbb{E}_{k-1}}\mathrm{Mod}_{\mathbb{F}_2}^{gr}).$$
   Here, by “connected", we mean “non-positively".

2. Some spectral sequence arguments

It turns out from this specific Koszul duality, we get a nice formula as follows.

Corollary 12.3. Bar $\mathrm{Free}_{\mathbb{E}_k}(M(n)) = \mathrm{Free}_{\mathbb{E}_{k-1}}(\Sigma M(n))$, where $M(n)$ is the graded $\mathbb{F}_2$-module M in degree n.

Suppose we knew $\mathrm{Free}_{\mathbb{E}_k}(\mathbb{F}_2(1))$ (the free $\mathbb{E}_k$-algera of the unit $\mathbb{F}_2$ at degree 1) with co-algebra structure, then it turns out that
$$\text{Cobar}\ \mathrm{Free}_{\mathbb{E}_k}(\Sigma\mathbb{F}_2(1)) = \mathrm{Free}_{\mathbb{E}_{k+1}}(\mathbb{F}_2(1)).$$

Therefore, we see there is sort of an inductive procedure to go higher.
Our Goal is to explain the specific example of the passage from
$$\mathrm{Free}_{\mathbb{E}_1}(-) \to \mathrm{Free}_{\mathbb{E}_2}(-)$$
using this sort of an **inductive argument**. However, we will try to set-up the proof to work for arbitrary $E_n \to E_{n+1}$, up to some minor modifications.

Remark 12.4. We can check the following fact:
$$[\mathrm{Free}_{\mathbb{E}_k}(\Sigma^t\mathbb{F}_2(1))]_w = (\mathbb{E}_{k(w)} \otimes \Sigma^{tw}\mathbb{F}_2)_{h\Sigma_w} =: \mathrm{Sym}_w^{\mathbb{E}_k}(\Sigma^t\mathbb{F}_2).$$
What we want to convey is that understanding the LHS (the free graded ones) is the same as understanding the ungraded RHS. No information about these free algebras is lost or gained when passing to the graded setting.
Note: In the equation above, $\Sigma^{tw}\mathbb{F}_2$ could be interpreted by $(\Sigma^t\mathbb{F}_2)^{\otimes w}$.

Let us now look at $\mathbb{E}_1 \to \mathbb{E}_2$ and specifically $\mathrm{Free}_{\mathbb{E}_2}(\mathbb{F}_2(1))$.

1. Now firstly, by the **Bar-Cobar equivalence**, we need to know something about $\mathrm{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}2(1))$, which really does become
   $$\mathrm{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}2(1)) = \bigoplus_{n\geq 0}\Sigma^n\mathbb{F}_2(n)$$

which we should think of the RHS as a **polynomial algebra**. That is, we have that

$$\pi_* \operatorname{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}_2(1)) = \mathbb{F}_2[x], |x| = (1,1)$$

Here x has a bi-degree (w,t) where w is the weight of x and t is its topological degree, we will stick with **bi-degree convention** for the rest of the lecture.

2. From the cobar complex $\operatorname{Cobar}(\operatorname{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}_2(1))$, there is a spectral sequence given

$$\operatorname{Cotor}^i_{\mathbb{F}_2[x]}(\mathbb{F}_2,\mathbb{F}_2) \implies \pi_{*-i}\operatorname{Cobar}(\operatorname{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}_2(1))).$$

Question: What is a CoTor?

> **Definition 12.5.** Let M, N be modules and consider $M \otimes_{\mathbb{Z}} N$. Now tensor product is right exact, so we can find the **left derived functors** $\operatorname{Tor}_A(-,-)$. Similarly, let A, B be **comodules**, then we can consider a **comodule tensor product** (called cotensoring) $M \square_{\mathbb{Z}} N$. It turns out this is left exact, so there exists **right derived functors** $\operatorname{Cotor}^i(-,-)$.

On $\mathbb{F}_2[x] = \pi_* \operatorname{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}_2(1))$, there is a co-algebra Δ given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \Delta(x^n) = \Delta(x)^n, \text{ and } \Delta(x^{2n}) = x^{2n} \otimes 1 + 1 \otimes x^{2n}.$$

(The last line follows from binomial expansion and the fact that this is in characteristic 2). Because of this, we can write, as a co-algebra,

$$\mathbb{F}_2[x] = \bigotimes_{n\geq 0}^{\infty} \mathbb{F}_2[x^{2n}]/x^{2n+1}, \text{ each } x^{2m} \text{ is primative.}$$

Given this decomposition, we get that

$$\operatorname{Cotor}_{\mathbb{F}_2[x]}(\mathbb{F}_2,\mathbb{F}_2) = \bigotimes_{n\geq 0} \operatorname{Cotor}_{\mathbb{F}_2[x^{2m}]/x^{2m+1}}(\mathbb{F}_2,\mathbb{F}_2).$$

The term $\operatorname{Cotor}_{\mathbb{F}_2[x^{2m}]/x^{2m+1}}(\mathbb{F}_2,\mathbb{F}_2)$ is in fact polynomial, and we write it as $\mathbb{F}_2[y_m]$ instead.

3. Thus, we have that the E_2-page of spectral sequence can be described as $\mathbb{F}_2[y_0, y_1, ...]$.

> **Remark 12.6.** If y_m survives the spectral sequence, it would give some class in $\pi_*\operatorname{Cobar}$ in bi-degree $(2^m, 2^m - 1)$.

4. **Claim:** There are no differentials in this spectral sequence, ie. $E_2 = E_\infty$. To do this, our **strategy** is to look at what needs to appear in weight 2 and propagate using some "naturality". Now consider $\operatorname{wt}(2)$ of $\operatorname{Free}_{\mathbb{E}_2}(\mathbb{F}_2(1))$. In this case we have that

$$\begin{aligned}(\mathbb{E}_2(2) \otimes \mathbb{F}_2)_{h\Sigma_2} &\simeq \mathbb{E}_2(2)_{h\Sigma_2} \otimes \mathbb{F}_2\\ &\simeq (\operatorname{Conf}_2(\mathbb{R}^2)_{\text{ord}})_{h\Sigma_2} \otimes \mathbb{F}_2\\ &\simeq \mathbb{R}P^{2-1} \otimes \mathbb{F}_2\end{aligned}$$

Note that power operations do play a role in this:

$$\begin{tikzcd} \mathbb{F}_2 & \mathbb{F}_2 \\ 0\,\bullet \arrow[d] & \bullet\,1 \arrow[d] \\ y_0^2 & y_1 \\ \text{“}Q_0(y_0)'' & \text{“}Q_1(y_0)'' \end{tikzcd}$$

It turns out in this case that $E_2 = \mathbb{F}_2[y_0, Q_1 y_0, y_2, ...]$, and actually $y_i = Q_1^{(i)} y_0$ more generally. To see why, one can "upack" the spectral sequence and see it! Indeed, $x^2 \in \mathbb{F}_2[x]$ in the cobar complex really needed to represent $Q_1 y_0$ (that this notion of squaring in algebra essentially gives the Q_1's).

Now we use a **naturality trick as follows**. We identify $y_2 = Q_1 Q_1 y_0$. From here we can consider the free $\mathbb{E}_2$-algebra $\mathrm{Free}_{\mathbb{E}_2}(Q_1(x)(2))$ and map it to $\mathrm{Free}_{\mathbb{E}_2}(x(1))$. Now we apply the bar construction eveywhere to get a comodule map which on π_* sends

$$Q_1 Q_1(x) \mapsto y_2.$$

Moreover, by naturality, y_2 cannot admint any more differentials.

Given this decomposition, we get that

$$\mathrm{Cotor}_{\mathbb{F}_2[x]}(\mathbb{F}_2, \mathbb{F}_2) = \bigotimes_{n \geq 0}^{\infty} \mathrm{Cotor}_{\mathbb{F}_2[x^{2m}]/x^{2m+1}}(\mathbb{F}_2, \mathbb{F}_2) = \bigotimes_{n \geq 0}^{\infty} \mathbb{F}_2[y_m]$$

where $y_i = Q_1^{(i)}(y_0)$. From here, we have that

$$E_2 = E_\infty = \mathbb{Z}[y_0, Q_1(y_0), Q_1^{(2)}(y_0), ...].$$

5. Finally, from here we recover the result - $\mathrm{Free}_{\mathbb{E}_2}(\mathbb{F}_2)$ is the **polynomial algebra on so-called "admissible sequences"** (ie. $Q_1 \circ Q_1 \circ ...$) and $\mathrm{Free}_{\mathbb{E}_n} 9\mathbb{F}_2)$ is the polynomial algebra on admissible sequences of the form $Q_1^{(e_1)} \circ Q_2^{(e_2)} \circ ... \circ Q_{n-1}^{(e_{n-1})}$).

This is the entire structure of the Dyer-Lashoff algebra! At least at prime 2, and low n.

13 Lecture 10.5: Multiplicative structure on (truncated) Brown–Peterson spectra (by Ryan Quinn)

Recall we saw that BP is some spectrum such that

$$\pi_*(\mathrm{BP}) = \mathbb{Z}_{(p)}[v_1, v_2, ...].$$

It is made specifically so that $H_*(\mathrm{BP}) = P_* \subset A_*$ is only the **polynomial part** of the Steenrod algebra. The idea is that BP is a spectrum with nice homotopy and nice homology.

This talk is about $\mathrm{BP}\langle n\rangle$'s, the truncated Brown-Peterson spectra. Their constructions are super not canonical. Essentially, they are given by

$$MU \to \mathrm{BP} \to \mathrm{BP}\langle n\rangle$$

such that $\pi_*(\mathrm{BP}\langle n\rangle) = \mathbb{Z}_{(p)}[v_1, ..., v_n]$.

Example 13.1. Here are some examples,

1. $\mathrm{BP}\langle -1\rangle = \mathbb{F}_p$.
2. $\mathrm{BP}\langle 0\rangle = \mathbb{Z}_{(p)}$.
3. $\mathrm{BP}\langle 1\rangle = ku_{(p)} = ko \otimes (\eta)$.
4. $\mathrm{BP}\langle 2\rangle = tmf(3) = tmf \otimes DA(1)$

Here we might have omitted for prime corrections. The upshot is to observe that all these have finitely presented homologies.

Conjecture 13.2 (Hahn-Wilson, Roughly Speaking)**.** $\mathrm{BP}\langle n\rangle$ can be obtained from smashing a spectrum of type n with finitely presented homologies.

All of these examples so far are $\mathbb{E}_\infty$, but this is not very typical, which we will see.

13.1 Multiplicative Structures on $\mathrm{BP}\langle n\rangle$

To give a brief historical account:

Go	No Go
[Chadwick-Mandell [CM15]] BP is $\mathbb{E}_2$	[Hu-May-Kriz [HKM01]] BP is not $\mathbb{E}_\infty$-MU
[Basterra-Mandell [BM12]] BP is $\mathbb{E}_4$	[Lawson [Law18]] At $p = 2$, BP is not $\mathbb{E}_{12}$.
[Hahn Wilson [HW22]] $\mathrm{BP}\langle n\rangle$ is $\mathbb{E}_3$-MU-algebra	[Senger [Sen24]] For $p > 2$, $\mathrm{BP}\langle n\rangle$ is not $2(p^2+2)$

Remark 13.3. Here we remark on two application:

1. (Hahn-Wilson [HW22]): Redshift at certain heights.
2. (Burkland-Hahn-Levy-Schlank [BHLS23]): $L_{T(n)} \neq L_{K(n)}$ for $n \geq 2$.

Here are some pre-requisites: Consider the ring $k[x]$ and $\mathrm{Tor}_*^{k[x]}(k,k)$, this will give the exterior algebra $\Lambda(\sigma x)$. Doing $\mathrm{Tor}_*^{\Lambda(\sigma x)}(k,k)$ gives $\Gamma(\sigma^2 x)$. Altogether, we have a sequence:

$$\begin{tikzcd}
k[x] \arrow[r, shift left, "\mathrm{Tor}_*^{k[x]}(k,k)"] & \Lambda(\sigma x) \arrow[l, shift left, "\mathrm{Ext}_{\Lambda(\sigma x)}(k,k)"] \arrow[r, shift left, "\mathrm{Tor}_*^{\Lambda(\sigma x)}(k,k)"] & \Gamma(\sigma^2 x) \arrow[l, shift left, "\mathrm{Ext}_{\Gamma(\sigma^2 x)}(k,k)"]
\end{tikzcd}$$

Example 13.4. We have that

$$H_* \operatorname{Free}^{\mathbb{E}_1}_*(S^2)) = \mathbb{Z}[x], H_*(\Sigma S^2) = \Lambda(\sigma x), H_*(H\mathbb{P}^\infty) = \Gamma(\sigma^2 x)$$

We also have a similar sequence $H_*(BU), H_*(SU), H_*(BSU)$.

Here is a **rough sketch** of the Hahn-Wilson result:

- The idea is to inductively build BP$\langle n+1\rangle$ from BP$\langle n\rangle$ with

$$\mathrm{BP}\langle n+1\rangle = \operatorname{map}_{\widehat{MU}(SU_{n+1})}(MU, \mathrm{BP}\langle n\rangle)$$

$$\mathbb{Z}[v_1, ..., v_n, v_{n+1}] = \operatorname{Ext}_{\widehat{MU_*}(SU_{n+1})}(MU_*, \mathbb{Z}[v_1, ..., v_n]).$$

- Out goal is to first make the notation $\widehat{MU}(SU_{n+1})$ precise and then to give $\mathbb{E}_n$-A-algebra structures.

Definition 13.5 (Enveloping Algebras). Let B be an $\mathbb{E}_n - A-$algebra (ie. $B \in \operatorname{Alg}^{\mathbb{E}_n}(\operatorname{LMod}_A)$. From here we define

$$U^{(1)}_A(B) = B \otimes_A B^{op}, U^{(n)}_A = B \otimes_{U^{(n-1)}_A} B^{op}.$$

Remark 13.6. In terms of factorization homology, $U^{(n)}_A(B) = \int_{\mathbb{R}^n - \{0\}} B$. When $n = 2$, we have that

$$U^{(2)}_A(B) = \mathrm{THH}(B/A) = B \otimes_{B \otimes_A B^{op}} B^{op} = \int_{S^1} B.$$

Definition 13.7 ($\mathbb{E}_k$-centers). Recall that if M is an R-module, then it is equivalent to the data of a ring homomorphism $R \to \operatorname{End}(M)$. Analogously, we specify that the $\mathbb{E}_n - A$-center of B is

$$\mathfrak{Z}_{\mathbb{E}_n - A}(B) \in \operatorname{Alg}^{\mathbb{E}_{nm}}(\mathbb{C}).$$

Defined by the universal property that

$$\mathbb{E}_{nm}\text{-map } R \to \mathfrak{Z}_{\mathbb{E}_n - A}(B) \iff \text{ B is an } \mathbb{E}_n - R - \text{algebra}.$$

Construction: If $f : B_i \to B_j$, then define $\mathfrak{Z}_{\mathbb{E}_n - A}(f)$:

1. If $f = id_B$, then $\mathfrak{Z}_{\mathbb{E}_n - A}(f) = \mathfrak{Z}_{\mathbb{E}_n - A}(B)$.
2. $\mathfrak{Z}_{\mathbb{E}_n - A}(f)$ has the underlying structure of $\operatorname{map}_{U^{(n)}_{A(B_j)}}(B_i, B_j)$.

Now we will discuss the **designer polynomial algebras**. As a nice pre-cursor to this, we first outline a proposition by Lurie.

Proposition 13.8 (Lurie). $\operatorname{Free}^{\mathbb{E}_1}(S^{2n})$ admits an $\mathbb{E}_2$-structure.

The main idea behind Lurie's proof was the diagram:

$$\begin{tikzcd}
\mathbb{Z}_{\geq 0} \arrow[r] & \mathbb{Z} \arrow[r, "\cdot n"] & \mathbb{Z} \arrow[r] & \operatorname{Pre}(\mathrm{Sp})
\end{tikzcd}$$

$$\begin{tikzcd}
k \arrow[r] & S^{2k}
\end{tikzcd}$$

We furthermore have that relim $= \bigoplus S^{2nk}$

Theorem 13.9 (Hahn-Wilson)**.** We have a diagram

$$\begin{tikzcd}
 & & \mathbb{Z} \times BU \arrow[r] \arrow[d] & \mathrm{Pic}(Sp) \\
\mathbb{Z}_{\geq 0} \arrow[r] & \mathbb{Z} \arrow[r, "\cdot n"] & \mathbb{Z} \arrow[ur, dashed] &
\end{tikzcd}$$

where $MU[y]$ is a free $\mathbb{E}_1 - MU$-algebra c

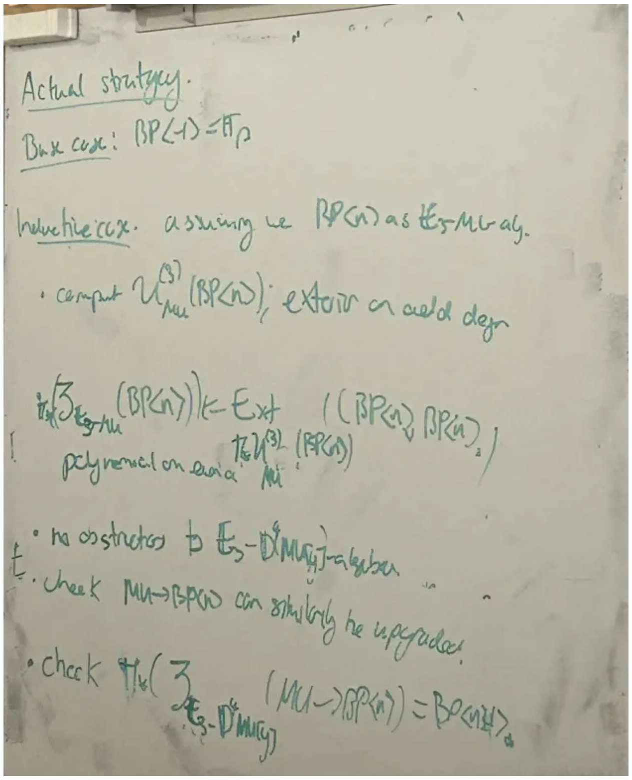
To proceed, we quickly need to inquire two facts from Koszul duality. Let $\mathbb{D}$ denote the Koszul dual.

1. $\mathbb{D}^{n1}\,\mathrm{MU}[y] = \mathrm{free}_{\mathbb{E}_{n-1}}(\Sigma^{-2k-n}MU)$ and it admits an $\mathbb{E}_n - MU$-algebra structures (has even cells as $\mathbb{E}_n$-algebra for n even).
2. $U^{(n-1)})_{\mathbb{D}^n MU[y]}(MU) = \mathrm{B}\,u(MU[y])$.

Now we will **proceed to the actual strategy** by induction.

- Base Case; $\mathbb{F}_p = BP\langle -1 \rangle$ is clearly $\mathbb{E}_3$.
- Inductive Case: Assuming BP$\langle n \rangle$ is an $\mathbb{E}_3$-MU-algebra.

♣♣♣ Mattie: [See picture:]



Actual strategy.

Base case: $BP\langle -1 \rangle = \mathbb{F}_p$

Inductive case. assuming we $BP\langle n \rangle$ as $\mathbb{E}_3$-MU-alg.

