UiO : University of Oslo

Alice Petronella Hedenlund

Multiplicative Tate Spectral Sequences

Thesis submitted for the degree of Philosophiae Doctor

Department of Mathematics
Faculty of Mathematics and Natural Sciences

2020

To my mother,
for introducing me to the wonderful world of mathematics.

Acknowledgements

Newton famously said

> *"If I have seen further than others, it is by standing upon the shoulders of giants."*

and while it is true that we owe a huge mathematical debt to the "giants" before us, we sometimes forget the debt we owe to the equals that we stand beside. Contrary to how it is often portrayed in popular media, mathematics is not a solitary activity, and the biggest portion of mathematical research comes out of discussion and collaboration. With this is mind, I would like to express my sincerest gratitude to the homotopy theory community in general. The homotopy theory community is warm, inclusive, and generous in its mathematical advice, and this thesis would not be even half of what it is without the support this community offers. If you have ever organised a conference I have been to, held a talk I have seen, written an article I have read, offered advice big or small, or just listed to me rant or rave about maths, then know that I am grateful to you. That being said, there are a couple of people I want to thank by name.

The first person I would like to thank specifically is John Rognes for taking me on as his PhD student. As an advisor, John is demanding in his constructive criticism, and while this can occasionally be daunting, I know that all his remarks and suggestions has helped me to grow tremendously as a mathematician these last couple of years. He is patient, always takes time for his students, and is extremely knowledgeable. In short, he is always ready to have a three hour monologue about whatever mathematical subject you are wondering about. I would consider it a success if I turned out to be even half of the mathematician he is.

The second person I would like to thank is Thomas Nikolaus for mathematical advice, but maybe even more for being my friend. It is quite hard to imagine what this thesis (and my life in general) would have looked like if I hadn't decided that: "Fine, I can have *one* drink with him at Oslo Mikrobryggeri". Thomas is open and generous with his math advice, and he is also encouraging and good at cheering me up when I am "whining". He also taught me about symmetric monoidal localisations.

Finally, I would like to thank my family for supporting me and always encouraging me to pursue my dreams. Writing this thesis would have been near impossible without their unconditional love. In particular, I would like to thank my mum Jeanette Hedenlund who was the first person to introduce me to the wonderful world of mathematics, and to whom this thesis is dedicated.

Contents

Chapter 1

Introduction

The thesis you are reading is the result of research conducted by the author during her employment as a PhD student supervised by John Rognes at the Department of Mathematics at the University of Oslo. The thesis consists of two papers:

Paper I A. Hedenlund and J. Rognes. *A multiplicative Tate spectral sequence for compact Lie group actions.* 2020.

Paper II A. Hedenlund. *Multiplicative spectral sequences via décalage.* 2020.

This first chapter is meant as both a historical background and a larger context for the two papers included in this thesis. We will try to keep things conceptual, focusing on *why* rather than the more technical *how*, which we leave to the two papers and the other references provided in the bibliography. My hope is that the reader, after finishing this introduction, will have an understanding for what results to expect in the two papers, and why the author saw the importance in pursuing them, even if the reader is not necessarily a member of the same field of mathematics as herself and might not care about the same type of questions. We will successively narrow down the main themes of this thesis, starting from a very broad context, and ending in the more specific topics to be covered.

Section 1 We introduce the context in which this thesis is written: homotopical algebra. We look at two frameworks that have historically been used to deal with homotopical phenomena: model categories and ∞-categories. We introduce the main mathematical objects that we study: spectra.

Section 2 We introduce the technical tools that we will use and study in this thesis: spectral sequences. We discuss what we mean by convergence of spectral sequences, and how to to think about multiplicative structures in spectral sequences.

Section 3 We introduce the specific topic in homotopical algebra we would like to study, namely Tate constructions. We start with an introduction to the classical concept of Tate cohomology, go on to discuss its generalisation to homotopical algebra, and introduce the Tate spectral sequence. We explain what technical difficulties one might expect when considering such a spectral sequence. We end with a short digression where the author explains her own personal reasons for studying the Tate construction, which comes from a background in algebraic K-theory and topological Hochschild homology.

1.1 Homotopical algebra

The broad context in which this thesis takes place is within the mathematical subject known as homotopical algebra. I could call the context algebraic topology, or homotopy theory, but I feel like that gives a wrong indication of the flavour of the results contained in this thesis, which are often more algebraic and less topological and/or geometrical. While an aspiring mathematician's first exposure to "homotopy" often comes packaged in a topology course, one could argue that this is mostly a historical feature, and that there is nothing intrinsically topological about the concept, at all. In this first section, I hope to convince the reader that homotopical algebra is so much more than just a subset of algebraic topology, with applications going far beyond topological questions. It is often more fruitful to think of homotopical algebra as a natural extension of algebra; what Waldhausen envisioned with his "brave new rings"-paradigm. From this point of view, homotopical algebra is like doing algebra over a "deeper" base ring; while $\mathbb{Z}$ is the initial commutative ring in classical algebra, the so-called sphere spectrum $\mathbb{S}$ is the initial "commutative" ring in homotopical algebra. We can think of homotopical algebra as what we get if we base-change classical algebra along the Hurewicz homomorphism $\mathbb{S} \to \mathbb{Z}$.

In this section, we start with a motivation for thinking "homotopically". We go on to discuss two of the (many) approaches to homotopical algebra that exist: model categories and ∞-categories. Lastly we discuss the main mathematical objects that we will study in this thesis: spectra, the abelian groups of homotopical algebra.

Homotopical algebra	Algebra
space/∞-groupoid/anima	set
spectrum	abelian group
$\mathbb{E}_1$-ring spectrum	associative and unital ring
$\mathbb{E}_\infty$-ring spectrum	commutative ring
$\mathrm{Fin}^{\simeq}$ - the underlying space/∞-groupoid/anima of the category of finite sets	$\mathbb{N}$ - the monoid of natural numbers
$\mathbb{S}$ - the sphere spectrum	$\mathbb{Z}$ - the ring of integers

1.1.1 Homotopical algebra: A motivation

Many important results in mathematics deal with the question of "figuring out what objects are the same". This can be something as rudimentary as stating that $1 + 1$ is the same thing as 2, or something as advanced as stating that:

> *A simply connected closed 3-manifold is "the same thing" as a 3-dimensional sphere.*

One standard way of rigorously dealing with the concept of "sameness" in mathematics is through the very useful and powerful language of category theory. In the sense of category theory, we talk about two mathematical objects x and y of the same category as "the same" (or: **isomorphic**) if we can find morphisms $f : x \to y$ and $g : y \to x$ whose two compositions gf and fg are equal to the identity morphisms on x and y, respectively [Mac98; Bor94]. For example, in the statement above, the category we are considering is that of topological spaces with continuous maps between them. Two spaces that are "the same" in the above sense are referred to as being **homeomorphic**, and a more mathematically rigorous statement of what we have written above becomes:

> *Every simply connected closed 3-manifold is homeomorphic to the 3-sphere.*

This statement is known as the Poincaré conjecture, and is to this day the only one of the seven Millennium Prize Problems to have been solved. The formulation of the Poincaré conjecture coincides with the conception of the mathematical subject known as algebraic topology, where the rough aim can be expressed as "finding algebraic invariants that classify topological spaces up to homeomorphism". One of the main branches of algebraic topology is homotopy theory, where the idea is to use a weaker notion of "sameness" called homotopy equivalences, in order to approach this aim. Explicitly, two topological spaces X and Y are said to be **homotopy equivalent** if there is a homotopy equivalence between them. This means that we can find continuous maps $f : X \to Y$ and $g : Y \to X$ such that gf and fg are homotopic to the identity maps. Here, two maps $\phi, \psi : A \to B$ are homotopic if there is **homotopy** between them; a continuous map

$$H : I \times A \to B \quad \text{such that} \quad \begin{cases} H(0,x) = \phi(x) \\ H(1,x) = \psi(x)\,. \end{cases}$$

One equivalent formulation of the Poincaré conjecture says, in layman's terms, that this weaker form of "sameness", is enough to guarantee the stronger form, when we are dealing with the class of topological spaces known as "3-manifolds":

> *A 3-manifold which is homotopy equivalent to the 3-sphere is also homeomorphic to the 3-sphere.*

At this point it is worth taking a step back and note that similar notions of this "weaker sameness" can be found in many other fields of mathematics, as the following examples show.

Example 1.1.1. Two chain complexes C_* and D_* are said to be chain homotopic if there is a chain homotopy equivalence between them. This means that we can find chain maps $f : C_* \to D_*$ and $g : D_* \to C_*$ such that gf and fg are chain homotopic to the identities on C_* and D_*, respectively. Here, two chain maps $\phi, \psi : A_* \to B_*$ are called chain homotopic if there is a **chain homotopy** between them; a collection of maps $s_n : A_n \to B_{n+1}$ satisfying

$$\phi_n - \psi_n = s_{n-1}\partial_n^A + \partial_{n+1}^B s_n\,.$$

Example 1.1.2. Even in category theory itself we do not care about categories up to isomorphism, but rather, only up to equivalence. Recall that two categories $\mathscr{C}$ and $\mathscr{D}$ are called equivalent if there are functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that GF and FG are naturally isomorphic to the identity functors. Here, a **natural transformation** between two functors $\Phi, \Psi : \mathscr{A} \to \mathscr{B}$ is a class of morphisms $\tau_a : \Phi a \to \Psi a$ in $\mathscr{B}$ indexed by the objects of $\mathscr{A}$ and such that for every morphism $f : a \to a'$ in $\mathscr{A}$, the diagram

$$\begin{tikzcd}
\Phi a \arrow[r, "\tau_a"] \arrow[d, "\Phi f"] & \Psi a \arrow[d, "\Psi f"] \\
\Phi a' \arrow[r, "\tau_{a'}"] & \Psi a'
\end{tikzcd}$$

commutes. We say that we have a **natural isomorphism** if all the morphisms $\tau_a : \Phi a \to \Psi a$ are isomorphisms.

The context of the above examples are different: the first example can be placed under homological algebra, while the second example belongs to category theory. However, we can note that all of these examples describe essentially the same concept of "sameness", a concept which involved some notion of morphisms between morphisms. We called these by different names (homotopy, chain homotopy, and natural transformation) depending on the context, but collectively these morphisms between morphisms can be referred to as 2-morphisms. It turns out that it is often useful to also think about morphisms between these 2-morphisms, what we would call 3-morphisms, and so on. The study of such structures can have many different names depending on the direction you approach them from, but in this thesis, we will use the terminology *higher category theory* and make the following distinctions:

Higher category theory The study of the structures involving an infinite hierarchy of morphisms. Here, the structures, the ∞-categories, are themselves the objects of interest.

Homotopy theory The classical study of spaces as living in the context of higher category theory, rather than in ordinary category theory, via continuous maps, homotopies between continuous maps, homotopies between homotopies between continuous maps, and so on.

Homotopical algebra The study of mathematical objects that live naturally in the context of higher category theory. In particular, it is the study of those "homotopical" objects that posess some extra structure, like a multiplicative structure of some kind, as for example spectra[1].

We will return to the mathematical objects mentioned in the last point, spectra, in Section 1.1.3.

[1] There is not a clear consensus on what the term *homotopical algebra* should mean. Quillen was arguably the first one to use the terminilogy and explains it as "the generalization of homological algebra to arbitrary categories which results by considering a simplicial object as being a generalization of a chain complex" [Qui67]. In the light of the discussion in Section 1.1.3.3, we feel confident in claiming that the study of spectra are at the heart of homotopical algebra,

1.1.2 Approaches to homotopical algebra

Dealing with homotopical phenomena can be quite complicated as it is often hard to get a concrete grasp on our objects. Indeed, as the above discussion shows, at its extreme end, homotopical algebra involves keeping track of an infinite hierarchy of morphisms, which is not always easy in practice. The hands-on approach usually does not cut it, so one of the main burning questions when working with some version of homotopical algebra becomes finding a framework that is rigid enough to grasp the objects we are working with, but flexible enough to actually prove things. In this section, we introduce two such prevalent frameworks, namely model categories and ∞-categories.

1.1.2.1 Model categories

Historically, one common way to deal with homotopical phenomena is to use model categories. This is especially true if one comes from the direction of homotopy theory, where the broad goal for a long time was, and still is to some degree, to "classify continuous maps between spaces up to homotopy". The set of homotopy classes of (based) continuous maps $Y \to X$ is typically denoted $[Y, X]$. The most relevant case is when Y is the n-sphere, in which case

$$\pi_n(X) := [S^n, X]$$

is referred to as the n**th homotopy group**[2] of X. The basic motivating question is this: We want to treat homotopy equivalences as if they are isomorphisms, so why not simply add formal inverses to them? The first big road-block with this line of thought is that what we get by inverting an arbitrary class of morphisms might not be a category. Model categories were introduced by Quillen and provide ways to deal with these sorts of set theoretical issues. Briefly, a **model category** $\mathcal{M}$ is a bicomplete category together with three distinguished classes of morphisms, weak equivalences, fibrations, and cofibrations, together with a bunch of axioms [Qui67; Hov99]. The **homotopy category** of $\mathcal{M}$ is the localisation of $\mathcal{M}$ with respect to the weak equivalences. Here, localisation simply means that we formally invert all the weak equivalences $\mathcal{W}$, forcing them to become isomorphisms:

$$\mathrm{Ho}(\mathcal{M}) = \mathcal{M}[\mathcal{W}^{-1}]$$

The model axioms make sure that this can be done without any set-theoretical problems. More specifically, they guarantee that

$$\mathrm{Ho}(\mathcal{M})(Y, X) := [Y, X]$$

as the term was originally indended to be used. Cisinski seems to use the term in a much wider sense in [Cis19] when he writes that it is "the study of the compatibility of localisations with (co)limits". This is certainly relevant to homotopical algebra, but one could argue that it describes homotopical behaviour in a larger generality. Our use of *homotopical algebra* is close to Lurie's use of *higher algebra*, with an affinity towards *homotopical* rather than *higher* as we feel that the former is more descriptive.

[2] When $n = 0$, this is just a set, so using the word "group" is perhaps not so appropriate. For $n \geq 1$ they are groups, though. For $n \geq 2$, they are additionally abelian by an Eckmann–Hilton argument.

is a set for all objects X and Y, so that $\mathrm{Ho}(\mathcal{M})$ is actually a category.

The employment of homotopical algebra in the framework of model categories has been shown to be quite powerful. Some notable examples are Quillen's work in rational homotopy theory [Qui69] and, more recently, Voevodsky's proof of the Milnor conjecture [Voe03b; Voe03a] and later the more general Bloch–Kato conjecture [Voe11], which heavily rely on model categorical methods. However, it is worth noting that a lot of possibly important information goes lost when we pass from a model category to its homotopy category. While you remember that two maps are homotopic, you lose the information on *how* they were homotopic in the first place. As in, you lose explicit information about the homotopy between the two homotopic maps.

Another problem is that homotopical phenomena interact quite badly with colimits and limits. To illustrate by an example, consider the category **CW** consisting of CW-complexes and continuous maps between them. Along the program described above, we can deal with homotopical questions in **CW** by endowing this category with a suitable model structure. We can then formally invert the homotopy equivalences (which are the weak equivalences in this model category) to obtain the category

$$\mathcal{H} = \mathrm{Ho}(\mathbf{CW})\,.$$

This is referred to as the **classical homotopy category**[3]; the objects are still CW-complexes, but the morphisms are now homotopy classes of continuous maps. For a simple example of how colimits interact badly with such constructions, consider the two diagrams

$$D = \left(\begin{array}{ccc} S^0 & \longrightarrow & * \\ \big\downarrow & & \\ * & & \end{array}\right) \quad \text{and} \quad D' = \left(\begin{array}{ccc} S^0 & \longrightarrow & I \\ \big\downarrow & & \\ I & & \end{array}\right),$$

where the two points of S^0 are sent to the endpoints of the intervals I in diagram D'. The two diagrams are levelwise homotopy equivalent, but note that their respective colimits, the pushouts, are not. The pushout of the left hand side is a point $*$, but the pushout of the right hand side is the circle S^1, which is certainly not contractible. Classically, the solution to this problem is to introduce homotopy colimits, but *a priori* this is just a method of constructing something that "behaves like a colimit" and is invariant under levelwise homotopy equivalences of diagrams. In particular, homotopy colimits do not have a similar universal property to that of ordinary colimits, neither in the category **CW**, nor in the category $\mathcal{H}$.

[3] Many different model categories can give rise to equivalent homotopy categories, and this is certainly true for the classical homotopy category. We can alternatively take the category of simplicial sets and give its standard Kan-Quillen model structure.

1.1.2.2 ∞-categories

To be able to use homotopical algebra to its best potential, you have to find a way to encode all the information on higher morphisms between the objects you want to study. One way of doing this, which seems to become more and more prevalent, is ∞-categories, as developed by the likes of Boardman–Vogt, Joyal, and Lurie [BV73; Joy02; Lur09; Lur17]. These are certain simplicial sets that can be said to behave like categories. We emphasize that the combinatorial behaviour of simplicial sets makes them extremely practical to work with in many situations.

Definition 1.1.3. An **∞-category**[4] is a simplicial set X in which every inner horn $\Lambda^n_k \to X$ can be extended to a simplex $\Delta^n \to X$.

To explain in which sense simplicial sets are category-like, we recall the nerve functor. The nerve functor is defined as evaluation at the fully faithful inclusion functor from the simplex category to the category of categories, in the sense that:

$$N : \mathrm{Cat} \longrightarrow \mathrm{sSet}\,, \quad \mathscr{C} \mapsto \mathrm{Hom}_{\mathrm{Cat}}(i(-), \mathscr{C})\,.$$

The reader can check for themself that the 0-simplices in the simplicial set $N\mathscr{C}$ are given by the objects in $\mathscr{C}$, the 1-simplices by the morphisms in $\mathscr{C}$, the 2-simplices by composable pairs of morphisms, and so on. We can let this serve as a paradigm when thinking of a simplicial set X as category-like:

- We can think of a 0-simplex $v : \Delta^0 \to X$ as an object of X.
- We can think of a 1-simplex $\phi : \Delta^1 \to X$ as a morphism[5] of X from the source $x = d_1(\phi)$ to the target $y = d_0(\phi)$.
- We can think of a 2-simplex $\sigma : \Delta^2 \to X$ as witnessing that $d_1(\sigma)$ is the composition of the map $\phi = d_2(\sigma)$ and $\psi = d_0(\sigma)$, informally visualised as the diagram:

$$\begin{tikzcd} & d_0(\phi) = d_1(\psi) \arrow[dr, "\psi = d_0(\sigma)"] & \\ d_1(\phi) \arrow[ur, "\phi = d_2(\sigma)"] \arrow[rr, "\psi\phi = d_1(\sigma)"'] & & d_0(\psi) \end{tikzcd}\,.$$

Nerves of categories are simplicial sets exhibiting a certain specific property: every inner horn $\Lambda^n_k \to N\mathscr{C}$ can be *uniquely* extended to a simplex $\Delta^n \to N\mathscr{C}$; see [Lur09, Proposition 1.1.2.2]. In particular, note that nerves of categories are examples of ∞-categories. However, there are many examples of ∞-categories that cannot be written as the nerve of an ordinary category. Using the category-like description of the simplicial set X, the inner 2-horn condition of Definition 1.1.3 is supposed to tell us that any pair of composable morphisms in X

[4] Also called quasi-category or weak Kan complex.

[5] From this point of view, it makes sense to think of the degenerate 1-simplex $\mathrm{id}_v = s_0(v)$ as the identity morphism of the object $v : \Delta^0 \to X$.

has a composite. Indeed, any pair of composable morphisms $\phi, \psi : \Delta^1 \to X$ determines an inner 2-horn $\Lambda^2_1 \to X$, informally visualised as:

$$\begin{tikzcd} & d_0(\phi) = d_1(\psi) \arrow[dr, "\psi"] & \\ d_1(\phi) \arrow[ur, "\phi"] \arrow[rr, dotted] & & d_0(\psi) \end{tikzcd}.$$

By the inner 2-horn condition, the dotted map in the diagram exists. While this composite is not unique in a strict sense, the rest of the inner horn conditions guarantee that it is unique in the sense of higher category theory. Indeed, possible compositions of ϕ and ψ form a simplicial set, and the condition that we have lifts for *all* inner horns guarantees that this simplicial set is actually a Kan complex, and even more, that this Kan complex is contractible. If we want to emphasize this higher categorical view of uniqueness we usually speak of something "being unique up to contractible choice"[6].

The concept of ∞-categories solves the conundrum regarding homotopy and colimits by adding something lying in between the category of CW-complexes and the classical homotopy category:

$$\mathbf{CW} \longrightarrow \mathcal{S} \longrightarrow \mathcal{H}$$

This something is the ∞-category of spaces[7]. There are ∞-categorical interpretations of colimits and limits, with suitable universal properties [Lur09, Chapter 4]. The ∞-category $\mathcal{S}$ has all of these, and what is classically referred to as a homotopy (co)limit in $\mathbf{CW}$ is mapped to such a (co)limit in $\mathcal{S}$.

1.1.3 Spectra

One common description of higher category theory is that it is category theory not built on sets, but on spaces. From this point of view, we can understand

[6] A way to rigorously state this sort of uniqueness of compositions as the characterising feature of an ∞-category is given by the following result.

Theorem 1.1.4 (Joyal). *A simplicial set X is an ∞-category if and only if the restriction map* $\mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$ *is an acyclic Kan fibration.*

We can think of $\mathrm{Map}(\Lambda^2_1, X)$ as "the space composition problems in X" and of $\mathrm{Map}(\Delta^2, X)$ as the "the space of solutions to composition problems". The theorem above tells us that the characterizing property of an ∞-category is that these two spaces are the same, from a homotopical point of view.

[7] Whether it is reasonable to call the objects in this ∞-category "spaces" is up for debate. From this historical account it seems reasonable, but it is worth noting that both the ∞-category $\mathcal{S}$ and the category $\mathcal{H}$ can be constructed in many other ways, that do not necessarily make use of spaces, as we usually think of them. Perhaps a better term for the objects in $\mathcal{S}$ is "homotopy types" or "∞-groupoids". In [CS19], the authors argue for the terminology "anima", in the sense of the "soul" of a space. Indeed, the ∞-category $\mathcal{S}$ can be obtained from the category of sets in a process of freely adding sifted colimits, which the authors refer to as "animation". We leave it to the reader to make up their own mind on what they think is the best word, but stick with "spaces" in this thesis, as it is the most well-used terminology at the time of writing.

homotopical algebra as algebra not built on sets, but on spaces. It turns out that the world of classical algebra can be embedded faithfully into the world of homotopical algebra and this allows for several interesting generalisations of algebra into a homotopical setting. The abelian groups of homotopical algebra are referred to as spectra. In the next couple of sections we explain what spectra are from various points of view.

1.1.3.1 Spectra as stable spaces

The earliest motivation behind spectra is that they describe a relatively well-behaved part of homotopy theory, namely stable behaviours. This motivation can be traced back to Freudenthal's suspension theorem, which tells us that the sequence

$$[Y, X] \longrightarrow [\Sigma Y, \Sigma X] \longrightarrow \cdots \longrightarrow [\Sigma^n Y, \Sigma^n X] \longrightarrow \cdots$$

of homotopy classes of maps will eventually stabilise [Fre38]. In the case that Y is an n-sphere, we write

$$\pi_n^{\mathrm{st}}(X) = \operatorname*{colim}_k \pi_{n+k}(\Sigma^k X)$$

for this stabilised value and call it the n**th stable homotopy group** of X. The subfield of **stable homotopy theory** can roughly be understood as the study of stable homotopy groups. Freudenthal's suspension theorem suggests the idea of introducing a category with objects that reflect this stable phenomenon and in which it is natural to study these types of stable behaviours. This category is known as the category of spectra, and was first introduced by Lima [Lim60]. In his sense, a **spectrum** X is simply an infinite sequences of pointed spaces $\{X_n\}_{n=0}^{\infty}$ equipped with continuous maps $\Sigma X_n \to X_{n+1}$ from the suspension of the nth space to the $(n+1)$th space. Note that every space K naturally gives rise to a spectrum, the **(unreduced) suspension spectrum** $\Sigma^\infty_+ K$, where the nth space is the nth suspension of K_+, with the obvious continuous maps between the different levels. Arguably, the most important example is the **sphere spectrum** which is simply defined as the suspension spectrum of the point:

$$\mathbb{S} = \Sigma^\infty_+ * \ .$$

The sphere spectrum $\mathbb{S}$ plays the same role in homotopical algebra, as the integers $\mathbb{Z}$ play in classical algebra; it is the initial "commutative" ring of homotopical algebra. For a more algebraic example of spectra, recall that, given an abelian group A and non-negative integer n, it is always possible to construct a space $K(A, n)$, the nth Eilenberg–Mac Lane space of A, in such a way that

$$\pi_k K(A, n) = \begin{cases} A & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

It is known that there are homotopy equivalences $\Omega K(A, n+1) \simeq K(A, n)$, so that we have left adjoint maps $\Sigma K(A, n) \to K(A, n+1)$. These make sure that the Eilenberg–Mac Lane spaces assemble into a spectrum, called the **Eilenberg–Mac Lane spectrum** of A and denoted HA.

1.1.3.2 Spectra as cohomology theories

The example of Eilenberg–Mac Lane spectra at the end of the previous section shows how spectra can be algebraic in nature. However, *a priori* there is nothing intrinsically algebraic about them, in general. Possibly, a better explanation for in what way they are algebraic is through Atiyah–Whitehead work on generalised cohomology theories. Recall that a **generalised cohomology theory** h^* is sequence of functors $h^n : \mathcal{H}^{\mathrm{op}} \to \mathrm{Ab}$ together with natural isomorphisms $\sigma : h^n \to h^{n+1} \circ \Sigma$, satisfying exactness and additivity [Whi62]. These conditions guarantee, via Brown's representability theorem [Bro62], that the functors $h^n : \mathcal{H}^{\mathrm{op}} \to \mathrm{Set}$ are all representable. That is, we can find pointed CW-complexes E_n such that

$$h^n(X) \cong [X, E_n].$$

Using the suspension isomorphism for generalised cohomology theories we have bijections

$$[X, E_n] \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}]\,,$$

and by full faithfulness of the Yoneda embedding we thus have homotopy equivalences $E_n \to \Omega E_{n+1}$. The left adjoint $\Sigma E_n \to E_{n+1}$ of this map is the wanted structure map in our spectrum $\{E_n\}_{n=0}^{\infty}$. Hence, the spaces E_n assemble into a spectrum. The converse is also true: every spectrum gives rise to a homology and cohomology theory. This suggests that we want to put a model structure on the category of spectra such that two spectra are equivalent if and only if they give rise to isomorphic (co)homology theories. This leads us to the definition of the **stable homotopy category** $\mathcal{SHC}$; what we get if we localise with respect to the weak equivalences in that model structure [BF78].

Ordinary cohomology with coefficients in the abelian group A gives us back the Eilenberg–Mac Lane spectrum via the above discussion. However, there are many examples of more exotic versions of cohomology theories, each of which gives rise to its own spectrum. Some examples are:

- Various flavours of topological K-theory: complex topological K-theory KU, real topological K-theory KO, and quaternionic topological K-theory KSp, ...

- Various flavours of cobordism[8]: complex cobordism MU, unoriented cobordism MO, oriented cobordism MSO,...

- Elliptic cohomology: elliptic curves gives rise to formal groups which in turn give rise to cohomology theories via Landweber's exact functor theorem [Lan76].

[8]The sphere spectrum also belongs here and corresponds to framed cobordism via the Pontryagin–Thom theorem.

1.1.3.3 Spectra as an analogy of unbounded chain complex

One alternative way of viewing spectra, which might be useful for people that care neither about stable behaviour of spaces nor generalised cohomology theories, is as follows:

> *Spectra are to spaces, what unbounded chain complexes are to non-negatively graded chain complexes*[9].

This can give a reason for working with spectra, at least if the reader sees the point in working with chain complexes and agrees that it is generally a stupid idea to restrict mathematics to the setting of non-negatively graded ones. Let us imagine a world where we only have access to non-negatively graded chain complexes. How would we construct the category of unbounded chain complexes from this? By thinking backwards, we could start by noting that every unbounded chain complex can be written as a colimit of bounded below ones simply by truncating:

$$C = \operatorname*{colim}_{n\to\infty} \tau_{\geq -n} C\,.$$

This allows us to describe unbounded chain complexes as colimits of bounded below chain complexes. A bounded below chain complex can be made into a non-negatively graded chain complex by suspending it enough times. Indeed, let us write

$$C^{(n)} = (\tau_{\geq -n} C)[n]\,, \quad n \geq 0\,,$$

and notice that this is always a non-negatively graded chain complex. In terms of non-negative chain complexes, suspensions, and colimits, our original chain complex can be written as the colimit of the system

$$C^{(0)} \longrightarrow C^{(1)}[-1] \longrightarrow C^{(2)}[-2] \longrightarrow \cdots\,.$$

This data could alternatively be phrased as:

1. A sequence $\{C^{(n)}\}_{n=0}^{\infty}$ of non-negative chain complexes.
2. A chain map $C^{(n)}[1] \to C^{(n+1)}$ for every non-negative integer n.

Compare this to the definition of a spectrum from Section 1.1.3.1. This might provide meaning to the concept of spectra, especially if the reader is already using simplicial methods to deal with questions concerning the derived category of chain complexes, via the Dold–Kan correspondence

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}) \simeq \mathrm{Ch}(\mathrm{Ab})_{\geq 0}\,.$$

A simplicial abelian group is in particular a Kan complex, which is what is usually interpret as a "space" in the theory of ∞-categories. From this point of view, it seems reasonable to make the switch from simplicial methods to homotopical algebra when you want to better understand unbounded chain complexes and the derived category of such.

[9] Thank you to Fabian Hebestreit for calling me in the middle of the night to explain this point of view when he had his eureka moment on spectra. It was as enlightening to me.

1.2 Spectral sequences

If spectra are the mathematical objects that we are interested in, then spectral sequences are the tools that we will use to study them with[10]. Both of the papers included in this thesis deal with spectral sequences, in some way or another. We discuss in order: what spectral sequences are, different ways they arise, what we mean by convergence of spectral sequences, and how to deal with multiplicative structures. In particular, although it is of course an important part of the subject, we will *not* discuss how to work with, manipulate, and compute with spectral sequences. This often depends very heavily on the spectral sequence in question, and it is very hard to say something in general. Instead, we focus on the aim of giving the reader a feel for what they are and how to think about them, and refer the reader who is hungry for more to [McC01]

1.2.1 Spectral sequences

Since their conception by Leray [Ler46], spectral sequences have proven to be incredibly useful tools in various subjects of mathematics. One can view them as a generalisation of the concept of an exact sequence, and they are primarily used for the same purpose, namely for computations of homotopy and/or homology groups. Let us start with the most basic definition. In what follows, we will consider the category of abelian groups, although similar definitions can be made in any abelian category. A **spectral sequence** (of abelian groups) consists of the following data:

1. for every integer $r \geq 1$, a bigraded abelian group $E^r = E^r_{p,q}$;
2. for every integer $r \geq 1$, a map $d^r : E^r \to E^r$ of bidegree $(-r, r-1)$ such that $d^r \circ d^r = 0$;
3. for every integer $r \geq 1$, an isomorphism $E^{r+1} \cong H(E^r, d^r)$ of bigraded abelian groups, where H refers to taking homology.

A morphism of spectral sequences is a collection of morphisms of bigraded abelian groups compatible with the differentials and with the isomorphisms $E^{r+1} \cong H(E^r, d^r)$, in the obvious way. This makes spectral sequences into a category which we denote as SSEQ. It is common to refer to the bigraded abelian group $E^r_{*,*}$ as the rth page of the spectral sequence, and to visualise it as a page in an imagined book, where we pass from one page to the next by taking homology.

Remark 1.2.1. There are many other grading conventions for spectral sequences. The one described above is called homological Serre grading. Another grading convention that is used is homological Adams grading $E^r_{n,s}$. To go between these two grading conventions we can use the linear transformations

$$(n, s) \mapsto (-s, n+s) \quad \text{and} \quad (p, q) \mapsto (p+q, -p)\,.$$

[10]Disclaimer: there is really no etymological connection between *spectra* and *spectral sequences* other than derivatives of the word *spectrum* being overused in mathematics.