- compute $\mathcal{U}^{(3)}_{MU}(BP\langle n \rangle)$; exterior on odd degr

$\pi_*(\mathfrak{Z}_{\mathbb{E}_3\text{-}MU}(BP\langle n \rangle)) \Leftarrow \mathrm{Ext}_{\pi_*\mathcal{U}^{(3)}_{MU}(BP\langle n \rangle)}((BP\langle n \rangle_*, BP\langle n \rangle_*))$

polynomial on even

- no obstruction to $\mathbb{E}_3$-$\mathbb{D}^4 MU[y]$-algebra
- check $MU \to BP\langle n \rangle$ can similarly be upgraded.
- check $\pi_*(\mathfrak{Z}_{\mathbb{E}_3\text{-}\mathbb{D}^4 MU[y]}(MU \to BP\langle n \rangle)) = BP\langle n+1 \rangle_*$

Base Case: For $\mathbb{F}_p$.

Proposition 13.10. $U^{(2)}_{MU}(\mathbb{F}_p) = \mathrm{THH}(\mathbb{F}_p/MU)$ is polynomial on even degrees.

Proof. Recall that the THH of the Thom spectra is given by

$$\begin{aligned}\mathrm{THH}(MU) &= MU \otimes \Sigma^\infty_+ SU \\ &= MU[SU].\end{aligned}$$

On the other hand, we have that $\mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p[\Omega S^3]$. By Hopkins-Malhowald, we have that $\mathbb{F}_p = \mathrm{Th}(\Omega^2 S^2 \to \mathrm{BGL}(\mathbb{S}_p))$.

We can compute $\mathrm{THH}(\mathbb{F}_p/MU)$ as the base change

$$\mathrm{THH}(\mathbb{F}_p) \otimes_{\mathrm{THH}(MU)} MU.$$

Expanding the relevant terms as above out, we have that

$$\begin{aligned}\mathrm{THH}(\mathbb{F}_p/MU) &= \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathrm{THH}(MU)} MU \\ &= (\mathbb{F}_p[\Omega S^3] \otimes_M UMU) \otimes_{MU[SU]} MU \\ &= \mathbb{F}_p[\Omega S^3] \otimes_M UBSU.\end{aligned}$$

■

For the inductive steps, we have that

$$U^{(1)}_{MU} BP\langle n\rangle = \Lambda(\sigma ...)$$

and $U^{(2)}_{MU} BP\langle n\rangle$ follows from there.

14 Lecture 11: T-structures, Filtered Objects, and Spectral Sequences (by Lucy Grossman)

A recommend reference for this talk is [Ant24].

14.1 T-structures

Recall the notion of t-structure came from a triangulated category.

Definition 14.1. Consider triangulated category $\mathcal{D}$, a t-structure is a pair of subcategories $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ such that:

1. For $X \in \mathcal{D}_{\geq 0}, Y \in \mathcal{D}_{\leq 0}$, $\mathrm{Hom}(X, Y[-1]) = 0$.
2. There are inclusions $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\geq 0}[-1] \subseteq \mathcal{D} \leq 0$.
3. For all $X \in \mathcal{D}$, there exists a cofiber sequence $X' \in \mathcal{D}_{\geq 0} \to X \to X'' \in \mathcal{D}_{\leq 0}[-1]$.

This is a concept for 1-categories, but since stable ∞-categories are supposed to be the analog, we can generalize as follows.

Definition 14.2. Let $\mathcal{C}$ be a stable ∞-category, a t-strucure on $\mathcal{C}$ is a t-structure on $h\mathcal{C}$, notationally, we write

$$\mathcal{C}_{\geq 0} := (h\mathcal{C})_{\geq 0} \text{ and } \mathcal{C}_{\leq 0} := (h\mathcal{C})_{\leq 0}$$

. This gives us a pair of truncation functors

$$\text{inclusion} : \mathcal{C}_{\geq 0} \leftrightarrow \mathcal{C} : R := \tau_{\geq 0}$$

Lemma 14.3. For any $m \leq n$, write $\mathcal{C}_{[m,n]} = \mathcal{C}_{\geq m} \cap \mathcal{C}_{\leq n}$. This there exists a natural equivalence between $\tau_{\geq m} \circ \tau_{\leq n} \cong \tau_{\leq n} \circ \tau_{\geq m}$ (ie. the order of truncation does not matter). This also commutes with Σ in the sense that

$$\Sigma \circ \tau_{\geq n} \simeq \tau_{n+1} \circ \Sigma.$$

Definition 14.4. Let $\mathcal{C}$ have a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, the **heart of this t-structure** is $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$.

Definition 14.5. Define a functor $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ by

$$\begin{aligned}\Sigma^{-1} \circ \tau_{\geq n} \circ \tau_{\leq n} &\simeq \Sigma^{-n} \circ \tau_{\leq n} \circ \tau_{\geq n} \\ &\simeq \tau_{\leq 0} \circ \tau_{\geq 0} \circ \Sigma^{-n} \\ &\simeq \tau_{\geq 0} \circ \tau_{\leq 0} \circ \Sigma^{-1}\end{aligned}$$

Recall if we have a fiber sequence $X \to Y \to Z$, we have tan induced long exact sequence

$$\ldots \to \pi_n X \to \pi_n Y \to \pi_n Z \to \pi_{n-1} X \to \ldots$$

14.2 Filtered Objects

Definition 14.6. Let $\mathcal{C}$ be a stable ∞-category, a **filtered object** of $\mathcal{C}$ is a functor $X : N(\mathbb{Z}) \to \mathcal{C}$. Here $N(\mathbb{Z})$ is the nerve of $\mathbb{Z}$ where $\mathbb{Z}$ is viewed as the poset category.

Definition 14.7. The ∞-category of **decreasing filtrations** in $\mathcal{C}$ is $F\mathcal{C} := \mathrm{Fun}(\mathbb{Z}^{op}, \mathcal{C})$. The ∞-category of **increasing filtrations** in $\mathcal{C}$ is $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$.

From **Antiieau**, we write a decreasing filtration as an infinite sequence

$$F^* : ... \to F^{s+1} \to F^s \to F^{s-1} \to ...$$

For a filtered object F^*, we say:

1. It is **exhaustive** if X is a colimit of F^*.
2. It is **complete** if $F^\infty := \lim_s F^s \simeq 0$.

If $\mathcal{C}$ admits sequential colimits, then any F^* can be viewed as giving a filtration $F^{-\infty} := \mathrm{colim}_s F^s$ - this is often also written as $|F^*|$.

Definition 14.8. Let $\mathcal{C}$ be an ∞-category with final object that admits cofibers, and let F^* be a decreasing filtration on $\mathcal{C}$. The **associated graded** pieces are

$$gr_F^s = \mathrm{cofiber}(F^{s+1} \to F^s).$$

The associated graded is suggestively written as F^s/F^{s+1}.

14.3 Formalism in Higher Algebra

Definition 14.9. Let $\mathcal{C}$ be a pointed ∞-category and consider 2-linearly-ordered sets. $J^{[i]}$ is the poset of pairs of elements $i \leq j$ in J, with poset structure $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$.

A J-complex in $\mathcal{C}$ is a functor

$$F : N(J^{[i]}) \to \mathcal{C}$$

such that:

1. For each $i \in J$, $F(i, i)$ is a 0-object of $\mathcal{C}$.
2. For $i \leq j \leq k$,

$$\begin{tikzcd} F(i,j) \arrow[r] \arrow[d] & F(i,k) \arrow[d] \\ F(j,j) \arrow[r] & F(j,k) \end{tikzcd}$$

   is a point.

We use $\mathrm{Gap}(J, \mathcal{C})$ to denote the full subcategory of $\mathrm{Fun}(N(J^{[i]}), \mathcal{C})$ spanned by J-complexes.

Definition 14.10. Let $\mathcal{C}$ be a stable ∞-category with t-structures. Let $X - t \in \mathrm{Gap}(\mathbb{Z}, \mathcal{C})$, we have for all triples in $\mathbb{Z}$ with $i \leq j \leq k$, a long exact sequence in $h\mathcal{C}^{\heartsuit}$:

$$... \to \pi_n(X(i,j)) \to \pi_n(X(i,k)) \to \pi_n(X(j,k)) \to \pi_{n-1}(X(j,j)) \to ...$$

For every $p, q \in \mathbb{Z}$, $r \geq 1$, we define

$$E_r^{p,q} = \mathrm{im}(\pi_{p+q}(X(p-r,p)) \to \pi_{p+q}X(p-1,p+r-1))$$

with differentials $E_r^{p,q} \to E_r^{p-r,q+r-1}$ from the following commutative diagram

$$\begin{tikzcd}
\pi_{p+q}X(p-r,p) \arrow[r] \arrow[d] & E_r^{p,q} \arrow[r] \arrow[d, "d_r"] & \pi_{p+q}(X(p-1,p+r-1)) \arrow[d] \\
\pi_{p+q-1}(X(p-\partial r,p-R)) \arrow[r] & E_r^{p-r,q+r-1} \arrow[r] & \pi_{p+q-r}(X(p-r-1,p-1))
\end{tikzcd}$$

Lemma 14.11. Let $\mathcal{C}$ be a pointed ∞-category admitting pushouts. Let $J = J_0 \cup \{-\infty\}$, a linearly-ordered set containing $\{-\infty\}$-least elements. We can regard J_0 as a linearly ordered subset of $J^{[i]}$ via

$$i \mapsto (-\infty, i)$$

have equivalence of ∞-categories between $\mathrm{Gap}(J\mathcal{C}) \to \mathrm{Fun}(N(J_0), \mathcal{C})$.

Now consider $\mathcal{C}$ a stable ∞-category with t-structure, we have a filtered object

$$X : N(\mathbb{Z}) \to \mathcal{C}$$

We can extend X to a complex in $\mathrm{Gap}(\mathbb{Z} \cup \{-\infty\}, \mathcal{C})$. The spectral srquence $\{E_r^{p,q}, d_r\}_{r \geq 1}$ is the spectral sequence associated to the filtered object X (via the Gap construction).

There is an alternative way to reach this construction. Consider $\mathrm{gr}_F^{[i,j]}$ as admitting a filtration

$$... \to 0 \to F^{j-1}/F_j \to F^{j-2}/F_j \to ... \to F^i/F_j \to ...$$

This turns out to be a complete and exhaustive filtration on $gr_F^{[i,j]}$.

If F^A is filtration, then the graded objects form a cochain complex (where the differential comes from). In this case, we have

$$\mathrm{gr}^{s+1} \to F^s/F^{s+2} \to \mathrm{gr}^s .$$

This gives $\delta : \mathrm{gr}^s \to \mathrm{gr}^{s+1}[1]$ with $\delta \circ \delta = 0$.

Associated to F^*, we have cochain complex

$$... \to \mathrm{gr}_F^{-s-1}[-s-1] \to \mathrm{gr}_F^{-s}[-s] \to \mathrm{gr}_F^{-s+1}[-s+1] \to ...$$

Applying π_* with respect to the t-structure on $\mathcal{C}$ to get a cochain complex in $\mathcal{C}^{\heartsuit}$. Hence we have

$$... \to \pi_t(\mathrm{gr}_F^{-s-1}[-s-1]) \to \pi_t(\mathrm{gr}_F^{-s}[-s]) \to \pi_t(\mathrm{gr}_F^{-s+1}[-s+1]) \to ...$$

The following theorem describes the passage between the E_1-page and cochain complexes.

Theorem 14.12 (3.21 (Ariotta's E_1-page Theorem))**.** Let $\mathcal{C}$ be a stable ∞-category with sequential colimits, consider the associated graded functor

$$\mathrm{gr}^F : F\mathcal{C} \to \mathrm{Gr}\,\mathcal{C},$$

factors through the forgetful functor $\mathrm{Ch}^\bullet(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$. It induces an equivalence

$$\hat{F} : \mathcal{C} \to \mathrm{Ch}^\bullet(\mathcal{C}).$$

Definition 14.13. $\mathrm{Ch}(\mathcal{C})$ inherits a pointwise t-structure with $X^\bullet$ is connective if $X^n \in \mathcal{C}_{\geq 0}$ for each n. We can use Theorem 3.21 to put this t-structure onto $F\mathcal{C}$. This t-structure is called the **Beilinson t-structure**.

Definition 14.14 (Decalage)**.** Let $\mathcal{C}$ be a stable ∞-category with sequential limits and colimits admitting a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. F^* a filtered object of $\mathcal{C}$. Then taking $\tau^B_{\geq \bullet} F$ (the Whitehead tower of F^* w.r.t to the Beilinson t-structure on $F\mathcal{C}$). One obtains a realization

$$...|\tau^B_{\geq n+1}(F)| \to |\tau^B_{\geq n}(F)| \to ...$$

a filtered object of $\mathcal{C}$. This is called $\mathrm{Dec}(F^*)$, the decalage of F.

Remark 14.15. To get the Adams spectral sequence, one requires

$$\mathrm{Ass}_E(X) := X \otimes E^{\otimes \bullet + 1}.$$

15 Lecture 12: Synthetic Spectra (by Jonathan Pederson)

The outline of this talk is as follows:

1. Discussions on the Adams Spectral Sequence (classically and tie-in with filtered spectra)
2. Synthetic category whose objects are the synthetic spectrum.
3. Try to say something about the synthetic homotopy groups and the Adams spectral sequence computations.

The main goal of this talk is to convince the audience of the following slogan:

"**Synthetic spectra (E) are a categorification of the Adams spectral sequence.**"

Two main references for the talk are [Pst23] and the appendix of [BHS22].

15.1 The Adams Spectral Sequence

Let $\mathcal{C}$ be a symmetric monoidal ∞-category, then $\mathcal{C}^{\text{fil}}$, the category of filtered objects in $\mathcal{C}$, is also **symmetric monoidal** given by the **Day-Convolution**. There is a unit $\mathbb{1}_{\mathcal{C}^{fil}} \in \mathcal{C}^{fil}$ given by

$$0 \to 0 \to \dots \to \mathbb{1}_{\mathcal{C}} \xrightarrow{id} \mathbb{1}_{\mathcal{C}} \xrightarrow{id} \dots$$

Now consider $\mathbb{1}_{\mathcal{C}^{fil}}(1)$ shifted by 1, this gives a map $\tau : \mathbb{1}_{\mathcal{C}^{fil}}(1) \to \mathbb{1}_{\mathcal{C}^{fil}}$

$$\begin{tikzcd}
0 \arrow[r] & \dots \arrow[r] & \mathbb{1}_{\mathcal{C}} \arrow[r] & \mathbb{1}_{\mathcal{C}} \arrow[r] & \mathbb{1}_{\mathcal{C}} \arrow[r] & \dots \\
0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] \arrow[u] & \mathbb{1}_{\mathcal{C}} \arrow[r] \arrow[u] & \mathbb{1}_{\mathcal{C}} \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

In particular, we observe that $\mathcal{C}^{fil}$ admits a splitting:

- $\tau^{-1} : \mathcal{C}^{fil} \to \mathcal{C}$ (colim/lim).
- $\mathcal{C}^{fil} \xrightarrow{-\otimes c\tau} \mathcal{C}^{gr}$ where $x \otimes c\tau = \bigoplus_{i \in \mathbb{Z}} X_i / X_{i+1}$.

Heuristic: Let $c\tau$ be the cofiber of τ. $- \otimes c\tau$ extracts a E_1-page and τ^{-1} extracts the E_∞-page.

Now suppose E is a $\mathbb{E}_1$-ring-spectrum. Consider the cobar complex

$$CB^\bullet_E(X) = X \otimes E^{\otimes n+1}.$$

and a totalization $\text{Tot}(\tau_{\geq n} CB^\bullet_E(X)) \in \text{Sp}^{fil}$.

Proposition 15.1. The associated spectral sequence to this construction is the Adams spectral sequence (also called the Bousfield-Kahn spectral sequence in this context).

Recall, **classically**, the E-based Adams tower is given by

$$\begin{tikzcd}
X = & X_0 \arrow[d, "f_0"] & X_1 \arrow[l] \arrow[d, "f_1"] & \dots \arrow[l] \\
 & X_0 \otimes E & X_1 \otimes E & \dots
\end{tikzcd}$$

Write $K_i = X_i \otimes E$, the tower satisfies the properties:

1. X_{s+1} is the fiber of f_s.

2. $E \otimes X_s$ is a retract of $E \otimes K_S$.
3. K_S is a retract of $E \otimes K_S$.
4. $\mathrm{Ext}^{t,u}(E_*(k_s)) = \begin{cases} \pi_u(K_s), t = 0 \\ 0, \text{ otherwise} \end{cases}$

Putting this together, under nice conditions (some flatness), we have a spectral sequence

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*) \implies \pi_{t-s}(\mathbb{S}^{\wedge}_E).$$

Observation: If we want to categorify the E-based Adams spectral sequence, we should think very carefully about exact sequences $A \to B \to C$ such that the resulting sequence

$$E_*(A) \to E_*(B) \to E_*(C)$$

is short exact sequence.

15.2 The Synthetic Category

We want a nice category to categorify the E-based Adams spectral sequence. Here by “nice", we want it to be stable, presentable, symmetric monoidal, has a t-structure, etc, and we want to **force certain sequences to be exact**. In summary, we want:

1. **Niceness Part**.
2. **The Behavior of Exact Sequences**.

For the niceness part, one might think we can take the colimit completion or the K-colimit completion for a nice set K. One might think we could take K to be the collection of simplicial sets, the problem is this does not really admit a nice description of the presheaf category.

Theorem 15.2. Let $\mathcal{C}$ be an ∞-category with finite coproducts and let K be the collection of filtered simplicial sets + Δ^{op}, then $\mathcal{P}^k(\mathcal{C}0 \subseteq P(\mathcal{C})$ are exactly the **finite product preserving presheaves**. In this case, we write $\mathcal{P}^K(\mathcal{C})$ as $P_\Sigma(\mathcal{C})$.

Now we start with $P_\Sigma(\mathrm{Sp}^\omega)$ (on compact objects). We can **stabilize this** as $P_\Sigma^{Sp}(\mathrm{Sp}^\omega)$ (ie. just presheaves valued in spectra). We also want a **notion of symmetric monoidality** such that given a fiber sequence $A \to B \to C$, we have a SES $E_*(A) \to E_*(B) \to E_*(C)$ (ie. closed under tensor product).

However, we actually have a problem here.

Example 15.3. Consider the map $\mathbb{S} \xrightarrow{\cdot 2} \mathbb{S} \to \mathbb{S}/2$, which is $H\mathbb{Z}_*$-surjection, but after $- \otimes \mathbb{S}/2$ it is not.

This suggests we should look at a sub-collection of spectra in the compact objects instead.

Definition 15.4. The finite E_*-projective spectra Sp^{fp} is the full subcategory formed by P such that E_*P is projective.

One can check this is now closed under tensor products.

From now we define:

Definition 15.5. Syn_E is the full subcategory of $P_\Sigma^{Sp}(\mathrm{Sp}^{fp})$ consisting of those $X : (\mathrm{Sp}^{fp})^{op} \to \mathrm{Sp}$ such that whenever $A \to B \to C$ is a E_*-SES, then $X(C) \to X(B) \to X(A)$ is a fiber sequence.

Remark 15.6. This Syn_E can also be obtained from a **Grothendieck topology** on Sp^{fp} generated by single coverings composing of E_*-surjections, then Syn_E consists of presheaves that are sheaves under this topology.

What are some examples of synthetic spectra?

There is a “less useful one" coming from the **Yoneda embedding**. This is an example, but it does not use anything from E.

The examples that take more advantage of E is the following collection of presheaves:

$$Y : \mathrm{Sp} \to P_\Sigma^{Sp}(Sp^{fp}), X \mapsto \tau_{\geq 0}\,\mathrm{map}_{Sp^{fp}}(-, X)$$

This is NOT always a synthetic spectrum! However, it is a synthetic spectrum if for any given $A \to B \to C$,

$$\tau_{\geq 0}\,\mathrm{map}(C, X) \to \tau_{\geq 0}\,\mathrm{map}(B, X) \to \tau_{\geq 0}\,\mathrm{map}(A, X)$$

is a fiber sequence exactly when $[B, X] \to [A, X]$ is surjective.

Definition 15.7. Let $\nu : \mathrm{Sp} \to \mathrm{Syn}_E$ be the **sheafification** of Y defined above. This is called the **synthetic analogue**.

Lemma 15.8. Let $A \to B \to C$ be a E_* short exact sequence, then $vA \to vB \to vC$ is a fiber sequence.

This is great for us! And we have the **famous diagram**:

$$\begin{tikzcd}
 & \mathrm{Sp} \arrow[d, "\nu"] \arrow[dr, "\pi_*(-\otimes E)"] & \\
\mathrm{Sp} \arrow[ur, "="] & \mathrm{Syn}_E \arrow[l, "\tau^{-1}"'] \arrow[r, "-\otimes c\tau"'] & \mathrm{Stable}_{E_*E}
\end{tikzcd}$$

Notation: Write $\mathbb{S}^{a,b} = \Sigma^{-b}\nu\mathbb{S}^{a+b}$ (this notation is from Burkland’s notes).

- The unit is given by $\mathbb{S} = \nu\mathbb{S} = \mathbb{S}^{0,0}$ (slight abuse of notation here).
- The τ map is interpreted as $\tau : S^{0,-1} = \Sigma\nu(\Sigma^{-1}\mathbb{S}) \to S^{0,0} = \nu(\mathbb{S})$.
- Syn_E is stable, presentable, and symmetric monoidal with the unit given above.
- $\nu : \mathrm{Sp} \to \mathrm{Syn}_E$ is a symmetric monoidal functor.
- $c\tau$ admits an $\mathbb{E}_\infty$-ring structure.
- There is a canonical t-structure on Syn_E given by the connective parts.

A key part of the set-up here is that a synthetic spectrum is equipped with an Adams filtration.

Definition 15.9. Let $f : X \to Y$ be a map. An E_*-Adams filtration $\geq s$ is

$$\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[d, "="'] & & & Y \arrow[d, "f_s"] \\
X_0 \arrow[r, "f_0"'] & X_1 \arrow[r, "f_1"'] & \dots \arrow[r] & X_{s-1}
\end{tikzcd}$$

such that $E_*(f_i) = 0$ for all $0 \leq i \leq s$!

Proposition 15.10. Suppose $f : X \to Y$ has an adams filtration $\geq k$ if there is a lift

$$\begin{tikzcd}
& \Sigma^{0,-k}\nu(Y) \arrow[d, "\tau^k"] \\
\nu(X) \arrow[ur, dashed] \arrow[r, "\nu(f)"'] & \nu(Y)
\end{tikzcd}$$

Proof Sketch. Now consider the sequence

$$\Sigma^{-1}Y \xrightarrow{g} Z \xrightarrow{h} X \xrightarrow{f} Y$$

And there is a sequence that looks like:

$$\begin{tikzcd}
\nu(\Sigma^{-1}Y) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \Sigma\nu(\Sigma^{-1}Y) \arrow[r, "\tau"] \arrow[d] & \nu(Y) \arrow[d] \\
\nu(Z) \arrow[r, "\nu(h)"'] & \nu(X) \arrow[r] \arrow[rr, bend right, "\nu(f)"'] & \mathrm{cofib}(\nu(h)) \arrow[r] & \nu(Y)
\end{tikzcd}$$

■

We did not end up getting to the computation examples.

16 Lecture 13: Chromatic homotopy theory is algebraic when $p > n^2+n+1$ (By Mattie Ji)

In this lecture, the presenter (Mattie Ji) will make an attempt to explain Piotr Pstrągowski's theorem that "chromatic homotopy theory is algebraic when $p > n^2+n+1$" [Pst21], and its application to the algebraicity of chromatic picard groups [Pst22]. These notes might contain errors as the presenter's understanding of the subject is limited.

16.1 The Main Theorems

The **chromatic convergence theorem** tells us for the p-local finite spectrum X, it can be recovered as the **homotopy limit**

$$... \to L_{E_n}X \to ... \to L_{E_1}X \to L_{E_0}X.$$

This informally suggests that the E_n-local category Sp_{E_n} should become more and more complicated (ie. topological) as $n \to \infty$. On the opposite direction, one question we could ask is the following.

Question 16.1. If the height h is sufficiently small compared to the prime p, is Sp_{E_h} easier to understand?

In this note, p will always mean a prime number. One answer to this question is given as follows.

Theorem 16.2. Let E be a p-local Landweber exact homology theory of height n. Suppose $p > n^2+n+1$, then there is an equivalence of category between:

1. $h(\mathrm{Sp}_E)$ - the homotopy category of E-local spectrum.
2. $h\mathcal{D}(E_*E)$ - the category of differential E_*E-comodules. This is the homotopy category of the derived ∞-category $\mathcal{D}(E_*E)$.

Recall a differential E_*E-comodule is a pair (M, d) where M is a E_*E-comodule and $d : M \to M$ is a morphism of comodules of degree 1 such that $d^2 = 0$. Here $\mathcal{D}(E_*E)$ may be constructed by considering a suitable **model structure** on the category of differential E_*E-comodules. Note that the homology functor gives a natural map $\mathrm{Sp}_E \to \mathrm{Comod}_{E_*E}$ to the abelian category.

Remark 16.3. $\mathcal{D}(E_*E)$ should not be confused with $\mathcal{D}(\mathrm{Comod}_{E_*E})$, which is the derived ∞-category of E_*E-comodules.

Remark 16.4. The case for when $n = 1$ was first proven by Bousfield.

Theorem 16.2 is in fact a special case of a more general theorem between the k-th truncations of the two ∞-categories.

Definition 16.5. Let $k \geq 1$ and $\mathcal{C}$ be an ∞-category. For each simplicial set K, we write $[K, \mathcal{C}]_k$ as the subset of maps maps between simplicial complexes

$$\phi : \mathrm{sk}^k K \to \mathcal{C} \text{ such that } \phi = \mathrm{res} \circ \psi$$

where $\psi \in \mathrm{Fun}(sk^{k+1}K, \mathcal{C})$ and res is the restriction map $\mathrm{Fun}(sk^{k+1}K, \mathcal{C}) \to \mathrm{Fun}(sk^k K, \mathcal{C})$.

We can consider a notion of **"k-homotopy classes of maps"** from $K \to \mathcal{C}$ as follows - for two maps $f, g \in [K, \mathcal{C}]_k$, we say $f \sim g$ if f and g are homotopic relative to $\mathrm{sk}^{k-1} K$ (in other words, they are equivalent as objects in $\mathrm{Fun}(\mathrm{sk}^k K, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sk}^{k-1} K, \mathcal{C})} \{f|_{\mathrm{sk}^{k-1} K}\}$.

For each ∞-category $\mathcal{C}$, we can associate a **homotopy k-category** $h_k\mathcal{C}$ (which is a k-category itself) such that

1. For any simplicial set K, $[K, h_k\mathcal{C}] \cong [K, \mathcal{C}]_k / \sim$.
2. If $\mathcal{C}$ is a k-category, the canonical map $\theta : \mathcal{C} \to h_k\mathcal{C}$ is an isomorphism.
3. If $k = 1$, $h_1\mathcal{C}$ is the homotopy category of $\mathcal{C}$.
4. For any n-category $\mathcal{D}$, taking precomposition with θ gives an isomorphism of simplicial sets $\mathrm{Fun}(h_k\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Alternatively, we can think of $h_k\mathcal{C}$ as “having the same objects as $\mathcal{C}$ but the mapping spaces are $(k-1)$-truncated".

Theorem 16.6. Let E be a p-local Landweber exact homology theory of height n. Suppose $p > n^2+n+1+\frac{k}{2}$, then $h_k \mathrm{Sp}_E$ and $h_k\mathcal{D}(E_*E)$ are equivalent.

Theorem 16.2 can be recovered in the special case when $k = 1$.

Remark 16.7. Note that by a prior deep result of **Hovey and Strickland**, the categories Sp_E and Comod_{E_*E} does not depend on the underlying choice of the Landweber exact homology theory E, as long as we do not change the choice of (p, n). Thus, to prove Theorem 16.6, it suffices for us to check this over a particular choice of homology theory. For technical reason, it is best to choose E such that its homotopy groups (ie. E_*) is concentrated in degrees divisble by $2p - 2$.

For the rest of this note, we assume we work with the **Johnson-Wilson theory** $E(n)$, whose homotopy groups are given by

$$E(n)_* \cong \mathbb{Z}_{(p)}[v_1, ..., v_{n-1}, v_n^{\pm 1}],$$

where $|v_i| = 2p^i - 2$. Unless mentioned otherwise, E means $E(n)$.

What is remarkable about Theorem 16.2 and Theorem 16.6 is that the two ∞-categories Sp_E and $\mathcal{D}(E_*E)$ are **known to not be equivalent** at any p! Thus, in order to prove these two theorems, we may not construct any equivalence of ∞-categories between them directly.

16.2 The Thematic Overview

Question 16.8. Why would we intuitively expect Theorem 16.2 and Theorem 16.6 to hold?

As far as the presenter knows, the main intuition between the two theorems amounted down to the following specific observations, which was then expanded upon by Piotr Pstrągowski using the theory of **synthetic spectra**. We will focus on the following specific observation first, and fill in some details later if time allotted (or in an accompanied notes such as this later). Therefore, we will not start with an outline of the proof but rather go inductively from an observation below.

Observation 1: A vanishing line result in Sp_E.

Theorem 16.9 (Folklore, Theorem 2.4 of [Pst21])**.** Let E be a Landweber exact homology theory of height n and $p > n + 1$, then for any comodules $M, N \in \mathrm{Comod}_{E_*E}$, $\mathrm{Ext}^{s,t}_{E_*E}(M, N) = 0$ for $s > n^2 + n$.

Remark 16.10. The index t gives a shift in M in this case. Recall that for M, N as E_*E-comodules, we write

$$\mathrm{Ext}^{s,t}_{E_*E}(M,N) = \mathrm{Ext}^s(M[t],N), (M[t])_a = M_{a-t}.$$

Lemma 16.11. Suppose $p > n+1$. $\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*) = 0$ for $s > n^2 + n$.

Proof Idea. This is allegedly a consequence of examining the chromatic spectral sequence and Morava' vanishing theorem. The idea is that $p > n+1$ implies $p-1$ does not divide n! In particular, this implies that the Morava stabilizer group $\mathbb{S}_n$ does not have any p-torsion, which implies that its virtual cohomological dimension (which is known to be n^2) and its actual cohomological dimension coincides. This contributes to the n^2 term, and the $+n$ comes from shifts in the spectral sequence. ■

Proof of Theorem 16.9. Let us say a E_*E comodule N is a **good target** if $\mathrm{Ext}^{s,t}_{E_*E}(E_*, N)$ for all $s > n^2+n$. By Lemma 16.11, E_* and its shifts are **good targets**. Let $0 \to M \to N \to P \to 0$ be a short exactly sequence of E_*E-comodules, this gives a long exact sequence

$$... \to \mathrm{Ext}^{s,t}(E_*, M) \to \mathrm{Ext}^{s,t}(E_*, N) \to \mathrm{Ext}^{s,t}(E_*, P) \to \mathrm{Ext}^{s+1,t}(E_*, M) \to ...$$

Suppose M is a **good target**, then we see clearly that $\mathrm{Ext}^{s,t}(E_*, N) \cong \mathrm{Ext}^{s,t}(E_*, P)$ for $s > n^2+n$, and hence N is a good target if and only if P is a good target. This implies that all finitely generated free E_*E-comodules are good targets, and hence all finitely generated comodules are **good targets**.

Any E_*E-comodule M is a filtered colimit of N_α where N_α is a finitely generated E_*E-comodule. One can check that the **filtered colimit actually commutes with** $\mathrm{Ext}^{s,t}_{E_*E}(E_*, \bullet)$. Indeed, this is because (E_*, E_*E) is a Hopf algebroid, so we can compute in particular

$$Ext^\bullet_{E_*E}(E_*, M) = H^\bullet(C_{E_*E}(M))$$

where $C_{E_*E}(M)$ is a **cobar complex** given by $M \to (E_*E) \otimes M \to (E_*E)^{\otimes 2} \otimes M \to ...$, and clearly the tensor product here commutes with filterd colimits, so the cobar complex computes with filtered colimits. This is still good up to reindexing by t, so we conclude that filtered colimit actually commutes with $\mathrm{Ext}^{s,t}_{E_*E}(E_*, \bullet)$.

Thus, this implies that M is a good target.

Now say an E_*E-comodule M is a **good source** if $\mathrm{Ext}^{s,t}_{E_*E}(M,N) = 0$ for all $s > n^2+n$ and all E_*E-comodules M. Since all E_*E-comodules are good targets, we have that E_* itself is a **good source**. We can again use the long exact sequence to check that being a good source is closed under direct sums and quotients. It turns out that E_* can generate the entire category of E_*E-comodules under those two operations, so we have that every E_*E-comodule M is a good source.

This observation translates for the statement that $\mathrm{Ext}^{s,t}_{E_*E}(M,N) = 0$ for $s > n^2+n$. ■

Observation 2: The (E-based) **Adams spectral sequence** collapses for low height.

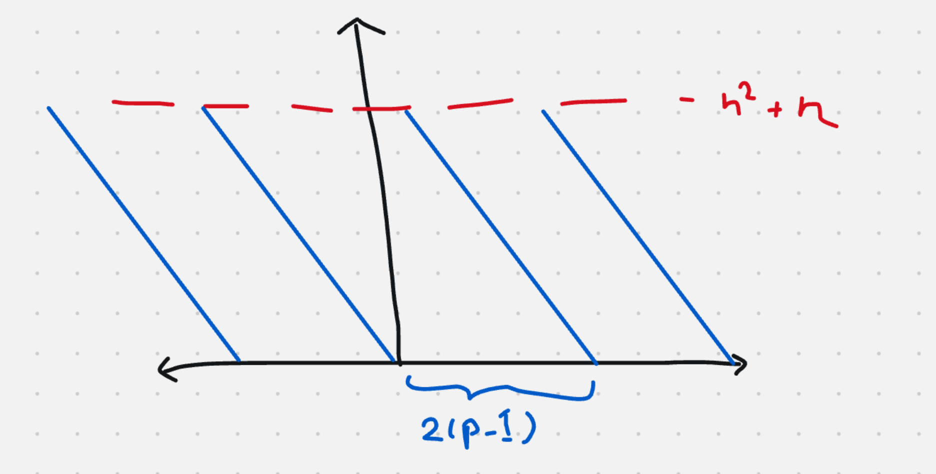
In particular, Theorem 16.9 indicates that the **Adams-Novikov spectral sequence** (ANSS) computing $L_{E(n)}\mathbb{S}$ would have a certain **vanishing line** where all the items above height n^2+n are 0. Given this **vanishing line result** in Theorem 16.9, the early observations for why we might expect chromatic homotopy theory to be algebraic at small heights compared to the primes is as follows.

Proposition 16.12. Suppose $p > n+1$ and $2p-2 > n^2+n$. the ANSS computing the homotopy groups of $L_{E(n)}\mathbb{S}$ collapses at the E_2-page.

Proof. We recall that the ANSS gives a spectral sequence gives a convergence

$$\mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, E(n)_*) \implies \pi_*(L_{E(n)}\mathbb{S}).$$

What would the E_2 page actually look like, in the Adams grading, though? Well recall from how $E(n)_*$ is defined, the degree of the smallest non-constant degree v_1 is $2(p-1) = 2p-2$. This, combined with the vanishing line result, suggests that the E_2-page of this spectral sequence could look like:



In particular, if we look at all the d_r differentials from $r = 2$ to $2(p-1)$, none of them can actually hit anything non-zero. When we get past $2(p-1) > n^2+n$, all the d_r-differentials would go above the vanishing line, which also only hits zero. Thus, the E_2-page survives to E_∞. ■

Therefore, we see that the homotopy groups of $L_{E(n)}\mathbb{S}$ (which is topological information) can be reduced down to a purely algebraic computation! (This can be done similarly for any such E).

Now to think more abstractly, we would like to extract out some more general properties of what made the proof of Proposition 16.12 worked.

- **Obervation 1:** There is a more general Adams-Novikov spectral sequence relating
$$\mathrm{Ext}^{t,s}_{E_*E}(E_*X, E_*Y) \implies [X,Y]^{t-s}.$$
- **Observation 2:** If E_*X, E_*Y are both **concentrated in** $\varphi \mod (2p-2)$ where $\varphi \in (2p-2)$ (we call such E_*E-comodules **pure of weight/phase** φ), then the same proof as in Proposition 16.12 would show that the E_2-page collapses!
- For convenience, we say a spectrum is **pure of weight/phase** φ if its homology E_*X is.

Putting the two observations more formally, we have shown that:

Lemma 16.13 (Equivalence between mapping spaces after (k-1) truncation)**.** Suppose $2p-2 > n^2+n$. Suppose X, Y are pure E-local spectra of the same phase φ, then the map $\mathrm{map}_{Sp_E}(X,Y) \to \mathrm{Hom}_{E_*E}(E_*X, E_*Y)$ induced by the homology functor is a $(2p-2-n^2-n)$-connected map of spaces.

Ideally we could like to strength the previous lemma to some statement like - there is an equivalence of category between $\mathrm{Comod}^{\varphi}(E_*E)$ (the subcategory of pure E_*E-comodules of phase φ) and $h_k\,\mathrm{Sp}^{\varphi}_E$ (the subcategory of E-local spectrum of pure weight/phase φ).

The previous lemma gives the higher category notion of **fully faithful**-ness. We would like a correspondent statement of essential sujectivity. This is done using **Goerss-Hopkins ohbstruction theory**.

Definition 16.14. A E-local spectrum X is **split** if it is a finite direct sum of pure spectrums X_i of some phases.

Proposition 16.15. Let $2p > n^2 + n$ and $p > n + 1$, a E_*E-comodule M can be realized as $E_*X \simeq M$ where X is split.

Proof. Since we are choosing $E(n)$ in particular, which is concentrated in degrees divisible by $q = 2(p-1)$. For purely algebraic reasons, every E_*E-comodule M can be written as a finite direct sum of pure comodules. Thus, without loss we can assume M is pure. Now, by general **Goerss-Hopkins obstruction theorem**, the obstructions for realizing M as a homology of some spectrum are in the Ext groups

$$\operatorname{Ext}^{k+2,k}_{E_*E}(M, M), k \geq 1.$$

Now by the vanishing line result in Proposition 16.12, when $k + 2 \geq (2p-2) + 2 = 2p > n^2 + n$, the Ext groups all vanish. When $k < 2p - 2$, these Ext groups vanish because M is pure. This concludes the proof. ■

This shows the **essential surjectivity part of the result**! Putting them together, we have that

Theorem 16.16. Let $\varphi \in \mathbb{Z}/q$ with $q = 2(p-1)$ and $k = 2p - 2 - n^2 - n$. The homology functor $E_* : \mathrm{Sp}_E \to \mathrm{Comod}_{E_*E}$ induces an equivalence between $h_k \mathrm{Sp}^{\varphi}_E \simeq \mathrm{Comod}^{\varphi}_{E_*E}$.