In homological Adams grading the d^r-differentials would go

$$d^r_{n,s} : E^r_{n,s} \to E^r_{n-1,s+r} .$$

Note that the grading conventions are not consistent between the papers contained in this thesis!

A common situation where spectral sequences arise is when considering filtrations of mathematical objects. In this way, spectral sequences provide means to translate homotopical information into algebraic information, that can then be processed in the standard fashion of homological algebra. This will be our main point of view on spectral sequences in this thesis; that they are convenient ways to store and process large amounts of mathematically information. One could argue that this is the source of both their their usefulness and difficulty. There are a number of convenient stepping-stones when passing from a filtration to a spectral sequence, which we now cover briefly. In all the sections below, $\mathscr{A}$ denotes the graded abelian category of abelian groups.

1.2.1.1 Exact couples

After Massey [Mas52, Section 1.4], we define an **(unrolled) exact couple** as a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A_{s-2} \arrow[rr, "\alpha_{s-1}"] & & A_{s-1} \arrow[rr, "\alpha_s"] \arrow[dl, "\beta_{s-1}"] & & A_s \arrow[r] \arrow[dl, "\beta_s"] & \cdots \\
& & E_{s-1} \arrow[ul, "\gamma_{s-1}"] & & E_s \arrow[ul, "\gamma_s"] & &
\end{tikzcd}$$

in $\mathscr{A}$, in which every triangle is exact. Here, the internal degrees of the maps α_s, β_s, and γ_s are 0, 0, and -1, respectively. Such an object gives rise to a spectral sequence by setting the E^1-page and d^1-differential to be

$$E^1_{s,*} = E_s \quad \text{and} \quad d^1 = \beta_{s-1} \circ \gamma_s .$$

The higher pages are given by considering the part

$$\begin{tikzcd}
\cdots \arrow[r] & A_{s-r-1} \arrow[rr, "\alpha_{s-r}"] & & A_{s-r} \arrow[r] \arrow[dl, "\beta_{s-r}"] & \cdots \arrow[r] & A_{s+1} \arrow[rr, "\alpha_s"] & & A_s \arrow[r] \arrow[dl, "\beta_s"] & \cdots \\
& & E_{s-r} \arrow[ul, "\gamma_{s-r}"] & & & & E_s \arrow[ul, "\gamma_s"] & &
\end{tikzcd}$$

of the exact couple; see [HR19, Lemma 3.4]. While exact couples are useful for building additive spectral sequences, they have the disadvantage in that there is no useful notion of a pairing of exact couples; we will return to this point in the section dealing with multiplicative structures on spectral sequences.

1.2.1.2 Cartan–Eilenberg systems

For multiplicative considerations it is often convenient to work with Cartan–Eilenberg systems [CE56, Section XV.7]. Consider the following two categories:

- The category $\mathbb{Z}^{[1]}$ whose objects are pairs (i,j) of integers with $i \leq j$, and where have a single morphism $(i,j) \to (i',j')$ precisely when $i \leq i'$ and $j \leq j'$.
- The category $\mathbb{Z}^{[2]}$ whose objects are triples (i,j,k) of integers with $i \leq j \leq k$, and where have a single morphism $(i,j,k) \to (i',j',k')$ precisely when $i \leq i'$, $j \leq j'$, and $k \leq k'$.

Note that we have three obvious functors $d_0, d_1, d_2 \colon \mathbb{Z}^{[2]} \longrightarrow \mathbb{Z}^{[1]}$ and two natural transformations $\iota \colon d_2 \longrightarrow d_1$ and $\pi \colon d_1 \longrightarrow d_0$. We define an **finite Cartan–Eilenberg system** as a pair (H, ∂) where $H \colon \mathbb{Z}^{[1]} \to \mathscr{A}$ is a functor and $\partial \colon Hd_0 \to Hd_2$ is a natural transformation, such that the triangle

$$\begin{tikzcd} Hd_2 \arrow[rr, "H\iota"] & & Hd_1 \arrow[dl, "H\pi"] \\ & Hd_0 \arrow[ul, "\partial"] & \end{tikzcd}$$

is exact. Adding an initial object $-\infty$ and terminal object ∞ to the poset $\mathbb{Z}$ gives us the notion of an **extended Cartan–Eilenberg** system. An extended Cartan–Eilenberg system thus associates to each pair (i,j) with $-\infty \leq i \leq j \leq \infty$ a graded abelian group $H(i,j)$, in a functorial way. Furthermore, it associates to each triple (i,j,k) with $-\infty \leq i \leq j \leq k \leq \infty$ a long exact sequence

$$\ldots \longrightarrow H(i,j) \longrightarrow H(i,k) \longrightarrow H(j,k) \xrightarrow{\partial} H(i,j) \longrightarrow \ldots,$$

where ∂ is a natural transformation of total degree -1.

Remark 1.2.2. The notion of a finite Cartan–Eilenberg system is connected to the notion of a $\mathbb{Z}$-complex in the ∞-category of spectra, as in [Lur17, Definition 1.2.2.2]. Indeed, given a $\mathbb{Z}$-complex $X : \mathbb{Z}^{[1]} \to \mathrm{Sp}$ the composition

$$\mathbb{Z}^{[1]} \xrightarrow{X} \mathrm{Sp} \xrightarrow{\pi_*} \mathscr{A}$$

forms a finite Cartan–Eilenberg system.

An extended Cartan–Eilenberg system gives rise to an exact couple by setting

$$A_s = H(-\infty, s) \quad \text{and} \quad E_s = H(s-1, s)$$

and

$$\begin{aligned} \alpha_s &: H(-\infty, s-1) \longrightarrow H(-\infty, s) \\ \beta_s &: H(-\infty, s) \longrightarrow H(s-1, s) \\ \gamma_s &: H(s-1, s) \longrightarrow H(-\infty, s-1) \end{aligned}$$

where the first two maps are induced by the maps $s-1 \leq s$ and $-\infty \leq s-1$ in the poset $\mathbb{Z}$, while the last map is induced by the natural transformation ∂ in the Cartan–Eilenberg system [HR19, Section 7].

1.2.1.3 Décalage

Informally, décalage is a way to make sense of "turning a page in the spectral sequence" on the level of filtrations. This was first introduced by Deligne in relation to his studies on Hodge structures [Del71]. Without going into too much detail: given a filtered chain complex (K, F), the associated decalée $\mathrm{Déc}(K)$ is a new filtered chain complex $(K, \mathrm{Déc}(F))$. This new filtered chain complex is constructed in such a way that the spectral sequence associated to $(K, \mathrm{Déc}(F))$ is isomorphic, after reindexing, to the spectral sequence associated to (K, F), but shifted forward one page:

$$E^r_{n,s}(\mathrm{Déc}(K)) \cong E^{r+1}_{n,s-n}(K)$$

in homological Adams grading. Although not originally phrased in this language, we can make sense of décalage using a t-structure on the derived filtered category, called the Beilinson t-structure [Beĭ87; BMS19]. We will study this approach to spectral sequences in Paper II.

1.2.2 Convergence

One of the main questions when working with spectral sequences is:

Is the spectral sequence computing what we want it to compute?

This is the question of convergence of spectral sequences. At the inception of the subject of spectral sequences, dealing with this question usually involved imposing quite severe finiteness conditions on the objects, but as the subject developed it became apparent that better considerations were needed. One groundbreaking article is [Boa99] and its introduction of the notion of conditional convergence.

In order to talk about convergence, we need to first establish some terminology regarding filtrations of abelian groups. For us, a **filtration** is simply a sequence of injective homomorphisms of abelian groups

$$\cdots \longrightarrow F^{q+1} \longrightarrow F^q \longrightarrow F^{q-1} \longrightarrow \cdots .$$

We consider this as an abstract filtration, and not as a filtration of a specific group, though we could of course say that is a filtration of $\mathrm{colim}_q F^q$. In this sense, all filtrations we work with are exhaustive, in Boardman's terminology. A filtration is called **derived complete** if the total derived inverse limit

$$\operatorname*{Rlim}_q F^q \simeq 0$$

vanishes.

Remark 1.2.3. We warn the reader that we use the symbol Rlim differently from Boardman here. What we mean by Rlim is the total right derived functor of the limit, and not just its first right derived functor. To clarify, for us, the derived inverse limit $\mathrm{Rlim}_q F^q$ is an object of $\mathcal{D}\mathbb{Z}_{[-1,0]}$, since the sequential limit functor only has two non-vanishing right derived functors. What Boardman writes as $\mathrm{Rlim}_q F^q$, we would instead write as

$$H_{-1}(\operatorname*{Rlim}_q F^q) = \lim_q{}^1 F^q \,.$$

Hence, being derived complete is equivalent to being complete and Hausdorff, in Boardman's terminology.

Given a spectral sequence $E^{\star}_{*,*}$, the E^2-page has a filtration

$$0 = B^2_{p,q} \subset B^3_{p,q} \subset \cdots \subset B^r_{p,q} \subset \cdots \subset Z^r_{p,q} \subset \cdots \subset Z^3_{p,q} \subset Z^2_{p,q} = E^2_{p,q}$$

of abelian groups in such a way that

$$E^r_{p,q} \cong Z^r_{p,q}/B^r_{p,q} \,.$$

We write

$$Z^\infty_{p,q} = \lim_r Z^r_{p,q} \quad \text{and} \quad B^\infty_{p,q} = \operatorname*{colim}_r B^r_{p,q}$$

and call these the **infinite cycles** and the **infinite boundaries**, respectively. The abelian groups

$$E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q} \quad \text{and} \quad RE^\infty_{p,q} = \lim_r{}^1 Z^r_{p,q}$$

are referred to as the **limit page** and the **derived limit page** of the spectral sequence, respectively. The point of convergence is to connect these objects, which are internal to the spectral sequence, to some filtration on the wanted target of the spectral sequence. We start with arguably the most useful notion of convergence. **Strong convergence** of a spectral sequence $E^{\star}_{*,*}$ to a graded group G_* consists of:

1. A derived complete filtration $F^{\star}_*$ for every integer $*$.

2. An isomorphism
$$E^\infty_{p,q} \cong F^q_{p+q}/F^{q+1}_{p+q}$$
for every pair of integers p and q.

3. An isomorphism
$$G_* \cong \operatorname*{colim}_q F^q_*$$
for every integer $*$.

We will often abusively say "the spectral sequence converges strongly to G_*" even though strong convergence is technically not a property of a spectral sequence, but rather extra structure. If our spectral sequence is strongly convergent[11], then G_* can be recovered from the spectral sequence in question in the strongest possible sense via [Boa99, Proposition 2.5]. However, there are a lot of spectral sequences that are not *a priori* strongly convergent. For those, Boardman introduced the notion of conditional convergence. If we are given an exact couple such that

$$\operatorname{Rlim}_s A_s \simeq 0\,,$$

we say that the associated spectral sequence **converges conditionally**. Note that conditional convergence is slightly unsatisfactory from a structural point of view in the sense that conditional convergence technically is a property of an exact couple, and not a property or structure on the spectral sequence itself. In other words, given a spectral sequence, with no information on how it arose, the question "Does the spectral sequence converge conditionally?" does not even makes sense. Nevertheless, the concept of conditional convergence is very useful in that it allows one to deduce strong convergence from conditions that are entirely internal to the spectral sequence in question, and which in many cases are easy to check. Indeed, depending on what sort of spectral sequence you have, the following table summarises sufficient criteria for strong convergence [Boa99, Theorem 6.1, Theorem 7.3, Theorem 8.2]:

half-plane with exiting differentials	conditional convergence
half-plane with entering differentials	conditional convergence + vanishing of the derived limit page RE^∞
whole-plane	conditional convergence + vanishing of the derived limit page RE^∞ + vanishing of Boardman's whole-plane obstruction W

Let us end this section by discussing the term W appearing in the last entry, which we have yet to explain. Instead of introducing the necessary terminology for introducing it in Boardman's language we refer to [HR19] where the authors give a simplified description of Boardman's obstruction group in terms of Cartan–Eilenberg systems. They show that it can be expressed as the kernel

$$W \cong \ker(\kappa)$$

of the canonical colimit-limit exchange map

$$\kappa : \operatorname*{colim}_i \lim_j H(i,j) \longrightarrow \lim_j \operatorname*{colim}_i H(i,j)\,.$$

[11]Or more correctly: "is endowed with the structure of strong convergence".

We note that although Boardman's obstruction group depends on the exact couple and/or the Cartan–Eilenberg system, there are criteria internal to the spectral sequence itself that guarantee the vanishing of Boardman's obstruction group [Boa99, Lemma 8.1].

Remark 1.2.4. As mentioned in the previous section, the notion of a spectral sequence makes sense more generally in an abelian category $\mathscr{A}$. To make sense of convergence for such spectral sequences, we need some assumptions on the abelian category, though. Assuming that sequential colimits and limits in $\mathscr{A}$ behaves as in the category of abelian groups will do the trick. What "behaving as" should mean is subtle, though. One might expect that we want sequential colimits and infinite countable products to be exact. In the terminology of Grothendieck's Tohoku paper [Gro57], we should require the abelian category $\mathscr{A}$ to satisfy AB5 and AB4*. However, in light of Neeman's counterexample to AB4* being sufficient to guarantee that $\lim^1$ vanishes on Mittag–Leffler sequences [Nee02], and Boardman heavily making use of Mittag–Leffler short exact sequences in his paper, we refrain from making any definite claims on this matter.

1.2.3 Multiplicative structures

One of the main foci of this thesis is multiplicative structures on spectral sequences. When the object we want to study has some extra structure, like some sort of pairing, it is useful, if not often essential, to incorporate this structure in the spectral sequence used to study the object. Such a structure can then be heavily exploited in computations.

The category of spectral sequences is not a symmetric monoidal category, so it does not make sense to talk about the tensor product of two spectral sequences. However, it does make sense to talk about multilinear maps of spectral sequences. This makes the category of spectral sequences into a multicategory, or, as it is also called, a coloured operad. Given spectral sequences (C^r, d^r), (D^r, d^r), and (E^r, d^r), a **bilinear map** (or **pairing**)

$$\phi : (C^\star_{*,*}, D^\star_{*,*}) \longrightarrow E^\star_{*,*}$$

is a collection of morphisms

$$\phi^r : C^r_{p,q} \otimes D^r_{p',q'} \longrightarrow E^r_{p+p',q+q'}$$

such that the following conditions hold:

1. We have
$$d^r\phi^r = \phi^r(d^r \otimes 1 + 1 \otimes d^r)$$
as morphisms $C^r_{p,q} \otimes D^r_{p',q'} \to E^r_{p+p'-r,q+q'+r-1}$.

2. The diagram

$$\begin{tikzcd} C^{r+1}_{p,q} \otimes D^{r+1}_{p',q'} \arrow[rr, "\phi^{r+1}"] \arrow[d] & & E^{r+1}_{p+p',q+q'} \arrow[d, "\cong"] \\ H_{p+p',q+q'}(C^r_{*,*} \otimes D^r_{*,*}) \arrow[rr, "H(\phi^r)"] & & H(E^r_{n+n',s+s'}) \end{tikzcd}$$

commutes.

The three ways of passing from filtrations to spectral sequences that we covered in Section 1.2.1 are more and less suitable for dealing with multiplicative structures on the associated spectral sequences. As already mentioned, exact couples are at a disadvantage in that there is no useful notion of a pairing of exact couples[12]. To deal with multiplicative questions it is therefore better to use one of the two other constructions: Cartan–Eilenberg systems or décalage. There is a suitable definition for a pairing of a finite Cartan–Eilenberg system. Indeed, given finite Cartan–Eilenberg systems (H', ∂), (H'', ∂) and (H, ∂), a **pairing**

$$\phi : (H', H'') \to H$$

is a collection of homomorphisms

$$\phi_r \colon H'(i-r, i) \otimes H''(j-r, j) \longrightarrow H(i+j-r, i+j)$$

of total degree 0, for all integers i and j and $r \geq 1$. These are required to satisfy the following two conditions:

1. Each square

$$\begin{tikzcd} H'(i-r, i) \otimes H''(j-r, j) \arrow[r, "\phi_r"] \arrow[d] & H(i+j-r, i+j) \arrow[d] \\ H'(i'-r', i') \otimes H''(j'-r', j') \arrow[r, "\phi_{r'}"] & H(i'+j'-r', i'+j') \end{tikzcd}$$

commutes, for all integers i, j, i', j' and $r, r' \geq 1$ with $i \leq i'$, $i - r \leq i' - r'$, $j \leq j'$ and $j - r \leq j' - r'$.

[12] The reader might disagree by referring to the paper [Mas54]. However, the structure and properties involved in the notion of a "pairing of exact couples" according to Massey are abundant enough to the point that the claim that a pairing of exact couples leads to a pairing of spectral sequences is essentially a tautology. In practice, the conditions one would need to check in order to show that one has a pairing of exact couples are essentially the same ones one would need to check in order to show that one has a pairing of the associated spectral sequences, rendering the use of exact couples as a stepping stone between filtrations and spectral sequences pointless when dealing with multiplicative questions. Hence our phrasing "no useful notion".

2. In the (non-commutative) diagram

$$\begin{tikzcd}
H'(i-r,i)\otimes H''(j-r,j) \arrow[rr, "\eta\otimes\partial"] \arrow[d, "\partial\otimes\eta"'] \arrow[dr, "\phi_r"] & & H'(i-1,i)\otimes H''(j-r-1,j-r) \arrow[d, "\phi_1"] \\
 & H(i+j-r,i+j) \arrow[dr, "\partial"] & \\
H'(i-r-1,i-r)\otimes H''(j-1,j) \arrow[rr, "\phi_1"] & & H(i+j-r-1,i+j-r)
\end{tikzcd}$$

the inner composition is the sum of the two outer ones:

$$\partial\phi_r = \phi_1(\partial\otimes\eta) + \phi_1(\eta\otimes\partial)\,.$$

In terms of elements, this identity in $H(i+j-r-1,i+j-r)$ can be written

$$\partial\phi_r(x\otimes y) = \phi_1(\partial x\otimes\eta y) + (-1)^{\|x\|}\phi_1(\eta x\otimes\partial y)$$

for $x\in H'(i-r,i)$ of total degree $\|x\|$ and $y\in H''(j-r,j)$.

This definition was exploited by Douady to show that such a pairing of Cartan–Eilenberg systems gives rise to a pairing of the associated spectral sequences [Dou59a; Dou59b].

1.3 The Tate construction

The specific topic in homotopical algebra that this thesis is concerned about is the Tate construction. This construction was first introduced by Greenlees and should be seen as a generalisation of Tate cohomology to the setting of spectra. In particular, the research in this thesis concerns the Tate spectral sequence, which is a spectral sequence that computes the homotopy groups of the Tate construction on some spectrum with group action. We start by discussing Tate cohomology, to explain the classical context, and go on to define the Tate construction in the setting of G-spectra. Next, we explore the Tate spectral sequence and what issues one might expect to pop up when studying this spectral sequence. Lastly, I explain my own personal reason for studying the Tate spectral sequence, which is connected to my interest in topological Hochschild homology and algebraic K-theory.

1.3.1 Tate cohomology

Tate cohomology was first introduced by Tate in his study of class field theory [Tat52]. We give a very brief introduction to the subject following [Bro82, Chapter VI] and [CE56, Chapter XII]. In what follows, G will be a finite group and M a G-module. The basic idea is that Tate cohomology is a way to splice together group homology and group cohomology of G with coefficients in M into a single cohomology theory. We usually do this via the so-called norm map: the map from the G-orbits to the G-fixed points of M defined as

$$\mathrm{Nm}_G : M_G \to M^G\,, \quad m \mapsto \sum_{g\in G} g\cdot m\,.$$

The **Tate cohomology groups** of G with coefficients in M can then be defined as

$$\hat{H}^i(G;M) = \begin{cases} H^i(G;M) & i \geq 1, \\ \operatorname{coker}(\mathrm{Nm}_G) & i = 0, \\ \ker(\mathrm{Nm}_G) & i = -1, \\ H_{-i-1}(G;M) & i \leq -2\,. \end{cases}$$

It turns out that the Tate cohomology groups can also be phrased as the (co)homology groups of a (co)chain complex of G-modules. Indeed, let P_* denote a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module, and note that the dual resolution $\mathrm{Hom}(P,\mathbb{Z})_*$ is a 'coresolution' of projective modules since $\mathbb{Z}[G]$ is a quasi-Frobenius algebra: projective and injective modules over $\mathbb{Z}[G]$ coincide. The spliced resolution $\hat{P}_*$, informally visualised as

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[rr, "\mathrm{Nm}_G"] \arrow[dr, "\epsilon"'] & & \mathrm{Hom}(P_0,\mathbb{Z}) \arrow[r] & \mathrm{Hom}(P_1,\mathbb{Z}) \arrow[r] & \cdots\,, \\ & & & & \mathbb{Z} \arrow[ur, "\epsilon^*"'] & & & \end{tikzcd}$$

is referred to as a **complete resolution**. We can then define Tate cohomology as the cohomology groups

$$\hat{H}^i(G;M) = H^i\,\mathrm{Hom}_{\mathbb{Z}[G]}(\hat{P}_*, M)\,.$$

One thing that is worth noting is that Tate cohomology is indeed a multiplicative cohomology theory in the sense that we can define a cup product

$$\smile : \hat{H}^i(G;M) \otimes \hat{H}^j(G;N) \longrightarrow \hat{H}^{i+j}(G;M\otimes N)$$

that extends the cup product on group cohomology, in a suitable sense. Although this can be done using complete resolutions and completed tensor products, as in [Bro82, Section VI.6], we here take the opportunity to introduce another perspective on Tate cohomology that is useful when thinking of the cup product. Given a projective resolution P_*, consider the mapping cone $\widetilde{P}_* = \mathrm{cone}(\epsilon : P_* \to \mathbb{Z})$ of the augmentation. After [Gre95], we define the **Tate complex** via the tensor product

$$T_*(M) = \widetilde{P}_* \otimes \mathrm{Hom}(P_*,\mathbb{Z})\,;$$

this chain complex is quasi-isomorphic to $\mathrm{Hom}(\hat{P}_*, M)$ after taking G-fixed points, by a zig-zag

$$\Big(\widetilde{P}_* \otimes \mathrm{Hom}(P_*,M)\Big)^G \xrightarrow{\sim} \Big(\widetilde{P}_* \otimes \mathrm{Hom}(\hat{P}_*,M)\Big)^G \xleftarrow[\sim]{} \mathrm{Hom}_{\mathbb{Z}[G]}(\hat{P}_*, M)\,.$$

This shows that Tate cohomology can alternatively be described as the homology of the G-fixed points of the Tate complex:

$$\hat{H}^i(G;M) = H_{-i}\left(\widetilde{P}_* \otimes \mathrm{Hom}(P,M)_*\right)^G$$

The advantage with this perspective is that the cup product on Tate cohomology can be described by using G-chain maps

$$\Psi : P_* \to P_* \otimes P_* \quad \text{and} \quad \Phi : \widetilde{P}_* \otimes \widetilde{P}_* \to \widetilde{P}_*$$

lifting the identity map $\mathrm{id} : \mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}$ and extending the fold map $\widetilde{P}_* \oplus_{\mathbb{Z}} \widetilde{P}_* \to \widetilde{P}_*$, respectively. Indeed, for G-modules M and N, the composite pairing

$$\begin{aligned}
\widetilde{P}_* \otimes \mathrm{Hom}(P_*,M) \otimes \widetilde{P}_* &\otimes \mathrm{Hom}(P_*,N) \\
&\longrightarrow \widetilde{P}_* \otimes \widetilde{P}_* \otimes \mathrm{Hom}(P_*,M) \otimes \mathrm{Hom}(P_*,N) \\
&\longrightarrow \widetilde{P}_* \otimes \widetilde{P}_* \otimes \mathrm{Hom}(P_* \otimes P_*, M \otimes N) \\
&\xrightarrow{\Phi\otimes\Psi^*} \widetilde{P}_* \otimes \mathrm{Hom}(P_*, M \otimes N)
\end{aligned}$$

is G-equivariant, and so induces an associative, unital, and graded commutative pairing

$$\smile : \hat{H}^i(G;M) \otimes \hat{H}^j(G;N) \longrightarrow \hat{H}^{i+j}(G;M \otimes N)$$

after passing to homology, which we refer to as the cup product on Tate cohomology. This extends the cup product on ordinary group cohomology in the obvious way.

1.3.2 The Tate construction

The Tate construction on a G-spectrum is the incarnation of Tate cohomology in homotopical algebra. Although it was first defined in the setting of genuine equivariant homotopy theory, it turns out that the Tate construction only depends on the naïve equivariant homotopy type of our spectrum, so to simplify the discussion, we here give an account in a much more naïve setting. We return to the classical point of view at the end of the section, in Remark 1.3.2.

Let G be a topological group, and let BG denote a fixed classifying space for it. We usually refer to the stable ∞-category

$$\mathrm{Sp}^{BG} = \mathrm{Fun}(BG, \mathrm{Sp})$$

as the ∞-category of **G-spectra**[13]. Here Sp denotes the ∞-category of spectra and the reader is encouraged to think of G-spectra as analogues to G-modules. We will follow ordinary ∞-categorical notation; in particular, all colimits and

[13] Again, we point out that these are not genuine G-spectra in the sense of equivariantly homotopy theory, but rather a naïve version. However, all constructions we consider in this thesis depend only on the naïve homotopy type of our spectra, so this is a point that we will sweep under the rug for now.

limits are implicitly derived, and we denote the smash product of spectra by $\otimes$. Given a G-spectrum X, there a two obvious things we can do with it, namely take the (homotopy) colimit or limit of our functor:

$$X_{hG} = \operatorname*{colim}_{BG} X \quad \text{and} \quad X^{hG} = \lim_{BG} X\,.$$

These spectra are referred to as the **homotopy orbits** and **homotopy fixed points** of X, respectively. One can view these as spectrum level versions of group homology and group cohomology, respectively. Indeed, if M is a G-module for some finite group G, then the homotopy orbits and homotopy fixed points of the Eilenberg–Mac Lane spectrum HM recover group homology and group cohomology on homotopy groups:

$$\pi_*(HM_{hG}) \cong H_*(G;M) \quad \text{and} \quad \pi_*(HM^{hG}) \cong H^{-*}(G;M)\,.$$

As mentioned, the Tate construction is the homotopical algebra version of Tate cohomology and, as in the classical case, we can access it by defining a suitable norm map between orbits and fixed points. To this end, we follow the discussion in [Rog08, Section 5.2], and consider $G\times G$-spectra, that is, functors $BG\times BG \to \mathrm{Sp}$. Note that a G-spectrum X can always be considered as a $G \times G$-spectrum by adding an extra trivial G-action. Another important $G \times G$-spectrum is the spherical group ring

$$\mathbb{S}[G] = \Sigma^\infty_+ G$$

with its obvious left and right G-actions coming from left and right multiplication of G on itself. Note that we have a canonical colimit-limit exchange map

$$\kappa : \operatorname*{colim}_{BG} \lim_{1\times BG} (X \otimes \mathbb{S}[G]) \to \lim_{BG} \operatorname*{colim}_{BG\times 1} (X \otimes \mathbb{S}[G])\,.$$

Simply unravelling the source and the target, we see that this identifies to a map

$$\mathrm{Nm}_G : (X \otimes \mathbb{S}[G]^{h(1\times G)})_{hG} \to X^{hG}$$

which we refer to as the **norm map**. The G-spectrum

$$D_G = \mathbb{S}[G]^{h(1\times G)}$$

appearing in the source is referred to as the **dualising spectrum** of G. The **Tate construction** on the G-spectrum X can be defined as the cofiber of the norm map

$$X^{tG} = \operatorname{cofib}(\mathrm{Nm}_G : (X \otimes D_{BG})_{hG} \longrightarrow X^{hG})\,,$$

and recovers Tate cohomology on homotopy groups, in the sense that

$$\pi_*(HM^{tG}) \cong \hat{H}^{-*}(G;M)$$

whenever M is a G-module for some finite group G.

Remark 1.3.1. In general, the dualising spectrum can be quite hard to understand. However, there is a very nice description of the dualising spectrum when G is a compact Lie group, due to Klein. In [Kle01], he identifies the dualising spectrum of a compact Lie group G with the representation sphere on the adjoint representation of G:

$$D_G = \mathbb{S}^{\mathrm{Ad}(G)}\,.$$

In particular, note that when G is a finite discrete group, which is a compact Lie group of dimension 0, the dualising module is just the sphere spectrum with trivial action, which explains why the dualising spectrum does not appear in the classical situation.

The homotopy fixed point functor $(-)^{hG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$ is lax symmetric monoidal, in the same way as group cohomology can be endowed with the graded commutative cup product. In the previous section, we saw that the cup product could be extended to Tate cohomology, and the same is true in homotopical algebra: the Tate construction functor can be endowed with the structure of a lax symmetric monoidal functor in such a way that the natural transformation

$$(-)^{hG} \longrightarrow (-)^{tG}$$

is symmetric monoidal, at least when G is a compact Lie group. In fact, this lax symmetric monoidal structure is unique up to contractible choice [NS18, Theorem I.3.1, Theorem I.4.1].

So far, we have not motivated why we should be interested in the Tate construction, so let us now connect it to something quite fundamental in algebraic topology. Firstly, note that we are not restricted to working with the classifying space of a group. Indeed, we may replace BG with some general space B all the way through[14]. This provides us with a perfectly good definition of Tate cohomology of the space B. For example, if $H\mathbb{Z}$ denotes the Eilenberg–Mac Lane spectrum of the integers, let us write $\underline{H\mathbb{Z}}$ for the same spectrum viewed as trivially parametrised over B. Then it seems reasonable to define Tate cohomology of B with coefficients in $\mathbb{Z}$ as the homotopy groups

$$\hat{H}^{-*}(B;\mathbb{Z}) = \pi_*\left(\mathrm{cofib}\left(\mathrm{Nm}_B : \operatorname*{colim}_B(\underline{H\mathbb{Z}} \otimes D_B) \to \lim_B \underline{H\mathbb{Z}}\right)\right)\,.$$

One could speculate that the main reason this has not been studied in the past is that it is trivial on a very big and important class of spaces, namely finite dimensional manifolds. If $B = M$ is an n-manifold, then the dualising spectrum is fibre-wise a sphere shifted into degree $-n$, and the norm map

$$\mathrm{Nm}_M : (\underline{H\mathbb{Z}} \otimes D_M)_{hM} \to \underline{H\mathbb{Z}}^{hM}$$

[14] As is common when working with ∞-categories, we mean "space" as "Kan complex" here. Note that $\delta_!(\mathbb{S}) = \mathbb{S}[G]$ where $\delta_!$ denotes induction along the diagonal map $\delta : BG \to BG\times BG$, so the obvious replacement for the spherical group ring will be the parametrised spectrum we get when inducing up the trivial sphere spectrum along the diagonal map $\delta : B \to B \times B$.

precisely induces the map

$$H_{*+n}(M;\mathbb{Z}_\omega) \to H^{-*}(M;\mathbb{Z})$$

appearing in the statement of generalised Poincaré duality, where $\mathbb{Z}_\omega$ is the orientation module of M. In this case, the norm map is an equivalence; this is quite literally the statement of the generalised Poincaré duality. Of course, this is a very special situation, and for general B, for example for classifying spaces of groups, we cannot expect the norm map to be an equivalence. The above discussion tells us that we could think of the dualising spectrum D_B as a spectrum level analogue of an orientation module for the space B, and of the Tate construction as a measure for the failure of a generalised version of Poincaré duality to hold [Kle07].

Remark 1.3.2. The Tate construction was first introduced by Greenless in the setting of genuine equivariant stable homotopy theory [Gre87; GM95]. This relies on explicit point-set models of spectra. We have already seen one such model; the one given by Lima that we recalled in Section 1.1.3.1. The issue with this model is that, although the stable homotopy category $\mathcal{SHC}$ is symmetric monoidal via the smash product [Ada74, Section III.4], we cannot endow the category that Lima sets up with a reasonable symmetric monoidal structure. This is less than stellar; if you want to do any type of algebra involving spectra, you better have access to some sort of tensor product on them. In the 90's, there was a boom of modified symmetric monoidal models of spectra, including, but not limited to: S-modules [EKMM97], symmetric spectra [HSS00], and orthogonal spectra [MMSS01]. We focus on orthogonal spectra, as it is the most well-used when passing to the equivariant setting. Roughly, an **orthogonal spectrum** is a spectrum X where the nth space X_n is endowed with an action of the orthogonal group $O(n)$. The category of orthogonal spectra is indeed closed symmetric monoidal and can be endowed with a model structure in such a way that its homotopy category is equivalent to the stable homotopy category.