Question 16.17. This establishes an equivalence between each phase, how do we build them together?

To do this, we consider a construction known as **Bousfield splitting**.

Definition 16.18. Let $R^{\varphi} : \mathrm{Comod}^{\varphi}_{E_*E} \to h_k \mathrm{Sp}^{\varphi}_E$ denote the inverse of the previously constructed equivalence. We define the **Bousfield splitting functor** as

$$\beta : \mathrm{Comod}_{E_*E} \to h_k \mathrm{Sp}_E, \beta(\bigoplus_{\varphi \in \mathbb{Z}/q} M^{\varphi}) = \bigvee_{\varphi \in \mathbb{Z}/q} R^{\varphi}(M^{\varphi})$$

(Here M^{φ} is the phase φ part of M). The functor β defines a partial inverse to E_* in the sense that $E_*(\beta M) = M$, but the image of β only contains split spectra.

Remark 16.19. Although we have not explained how to prove the theorem yet, the Bousfield splitting functor β itself is already quite useful. For example, it was used to show that (see Section 16.5):

If $2p - 2 > n^2 + n$, then $\mathrm{Pic}(\mathrm{Sp}_{K(n)})$ is isomorphic to $\mathrm{Pic}(E^{\vee}_* E)$ where $E^{\vee}_* E = \pi_* L_{K(n)}(E \wedge E)$. Here E is the Morava E-theory.

16.3 Goerss Hopkins Towers and Intermediate Categories

So far, we have seen the construction of a Bousfield splitting functor $\beta : \mathrm{Comod}_{E_*E} \to h_t \mathrm{Sp}_E$, which tried to generalize some of the preliminary observations we made about algebraicity. The next question is - how do we prove this in general?

Again, we cannot make any direct comparisions between Sp_E and $\mathcal{D}(E_*E)$, as the two categories are not related. What Piotr Pstrągowski considered is to use Goerss-Hopkins theory and synthetic spectra to construct the following:

1. For Sp_E, a tower of ∞-categories:

$$Sp_E = \mathcal{M}_\infty^{top} \to ... \to \mathcal{M}_1^{top} \to \mathcal{M}_0^{top}$$

such that $Sp_E \simeq \lim_{\leftarrow} M_\ell^{top}$, M_ℓ^{top} is an $(\ell+1)$-category, $M_0^{top} \simeq \mathrm{Comod}_{E_*E}$, and the functor $Sp_E \to \mathcal{M}_0^{top}$ can be identified as the usual homology functor.

2. For $\mathcal{D}(E_*E)$, a tower of ∞-categories

$$\mathcal{D}(E_*E) = \mathcal{M}_\infty^{alg} \to ... \to \mathcal{M}_1^{alg} \to \mathcal{M}_0^{alg}$$

such that it has similarly properties - namely. $\mathcal{D}(E_*E) \simeq \lim_{\leftarrow} M_\ell^{alg}$, M_ℓ^{alg} is an $(\ell+1)$-category, $M_0^{alg} \simeq \mathrm{Comod}_{E_*E}$, as well as identifications with the usual homology functor.

3. (**Obstructions to Lifting**) Furthermore, for $X \in \mathcal{M}_{\ell-1}$ (here we omit the superscript to indicate it works for both), there are obstruction in $\mathrm{Ext}_{E_*E}^{\ell+2,\ell}(u_0X, u_0X)$ which vanishes if and only if X can be lifted to $\mathcal{M}_\ell^{top}$ (here u_0 is the map given in the tower from $\mathcal{M}_{\ell-1} \to \mathcal{M}_0$).

4. (**Fiber Sequences**) For $X, Y \in \mathcal{M}_\ell$ with $\ell \geq 1$, there is a fiber sequence

$$\mathrm{map}_{\mathcal{M}_\ell}(X,Y) \to \mathrm{map}_{\ell-1}(u_{\ell-1}X, u_{\ell-1}Y) \to \mathrm{map}_{\mathcal{D}(\mathrm{Comod}_{E_*E})}(u_0X, \Sigma^{\ell+1}u_0Y[-\ell]).$$

Some helpful sources for the tower constructions are in the original papers of Goerss-Hopkins, and a reformulation/enhancement of the Goerss-Hopkins obstruction theory in a prestable ∞-category equipped with a suitable periodicity operator [PV22], which recovers the results of Goerss-Hopkins in the ∞-category of synthetic spectra.

Proof of Theorem 16.6. The proof of Theorem 16.6 is given in the following two steps.

1. When $p > n+1$ and fix k, write $\ell = n^2+n+k-1$, there is an equivalence between the **homotopy k-categories** of $\mathcal{M}_\infty^{top}$ (resp. $\mathcal{M}_\infty^{alg}$) with $\mathcal{M}_\ell^{top}$ (resp. $\mathcal{M}_\ell^{alg}$). In other words, we have equivalences

$$h_k \mathrm{Sp}_E \simeq h_k\mathcal{M}_\ell^{top} \text{ and } h_k\mathcal{D}(E_*E) \simeq h_k\mathcal{M}_\ell^{\mathrm{alg}}.$$

2. For $p > n^2+n+1+\frac{k}{2}$, the Bousfield splitting functor β induces an equivalence of categories between $\mathcal{M}_\ell^{\mathrm{top}}$ and $\mathcal{M}_\ell^{\mathrm{alg}}$.

These chains of equivalences would give a chain of equivalences resulting in $h_k \mathrm{Sp}_E \simeq h_k\mathcal{D}(E_*E)$. ■

In this section, let us examine how Step 1 works.

Proof of Step 1. Write $\mathcal{M}_\ell$ to either indicate the topological or the algebraic tower. Observe that since $\mathcal{M}_\infty = \lim_{\leftarrow} \mathcal{M}_\ell$, it suffices for us to check that $\mathcal{M}_{\ell'+1} \to \mathcal{M}_{\ell'}$ is an equivalence on h_k for all $\ell' \geq \ell$. Recall what this means is again - **we need an analog of - 1. fully faithfulness after appropriate truncation, and 2. essential surjecitvity.**

Rewriting k in terms of ℓ and n, we have that

$$k = (\ell + 1 - n^2 - n).$$

Consequently, for the theorem to be well-defined, we must have that $k \geq 1$, and hence $\ell \geq n^2+n$.

Let $X \in \mathcal{M}_{\ell'}$ be an object. The obstructions of lifting X to an object $X' \in \mathcal{M}_{\ell'+1}$ lies in the Ext group $\mathrm{Ext}_{E_*E}^{\ell'+3,\ell'+1}(u_0X, u_0X)$. However, we observe that $\ell'+3 \geq \ell+3 \geq (n^2+2)+3 > n^2+n$, thus Theorem 16.9 implies that the Ext group is zero! This means that we can lift X, so the following maps are all **essentially surjective**

$$\mathcal{M}_{\ell'+1} \to \mathcal{M}_{\ell'}, \ell' \geq \ell.$$

Now for the appropriate analog of **fully faithfulness**, it suffices for us to show that the map between mapping spaces $\mathrm{map}_{M_{\ell'+1}}(X, Y) \to \mathrm{map}_{M_{\ell'}}(u_{\ell'}X, u_{\ell'}Y)$ is a k-**connected map** (which removes the homotopical information below). This is because if we take a $(k-1)$-truncation map (which removes all the homotopical information above), this would be a genuine equivalence (ie. fully faithfullness). To do this, we need to invoke the fiber sequence axiom, which to recall, gives us

$$\mathrm{map}_{\mathcal{M}_{\ell'+1}}(X, Y) \to \mathrm{map}_{\ell'}(u_{\ell'}X, u_{\ell'}Y) \to \mathrm{map}_{\mathcal{D}(\mathrm{Comod}_{E_*E})}(u_0X, \Sigma^{\ell'+2}u_0Y[-\ell'-1]).$$

Now recall the derived ∞-category $\mathcal{D}(\mathrm{Comod}_{E_*E})$ have mapping spaces whose homotopy groups are exactly the Ext-groups, ie.

$$\pi_s(\mathrm{map}_{\mathcal{D}(\mathrm{Comod}_{E_*E})}(u_0X, \Sigma^{\ell'+2}u_0Y[-\ell'-1])) = \mathrm{Ext}_{E_*E}^{\ell'+2-s,-\ell'-1}(u_0X, u_0Y)$$

(after accounting for appropriate shifting).Now for $s \leq k = (\ell + 1 - n^2 - n)$, we have that

$$\begin{aligned} \ell' + 2 - s &\geq \ell' + 2 - k \\ &= \ell' + 2 - (\ell + 1 - n^2 - n) \\ &= \ell' - \ell + 1 + n^2 + n \\ &\geq n^2 + n + 1 \end{aligned}$$

Hence it is above the vanishing line, so the **Ext group vanishes**! This means that the map $\mathrm{map}_{\mathcal{M}_{\ell'+1}}(X, Y) \to \mathrm{map}_{\ell'}(u_{\ell'}X, u_{\ell'}Y)$ is k-connected!

This concludes the proof of this step. ■

16.4 Equivalence of Intermediate Categories

It remains for us to prove the equivalence between the two intermediate categories (Step 2). The proof uses synthetic spectra, and we will give an overview of the specific proof of Step 2 without being too detailed.

Here are some facts about these synthetic spectra. Here, a **synthetic spectrum** refers to a hypercomplete connective synthetic spectrum based on E, so it is a hypercomplete spherical sheaf of spaces on the site Sp_E^{fp} of finite E_*-projective spectra. We use Syn denotes the ∞-category of synthetic spectrum.

Definition 16.20. Let $\mathrm{Syn}^{\heartsuit}$ denote the heart of Syn (wrt to its canonical t-structure). This is also called the subcategory of discrete objects in Syn. one can check that $\mathrm{Syn}^{\heartsuit}$ is equivalent to Comod_{E_*E}.

The ∞-category Syn is graded symmetric monoidal with monoidal unit $\mathbb{1}$, and admits a tower of module ∞-categories

$$\mathrm{Syn} \to \ldots \to \mathrm{Mod}_{\mathbb{1}_{\leq 1}}(\mathrm{Syn}) \to \mathrm{Mod}_{\mathbb{1}_{\leq 0}}(\mathrm{Syn})$$

induced by the tower of commutative algebras $\mathbb{1} \to \ldots \to \mathbb{1}_{\leq 1} \to \mathbb{1}_{\leq 0}$ (which is the Postnikov truncation of the unit). Note that by construction $\mathbb{1}_{\leq t}$ is the **cofiber** of τ^{t+1} (ie. $c\tau^{t+1}$), and $\mathrm{Mod}_{\mathbb{1}_{\leq t}}(\mathrm{Syn})$ can be referred to as "modules over the cofiber of τ^{t+1}".

Proposition 16.21. One can check that the equivalence between $\mathrm{Syn}^{\heartsuit}$ and Comod_{E_*E} can in fact be extended to an equivalence $\mathrm{Mod}_{\mathbb{1}_{\leq 0}}(\mathrm{Syn})$ and $\mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0}$, the connective derived ∞-category of comodules.

Definition 16.22. There is a functor $\nu : \mathrm{Sp}_E \to \mathrm{Syn}$ called the **synthetic analogoue** that is a fully-faithful embedding of ∞-categories. The essential image are synthetic spectra X such that $\mathbb{1}_{\leq 0} \otimes X$ is discrete.

Proposition 16.23. $\mathcal{M}_\ell^{\mathrm{top}}$ is the ∞-category of $\mathbb{1}_{\leq \ell}$-modules X in Syn such that $\mathbb{1}_{\leq 0} \otimes_{\mathbb{1}_{\leq \ell}} X$ is discrete. (X is called a topological potential ℓ-stage).

Similarly, we can reformulate the algebraic towers as follows. It turns out there exists a certain commutative algebra $P(\mathbb{1})$, constructed by Barnes and Roitzheim, such that

$$\mathcal{D}(E_*E) \simeq \mathrm{Mod}_{P(\mathbb{1})}(\mathcal{D}(\mathrm{Comod}_{E_*E}))$$

Here $\mathcal{D}(\mathrm{Comod}_{E_*E})$ refers to the derived ∞-category of the abelian category Comod_{E_*E}. $\mathbb{P}(1)$ is called the **periodicized unit** with $P(\mathbb{1}) \simeq E_*[\tau^{\pm 1}]$ with $|\tau| = (1, -1)$..

Proposition 16.24. Let $P := P(\mathbb{1})_{\geq 0}$. M_ℓ^{alg} is the ∞-category of $P_\ell = P \otimes \mathbb{1}_{\leq \ell}$-modules M such that $P_{\leq 0} \otimes_{P_{\leq \ell}} M$ is discrete (such M is called an algebraic potential ℓ-stage).

Now we start constructing an equivalence of towers using the Bousfield splitting technology.

Proposition 16.25. There is a **monadic adjunction** (β^*, β_*) between $\mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0}$ and $\mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn})$ induced by the Bousfield splitting functor β when $2p - 2 > n^2 + n$ and $\ell \leq 2p - 3 - n^2 - n/$

Proof Sketch. Now when $2p-2 > n^2+n$ and $\ell \leq 2p-3-n^2-n$, we have constructed a Bousfield splitting functor

$$\beta : \mathrm{Comod}_{E_*E} \to h_{\ell+1}\,\mathrm{Sp}_E$$

(here we slightly shift the indices). Without loss, we restrict β to $\mathrm{Comod}_{E_*E}^{fp}$. Now, consider the functor

$$(\mathbb{1}_{\leq 0} \otimes -) \circ \nu \circ \beta : \mathrm{Comod}_{E_*E}^{fp} \to \mathrm{Mod}_{\mathbb{1}_{\leq 0}}(\mathrm{Syn}).$$

One can check this is **symmetric monoidal** and **uniquely extends** to a symmetric monoidal equivalence between $\mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0}$ and $\mathrm{Mod}_{\mathbb{1}_{\leq 0}}(\mathrm{Syn})$, and also a **symmetric monoidal cocontinuous functor**

$$\beta^* : \mathcal{D}(\mathrm{Comod}_{E_*E}) \to \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}).$$

From here we use β^* to denote the **unique cocontinuous extension of the functor** $(\mathbb{1}_{\leq \ell} \otimes -) \circ \nu \circ \beta$, which formally admits a right adjoint β_*. The pair (β^*, β_*) is called the **Bousfield adjunction**. The monadicity of this adjunction follows from an application of the **Barr-Beck-Lurie theorem**. ■

Theorem 16.26. The functor $\beta_* : \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) \to \mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0}$ lifts to a symmetric monoidal adjoint equivalence (γ^*, γ_*) between $\mathrm{Mod}_{\beta_*\mathbb{1}_{\leq \ell}}(\mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0})$ and $\mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn})$.

We omit the proof here, but the monadicity of (β^*, β_*) is crucial in the proof!

Finally, we will now prove Step 2 of the construction.

Theorem 16.27. For $p > n^2 + n + 1 + \frac{k}{2}$, there is an equivalence between M_ℓ^{top} and M_ℓ^{alg}, where $\ell = n^2 + n + k - 1$.

Proof Sketch. The condition on the prime allows us to construct the initial map β, so we can use the previous theorem. In the construction of the equivalence γ^*'s, we have a map

$$\gamma^* : \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) \to \mathrm{Mod}_{\beta_*(\mathbb{1}_{\leq \ell})}(\mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0})$$

which is an equivalence. If we tensor by the discretization of the unit, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) \arrow[r, "\gamma^*"] \arrow[d, "\mathbb{1}_{\leq 0}\otimes_{\mathbb{1}_{\leq \ell}} -"'] & \mathrm{Mod}_{\beta_*(\mathbb{1}_{\leq \ell})}(\mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0}) \arrow[d, "(\beta_*\mathbb{1}_{\leq \ell})_{\leq 0}\otimes_{\beta_*(\mathbb{1}_{\leq t})} -"] \\
\mathrm{Mod}_{\mathbb{1}_{\leq 0}}(\mathrm{Syn}) \arrow[r] & \mathrm{Mod}_{(\beta_*(\mathbb{1}_{\leq \ell}))_{\leq 0}}(\mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0})
\end{tikzcd}$$

It turns out that from **the theory of synthetic spectrum**, the associative algebras $\beta_*\mathbb{1}_{\leq \ell}$ and $P_{\leq \ell}$ are actually "equivalent" in the following sense. The fact that (β^*, β_*) is **monadic** intuitively means that we can think of $\mathbb{1}_{\leq \ell}$ as some module over the algebra in $\mathbb{P}(1)$ via β_*. Now, we see that both $\beta_*\mathbb{1}_{\leq \ell}$ and $P_{\leq \ell}$ are both truncations of the free associative algebra on $\Sigma E_*[-1]$.

$\tau_{\leq \ell+1}(\mathbb{1}_{\leq 0}\{\tau\})$ is the free associative algebra, there is only one object with π_* equal to

$$\mathbb{1}_{\leq 0}[\tau]/\tau^{\ell+1}.$$

So in particular, we can without loss do some replacement to get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) \arrow[r, "\delta^*"] \arrow[d, "\mathbb{1}_{\leq 0}\otimes_{\mathbb{1}_{\leq \ell}} -"'] & \mathrm{Mod}_{P\leq \ell}(\mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0}) \arrow[d, "P_{\leq 0}\otimes_{P_{\leq \ell}} -"] \\
\mathrm{Mod}_{\mathbb{1}_{\leq 0}}(\mathrm{Syn}) \arrow[r] & \mathrm{Mod}_{P_{\leq 0}}(\mathcal{D}(\mathrm{Comod}_{E_*E})_{\geq 0})
\end{tikzcd}$$

where δ^* is also an equivalence.

Now to show that there is an equivalence between M_ℓ^{top} and M_ℓ^{alg}, we recall that they are exactly the discrete objects after appropriate tensoring on the vertical maps of both sides. Thus, we conclude that they are equivalent. ■

16.5 Applications: The Algebraicity of Chromatic Picard Groups

This section was not covered in the lecture, nor was it expected. The presenter (who is the also the notetaker) added this in to show an application of the main theorem.

Definition 16.28. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, the equivalence classes of invertible objects (w.r.t to the tensor product) often forms a set, which then has a group operation given by $\otimes$. This group is called the **Picard group** $\mathrm{Pic}(\mathcal{C})$.

In this section, we use E to denote the Lubin-Tate Morava E-theory E_n of height n at prime p, ie.

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}], |u_i| = 0, |u| = 2,$$

where $W(\mathbb{F}_{p^n})$ denotes the ring of Witt vectors. Note that $\pi_0(E_n) = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]]$ is a complete local ring with unique maximal ideal $\mathfrak{m} = (p, u_1, ..., u_{n-1})$. The associated Morava K-theory K of E_n is the unique E_*-module such that

$$\pi_*(K) = \pi_*(E)/\mathfrak{m}.$$

Note that K is Bousfield equivalent to the usual $K(n)$.