We pass to equivariant stable homotopy theory when we add a group action to the picture. We will try to stay informal in this discussion and refer the reader to [Sch18, Section 3.1] for a more thorough discussion. Roughly, an **orthogonal G-spectrum** is an orthogonal spectrum with an action of a compact Lie group G. This is a closed symmetric monoidal category via the underlying closed symmetric monoidal structure on orthogonal spectra, equipped with diagonal and conjugate G-action. The distinction between "genuine" and "naïve" equivariant homotopy theory comes in when we define the weak equivalences; the notion of a weak equivalence in genuine equivariant homotopy theory is significantly stronger than simply asking for a G-equivariant map that is a weak equivalence in the underlying model category of non-equivariant orthogonal spectra. This inevitably leads to some homotopy theoretical difficulties when one studies fixed points. There is an obvious fixed point functor that takes an orthogonal G-spectrum X to the orthogonal spectrum whose nth level is given by the set-theoretic fixed points X_n^G, but this does not necessarily preserve (genuine) weak equivalences of G-spectra. Instead, one needs to derive this functor, and we so obtain a homotopically meaningful functor $(-)^G$ referred to as the **genuine fixed points**.

In genuine stable homotopy theory, a point-set model for the homotopy fixed points is given as

$$X^{hG} = F(EG_+, X)^G\,,$$

where EG denotes a free (non-equivariantly) contractible G-space and $F(-,-)$ refers to the function objects in the closed monoidal structure on orthogonal G-spectra. Similarly, a point-set model for the Tate construction is

$$X^{tG} = \left(\widetilde{EG} \wedge F(EG_+, X)\right)^G$$

where $\widetilde{EG}$ is the mapping cone of the collapse map $EG_+ \to S^0$. In this setting, the multiplicative structure on homotopy fixed points and the Tate construction relies on the existence of G-equivariant maps

$$EG_+ \longrightarrow EG_+ \wedge EG_+ \quad \text{and} \quad \widetilde{EG} \wedge \widetilde{EG} \longrightarrow \widetilde{EG}\,.$$

Such maps exist due to obstruction theory and are unique up to homotopy.

1.3.3 The Tate spectral sequence

The main common thread through this thesis is the Tate spectral sequence. This is a spectral sequence which is supposed to compute the homotopy groups of the Tate construction on a G-spectrum[15] for some topological group G. We will sketchily refer to this spectral sequence, to be constructed in various different ways, as

$$\hat{H}^{-p}(G, \pi_q(X)) \Longrightarrow \pi_{p+q}(X^{tG})\,.$$

There are essentially three questions to consider here:

1. How do we algebraically make sense of the left hand side?

2. How do we make sure that the spectral sequence is multiplicative?

3. How do we make sure that the spectral sequence converges?

We note that none of these questions are particularly straight-forward. Regarding the first question: we do have a good algebraic understanding for Tate cohomology when G is a finite group. However, what "Tate cohomology of a compact Lie group G" should mean is less clear, for example. Multiplicativity of the Tate spectral sequence is a technical question that involves homotopical control of the maps of our potential filtrations. Finally, the third question is made extra tricky by the fact that the Tate spectral sequence is generally a whole-plane spectral sequence, so we need to take Boardman's whole-plane obstruction into account.

[15] Or more generally, a parametrised spectrum.

Tate spectral sequences for actions of finite groups are relatively well understood and can be constructed in a number of different ways. Below, we outline three approaches and point out some advantages and disadvantages with each method, with indications where the methods used for finite groups are insufficient for more general topological groups.

1.3.3.1 The Greenlees–May construction

Arguably the first construction of the Tate spectral sequence is due to Greenlees–May and works in analogy to the complete resolutions view of Tate cohomology [Gre87; GM95]. Here, we work in the context of genuine equivariant stable homotopy theory, and so take our model for the Tate construction as

$$X^{tG} = \left(\widetilde{EG} \wedge F(EG_+, X)\right)^G .$$

We construct a filtration of $\widetilde{EG}$ by using its G-CW structure, dualise this filtration by taking Spanier–Whitehead duals, and splice the two filtration together to obtain a bi-infinite filtration of $\widetilde{EG}$. In turn, this induces a bi-infinite filtration on the Tate construction which we refer to as the **Greenlees filtration**. The first page of the spectral sequences can be expressed as

$$E^1_{*,*} \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\hat{P}_*, \pi_*(X)) ,$$

where $\hat{P}_*$ is a complete resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module. We conclude that the second page of this spectral sequence is given by the Tate cohomology groups:

$$E^2_{*,*} \cong \hat{H}^{-*}(G; \pi_*(X)) .$$

It is straight-forward to show that the Tate spectral sequence constructed in this way is conditionally convergent. The problem with this filtration is that it is not clear how to endow the resulting spectral sequence with a multiplicative structure. Indeed, you tend to run into difficult technical problems regarding the homotopies once you start mixing negative and positive indices in the Greenlees-filtration. One way of avoiding this problem is to construct a filtration on the Tate construction that does not involve dualising.

1.3.3.2 The Hesselholt–Madsen construction

The Hesselholt–Madsen construction of the Tate spectral sequence works in analogy to the Tate complex view of Tate cohomology, and so circumvents the need to dualise [HM03, Section 4]. Here, we work with the filtrations EG and $\widetilde{EG}$ coming from the G-CW structure on the two spaces, simultaneously. These filtrations induces a filtration on the Tate construction by smashing them together and totalising the obtained bigraded filtration. The first page of the Tate spectral sequence obtained in this way is given by

$$E^1_{*,*} = \left(\widetilde{P}_* \otimes \mathrm{Hom}(P_*, \pi_*(X))\right)^G .$$

Since this chain complex is quasi-isomorphic to $\mathrm{Hom}_{\mathbb{Z}[G]}(\hat{P}_*, \pi_*(X))$ (see the discussion in Section 1.3.1), we know that the second page of this spectral sequence is also given by the Tate cohomology groups:

$$E^2_{*,*} \cong \hat{H}^{-*}(G; \pi_*(X)) .$$

The Hesselholt–Madsen construction of the Tate spectral sequence can be endowed with a multiplicative structure. To deal with homotopical issues Hesselholt and Madsen employ functorial G-CW replacements to convert the G-spectra to G-CW spectra. One can also find functorial G-CW replacements when G is a non-finite compact Lie group, but the monoidal properties of such are less clear, which is an issue when moving from the setting of finite groups to compact Lie groups. We saw before that the multiplicative structure on the Tate construction relied on the existence of G-equivariant maps $EG_+ \to EG_+ \wedge EG_+$ and $\widetilde{EG} \wedge \widetilde{EG} \to \widetilde{EG}$. For finite groups these can be chosen to be cellular, so that they preserve the chosen filtration. On the first page of the Hesselholt–Madsen Tate spectral sequence, the pairings of filtrations correspond to the maps

$$P_* \longrightarrow P_* \otimes P_* \quad \text{and} \quad \widetilde{P}_* \otimes \widetilde{P}_* \longrightarrow \widetilde{P}$$

which precisely induce the cup product on Tate cohomology.

1.3.3.3 The Postnikov tower construction

Given a space or a spectrum there is always a way to kill off the homotopy groups over or under a certain degree. The filtrations so obtained are referred to as the Postnikov and Whitehead towers, respectively, and it is possible to construct the Tate spectral sequence also using these constructions. We focus on the Whitehead tower construction, since this has better multiplicative properties than the Postnikov one, see for example [Dug03]. Roughly, the construction of the Tate spectral sequence constructed in this way proceeds by taking the G-equivariant Whitehead (or Postnikov) tower of a G-spectrum X and then taking the G-Tate construction on each level. This construction is also covered in [GM95]. The fundamental difficulty with this construction, when moving from finite groups to compact Lie groups, is that is not straight-forward to access the effect of the action of G on X on the level of the homotopy groups of X.

Note that the construction illustrated here is a fundamentally different view on the Tate spectral sequence; for both the Greenlees–May and the Hesselholt–Madsen filtration we start with a cellular filtration related to the group G, while in the Postnikov/Whitehead filtration construction we consider filtrations of the spectrum X. This is reminiscent to various constructions of the Atiyah–Hirzebruch spectral sequence.

1.3.4 Digression: Topological periodic homology

The author came to study the Tate construction via an interest in topological Hochschild homology, which can be viewed as a lift of Hochschild homology to the setting of homotopical algebra. Topological Hochschild homology was first introduced by Bökstedt in the unpublished manuscript [Bök85b] as a tool to study algebraic K-theory. He was motivated by the 'brave new rings' paradigm of Waldhausen. One of the main advantages of topological Hochschild homology is that it is a lot more amenable to computations than algebraic K-theory, yet still allows access to a significant portion of information on the former via trace methods. Some of the first computations on topological Hochschild homology are due to Bökstedt and coauthors; most notable are the articles [Bök85a] and [BHM93]. In the first of these articles Bökstedt computed topological Hochschild homology of the prime fields $\mathbb{F}_p$, as well as of the integers $\mathbb{Z}$. The computation of $\mathrm{THH}(\mathbb{Z})$ allowed mathematicians to stretch the boundaries for what was know about algebraic K-theory of $\mathbb{Z}$, which is in turn intimately connected to the Kummer–Vandiver conjecture in algebraic number theory.

As a cyclic object, topological Hochschild homology has the structure of a $\mathbb{T}$-spectrum, where $\mathbb{T}$ denotes the circle group. The Tate construction on topological Hochschild homology with respect to the entire $\mathbb{T}$-action is referred to as topological periodic homology, and denoted

$$\mathrm{TP}(R) = \mathrm{THH}(R)^{t\mathbb{T}}\,.$$

While this construction had been studied before, it was put in the spotlight by Hesselholt who showed that it has important connections to Hasse–Weil zeta functions [Hes18]. Work pioneered by Hesselholt and coauthors, had previously understood the importance of topological Hochschild homology, and its various refinements, in arithmetic contexts (see for example [Hes96; GH99; HM03; Hes06]). The research field is still a very active one, especially after the recent simplified reformulation of cyclotomic spectra in the ∞-categorical framework by Nikolaus–Scholze [NS18] motivated by research of Bhatt–Morrow–Scholze on integral p-adic Hodge theory [BMS18; BMS19].

Chapter 2

Summary of papers

In this chapter, we give summary of the papers found in this thesis. As the relevant background and history can be found in the previous chapter, we focus on the main results.

2.1 A multiplicative Tate spectral sequence for compact Lie group actions

The aim of this paper, which is joint with John Rognes, is to construct a multiplicative Tate spectral sequence when we are dealing with actions of compact Lie groups. Here, we work in the setting of genuine equivariant stable homotopy theory and so use equivariant orthogonal spectra as our model for G-spectra, as described in Remark 1.3.2. Given a commutative (non-equivariant) orthogonal ring spectrum R we consider the group ring

$$R[G] = R \wedge G_+ \, .$$

If the homotopy groups $R[G]_* = \pi_*(R[G])$ are flat over $R_* = \pi_*(R)$, then the group structure on G makes sure that $R[G]_*$ is a Hopf algebra over R_*. If we moreover assume that $R[G]_*$ is finitely generated projective over R_*, we show that we have access to a multiplicative G-Tate spectral sequence

$$E^2_{p,*} = \widehat{\mathrm{Ext}}^{-p}_{R[G]_*}(R_*, \pi_*(X)) \Longrightarrow \pi_{p+*}(X^{tG})$$

where the E^2-page is given by the complete Ext groups of R_* over $R[G]_*$ with coefficients in the $R[G]_*$-module $\pi_*(X)$. This spectral sequence will be strongly convergent under mild hypotheses, such as for instance in the case when the RE^∞-page vanishes and the spectrum X is bounded below. The paper can be said to consist of two parts: an algebraic and a topological one. The algebraic part consists of giving an algebraic formulation for the E^2-page of the spectral sequence described above, while the topological part consists of actual constructions of the spectral sequences from sequences of orthogonal G-spectra.

Note that the topological framework allows for Hopf algebras over very complicated rings, like $\mathbb{S}_*$, so we do not want to restrict ourselves to an oversimplified algebraic setting. Given a Hopf algebra Γ over a (possibly graded) commutative ring k, we study the Tate complex

$$T_*(M) = \widetilde{P}_* \otimes \mathrm{Hom}(P_*, M)$$

for a Γ-module M, where $P_* \to k$ is a projective resolution of k as a trivial Γ-module and $\widetilde{P}_*$ is its mapping cone. In particular, if Γ is finitely generated projective over k, then we show that the homology of the Γ-invariants of the Tate complex is isomorphic to what is referred to as complete Ext [CK97; Mis94] of Γ with coefficients in M:

$$\widehat{\operatorname{Ext}}^n_\Gamma(k, M) \cong H_{-n} \operatorname{Hom}_\Gamma(k, T_*(M))$$

The key ingredient is a result by Pareigis [Par71] which identifies the k-dual of a finitely generated Hopf algebra with the induced Γ-module

$$\operatorname{Hom}_k(\Gamma, k) \cong \operatorname{Ind}_{\mathrm{k}}^{\Gamma} P(\operatorname{Hom}_k(\Gamma, k)),$$

where $P(\operatorname{Hom}_k(\Gamma, k))$ is the primitives for the Γ-coaction on $\operatorname{Hom}_k(\Gamma, k)$, which is a finitely generated projective k-module of constant rank 1. In particular, this result implies that induced modules over Γ are coinduced, and vice versa.

The main motivation behind working with Tate complexes, as opposed to complete resolutions, has to do with multiplicative structures on the Tate spectral sequence. In the algebraic setting, we show that there is an associative, unital, and graded commutative pairing on complete Ext

$$\smile\colon \widehat{\operatorname{Ext}}_\Gamma(k, M) \otimes \widehat{\operatorname{Ext}}_\Gamma(k, N) \longrightarrow \operatorname{Ext}_\Gamma(k, M \otimes N)$$

which we refer to simply as the cup product. This extends the ordinary cup product on Ext, in a suitable sense.

The topological part of the paper can be said to contain two sub-parts: a more general consideration of spectral sequences coming from filtrations of orthogonal G-spectra via Cartan–Eilenberg systems, and the construction of the Tate spectral sequence, more specifically. In the more general part, we show that a pairing of sequences of orthogonal G-spectra gives rise of a pairing of the corresponding spectral sequences, via the use of Cartan–Eilenberg systems. This can be regarded as folklore, but we felt that an explicit reference for this fact was not available at the time of writing, so we decided to give a complete proof. Here, we use hands-on methods to handle homotopy theoretical issues. In particular, we use the classical mapping telescope construction, which has convenient monoidal properties, to deal with sequential homotopy colimits of spectra. As mentioned in Section 1.3.3.2, in the case of finite groups Hesselholt and Madsen [HM03, Section 4.3] instead use a functorial G-CW replacement deal with these sorts of issues in the Tate spectral sequence. There exists a functorial G-CW replacement also for compact Lie groups [Sey83], but its monoidal properties are less clear.

To construct the G-Tate spectral sequence, we use the point-set model

$$X^{tG} = \Big(\widetilde{EG} \wedge F(EG_+, X)\Big)^G \simeq \Big((R \wedge \widetilde{EG}) \wedge_R F_R(R \wedge EG_+, X)\Big)^G .$$

The idea is to start by giving the free G-space EG the simplicial skeletal filtration coming from the construction of EG using the simplicial bar construction [May72]. This induces filtrations on $F_R(R \wedge EG_+, X)$ and $R \wedge \widetilde{EG}$, which can be combined into a filtration called the Hesselholt–Madsen filtration:

$$HM_c(X) = \operatorname*{hocolim}_{a+b \le c} \widetilde{E}_a \wedge F(E/E_{-b-1}, X).$$

As we have mentioned before, the existence of a multiplicative structure on the Hesselholt–Madsen G-Tate spectral sequence relies on the existence of filtration-preserving maps $EG_+ \to EG_+ \wedge EG_+$ and $\widetilde{EG} \wedge \widetilde{EG} \to \widetilde{EG}$. The first is known to exist, and we prove by obstruction theory that the second one exists under the assumption that $R[G]_*$ is projective over R_*. The work we have done in the more general setting of Cartan–Eilenberg systems from sequences of orthogonal G-spectra then guarantees that the Tate spectral sequence is multiplicative. However, convergence is much less clear.

To settle questions about convergence, we compare the Hesselholt–Madsen filtration to another possible filtration of the Tate construction, that we dub the Greenlees–May filtration and refer to as $GM_\star(X)$. It is not hard to show that the G-Tate spectral sequence associated to this filtration is conditionally convergent. By showing that there is a map of filtrations

$$\alpha : GM_\star(X) \longrightarrow HM_\star(X)$$

which induces an isomorphism of spectral sequences from the E^2-page and on, we can then deduce convergence results for the Hesselholt–Madsen G-Tate spectral sequence under mild conditions, such as when the spectrum X is bounded below and the derived limit term RE^∞ vanishes.

2.2 Multiplicative spectral sequences via décalage

The aim of this article is to give an clear account of the subject of multiplicative spectral sequences using the modern language of ∞-categories, and from this access highly structured results regarding the passage from filtrations to spectral sequences. I am the sole author of this paper, but it is worth noting that it builds on joint work together with Achim Krause and Thomas Nikolaus. I hope to eventually publish the results of the article in a joint paper that also includes (yet unfinished) considerations of conditional convergence from the décalage point of view.

As we explained in Section 1.2.1.3, décalage is a way to make sense of “turning the page in a spectral sequence” on the level of filtrations. Although it was initially defined quite hands-on in the context of filtered chain complexes, one can make sense of décalage in the language of the Beilinson t-structure. This allows one to generalise the construction also to spectra. Indeed, the stable ∞-category $\mathrm{Tow}(\mathrm{Sp}) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp})$, whose objects we refer to as filtrations, can be equipped with a t-structure by declaring the Beilinson n-connective filtrations to be the objects in the subcategory

$$\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei}}_{\geq n} = \{X \in \mathrm{Tow}(\mathrm{Sp}) \mid \mathrm{Gr}^q(X) \in \mathrm{Sp}_{\geq n-q} \text{ for all } n\}\,,$$

where

$$\mathrm{Gr}^q(X) = X(q)/X(q+1) = \mathrm{cofib}(X(q+1) \to X(q))\,,$$

and the cofibre is meant in the ∞-categorical sense, i.e. as a homotopy colimit. The heart of this t-structure is the abelian category of chain complexes of abelian groups. This is shown in [BMS19, Section 5], and we take care to show that this equivalence of categories is also compatible with all the multiplicative structures involved. It is an observation by Antieau that décalage can be phrased using the cover functors in the Beilinson t-structure, and that this extends the notion of décalage to the setting of filtered spectra. In the paper, we define the décalée of the filtration X as the filtration $\mathrm{Déc}(X)$ given by

$$\cdots \longrightarrow \operatorname*{colim}_q(\tau^{\mathrm{Bei}}_{\geq n+1}X)(q) \longrightarrow \operatorname*{colim}_q(\tau^{\mathrm{Bei}}_{\geq n}X)(q) \longrightarrow \operatorname*{colim}_q(\tau^{\mathrm{Bei}}_{\geq n-1}X)(q) \longrightarrow \cdots,$$

where the middle term is placed in filtration degree n. The above construction gives us a functor

$$\mathrm{Déc} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \mathrm{Tow}(\mathrm{Sp})$$

that we refer to as décalage. If X is a complete filtration of A, then $\mathrm{Déc}(X)$ is a complete filtration of A, as well, and we can use iterated décalage to build a spectral sequence associated to X. Explicitly, we show that the assignment

$$E^r_{n,s}(X) = \pi_n(\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)))$$

determines a spectral sequence (in homological Adams grading). Here, the differential $d^r_{n,s} : E^r_{n,s} \to E^r_{n-1,s+r}$ is induced by the connecting homomorphism in the pushout square

$$\begin{tikzcd}
\mathrm{Gr}^{(r-1)n+s+1}(\mathrm{Déc}^{r-1}(X)) \arrow[r] \arrow[d] & \dfrac{\mathrm{Déc}^{r-1}(X)((r-1)n+s)}{\mathrm{Déc}^{r-1}(X)((r-1)n+s+2)} \arrow[d] \\
0 \arrow[r] & \mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)).
\end{tikzcd}$$

This is essentially trivial once we prove that the nth associated graded of the décalée of X can be expressed as the Eilenberg–Mac Lane spectrum of the nth Beilinson homotopy groups of X:

$$\mathrm{Gr}^n \mathrm{Déc}(X) \simeq H\pi^{\mathrm{Bei}}_n(X)[n].$$

Indeed, the isomorphism between the $r+1$th page of the associated spectral sequence and the homology of the rth page is precisely induced by this equivalence. Careful considerations shows that this equivalence is symmetric monoidal in a suitable sense, and this allows us to prove that the functor

$$E^\star_{*,*} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \mathrm{SSEQ}$$

from filtrations to spectral sequences admits the structure of a map of ∞-operads. Let us elaborate on this statement. The ∞-category $\mathrm{Tow}(\mathrm{Sp})$ can be endowed with a symmetric monoidal structure via Day convolution. The category of spectral sequences is not symmetric monoidal, however. But while cannot technically speak of the tensor product of two spectral sequences, it is possible to speak of multilinear maps of such, and this makes spectral sequences into a coloured operad. Hence Tow(Sp) and SSEQ both fit into the framework of ∞-operads, and the above statement has meaning.

In an appendix, we describe how to prove the Tate spectral sequence is multiplicative, using the machinery developed in the rest of the paper. In particular, we look closer at the Tate spectral sequence for topological Hochschild homology and topological restriction homology at the prime p, and describe in what sense topological periodic homology is a version of 2-periodic crystalline cohomology.

Bibliography

[Ada74] J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, 1974.

[Beĭ87] A. A. Beĭlinson. "On the derived category of perverse sheaves". In: *K-theory, arithmetic and geometry (Moscow, 1984–1986)*. Vol. 1289. Lecture Notes in Math. Springer, Berlin, 1987, pp. 27–41.

[BF78] A. K. Bousfield and E. M. Friedlander. "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets". In: *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*. Vol. 658. Lecture Notes in Math. Springer, Berlin, 1978, pp. 80–130.

[BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen. "The cyclotomic trace and algebraic K-theory of spaces". In: *Invent. Math.* 111.3 (1993), pp. 465–539.

[BMS18] B. Bhatt, M. Morrow, and P. Scholze. "Integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397.

[BMS19] B. Bhatt, M. Morrow, and P. Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310.

[Boa99] J. M. Boardman. "Conditionally convergent spectral sequences". In: *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*. Vol. 239. Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 49–84.

[Bök85a] M. Bökstedt. "The topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$". Preprint. 1985.

[Bök85b] M. Bökstedt. "Topological Hochschild homology". Preprint. 1985.

[Bor94] F. Borceux. *Handbook of categorical algebra. 1*. Vol. 50. Encyclopedia of Mathematics and its Applications. Basic category theory. Cambridge University Press, Cambridge, 1994.

[Bro62] E. H. Brown Jr. "Cohomology theories". In: *Ann. of Math. (2)* 75 (1962), pp. 467–484.

[Bro82] K. S. Brown. *Cohomology of groups*. Vol. 87. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[CE56] H. Cartan and S. Eilenberg. *Homological algebra.* Princeton University Press, Princeton, N. J., 1956.

[Cis19] D.-C. Cisinski. *Higher categories and homotopical algebra.* Vol. 180. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2019.

[CK97] J. Cornick and P. H. Kropholler. "On complete resolutions". In: *Topology Appl.* 78.3 (1997), pp. 235–250.

[CS19] K. Cesnavicius and P. Scholze. *Purity for flat cohomology.* `https://arxiv.org/abs/1912.10932`. 2019.

[Del71] P. Deligne. "Théorie de Hodge. II". In: *Inst. Hautes Études Sci. Publ. Math.* 40 (1971), pp. 5–57.

[Dou59a] A. Douady. "La suite spectrale d'Adams". In: *Séminaire Henri Cartan* 11.2 (1959).

[Dou59b] A. Douady. "La suite spectrale d'Adams : structure multiplicative". In: *Séminaire Henri Cartan* 11.2 (1959).

[Dug03] D. Dugger. *Multiplicative structures on homotopy spectral sequences II.* `https://arxiv.org/abs/math/0305187`. 2003.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory.* Vol. 47. Mathematical Surveys and Monographs. With an appendix by M. Cole. American Mathematical Society, Providence, RI, 1997.

[Fre38] H. Freudenthal. "Über die Klassen der Sphärenabbildungen I. Große Dimensionen". In: *Compositio Math.* 5 (1938), pp. 299–314.

[GH99] T. Geisser and L. Hesselholt. "Topological cyclic homology of schemes". In: *Algebraic K-theory (Seattle, WA, 1997).* Vol. 67. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1999, pp. 41–87.

[GM95] J. P. C. Greenlees and J. P. May. "Generalized Tate cohomology". In: *Mem. Amer. Math. Soc.* 113.543 (1995).

[Gre87] J. P. C. Greenlees. "Representing Tate cohomology of G-spaces". In: *Proc. Edinburgh Math. Soc. (2)* 30.3 (1987), pp. 435–443.

[Gre95] J. P. C. Greenlees. "Commutative algebra in group cohomology". In: *J. Pure Appl. Algebra* 98.2 (1995), pp. 151–162.

[Gro57] A. Grothendieck. "Sur quelques points d'algèbre homologique". In: *Tohoku Math. J. (2)* 9 (1957), pp. 119–221.

[Hes06] Lars Hesselholt. "On the topological cyclic homology of the algebraic closure of a local field". In: *An alpine anthology of homotopy theory.* Vol. 399. Contemp. Math. Amer. Math. Soc., Providence, RI, 2006, pp. 133–162.

[Hes18] L. Hesselholt. "Topological Hochschild homology and the Hasse–Weil zeta function". In: (2018), pp. 157–180.

[Hes96] L. Hesselholt. “On the p-typical curves in Quillen’s K-theory”. In: *Acta Math.* 177.1 (1996), pp. 1–53.

[HM03] L. Hesselholt and I. Madsen. “On the K-theory of local fields”. In: *Ann. of Math. (2)* 158.1 (2003), pp. 1–113.

[Hov99] M. Hovey. *Model categories.* Vol. 63. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999.

[HR19] G. O. Helle and J. Rognes. “Boardman’s whole-plane obstruction group for Cartan–Eilenberg systems”. In: *Doc. Math.* 24 (2019).

[HSS00] M. Hovey, B. Shipley, and J. Smith. “Symmetric spectra”. In: *J. Amer. Math. Soc.* 13.1 (2000), pp. 149–208.

[Joy02] A. Joyal. “Quasi-categories and Kan complexes”. In: vol. 175. 1-3. Special volume celebrating the 70th birthday of Professor Max Kelly. 2002, pp. 207–222.

[Kle01] J. R. Klein. “The dualizing spectrum of a topological group”. In: *Math. Ann.* 319.3 (2001), pp. 421–456.

[Kle07] J. R. Klein. “The dualizing spectrum. II”. In: *Algebr. Geom. Topol.* 7 (2007), pp. 109–133.

[Lan76] P. S. Landweber. “Homological properties of comodules over $M\mathrm{U}_*(M\mathrm{U})$ and $\mathrm{BP}_*(\mathrm{BP})$”. In: *Amer. J. Math.* 98.3 (1976), pp. 591–610.

[Ler46] J. Leray. “Structure de l’anneau d’homologie d’une représentation”. In: *C. R. Acad. Sci. Paris* 222 (1946), pp. 1419–1422.

[Lim60] E. L. Lima. “Stable Postnikov invariants and their duals”. In: *Summa Brasil. Math.* 4 (1960), pp. 193–251.

[Lur09] J. Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.

[Lur17] J. Lurie. “Higher Algebra”. `www.math.ias.edu/ lurie/papers/HA.pdf`. 2017.

[Mac98] S. Mac Lane. *Categories for the working mathematician.* Second. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[Mas52] W. S. Massey. “Exact couples in algebraic topology. I, II”. In: *Ann. of Math. (2)* 56 (1952), pp. 363–396.

[Mas54] W. S. Massey. “Products in exact couples”. In: *Ann. of Math. (2)* 59 (1954), pp. 558–569.

[May72] J. P. May. *The geometry of iterated loop spaces.* Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972.

[McC01] J. McCleary. *A user’s guide to spectral sequences.* 2nd ed. Vol. 58. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2001.

[Mis94] G. Mislin. “Tate cohomology for arbitrary groups via satellites”. In: *Topology Appl.* 56.3 (1994), pp. 293–300.

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. “Model categories of diagram spectra”. In: *Proc. London Math. Soc. (3)* 82.2 (2001), pp. 441–512.

[Nee02] A. Neeman. “A counterexample to a 1961 “theorem” in homological algebra”. In: *Invent. Math.* 148.2 (2002). With an appendix by P. Deligne, pp. 397–420.

[NS18] T. Nikolaus and P. Scholze. “On topological cyclic homology”. In: *Acta Math.* 221.2 (2018), pp. 203–409.

[Par71] B. Pareigis. “When Hopf algebras are Frobenius algebras”. In: *J. Algebra* 18 (1971), pp. 588–596.

[Qui67] D. G. Quillen. *Homotopical algebra.* Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[Qui69] D. Quillen. “Rational homotopy theory”. In: *Ann. of Math. (2)* 90 (1969), pp. 205–295.

[Rog08] J. Rognes. “Galois extensions of structured ring spectra. Stably dualizable groups”. In: *Mem. Amer. Math. Soc.* 192.898 (2008).

[Sch18] S. Schwede. *Global homotopy theory.* Vol. 34. New Mathematical Monographs. Cambridge University Press, Cambridge, 2018.

[Sey83] R. M. Seymour. “Some functional constructions on G-spaces”. In: *Bull. London Math. Soc.* 15.4 (1983), pp. 353–359.

[Tat52] J. Tate. “The higher dimensional cohomology groups of class field theory”. In: *Ann. of Math. (2)* 56 (1952), pp. 294–297.

[Voe03a] V. Voevodsky. “Motivic cohomology with $\mathbf{Z}/2$-coefficients”. In: *Publ. Math. Inst. Hautes Études Sci.* 98 (2003), pp. 59–104.

[Voe03b] V. Voevodsky. “Reduced power operations in motivic cohomology”. In: *Publ. Math. Inst. Hautes Études Sci.* 98 (2003), pp. 1–57.

[Voe11] V. Voevodsky. “On motivic cohomology with $\mathbf{Z}/l$-coefficients”. In: *Ann. of Math. (2)* 174.1 (2011), pp. 401–438.

[Whi62] G. W. Whitehead. “Generalized homology theories”. In: *Trans. Amer. Math. Soc.* 102 (1962), pp. 227–283.

Papers

Paper I

A multiplicative Tate spectral sequence for compact Lie group actions

I

Alice Hedenlund and John Rognes

Abstract

Given a compact Lie group G and a commutative orthogonal ring spectrum R such that $R[G]_* = \pi_*(R \wedge G_+)$ is finitely generated and projective over $\pi_*(R)$, we construct a multiplicative G-Tate spectral sequence for each R-module X in orthogonal G-spectra, with E^2-page given by the Hopf algebra Tate cohomology of $R[G]_*$ with coefficients in $\pi_*(X)$. Under mild hypotheses, such as X being bounded below and the derived page RE^∞ vanishing, this spectral sequence converges strongly to the homotopy $\pi_*(X^{tG})$ of the G-Tate construction $X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G$.

I.1 Introduction

This paper grew out of an attempt to spell out the details for the Tate spectral sequence for the circle group $\mathbb{T}$. The construction of a multiplicative Tate spectral sequence for finite groups has been around for a while now: the first construction, due to Greenlees–May, can be found in [GM95], and another one, due to Hesselholt–Madsen, which makes the multiplicative properties of the spectral sequence more transparent, can be found in [HM03]. However, while multiplicativity of the $\mathbb{T}$-Tate spectral sequences has been used in computations, the authors of this paper have found references discussing the details for how such a spectral sequence is constructed surprisingly lacking. We hope that this paper will fill that gap in the literature.

The authors' motivation for considering the $\mathbb{T}$-Tate spectral sequence comes from the study of topological Hochschild homology and its refinements, such as topological cyclic homology. Given an E_1-ring spectrum B, topological Hochschild homology $\mathrm{THH}(B)$, first defined in the unpublished manuscript [Bök85], is a genuine $\mathbb{T}$-equivariant spectrum. The study of the Tate construction on this spectrum using the entire circle action goes back to [BM94] and [AR12], and was put in the spotlight by Hesselholt in [Hes18] under the name of topological periodic cyclic homology:

$$\mathrm{TP}(B) = \mathrm{THH}(B)^{t\mathbb{T}}.$$

Recently, Bhatt–Morrow–Scholze showed that there is a tight connection between topological periodic cyclic homology and crystalline cohomology [BMS19].

Background and aim

Classically, Tate cohomology is a way to combine group homology and group cohomology into a single multiplicative cohomology theory, and was first introduced by Tate in his study of class field theory [Tat52]. We sketch the main ideas involved following [CE56, Section XII.3] and [Bro82]. Given a finite group G, the main observation of Tate cohomology is this: if we dualise a projective resolution of $\mathbb{Z}$ as a trivial module over $\mathbb{Z}[G]$, we end up with a "coresolution" of $\mathbb{Z}$ by projective $\mathbb{Z}[G]$-modules. This "coresolution" $\mathrm{Hom}_{\mathbb{Z}}(P_*, \mathbb{Z})$ can be spliced with the original projective resolution P_*, and we so obtain a bi-infinite resolution $\hat{P}_*$ of $\mathbb{Z}$ called a complete resolution. Tate cohomology of G with coefficients in a G-module M is defined as

$$\hat{H}^n(G, M) = H^n(\mathrm{Hom}_G(\hat{P}_*, M))\,.$$

The Tate construction in the category of G-spectra can be seen as a generalisation of Tate cohomology in the context of higher algebra. Given a compact Lie group G and orthogonal G-spectrum X, we define the G-homotopy orbits and G-homotopy fixed points of X as

$$X_{hG} = EG_+ \wedge_G X \quad \text{and} \quad X^{hG} = F(EG_+, X)^G\,,$$

respectively. Here EG denotes a free contractible G-space. These can be regarded as generalisations of group homology and group cohomology. Indeed, if G is a finite group and $X = HM$ is the Eilenberg–Mac Lane spectrum on the G-module M, then the homotopy groups of the G-homotopy orbits and G-homotopy fixed points of HM recover group homology and group cohomology of G with coefficients in M, respectively. Following [Gre87; GM95], we define the G-Tate construction on X as the G-fixed point spectrum

$$X^{tG} = \left(\widetilde{EG} \wedge F(EG_+, X)\right)^G$$

with respect to the diagonal G-action. Here, $\widetilde{EG}$ denotes the mapping cone of the collapse map $c\colon EG_+ \to S^0$. This is a generalisation of Tate cohomology in the sense that the homotopy groups of the Tate construction on HM for a G-module M recover the Tate cohomology groups of the finite group G with coefficients in M.

One important property of the Tate construction is that it is multiplicative in the sense that any pairing $X \wedge Y \to Z$ of orthogonal G-spectra gives rise to a pairing $X^{tG} \wedge Y^{tG} \longrightarrow Z^{tG}$ of their Tate constructions. This relies on the existence of G-maps $EG_+ \to EG_+ \wedge EG_+$ and $\widetilde{EG} \wedge \widetilde{EG} \to \widetilde{EG}$. It is well-known that the diagonal map $EG_+ \to EG_+ \wedge EG_+$ induces a pairing

$$X^{hG} \wedge Y^{hG} \longrightarrow Z^{hG},$$

making the G-homotopy fixed points construction a lax symmetric monoidal functor. The inclusion $S^0 \to \widetilde{EG}$ and the canonical identifications $S^0 \wedge \widetilde{EG} \cong \widetilde{EG} \cong \widetilde{EG} \wedge S^0$ induce a natural map

$$X^{hG} \longrightarrow X^{tG}$$

and pairings $X^{hG} \wedge Y^{tG} \to Z^{tG}$ and $X^{tG} \wedge Y^{hG} \to Z^{tG}$. There is a G-map $N\colon \widetilde{EG} \wedge \widetilde{EG} \to \widetilde{EG}$ extending the canonical identifications, and any two such extensions are homotopic. Any choice of extension then induces a pairing

$$X^{tG} \wedge Y^{tG} \longrightarrow Z^{tG}$$

compatible with the above-mentioned map and pairings[1]. In general, the extension N will only be commutative and associative up to (coherent) homotopy, so $X \mapsto X^{tG}$ is not a lax symmetric monoidal functor to the category of orthogonal spectra, but only satisfies a homotopy coherent version of this property, which could be made precise using operad actions. For our purposes it suffices to note that it is lax symmetric monoidal as a functor to the stable homotopy category.

Given an orthogonal G-spectrum X, the aim of the present paper is to construct a G-Tate spectral sequence

$$\hat{E}^r_{s,t}(X) \Longrightarrow \pi_{s+t}(X^{tG}),$$

with an algebraically specified E^2-page, converging, in some suitable sense, to the homotopy groups of the G-Tate construction on X. Moreover, we would like this spectral sequence to be multiplicative, in the sense that a pairing $X \wedge Y \to Z$ of orthogonal G-spectra should induce a pairing

$$(\hat{E}^r(X), \hat{E}^r(Y)) \longrightarrow \hat{E}^r(Z)$$

of G-Tate spectral sequences. Finally, we want the pairing of E^∞-pages to be compatible with the pairing

$$\pi_*(X^{tG}) \otimes \pi_*(Y^{tG}) \longrightarrow \pi_*(Z^{tG})$$

of abutments. In particular, if X is an orthogonal G-ring spectrum, then the G-Tate spectral sequence of X should be an algebra spectral sequence converging multiplicatively to $\pi_*(X^{tG})$. As already mentioned, how to construct such spectral sequences is well–known in the situation of G being a finite group. Our goal is to generalise this to higher dimensional compact Lie groups.

[1] Work by Nikolaus–Scholze shows that this multiplicative structure is actually unique, in a homotopy theoretical sense; see [NS18, Theorem I.3.1]. This will not be important for our work, though.

Main results

Let us start by describing roughly, without going into too much detail, what we will do in this paper. We will carry out the construction of multiplicative and conditionally convergent Tate spectral sequences for compact Lie groups G such that $\mathbb{S}[G]_* = \pi_*(\mathbb{S}[G])$ is finitely generated projective as a module over $\mathbb{S}_* = \pi_*(\mathbb{S})$. Here $\mathbb{S}$ denotes the sphere spectrum and

$$\mathbb{S}[G] = \mathbb{S} \wedge G_+$$

is the unreduced suspension spectrum of G. Under these assumptions, $\mathbb{S}[G]_*$ is a finitely generated and projective cocommutative Hopf algebra over $\mathbb{S}_*$, and we will show that we have access to a multiplicative G-Tate spectral sequence with E^2-page given by the complete Ext-groups

$$\hat{E}^2_{s,*}(X) = \widehat{\mathrm{Ext}}^{-s}_{\mathbb{S}[G]_*}(\mathbb{S}_*, \pi_*(X))$$

of $\mathbb{S}_*$ over $\mathbb{S}[G]_*$ with coefficients in the $\mathbb{S}[G]_*$-module $\pi_*(X)$. The multiplicative structure in complete Ext is given by a graded commutative and associative cup product, and this will serve as a substitute for the failure of $X \mapsto X^{tG}$ to be lax symmetric monoidal. This spectral sequence will be strongly convergent under mild hypotheses, such as for instance in the case when the derived E^∞-page RE^∞ vanishes and the spectrum X is bounded below.

We note that this generality includes the case where $G = \mathbb{T}$ is the circle group, our main interest, but does not cover cases such as $G = SO(3)$. We therefore broaden our scope by considering a commutative[2] 'ground' orthogonal ring spectrum R and a compact Lie group G such that $R[G]_* = \pi_*(R[G])$ is finitely generated and projective over $R_* = \pi_*(R)$, where

$$R[G] = R \wedge G_+ \,.$$

This then includes cases such as $R = \mathbb{S}[1/2]$ and $R = H\mathbb{F}_2$, with $G = SO(3)$. Given an R-module X in orthogonal G-spectra we shall construct a multiplicative G-Tate spectral sequence

$$\hat{E}^2_{s,*}(X) = \widehat{\mathrm{Ext}}^{-s}_{R[G]_*}(R_*, \pi_*(X)) \Longrightarrow \pi_{s+*}(X^{tG})$$

where the E^2-page is now given as complete Ext of R_* over $R[G]_*$ with coefficients in $\pi_*(X)$. This will be strongly convergent under the same conditions as before.

[2]For somewhat technical reasons, it is not sufficient for us to assume that R is homotopy commutative. We analyse the product in the filtered R-module G-spectrum

$$\widetilde{EG} \wedge F(EG_+, R \wedge X) \cong L \wedge_R M \,,$$

with $L = R \wedge \widetilde{EG}$ and $M = F(EG_+, R \wedge X)$, as a composition

$$L \wedge_R M \wedge_R L \wedge_R M \xrightarrow{(23)} L \wedge_R L \wedge_R M \wedge_R M \xrightarrow{\phi \wedge \psi} L \wedge_R M$$

for filtered products $\phi \colon L \wedge_R L \to L$ and $\psi \colon M \wedge_R M \to M$. Homotopy commutativity is not sufficient to ensure that the twist map $\tau \colon M \wedge_R L \to L \wedge_R M$ implicit in the definition of (23) is an R-R-bimodule map.

Tate cohomology of Hopf algebras

In Section I.2 we develop a theory of Tate cohomology of a finitely generated and projective Hopf algebra Γ over a (possibly graded) commutative ring k, with the aim being to algebraically describe the E^2-page of a suitable Tate spectral sequence. Our approach will be different from the complete resolution approach, and we instead rely on the so–called Tate complex. Given a projective Γ-resolution P_* of k, we will denote the mapping cone of the augmentation map $\epsilon : P_* \to k$ as $\widetilde{P}_*$. The Tate complex of a Γ-module M, first defined in [Gre95], is the Γ-chain complex

$$\mathrm{hm}_*(M) = \widetilde{P}_* \otimes_k \mathrm{Hom}_k(P, M)_*$$

where Γ acts diagonally on the tensor product and by conjugation on $\mathrm{Hom}(P, M)_*$. In the aforementioned paper, the author shows that in the classical case, meaning $k = \mathbb{Z}$ and $\Gamma = \mathbb{Z}[G]$ for a finite group G, there is a zig-zag of maps

$$\widetilde{P}_* \otimes_k \mathrm{Hom}_k(P, M)_* \longrightarrow \widetilde{P}_* \otimes_k \mathrm{Hom}_k(\hat{P}, M)_* \longleftarrow \mathrm{Hom}_k(\hat{P}, M)_*$$

which become quasi-isomorphisms after taking G-invariants. The conclusion is that Tate cohomology can also be computed as the (co)homology groups of the G-invariants of the Tate complex. Recall that $\hat{P}_*$ denoted a complete resolution. We show that a similar result holds true in our setting: under the assumption that Γ is a finitely generated and projective Hopf algebra over k, the homology of the Γ-invariants of $\mathrm{hm}_*(M)$, which we can reasonably refer to as the Tate cohomology of Γ with coefficients in M, is isomorphic to the complete Ext of k over Γ with coefficients in M.

Theorem I.1.1. *If Γ is a finitely generated and projective Hopf algebra over k, then*

$$\widehat{\mathrm{Ext}}^n_\Gamma(k, M) \cong H_{-n}(\mathrm{Hom}_\Gamma(k, \mathrm{hm}_*(M)))\,.$$

The above result, which in the text corresponds to Theorem I.2.26 and Remark I.2.27, relies crucially on a result by Pareigis which exhibits the k-dual of a Hopf algebra Γ as an induced Γ-module.

Theorem I.1.2 ([Par71, Lemma 2, Proposition 3])**.** *Let Γ be a finitely generated projective Hopf algebra over k. Then there is an isomorphism*

$$\mathrm{Hom}_k(\Gamma, k) \cong \mathrm{Ind}_{\mathrm{k}}^{\Gamma} P(\mathrm{Hom}_k(\Gamma, k))$$

of right Γ-modules, where $P(\mathrm{Hom}_k(\Gamma, k))$ is a finitely generated projective k-module of constant rank 1, given as the primitives for the right Γ-coaction on $\mathrm{Hom}_k(\Gamma, k)$.

Our main reason for working primarily with Tate complexes, as opposed to complete resolutions, has to do with multiplicative structures. Recall that the cup product

$$\smile : \mathrm{Ext}^*_\Gamma(k, M) \otimes_k \mathrm{Ext}^*_\Gamma(k, N) \longrightarrow \mathrm{Ext}^*_\Gamma(k, M \otimes_k N)$$

relies on the existence of a Γ-linear chain map $\Psi : P_* \to P_* \otimes_k P_*$ covering the identity map $\mathrm{id} : k \to k \otimes_k k$. Such a chain map exists and is unique up to chain homotopy, by elementary homological algebra. One can extend this cup product to a product on Hopf algebra Tate cohomology by the existence of a Γ-linear chain map $\Phi : \widetilde{P}_* \otimes_k \widetilde{P}_* \to \widetilde{P}_*$ extending the fold map $\widetilde{P}_* \oplus_k \widetilde{P}_* \to \widetilde{P}_*$. For Γ-modules M and N the composite pairing

$$\begin{aligned}
&\widetilde{P}_* \otimes_k \mathrm{Hom}_k(P_*, M) \otimes_k \widetilde{P}_* \otimes_k \mathrm{Hom}_k(P_*, N) \\
&\qquad \xrightarrow{1\otimes\tau\otimes 1} \widetilde{P}_* \otimes_k \widetilde{P}_* \otimes_k \mathrm{Hom}_k(P_*, M) \otimes \mathrm{Hom}_k(P_*, N) \\
&\qquad \xrightarrow{1\otimes 1\otimes\alpha} \widetilde{P}_* \otimes_k \widetilde{P}_* \otimes_k \mathrm{Hom}_k(P_* \otimes_k P_*, M \otimes_k N) \\
&\qquad \xrightarrow{\Phi\otimes\Psi^*} \widetilde{P}_* \otimes_k \mathrm{Hom}_k(P_*, M \otimes_k N)
\end{aligned}$$

is Γ-linear, and it induces an associative, unital, and graded commutative pairing

$$\smile : \widehat{\mathrm{Ext}}^*_\Gamma(k, M) \otimes_k \widehat{\mathrm{Ext}}^*_\Gamma(k, N) \longrightarrow \widehat{\mathrm{Ext}}^*_\Gamma(k, M \otimes_k N)$$

after passing to homology, which we refer to as the cup product on Tate cohomology. This extends the cup product on ordinary Ext, in a suitable sense. See Proposition I.2.31.

Finally, in Section I.2.6, we do a full computation of the Tate cohomology, together with the cup product, of the Hopf algebra

$$\Gamma = k[s]/(s^2 = \eta s)\,, \quad |s| = 1\,.$$

where s is a primitive element and k is a graded commutative ring with an element η in internal degree 1 satisfying $2\eta = 0$. This has relevance in the situation $G = \mathbb{T}$, which is our main case of interest. Indeed, we have

$$\pi_*(\mathbb{S}[\mathbb{T}]) \cong \pi_*(\mathbb{S})[s]/(s^2 = \eta s)$$

where η is the image of the complex Hopf map in $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$. See Proposition I.3.3. The conclusion of the computation is the following theorem, which in the text is Theorem I.2.52 and Remark I.2.54.

Theorem I.1.3. *Tate cohomology of $\Gamma = k[s]/(s^2 = \eta s)$ with coefficients in the Γ-module M is isomorphic to the homology of the differential graded Γ-module*

$$M[t, t^{-1}]$$

with differential

$$d(m) = tms \quad \textit{and} \quad d(t) = t^2\eta\,,$$

where m is an element of M and t has homological degree -1, internal degree $|t| = -1$ and total degree $\|t\| = -2$. If $\mu : M \otimes N \to L$ is a pairing of Γ-modules, then the cup product

$$\smile : \widehat{\mathrm{Ext}}^{c_1}_\Gamma(k, M) \otimes \widehat{\mathrm{Ext}}^{c_2}_\Gamma(k, N) \longrightarrow \widehat{\mathrm{Ext}}^{c_1+c_2}_\Gamma(k, M \otimes N) \longrightarrow \widehat{\mathrm{Ext}}^{c_1+c_2}_\Gamma(k, L)$$

is precisely the one induced by the obvious pairing

$$M[t,t^{-1}] \otimes N[t,t^{-1}] \longrightarrow L[t,t^{-1}]$$

on homology.

Sequences of spectra and spectral sequences

The main difficulty of the paper lies in verifying that there is a construction of the Tate spectral sequence that is multiplicative. To deal with multiplicative structures on spectral sequences we have decided to employ Cartan–Eilenberg systems. These are mathematical gadgets, first introduced in [CE56], which determine a spectral sequence. For us, the advantage is that there is a useful notion of pairings of Cartan–Eilenberg systems, and that one can prove that a pairing of Cartan–Eilenberg systems gives rise to a pairing of the associated spectral sequences. Our contribution is a detailed and explicit proof that a pairing of sequences of orthogonal G-spectra gives rise to a pairing of Cartan–Eilenberg systems. Here, **sequence** simply means a sequential diagram

$$\cdots \longrightarrow X_{i-1} \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots$$

of maps of orthogonal G-spectra, and **pairing** $\phi\colon (X_\star, Y_\star) \to Z_\star$ refers to a collection of G-maps

$$\phi_{i,j}\colon X_i \wedge Y_j \longrightarrow Z_{i+j}$$

for all integers i and j, making the squares

$$\begin{tikzcd}
X_{i-1} \wedge Y_j \arrow[r, "\phi_{i-1,j}"] \arrow[d] & Z_{i+j-1} \arrow[d] & X_i \wedge Y_{j-1} \arrow[l, "\phi_{i,j-1}"'] \arrow[d] \\
X_i \wedge Y_j \arrow[r, "\phi_{i,j}"] & Z_{i,j} & X_i \wedge Y_j \arrow[l, "\phi_{i,j}"']
\end{tikzcd}$$

commute strictly. It is well-known that a sequence of orthogonal G-spectra gives rise to an unrolled exact couple on equivariant homotopy groups, which in turn gives rise to a spectral sequence. That a pairing of sequences gives rise of a pairing of the corresponding spectral sequences can also reasonably be regarded as folklore, but as the authors feel that an explicit reference for this is not available at the time of writing, we have decided to give a complete proof of this fact.

For homotopical control in the proofs, some sort of ‘cofibrant replacement’ of the sequence $X_\star$ is needed. In this paper we have chosen to use the classical telescope construction to deal with these sorts of issues. See Section I.4.3. Our main reason for this is that these ‘cofibrant replacements’ behave well with respect to monoidal properties. This allows us to always approximate a sequence $X_\star$ with an equivalent sequence $T_\star(X)$ in a way that will make our analysis of multiplicative structures more manageable.

The main result of Section 4 of the paper is the following, which in the text corresponds to Theorem I.4.27.

Theorem I.1.4. *A pairing $\phi\colon (X_\star, Y_\star) \to Z_\star$ of sequences of orthogonal G-spectra gives rise to a pairing $\phi : (E^*(X_\star), E^*(Y_\star)) \to E^*(Z_\star)$. Explicitly, we have access to a collection of homomorphisms*

$$\phi^r : E^r(X_\star) \otimes E^r(Y_\star) \longrightarrow E^r(Z_\star)$$

for all $r \geq 1$, such that:

1. *The Leibniz rule*
$$d^r\phi^r = \phi^r(d^r \otimes 1) + \phi^r(1 \otimes d^r)$$
holds as an equality of homomorphisms $E^r_i(X_\star) \otimes E^r_j(Y_\star) \longrightarrow E^r_{i+j-r}(Z_\star)$ for all $i, j \in \mathbb{Z}$ and $r \geq 1$.

2. *The diagram*
$$\begin{array}{ccc} E^{r+1}(X_\star) \otimes E^{r+1}(Y_\star) & \xrightarrow{\phi^{r+1}} & E^{r+1}(Z_\star) \\ \downarrow & & \downarrow{\scriptstyle\cong} \\ H(E^r(X_\star) \otimes E^r(Y_\star)) & \xrightarrow{H(\phi^r)} & H(E^r(Z_\star)) \end{array}$$
commutes for all $r \geq 1$.

Moreover, the induced pairing ϕ_* on filtered abutments is compatible with the pairing ϕ^∞ of E^∞-pages in the sense of Proposition I.4.12. Explicitly, the diagram

$$\begin{array}{ccc} \dfrac{F_iA_\infty(X_\star)}{F_{i-1}A_\infty(X_\star)} \otimes \dfrac{F_jA_\infty(Y_\star)}{F_{j-1}A_\infty(Y_\star)} & \xrightarrow{\bar\phi_*} & \dfrac{F_{i+j}A_\infty(Z_\star)}{F_{i+j-1}A_\infty(Z_\star)} \\ {\scriptstyle\beta\otimes\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ E^\infty_i(X_\star) \otimes E^\infty_j(Y_\star) & \xrightarrow{\phi^\infty} & E^\infty_{i+j}(Z_\star) \end{array}$$

commutes, for all $i, j \in \mathbb{Z}$. Here the abutments are given as

$$A_\infty(X_\star) \cong \pi^G_* \operatorname{Tel}(X_\star)$$
$$A_\infty(Y_\star) \cong \pi^G_* \operatorname{Tel}(Y_\star)$$
$$A_\infty(Z_\star) \cong \pi^G_* \operatorname{Tel}(Z_\star)$$

with filtrations by the images

$$F_iA_\infty(X_\star) = \operatorname{im}(\pi^G_*(X_i) \longrightarrow A_\infty(X_\star))$$
$$F_jA_\infty(Y_\star) = \operatorname{im}(\pi^G_*(Y_j) \longrightarrow A_\infty(Y_\star))$$
$$F_kA_\infty(Z_\star) = \operatorname{im}(\pi^G_*(Z_k) \longrightarrow A_\infty(Z_\star))\,,$$

respectively.

The G-Tate spectral sequence

Given an R-module X in orthogonal G-spectra, there are a number of ways of constructing Tate spectral sequences additively; as mentioned, the difficulty lies in establishing multiplicative properties of the constructions. The standard way of constructing a Tate spectral sequence seems to be by filtering the Tate construction

$$X^{tG} = \left(\widetilde{EG} \wedge F(EG_+, X)\right)^G \simeq \left((R \wedge \widetilde{EG}) \wedge_R F_R(R \wedge EG_+, X)\right)^G$$

by filtering $\widetilde{EG}$, in some suitable sense, dualising this filtration, and splicing, in analogy with the construction of complete resolutions by dualising and splicing projective resolutions. This is far from ideal if one aims to prove any multiplicative properties of the Tate spectral sequence. We will instead prove multiplicativity of the Tate spectral sequence using a construction along the lines of [HM03]. In this construction, we filter $F(EG_+, X)$ and $\widetilde{EG}$ separately, and totalise to get a filtration on the Tate construction.

In more detail, we proceed as follows in Section 6. We start by giving the free G-space EG the simplicial skeletal filtration $F_\star EG$ coming from the construction of EG using the simplicial bar construction. This induces a filtration

$$E_\star = R \wedge F_\star EG$$

on $R \wedge EG_+$, which in turn induces a filtration

$$M_\star(X) = F_R(E/E_{-\star-1}, X)$$

on $F_R(R \wedge EG_+, X)$, and a filtration

$$\widetilde{E}_\star = \mathrm{cone}(E_{\star-1} \longrightarrow R)$$

on $R \wedge \widetilde{EG}$. The convolution filtration

$$HM_\star(X) = (\widetilde{E} \wedge T(M(X)))_\star = \operatorname*{colim}_{i+j \leq \star} \widetilde{E}_i \wedge_R T(M(X))_j$$

is referred to as the Hesselholt–Madsen filtration. For homotopical control we have ‘cofibrantly replaced’ the filtration $M_\star(X)$ with its telescopic approximation $T_\star(M(X))$. Under our projectivity assumptions, we show that the E^1-page of the spectral sequence arising from the Hesselholt–Madsen filtration is given by

$$\hat{E}^1_{c,*} \cong \mathrm{Hom}_{R[G]_*}(R_*, \mathrm{hm}_c(\pi_*(X)))\,,$$

so that the E^2-page is given as the Hopf algebra Tate cohomology groups

$$\hat{E}^2_{c,*} \cong \widehat{\mathrm{Ext}}^{-c}_{R[G]_*}(R_*, \pi_*(X))\,,$$

as defined in Section 2. See Proposition I.6.16 and Theorem I.6.17. We note that the Hesselholt–Madsen G-Tate spectral sequence is not obviously conditionally convergent, so for convergence issues we need to do some additional work.

The existence of a multiplicative structure on the Hesselholt–Madsen G-Tate spectral sequence relies on the existence of filtration-preserving maps

$$EG_+ \longrightarrow EG_+ \wedge EG_+ \quad \text{and} \quad \widetilde{EG} \wedge \widetilde{EG} \longrightarrow \widetilde{EG}\,.$$

The first is known to exist, and we prove by obstruction theory that the second one exists under the assumption that $R[G]_*$ is projective over R_*. See Proposition I.6.9. This guarantees that a pairing $X \wedge_R Y \to Z$ of R-modules in orthogonal spectra induces a pairing

$$(HM_\star(X), HM_\star(Y)) \longrightarrow HM_\star(Z)$$

of the corresponding Hesselholt–Madsen filtrations. The work done in Section 4 then guarantees that the G-Tate spectral sequence constructed from the Hesselholt–Madsen filtration has a multiplicative structure. Moreover, we show that the multiplicative structure on the E^2-page agrees with the one given by cup product on Tate cohomology. See Theorem I.6.18 and Theorem I.6.21.

To settle questions about convergence we compare the Hesselholt–Madsen filtration to another possible filtration of the Tate construction. The filtration we are referring to is the filtration $GM_\star(X)$ given in each degree as

$$GM_k(X) = \begin{cases} \widetilde{E}_k \wedge_R T_0(M(X)) & \text{for } k \geq 0, \\ \widetilde{E}_0 \wedge_R T_k(M(X)) & \text{for } k \leq 0. \end{cases}$$

Here, the structure maps $GM_{k-1}(X) \to GM_k(X)$ for $k \geq 1$ are induced by the maps $\widetilde{E}_{k-1} \to \widetilde{E}_k$ in the filtration $\widetilde{E}_\star$, while the maps for $k \leq 0$ are those of $T_\star(M(X))$. This filtration is referred to as the **Greenlees–May filtration**. It is straight-forward to show that the spectral sequence arising from the Greenlees–May filtration is conditionally convergent; see Lemma I.6.37. Moreover, in Lemma I.6.25 we show that there is a map of filtrations

$$\alpha : GM_\star(X) \longrightarrow HM_\star(X)\,,$$

which induces an isomorphism of spectral sequences from the E^2-page and on. See Proposition I.6.31. We can then deduce convergence results for the Hesselholt–Madsen G-Tate spectral sequence in certain favourable situations, such as in the case when the spectrum X is bounded below and the derived E^∞-page RE^∞ vanishing. In particular, we have the following result, which in the text corresponds to Theorem I.6.43.

Theorem I.1.5. *If the Greenlees–May G-Tate spectral sequence for X is strongly convergent, then so is the Hesselholt–Madsen G-Tate spectral sequence for X.*

Organisation of the paper

Let us discuss the various sections contained in this paper, and how they relate to one another.

Section 2 In this section we develop a theory of Tate cohomology for finitely generated projective Hopf algebras, with a view toward being able to satisfactorily describe the E^2-page of a G-Tate spectral sequence for compact Lie groups.

Section 3 In this section we do a quick review of orthogonal G-spectra. Most of this section can be regarded as well–known to people working in genuine equivariant stable homotopy theory. However, we want to highlight Proposition I.3.6, for which we have not found a reference, and which will be important in later parts of the paper.

Section 4 In this section we discuss sequences of orthogonal G-spectra, Cartan–Eilenberg systems, and spectral sequences, with a special focus on multiplicative structures. This section may well be read separately from the rest of the paper, possibly in addition to Section 3.1, which contains a quick recap on orthogonal G-spectra. We hope it can be of use as a reference for multiplicative structures on spectral sequences coming from sequences of spectra.

Section 5 In this section we discuss the G-homotopy fixed point spectral sequence for an orthogonal G-spectrum. This is meant as a warm-up to the G-Tate spectral sequence, but can absolutely be read in its own right.

Section 6 In this section we discuss various constructions of the G-Tate spectral sequence of an orthogonal G-spectrum. The reader who only cares for the $\mathbb{T}$-Tate spectral sequence will find a summary of the relevant results at the very end of the paper, in Section I.6.7.

I.2 Tate cohomology for Hopf algebras

The algebraic objects that we are led to work with when constructing the Tate spectral sequence are Hopf algebras and chain complexes of modules over these. The topological context we will discuss later in the paper allows for Hopf algebras over fairly complicated rings, which forces us to work in the generality of Hopf algebras over arbitrary, possibly graded, commutative rings. We give a brief account of this in Section 2.1 and Section 2.2. We go on to give a suitable definition of Tate cohomology of Hopf algebras via the so-called Tate complex in Section 2.3. In Section 2.4 we relate this definition to the ordinary definition of Tate cohomology in terms of complete resolutions. In particular, we show in Theorem I.2.26 that our definition agrees with what is traditionally referred to as Tate cohomology or complete Ext, in the case when our Hopf algebra Γ is finitely generated and projective over its base ring k. The crucial point that allows us to do this is a result of Pareigis, which in particular forces the k-dual of Γ to be finitely generated and projective over Γ, under the same hypotheses. We discuss the multiplicative structure of Tate cohomology in Section 2.5, and finish with an explicit computation in Section 2.6.

I.2.1 Modules over Hopf algebras

Let k be a graded commutative ring, where we mean commutative in the graded sense. All unlabelled tensors and homs are to be taken over k. We denote the closed symmetric monoidal category of right k-modules by $\mathrm{Mod}(k)$. Note that such modules are implicitly graded, and that morphisms of such modules, which we will refer to as k-linear homomorphisms, are degree-preserving.

Definition I.2.1. A **Hopf algebra** Γ **over** k is a k-module equipped with five k-linear homomorphisms: **multiplication** $\phi : \Gamma \otimes \Gamma \to \Gamma$, **comultiplication** $\psi : \Gamma \to \Gamma \otimes \Gamma$, **unit** $\eta : k \to \Gamma$, **counit** $\epsilon : \Gamma \to k$, and **antipode** $\chi : \Gamma \to \Gamma$. These are subject to the following conditions:

1. Multiplication and unit provide Γ with the structure of a k-algebra.

2. Comultiplication and counit provide Γ with the structure of a k-coalgebra.

3. Comultiplication and counit are k-algebra morphisms, or equivalently, multiplication and unit are k-coalgebra morphisms.

4. The antipode satisfies the formulae $\phi(1 \otimes \chi)\psi = \eta\epsilon = \phi(\chi \otimes 1)\psi$.

We say that a Hopf algebra is **cocommutative** if the comultiplication satisfies $\tau\psi = \psi$, where τ denotes the twist in $\mathrm{Mod}(k)$. We are going to assume that all Hopf algebras we work with are cocommutative in this paper.

A module over a Hopf algebra is just a module over the underlying k-algebra. For a right Γ-module M we denote the right action by $\rho_M : M \otimes \Gamma \to M$. We denote the category of right Γ-modules by $\mathrm{Mod}(\Gamma)$. This is a closed symmetric[3] monoidal category if we endow the category with the tensor products and internal homs over k together with appropriate Γ-actions on these objects. Here let M, N, and L be Γ-modules. The tensor product $M \otimes N$ is endowed with the diagonal Γ-action. This is the composition

$$M \otimes N \otimes \Gamma \xrightarrow{1\otimes 1\otimes\psi} M \otimes N \otimes \Gamma \otimes \Gamma \xrightarrow{1\otimes\tau\otimes 1} M \otimes \Gamma \otimes N \otimes \Gamma \xrightarrow{\rho_M\otimes\rho_N} M \otimes N \ .$$

The unit of the tensor product is k regarded as a trivial Γ-module via the counit:

$$k \otimes \Gamma \xrightarrow{1\otimes\epsilon} k \otimes k = k \ .$$

The internal hom $\mathrm{Hom}(N, L)$ becomes a Γ-module by giving it the conjugate Γ-action. This is the Γ-action that needs to be on the internal hom to make sure that $\mathrm{Hom}(N, -)$ is right adjoint to $(-) \otimes N : \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(\Gamma)$. In other words, the characterising feature of the conjugate Γ-action is that it is the Γ-action on $\mathrm{Hom}(N, L)$ that makes the counit $\mathrm{Hom}(N, L) \otimes N \to L$ and the unit

[3] Symmetry uses that Γ is cocommutative.