Definition 16.29. Let X be a spectrum, its completed E-homology is

$$E_*^\vee(X) := \pi_* L_{K(n)}(E \wedge X).$$

The natural map $X \mapsto E_*^\vee(X)$ sends $K(n)$-invertible spectrum to $E_*^\vee(E)$-invertible comodules. Thus, we have a map

$$\phi : \mathrm{Pic}(\mathrm{Sp}_{K(n)}) \to \mathrm{Pic}(E_*^\vee E).$$

Classically, due to sparsitiy in the K-local ANSS, it is already known that:

Theorem 16.30. If $2p - 2 > n^2$ and $(p-1)$ does not divide n, then the map ϕ is injective.

The theorem that Piotr Pstrągowski proved is as follows.

Theorem 16.31 (Theorem 1.1 of [Pst22])**.** If $2p - 2 > n^2 + n$, the map ϕ is an isomorphism.

Note that theorem does not take full use of Theorem 16.6, so the bounds on p and n are less restrictive. The key construction used in the proof is instead the Bousfield splitting functor $\beta : \mathrm{Comod}_{E_*E} \to h\,\mathrm{Sp}_E$.

Proof Sketch. Note that $2p-2 > n^2+n$ implies $2p-2 > n^2$ and $(p-1)$ does not divide n, so the map ϕ is injective. It remains to prove surjectivity.

Let $M \in \mathrm{Pic}(E_*^\vee E)$. Note that for technical reasons we omit here, such M, viewed as an E_*-module, must always be free of rank 1. Now define $X_k := L_K(\beta M/\mathfrak{m}^k M)$ (recall $\mathfrak{m}$ is the unique maximal ideal in $\pi_0(E)$). Since β is a functor, we can define a map $X_k \to X_{k-1}$ that corresponds to the quotient map $M/\mathfrak{m}^k \to M/\mathfrak{m}^{k-1}$. Now we take X to be the limit $X = \lim_{\leftarrow} X_i$ (specifically, we take a lift of the tower into Sp_K and hence compute its limit, one can check the limit is independent of such lifts).

This X is the desired element in $\mathrm{Pic}(\mathrm{Sp}_{K(n)})$ such that $E_*^\vee(X) = M$! Indeed, the intermediate X_k's are dualizable, and one can check that $X = D(\lim_{\to} DX_k)$, where D denotes the $K(n)$-local Spanier-Whitehead dual. The invertibility of $\lim_{\to} DX_k$ implies X is invertible, the fact that M, viewed as an E_*-module, is free of rank 1, implies that $K_*(\lim_{\to} DX_k) \simeq K_*$, which will imply invertibility of the limit.

Now finally, since X_k and X are both invertible after localizing at K, we have that $L_k(E \wedge X) \simeq \lim_{\leftarrow} L_K(E \wedge X_k)$. In homotopy groups, this gives a Milnor exact sequence of the form

$$0 \to \underset{\leftarrow}{\overset{1}{\lim}}(M/\mathfrak{m}^k M)[-1] \to E_*^\vee(X) \to \underset{\leftarrow}{\lim}\, M/\mathfrak{m}^k M \to 0,$$

where the $\lim^1$ term vanish since M is free of rank 1. Thus, we have that $E_*^\vee(X) \cong \lim_{\leftarrow} M/\mathfrak{m}^k M \cong M$. This proves surjectivity. ■

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17 Lecture 14: Multiplicative Structure on Quotient Ring Spectra (by Emma Brink)

In this talk, we will present a result of Burkland [Bur22].

Theorem 17.1. Let $R \in \mathrm{CAlg}(Sp)$ (ie. an $\mathbb{E}_\infty$-ring spectrum). Let $x \in R$ and suppose R/x admits a left unital multiplication, then R/x^{n+1} has the structure of an $\mathbb{E}_n$-algebra for $n \geq 1$.

There is indeed a slight more general theorem as follows.

Theorem 17.2. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ (ie. presentably symmetric monoidal stable ∞-category) such that $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^\omega)$ (where $\mathcal{C}^\omega \subseteq \mathcal{C}$ is a monoidal subcategory). Consider the map $x \to 1_\mathcal{C} \in \mathcal{C}^\omega$ such that $1_\mathcal{C}/x$ has left unital multiplication, then $1_\mathcal{C}/x^{n+1}$ admits an $\mathbb{E}_n$-algebra structure for $n \geq 1$.

To derive the first theorem, we take the case where $\mathcal{C} = \mathrm{LMod}_R(\mathrm{Sp})$.

The general **strategy** for prove the second theorem is as follows:

- One exames the map $1_\mathcal{C}/x^{n+1} \to 1_\mathcal{C}/^{\mathbb{E}_n}x^{n+1}$.
- We would like to "attach cells" to kill difference (ie. form pushouts with free algebras in $\mathbb{E}_n$-algebras here).
- To do this, we describe conditions on $\mathcal{C}$ under which this is possible (this involves an obstruction theory), and then "deform" $\mathcal{C}$ with categories satisfying the aforementioned conditions.

17.1 Obstruction Theory

In this section we fix $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ and consider the map $x : X \to 1_\mathcal{C}$. We saw this morning at there is a functor

$$(-)_{\tau^{-1}} : \mathcal{C}^{\mathrm{fil}} \to \mathcal{C}, x \mapsto \operatorname{colim}_n X_n.$$

and consider the diagram:

$$\begin{tikzcd}
& \mathcal{C}^{fil} \arrow[ld] \arrow[rd, "(-)_{\tau^{-1}}"] & \\
\mathcal{C}^{Gr} & X_\bullet \arrow[ld] \arrow[rd] & \mathcal{C} \\
(\frac{X_n}{X_{n-1}})_n & 1_\mathcal{C}/\tau X \arrow[ld] \arrow[rd] & \mathrm{colimit}_n\, X_n \\
1_\mathcal{C} \oplus \Sigma X(1) & 1_\mathcal{C}/^{\mathbb{E}_n}\tau X \arrow[ld, mapsto] \arrow[rd] & 1_\mathcal{C}/X \\
1_{\mathcal{C}^{Gr}}\{\Sigma X(1)\} = 1_\mathcal{C} \oplus \Sigma X(1) \oplus (\text{something with } \deg \geq 2) & & 1_\mathcal{C}/^{\mathbb{E}_n}X
\end{tikzcd}$$

Note here that $1_\mathcal{C} \oplus \Sigma X(1)$ admits an $\mathbb{E}_n$-algebra structure, and this can obtained by attaching cells to $1_{\mathcal{C}^{\mathrm{Gr}}}\{\Sigma X(1)\}$ to kill differences in the incoming degrees successively.

Now we outline a construction as follows.

Definition 17.3. Let $R^1 := 1_{\mathcal{C}^{\mathrm{Gr}}}\{\Sigma X(1)\}$. By the diagram above this has a natural map $R^1 \to 1_{\mathcal{C}} \oplus \Sigma X(1)$. Suppose by induction that we have constructed some tower

$$(R^1 \to R^2 \to \dots \to R^{k-1}) \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}^{Gr})/(1_{\mathcal{C}} \oplus \Sigma X(1))$$

and $R^{k-1} \to 1_{\mathcal{C}} \oplus \Sigma X(1)$ is an equivalence in degree $\leq k-1$. We construct the fiber

$$X_k := \mathrm{Fib}((R^{k-1})_k \to (1_{\mathcal{C}} \oplus \Sigma X(1))_k).$$

From here we define R^k and the specified map $R^k \to 1_{\mathcal{C}} \oplus \Sigma X(1)$ as the following pushout diagram and the map is given by universal property:

$$\begin{tikzcd}
1_{\mathcal{C}^{Gr}}\{X_k(k)\} \arrow[r] \arrow[d] & 1_{\mathcal{C}^{Gr}} \arrow[d] \arrow[rdd] & \\
R^{k-1} \arrow[r] \arrow[rrd] & R^k \arrow[rd, dashed] & \\
& & 1_{\mathcal{C}} \oplus \Sigma X(1)
\end{tikzcd}$$

Now repeating this forever, we get a map

$$\mathrm{colim}_k\, R^k \to 1_{\mathcal{C}} \oplus \Sigma X(1).$$

Furthermore, based on how we constructed, this map is evidently an equivalence.

Now we would like to lift this to the setting of $\mathcal{C}^{\mathrm{Fil}}$. Let

$$\tilde{R}^1 := 1_{\mathcal{C}}^{fil_{\mathbb{E}_n}}/\tau X.$$

We seek to construct a sequence of maps

$$\tilde{R}^1 \to \tilde{R}^2 \to \dots \tilde{R}^k \to \dots$$

such that for all k we have an induced map of pushout squares

$$\begin{tikzcd}
1_{\mathcal{C}^{Fil}}\{X_k(k)\} \arrow[r] \arrow[d, "\tilde{S}_k"] & 1_{\mathcal{C}^{Fil}} \arrow[d] & {} \arrow[r, "gr_*"'] & {} & 1_{\mathcal{C}^{Gr}}\{X_k(k)\} \arrow[r] \arrow[d] & 1_{\mathcal{C}^{Gr}} \arrow[d] \\
\tilde{R}^{k-1} \arrow[r] & \tilde{R}^k & & & R^{k-1} \arrow[r] & R^k
\end{tikzcd} .$$

This is constructed in a way such that

$$\mathrm{gr}_*(\mathrm{colim}_k\, \tilde{R}^k) = \begin{cases} 1 \oplus \Sigma X(1), * = 1 \\ 0, * \geq 2 \end{cases} .$$

This implies that $\mathrm{colim}_k\, \tilde{R}^k$ is an $\mathbb{E}_n$-algebra of $1_{\mathcal{C}}/\tau v$ and note also that $\mathrm{gr}_0(\mathrm{colim}_k\, \tilde{R}^k) = 1_{\mathcal{C}}$.

Now we describe how to build the tower.

Definition 17.4. Suppose by induction that we have constructed the sequence

$$\tilde{R}^1 \to \dots \to \tilde{R}^{k-1}.$$

We now consider a lifting problem

$$\begin{tikzcd}
X_k \arrow[r, dashed, "\exists \tilde{s}_k?"] \arrow[dr, "s_k"] & (\tilde{R}^{k-1})_k \arrow[d] \\
& (R^{k-1})_k = \frac{(\tilde{R}^{k-1})_k}{(\tilde{R}^{k-1})_{k-1}} \arrow[d, "\partial"] \\
& \Sigma(\tilde{R}^{k-1})_{k-1}
\end{tikzcd}$$

It turns out that we have an obstruction criterion as follows:

Proposition 17.5. A lift of such $\tilde{s}_k$ exists if and only if $\partial \circ s_k = 0$ as an element in $[X_k, \Sigma(\tilde{R}^{k-1})_{k-1}]$.

Warning: The map s_k depends on $\tilde{s}_i$ for $i \leq i - 1$.

Thus, in order to produce this valid lift, we ought to examine the term $[X_k, \Sigma(\tilde{R}^{k-1})_{k-1}]$:

- For $j \leq k - 1$, one has that $\mathrm{gr}_j(\tilde{R}^{k-1}) = R_j^{k-1} = (1 \oplus \Sigma X(1))_j$, which is $= 0$ when $2 \leq j \leq k - 1$. This implies that
$$(\tilde{R}^{k-1})_{k-1} = (\tilde{R}^{k-1})_1 = 1_{\mathcal{C}}/X.$$
- Write X_k now as $X_k = \Omega^{1+n} D_k^n({}^{n+1}X)$, where D_k^n is defined as the composition $\mathcal{C} \to \mathrm{Fun}(\mathbb{E}_n(k)//\Sigma_k, \mathcal{C}) \xrightarrow{\lim} \mathcal{C}$, where $\mathbb{E}_n(k)//\Sigma_k$ is the unordered configuration space of k-points in $\mathbb{R}^n$.

By these two bullet points in mind, we see that

$$[X_k, \Sigma(\tilde{R}^{k-1})_{k-1}] \cong [\Omega^{1+n} D_k^n({}^{n+1}X), \Sigma 1_{\mathcal{C}}/X].$$

Now the idea is to use some creative Bar and CoBar arguments to:

- Show inductively $(\mathrm{Bar}^{(n)} R^{k-1})_k = 0$.
- If $x \to y$ is a morphism in $\mathrm{CAlg}(\mathcal{C}_{\geq 0}^{Gr})$ such that $x \to y$ is an euqivalence in degree $\leq k$, then the map
$$\mathrm{Bar}^{(n)} x \to \mathrm{Bar}^{(n)} y$$
is an equivalence in degree $\leq k$.
- and other technical arguments

Eventually the point is that the one can show the following quantities are all zero for $0 \leq s \leq t := (k-1)(n-1)$,

$$\pi_0 \, \mathrm{map}(\Omega^{1+n+t-s}(\Sigma^{n+1}X)^{\otimes k}, 1_{\mathcal{C}}/x] = 0$$

then there is a surjection

$$0 \to [\Omega^{1+n} D_k^n(\Sigma^{n+1}X), 1_{\mathcal{C}}/x],$$

so the term is 0, and hence the term $1_{\mathcal{C}}/x$ has an $\mathbb{E}_n$-algebra structure. Here the number $(k-1)(n-1)$ comes from the **Fox-Neuwirth cell decomposition structure of** $\mathbb{E}_n(k)//\Sigma_k$.

Summarzing the arguments above, we have:

Theorem 17.6. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ and $x : X \to 1_{\mathcal{C}}$. Let $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}_{\geq 2}$ and for all $t \in \{0, ..., (n-1)(k-1)\}$, the following term

$$[\Omega^{1+n+t}(\Sigma^{n+1}X)^{\otimes k}, \Sigma 1_{\mathcal{C}}/X] = 0,$$

then $1_{\mathcal{C}}/X$ admits an $\mathbb{E}_n$-algebra structure.

Remark 17.7. The structure here is also unique.

Remark 17.8. Following Emma's notes faithfully, she then stated, quote "vanishing is very restrictive ☹!", refering to the vanishing condition in Theorem 17.6.

17.2 Deforming $\mathcal{C}$

This leads to our next section, where we would like to try to "deform" the set-up. Now we suppose $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^\omega) \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ where $\mathcal{C}^\omega \subseteq \mathcal{C}$ admits a monoidal ∞-subcategory structure, and we have a map $x : X \to 1_{\mathcal{C}} \in \mathcal{C}^\omega$ where $1_{\mathcal{C}}/X$ admits a left unital multiplication.

We incite the following theorem.

Theorem 17.9 (Burkland, Patchkoria-Pstragowski [PP23])**.** Let $\mathcal{C}$ be as above, then there exists a commutative diagram

$$\begin{tikzcd} & \mathrm{Def}(\mathcal{C}, x) \arrow[dr, "\tau^{-1}"] & & \in \mathrm{CAlg}(\mathrm{Cat}) \\ \mathcal{C} \arrow[ur, "v"] \arrow[rr, "id"'] & & \mathcal{C} & \end{tikzcd}$$

such that the following holds:

1. The map v preserves filtered colimits and is fully faithful on $\mathcal{C}^\omega$.
2. τ^{-1} preserves colimits.
3. $\mathrm{Def}(\mathcal{C}, x) \in \mathrm{Pr}^L_{st}$.
4. And very importantly, $v(1_{\mathcal{C}}/x^q)$ admits an $\mathbb{E}_n$-algebra structure for all $q > n$ (more refinedly, $v(1_{\mathcal{C}}, x^q) = 1/\tilde{x}^q$ where the conditions in Theorem 17.6 vanishes).

Proof. (Note: Burkland constructs a different deformation).

Write $\mathcal{P}^X_\Sigma(\mathcal{C}^\omega)$ as the full ∞-subcategory (in $\mathcal{P}(\mathcal{C}^\omega)$) of functors $F : (\mathcal{C}^\omega)^{op} \to \mathrm{An}$ such that

- $\mathcal{F}$ peserves finite products,
- For all $p : a \to b$ in $\mathcal{C}^\omega$ such that $p \otimes 1/X$ splits, we have that

$$F(b) = \lim_\Delta F(\check{\mathcal{C}}(p)).$$

Now consider the diagram

$$\begin{tikzcd}
\mathcal{P}^X_\Sigma(\mathcal{C}^\omega) \arrow[dr, "\subseteq"] & & \\
& \mathcal{P}(\mathcal{C}^\omega) \arrow[ul, dashed, bend right, "\text{symmetric localization}"'] \arrow[dr] & \\
\mathcal{C}^\omega \arrow[uu, dashed] \arrow[ur] \arrow[rr, hook] & & \mathcal{C}
\end{tikzcd}$$

This induces a diagram

$$\begin{tikzcd}
& \mathrm{Sp}(\mathcal{P}^X_\Sigma(\mathcal{C}^\omega)) \arrow[ddr, "\tau^{-1}"] & \\
& \mathcal{P}^X_\Sigma(\mathcal{C}^\omega) \arrow[u, hook, "\phi"] & \\
\mathcal{C} = \mathrm{Ind}(\mathcal{C}^\omega) \arrow[uur, "v"] \arrow[rr, bend right, "id"'] & \mathcal{C}^\omega \arrow[l] \arrow[u, hook] \arrow[r] & \mathcal{C}
\end{tikzcd}$$

such that ϕ is fully-faithful, and hence $\mathcal{P}^X_\Sigma(\mathcal{C}^\omega)$ is presentable and the map preserves colimits.

Now by construction, if $a \to b \xrightarrow{p} c$ is a cofiber sequence in $\mathcal{C}^\omega$ such that $p \otimes 1/x$ splits, then $va \to vb \to vc$ is also a cofiber sequence. Since $1_\mathcal{C}/x$ has a left unital multiplication, the special case of the previous statement shows us that

$$v1_\mathcal{C} \to v1_\mathcal{C}/x \to v\Sigma X \text{ is a cofiber sequence .}$$

From here we consider $\tilde{x} : \tilde{X} \coloneqq \Omega(v\Sigma X) \to v1_\mathcal{C}$. By stability, we have that $1_\mathcal{C}/\tilde{x} \simeq v(1_\mathcal{C}/x)$ and

$$1_\mathcal{C}/\tilde{x}^q \simeq v(1_\mathcal{C}/x^q).$$

It remains for us to verify the following claim.

Claim: For $q > n$, we have that $1_\mathcal{C}/\tilde{x}^q$ has an $\mathbb{E}_n$-algebra structure.

Indeed, we seek to apply Theorem 17.6 to this set-up. Thus, we should like to verify that for $0 \le r \le (n-1)(k-1)$, we have that

$$[\Omega^{2+n+r}(\Sigma^{n+1}\Omega^q(v\Sigma^q X))^{\otimes k}, 1_\mathcal{C}/\tilde{x}^q] = 0.$$

Now observe that we can rewrite the left term as

$$\Omega^{2+n+r-k(n+1-q)} v(\Sigma^q X)^{\otimes k}.$$

The obstruction vanishing actually follows from a more general fact we will show here - that is, for any $y \in \mathcal{C}^\omega$ and $s \ge q$, we have

$$[\Omega^s vy, 1_\mathcal{C}/\tilde{x}^q] = 0.$$

Indeed, for any $p : a \to b$ in $\mathcal{C}^\omega$ such that $\rho \otimes 1/x$ admits a section. It this case, for all $T \in \mathcal{C}^\otimes$, the induced map by p yields a surjection

$$p^* : [-, 1/x \otimes T] \to [-, 1/x \otimes T].$$

To see why, we note that for the map $a \otimes 1/x \to b \otimes 1/x$, we have a section s going back. Given a map $t : a \to 1/x \otimes T$, consider the diagram

$$\begin{tikzcd}
a \arrow[r, "p"] \arrow[d] & b \arrow[d] & & \\
a \otimes 1/x \arrow[r] & b \otimes 1/x \arrow[r, "s"'] & a \otimes 1/x \arrow[r, "t\otimes 1/x"] & (1/x \otimes T) \otimes 1/x \arrow[d, "\text{swap}"'] \\
& & & T \otimes (1/x \otimes 1/x) \arrow[d, "m"'] \\
& & & T \otimes 1/x
\end{tikzcd}$$

We see that the composition here is **actually** t **again**, and because we had the term p at the beginning, this verified surjectivity.

Now applying it to p arising from the left unital multiplication and using the surjection above, we necessarily have that for all $y, T \in \mathcal{C}^\omega$ and $s > 0$,

$$\Omega^s vy, v(1/x) \otimes vT] = 0.$$

The proof now following from induction using the cofiber sequence

$$[(\Omega v\Sigma X)^{\otimes q-1} \otimes v1_{\mathcal{C}}/\tilde{x} \to 1_{\mathcal{C}}/\tilde{x}^q \to 1_{\mathcal{C}}/\tilde{x}^{q-1}.$$

■

18 The Second Q and A Session (by Gijs Heuts and Ishan Levy)

Question 18.1. How do we translate behind the language of stacks and Hopf algebroids?

The most common pairs of Hopf algebroids we have in homotopy theory as (MU_*, MU_*MU) and (E_*, E_*E). We should really think of them as the beginnings of some simplicial object, ie.

$$\begin{tikzcd} MU_* \arrow[r, shift left=2] \arrow[r, shift right=2] & MU_*(MU) \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \ldots \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

Formally Hopf algebroids are the co-groupoid objects in the category of rings. The framework of stacks are really interchangeable with Hopf algebroids. Classically many constructions were said in the setting of Hopf algebroids, now they are mostly formulated in a stacky context. The reason why is because when we want to extend elliptic cohomology over non-affine covers, the language of Hopf algebroids become quite cumbersome, and hence we need the language of stacks.

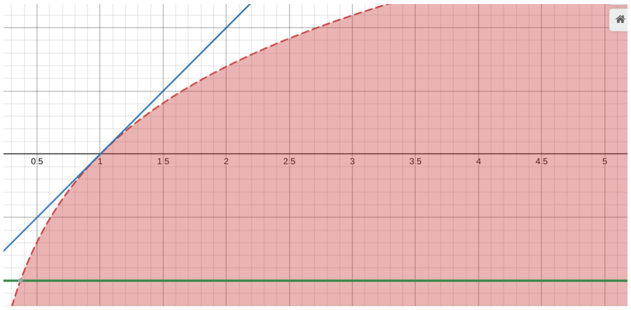
In particular $(MU_*, MU_*(MU))$ corepresents a functor $\mathrm{CRing} \to \mathrm{Gpd}$. Just like how modules over a ring correspond to quasicoherent sheaves over it, we can take that as the definition of quasicoherent sheaves over stacks in this context. At the end of the day, the stacky language is just a language. (COCTALOS [Hop99] is a good reference).

Question 18.2. What are the vanishing curves for the Adams spectral sequence?

Instead of answering this question, we will look at some context around this, which is - **what is the relationship between the ANSS and the nilpotence theorem**? Well the ANSS is a spectral sequence

$$E_2 = H^\bullet(M_{fg}; \omega^\bullet) \implies \pi_\bullet \mathbb{S}.$$

If we look at the E_∞-page of the ANSS, because MU detects nilpotence, it tells us that there is a **sublinear** vanishing curve (here sublinear means the curve grows slower under a line) on the E_∞-page such that everything above it is zero! Indeed, in this case we have everything above the red line is zero:

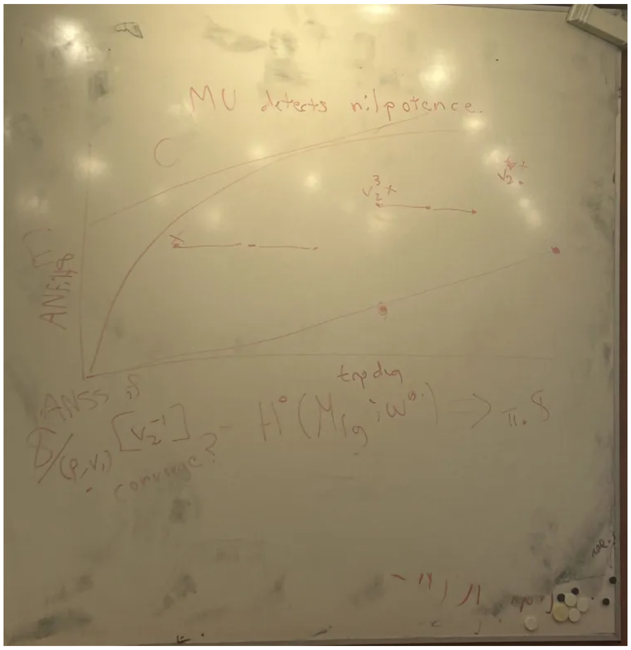


Note that the nilpotence theorem can also be deduced from this sublinear vanishing curve, but the **vanishing line result should be thought of as a more refined statement**.

1. Hopkins conjecture that the vanishing line is square root.
2. It is conjectured that the failure of the telescope conjecture may be lying on a similar curve.

On a related note, we still have a vanishing line result for $\mathbb{S}/(p, v_1)$. One can ask what about the ANSS for $\mathbb{S}/(p, v_1)[v_2^{-1}]$?

Since v_2 has Adams filtration zero, but we have these v_2 jumps that look like:



This is a **sideway parabola!**

Question 18.3. Non-examples of multiplicative structure on some Moore spectra.

Take $\mathbb{S}/2$. We claim it does not admit a left unital multiplication (and therefore. no multiplicative structure whatsoever). The reason why is because $id_{\mathbb{S}/2}$ has order 4!

If it has a left unital multiplication, we we consider given by the inclusion of a piece and multiplication:

$$\mathbb{S}/2 \to \mathbb{S}/2 \otimes \mathbb{S}/2 \to \mathbb{S}/2$$

One can check this doesn't split as $\mathbb{S}/2 \oplus \Sigma^1\mathbb{S}/2$ by checking on $H^*(-; \mathbb{F}_p)$ (by Bockstein and Cartain).

Another example is $\mathbb{S}/p^k$ are never $\mathbb{E}_\infty$. If this is $\mathbb{E}_\infty$, this implies KU/p^k is $\mathbb{E}_\infty$, which contracts the presense of a delta structure on its homotopy groups.

Question 18.4. Examples of filtrations for spectral sequences?

Let R be a ring spectrum and M, N be module spectra over R. How can we compute the homotopy groups of $M \otimes_R N$? Well, one way to do this is to consider the Postnikov truncation and look at

$$(\tau_{\geq \bullet} M) \otimes_{\tau_{\geq \bullet} R} (\tau_{\geq \bullet} N)$$

This filtration gives $\mathrm{Tor}_{\pi_\bullet R}(\pi_* M, \pi_* N) = \Sigma\pi_\bullet M \otimes_{\Sigma\pi_\bullet R} \Sigma\pi_\bullet N$.

Another good example is the homotopy fixed points spectral sequence.

Question 18.5. More discussion on operads?

The most fundamental example of operads that DOES NOT arise in spaces is that of a **spectral Lie operad** Lie. Here are a few ways to interpret this:

1. They are **Goodwillie derivatives** of the identity functor on S_* (pointed spaces). (Recall in rational homotopy theory, Quillen proved they are roughly equivalent to dg Lie algebra, but the point here is that they always show up, regardless).
2. Koszul duality: in this case we actually have that $\mathrm{Lie} \cong \mathrm{Bat}(\mathrm{Com})^\vee$ (Here $\vee$ is the Spanier Whitehead dual).
3. The terms of the operad have a concrete description with $\mathrm{Lie}(m)$ being $(\Sigma\,\mathrm{Part}^{\pm}(m)^{\diamond})^{v}$. Here $\mathrm{Part}^{\pm}(m)$ is the partition poset of $\{1, ..., n\}$ minus the trivial partition and the discrete partition. $\diamond$ denotes unreduced suspension, so then we can suspend, and then take its Spanier-Whitehead dual.

Question 18.6. Fancy way to see $\mathbb{E}_\infty$ structure on Lubin-Tate Morava E-theory E_n?

We saw this category of syntehtic spectra Syn_E for a general E. Let us specialize to the case $E = MU$. In this case we admits maps

$$\mathrm{Sp} \xleftarrow{\tau^{-1}} \mathrm{Syn}_{MU} \xrightarrow{-\otimes c\tau} \mathrm{IndCoh}(M_{fg})$$

To construct Morava E-theory, what we can try is to construct an $\mathbb{E}_\infty$-ring in Syn_{MU} and send it through τ^{-1} to get to Morava E-theory.

To do this, we can make a stack on connected $\mathbb{E}_\infty$-rings, what we mean is a functor

$$\text{Connected } \mathbb{E}_\infty - \text{rings} \to \mathrm{Spc}, R \mapsto \mathrm{Fun}^{\otimes}(\mathrm{Syn}_{MU}, \mathrm{Mod}_R).$$

There is a periodic version of MU, called MUP which is the **Thom spectrum** of $BU \times \mathbb{Z} \to \mathrm{Pic}(\mathbb{S})$, and it gives a cosimplicial object

$$\tau_{\geq \bullet}\,\mathrm{MUP} \text{ three arrows } \tau_{\geq \bullet} MUP \otimes MUP...$$

Let us call this stack $\tilde{M}_{fg}$. It turns out there is a line bundle over $\tilde{M}_{fg}$. Note that the cofiber of τ gives $\tilde{M}_{fg}/\tau = M_{fg}$. Now we look at maybe an open substack $\tilde{M}_{fg}^{\leq n}$ for which we have

$$\tilde{M}_{fg}^{\leq n}/\tau = M_{fg}^{\leq n}.$$

Note that we can get

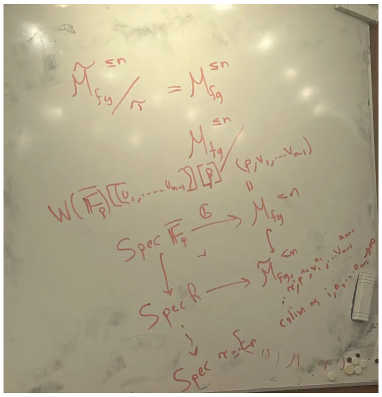
$$M_{fg}^{=n} = M_{fg}^{\leq n}/(p, v_1, ..., v_{n-1}).$$

Now we can construct a stack by considering a classifying map

$$\mathbb{G} : \mathrm{Spec}\,\overline{\mathbb{F}_p} \to M_{fg}^{=n}$$

given by the unique formal group of height n over this. It turns out that this map is actually **formall étale**!

Now whenever we have a formally étale map and there is a deformation of the target, this lifts to a deformation of the source. We can do this for $(M_{fg})^{\leq n}_{p^{a_0}, v_1^{a_1}, ..., v_{n-1}^{a_{n-1}}}$, and see the following picture to how to construct the connective Morava E-theory:



19 Lecture 15: McClure's Theorem (by Preston Cranford)

We will begin by stating the theorem. We take the free p-complete KU-algebra on a set X with one-element, ie. $\text{Free}_{KU_p^\wedge}(X)$. Now we take the $\mathbb{E}_\infty$-version of this with another p-completion.

Theorem 19.1 (McClure's Theorem)**.** $(\text{Free}^{\mathbb{E}_\infty}_{KU_p^\wedge}(X))_p^\wedge$ is a free p-complete δ-ring.

Remark 19.2. As far as the speaker knows, there is no good / systematic proof of this written down anywhere/ The speaker is taught by Ishan Levy of the proof of this theorem.

Before we start discussing how to prove this, we first want to justify why McCLure's theorem is very useful. Indeed, to know the $K(1)$-local power operations, this is equivalent to understanding the homotopy groups of $(\text{Free}^{\mathbb{E}_\infty}_{KU_p^\wedge}(X))_p^\wedge$, which McClure's theorem tells us how.

Let us recall - **Fact:** Let R be p-torsion free, then there is a bijection between

$$\{\delta-\text{structures}\} \leftrightarrow \{\varphi : R \to R, \text{lift of Frobenius}\}$$

Here a Frobenius refers to a map $R/p \to R/p$ with $x \mapsto x^p$.

Now we take $\mathbb{Z}$ and adjoint $X, \delta(X), \delta^2(X), ...$, we claim that:

Proposition 19.3. $\mathbb{Z}_p^\wedge[X, \delta(X), \delta^2(X), ...]$ is $\text{Free}_\delta(X)_p^\wedge$ (with 1 generator). The δ-structure is implicitly written into this, with the understanding that the ring is $\mathbb{Z}_p^\wedge[x_0, x_1, x_2, ...]$ with the identification $x_0 = x, x_1 = \delta(x), x_2 = \delta^2(x), ...$

From now on, for the ease of notations, we drop $\mathbb{E}_\infty$ from $\text{Free}^{\mathbb{E}_\infty}_{KU_p^\wedge}(X)$. Now we have that

$$\text{Free}_{KU_p^\wedge}(X) \cong \bigoplus_{n\geq 0}((KU_p^\wedge)_{h\Sigma n}^{\otimes_{KU_p^\wedge} n})_p^\wedge \cong \bigoplus_{n\geq 0}(KU_p^\wedge \otimes (B\Sigma n))_p^\wedge.$$

Here the second isomorphism is because the action of Σ_n on the n tensor products has stabilizer being Σn.

We have not yet calculated any of the terms on the RHS, but when we do it we will need to use group cohomology. Therefore, let us now recall some facts from group cohomology. Recall that for a subgroup $H \subset G$, we have

$$\begin{tikzcd} H^\bullet(BG, M) \arrow[r, "\text{res}" description] & H^\bullet(BH, M) \arrow[l, bend right=20, "\rho"'] \arrow[l, bend left=20, "\text{Tr}"] \end{tikzcd}$$

where res is the restriction map and Tr is the "wrong way" transfer map. They in particular satisfy:

$$\text{Tr} \circ \mathcal{E} = [G : H].$$

Now for $n < p$, (so for now let us operate under this) let us look at $(\text{KU}\,/p)^*(B\Sigma n)$. We make two claims:

1. $(KU/p)^*(B\bullet) \cong KU/p$.
2. $(KU/p)_\bullet(B\Sigma n) \cong_{\text{rank}\,\mathbb{Z}_p} KU_p^\wedge(B\Sigma n)$.

Proof Idea. For Claim 2, we look at the cofiber sequence

$$KU_p^\wedge \xrightarrow{\times p} KU_p^\wedge \to KU/p.$$

One can check that $(KU/p)_\bullet(B\Sigma_n)$ is even. ∎

Proposition 19.4. **Claim:** From here, we have that

$$(KU/p)^\bullet(B\Sigma_p) \simeq KU/p \oplus KU/p\{\delta x\}$$

where the equivalence is induced by $\mathcal{E} \circ \mathrm{Tr}$.

Proof Idea. We clearly have that $BC_p \subset B\Sigma_p$. Now we use another group cohomology fact - when $H \subset G$ is a subgroup and $[G : H]$ is coprime to p, and for $\mathbb{F}_p$-(possible grouo ring)-modules M, we have $H^\bullet(G, M) \cong H^\bullet(H, M)^{G/H} \hookrightarrow H^\bullet(H, M)$.

We have not finished the proof yet, for which we will finish later. ■

Proposition 19.5. **Claim:** $KU/p^\bullet(BC_p) \cong \mathbb{F}_p[t]/t^p \otimes \mathbb{F}[\beta^{\pm 1}]$ with $|t| = 2$ and $|\beta| = 2$.

Proof. Indeed, we have the Atiyah-Hirzeburch spectral sequence

$$H^p(BC_p, \pi_q(KU/p)) \implies KU/p^{p+q}(-).$$

Now we have that $H^\bullet(BC_p; \mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda_{\mathbb{F}_p}(\mathcal{C})$ where $|t| = 2$ and $|\mathcal{C}| = 1$. From here one can check that

$$d\mathcal{E} = [p]_F(t) = v_1 t^p = \beta^{p-1}t^p$$

(the p-series of the FGL on KU/p). Why is this? Well consider the induced map $BS^1 \to BS^1$ by $p : S^1 \to S^1$. The fiber of this is given by BC_p, so we extend this to

$$S^1 \to BC_p \to BS^1 \to BS^1$$

Now $S^1 \to BC_p \to BS^1$ is an S^1-bundle, which a Gysin sequence arguement can be used to figure out the differential. ■

Now consider the semi-direct product $C_p \rtimes \mathrm{Aut}(C_p)$, clearly $B(C_p \rtimes \mathrm{Aut}(C_p)) \subset B\Sigma_p$.

Proposition 19.6. **Claim:** $KU/p^\bullet(B(C_p \rtimes \mathrm{Aut}(C_p)))$ is rank 2.

Proof Idea. In this case we consider the SES $0 \to C_p \to C_p \rtimes \mathrm{Aut}(C_p) \to \mathrm{Aut}(C_p) \to 0$. Apply the Lyndon-Hochschild-Serre spectral sequence to this SES, we can deduce that $KU/p^\bullet(B(C_p \rtimes \mathrm{Aut}(C_p))) \cong (KU/p^\bullet(BC_p))^{h\,\mathrm{Aut}(C_p)}$, and the action sends, well

$$1 \to 1, t \to ut, ..., t^{p-1} \to t^{p-1}$$

which we see only 1 and t^{p-1} gets fixed, and hence there is only rank 2. ■

Now we revisit Proposition 19.4.

Proposition 19.7. **Claim:** We have that $KU/p(B\Sigma_p) \simeq KU/p \oplus KU/p$.

Proof Continued. We have a sequence

$$KU/p \xrightarrow{\mathcal{E}} K(1)_{h\Sigma_p} \xrightarrow{\sim} K(1)^{h\Sigma_p} \hookrightarrow K(1) = KU/p$$

The map $K(1)_{h\Sigma_p} \to K(1)$ is really the transfer map, for which we know is the index. ■

To summarize, currently we have figured out that $(\mathrm{Free}^{\mathbb{E}_\infty}_{KU^\wedge_p}(X))^\wedge_p$ is direct sum of $x, x^2, ..., x^{p-1}, x^p.\delta(x), ...$ where $|x| = 1$. Now we wish to actually give a δ-structure on this. Indeed, we define φ given by the relations that

$$\varphi(\mathcal{E}) = 1 \text{ and } \varphi(\mathrm{tr}) = 0.$$

It is sufficient to define them on these two terms. From here, we claim that φ is indeed a ring homomorphism. We look at this in a sequence of arguments.

Proposition 19.8. $\varphi(x + y) = \varphi(x) + \varphi(y)$.