$M \to \mathrm{Hom}(M \otimes N, N)$ into Γ-linear maps. Explicitly, the conjugate action is adjoint to the composition

$$\begin{aligned}\mathrm{Hom}(N, L) \otimes \Gamma \otimes N &\xrightarrow{1\otimes\tau} \mathrm{Hom}(N, L) \otimes N \otimes \Gamma \xrightarrow{1\otimes 1\otimes\psi} \mathrm{Hom}(N, L) \otimes N \otimes \Gamma \otimes \Gamma \\ &\xrightarrow{1\otimes 1\otimes\chi\otimes 1} \mathrm{Hom}(N, L) \otimes N \otimes \Gamma \otimes \Gamma \\ &\xrightarrow{1\otimes\rho_N\otimes 1} \mathrm{Hom}(N, L) \otimes N \otimes \Gamma \xrightarrow{\mathrm{ev}\otimes 1} L \otimes \Gamma \xrightarrow{\rho_L} L\,.\end{aligned}$$

These actions on tensor and hom-objects ensure that the forgetful functor $U\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(k)$ is strict closed monoidal.

Lemma I.2.2. *Let M and N be Γ-modules, where we assume that M is projective over Γ and N is projective over k. Then $M \otimes N$ is projective over Γ.*

Proof. By the tensor-hom adjunction we have a natural isomorphism

$$\mathrm{Hom}_\Gamma(M \otimes N, -) \cong \mathrm{Hom}_\Gamma(M, \mathrm{Hom}(N, -))$$

of functors. Since N is projective over k the functor $\mathrm{Hom}(N, -)$ is exact, and since M is projective over Γ the functor $\mathrm{Hom}_\Gamma(M, -)$ is exact. The functor $\mathrm{Hom}_\Gamma(M \otimes N, -)$ is then also exact, being naturally isomorphic to the composition of two exact functors. This is equivalent to the assertion that $M \otimes N$ is projective over Γ. $\square$

The forgetful functor U admits a left adjoint

$$\mathrm{Ind}_{\mathrm{k}}^{\Gamma} : \mathrm{Mod}(k) \longrightarrow \mathrm{Mod}(\Gamma)\,,$$

which we refer to as **induction**. This functor sends a k-module C to $C \otimes \Gamma$ with the Γ-action given by

$$C \otimes \Gamma \otimes \Gamma \xrightarrow{1\otimes\phi} C \otimes \Gamma\,.$$

The forgetful functor U also admits a right adjoint

$$\mathrm{Coind}_{\mathrm{k}}^{\Gamma} : \mathrm{Mod}(k) \longrightarrow \mathrm{Mod}(\Gamma)\,,$$

which is referred to as **coinduction**. This functor sends a k-module C to $\mathrm{Hom}(\Gamma, C)$ with Γ-action given as the adjoint of

$$\begin{aligned}\mathrm{Hom}(\Gamma, C) \otimes \Gamma \otimes \Gamma &\xrightarrow{1\otimes\tau} \mathrm{Hom}(\Gamma, C) \otimes \Gamma \otimes \Gamma \xrightarrow{1\otimes 1\otimes\chi} \mathrm{Hom}(\Gamma, C) \otimes \Gamma \otimes \Gamma \\ &\xrightarrow{1\otimes\phi} \mathrm{Hom}(\Gamma, C) \otimes \Gamma \xrightarrow{\mathrm{ev}} C\,.\end{aligned}$$

The fact that the forgetful functor is strict monoidal makes sure that induction and coinduction interact with the forgetful functor in various useful ways. In [LMSM86, Section 2.4] the following formulae, in the context of equivariant stable homotopy theory, are called **untwisting isomorphisms**, and we will refer to them as such also in this paper.

Proposition I.2.3. *Let M be a Γ-module and let C be a k-module. There are natural Γ-module isomorphisms:*

1. $$\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(C \otimes U(M)) \cong \mathrm{Ind}_{\mathrm{k}}^{\Gamma}(C) \otimes M$$

2. $$\mathrm{Hom}(M, \mathrm{Coind}_{\mathrm{k}}^{\Gamma}(C)) \cong \mathrm{Coind}_{\mathrm{k}}^{\Gamma}(\mathrm{Hom}(U(M), C))$$

3. $$\mathrm{Hom}(\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(C), M) \cong \mathrm{Coind}_{\mathrm{k}}^{\Gamma}(\mathrm{Hom}(C, U(M)))\,.$$

Proof. The result follows formally from the Yoneda lemma together with the fact that $\mathrm{Mod}(\Gamma) \to \mathrm{Mod}(k)$ is strict closed monoidal. Let us show the first isomorphism, the other two are proven in a similar manner.

Consider the functor corepresented by $\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(C \otimes U(M))$. By adjunctions we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\Gamma}(\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(C \otimes U(M)), -) &\cong \mathrm{Hom}(C \otimes U(M), U(-)) \\ &\cong \mathrm{Hom}(C, \mathrm{Hom}(U(M), U(-)))\,. \end{aligned}$$

Since the forgetful functor is strict closed monoidal we have the identity

$$\mathrm{Hom}(C, \mathrm{Hom}(U(M), U(-))) = \mathrm{Hom}(C, U(\mathrm{Hom}(M, -)))$$

and by adjunctions again

$$\begin{aligned} \mathrm{Hom}(C, U(\mathrm{Hom}(M, -))) &\cong \mathrm{Hom}_{\Gamma}(\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(C), \mathrm{Hom}(M, -)) \\ &\cong \mathrm{Hom}_{\Gamma}(\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(C) \otimes M, -)\,. \end{aligned}$$

The Yoneda lemma now asserts that we have a natural isomorphism, as wanted. □

Corollary I.2.4. *Let M be a Γ-module. There are natural isomorphisms*

$$\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(U(M)) \cong \Gamma \otimes M \quad \textit{and} \quad \mathrm{Coind}_{\mathrm{k}}^{\Gamma}(U(M)) \cong \mathrm{Hom}(\Gamma, M)$$

where the Γ-action on the right hand sides are the ordinary diagonal and conjugate actions, respectively.

Proof. Use that $\Gamma = \mathrm{Ind}_{\mathrm{k}}^{\Gamma}(k)$. □

We will also deal a lot with functional duals of modules over Hopf algebras, so let us now recall this story.

Definition I.2.5. For each Γ-module M let

$$DM = \mathrm{Hom}(M, k)$$

be its **functional dual**. This is a Γ-module by using the usual conjugate Γ-action.

Note that the evaluation pairing ev : $\mathrm{Hom}(N, L) \otimes N \to L$ gives rise to a natural Γ-linear pairing

$$\alpha : \mathrm{Hom}(N, L) \otimes \mathrm{Hom}(N', L') \longrightarrow \mathrm{Hom}(N \otimes N', L \otimes L')$$

adjoint to the composition

$$\begin{aligned}\mathrm{Hom}(N, L) \otimes \mathrm{Hom}(N', L') \otimes N \otimes N' &\xrightarrow{1\otimes\tau\otimes 1} \mathrm{Hom}(N, L) \otimes L \otimes \mathrm{Hom}(N', L') \otimes N' \\ &\xrightarrow{\mathrm{ev}\otimes\mathrm{ev}} L \otimes L' \,.\end{aligned}$$

In the case $N' = L = k$ this specialises to a natural Γ-linear homomorphism

$$\nu : DN \otimes L' \longrightarrow \mathrm{Hom}(N, L') \,.$$

This map is an isomorphism when N is finitely generated and projective over k.

So far we have only discussed Γ-modules, but we can also talk about (right) comodules over Γ, by which we mean comodules over the underlying coalgebra structure of Γ. If Γ is finitely generated projective over k then we can endow its functional dual $D\Gamma$ with such a Γ-coaction. Moreover, this Γ-coaction is compatible with the Γ-action in a suitable way. See [Par71, Prop. 2]. This allows us to conclude the following.

Theorem I.2.6 ([Par71, Lemma 2, Proposition 3])**.** *Let Γ be a finitely generated projective Hopf algebra over k. Then there is an isomorphism*

$$D\Gamma \cong \mathrm{Ind}_{\mathrm{k}}^{\Gamma} P(D\Gamma)$$

of right Γ-modules, where $P(D\Gamma)$ is a finitely generated projective k-module of constant rank 1, given as the primitives for the right Γ-coaction on $D\Gamma$.

Note in particular that a direct consequence of Γ being finitely generated and projective over k is that $D\Gamma$ is itself finitely generated and projective over Γ. This result will be crucial in our treatment of Tate cohomology of Hopf algebras. For now, let us simply note that the result implies that we have a 'Wirthmüller isomorphism'.

Corollary I.2.7. *Let Γ be a finitely generated and projective Hopf algebra over k and let C be a k-module. There is a natural isomorphism*

$$\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(P(D\Gamma) \otimes C) \cong \mathrm{Coind}_{\mathrm{k}}^{\Gamma}(C)$$

of Γ-modules.

Proof. We can assume that C is obtained from a Γ-module M by forgetting the Γ-action, as in $C = UM$. By the first untwisting isomorphism of Proposition I.2.3 we have

$$\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(P(D\Gamma) \otimes U(M)) \cong \mathrm{Ind}_{\mathrm{k}}^{\Gamma}(P(D\Gamma)) \otimes M \,.$$

By Pareigis' result it follows that

$$\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(P(D\Gamma)) \otimes M \cong D\Gamma \otimes M\,.$$

Finally, by Γ being finitely generated projective over k and untwisting, more specifically Corollary I.2.4, we have

$$D\Gamma \otimes M \cong \mathrm{Hom}(\Gamma, M) \cong \mathrm{Coind}_{\mathrm{k}}^{\Gamma}(U(M))\,.$$

□

Since $P(D\Gamma)$ is tensor-invertible over k the above tells us that induced modules are the same things as coinduced modules. We also note that that duals of finitely generated projectives over Γ are themselves finitely generated projective over Γ.

Corollary I.2.8. *Let Γ be a finitely generated projective Hopf algebra over k and let M be a finitely generated projective Γ-module. Then its dual DM is also finitely generated projective over Γ.*

Proof. Since M is finitely generated, we can find a short exact sequence

$$0 \longrightarrow \ker(r) \longrightarrow \textstyle\bigoplus_{i\in I} \Sigma^{n_i}\Gamma \xrightarrow{\ r\ } M \longrightarrow 0$$

of Γ-modules, where I is a finite indexing set. Since M is projective, this short exact sequence splits, so that we can find a Γ-linear map $u\colon M \to \bigoplus_{i\in I} \Sigma^{n_i}\Gamma$ such that $ru = \mathrm{id}_M$. Consider the k-dual picture. Applying $\mathrm{Hom}(-,k)$ gives us a long exact sequence of Γ-modules:

$$\cdots \longleftarrow \mathrm{Ext}_k^1(M,k) \longleftarrow D\ker(r) \longleftarrow D\left(\textstyle\bigoplus_{i\in I} \Sigma^{n_i}\Gamma\right) \xleftarrow{\ r^*\ } DM \longleftarrow 0\,.$$

Since M is projective over Γ, which is in turn projective over k, it follows that M is projective over k, from which we conclude that $\mathrm{Ext}_k^1(M,k) \cong 0$. We are left with a short exact sequence, which is also split since $u^*r^* = \mathrm{id}_{DM}$. We conclude that DM is finitely generated projective over Γ since it is a retract of the finitely generated projective Γ-module

$$D\left(\bigoplus_{i\in I} \Sigma^{n_i}\Gamma\right) \cong \bigoplus_{i\in I} \Sigma^{-n_i} D\Gamma \cong \bigoplus_{i\in I} \Sigma^{-n_i}\,\mathrm{Ind}_{\mathrm{k}}^{\Gamma} P(D\Gamma)\,.$$

□

I.2.2 Chain complexes of Γ-modules

In this section we give the conventions for chain complexes of Γ-modules. A lot is standard; the category $\mathrm{Mod}(\Gamma)$ is an abelian category and what we mean by chain complexes of Γ-modules is nothing more than the ordinary category of chain complexes in this abelian category. However, we want to make a point of clarifying certain subtle points, especially related to grading and signs.

Definition I.2.9. A **chain complex X_* of Γ-modules** is a family $(X_n)_{n\in\mathbb{Z}}$ of Γ-modules together with morphisms of Γ-modules $\partial : X_n \to X_{n-1}$, called **boundaries**, such that $\partial^2 = 0$. A **chain map** $f : X_* \to Y_*$ is a family of Γ-module homomorphisms $f_n : X_n \to Y_n$ that commute with the boundaries.

The Γ-module X_n in the chain complex is of course implicitly graded:

$$X_n = \bigoplus_{\ell\in\mathbb{Z}} X_{n,\ell},$$

and if we want to emphasize the bigrading we will write $X_{*,*}$ for the complex. We remark on the following point: as the boundaries are morphisms of Γ-modules they preserve the Γ-module grading in the sense that they are maps $\partial : X_{n,\ell} \to X_{n-1,\ell}$. We use the following terminology for the different degrees.

Definition I.2.10. Let X_* be a chain complex of Γ-modules. If x is an element in $X_{n,\ell}$ we say that x has **homological degree** n, **internal degree** $|x| = \ell$, and **total degree** $\|x\| = n + \ell$.

As indicated, we will use the notations $\|x\|$ and $|x|$ for the total and internal degree of x, respectively. The category of chain complexes of Γ-modules is closed symmetric monoidal. If X_*, Y_*, and Z_* are chain complexes of Γ-modules then the tensor product $X_* \otimes Y_*$ is defined in each degree as

$$(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes Y_j, \quad \partial(x \otimes y) = \partial(x) \otimes y + (-1)^{||x||} x \otimes \partial(y).$$

The unit for the tensor product is k concentrated in homological degree 0. Note that the twist isomorphism is given as

$$\tau : (X \otimes Y)_* \longrightarrow (Y \otimes X)_*\,, \quad x \otimes y \mapsto (-1)^{\|x\|\|y\|} y \otimes x\,.$$

The hom complex $\mathrm{Hom}(Y, Z)_*$ is defined as

$$\mathrm{Hom}(Y, Z)_n = \prod_{i+j=n} \mathrm{Hom}(Y_{-i}, Z_j), \quad (\partial f)(x) = \partial(f(x)) - (-1)^{||f||} f(\partial(x))\,.$$

We will in particular be interested in the case when Z_* is a Γ-module M, regarded as a chain complex concentrated in homological degree 0. In this case, we will often denote the differential in the resulting function complex as $\partial^* = \mathrm{Hom}(\partial, 1)$. Explicitly, we have

$$\mathrm{Hom}(Y, M)_n = \mathrm{Hom}(Y_{-n}, M)\,, \quad (\partial^* f)(x) = -(-1)^{||f||} f(\partial(x))\,.$$

As before, the evaluation pairing $\mathrm{ev} : \mathrm{Hom}(Y, Z)_* \otimes Y_* \to Z_*$ gives rise to a natural Γ-chain map

$$\alpha : \mathrm{Hom}(Y, Z)_* \otimes \mathrm{Hom}(Y', Z')_* \longrightarrow \mathrm{Hom}(Y_* \otimes Y'_*, Z_* \otimes Z'_*)$$

adjoint to the composition

$$\begin{aligned}\mathrm{Hom}(Y,Z)_* \otimes \mathrm{Hom}(Y',Z')_* \otimes Y_* \otimes Y'_* &\xrightarrow{1\otimes\tau\otimes 1} \mathrm{Hom}(Y,Z)_* \otimes Y_* \otimes \mathrm{Hom}(Y',Z')_* \otimes Y'_* \\ &\xrightarrow{\mathrm{ev}\otimes\mathrm{ev}} Z_* \otimes Z'_* \,.\end{aligned}$$

Note that this introduces a sign in the formula for α coming from the twist τ. Explicitly,

$$\alpha(f \otimes g)(x \otimes y) = (-1)^{\|g\|\|x\|} f(x) \otimes g(y)\,. \tag{I.1}$$

We now turn to suspensions and mapping cones. These are determined by specifying a 'circle chain complex' and an 'interval chain complex'. The interval object is the chain complex I_* given as

$$0 \longrightarrow k\{i_1\} \xrightarrow{\partial} k\{i_0\} \longrightarrow 0\,, \quad \partial(i_1) = i_0\,.$$

Both of the generators i_0 and i_1 are regarded as a having internal degree 0 and the subscripts indicate the homological degrees. The circle object is the chain complex C_* given as

$$0 \longrightarrow k\{c_1\} \longrightarrow 0\,,$$

again with c_1 regarded as having internal degree 0 and with the subscript indicating the homological degree.

The convention we will use in this paper is that chain complexes are suspended on the left. In more precise terms, the suspension of a chain complex X_* is the chain complex $X[1]_* = C_* \otimes X_*$. From the definition of the symmetric monoidal structure and the appropriate identifications we get

$$X[1]_n \cong X_{n-1}\,, \quad \partial_{X[1]}(x) = -\partial_X(x)\,.$$

Definition I.2.11. The **mapping cone** of a chain map $f : X_* \to Y_*$ is the chain complex $\mathrm{cone}(f)_*$ given as the pushout in the diagram

$$\begin{tikzcd} X_* \arrow[r, "f"] \arrow[d, "i_0"'] & Y_* \arrow[d] \\ I_* \otimes X_* \arrow[r] & \mathrm{cone}(f)_* \end{tikzcd}$$

where $i_0(x) = i_0 \otimes x$.

Explicitly, we have

$$\mathrm{cone}(f)_n \cong X_{n-1} \oplus Y_n\,, \quad \partial(x,y) = (-\partial(x), \partial(y) + f(x))\,.$$

We have a short exact sequence of chain complexes

$$0 \longrightarrow Y_* \longrightarrow \mathrm{cone}(f)_* \longrightarrow X[1]_* \longrightarrow 0$$

where the first map is $y \mapsto (0,y)$ and the second one is $(x,y) \mapsto x$. We leave it to the reader to convince themself that these are indeed chain maps.

I.2.3 Tate complexes

Consider a projective Γ-resolution $\epsilon : P_* \to k$ of the trivial Γ-module k. We will denote the mapping cone of the map ϵ as $\widetilde{P}_*$. With the conventions from before we hence have

$$\widetilde{P}_n \cong \begin{cases} k & \text{if } n = 0 \\ P_{n-1} & \text{otherwise} \end{cases}$$

with boundary $\tilde{\partial}\colon \widetilde{P}_n \to \widetilde{P}_{n-1}$ given as

$$\tilde{\partial}(x) = \begin{cases} -\partial(x) & \text{if } n \geq 2 \\ \epsilon(x) & \text{if } n = 1. \end{cases}$$

Let us use the notation $i\colon k \to \widetilde{P}_*$ for the inclusion. We now define the so-called Tate complex.

Definition I.2.12. For each Γ-module M let

$$\mathrm{hm}_*(M) = \widetilde{P}_* \otimes \mathrm{Hom}(P, M)_*$$

be the **Tate complex** of [Gre95, §3].

Complexes of this type arise from a filtration of the Tate construction on a G-spectrum that we call the Hesselholt–Madsen filtration, which is adapted from [HM03], and this explains the notation "hm". See Section I.6.3. Explicitly, the Tate complex is given in each homological degree by

$$\mathrm{hm}_n(M) = \bigoplus_{i+j=n} \widetilde{P}_i \otimes \mathrm{Hom}(P_{-j}, M)$$

with boundary given as

$$\partial_{\mathrm{hm}}(x \otimes f) = \tilde{\partial}(x) \otimes f + (-1)^{||x||} x \otimes \partial^*(f)\,.$$

Definition I.2.13. For an integer n let

$$\widehat{\mathrm{Ext}}^n_\Gamma(k, M) = H_{-n}(\mathrm{Hom}_\Gamma(k, \mathrm{hm}_*(M)))$$

be the k-module given by the $(-n)$th homology of the chain complex

$$\mathrm{Hom}_\Gamma(k, \mathrm{hm}_*(M)) = \mathrm{Hom}_\Gamma(k, \widetilde{P}_* \otimes \mathrm{Hom}(P, M)_*)\,.$$

We call this the nth Tate cohomology group of Γ with coefficients in the Γ-module M.

To be able to compare this definition to the standard definition of Tate cohomology in terms of complete resolutions, it is convenient to introduce an alternative, quasi-isomorphic chain complex.

Definition I.2.14. For each Γ-module M, let $\mathrm{gm}_*(M)$ be the pushout in the diagram

$$\begin{array}{ccc} M & \xrightarrow{\epsilon^*} & \mathrm{Hom}(P, M)_* \\ \downarrow{\scriptstyle i\otimes 1} & & \downarrow \\ \widetilde{P}_* \otimes M & \longrightarrow & \mathrm{gm}_*(M)\,. \end{array}$$

Here the top horizontal morphism is the map

$$\epsilon^* = \mathrm{Hom}(\epsilon, 1)\colon M \cong \mathrm{Hom}(k, M) \longrightarrow \mathrm{Hom}(P, M)_*$$

contravariantly induced by the augmentation, and the left hand vertical morphism is the map

$$i \otimes 1 : M \cong k \otimes M \longrightarrow \widetilde{P}_* \otimes M$$

induced by the inclusion of k into the mapping cone $\widetilde{P}_* = \mathrm{cone}(\epsilon)$.

Complexes of this type arise from a filtration of the Tate construction on a G-spectrum that we call the Greenlees–May filtration, which is adapted from [GM95], and this explains the notation "gm". See Section I.6.5.

Proposition I.2.15. *Explicitly, the complex* $\mathrm{gm}_*(M)$ *is given in each homological degree as*

$$\mathrm{gm}_n(M) \cong \begin{cases} \widetilde{P}_n \otimes M & \text{if } n \geq 1 \\ \mathrm{Hom}(P_{-n}, M) & \text{if } n \leq 0 \end{cases}$$

and under these identifications the boundary $\partial_{\mathrm{gm}}\colon \mathrm{gm}_n(M) \to \mathrm{gm}_{n-1}(M)$ *is given as*

$$\partial_{\mathrm{gm}} = \begin{cases} \tilde{\partial} \otimes 1 & \text{if } n \geq 2 \\ \partial^* & \text{if } n \leq 0 \\ \widetilde{P}_1 \otimes M \xrightarrow{\epsilon \otimes 1} M \xrightarrow{\epsilon^*} \mathrm{Hom}(P_0, M) & \text{if } n = 1. \end{cases}$$

Proof. The only non-trivial case happens when the homological degree is $n = 0$. In this case we have a pushout square

$$\begin{array}{ccc} M & \xrightarrow{\epsilon^*} & \mathrm{Hom}(P_0, M) \\ \downarrow{\scriptstyle i\otimes 1} & & \downarrow \\ \widetilde{P}_0 \otimes M & \longrightarrow & \mathrm{gm}_0(M)\,. \end{array}$$

Since $i \otimes 1$ is an isomorphism in this homological degree it follows that so is the map $\mathrm{Hom}(P_0, M) \to \mathrm{gm}_0(M)$.

It is straight-forward to see that the boundary $\partial\colon \mathrm{gm}_n(M) \to \mathrm{gm}_{n-1}(M)$ is given by $\tilde{\partial} \otimes 1$ and ∂^* when $n \geq 2$ and $n \leq 0$, respectively. For the remaining boundary, note that the element $1 \otimes m$ in $\widetilde{P}_0 \otimes M$ is identified with the element $f : y \mapsto m\epsilon(y)$ in $\mathrm{Hom}(P_0, M)$ when both are viewed as elements of the pushout $\mathrm{gm}_0(M)$. The boundary wants to take the element $x \otimes m$ in $\widetilde{P}_1 \otimes M \cong \mathrm{gm}_1(M)$ to $\epsilon(x) \otimes m$ in $\widetilde{P}_0 \otimes M$. This is identified with the map $y \mapsto (-1)^{|m||x|} m\epsilon(x)\epsilon(y)$ in $\mathrm{Hom}(P_0, M)$. Schematically, we are taking the composite

$$\widetilde{P}_1 \otimes M \xrightarrow{\epsilon\otimes 1} \widetilde{P}_0 \otimes M \cong M \cong \mathrm{Hom}(k, M) \xrightarrow{\epsilon^*} \mathrm{Hom}(P_0, M)\,. \qquad \square$$

Visually, $\mathrm{gm}_*(M)$ is the complex

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{P}_2 \otimes M \arrow[r, "\widetilde{\partial}\otimes 1"] & \widetilde{P}_1 \otimes M \arrow[rr] \arrow[dr, "\epsilon\otimes 1"'] & & \mathrm{Hom}(P_0, M) \arrow[r, "\partial^*"] & \mathrm{Hom}(P_1, M) \arrow[r] & \cdots\,. \\
& & & M \arrow[ur, "\epsilon^*"'] & & &
\end{tikzcd}$$

We will often refer to the complex $\mathrm{gm}_*(M)$ as being obtained by 'splicing' $\widetilde{P}_* \otimes M$ and $\mathrm{Hom}(P, M)_*$ together.

Note that the universal property of the pushout ensures that we have an induced Γ-chain map $\theta : \mathrm{gm}_*(M) \to \mathrm{hm}_*(M)$ in the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\epsilon^*"] \arrow[d, "i\otimes 1"] & \mathrm{Hom}(P, M)_* \arrow[d] \arrow[ddr, bend left, "i\otimes 1"] & \\
\widetilde{P}_* \otimes M \arrow[r] \arrow[drr, bend right, "1\otimes\epsilon^*"'] & \mathrm{gm}_*(M) \arrow[dr, "\theta"] & \\
& & \mathrm{hm}_*(M)\,.
\end{tikzcd}$$

Here the 'bendy' map

$$1 \otimes \epsilon^* : \widetilde{P}_* \otimes M \cong \widetilde{P}_* \otimes \mathrm{Hom}(k, M) \longrightarrow \widetilde{P}_* \otimes \mathrm{Hom}(P_*, M)$$

is again the map contravariantly induced by the augmentation, and the other 'bendy' map

$$i \otimes 1 : \mathrm{Hom}(P, M)_* \cong k \otimes \mathrm{Hom}(P, M)_* \longrightarrow \widetilde{P}_* \otimes \mathrm{Hom}(P, M)_*$$

is induced by the inclusion $i : k \to \widetilde{P}_*$.

Proposition I.2.16. *The k-linear chain map*

$$\mathrm{Hom}(1, \theta) \colon \mathrm{Hom}_\Gamma(k, \mathrm{gm}_*(M)) \longrightarrow \mathrm{Hom}_\Gamma(k, \mathrm{hm}_*(M))$$

is a quasi-isomorphism, inducing isomorphisms

$$H_n(\mathrm{Hom}_\Gamma(k, \mathrm{gm}_*(M))) \xrightarrow{\cong} \widehat{\mathrm{Ext}}_\Gamma^{-n}(k, M)$$

for all integers n.

Proof. We compatibly filter $\mathrm{gm}_*(M)$ and $\mathrm{hm}_*(M)$, setting

$$F_s\,\mathrm{gm}_k(M) = \begin{cases} 0 & \text{for } k > s, \\ \mathrm{gm}_k(M) & \text{for } k \leq s \end{cases}$$

and

$$F_s\,\mathrm{hm}_k(M) = \bigoplus_{\substack{i+j=k \\ i \leq s}} \widetilde{P}_i \otimes \mathrm{Hom}(P_{-j}, M)\,.$$

We obtain a vertical map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & F_{s-1}\,\mathrm{gm}_*(M) \arrow[r] \arrow[d, "\theta_{s-1}"'] & F_s\,\mathrm{gm}_*(M) \arrow[r] \arrow[d, "\theta_s"'] & \dfrac{F_s\,\mathrm{gm}_*(M)}{F_{s-1}\,\mathrm{gm}_*(M)} \arrow[r] \arrow[d, "\bar\theta_s"'] & 0 \\ 0 \arrow[r] & F_{s-1}\,\mathrm{hm}_*(M) \arrow[r] & F_s\,\mathrm{hm}_*(M) \arrow[r] & \dfrac{F_s\,\mathrm{hm}_*(M)}{F_{s-1}\,\mathrm{hm}_*(M)} \arrow[r] & 0\,. \end{tikzcd}$$

Each horizontal short exact sequence is degree-wise split as an extension of Γ-modules, hence remains short exact after applying $\mathrm{Hom}_\Gamma(k, -)$.

For $s = 0$, the map $F_0\,\mathrm{gm}_*(M) \to F_0\,\mathrm{hm}_*(M)$ is an isomorphism. We claim for each $s \geq 1$ that the map of filtration subquotients

$$\bar\theta_s \colon \frac{F_s\,\mathrm{gm}_*(M)}{F_{s-1}\,\mathrm{gm}_*(M)} \longrightarrow \frac{F_s\,\mathrm{hm}_*(M)}{F_{s-1}\,\mathrm{hm}_*(M)}$$

induces a quasi-isomorphism $\mathrm{Hom}(1, \bar\theta_s)$ after applying $\mathrm{Hom}_\Gamma(k, -)$. It follows by induction that $\mathrm{Hom}(1, \theta_s)$ is a quasi-isomorphism for each $s \geq 0$. Passing to colimits over s it follows that $\mathrm{Hom}(1, \theta)$ is a quasi-isomorphism.

It remains to prove the claim. We can rewrite $\bar\theta_s$ for $s \geq 1$ as

$$1 \otimes \epsilon^* \colon \widetilde{P}_s \otimes M \longrightarrow \widetilde{P}_s \otimes \mathrm{Hom}(P, M)_*\,.$$

Here $\widetilde{P}_s$ is Γ-projective, so it suffices to prove that

$$\mathrm{Hom}_\Gamma(1, 1 \otimes \epsilon^*) \colon \mathrm{Hom}_\Gamma(k, L \otimes M) \longrightarrow \mathrm{Hom}_\Gamma(k, L \otimes \mathrm{Hom}(P, M)_*)$$

is a quasi-isomorphism for any projective Γ-module L. By preservation of quasi-isomorphisms under passage to retracts, we may assume that L is free. Since $\mathrm{Hom}_\Gamma(k, -)$ commutes with direct sums, it suffices to consider the case $L = \Gamma$. Using the Hopf algebra structure of Γ, there is a natural untwisting isomorphism

$$\Gamma \otimes N \cong \mathrm{Ind}_{\mathrm{k}}^{\Gamma}(N)$$

for any Γ-module N, where Γ acts diagonally on the left hand side and we use the induced Γ-action on the right hand side. See Corollary I.2.4. The augmentation $\epsilon\colon P_* \to k$ admits a k-linear chain homotopy inverse. Hence $\epsilon^*\colon M \to \mathrm{Hom}(P,M)_*$ also admits such a chain homotopy inverse, and

$$\mathrm{Ind}_{\mathrm{k}}^{\Gamma}(\epsilon^*)\colon \mathrm{Ind}_{\mathrm{k}}^{\Gamma}(M) \longrightarrow \mathrm{Ind}_{\mathrm{k}}^{\Gamma}(\mathrm{Hom}(P,M)_*)$$

admits a Γ-linear chain homotopy inverse. By naturality of the untwisting isomorphism,

$$1 \otimes \epsilon^*\colon \Gamma \otimes M \longrightarrow \Gamma \otimes \mathrm{Hom}(P,M)_*$$

admits a Γ-module chain homotopy inverse, and therefore induces a k-module chain homotopy equivalence after applying $\mathrm{Hom}_\Gamma(k,-)$. This proves the claim that $\mathrm{Hom}_\Gamma(1, 1\otimes\epsilon^*)$ is a quasi-isomorphism. □

Corollary I.2.17. *The inclusion* $\mathrm{Hom}(P,M)_* \to \mathrm{gm}_*(M)$ *induces an isomorphism*

$$\gamma\colon \mathrm{Ext}^n_\Gamma(k,M) \longrightarrow \widehat{\mathrm{Ext}}^n_\Gamma(k,M)$$

for each $n \geq 1$, *and a surjection for* $n = 0$.