Proof Sketch. Consider $\mathrm{Free}_{KU^\wedge_p}(z)$ and $\mathrm{Free}_{KU^\wedge_p}(x, y)$. Consider a map

$$\mathrm{Free}_{KU^\wedge_p}(z) \to \mathrm{Free}_{KU^\wedge_p}(x, y), z \mapsto x + y.$$

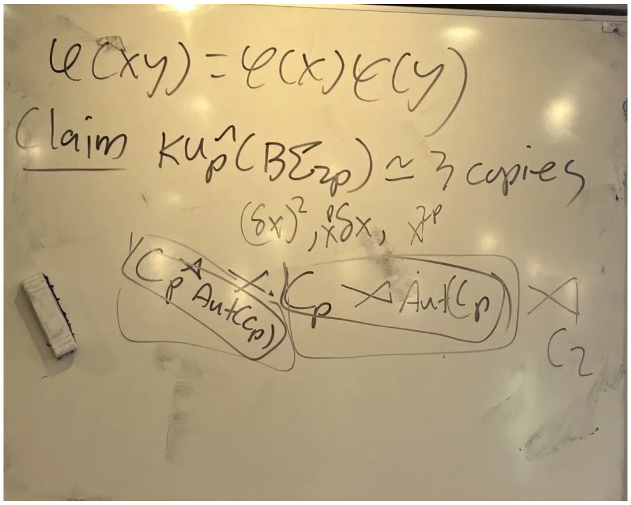
The free $KU^\wedge_p$-algebra on x, y, recall, gives a decomposition

$$\mathrm{Free}_{KU^\wedge_p}(x, y) \cong \bigoplus_{n>0} (KU^\wedge_p \oplus KU^\wedge_p)^{\otimes^n_{KU^\wedge_p}}_{h\Sigma n}.$$

Now Σ_n permutes $KU^\wedge_p \times \{x, y\}^p$ and looking at an extension of the map concludes the proof. ■

Proposition 19.9. $\varphi(xy) = \varphi(x)\varphi(y)$.

Proof. See picture:



■

Now we claim that it satisfies the universal property making it into a free delta ring, for surjectivity pf $\mathrm{Free}_\delta(x) \to \mathrm{Free}_{KU^\wedge_p}(w)$. We can do this by considering the term $KU^\wedge_p(B\Sigma_n)$ with $n = \sum_{i=1}^k a_i p^i$ with $a_i < p$ and then use something about $BC_p^{\rtimes a_0} X$.

For injectivity, consider $\mathrm{Free}_\delta(x) \to \mathrm{Free}_{KU^\wedge_p}(w)$ and lift to a diagram of the form

$$\begin{tikzcd} & S \\ \mathrm{Free}_\delta(x) \arrow[ur, hook] \arrow[r] & \mathrm{Free}_{KU^\wedge_p}(w) \arrow[u, hook] \end{tikzcd}$$

Then one can conclude this apparently by considering $(\Sigma^\infty_+ \mathbb{N}[1/p])^{\otimes\infty} \otimes KU$ and $\mathbb{F}_p[x_0^{1/p}, x_1^{1/p}, ...]$.

20 Lecture 16: The Chromatic Nullstellensatz (by Max Blans)

Today we will be talking about the **Chromatic Nullstellensatz** [BSY25]! To set the stage, let us first talk about the classical Nullstellensatz.

Theorem 20.1 ((Weak Nullstellensatz) David Hilbert)**.** Let L be an algebraically closed field. If $(f_1, ..., f_k) \subsetneq L[x_1, ..., x_n]$ (ie. the ideal generated by $f_1, ..., f_k$ does not generate the entire ring), then the ideal has a common zero. An alternative formulation of this theorem is that the L-algebra $L[x_1, ..., x_n]/(f_1, ..., f_k)$ admits a map to L.

To make this more chromatic, we want a more categorical formulation.

Observation: $L[x_1, ..., x_n]/(f_1, ..., f_k)$ is a **compact object** in the category of L-algebras!

Precisely, we could rephrase the Nullstellensatz as every non-terminal compact object admitting a map to L.

Theorem 20.2 (Chromatic Nullstellensatz, Bukland-Schlank-Yuan)**.** Let $E(L)$ be Morava E-theory. Every compact non-terminal object in $\mathrm{CAlg}^n_{E(L)} := \mathrm{CAlg}_{E(L)}(\mathrm{Sp}_{T(n)})$ admits a map to $E(L)$.

This theorem is a consequence of the following theorem.

Theorem 20.3. If $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ that is not zero, then there exists a map $R \to E(L)$ that is non-zero.

Goal: The proof proceeds more-or-less the same at each height, but there is a statement about power operations that is harder to prove at higher heights. Therefore, our goal is to give a proof at height 1.

20.1 Morava E-theory and Tilting

We have already seen that Morava E-theories can be constructed over any perfect field of characteristic p with its associated formal group law. Recall an $\mathbb{F}_p$-algebra is perfect if its Frobenius automorphism is an isomorphism. Lurie was able to extend the Morava E-theory construction to all perfect algebras.

Theorem 20.4 (Goerss-Hopkins-Miller, Lurie)**.** Let A be a perfect $\mathbb{F}_p$-algebra with H_0 formal group over A of height n, then there exists $E(A, H_0) \in \mathrm{CAlg}(\mathrm{Sp}_{K(n)})$ such that

$$\pi_* E(A; H_0) = W(A)[[u_1, ..., u_{n-1}]][u^{\pm 1}], |u| = 2.$$

This construction is functorial in (A, H_0).

In our case we want to work with $E(\mathbb{F}_p; \mathbb{G}_m) = \mathrm{KU}_p^\wedge$ - also known as Morava E-theory of height 1.

Theorem 20.5. There is an adjunction:

$$E(-) : \mathrm{Perf}_{\mathbb{F}_p} \leftrightarrow \mathrm{CAlg}^\wedge_{KU_p^\wedge} : (-)^b.$$

such that:

1. $E(A) = E(A; \mathbb{G}_m)$.
2. $E(-)$ is fully faithful (in other words, this is a co-localization).

3. $R^b = ((\pi_0 R)/p)^b := \lim(\pi_0 R/p \xleftarrow{x\mapsto x^p} \pi_0 R/p \xleftarrow{x\mapsto x^p} ...)$

By $\mathrm{CAlg}^{\wedge}_{KU_p^\wedge}$, we mean $\mathrm{CAlg}_{KU_p^\wedge}(\mathrm{Sp}_{T(1)})$.

20.2 The Proof in a Nutshell

Now suppose $R \neq 0 \in \mathrm{CAlg}(\mathrm{Sp}_{T(1)})$. We want a map $R \to E(L)$:

1. Since $KU_p^\wedge$ is conservative on $T(1)$-local spectra, we we have a natural map $R \to R \otimes KU_p^\wedge \neq 0$, so we can assume R is a $KU_p^\wedge$-algebra.
2. It suffices to give $R \to E(A)$ for $A \neq 0$.

Idea of the Proof: We have a map $E(R^b) \to R$, for which can we can modify R until this becomes an equivalence.

Here we give an example:

Example 20.6. Suppose $\alpha \in \pi_1(R)$ and consider the pushout diagram

$$\begin{tikzcd}
KU_p^\wedge\{z'\} \arrow[r, "z'\mapsto\alpha"] \arrow[d, "z'\mapsto 0"'] & R \arrow[d] \\
KU_p^\wedge \arrow[r] & R'
\end{tikzcd}$$

This will ensure that $\alpha = 0$ in R'.

The idea is we now want to iterate this square.

Definition 20.7. Let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(1)}$. We say a map $f : M \to N \in \mathrm{Mod}_R^w(\mathrm{Sp}_{T(1)})$ is **nilpotent** if $f^{\otimes_R k} = 0$ for $k >> 0$. We say that $R \to S$ in $\mathrm{CAlg}(\mathrm{Sp}_{T(1)})$ **detects nilpotence** if - $f : M \to N$ is nilpotent if and only if $f \otimes_R s$ is nilpotent.

The following will be the most important example of nilpotent-detecting maps.

Example 20.8. If the map $R \to S$ is **conservative**, meaning that $M \otimes_R S = 0$ implies $M = 0$, then $R \to S$ detects nilpotence.

The maps detecting nilpotence will be important to us for 2 reasons:

1. If $R \neq 0$, and $R \to S$ detects nilpotence, then $S \neq 0$.
2. Nilpotence detecting maps have great closure properties. Indeed, nilpotence detecting maps are closed under base change, retracts, and transfinite compositions. In other words, they form a **weakly saturated class**. (This is exactly what we need to run a small object argument).

Strategy: We will construct three maps:

1. $f : E(\mathbb{F}_p[t^{1/p^\infty}]) \to E(\mathbb{F}_p[t^{\pm 1/p^\infty}]) \times E(\mathbb{F}_p)$.
2. $g : KU_p^\wedge\{z^0\} \to E(A)$ (for A to be determined).
3. $h : KU_p^\wedge\{z^1\} \to KU_p^\wedge$.

We show that if R has the **right lifting property** with respect to these maps, then $R \cong E(R^b)$. And that these maps are nilpotence detecting. Then we run a **small object argument** to produce a map $R \to S$ such that it is nilpotence detecting and S has the right lifting property w.r.t to f, g, h, meaning S becomes a Morava-E-theory.

The rest of the talk consists of constructing f, g, h, showing they are nilpotence detecting and RLP with respect to f, g, h implies isomorphism to Morava E-theory. In particular,

1. RLP wrt to f implies $E(R^b) \to R$ is injective on π_0.
2. RLP wrt to g implies $E(R^b) \to R$ is surjective on π_0.
3. RLP wrt to h implies $\pi_1 R = 0$.
4. This is enough since both sides are even periodic.

20.3 The Map h

It is clear that if R has RLP to h, then $\pi_1 R = 0$. Indeed, we can just consider the lift

$$\begin{tikzcd} KU_p^\wedge\{z'\} \arrow[r] \arrow[d, "z' \mapsto 0"] & R \\ KU_p^\wedge \arrow[ur, dashed] & \end{tikzcd}$$

Now we want to show this is nilpotence detecting.

Example 20.9. Consider $\pi_* \mathbb{F}_p^{tC_p} = \mathbb{F}[t_2^{\pm}] \otimes \Lambda(\alpha_{-1})$ where $\beta\alpha_{-1} = t_2^{-1}$. Killing π_1 here is very destructive.

Example 20.10. $\mathbb{Q}\{z^1\} = \mathbb{Q} \oplus \mathbb{Q}[1] \to \mathbb{Q}$ (this is conservative and therefore nilpotence detecting).

In our case, let us consider $\pi_* KU_p^\wedge\{z^1\}$. One can compute using a bar spectral sequence that

$$\pi_* KU_p^\wedge\{z^1\} = \Lambda(z', \psi(z'), \psi^2(z'), ...)$$

which implies that h is **nilpotence detecting**.

20.4 The Map g

This is the "mystery map" that we did not specify. Now we said it looks like

$$g : \mathrm{KU}_p^\wedge\{z^0\} \to E(A)$$

The reason why this is the hardest map is because we are directly constructing a map into Morava E-theory. If our goal is to use g to induce surjection on π_0, we really need to look into **power operations**.

We also from the **last lecture** that $\pi_0 \, \mathrm{KU}_p^\wedge\{z^0\}$ is a free δ-ring on 1-generator! In other words,

$$\pi_0 \, \mathrm{KU}_p^\wedge\{z^0\} = \mathbb{Z}_p[z^0, \delta(z^0), ...]^\wedge - p = \mathrm{free}_\delta(z^0)_p^\wedge.$$

On the other hand, $\pi_0(A) = W(A)$ (recall we are at height 1).

Theorem 20.11 (Joyal). Let $A \in \mathrm{CRing}$, then $W(A)$ is the **cofree δ-rings** on A.

Remark 20.12. At higher heights, we no longer have $W(A)$, but it turns out the theorem of Joyal is still true if we replace thee word δ-ring with t-algebra. This is much harder to prove at higher heights.

The adjunction earlier makes it easier to construct maps out of Morava E-theory, but it turns out this cofreeness also makes it manageable to construct maps into Morava E-theory.

Proposition 20.13. Let $A \in \mathrm{Perf}_{\mathbb{F}_p}$. The following map is a bijection

$$\pi_0 \operatorname{Map}_{CAlg^{\wedge}_{KU^{\wedge}_p}}(KU^{\wedge}_p\{z^0\}, E(A)) \xrightarrow{(-/p)^{\#}} \operatorname{Hom}_{Perf_{\mathbb{F}_p}}((\mathrm{free}_{\delta}(z^0)/p)^{\#}, A).$$

where $B^{\#} := \lim_{\rightarrow}(B \xrightarrow{x \mapsto x^p} B \xrightarrow{x \mapsto x^p} ...)$

Proof. Consider $\pi_0 \operatorname{Map}_{CAlg^{\wedge}_{KU^{\wedge}_p}}(KU^{\wedge}_p\{z^0\}, E(A))$ on the LHS, this is the same as $\pi_0\Omega^{\infty}E(A)$, and we have a sequence

$$\begin{aligned}
\pi_0\Omega^{\infty}E(A) &\simeq W(A) \\
&\simeq \operatorname{Hom}_{\delta}(\mathrm{free}_{\delta}(z^0), W(A)) \\
&\simeq \operatorname{Hom}_{CRing}(\mathrm{free}_{\delta}(z^0), A) \\
&\simeq \operatorname{Hom}_{\mathrm{Perf}_{\mathbb{F}_p}}((\mathrm{free}_{\delta}(z^0)/p)^{\#}, A)
\end{aligned}$$

One can check that by diagram chase the bijection ehre is given by the map outlined. ■

Finally, we define g as follows.

Definition 20.14. Let $g : KU^{\wedge}_p\{z^0\} \to E((\mathrm{free}_{\delta}(z^0)/p)^{\#})$ correspond to the identity under the bijection of thew previous proposition.

Proposition 20.15. If R has the RLP to g, then $E(R^b) \to R$ is a surjection on π_0.

Proof. Let $x \in \pi_0(R)$. Consider the lift

$$\begin{tikzcd}
KU^{\wedge}_p\{z^0\} \arrow[r, "z^0 \mapsto x"] \arrow[d, "g"'] & R & E(R^b) \arrow[l, dashed] \\
E(A) \arrow[ur, dashed, "\text{by RLP}"'] & &
\end{tikzcd}$$

Because the map $E(R^b) \to R$ corresponds to co-localization, we can lift this map further to give

$$\begin{tikzcd}
KU^{\wedge}_p\{z^0\} \arrow[r, "z^0 \mapsto x"] \arrow[d, "g"'] & R & E(R^b) \arrow[l, dashed] \\
E(A) \arrow[ur, dashed, "\text{by RLP}"'] \arrow[urr, dashed] & &
\end{tikzcd}$$

This proves surjecitvity. ■

Proposition 20.16. g detects nilpotence.

Proof. Since $\mathrm{Mod}^{\wedge}_{KU^{\wedge}_p} \xrightarrow{-\otimes_{KU^{\wedge}_p} KU/p} \mathrm{MU}^{\wedge}_{KU^{\wedge}_p}$ is conservative, it suffices to check this after modding out by p. On the level of π_0, the term g/p is given by

$$\mathrm{free}_\delta(z^0)/p \hookrightarrow (\mathrm{free}_\delta(z^0)/p)^{\#}$$

by sending

$$\mathbb{F}_p[z^0, \delta(z), ...] \hookrightarrow \mathbb{F}_p[(z)^{1/p^\infty}, ...]$$

This is a faithfully flat map, and an argument with the Tor spectral sequence implies it is conservative, and hence nilpotent. ■

20.5 The Map f

f is the map given by

$$E(\mathbb{F}_p[t^{1/p^\infty}]) \to E(\mathbb{F}_p[t^{\pm 1/p^\infty}]) \times E(\mathbb{F}_p)$$

by $t \mapsto (t, 0)$. f should make sure that it corresponds to injection, but it turns out having a RLP to f gives something stronger.

Proposition 20.17. R has RLP with respect to f if and only if $(\pi_0 R/p)^b$ is of Krull Dimension 0.

Proof. The proof is entirely by commutative algebra. By adjunction R having RLP with respect to f is equivalent to R^b having RLP with respect to the map

$$\mathbb{F}_p[t] \to \mathbb{F}_p[t^{\pm}] \times \mathbb{F}_p.$$

Claim: This is equivalent to R^b being reduced (because R^b is perfect) and Krull dimension 0 (entirely by commutative algebra). ■

Remark 20.18. $(-)^b$ is called the flat.

Proposition 20.19. R^b of Krull dimension 0 implies $E(R^b) \to R$ is injective on π_0.

Proof. We claim it suffices to check this on $\pi_0(-)/p$. Indeed,

$$\begin{tikzcd}
\pi_0(E(R^b)) \arrow[r] \arrow[d] \arrow[rr, bend left, "\text{identity}"] & W(\pi_0(E(R^b))) \arrow[r, "W(-/p)"] \arrow[d] & W(\pi_0(E(R^b))) \arrow[d] \\
\pi_0(R) \arrow[r] & W(\pi_0(R)) \arrow[r, "W(-/p)"'] & W(\pi_0 R/p)
\end{tikzcd}$$

Now the right most vertical map - $W(-)$ sends injective maps to injective maps. Therefore, we see that by diagram chasing $\pi_0(E(R^b)) \to \pi_0(R)$ being injective is implies by $\pi_0 E(R^b)/p \to \pi_0 R/p$ being injective.

Why is the map injective mod p? Well we have that $\pi_0(E(R^b)) = R^b/p$. Suppose $y \in R^b$ is in the kernel. Since R^b has Krull dimension 0, y is generated by an idempotent element e. All components of e in the limit are idempotent and nilpotent, so $e = y = 0$. ■

We have now proven everything except f is nilpotence detecting. This is not hard, and the speaker gives it as an amusing exercise.

Exercise 20.20. Show that f is nilpotence detecting.

Remark 20.21. The converse of the chromatic Nullstellensatz is also true. In other words, if the spectrum satisfies the conlusion, it is Morava E-theory.

21 Lecture 17: Applications of the Chromatic Nullstellensatz (by Vignesh Subramanian)

Otherwise known as **Examples of $T(n)$-local $\mathbb{E}_\infty$-rings**.

At some point, we decided that we want to study spaces Spc, and then we moved on the Sp. Spectra is in some sense a linear approximation of Spc, bjt we can also think about quadratic, cubic, ... approximations of Spc:

$$\begin{tikzcd}
\mathrm{Spc} \arrow[d] \arrow[dr] \arrow[drr] & & \\
\mathrm{Sp} & \mathcal{P}_2(\mathrm{Spc}) \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

Theorem 21.1 (Quillen/Sulliven)**.** The rational homotopy groups can be studied by differential $\mathbb{Q}$-lie algebras.

Theorem 21.2 (Mandell's Theorem)**.** The functor $X \mapsto \overline{\mathbb{F}_p}^X$ from $\overline{(\mathcal{S}^{\geq 1}_{p-complete,ft})}^{op}$ into $\mathrm{CAlg}_{\overline{\mathbb{F}}_p}$ is an embedding. (Here ≥ 1 means simply connected)

In today's talk, the theorem we want to prove is:

Theorem 21.3 (Hopkins-Lurie + Burkland-Schlank-Yuan)**.** The functor $X \mapsto E_n^X$ gives an embedding

$$(\mathcal{S}^{\leq n}_{p-fin})^{op} \hookrightarrow \mathrm{CAlg}_{E_n}(\mathrm{Sp}_{K(n)}) =: \mathrm{CAlg}^{\wedge}_{E_n}.$$

Now consider a $E_\infty - K(n)-$local functor

$$\pi_*(\mathrm{Mod}_A(\mathrm{Sp}_{K(n)})^X) \to \mathrm{Mod}_A(\mathrm{Sp}_{K(n)})$$

$$F \mapsto C^*(X;\mathcal{F}) \text{ with an action of } C^\infty(X;A).$$

In general we always have a diagram of the following form

$$\begin{tikzcd}
\mathrm{Mod}_A(\mathrm{Sp}_{K(n)})^X \arrow[r] \arrow[d] & \mathrm{Mod}_A(\mathrm{Sp}_{K(n)}) \\
\mathrm{Mod}_{C^*(X;A)}(\mathrm{Sp}_{K(n)}) \arrow[ur] &
\end{tikzcd}$$

Question: Is the vertical map going down an equivalence?