I.2.4 Complete resolutions

In this section, we make the standing assumption that the cocommutative Hopf algebra Γ is finitely generated and projective over k. Let us now relate the complex $\mathrm{gm}_*(M)$ to the complete resolutions often used when defining Tate cohomology.

Definition I.2.18. Let $\hat{P}_*$ be the pullback in the diagram

$$\begin{tikzcd} \hat{P}_* \arrow[r] \arrow[d] & P_* \arrow[d, "\epsilon"] \\ D\widetilde{P}_* \arrow[r, "i^*"] & k \end{tikzcd}$$

where $D\widetilde{P}_* = \mathrm{Hom}(\widetilde{P}_*, k)$.

Proposition I.2.19. *Explicitly, the chain complex* $\hat{P}_*$ *is given in each homological degree as*

$$\hat{P}_n \cong \begin{cases} P_n & \text{for } n \geq 0, \\ D(\widetilde{P}_{-n}) & \text{for } n < 0 \end{cases}$$

with boundary given as

$$\hat{\partial}_n = \begin{cases} \partial_n & \text{for } n > 0, \\ \epsilon^* \circ \epsilon & \text{for } n = 0, \\ D(\tilde{\partial}_{1-n}) & \text{for } n < 0 \end{cases}$$

under these identifications.

Proof. The only non-trivial case is when we are dealing with something involving homological degree 0. Since $D\widetilde{P}_0 \to k$ is an isomorphism, it follows that the projection $\hat{P}_0 \to P_0$ is one, as well. This shows that the chain complex is given in each homological degree as asserted. The only thing left to prove is that the boundaries are given as claimed. To do so, note that the inverse to the projection is the map $P_0 \to \hat{P}_0$ given by

$$P_0 \longrightarrow \hat{P}_0 = P_0 \times_k D\widetilde{P}_0 \,, \quad x \mapsto (x, \epsilon(x))\,.$$

It is clear that boundary $\hat{\partial} : \hat{P}_n \to \hat{P}_{n-1}$ is given by ∂_n and $D(\tilde{\partial}_{1-n})$ when $n > 0$ and $n < 0$, respectively. When $n = 0$, we are looking at the boundary

$$P_0 \times_k D\widetilde{P}_0 \longrightarrow D\widetilde{P}_1 \,, \quad (x, f) \mapsto \epsilon^*(f)$$

which under the identifications made above corresponds to the composition

$$P_0 \longrightarrow P_0 \times_k D\widetilde{P}_0 \longrightarrow D\widetilde{P}_1 \,, \quad x \mapsto (\epsilon^* \circ \epsilon)(x)\,. \qquad \square$$

Diagrammatically, we can visualise $\hat{P}_*$ as the "spliced" complex

$$\begin{tikzcd}
\cdots \arrow[r] & \hat{P}_1 \arrow[r] & \hat{P}_0 \arrow[r] \arrow[dr, "\epsilon"'] & \hat{P}_{-1} \arrow[r] & \hat{P}_{-2} \arrow[r] & \cdots \,. \\
 & & & k \arrow[u, "\epsilon^*"'] & &
\end{tikzcd}$$

We will show that if P_* is assumed to be a projective resolution of finite type[4], then this is a complete resolution. See Remark I.2.27. First we need a lemma.

Lemma I.2.20. *Let*

$$\begin{tikzcd}
Q_* \arrow[r, "g'"] \arrow[d, "f'"] & A_* \arrow[d, "f"] \\
C_* \arrow[r, "g"] & B_*
\end{tikzcd}$$

be a pullback diagram of chain complexes. Assume that there is some chain map $\phi : B_* \to A_*$ such that $f\phi = \mathrm{id}_{B_*}$ and $\phi f \simeq \mathrm{id}_{A_*}$ witnessed by a chain homotopy $H : A_n \to A_{n+1}$ satisfying $fH = 0$. Then there is a chain map $\phi' : C_* \to Q_*$ such that $f'\phi' = \mathrm{id}_{C_*}$ and $\phi' f' \simeq \mathrm{id}_{Q_*}$.

Proof. Consider the diagram

$$\begin{tikzcd}
C_* \arrow[dr, dotted, "\phi'"] \arrow[drr, bend left, "\phi g"] \arrow[ddr, bend right, "\mathrm{id}_{C_*}"'] & & \\
 & Q_* \arrow[r, "g'"] \arrow[d, "f'"] & A_* \arrow[d, "f"] \\
 & C_* \arrow[r, "g"] & B_*
\end{tikzcd}$$

[4]Recall that a chain complex C_* of projective modules is said to be of finite type if it is finitely generated in each homological degree.

in which we have an induced chain map $\phi' : C_* \to Q_*$ by the universal property of a pullback. This shows that $f'\phi' = \mathrm{id}_{C_*}$. Let us show that this constitutes a left homotopy inverse, as well.

Let $H_n : A_n \to A_{n+1}$ be the chain homotopy between ϕf and id_{A_*}. That is

$$\mathrm{id}_{A_*} - \phi f = \partial H_n + H_{n-1}\partial\,.$$

We want to use this data to build a chain homotopy between id_{Q_*} and $\phi' f'$. To do this, consider the diagram

$$\begin{tikzcd}
A_n \arrow[dr, dotted, "h"] \arrow[drr, bend left, "H_n"] \arrow[ddr, bend right, "0"'] & & \\
& Q_{n+1} \arrow[r, "g'"] \arrow[d, "f'"] & A_{n+1} \arrow[d, "f"] \\
& C_{n+1} \arrow[r, "g"] & B_{n+1}
\end{tikzcd}$$

which commutes since $fH_n = 0$, by assumption. Again, by the universal property of a pullback we have induced maps $h : A_n \to Q_{n+1}$. Let us set

$$H'_n = hg' : Q_n \longrightarrow Q_{n+1}\,.$$

We claim that these maps constitute a chain homotopy between id_{Q_*} and $\phi' f'$. That is, we claim that they satisfy

$$\mathrm{id}_{Q_*} - \phi' f' = \partial H'_n + H'_{n-1}\partial\,.$$

To show this, we appeal to the uniqueness of maps induced from pullbacks. Consider the diagram

$$\begin{tikzcd}
Q_n \arrow[dr, dotted, "?"] \arrow[drr, bend left, "\partial H_n g' + H_{n-1}\partial g'"] \arrow[ddr, bend right, "0"'] & & \\
& Q_n \arrow[r, "g'"] \arrow[d, "f'"] & A_n \arrow[d, "f"] \\
& C_n \arrow[r, "g"] & B_n
\end{tikzcd}\,.$$

This diagram commutes, since

$$\begin{aligned}
f\partial H_n g' + fH_{n-1}\partial g' &= \partial f H_n g' + fH_{n-1}\partial g' \\
&= \partial f g' h g' + f g' h \partial g' \\
&= \partial g f' h g' + g f' h \partial g' \\
&= 0\,,
\end{aligned}$$

so we do indeed have a unique induced map in the diagram. We claim that the question-mark in the diagram can be filled by both $\mathrm{id}_{Q_*} - \phi' f'$ and $\partial H'_n + H'_{n-1}\partial$, so they must agree by uniqueness of the induced map. Checking this claim is straight-forward. The checks

$$\begin{aligned} g'(\mathrm{id}_{Q_*} - \phi' f') &= g' - g'\phi' f' \\ &= g' - \phi g f' \\ &= g' - \phi f g' \\ &= (\mathrm{id}_{A_*} - \phi f) g' \\ &= (\partial H_n + H_{n-1}\partial) g' \,, \end{aligned}$$

and

$$\begin{aligned} f'(\mathrm{id}_{Q_*} - \phi' f') &= f' - f'\phi' f' \\ &= f' - f' \\ &= 0 \end{aligned}$$

show that the map $\mathrm{id}_{Q_*} - \phi' f'$ fits into the diagram. The checks

$$\begin{aligned} g'(\partial H'_n + H'_{n-1}\partial) &= g'\partial h g' + g' h g' \partial \\ &= \partial g' h g' + g' h g' \partial \\ &= \partial H_n g' + H_{n-1} g' \partial \\ &= \partial H_n g' + H_{n-1} \partial g' \\ &= (\partial H_n + H_{n-1}\partial) g' \,, \end{aligned}$$

and

$$\begin{aligned} f'(\partial H'_n + H'_{n-1}\partial) &= f'\partial H'_n + f' H'_{n-1}\partial \\ &= \partial' f' H'_n + f' H'_{n-1}\partial \\ &= \partial' f' h g' + f' h g' \partial \\ &= 0 \end{aligned}$$

show that the map $\partial H'_n + H'_{n-1}\partial$ also fits into the diagram, which concludes the proof. □

Proposition I.2.21. *Assume that P_* is of finite type over Γ. Then $\hat{P}_*$ is an acyclic complex of projective Γ-modules such that $\mathrm{Hom}_\Gamma(\hat{P}_*, Q)$ is acyclic for every coinduced Γ-module Q.*

Proof. Since P_n is finitely generated and projective over Γ in each homological degree n, it follows that $\hat{P}_n$ must be finitely generated and projective over Γ, as well, by Corollary I.2.8.

To show that $\hat{P}_*$ is acyclic, we show that it is k-linearly contractible. Since $\epsilon : P_* \to k$ is a chain homotopy equivalence, we can find a homotopy inverse $f : k \to P_*$. In this case, we can pick $\eta : k \to P_*$ so that $\epsilon\eta = \mathrm{id}_k$ on the nose. Since k is concentrated in homological degree 0 we know that the chain homotopy $\eta\epsilon \simeq \mathrm{id}_{P_*}$ is zero after post-composition with ϵ, so that Lemma I.2.20

applies. This shows that the map $\hat{P}_* \to D\widetilde{P}_*$ is a chain homotopy equivalence. Since $\widetilde{P}_*$ is chain contractible, we conclude that so is its dual $D\widetilde{P}_*$ and hence also $\hat{P}_*$.

If $Q = \mathrm{Coind}_{\mathrm{k}}^{\Gamma}(C)$ for some k-module C, then

$$\mathrm{Hom}_{\Gamma}(\hat{P}_*, \mathrm{Coind}_{\mathrm{k}}^{\Gamma}(C)) \cong \mathrm{Hom}(\hat{P}_*, C)\,.$$

Since $\hat{P}_*$ is k-linearly contractible it follows that $\mathrm{Hom}(\hat{P}_*, C)$ is contractible, and therefore acyclic. □

Let M be a Γ-module. The chain map $\hat{P}_* \to P_*$ induces a chain map

$$\mathrm{Hom}(P, M)_* \longrightarrow \mathrm{Hom}(\hat{P}, M)_*$$

which is an isomorphism in homological degrees $* \leq 0$. In addition to this map, we also have a chain map composition

$$\widetilde{P}_* \otimes M \longrightarrow DD\widetilde{P}_* \otimes M \xrightarrow{\nu} \mathrm{Hom}(D\widetilde{P}, M)_* \longrightarrow \mathrm{Hom}(\hat{P}, M)_*\,.$$

In particular, this is an isomorphism in homological degrees $* \geq 1$ under the assumption that $\widetilde{P}_*$ is of finite type over k. Note that the chain maps described above fit into the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\epsilon^*"] \arrow[d, "i\otimes 1"] & \mathrm{Hom}(P,M)_* \arrow[d] \arrow[ddr, bend left] & \\
\widetilde{P}_* \otimes M \arrow[r] \arrow[drr, bend right] & \mathrm{gm}_*(M) \arrow[dr, "\beta"] & \\
 & & \mathrm{Hom}(\hat{P}, M)_*\,,
\end{tikzcd}$$

so that we have an induced chain map β by the universal property of $\mathrm{gm}_*(M)$.

Proposition I.2.22. *Suppose that $\widetilde{P}_*$ is of finite type over k. Then the map*

$$\beta : \mathrm{gm}_*(M) \xrightarrow{\cong} \mathrm{Hom}(\hat{P}, M)_*$$

is a natural isomorphism of Γ-chain complexes.

Proof. The assertion is clear in homological degrees $n > 0$ and $n < 0$. If $n = 0$ we are looking at the diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(P,M)_0 \arrow[d, "\cong"] \arrow[ddr, bend left, "\cong"] & \\
\widetilde{P}_0 \otimes M \arrow[r] \arrow[drr, bend right] & \mathrm{gm}_0(M) \arrow[dr, "\beta"] & \\
 & & \mathrm{Hom}(\hat{P}, M)_0\,,
\end{tikzcd}$$

in which we have marked the obvious isomorphisms. It is then clear that β is an isomorphism, as well. □

Corollary I.2.23. *Suppose that the projective* Γ*-resolution* P_* *is of finite type over* k*. Then there is a natural isomorphism*

$$\widehat{\mathrm{Ext}}^n_\Gamma(k, M) \cong H^n(\mathrm{Hom}_\Gamma(\hat{P}, M)_*)$$

for all $n \in \mathbb{Z}$*.*

Proof. Combine Proposition I.2.16 and Proposition I.2.22. □

It turns out that, under the assumption that Γ is finitely generated projective over k, we can always construct the projective resolution P_* so that it is of finite type over Γ. It is then necessarily also of finite type over k. This can be done via the bar construction, which we now review. See ([May72, §9, §10, §11] and [GM74, App. A]) for more details.

Construction I.2.24 (The bar construction)**.** Let Γ be a k-algebra, M a right Γ-module, and N a left Γ-module. We form a simplicial object $B_\bullet(M, \Gamma, N)$: $\Delta^{\mathrm{op}} \to \mathrm{Mod}(k)$ as follows. In simplicial degree q we let

$$B_q(M, \Gamma, N) = M \otimes \Gamma^{\otimes q} \otimes N.$$

It is customary to write

$$m[\gamma_1|\cdots|\gamma_q]n = m \otimes \gamma_1 \otimes \cdots \otimes \gamma_q \otimes n$$

for an element in the qth simplicial degree; hence the terminology *bar* construction. In this notation, the face maps are given as

$$d_i(m[\gamma_1|\cdots|\gamma_q]n) = \begin{cases} m\gamma_1[\gamma_2|\cdots|\gamma_q]n & i = 0 \\ m[\gamma_1|\cdots|\gamma_i\gamma_{i+1}|\cdots|\gamma_q]n & 0 < i < q \\ m[\gamma_1|\cdots|\gamma_{q-1}]\gamma_q n & i = q \end{cases}$$

and the degeneracy maps are given by

$$s_i(m[\gamma_1|\cdots|\gamma_q]n) = m[\gamma_1|\cdots|\gamma_i|1|\gamma_{i+1}|\cdots|\gamma_q]n\,.$$

The simplicial k-module $B_\bullet(M, \Gamma, N)$ can be turned into a non-negative k-complex in essentially two ways.

- The most straight-forward way to turn $B_\bullet(M, \Gamma, N)$ into a Γ-chain complex $B_*(M, \Gamma, N)$ is by taking the Γ-module in homological degree n to be equal to the n-simplices of $B_\bullet(M, \Gamma, N)$ and to let the boundary in the chain complex be the alternating sum of the face maps:

  $$B_n = B_n(M, \Gamma, N) \quad \text{and} \quad \partial = \sum_{i=0}^{n} (-1)^i d_i : B_n \longrightarrow B_{n-1}\,.$$

  This is referred to as the **bar complex**.

- To get a smaller quasi-isomorphic chain complex, more convenient for computations, we can turn $B_\bullet(M, \Gamma, N)$ into a chain complex $NB_* = NB_*(M, \Gamma, N)$ quotienting out by the degenerate simplices. Explicitly, in homological degree n we have

$$\begin{aligned} NB_n &= B_n/(s_0 B_{n-1} + \cdots + s_{n-1} B_{n-1}) \\ &\cong M \otimes \overline{\Gamma}^{\otimes n} \otimes N\,, \end{aligned}$$

where

$$\overline{\Gamma} = \operatorname{coker}(\eta) \cong \ker(\epsilon)\,.^5$$

The boundary $\partial\colon NB_n \to NB_{n-1}$ is given by the same formula as before, which makes sense because $\partial(s_0 B_{n-1} + \cdots + s_{n-1} B_{n-1}) \subset (s_0 B_{n-2} + \cdots + s_{n-2} B_{n-2})$. We refer to (NB_*, ∂) as the **normalised bar complex**.

There is a natural Γ-action on the simplicial k-module $B_\bullet(M, \Gamma, \Gamma)$ arising from viewing $N = \Gamma$ as a Γ-Γ-bimodule. Explicitly, in each simplicial degree we have the right Γ-action $B_q(M, \Gamma, \Gamma) \otimes \Gamma \to B_q(M, \Gamma, \Gamma)$ given by

$$m[\gamma_1|\cdots|\gamma_q]\gamma_{q+1} \otimes \gamma \mapsto m[\gamma_1|\cdots|\gamma_q]\gamma_{q+1}\gamma$$

and this Γ-action commutes with the simplicial structure maps of $B_\bullet(M, \Gamma, \Gamma)$, so that $B_\bullet(M, \Gamma, \Gamma)$ extends to a simplicial Γ-module. It is a standard exercise in simplicial homotopy theory to check that $B_\bullet(M, \Gamma, \Gamma)$ is simplicially homotopy equivalent to M viewed as a constant simplicial Γ-module. As a consequence, the complexes

$$B_*(M, \Gamma, \Gamma) \quad \text{and} \quad NB_*(M, \Gamma, \Gamma)$$

are resolutions of M as a Γ-module. We refer to these as the **bar resolution** and **normalised bar resolution** of M as a Γ-module, respectively. See [May72, Prop. 9.9] and [GM74, Lem. A.8].

Proposition I.2.25. *Assume that Γ is finitely generated and projective over k. If M is finitely generated projective over k, then the bar resolution $B_*(M, \Gamma, \Gamma)$ and the normalised bar resolution $NB_*(M, \Gamma, \Gamma)$ are Γ-projective resolutions of M of finite type.*

Proof. Since $B_\bullet(M, \Gamma, \Gamma)$ is simplicially homotopy equivalent to M, the bar resolution is a resolution of M. It is finitely generated, and projective in each degree by an application of Lemma I.2.2. The proof in the normalised case is very similar. □

Theorem I.2.26. *When Γ is finitely generated and projective over k, each short exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

[5] This isomorphism follows from $\epsilon\eta = \mathrm{id}_k$.

of Γ*-modules induces a long exact sequence*

$$\ldots \longrightarrow \widehat{\operatorname{Ext}}^n_\Gamma(k, M') \longrightarrow \widehat{\operatorname{Ext}}^n_\Gamma(k, M) \longrightarrow \widehat{\operatorname{Ext}}^n_\Gamma(k, M'') \xrightarrow{\delta} \widehat{\operatorname{Ext}}^{n+1}_\Gamma(k, M') \longrightarrow \ldots .$$

Furthermore, if M *is an induced or coinduced* Γ*-module*[6]*, then* $\widehat{\operatorname{Ext}}^n_\Gamma(k, M) = 0$ *for all* $n \in \mathbb{Z}$*.*

Proof. If Γ is finitely generated projective over k, then the bar complex $B_*(k, \Gamma, \Gamma)$ constitutes a projective Γ-resolution of k of finite type, so that Proposition I.2.21 applies. The long exact sequence is then induced by the short exact sequence

$$0 \longrightarrow \operatorname{Hom}_\Gamma(\hat{P}_*, M') \longrightarrow \operatorname{Hom}_\Gamma(\hat{P}_*, M) \longrightarrow \operatorname{Hom}_\Gamma(\hat{P}_*, M'') \longrightarrow 0$$

of k-module chain complexes. Here we are using Corollary I.2.23 to identify the terms in the long exact homology sequence. That Tate cohomology vanishes on induced/coinduced modules is a direct consequence of condition (2) in Proposition I.2.21. □

Remark I.2.27. Note that if Γ is finitely generated projective, the chain complex $\hat{P}_*$ constructed from $P_* = B_*(k, \Gamma, \Gamma)$ is indeed a complete Γ-resolution of k in the sense of [CK97, Definition 1.1]. This uses Proposition I.2.21 and the fact that all projective Γ-modules are retracts of induced Γ-modules, which are coinduced Γ-modules by Corollary I.2.7. In particular, the results of [CK97] apply and we can conclude that our $\widehat{\operatorname{Ext}}_\Gamma(k, -)$ agrees with what is traditionally referred to as "complete Ext", in this case.

I.2.5 Multiplicative structure of Tate cohomology

We will now define a suitable pairing on Hopf algebra Tate cohomology. As before, we will assume that Γ is finitely generated and projective over k, so that this theory coincides with complete Ext.

Proposition I.2.28. *There is a unique, up to chain homotopy,* Γ*-linear chain map* $\Psi\colon P_* \to P_* \otimes P_*$ *covering the identity map* $\mathrm{id}\colon k \to k \otimes k$*.*

Proof. We first note that the chain complex $P_* \otimes P_*$, with diagonal Γ-action, is a Γ-resolution of $k \otimes k = k$. To see this, use the spectral sequence associated to $P_* \otimes P_*$ viewed as a double complex. This converges strongly to the homology of $P_* \otimes P_*$. The first page of the spectral sequence is given by

$$E^1_{s,t} \cong H_t(P_s \otimes P_*) \cong P_s \otimes H_t(P_*)\,,$$

since P_s is projective over Γ, and hence over k. This is zero unless $t = 0$, where it is P_s. The d^1-differential is induced by the horizontal differential in the double complex, so that the E^2-page is k concentrated at the origin.

[6]Due to the assumption that Γ is finitely generated projective induced modules are coinduced by Corollary I.2.7, and vice versa, so these are actually equivalent conditions.

Classical homological algebra then asserts that there is a unique (up to chain homotopy) Γ-linear chain map as asserted; see [Mac95, Chapter III Thm. 6.1]. □

The Γ-linear chain map $\Psi : P_* \to P_* \otimes P_*$ described above induces a product on $\mathrm{Ext}^*_\Gamma(k,-)$ via the pairing

$$\begin{aligned}\mathrm{Hom}_\Gamma(P_*,M)\otimes\mathrm{Hom}_\Gamma(P_*,N) &\xrightarrow{\alpha} \mathrm{Hom}_\Gamma(P_*\otimes P_*,M\otimes N)\\ &\xrightarrow{\Psi^*} \mathrm{Hom}_\Gamma(P_*,M\otimes N)\end{aligned}$$

of k-module complexes. By cocommutativity of Γ and uniqueness (up to chain homotopy) of Ψ, we have that $\Psi \simeq \tau\circ\Psi$. Passing to homology, this gives us an associative, unital, and graded commutative multiplication

$$\smile\colon\ \mathrm{Ext}^*_\Gamma(k,M)\otimes\mathrm{Ext}^*_\Gamma(k,N)\longrightarrow\mathrm{Ext}^*_\Gamma(k,M\otimes N)$$

that we will refer to as the **cup product**. In particular, $\mathrm{Ext}^*_\Gamma(k,k)$ is a k-algebra, and $\mathrm{Ext}^*_\Gamma(k,M)$ is an $\mathrm{Ext}_\Gamma(k,k)$-module for each Γ-module M. If M is a Γ-module algebra, then $\mathrm{Ext}^*_\Gamma(k,M)$ is an $\mathrm{Ext}^*_\Gamma(k,k)$-algebra.

We proceed to define the cup product in Tate cohomology for Hopf algebras. For this, we need a unique (up to chain homotopy) Γ-linear extension of the fold map in the category of chain complexes of Γ-modules under k. Explicitly, the fold map ∇ is the induced map in the commutative diagram

$$\begin{tikzcd}
k\otimes k \arrow[r, "i\otimes 1"] \arrow[d, "1\otimes i"] & \widetilde{P}_* \arrow[d] \arrow[ddr, bend left, "\mathrm{id}"] & \\
\widetilde{P}_* \arrow[r] \arrow[drr, bend right, "\mathrm{id}"] & \widetilde{P}_*\oplus_k\widetilde{P}_* \arrow[dr, dashed, "\nabla"] & \\
 & & \widetilde{P}_*\,,
\end{tikzcd}$$

where the inner square is a pushout diagram. Let us start with a more general result.

Lemma I.2.29. *Let A_*, B_*, and C_* be chain complexes of Γ-modules, where we assume that C_* is non-negative and exact. Let $i : A_* \to B_*$ be an injective chain map and assume that $Q_* = \mathrm{coker}(i)$ is projective over Γ in each homological degree. Then, for each chain map $f : A_* \to C_*$ there is a chain map $g : B_* \to C_*$ such that $gi = f$. Moreover, this chain map is unique up to a chain homotopy that is zero on the image of i.*

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "i"] \arrow[d, "f"'] & B_* \arrow[r, "r"] \arrow[dl, dashed, "g"] & Q_* \arrow[r] & 0\\
 & C_* & & &
\end{tikzcd}$$

where the top sequence is short exact in each homological degree. Since C_* is non-negative, we must have $g_n = 0$ for $n < 0$. To construct the rest of the chain map we proceed by induction. Assume inductively that we have constructed g_m satisfying

$$g_m i_m = f_m \quad \text{and} \quad g_{m-1}\partial = \partial g_m$$

for all $m < n$. Since

$$0 = g_{n-2}\partial^2 = \partial g_{n-1}\partial$$

and C_* is exact we know that $g_{n-1}\partial$ lands in $\partial(C_n)$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"] & B_n \arrow[r, "r_n"] \arrow[d, "g_{n-1}\partial"] \arrow[dl, dashed, "g_n"] & Q_n \arrow[r] & 0 \\ & C_n \arrow[r] & \partial(C_n) & & \end{tikzcd}$$

in which we want to find a dashed map $g_n : B_n \to C_n$ that makes both triangles commute. Since Q_n is projective, the short exact sequence at the top of the diagram splits, and we can find $s_n : Q_n \to B_n$ and $t_n : B_n \to A_n$ such that

$$i_n t_n + s_n r_n = \mathrm{id}_{B_n} \ .$$

Moreover, we can find a map $h_n : Q_n \to C_n$ such that $g_{n-1}\partial s_n = \partial h_n$. We define $g_n : B_n \to C_n$ by setting

$$g_n = f_n t_n + h_n r_n \ .$$

This map satisfies

$$g_n i_n = f_n t_n i_n + h_n r_n i_n = f_n + 0 = f_n$$

and

$$\begin{aligned} \partial g_n &= \partial f_n t_n + \partial h_n r_n \\ &= f_{n-1}\partial t_n + \partial h_n r_n \\ &= g_{n-1} i_{n-1} \partial t_n + g_{n-1}\partial s_n r_n \\ &= g_{n-1}\partial(i_n t_n + s_n r_n) \\ &= g_{n-1}\partial \, , \end{aligned}$$

which concludes the construction of g.

We now show that the map $g : B_* \to C_*$ is unique up to a chain homotopy that is zero on $i(A_*)$. Let $g' : B_* \to C_*$ be another chain map satisfying $f = g'i$. We want to show that we can find a chain homotopy between $k = g - g'$ and 0 that is zero on the image of i. That is, we want to find a collection of maps $H_n : B_n \to C_{n+1}$ such that

$$k_n = H_{n-1}\partial + \partial H_n \quad \text{and} \quad H_n i_n = 0$$

for all n. Again, we use induction. Since C_* is non-negative we must have $H_n = 0$ for $n < 0$. Assume inductively that we have constructed $H_m : B_m \to C_{m+1}$ satisfying

$$k_m = H_{m-1}\partial + \partial H_m \quad \text{and} \quad H_m i_m = 0$$

for all $m < n$. Consider the map $k_n - H_{n-1}\partial : B_n \to C_n$. Since

$$\begin{aligned}\partial(k_n - H_{n-1}\partial) &= \partial k_n - \partial H_{n-1}\partial \\ &= \partial k_n - (k_{n-1} - H_{n-2}\partial)\partial \\ &= \partial k_n - k_{n-1}\partial \\ &= 0\end{aligned}$$

and C_* is exact we know that $k_n - H_{n-1}\partial$ lands in $\partial(C_{n+1})$. Consider the diagram

$$\begin{tikzcd} & Q_n \arrow[dl, dashed, "\beta_n"'] \arrow[d, "k_n s_n - H_{n-1}\partial s_n"] & \\ C_{n+1} \arrow[r] & \partial(C_{n+1}) \arrow[r] & 0 \end{tikzcd}$$

in which we have a map β_n since Q_n is projective. We define

$$H_n = \beta_n r_n : B_n \longrightarrow C_{n+1} ,$$

which vanishes on the image of i_n since $r_n i_n = 0$. Furthermore, we have

$$\begin{aligned}H_{n-1}\partial + \partial H_n &= H_{n-1}\partial + \partial\beta_n r_n \\ &= H_{n-1}\partial + k_n s_n r_n - H_{n-1}\partial s_n r_n \\ &= H_{n-1}\partial + k_n - H_{n-1}\partial \\ &= k_n\end{aligned}$$

where the penultimate equality sign follows from the fact that k_n and H_{n-1} vanish on the image of i_n and i_{n-1}, respectively, so that $k_n = k_n(i_n t_n + s_n r_n) = k_n s_n r_n$ and

$$\begin{aligned}H_{n-1}\partial &= H_{n-1}\partial(i_n t_n + s_n r_n) \\ &= H_{n-1}\partial i_n t_n + H_{n-1}\partial s_n r_n \\ &= H_{n-1} i_n \partial t_n + H_{n-1}\partial s_n r_n \\ &= H_{n-1}\partial s_n r_n .\square\end{aligned}$$

Proposition I.2.30. *There is a unique, up to chain homotopy,* Γ*-linear chain map* $\Phi : \widetilde{P}_* \otimes \widetilde{P}_* \to \widetilde{P}_*$ *that makes the diagram*

$$\begin{tikzcd} \widetilde{P}_* \oplus_k \widetilde{P}_* \arrow[rr, "\nabla"] \arrow[dr] & & \widetilde{P}_* \\ & \widetilde{P}_* \otimes \widetilde{P}_* \arrow[ur, dashed, "\Phi"] & \end{tikzcd}$$

commute.

Proof. This is an application of Lemma I.2.29. The diagram we are considering is

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{P}_* \oplus_k \widetilde{P}_* \arrow[r, "i"] \arrow[d, "\nabla"'] & \widetilde{P}_* \otimes \widetilde{P}_* \arrow[r, "r"] \arrow[dl, dashed, "\Phi"] & Q_* \arrow[r] & 0 \\ & \widetilde{P}_* & & & \end{tikzcd}$$

We only need to check that the cokernel of the map i is projective over Γ in each homological degree. By construction, the cokernel is the total cokernel of the commutative diagram

$$\begin{tikzcd} k \otimes k \arrow[r, "i\otimes 1"] \arrow[d, "1\otimes i"'] & \widetilde{P}_* \otimes k \arrow[d, "1\otimes i"] \\ k \otimes \widetilde{P}_* \arrow[r, "i\otimes 1"] & \widetilde{P}_* \otimes \widetilde{P}_* \, . \end{tikzcd}$$

This can be calculated by computing the cokernels of the two horizontal maps followed by the cokernel of induced vertical map. Explicitly:

$$\begin{aligned} \operatorname{coker}(i) &\cong \operatorname{coker}(\operatorname{coker}(i\otimes 1) \longrightarrow \operatorname{coker}(i\otimes 1)) \\ &\cong \operatorname{coker}(1\otimes i : P[1]_* \otimes k \longrightarrow P[1]_* \otimes \widetilde{P}_*) \\ &\cong P[1]_* \otimes P[1]_* \, . \end{aligned}$$

In particular, we note that the cokernel is a complex of projective Γ-modules. $\square$

We can now define a pairing on $\mathrm{hm}_*(-)$ using Φ and Ψ. For Γ-modules M and N the composite pairing

$$\begin{aligned} &\widetilde{P}_* \otimes \operatorname{Hom}(P_*, M) \otimes \widetilde{P}_* \otimes \operatorname{Hom}(P_*, N) \\ &\qquad \xrightarrow{1\otimes\tau\otimes 1} \widetilde{P}_* \otimes \widetilde{P}_* \otimes \operatorname{Hom}(P_*, M) \otimes \operatorname{Hom}(P_*, N) \\ &\qquad \xrightarrow{1\otimes 1\otimes\alpha} \widetilde{P}_* \otimes \widetilde{P}_* \otimes \operatorname{Hom}(P_* \otimes P_*, M \otimes N) \\ &\qquad \xrightarrow{\Phi\otimes\Psi^*} \widetilde{P}_* \otimes \operatorname{Hom}(P_*, M \otimes N) \end{aligned}$$

is Γ-linear, so it induces a pairing

$$\operatorname{Hom}_\Gamma(k, \mathrm{hm}_*(M)) \otimes \operatorname{Hom}_\Gamma(k, \mathrm{hm}_*(N)) \longrightarrow \operatorname{Hom}_\Gamma(k, \mathrm{hm}_*(M \otimes N))$$

of k-module complexes. Note that the uniqueness of Φ up to chain homotopy guarantees that $\Phi \circ \tau \simeq \Phi$, and we have already observed that $\tau \circ \Psi \simeq \Psi$, which ensures that we get an associative, unital, and graded commutative pairing

$$\smile \colon \widehat{\operatorname{Ext}}^*_\Gamma(k, M) \otimes \widehat{\operatorname{Ext}}^*_\Gamma(k, N) \longrightarrow \widehat{\operatorname{Ext}}^*_\Gamma(k, M \otimes N)$$

after passing to homology. The inclusion $\operatorname{Hom}(P, M)_* \to \mathrm{hm}_*(M)$ provides us with a map

$$\operatorname{Ext}^*_\Gamma(k, M) \longrightarrow \widehat{\operatorname{Ext}}^*_\Gamma(k, M) \, .$$

This map is compatible with the multiplicative structures we have defined above, in the sense of the following proposition.

Proposition I.2.31. *The two diagrams*

$$\begin{array}{ccc}
\mathrm{Ext}_\Gamma^{b_1}(k,M)\otimes \mathrm{Ext}_\Gamma^{b_2}(k,N) & \xrightarrow{\smile} & \mathrm{Ext}_\Gamma^{b_1+b_2}(k,M\otimes N) \\
\downarrow & & \downarrow \\
\mathrm{Ext}_\Gamma^{b_1}(k,M)\otimes \widehat{\mathrm{Ext}}_\Gamma^{b_2}(k,N) & \longrightarrow & \widehat{\mathrm{Ext}}_\Gamma^{b_1+b_2}(k,M\otimes N) \\
\downarrow & & \| \\
\widehat{\mathrm{Ext}}_\Gamma^{b_1}(k,M)\otimes \widehat{\mathrm{Ext}}_\Gamma^{b_2}(k,N) & \xrightarrow{\smile} & \widehat{\mathrm{Ext}}_\Gamma^{b_1+b_2}(k,M\otimes N)
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Ext}_\Gamma^{b_1}(k,M)\otimes \mathrm{Ext}_\Gamma^{b_2}(k,N) & \xrightarrow{\smile} & \mathrm{Ext}_\Gamma^{b_1+b_2}(k,M\otimes N) \\
\downarrow & & \downarrow \\
\widehat{\mathrm{Ext}}_\Gamma^{b_1}(k,M)\otimes \mathrm{Ext}_\Gamma^{b_2}(k,N) & \longrightarrow & \widehat{\mathrm{Ext}}_\Gamma^{b_1+b_2}(k,M\otimes N) \\
\downarrow & & \| \\
\widehat{\mathrm{Ext}}_\Gamma^{b_1}(k,M)\otimes \widehat{\mathrm{Ext}}_\Gamma^{b_2}(k,N) & \xrightarrow{\smile} & \widehat{\mathrm{Ext}}_\Gamma^{b_1+b_2}(k,M\otimes N)
\end{array}$$

commute. In particular, it follows that $\widehat{\mathrm{Ext}}_\Gamma^*(k,k)$ *is an* $\mathrm{Ext}_\Gamma^*(k,k)$*-algebra, and* $\mathrm{Ext}_\Gamma^*(k,M)\to\widehat{\mathrm{Ext}}_\Gamma^*(k,M)$ *is an* $\mathrm{Ext}_\Gamma^*(k,k)$*-module homomorphism. If* M *is a* Γ*-module algebra, then* $\mathrm{Ext}_\Gamma^*(k,M)\to\widehat{\mathrm{Ext}}_\Gamma^*(k,M)$ *is an* $\mathrm{Ext}_\Gamma^*(k,k)$*-algebra homomorphism.*

Proof. This follows from the commutative diagrams

$$\begin{array}{ccc}
k\otimes k & = & k \\
\downarrow & & \downarrow \\
k\otimes \widetilde{P}_* & = & \widetilde{P}_* \\
\downarrow & & \| \\
\widetilde{P}_*\otimes\widetilde{P}_* & \xrightarrow{\Phi} & \widetilde{P}_*
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
k\otimes k & = & k \\
\downarrow & & \downarrow \\
\widetilde{P}_*\otimes k & = & \widetilde{P}_* \\
\downarrow & & \| \\
\widetilde{P}_*\otimes\widetilde{P}_* & \xrightarrow{\Phi} & \widetilde{P}_*\,.
\end{array}$$

□

I.2.6 Computation

In this section we look at a sample computation of the Tate cohomology of a Hopf algebra. Let k be a graded commutative ring with an element η in degree 1 such that $2\eta=0$. We will consider the Hopf algebra

$$\Gamma = k[s]/(s^2=\eta s)\,,\quad |s|=1\,.$$

Here s is a primitive element, so that comultiplication is given by $\psi = s \otimes 1 + 1 \otimes s$, counit by $\epsilon(s) = 0$, and antipode by $\chi(s) = -s$. To clarify: our goal is to compute $\widehat{\mathrm{Ext}}_{\Gamma}^{*}(k, M)$ where M is a Γ-module. This situation naturally appears when we consider the Tate construction on a spectrum X with an action of the circle $\mathbb{T}$. In this situation, we will have

$$\Gamma = \pi_*(\mathbb{S}[\mathbb{T}]) \qquad k = \pi_*(\mathbb{S}) \qquad M = \pi_*(X)\,.$$

See Proposition I.3.3 and Section I.6.

A projective resolution P_* of k as a trivial Γ-module is

$$\cdots \longrightarrow \Gamma\{p_3\} \xrightarrow{\partial_3} \Gamma\{p_2\} \xrightarrow{\partial_2} \Gamma\{p_1\} \xrightarrow{\partial_1} \Gamma\{p_0\} \longrightarrow 0$$

with the internal degree of the generator p_b being $|p_b| = b$ and the total degree being $\|p_b\| = 2b$. As (right) k-modules we have $P_b = \Gamma\{p_b\} = k\{p_b, p_b s\}$ where $\|p_b s\| = 2b + 1$. The boundary of the complex is given by

$$\partial_{b+1}(p_{b+1}) = \begin{cases} p_b s & b \geq 0 \text{ even} \\ p_b(s + \eta) & b \geq 1 \text{ odd} \end{cases}$$

and the augmentation $\epsilon \colon P_* \to k$ is given by $\epsilon(p_0) = 1$.

By definition, the mapping cone $\widetilde{P}_* = \mathrm{cone}(P_* \to k)$ is isomorphic to the complex

$$\cdots \longrightarrow \Gamma\{\tilde{p}_3\} \xrightarrow{\tilde{\partial}_3} \Gamma\{\tilde{p}_2\} \xrightarrow{\tilde{\partial}_2} \Gamma\{\tilde{p}_1\} \xrightarrow{\tilde{\partial}_1} k\{\tilde{p}_0\} \longrightarrow 0$$

where the internal degrees of the generators are $|\tilde{p}_0| = 0$ and $|\tilde{p}_a| = a - 1$ for $a \geq 1$, and the total degrees are $\|\tilde{p}_0\| = 0$ and $\|\tilde{p}_a\| = 2a - 1$ for $a \geq 1$. As before, we can also write this as a complex of (right) k-modules $\Gamma\{\tilde{p}_a\} = k\{\tilde{p}_a, \tilde{p}_a s\}$ for $a \geq 1$, where $|\tilde{p}_a s| = a$ and $\|\tilde{p}_a s\| = 2a$. The boundary is given by

$$\tilde{\partial}_a(\tilde{p}_a) = \begin{cases} \tilde{p}_0 & a = 1 \\ -\tilde{p}_{a-1} s & a \geq 2 \text{ even} \\ -\tilde{p}_{a-1}(s + \eta) & a \geq 3 \text{ odd.} \end{cases}$$

The chain complex we want to consider is the Tate complex $\mathrm{hm}_*(M)$, or rather, its Γ-invariants. Recall that the Tate complex $\mathrm{hm}_*(M)$ is given in each homological degree by

$$\mathrm{hm}_c(M) = \bigoplus_{a+b=c} \widetilde{P}_a \otimes \mathrm{Hom}(P_{-b}, M)$$

with boundary given as

$$\partial_{\mathrm{hm}}(x, f) = \tilde{\partial}(x) \otimes f + (-1)^{\|x\|} x \otimes \partial^*(f)\,.$$

When calculating the second term we also remember that

$$(\partial^* f)(v) = -(-1)^{\|f\|} f(\partial(v))$$

for an element $f \in \operatorname{Hom}(P_*, M)$ since M is concentrated in homological degree 0.

It will also be useful to consider the Tate complex in its bicomplex version. In this case, we will write $(\mathrm{hm}_{*,*}(M), \partial^h, \partial^v)$ where

$$\mathrm{hm}_{a,b}(M) = \widetilde{P}_a \otimes \operatorname{Hom}(P_{-b}, M)$$

and the horizontal and vertical boundaries are the first and the second term in the formula for the boundary in hm_*, respectively. The total complex of this bicomplex is equal to the Tate complex, by definition. Moreover, let us write $(U_{*,*}, \partial^h, \partial^v)$ for the restriction of the bicomplex to the Γ-invariants

$$U_{a,b} = \operatorname{Hom}_\Gamma(k, \widetilde{P}_a \otimes \operatorname{Hom}(P_{-b}, M)) .$$

We refer to the total complex of this bicomplex as $(U_*, \partial^h + \partial^v)$; it is isomorphic to $\operatorname{Hom}_\Gamma(k, \mathrm{hm}_*(M))$. Let us introduce some notation for the different elements in the bicomplex $U_{*,*}$ to keep our computations from becoming too messy.

Notation I.2.32. Let x be an element of M and write

$$f_b \cdot x = \tilde{p}_0 \otimes \begin{pmatrix} p_b \mapsto x \\ p_b s \mapsto xs \end{pmatrix}$$

for an element in $\operatorname{Hom}_\Gamma(k, \widetilde{P_0} \otimes \operatorname{Hom}(P_b, M))$ for $b \geq 0$.

Notation I.2.33. Let y be an element of M and write

$$g_{a,b} \cdot y = \tilde{p}_a \otimes \begin{pmatrix} p_b \mapsto y \\ p_b s \mapsto ys \end{pmatrix} + \tilde{p}_a s \otimes \begin{pmatrix} p_b \mapsto 0 \\ p_b s \mapsto (-1)^{|y|} y \end{pmatrix}$$

for an element in $\operatorname{Hom}_\Gamma(k, \widetilde{P_a} \otimes \operatorname{Hom}(P_b, M))$ for $a \geq 1$ and $b \geq 0$.

Notation I.2.34. Let z be an element of M and write

$$h_{a,b} \cdot z = \tilde{p}_a s \otimes \begin{pmatrix} p_b \mapsto z \\ p_b s \mapsto z(s + \eta) \end{pmatrix}$$

for an element in $\operatorname{Hom}_\Gamma(k, \widetilde{P_a} \otimes \operatorname{Hom}(P_b, M))$ for $a \geq 1$ and $b \geq 0$.

It is a straight-forward computation to check that these are indeed Γ-invariant elements, in the sense that

$$(g_{a,b} \cdot y) \cdot s = 0 ,$$

and analogously for $f_b \cdot x$ and $h_{a,b} \cdot z$, using the following lemma.

Lemma I.2.35. *The (right) conjugate action of s on an element f of* $\operatorname{Hom}(M, N)$ *is given by*

$$(fs)(m) = (-1)^{\|m\|}(f(m)s - f(ms)) .$$

Proof. Recall that the characterising property of the conjugate action is that it is the action on a function object $\mathrm{Hom}(M, N)$ such that the evaluation pairing $\mathrm{ev} : \mathrm{Hom}(M, N) \otimes M \to N$ is Γ-linear. Explicitly, the Γ-action on $\mathrm{Hom}(M, N)$ makes the diagram

$$\begin{tikzcd}
\mathrm{Hom}(M,N) \otimes M \otimes \Gamma \arrow[r, "\mathrm{ev}\otimes 1"] \arrow[d, "\rho_{\mathrm{Hom}(M,N)\otimes M}"'] & N \otimes \Gamma \arrow[d, "\rho_N"] \\
\mathrm{Hom}(M,N) \otimes M \arrow[r, "\mathrm{ev}"] & N
\end{tikzcd}$$

commute. The top composition sends a generic element to

$$f \otimes m \otimes s \mapsto f(m) \otimes s \mapsto f(m)s\,,$$

while the bottom composition sends the generic element to

$$f \otimes m \otimes s \mapsto f \otimes ms + (-1)^{\|m\|} fs \otimes m \mapsto f(ms) + (-1)^{\|m\|}(fs)(m)\,.$$

These must agree, which necessarily gives us the assertion. $\square$

Furthermore, these form an 'M-basis' of the Γ-invariants of the Tate complex in the sense of the following proposition.

Proposition I.2.36. *Let $b \geq 0$ and $a \geq 1$. There are k-module isomorphisms*

$$\begin{aligned}
\Sigma^{-b} M &\xrightarrow{\cong} \mathrm{Hom}_\Gamma(k, \widetilde{P}_0 \otimes \mathrm{Hom}(P_b, M)) \\
x &\mapsto f_b \cdot x
\end{aligned}$$

and

$$\begin{aligned}
\Sigma^{a-b-1} M \oplus \Sigma^{a-b} M &\xrightarrow{\cong} \mathrm{Hom}_\Gamma(k, \widetilde{P}_a \otimes \mathrm{Hom}(P_b, M)) \\
(y, z) &\mapsto g_{a,b} \cdot y + h_{a,b} \cdot z\,.
\end{aligned}$$

Proof. The maps are clearly injective, so we only need to show that they are surjective.

A general element in $\widetilde{P}_0 \otimes \mathrm{Hom}(P_b, M)$ is on the form

$$\tilde{p}_0 \otimes \begin{pmatrix} p_b \mapsto x \\ p_b s \mapsto y \end{pmatrix}.$$

By Lemma I.2.35, the right action of the primitive element s on such an element is

$$\tilde{p}_0 \otimes \begin{pmatrix} p_b \mapsto xs - y \\ p_b s \mapsto y\eta - ys \end{pmatrix}.$$

For our original element to be Γ-invariant this must be zero, which gives us $y = xs$. In other words, a Γ-invariant element in $\widetilde{P}_0 \otimes \mathrm{Hom}(P_b, M)$ can be written $f_b \cdot x$, where we let x range throughout M. The grading suspension appearing in the isomorphism makes sure that this is actually a map of graded k-modules. Indeed, the internal degree of our element is

$$|f_b \cdot x| = |\tilde{p}_0| + |x| - |p_b| = |x| - b\,.$$

A general element in $\widetilde{P}_a \otimes \mathrm{Hom}(P, M)_b$ is on the form

$$\tilde{p}_a \otimes \begin{pmatrix} p_b \mapsto x \\ p_b s \mapsto y \end{pmatrix} \;+\; \tilde{p}_a s \otimes \begin{pmatrix} p_b \mapsto z \\ p_b s \mapsto w \end{pmatrix}.$$

We assume that this is a homogeneous element, so that $|y| = |x|+1$, $|z| = |x|-1$, and $|w| = |x|$. Letting the primitive element s act on this element from the right we obtain

$$\begin{aligned} &(-1)^{|x|}\tilde{p}_a s \otimes \begin{pmatrix} p_b \mapsto x \\ p_b s \mapsto y \end{pmatrix} + \tilde{p}_a \otimes \begin{pmatrix} p_b \mapsto xs - y \\ p_b s \mapsto y\eta - ys \end{pmatrix} \\ &- \;(-1)^{|x|}\tilde{p}_a s\eta \otimes \begin{pmatrix} p_b \mapsto z \\ p_b s \mapsto w \end{pmatrix} \;+\; \tilde{p}_a s \otimes \begin{pmatrix} p_b \mapsto zs - w \\ p_b s \mapsto w\eta - ws \end{pmatrix}. \end{aligned}$$

For our element to be Γ-invariant we want this to add up to zero. In other words, we need to solve the following system of equations

$$\begin{cases} xs - y & = 0 \\ y\eta - ys & = 0 \\ (-1)^{|x|}x - (-1)^{|x|}z\eta + zs - w & = 0 \\ (-1)^{|x|}y - (-1)^{|x|}w\eta + w\eta - ws & = 0\,. \end{cases}$$

It is straight-forward to check that the solutions are given by the two independent equations

$$\begin{cases} y & = xs \\ w & = (-1)^{|x|}x - (-1)^{|x|}z\eta + zs, \end{cases}$$

which tells us that a Γ-invariant element can be written

$$g_{a,b} \cdot x + h_{a,b} \cdot z$$

where we are free to vary x and z in M. The suspensions in the source of the k-isomorphism are again there to make sure that the grading is preserved by the isomorphism. Indeed,

$$|g_{a,b} \cdot x| = |\tilde{p}_a| + |x| - |p_b| = a - 1 + |x| - b$$

and

$$|h_{a,b} \cdot z| = |\tilde{p}_a s| + |z| - |p_b| = a + |z| - b\,. \qquad \square$$

We now need to figure out what the boundary on these generic Γ-invariant elements looks like. Keeping track of all the signs we end up with the following description of the horizontal and vertical boundaries in terms of our 'M-basis'.

Lemma I.2.37. *The horizontal boundary on $f_b \cdot x$ is given by*

$$\partial^h(f_b \cdot x) = 0$$

and the vertical boundary is given by

$$\partial^v(f_b \cdot x) = \begin{cases} -(-1)^{|x|} f_{b+1} \cdot xs & b \geq 0 \text{ even} \\ -(-1)^{|x|} f_{b+1} \cdot x(s+\eta) & b \geq 1 \text{ odd.} \end{cases}$$

Lemma I.2.38. *The horizontal boundary on $g_{a,b} \cdot y$ is given by*

$$\partial^h(g_{a,b} \cdot y) = \begin{cases} f_b \cdot y & \text{for } a = 1 \\ -h_{a-1,b} \cdot y & \text{for } a \geq 2 \text{ even} \\ g_{a-1,b} \cdot y\eta - h_{a-1,b} \cdot y & \text{for } a \geq 3 \text{ odd} \end{cases}$$

and the vertical boundary is given by

$$\partial^v(g_{a,b} \cdot y) = \begin{cases} (-1)^{|y|} g_{a,b+1} \cdot ys \ + \ h_{a,b+1} \cdot y & \text{for } b \geq 0 \text{ even} \\ (-1)^{|y|} g_{a,b+1} \cdot y(s+\eta) \ + \ h_{a,b+1} \cdot y & \text{for } b \geq 1 \text{ odd.} \end{cases}$$

Lemma I.2.39. *The horizontal boundary on $h_{a,b} \cdot z$ is given by*

$$\partial^h(h_{a,b} \cdot z) = \begin{cases} h_{a-1,b} \cdot z\eta & \text{for } a \geq 2 \text{ even} \\ 0 & \text{for } a \geq 1 \text{ odd} \end{cases}.$$

and the vertical boundary is given by

$$\partial^v(h_{a,b} \cdot z) = \begin{cases} -(-1)^{|z|} h_{a,b+1} \cdot z(s+\eta) & \text{for } b \geq 0 \text{ even} \\ -(-1)^{|z|} h_{a,b+1} \cdot zs & \text{for } b \geq 1 \text{ odd.} \end{cases}$$

We calculate the homology of U_* by filtering the first tensor factor of $U_{*,*}$ and using the spectral sequence for the total complex of a bicomplex:

$$E^1_{a,-b} = H_{-b}(\mathrm{Hom}_\Gamma(k, \widetilde{P}_a \otimes \mathrm{Hom}(P_*, M))) \Rightarrow H_{a-b}(\mathrm{Hom}_\Gamma(k, \widetilde{P}_* \otimes \mathrm{Hom}(P_*, M)))\,.$$

The bicomplex $(U_{*,*}, \partial^h, \partial^v)$ is displayed in Figure I.1 for the convenience of the reader.

Remark I.2.40. Let us clarify how to interpret Figure I.1 and the matrix notation appearing in it. The horizontal and vertical boundaries are given in terms of the 'M-bases' $\{f_b\}$ for $U_{0,-b}$ and $\{g_{a,b}, h_{a,b}\}$ for $U_{a,-b}$. We record $f_b \cdot x$ and $g_{a,b} \cdot y + h_{a,b} \cdot z$ as the column vectors

$$\begin{bmatrix} x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y \\ z \end{bmatrix},$$

respectively. The boundaries are then indicated by multiplication with the corresponding matrices appearing in the figure. Multiplication is done, as is usual, with the matrix on the left hand side. So, to clarify: A vertical boundary $\partial^v : U_{a,-b} \to U_{a,-b-1}$ recorded as a 2×2-matrix (with entries in Γ) and multiplied with the relevant column vector (with entries in M)

$$\begin{bmatrix} i & j \\ k & \ell \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} iy + jz \\ ky + \ell z \end{bmatrix}$$

indicates that this boundary is given as

$$g_{a,b} \cdot y + h_{a,b} \cdot z \mapsto g_{a,b+1} \cdot (iy + jz) + h_{a,b+1} \cdot (ky + \ell z) \,.$$

Note that in this convention Γ ends up acting on M from the left, through the twist isomorphism followed by the right action. To see that the boundaries given in the matrix notation actually agree with the ones given in Lemma I.2.37, Lemma I.2.38, and Lemma I.2.39, we have to switch the position of the Γ-values (i, j, k, and ℓ) and the M-values (y and z), which typically introduces a sign. For example, the boundary $\partial^v : U_{a,-b} \to U_{a,-b-1}$ for even b is recorded in the figure as

$$\begin{bmatrix} s & 0 \\ 1 & -(s+\eta) \end{bmatrix} .$$

Left multiplication of this matrix with the column vector corresponding to $g_{a,b} \cdot y$ gives

$$\begin{bmatrix} s & 0 \\ 1 & -(s+\eta) \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} sy \\ y \end{bmatrix} ,$$

which tells us that this vertical boundary is given by

$$g_{a,b} \cdot y \mapsto g_{a,b+1} \cdot sy + h_{a,b+1} \cdot y = (-1)^{|y|} g_{a,b+1} \cdot ys + h_{a,b+1} \cdot y \,,$$

which is indeed in agreement with Lemma I.2.38.

Before we explicitly compute the first page of the spectral sequence for the bicomplex $U_{*,*}$, we again introduce some notation.

Notation I.2.41. Let z be an element of M and write

$$\begin{aligned} u_a \cdot z &= -(-1)^{|z|} g_{a,0} \cdot z(s+\eta) - h_{a,0} \cdot z \\ &= -(-1)^{|z|} \tilde{p}_a \otimes \begin{pmatrix} p_0 \mapsto z(s+\eta) \\ p_0 s \mapsto 0 \end{pmatrix} - \tilde{p}_a s \otimes \begin{pmatrix} p_0 \mapsto z \\ p_0 s \mapsto 0 \end{pmatrix} \end{aligned}$$

for the specified element in $\mathrm{Hom}_\Gamma(k, \widetilde{P_a} \otimes \mathrm{Hom}(P_b, M))$ for $a \geq 1$ and $b \geq 0$.

Proposition I.2.42. *The E^1-page of the bicomplex spectral sequence for $U_{*,*}$ is given as*

$$H_{-b}(\mathrm{Hom}_\Gamma(k, \widetilde{P}_0 \otimes \mathrm{Hom}(P, M)_*)) \cong \begin{cases} f_0 \cdot \ker(s) & \textit{for } b = 0, \\[2ex] f_b \cdot \dfrac{\ker(s+\eta)}{\mathrm{im}(s)} & \textit{for } b \geq 1 \textit{ odd}, \\[2ex] f_b \cdot \dfrac{\ker(s)}{\mathrm{im}(s+\eta)} & \textit{for } b \geq 2 \textit{ even}, \end{cases}$$

$$
\begin{tikzcd}
U_{0,0} \arrow[d, "{[\,-s\,]}"'] & U_{1,0} \arrow[l, "{[\,1\ 0\,]}"'] \arrow[d, "{\begin{bsmallmatrix} s & 0 \\ 1 & -(s+\eta) \end{bsmallmatrix}}"'] & U_{2,0} \arrow[l, "{\begin{bsmallmatrix} 0 & 0 \\ -1 & \eta \end{bsmallmatrix}}"'] \arrow[d, "{\begin{bsmallmatrix} s & 0 \\ 1 & -(s+\eta) \end{bsmallmatrix}}"'] & U_{3,0} \arrow[l, "{\begin{bsmallmatrix} \eta & 0 \\ -1 & 0 \end{bsmallmatrix}}"'] \arrow[d, "{\begin{bsmallmatrix} s & 0 \\ 1 & -(s+\eta) \end{bsmallmatrix}}"'] & \cdots \arrow[l, "{\begin{bsmallmatrix} 0 & 0 \\ -1 & \eta \end{bsmallmatrix}}"'] \\
U_{0,-1} \arrow[d, "{[\,-(s+\eta)\,]}"'] & U_{1,-1} \arrow[l, "{[\,1\ 0\,]}"'] \arrow[d, "{\begin{bsmallmatrix} s+\eta & 0 \\ 1 & -s \end{bsmallmatrix}}"'] & U_{2,-1} \arrow[l, "{\begin{bsmallmatrix} 0 & 0 \\ -1 & \eta \end{bsmallmatrix}}"'] \arrow[d, "{\begin{bsmallmatrix} s+\eta & 0 \\ 1 & -s \end{bsmallmatrix}}"'] & U_{3,-1} \arrow[l, "{\begin{bsmallmatrix} \eta & 0 \\ -1 & 0 \end{bsmallmatrix}}"'] \arrow[d, "{\begin{bsmallmatrix} s+\eta & 0 \\ 1 & -s \end{bsmallmatrix}}"'] & \cdots \arrow[l, "{\begin{bsmallmatrix} 0 & 0 \\ -1 & \eta \end{bsmallmatrix}}"'] \\
U_{0,-2} \arrow[d, "{[\,-s\,]}"'] & U_{1,-2} \arrow[l, "{[\,1\ 0\,]}"'] \arrow[d, "{\begin{bsmallmatrix} s & 0 \\ 1 & -(s+\eta) \end{bsmallmatrix}}"'] & U_{2,-2} \arrow[l, "{\begin{bsmallmatrix} 0 & 0 \\ -1 & \eta \end{bsmallmatrix}}"'] \arrow[d, "{\begin{bsmallmatrix} s & 0 \\ 1 & -(s+\eta) \end{bsmallmatrix}}"'] & U_{3,-2} \arrow[l, "{\begin{bsmallmatrix} \eta & 0 \\ -1 & 0 \end{bsmallmatrix}}"'] \arrow[d, "{\begin{bsmallmatrix} s & 0 \\ 1 & -(s+\eta) \end{bsmallmatrix}}"'] & \cdots \arrow[l, "{\begin{bsmallmatrix} 0 & 0 \\ -1 & \eta \end{bsmallmatrix}}"'] \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Figure I.1: The bicomplex $(U_{*,*}, \partial^h, \partial^v)$ for $\Gamma = k[s]/(s^2 = \eta s)$

when $a = 0$, and as

$$H_{-b}(\mathrm{Hom}_\Gamma(k, \widetilde{P}_a \otimes \mathrm{Hom}(P, M)_*)) \cong \begin{cases} u_a \cdot M & \textit{for } b = 0 \\ 0 & \textit{otherwise,} \end{cases}$$

when $a \geq 1$.

Proof. This is essentially an exercise in linear algebra using the matrices in Figure I.1. The kernels of the boundaries are computed by computing the nullspaces of the corresponding matrices. Similarly, the images of boundaries are computed by computing the column spaces of the matrices. Let us give the details for the $a \geq 1$ case, as the $a = 0$ case is directly visible by inspecting the figure. The nullspaces of the matrices

$$\begin{bmatrix} s & 0 \\ 1 & -(s+\eta) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s+\eta & 0 \\ 1 & -s \end{bmatrix}$$

are generated by the column vectors

$$\begin{bmatrix} s+\eta \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s \\ 1 \end{bmatrix},$$

respectively, and the column spaces are generated by the column vectors

$$\begin{bmatrix} s \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s+\eta \\ 1 \end{bmatrix},$$

respectively. From this is follows that the homology is concentrated in homological degree $b = 0$, in the $a \geq 1$ case. Here it consists of elements of the form

$$g_{a,0} \cdot (s+\eta)z + h_{a,0} \cdot z = (-1)^{|z|} g_{a,0} \cdot z(s+\eta) + h_{a,0} \cdot z$$

for varying z in M. For reasons concerning the multiplicative structure, we have decided to denote the above element by $-u_a \cdot z$. □

In particular, note that the above result tells us that the E^1-page of the spectral sequence is concentrated around the boundary of the fourth quadrant. The d^1-differential $d^1 : E^1_{a,-b} \to E^1_{a-1,-b}$ in the spectral sequence is induced by the horizontal boundary, and is by degree reasons only non-zero on the positive a-axis. There it is given by

$$d^1(u_a \cdot z) = \begin{cases} -(-1)^{|z|} f_0 \cdot z(s+\eta) & \text{for } a = 1 \\ -(-1)^{|z|} u_{a-1} \cdot zs & \text{for } a \geq 2 \text{ even} \\ -(-1)^{|z|} u_{a-1} \cdot z(s+\eta) & \text{for } a \geq 3 \text{ odd} \end{cases}$$

by using Lemma I.2.38 and Lemma I.2.39. We conclude that the second page of the spectral sequence is concentrated along the a- and b-axes and that

$$E^2_{a,-b} \cong \begin{cases} f_b \cdot \dfrac{\ker(s)}{\operatorname{im}(s+\eta)} & \text{for } a = 0 \text{ and } b \geq 0 \text{ even,} \\[2ex] f_b \cdot \dfrac{\ker(s+\eta)}{\operatorname{im}(s)} & \text{for } a = 0 \text{ and } b \geq 1 \text{ odd,} \\[2ex] u_a \cdot \dfrac{\ker(s)}{\operatorname{im}(s+\eta)} & \text{for } b = 0 \text{ and } a \geq 2 \text{ even,} \\[2ex] u_a \cdot \dfrac{\ker(s+\eta)}{\operatorname{im}(s)} & \text{for } b = 0 \text{ and } a \geq 1 \text{ odd.} \end{cases}$$

There is no room for further differentials, and the infinite cycles along the upper and left hand edges are necessarily also d^1-cycles in the total complex U_*. We conclude:

Proposition I.2.43.

$$\widehat{\operatorname{Ext}}^c_\Gamma(k, M) \cong \begin{cases} f_c \cdot \dfrac{\ker(s)}{\operatorname{im}(s+\eta)} & \text{for } c \geq 0 \text{ even,} \\[2ex] f_c \cdot \dfrac{\ker(s+\eta)}{\operatorname{im}(s)} & \text{for } c \geq 1 \text{ odd,} \\[2ex] u_{-c} \cdot \dfrac{\ker(s)}{\operatorname{im}(s+\eta)} & \text{for } c \leq -2 \text{ even,} \\[2ex] u_{-c} \cdot \dfrac{\ker(s+\eta)}{\operatorname{im}(s)} & \text{for } c \leq -1 \text{ odd.} \end{cases}$$

Now all that remains is to describe the multiplicative structure. That is, given two Γ-modules M and N we want to determine the cup product

$$\smile : \widehat{\operatorname{Ext}}^{c_1}_\Gamma(k, M) \otimes \widehat{\operatorname{Ext}}^{c_2}_\Gamma(k, N) \longrightarrow \widehat{\operatorname{Ext}}^{c_1+c_2}_\Gamma(k, M \otimes N).$$

In order to do so we need a Γ-linear chain map $\Psi : P_* \to P_* \otimes P_*$ covering the identity of k, and a Γ-linear chain map $\Phi : \widetilde{P}_* \otimes \widetilde{P}_* \to \widetilde{P}_*$ extending the fold map, as per Section I.2.5.