In general for X a p-finite space, there is a functor $F : X \to \mathrm{Sp}_{K(n)}$ with

$$\mathrm{Nm}_X : \mathrm{colim}_X F \to \lim_X F$$

that "roguhly looks like an analog of the norm map $A/G \to A^G$".

Theorem 21.4. Let $f : X \to Y$ be a map of spaces, $A \in \mathrm{Alg}(\mathrm{Sp}^X_{K(n)})$, and $G : \mathrm{LMod}_A(\mathrm{Sp}^X_{K(n)}) \to \mathrm{LMod}_{f_*\mathcal{A}}(\mathrm{Sp}^Y_{K(n)})$, then

1. Assume that $\mathrm{fib}(f)$ is m-truncated and p-finite, then f_* has a fully faithful left adjoint.

2. The $\mathrm{fib}(f)$ being n-truncated implies that G is an equivalence.

Proposition 21.5 (Push-Pull). If $f : X \to Y$ is m-truncated and p-finite, and $A \in \mathrm{Alg}(\mathrm{Sp}^{Y}_{K(n)})$ and $M \in \mathrm{RMod}_{f^*A}(\mathrm{Sp}^{X}_{K(n)})$. Then we have the following comparison map for $N \in \mathrm{LMod}_A(\mathrm{Sp}^{Y}_{K(n)})$:

$$B_{M,N} : f_*M \otimes_A N \to f_*(M \otimes_{f^*A} f^*N)$$

which is equivalence.

Proof. Reduce this to the case where $Y = *$. Now fix M and consider the category $\mathcal{C}$, which is a subcategory of $\mathrm{LMod}_A(\mathrm{Sp}_{K(n)})$ such that $B_{M,N}$ is an equivalence. We can check that $\mathcal{C}$ is closed under colimits. The left side of $B_{M,N}$ expands out just fine, the right hand side - f_* preserves colimits because f is required to be m-truncated and p-finite.

Observe that $A \in \mathcal{C}$. Since f_* preserves colimits, this implies that $\mathcal{C} = \mathrm{LMod}_A(\mathrm{Sp}_{K(n)})$. ■

Remark 21.6. We do have an equivalence $C^*(X; M) \cong C^*(X; A) \otimes M$ for $M \in \mathrm{LMod}_A(\mathrm{Sp}_{K(n)})$.

Theorem 21.7. Let A be a $K(n)$-local $\mathbb{E}_\infty$-ring, and consider the pullback

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[d] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

The functor $C^*(-, A)$ maps them to a square

$$\begin{tikzcd}
A^{X'} \arrow[r] \arrow[d] & A^{X} \arrow[d] & (*) \\
A^{Y'} \arrow[r] & A^{Y} &
\end{tikzcd}$$

If Y is n-truncated and p-finite, X is m-type, then the output square $(*)$ is a pushout in $\mathrm{CAlg}^{\wedge}_{A}$.

Recall we want to prove the following

Theorem 21.8. There is an embedding $(\mathcal{S}^{p-finite}_{\leq n})^{op} \hookrightarrow \mathrm{CAlg}_A(\mathrm{Sp}_{K(n)})$ with $A = E_n$.

Let us proceed in the following.

Proposition 21.9. Let A be a $\mathbb{E}_\infty - K(n)$-local ring, the following are equivalent:

1. $C^*(-; A) : (\mathcal{S}^{p-finite}_{\leq n})^{op} \to \mathrm{CAlg}^{\wedge}_{A}$ is fully faithful.
2. For all $X \in (\mathcal{S}^{p-finite}_{\leq n})$, the canonical map $X \to \mathrm{map}_{\mathrm{CAlg}_A}(A^X, X)$ is an equivalence.
3. Check the second item for the special case $X = K(\mathbb{Z}/p, n)$.

Clearly (1) implies (2) and (2) implies (3). Now let (1') be the condition that for any p-finite n-truncated X, we have equivalence

$$\mathrm{Map}_{\mathcal{S}}(Y, X) \to \mathrm{map}_{\mathrm{CAlg}_A}(A^X, A^Y).$$

For (3) implies (2), this comes from the Eilenberg-Moore spectral sequence. The idea is to define $\mathcal{C} \subseteq \mathcal{S}^{p-finite}_{\leq n}$ to be the subcategory given by X such that its canonical map α is an equivalence. (3) implies $K(\mathbb{Z}/p, n) \in \mathcal{C}$, and the Eilenberg-Moorse spectral sequence implies $\mathcal{C}$ is closed under finite limits. Thus, $K(\mathbb{Z}/p, m) \in \mathcal{C}$ for all $m \leq n$. When $m = 0$, this implies all finite sets in $\mathcal{C}$. Now since we are dealing for p-finite spaces, for $K(G, m)$ we can write an SES

$$0 \to G' = \mathbb{Z}/p \to G \to G'' \to 0$$

which gives a fiber sequence of the form

$$K(G', m) \to K(G, m) \to K(G'', m)$$

which lets us recover all n-truncated p-finite spaces.

We are interested in the specific case where $A = E_n$. Now, $\mathrm{map}_{\mathrm{CAlg}^{\wedge}_{E_n}}(E_n^X, E_n) \simeq \mathrm{Hom}_{\mathrm{Sp}_{\geq 0}}(\mathbb{Z}/p, \mathrm{GL}_1 E_n)$.

There is a notion of **strict elements** which are $\mathrm{Hom}_{\mathrm{Sp}_{\geq 0}}(\mathbb{N}, R)$. There is also a notion of $\mathbb{Z}/p$-**strict-units** given by $\mathrm{Hom}_{\mathrm{Sp}_{\geq 0}}(\mathbb{Z}/p, GL_1 R)$. Here by $GL_1 R$, we mean the pullback:

$$\begin{tikzcd}
GL_1 R \arrow[r] \arrow[d] & \Omega^\infty R \arrow[d] \\
\pi_*(R) \arrow[r] & \pi_0 R
\end{tikzcd}$$

Our theorem reduces to the relevant statement on strict units.

Now when L is algebraically closed, $L \to F$, and $R \in \mathrm{CAlg}_{E(L)}$, the **chromatic Nullstellensatz** implies that there exists a map $R \to E(F)$. In this case, if we look at the associated map in Picard space, we get

$$\mathrm{Pic}(\mathrm{Mod}^{\wedge}_{E(L)}) \to \mathrm{Pic}(\mathrm{Mod}^{\wedge}_{R}),$$

then the map admits a retract!

This allows us to have an $\mathbb{E}_\infty$-map $f : X \to \mathrm{Pic}(\mathcal{C})$ where $X \in \mathrm{Sp}_{\geq 0}$ and $Mf \in \mathrm{CAlg}(\mathcal{C})$.

For $\mathcal{C} \in \mathrm{CAlg}_{Mod_{E_n}}(Pr^{L,\otimes}_{st})$, we have now the following equivalent statements:

1. $Mf \neq 0$.
2. $Mf \to \mathbb{1}_{\mathcal{C}}$
3. The map $\mathrm{Pic}(\mathcal{C}) \to \mathrm{Pic}(\mathrm{Mod}_{Mf}(\mathcal{C}))$ has a retract.

Proposition 21.10. Let L be algebriacally closed and H a p-torsion abelian group, then

$$\mathrm{Hom}_{\mathrm{Sp}_{\geq 0}}(H, \mathrm{Pic}(\mathrm{Mod}^{\wedge}_{E(L)})) \simeq \Sigma^{n+1} H^*.$$

Proof. It it enough to prove this for the case where H is finite (we can do the general case by an Ind-like argument). Now consider $f \in \pi_m \mathrm{Hom}_{\mathrm{Sp}_{\geq 0}}(H, \mathrm{Pic}(\mathrm{Mod}^{\wedge}_{E(L)}))$. This map f can be viewed as a map

$$f : B^m H \to \mathrm{Pic}(\mathrm{Mod}^{\wedge}_{E(L)})$$

and we will have that M_f is a limit - $M_f \simeq \lim_{B^m H} f$. From here, we have

$$\lim_{X} \mathrm{Fun}(X; \mathrm{Mod}^{\wedge}_{E(L)}) \to \mathrm{Mod}^{\wedge}_{E(L)}$$

is conservative when $X = B^m H$ for $m \neq n+1$. Now we have an equivalence of functors between

$$H \mapsto H^* \text{ and } H \mapsto [\Sigma^{m+1} H, \mathrm{Pic}(\mathrm{Mod}^{\wedge}_{E(L)})].$$

■

22 Lecture 18: Conclusion and Outlook (by Gijs Heuts)

In this lecture, we want to give you some outlook on what you can do in the future with chromatic homotopy theory. The list is somewhat random and decided by the speaker. We will not go particularly deep into anything. It is mainly grouped around 2 themes:

1. Interaction between different heights.
2. Interaction between chromatic homotopy theory and geometry + homological stability.

Ordinary Algebra: Consider the derived category $\mathbb{Z}$ - $D(\mathbb{Z})$ and its comapct objects $D(\mathbb{Z})^\omega$. Here is a picture of $\mathrm{Spec}(\mathbb{Z})$ parametrized by $\mathbb{Q}$ at the bottom (here we draw the prime ideal in terms of their residue fields):

$$\begin{tikzcd}
& \mathbb{F}_2 & \mathbb{F}_3 & \mathbb{F}_5 & \cdots \\
\operatorname{Spec}\mathbb{Z} & & \mathbb{Q} \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Higher Algebra: Without being too formal, we also want heuristically a notion of derived category of the sphere spectrum where $\mathrm{Sp}^\omega = D(\mathbb{S})^\omega$. In this case, when we look at “Spec($\mathbb{S}$)" (really thick subcategories), the **thick subcategory theorem** tells us that there is a grading that looks like:

$$\begin{tikzcd}
\mathbb{F}_2 & \mathbb{F}_3 & \mathbb{F}_5 & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \\
K(2)_2 \arrow[u] & K(2)_3 \arrow[u] & K(2)_5 \arrow[u] & \cdots \\
K(1)_2 \arrow[u] & K(1)_3 \arrow[u] & K(1)_5 \arrow[u] & \cdots \\
& \mathbb{Q} \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

The homotopy theory, in the view of thcik subcategory, breaks into two parts:

1. **Local:** Understanding the “monochromatic pieces" - ie. $\mathrm{Sp}_{K(n)}$ or $\mathrm{Sp}_{T(n)}$.
2. **Global:** Glue together pieces into a global picture of Sp.

22.1 Blueshift and Redshift

One way for how these different pieces interact for a global picture is what is called **Blueshift** (go down in height) and **Redshift** (go up in height).

Question 22.1. What is up with blueshift?

The following theorem is an example of **blueshift**.

Theorem 22.2 (Kuhn). Let $X \in L_n^f\,\mathrm{Sp}$ and G is a finite group which acts on X, then we can form the **Tate construction** with respect to this action to get

$$X^{tG} := \mathrm{cofib}(X_{hG} \to X^{hG})$$

The theorem asserts that X^{tG} is L^f_{n-1}-local.

Question 22.3. What is up with redshift?

The following is an example of **redshift**, which is related to **algebraic K-theory**! The philosophy came out of Anson and Rognes is that they observed, in many computations, that K-theory typically “increases the height by 1".

Theorem 22.4 (Many People)**.** Let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, then $L_{T(n+1)}K(R)$ is non-zero. Therefore, there are some genuinely increase in height by algebraic K-theory (and the $L_{T(n+k)}$ for $k > 1$ would localize to 0).

This has been an extremely productive line of research in the last few years, and it played a big role in the resolution of the telescope conjecture. However, there are still many open questions in the area.

Quantitative Versions of Redshift:

1. We know $L_{T(n+1)}K(E_n)$ is non-zero, but what are the homotopy groups of $L_{T(n+1)}K(E_n)$?
2. What is $L_{T(2)}K(L_{K(1)}\mathbb{S})$? One thing we saw this week is that we have good ring spectra “approximating" the $K(n)$-local sphere spectrum, namely the Morava E-theories E_n. We don’t have a similar method for $L_{T(n)}\mathbb{S}$ for $n \geq 2$, yet.
3. **Challenge:** Find good ring spectra over $L_{T(n)}\mathbb{S}$.

22.2 Chromatic Splitting Conjecture

This is a conjecture due to Hopkins about **fracture squares**. Recall, the **arithmetic fracture square** is given as follows - for $X \in \mathrm{Sp}_{(p)}$ is p-local, then we can either rationalize or p-complete X, the point is that they actually determine X by a pullback diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^\wedge_p \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & (X^\wedge_p)_{\mathbb{Q}} \end{tikzcd}$$

The **chromatic fracture square** tells us that for $n \geq 1$, we have a pullback diagram where we can build L_nX from $L_{K(n)}X$ and $L_{n-1}X$ by

$$\begin{tikzcd} L_nX \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\ L_{n-1}X \arrow[r] & L_{n-1}L_{K(n)}X \end{tikzcd}$$

Conjecture 22.5 (Weak Chromatic Splitting Conjecture)**.** The **chromatic splitting conjecture** states that the map $L_{n-1}X \to L_{n-1}L_{K(n)}X$ is actually an inclusion as a summand.

There are many interesting corollaries if this conjecture is true:

1. If X is the p-completion of a finite spectra, then the map $X \to \prod_{n\geq 0} L_{K(n)}X$ is the retraction of a summand.
2. The conjecture is known to hold for $n \leq 2$ - this is done with explicit heavy computations.

There is a **strong form of this conjecture** that predicts exactly what this map is like, but Agnes Beaudry disproved this in the case $n = p = 2$.

22.3 Unstable Homotopy Theory

The homotopy theory of unstable spaces can very much be broken up similarly using chromatic methods. We saw this week that we have the **Bousfield-Kuhn functors**

$$\Phi_n : \mathcal{S}_* \to \mathrm{Sp}_{T(n)}, n \geq 0$$

which record the v_n-periodic homotopy types of spaces.

Question 22.6. How are the different Φ_n's related?

Note that Φ_n's lift to an equivalence between

$$\mathcal{S}_{v_n} \overset{\cong}{\longrightarrow} \mathrm{Lie}(\mathrm{Sp}_{T(n)}).$$

Question 22.7. Can we come up with a theory - which we might call "trans-chromatic Lie algebras" $\tilde{Lie}_{\leq n}$ - that combines the Φ_i's and $\mathrm{Lie}(\mathrm{Sp}_{T(i)})$ for $i \leq n$ into one object?

Warning: There is one category we can try which is $\mathrm{Lie}(L_n^f \, \mathrm{Sp})$, but this turns out to be the wrong answer.

The reason why is because $\mathcal{S}_*$ becomes "algebraic" when localized at a single height, but it can never be "algebraic" when localized at several heights at the same time. There is a quantifiable obstruction to this.

Conjecture 22.8. There is a strong relationship between

$$\Theta : \mathrm{coSp}(L_n^f \mathcal{S}_*^{p^\infty - torsion} \leftrightarrow L_n^f \mathcal{S}_*^{p^\infty - torsion} : \Phi$$

that is an adjuntion and gives a nice description.

Yuqing Shi considered and solved a monochromatic case of this, roughly speaking.

22.4 Connections to Geometry and Homological Stability

Here there are many open questions. Originally, algebraic topology was not distinct from geometry, it was a tool to study geometry, but later it went into another direction.

Question 22.9. Can we still give geometric descriptions of cochains for Morava E-theory, K(n)? Geometric description for tmf (this is called the Stolz-Teichner program)?

The idea is that tmf is supposed to be the height 2 version of K-theory. K-theory is about vector bundles, and tmf is suppsoed to be about 2d field theories. Another feature of K-theory is it connects really well to index theorems, but what about tmfs? tmf is originally supposed to be constructed to be related to index theory on the free loop space.

Another question is about **manifolds**! One thing that happened recently is that there is a lot of progress in computing $H^*(-;\mathbb{Q})$ and $\pi_*(-) \otimes \mathbb{Q}$ of $\mathrm{BDiff}_\partial(M)$! The methods are quite robust, and there is no real reason to expect why we only limit ourselves to these cohomology theories?

Question 22.10. Why not K-theory? Or higher Morava E-theories?

Finally, we want to say something about **homological stability**. The usual question for homological stability does not really make sense for periodic (co)homology theories ... but the questions for representation stability can be formed for generalized cohomology theories!

What we want to talk about here is not necessarily an application of traditional chromatic methods, but rather a parallel development in homological stability (by Randal-Williams) that mirrors that of chromatic homotopy theory.

Say R is an $\mathbb{E}_2$-algebra over a field k, graded. For example, we can take

$$R = \{C_*(\mathrm{Conf}_d(\mathbb{R}^2); k)\}_{d\geq 0}$$

or $R = \{C_*(G_d; k)\}_{d\geq 0}$ where the $\bigsqcup_{d\geq 0} BG_d$ form a braided monoidal groupoid.

Notationally, we write

$$\pi_{n,d}R := \pi_n R(d).$$

Stability often looks as follows - for $\sigma \in \pi_{0,1}R = \pi_0 R(1)$. Stabilization is concerned, equivalently, as multiplication by σ in this $\mathbb{E}_2$-algebra. Equivalently, a formulation of stability is that R/σ has a **vanishing line**, ie.

$$\pi_{n,d}(R/\sigma) = 0 \text{ for some } d < An + B.$$

(ie. σ is a homology isomorphism in some range).

This is just a translation so far, but if we think about it in this way, there is a notion of **secondary stabolity** we can think of (due to GKRW). They looked at R being the $\mathbb{E}_2$-algebra of $C_*(\mathrm{MCG}(S, \partial S))$ where S is a genus 2 surface with a disk cut out.

Theorem 22.11 (CGWK).

1. $\pi_{n,d}(R/\sigma) = 0$ for $d < 2n/3$.
2. (Secondary Stability): There are maps $\varphi : \mathbb{S}^{3,2} \otimes R/\sigma \to R/\sigma$ such that $\pi_{n,d}(R/(\sigma, \varphi)) = 0$ for $d < 3n/4$.

The second point is a statement about some stability for relative homology. If we work this oout, then equivalently this is saying

$$H_{d-2}(G_{n-3}, G_{n-4}) \cong H_d(G_n, G_{n-1}).$$

What Oscar Randal-Williams did is he packaged this phenonemon into what is called a “periodicity theorem".

Theorem 22.12 (Periodicity Theorem, VERY roughly). For R a good enough $\mathbb{E}_2$-ring, graded over $\mathbb{F}_p$, there are self-maps as follows:

$$\alpha_1 : \mathbb{S}^{n_1,d_1} \otimes R \to R$$

$$\alpha_2 : \mathbb{S}^{n_2,d_2} \otimes R/\alpha_1 \to R/\alpha_1$$

$$\vdots$$

$$\alpha_i : \mathbb{S}^{n_i,d_i} \otimes \mathbb{S}/(\alpha_1, ..., \alpha_{i-1}) \to R/(\alpha_1, ..., \alpha_{i-1})$$

$$\vdots$$

The following properties are true:

1. Each α_i is not nilpotent.
2. $R/(\alpha_1, ..., \alpha_i)$ has a vanishing line of slope < 1.

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