Lemma I.2.44. *A Γ-linear chain map $\Psi : P_* \to P_* \otimes P_*$ that covers the identity is given by*

$$\Psi(p_b) = \sum_{b_1+b_2=b} p_{b_1} \otimes p_{b_2}.$$

By Γ-linearity we have

$$\Psi(p_b s) = \sum_{b_1+b_2=b} p_{b_1} s \otimes p_{b_2} + p_{b_1} \otimes p_{b_2} s.$$

Proof. Note that

$$\partial(p_b) = p_{b-1}(s + (b-1)\eta)$$

for $b \geq 1$. To verify that Ψ, as specified in the statement of the lemma, is a chain map, we must show that

$$\begin{aligned} \partial(\Psi(p_b)) &= \sum_{b_1+b_2=b} \partial(p_{b_1}) \otimes p_{b_2} + p_{b_1} \otimes \partial(p_{b_2}) \\ &= \sum_{b_1+b_2=b} p_{b_1-1}(s + (b_1-1)\eta) \otimes p_{b_2} + p_{b_1} \otimes p_{b_2-1}(s + (b_2-1)\eta) \end{aligned}$$

is equal to

$$\Psi(\partial(p_b)) = \Psi(p_{b-1}s) + \Psi(p_{b-1})(b-1)\eta\,.$$

Here

$$\sum_{b_1+b_2=b} p_{b_1-1}s \otimes p_{b_2} + p_{b_1} \otimes p_{b_2-1}s = \Psi(p_{b-1})s\,,$$

so it remains to check that

$$\sum_{b_1+b_2=b} p_{b_1-1}(b_1-1)\eta \otimes p_{b_2} + p_{b_1} \otimes p_{b_2-1}(b_2-1)\eta = \Psi(p_{b-1})(b-1)\eta\,.$$

When b is odd the terms of the left hand side cancel in pairs, and the right hand side is zero. When b is even only the terms with b_1 and b_2 both even contribute to the left hand side, and these add up to $\Psi(p_{b-1})\eta$, as required. Finally,

$$(\epsilon \otimes \epsilon)(\Psi(p_0)) = 1 = \epsilon(p_0)\,,$$

so Ψ is indeed a chain map covering $k \otimes k = k$. □

Lemma I.2.45. *A Γ-linear chain map $\Phi\colon \widetilde{P}_* \otimes \widetilde{P}_* \to \widetilde{P}_*$ that extends the fold map is given by*

$$\begin{aligned}
\Phi(\tilde{p}_{a_1} \otimes \tilde{p}_{a_2}) &= 0 \\
\Phi(\tilde{p}_{a_1} \otimes \tilde{p}_{a_2} s) &= -\tilde{p}_a \\
\Phi(\tilde{p}_{a_1} s \otimes \tilde{p}_{a_2}) &= -\tilde{p}_a \\
\Phi(\tilde{p}_{a_1} s \otimes \tilde{p}_{a_2} s) &= -\tilde{p}_a(s+\eta)
\end{aligned}$$

for $a_1, a_2 \geq 1$ and $a = a_1 + a_2$. Furthermore,

$$\begin{aligned}
\Phi(\tilde{p}_0 \otimes \tilde{p}_{a_2}) &= \tilde{p}_{a_2} \\
\Phi(\tilde{p}_0 \otimes \tilde{p}_{a_2} s) &= \tilde{p}_{a_2} s \\
\Phi(\tilde{p}_{a_1} \otimes \tilde{p}_0) &= \tilde{p}_{a_1} \\
\Phi(\tilde{p}_{a_1} s \otimes \tilde{p}_0) &= \tilde{p}_{a_1} s
\end{aligned}$$

and $\Phi(\tilde{p}_0 \otimes \tilde{p}_0) = \tilde{p}_0$.

Proof. Note that the differential in the chain complex $\widetilde{P}_*$ can be described as

$$\tilde{\partial}(\tilde{p}_a) = \begin{cases} \tilde{p}_0 & \text{for } a = 1 \\ -\tilde{p}_{a-1}(s + a\eta) & \text{for } a \geq 2. \end{cases}$$

To check that Φ, as specified in the statement of the lemma, is Γ-linear, we observe that

$$\begin{aligned}
\Phi((\tilde{p}_{a_1} \otimes \tilde{p}_{a_2})s) &= \Phi(\tilde{p}_{a_1} \otimes \tilde{p}_{a_2} s - \tilde{p}_{a_1} s \otimes \tilde{p}_{a_2}) \\
&= -\tilde{p}_a + \tilde{p}_a \\
&= 0 \\
&= \Phi(\tilde{p}_{a_1} \otimes \tilde{p}_{a_2})s
\end{aligned}$$

and that

$$\begin{aligned}
\Phi((\tilde{p}_{a_1} \otimes \tilde{p}_{a_2} s)s) &= \Phi(\tilde{p}_{a_1} s \otimes \tilde{p}_{a_2} s + \tilde{p}_{a_1} \otimes \tilde{p}_{a_2} \eta s) \\
&= -\tilde{p}_a(s+\eta) \ + \tilde{p}_a \eta \\
&= -\tilde{p}_a s \\
&= \Phi(\tilde{p}_{a_1} \otimes \tilde{p}_{a_2} s)s\,.
\end{aligned}$$

The check that Φ is a chain map is contained in the computations

$$\begin{aligned}
\Phi(\partial_{\widetilde{P}_*\otimes\widetilde{P}_*}(\tilde{p}_{a_1}\otimes\tilde{p}_{a_2})) &= \Phi(\tilde{\partial}(\tilde{p}_{a_1})\otimes\tilde{p}_{a_2}) - \Phi(\tilde{p}_{a_1}\otimes\tilde{\partial}(\tilde{p}_{a_2}))\\
&= \begin{cases} -\Phi(\tilde{p}_{a_1-1}(s+a_1\eta)\otimes\tilde{p}_{a_2}) & \\ \qquad +\Phi(\tilde{p}_{a_1}\otimes\tilde{p}_{a_2-1}(s+a_2\eta)) & \text{for } a_1,\, a_2\geq 2,\\ \Phi(\tilde{p}_0\otimes\tilde{p}_{a_2}) & \\ \qquad +\Phi(\tilde{p}_1\otimes\tilde{p}_{a_2-1}(s+a_2\eta)) & \text{for } a_1=1,\, a_2\geq 2,\\ -\Phi(\tilde{p}_{a_1-1}(s+a_1\eta)\otimes\tilde{p}_1) & \\ \qquad -\Phi(\tilde{p}_{a_1}\otimes\tilde{p}_0) & \text{for } a_1\geq 2,\, a_2=1 \end{cases}\\
&= \tilde{p}_{a-1}-\tilde{p}_{a-1}\\
&= 0\\
&= \tilde{\partial}(\Phi(\tilde{p}_{a_1}\otimes\tilde{p}_{a_2}))
\end{aligned}$$

and

$$\begin{aligned}
\Phi(\partial_{\widetilde{P}_*\otimes\widetilde{P}_*}(\tilde{p}_{a_1}\otimes\tilde{p}_{a_2}s)) &= \Phi(\tilde{\partial}(\tilde{p}_{a_1})\otimes\tilde{p}_{a_2}s) - \Phi(\tilde{p}_{a_1}\otimes\tilde{\partial}(\tilde{p}_{a_2}s))\\
&= \begin{cases} -\Phi(\tilde{p}_{a_1-1}(s+a_1\eta)\otimes\tilde{p}_{a_2}s) & \\ \qquad +\Phi(\tilde{p}_{a_1}\otimes\tilde{p}_{a_2-1}(s+a_2\eta)s) & \text{for } a_1,\, a_2\geq 2,\\ \Phi(\tilde{p}_0\otimes\tilde{p}_{a_2}s) & \\ \qquad +\Phi(\tilde{p}_1\otimes\tilde{p}_{a_2-1}(s+a_2\eta)s) & \text{for } a_1=1,\, a_2\geq 2,\\ -\Phi(\tilde{p}_{a_1-1}(s+a_1\eta)\otimes\tilde{p}_1s) & \\ \qquad -\Phi(\tilde{p}_{a_1}\otimes\tilde{p}_0s) & \text{for } a_1\geq 2,\, a_2=1 \end{cases}\\
&= \tilde{p}_{a-1}(s+a\eta)\\
&= -\tilde{\partial}(\tilde{p}_a)\\
&= \tilde{\partial}(\Phi(\tilde{p}_{a_1}\otimes\tilde{p}_{a_2}s))\,.
\end{aligned}$$

□

Now we want to use the above chain maps to compute the multiplicative structure. Recall that the cup product is induced by the composite pairing

$$\begin{aligned}
\widetilde{P}_*\otimes\mathrm{Hom}(P,M)_*\otimes\widetilde{P}_*\otimes\mathrm{Hom}(P,N)_* &\xrightarrow{1\otimes\tau\otimes 1} \widetilde{P}_*\otimes\widetilde{P}_*\otimes\mathrm{Hom}(P,M)_*\otimes\mathrm{Hom}(P,N)_*\\
&\xrightarrow{1\otimes 1\otimes\alpha} \widetilde{P}_*\otimes\widetilde{P}_*\otimes\mathrm{Hom}(P\otimes P,M\otimes N)_*\\
&\xrightarrow{\Phi\otimes\Psi^*} \widetilde{P}_*\otimes\mathrm{Hom}(P,M\otimes N)_*.
\end{aligned}$$

There are two signs to be wary of here; the first one comes from twisting $\mathrm{Hom}(P,M)_*$ past the second $\widetilde{P}_*$-factor, and the second sign comes from using the canonical map α. Please refer to Equation I.1. The cup product computations, which can be found in the lemmas below, are straight-forward computations. We only include the verification of two of the lemmas, as the other three are very similar.

Lemma I.2.46. *Let $f_{b_1} \cdot m$ and $f_{b_2} \cdot n$ be cycles with homology classes in $\widehat{\operatorname{Ext}}_\Gamma^{b_1}(k, M)$ and $\widehat{\operatorname{Ext}}_\Gamma^{b_2}(k, N)$, respectively. The cup product of these is the cycle*

$$f_{b_1} \cdot m \smile f_{b_2} \cdot n = f_{b_1+b_2} \cdot m \otimes n$$

with homology class in $\widehat{\operatorname{Ext}}_\Gamma^{b_1+b_2}(k, M \otimes N)$.

Lemma I.2.47. *Let $u_{a_1} \cdot m$ and $u_{a_2} \cdot n$ be cycles with homology classes in $\widehat{\operatorname{Ext}}_\Gamma^{-a_1}(k, M)$ and $\widehat{\operatorname{Ext}}_\Gamma^{-a_2}(k, N)$, respectively. The cup product of these is the cycle*

$$u_{a_1} \cdot m \smile u_{a_2} \cdot n = u_{a_1+a_2} \cdot m \otimes n$$

with homology class in $\widehat{\operatorname{Ext}}_\Gamma^{-a_1-a_2}(k, M \otimes N)$.

Lemma I.2.48. *Let $f_0 \cdot m$ and $u_a \cdot n$ be cycles with homology classes in $\widehat{\operatorname{Ext}}_\Gamma^{0}(k, M)$ and $\widehat{\operatorname{Ext}}_\Gamma^{-a}(k, N)$, respectively. The cup product of these is the cycle*

$$f_0 \cdot m \smile u_a \cdot n = u_a \cdot m \otimes n$$

with homology class in $\widehat{\operatorname{Ext}}_\Gamma^{-a}(k, M \otimes N)$.

Proof. Since $f_0 \cdot m$ is assumed to be a cycle in U_* we know that m is an element in $\ker(s)$. An explicit description of this cycle is then

$$f_0 \cdot m = \tilde{p}_0 \otimes \begin{pmatrix} p_0 \mapsto m \\ p_0 s \mapsto 0 \end{pmatrix}.$$

The first map $1 \otimes \tau \otimes 1$ in the composite pairing twists the second tensor factor past the third one, so that

$$\begin{aligned}
\tilde{p}_0 \otimes \begin{pmatrix} p_0 \mapsto m \\ p_0 s \mapsto 0 \end{pmatrix} \otimes \left(-(-1)^{|n|} \tilde{p}_a \otimes \begin{pmatrix} p_0 \mapsto n(s+\eta) \\ p_0 s \mapsto 0 \end{pmatrix} - \tilde{p}_a s \otimes \begin{pmatrix} p_0 \mapsto n \\ p_0 s \mapsto 0 \end{pmatrix} \right) \\
\mapsto -(-1)^{|m|+|n|} \tilde{p}_0 \otimes \tilde{p}_a \otimes \begin{pmatrix} p_0 \mapsto m \\ p_0 s \mapsto 0 \end{pmatrix} \otimes \begin{pmatrix} p_0 \mapsto n(s+\eta) \\ p_0 s \mapsto 0 \end{pmatrix} \\
- \tilde{p}_0 \otimes \tilde{p}_a s \otimes \begin{pmatrix} p_0 \mapsto m \\ p_0 s \mapsto 0 \end{pmatrix} \otimes \begin{pmatrix} p_0 \mapsto n \\ p_0 s \mapsto 0 \end{pmatrix}.
\end{aligned}$$

The second map in the composite is $1 \otimes 1 \otimes \alpha$, so that

$$
\begin{aligned}
&- (-1)^{|m|+|n|}\tilde{p}_0 \otimes \tilde{p}_a \otimes \begin{pmatrix} p_0 \mapsto m \\ p_0 s \mapsto 0 \end{pmatrix} \otimes \begin{pmatrix} p_0 \mapsto n(s+\eta) \\ p_0 s \mapsto 0 \end{pmatrix} \\
&\qquad - \tilde{p}_0 \otimes \tilde{p}_a s \otimes \begin{pmatrix} p_0 \mapsto m \\ p_0 s \mapsto 0 \end{pmatrix} \otimes \begin{pmatrix} p_0 \mapsto n \\ p_0 s \mapsto 0 \end{pmatrix} \\
&\mapsto -(-1)^{|m|+|n|}\tilde{p}_0 \otimes \tilde{p}_a \otimes \begin{pmatrix} p_0 \otimes p_0 \mapsto m \otimes n(s+\eta) \\ p_0 s \otimes p_0 \mapsto 0 \\ p_0 \otimes p_0 s \mapsto 0 \\ p_0 s \otimes p_0 s \mapsto 0 \end{pmatrix} \\
&\qquad - \tilde{p}_0 \otimes \tilde{p}_a s \otimes \begin{pmatrix} p_0 \otimes p_0 \mapsto m \otimes n \\ p_0 s \otimes p_0 \mapsto 0 \\ p_0 \otimes p_0 s \mapsto 0 \\ p_0 s \otimes p_0 s \mapsto 0 \end{pmatrix}.
\end{aligned}
$$

Lastly, the computations of Ψ and Φ given in Lemma I.2.44 and Lemma I.2.45 tell us that the final map in the composite is such that

$$
\begin{aligned}
&- (-1)^{|m|+|n|}\tilde{p}_0 \otimes \tilde{p}_a \otimes \begin{pmatrix} p_0 \otimes p_0 \mapsto m \otimes n(s+\eta) \\ p_0 s \otimes p_0 \mapsto 0 \\ p_0 \otimes p_0 s \mapsto 0 \\ p_0 s \otimes p_0 s \mapsto 0 \end{pmatrix} \\
&- \tilde{p}_0 \otimes \tilde{p}_a s \otimes \begin{pmatrix} p_0 \otimes p_0 \mapsto m \otimes n \\ p_0 s \otimes p_0 \mapsto 0 \\ p_0 \otimes p_0 s \mapsto 0 \\ p_0 s \otimes p_0 s \mapsto 0 \end{pmatrix} \mapsto -(-1)^{|m|+|n|}\tilde{p}_a \otimes \begin{pmatrix} p_0 \mapsto m \otimes n(s+\eta) \\ p_0 s \mapsto 0 \end{pmatrix} \\
&\qquad - \tilde{p}_a s \otimes \begin{pmatrix} p_0 \mapsto m \otimes n \\ p_0 s \mapsto 0 \end{pmatrix}
\end{aligned}
$$

where the target can be identified with $u_a \cdot (m \otimes n)$, as wanted. □

Lemma I.2.49. *Let $u_a \cdot m$ and $f_0 \cdot n$ be cycles with homology classes in $\widehat{\mathrm{Ext}}_\Gamma^{-a}(k, M)$ and $\widehat{\mathrm{Ext}}_\Gamma^{0}(k, N)$, respectively. The cup product of these is the cycle*

$$u_a \cdot m \smile f_0 \cdot n = u_a \cdot m \otimes n$$

with homology class in $\widehat{\mathrm{Ext}}_\Gamma^{-a}(k, M \otimes N)$.

Lemma I.2.50. *Let $f_1 \cdot m$ and $u_1 \cdot n$ be cycles with homology classes in $\widehat{\mathrm{Ext}}_\Gamma^{1}(k, M)$ and $\widehat{\mathrm{Ext}}_\Gamma^{-1}(k, N)$, respectively. Then the two cycles*

$$f_1 \cdot m \smile u_1 \cdot n \simeq f_0 \cdot m \otimes n.$$

are homologous in the complex U_*, so that they determine the same class in $\widehat{\mathrm{Ext}}_\Gamma^{0}(k, M \otimes N)$.

Proof. Since $f_1 \cdot m$ and $u_1 \cdot n$ are assumed to be cycles we know that m and n are elements in $\ker(s+\eta)$, which directly implies that $m \otimes n$ is an element of $\ker(s)$ since

$$\begin{aligned}(m\otimes n)\cdot s &= m\otimes ns + (-1)^{|n|} ms \otimes n \\ &= m\otimes ns + m\otimes n\eta + m \otimes n\eta + (-1)^{|n|} ms\otimes n \\ &= m\otimes n(s+\eta) + (-1)^{|n|} m(s+\eta)\otimes n \\ &= 0\,.\end{aligned}$$

An explicit description of the two cycles $f_1 \cdot m$ and $u_1 \cdot n$ is

$$f_1 \cdot m = \tilde{p}_0 \otimes \begin{pmatrix} p_1 \mapsto m \\ p_1 s \mapsto ms \end{pmatrix} \quad \text{and} \quad u_1 \cdot n = -\tilde{p}_1 s \otimes \begin{pmatrix} p_0 \mapsto n \\ p_0 s \mapsto 0 \end{pmatrix}.$$

The first map $1 \otimes \tau \otimes 1$ in the composite pairing twists the second tensor factor past the third one, so that

$$-\tilde{p}_0 \otimes \begin{pmatrix} p_1 \mapsto m \\ p_1 s \mapsto ms \end{pmatrix} \otimes \tilde{p}_1 s \otimes \begin{pmatrix} p_0 \mapsto n \\ p_0 s \mapsto 0 \end{pmatrix} \mapsto -\tilde{p}_0 \otimes \tilde{p}_1 s \otimes \begin{pmatrix} p_1 \mapsto m \\ p_1 s \mapsto ms \end{pmatrix} \otimes \begin{pmatrix} p_0 \mapsto n \\ p_0 s \mapsto 0 \end{pmatrix}.$$

The second map in the composite is $1 \otimes 1 \otimes \alpha$, so that

$$-\tilde{p}_0 \otimes \tilde{p}_1 s \otimes \begin{pmatrix} p_1 \mapsto m \\ p_1 s \mapsto ms \end{pmatrix} \otimes \begin{pmatrix} p_0 \mapsto n \\ p_0 s \mapsto 0 \end{pmatrix} \mapsto -\tilde{p}_0 \otimes \tilde{p}_1 s \otimes \begin{pmatrix} p_1 \otimes p_0 \mapsto m \otimes n \\ p_1 s \otimes p_0 \mapsto (-1)^{|n|} ms \otimes n \\ p_1 \otimes p_0 s \mapsto 0 \\ p_1 s \otimes p_0 s \mapsto 0 \end{pmatrix}.$$

Lastly, the computations of Ψ and Φ given in Lemma I.2.44 and Lemma I.2.45 tells us that the final map in the composite is such that

$$-\tilde{p}_0 \otimes \tilde{p}_1 s \otimes \begin{pmatrix} p_1 \otimes p_0 \mapsto m \otimes n \\ p_1 s \otimes p_0 \mapsto (-1)^{|n|} ms \otimes n \\ p_1 \otimes p_0 s \mapsto 0 \\ p_1 s \otimes p_0 s \mapsto 0 \end{pmatrix} \mapsto -\tilde{p}_1 s \otimes \begin{pmatrix} p_1 \mapsto m \otimes n \\ p_1 s \mapsto (-1)^{|n|} ms \otimes n \end{pmatrix}.$$

The right hand term can be identified with $-h_{1,1} \cdot m \otimes n$. We conclude that

$$f_1 \cdot m \smile u_1 \cdot n = -h_{1,1} \cdot m \otimes n.$$

Note that the boundary of $g_{1,0} \cdot m \otimes n$ is

$$\partial(g_{1,0} \cdot m \otimes n) = f_0 \cdot m \otimes n + h_{1,1} \cdot m \otimes n\,,$$

which tells us that $f_0 \cdot m \otimes n$ and $-h_{1,1} \cdot m \otimes n$ are homologous in U_*, and hence represent the same class in $\widehat{\mathrm{Ext}}^0_\Gamma(k, M \otimes N)$. $\square$

We decide to make a final change of notation.

Notation I.2.51. Let $t^b \cdot m$ and $t^{-a} \cdot n$ denote the homology classes

$$t^b \cdot m = [f_b \cdot m] \quad \text{and} \quad t^{-a} \cdot n = [u_a \cdot n]$$

in $\widehat{\operatorname{Ext}}_\Gamma^b(k, M)$ for $b \geq 0$ and in $\widehat{\operatorname{Ext}}_\Gamma^{-a}(k, N)$ for $a \geq 1$, respectively.

Note that $t^b \cdot m$ has internal and total degrees equal to those of $f_b \cdot m$, so that

$$|t^b \cdot m| = |m| - b \quad \text{and} \quad \|t_b \cdot m\| = |m| - 2b\,.$$

Similarly, $t^{-a} \cdot n$ has internal and total degrees equal to that of $u_a \cdot n$, so that

$$|t^{-a} \cdot n| = a + |n| \quad \text{and} \quad \|t^{-a} \cdot n\| = 2a + |n|\,.$$

We conclude that, formally, the symbol t has homological degree -1, internal degree $|t| = -1$ and total degree $\|t\| = -2$. Using this new notation we have the following theorem.

Theorem I.2.52.

$$\widehat{\operatorname{Ext}}_\Gamma^c(k, M) \cong \begin{cases} t^c \cdot \dfrac{\ker(s)}{\operatorname{im}(s+\eta)} & \text{for } c \text{ even,} \\ \\ t^c \cdot \dfrac{\ker(s+\eta)}{\operatorname{im}(s)} & \text{for } c \text{ odd.} \end{cases}$$

The cup product

$$\widehat{\operatorname{Ext}}_\Gamma^{c_1}(k, M) \otimes \widehat{\operatorname{Ext}}_\Gamma^{c_2}(k, N) \longrightarrow \widehat{\operatorname{Ext}}_\Gamma^{c_1+c_2}(k, M \otimes N)$$

is given by

$$(t^{c_1} \cdot m) \smile (t^{c_2} \cdot n) = t^{c_1+c_2} \cdot m \otimes n\,.$$

Corollary I.2.53.

$$\widehat{\operatorname{Ext}}_\Gamma^c(k, k) \cong \begin{cases} t^c \cdot \operatorname{coker}(\eta) & \text{for } c \text{ even,} \\ t^c \cdot \ker(\eta) & \text{for } c \text{ odd.} \end{cases}$$

In this case, the cup product

$$\widehat{\operatorname{Ext}}_\Gamma^{c_1}(k, k) \otimes \widehat{\operatorname{Ext}}_\Gamma^{c_2}(k, k) \longrightarrow \widehat{\operatorname{Ext}}_\Gamma^{c_1+c_2}(k, k)$$

is given by

$$(t^{c_1} \cdot x) \smile (t^{c_2} \cdot y) = t^{c_1+c_2} \cdot xy$$

and makes $\widehat{\operatorname{Ext}}_\Gamma^*(k, k)$ into a k-algebra over which $\widehat{\operatorname{Ext}}_\Gamma^*(k, M)$ is a module for any Γ-module M.

Remark I.2.54. Note that in Theorem I.2.52 above, the answer is also the homology of the differential graded Γ-module

$$M[t, t^{-1}]$$

with differential

$$d(m) = tms \quad \text{and} \quad d(t) = t^2\eta\,,$$

where m is an element of M and t has homological degree -1 and internal degree -1. Indeed, by the Leibniz rule we then have

$$\begin{aligned} d(t^c m) &= d(t^c)m + t^c d(m) = ct^{c+1}\eta m + t^{c+1}ms \\ &= \begin{cases} t^{c+1}ms & \text{if } c \text{ is even} \\ t^{c+1}m(s+\eta) & \text{if } c \text{ is odd,} \end{cases} \end{aligned}$$

which gives us the same homology groups as in Theorem I.2.52. Note that this is also true multiplicatively: if $\mu : M \otimes N \to L$ is a pairing of Γ-modules, then the cup product

$$\widehat{\mathrm{Ext}}_\Gamma^{c_1}(k, M) \otimes \widehat{\mathrm{Ext}}_\Gamma^{c_2}(k, N) \longrightarrow \widehat{\mathrm{Ext}}_\Gamma^{c_1+c_2}(k, M \otimes N) \longrightarrow \widehat{\mathrm{Ext}}_\Gamma^{c_1+c_2}(k, L)$$

is precisely the one induced by the obvious pairing

$$M[t, t^{-1}] \otimes N[t, t^{-1}] \longrightarrow L[t, t^{-1}]$$

on homology.

I.3 Homotopy groups of orthogonal G-spectra

In this section we discuss some results regarding equivariant stable homotopy groups. Our chosen model for equivariant spectra is orthogonal G-spectra, and we recall some basic theory about these objects and their homotopy groups in Section I.3.1. In Section I.3.2 we define the main Hopf algebra that we will work with in this paper, namely the (non-equivariant) homotopy groups of the unreduced suspension spectrum of a compact Lie group, also referred to as the spherical group ring $\mathbb{S}[G]$ of that group. Since our main group of interest is the circle $\mathbb{T}$, we also give an explicit description of $\pi_*(\mathbb{S}[\mathbb{T}])$ as an algebra over $\pi_*(\mathbb{S})$. Lastly, in Section I.3.3 we show, under suitable projectivity assumptions, that we can sometimes describe the equivariant homotopy groups $\pi_*^G(X)$ of an orthogonal G-spectrum X as the '$\pi_*(\mathbb{S}[G])$-invariants' of the non-equivariant homotopy groups $\pi_*(X)$.

I.3.1 Equivariant homotopy groups

Let G be a compact Lie group, and let X be an orthogonal G-spectrum, as in [MM02, §II.2] and [Sch18, §3.1]. In what follows, we will always assume that G

acts from the right. Recall that, in particular, X associates to each (finite-dimensional, orthogonal) G-representation V a based G-space $X(V)$, and to each pair (U, V) of G-representations a G-equivariant structure map $\sigma\colon \Sigma^U X(V) \to X(U \oplus V)$.

We can define G-equivariant homotopy groups $\pi_*^G(X)$ associated to X. To do this, one fixes a complete G-universe[7] $\mathscr{U}$ containing a fixed copy of $\mathbb{R}^\infty$. Note that the set of finite-dimensional G-subrepresentations of $\mathscr{U}$ is partially ordered by inclusion. For non-negative integers $q \geq 0$, we define the qth G-equivariant homotopy group of X as the colimit, over this directed partially ordered set, of the sets of homotopy classes $[f]$ of G-maps $f\colon \Sigma^V S^q \to X(V)$:

$$\pi_q^G(X) = \operatorname*{colim}_V\, [\Sigma^V S^q, X(V)]^G\,.$$

Similarly, to define the non-positive G-equivariant homotopy groups, we let

$$\pi_{-q}^G(X) = \operatorname*{colim}_V\, [\Sigma^{V-\mathbb{R}^q} S^0, X(V)]^G$$

where $V - \mathbb{R}^q$ denotes the orthogonal complement of $\mathbb{R}^q$ in V. These definitions agree for $q = 0$. Each equivariant homotopy group $\pi_q^G(X)$ is naturally an abelian group.

The category of orthogonal G-spectra is symmetric monoidal, with the symmetric monoidal product being denoted $\wedge$ and referred to as the smash product. The unit of this symmetric monoidal structure is the sphere spectrum $\mathbb{S}$ with the trivial G-action. Any pairing $\phi\colon X \wedge Y \to Z$ of orthogonal G-spectra gives rise to a pairing of the corresponding equivariant homotopy groups. Consider classes $[f] \in \pi_p^G(X)$ and $[g] \in \pi_q^G(Y)$, represented by homotopy classes of G-maps $f\colon \Sigma^V S^p \to X(V)$ and $g\colon \Sigma^W S^q \to Y(W)$, respectively. The induced pairing

$$\phi_*\colon \pi_p^G(X) \otimes \pi_q^G(Y) \longrightarrow \pi_{p+q}^G(Z)$$

maps $[f]\otimes[g]$ to the element represented by the homotopy class of the composite

$$\Sigma^{V\oplus W} S^{p+q} \xrightarrow{\cong} \Sigma^V S^p \wedge \Sigma^W S^q \xrightarrow{f\wedge g} X(V) \wedge Y(W) \xrightarrow{\phi} Z(V \oplus W)\,.$$

Similar constructions can be carried out if p or q is negative, although this is a bit tricky. In this way, we obtain a pairing

$$\phi_*\colon \pi_*^G(X) \otimes \pi_*^G(Y) \longrightarrow \pi_*^G(Z)$$

of graded abelian groups.

Remark I.3.1. More generally, given a group homomorphism $\alpha\colon G \to H \times K$ and an α-equivariant map $X \wedge Y \to Z$, where X, Y and Z are orthogonal H-, K- and G-spectra, respectively, we obtain a pairing

$$\pi_*^H(X) \otimes \pi_*^K(Y) \longrightarrow \pi_*^G(Z)\,.$$

[7] A complete G-universe is an orthogonal representation of countably infinite dimension in which every finite dimensional G-representation, and their countably infinite direct sums, embeds.

Here, α-equivariant means that the diagram

$$\begin{tikzcd}
X \wedge Y \wedge G_+ \arrow[r, "1\wedge 1\wedge\alpha"] \arrow[d] & X \wedge Y \wedge H_+ \wedge K_+ \arrow[r, "1\wedge\tau\wedge 1"] & X \wedge H_+ \wedge Y \wedge K_+ \arrow[r] & X \wedge Y \arrow[d] \\
Z \wedge G_+ \arrow[rrr] & & & Z
\end{tikzcd}$$

commutes.

If R is a commutative (non-equivariant) orthogonal ring spectrum, with multiplication $\mu\colon R \wedge R \to R$, then the induced pairing

$$\mu_* \colon \pi_*(R) \otimes \pi_*(R) \longrightarrow \pi_*(R)$$

on (non-equivariant) homotopy groups makes $R_* = \pi_*(R)$ into a graded commutative ring. A right R-module in orthogonal G-spectra is an orthogonal G-spectrum X with an associative and unital action $\rho\colon X \wedge R \to X$, defined in the category of orthogonal G-spectra. Here R is regarded as a G-spectrum with trivial action. In this case, there is an induced pairing

$$\rho_* \colon \pi_*^G(X) \otimes R_* \longrightarrow \pi_*^G(X)$$

making $\pi_*^G(X)$ into a right R_*-module. If X and Y are two R-modules in orthogonal G-spectra, then the canonical map $X \wedge Y \to X \wedge_R Y$ induces a pairing

$$\pi_*^G(X) \otimes \pi_*^G(Y) \longrightarrow \pi_*^G(X \wedge_R Y)$$

that equalizes the two composites from $\pi_*^G(X) \otimes R_* \otimes \pi_*^G(Y)$, so that we have the induced dashed map making the diagram

$$\begin{tikzcd}
\pi_*^G(X) \otimes R_* \otimes \pi_*^G(Y) \arrow[r, shift left] \arrow[r, shift right] & \pi_*^G(X) \otimes \pi_*^G(Y) \arrow[r] \arrow[dr] & \pi_*^G(X) \otimes_{R_*} \pi_*^G(Y) \arrow[d, dashed] \\
 & & \pi_*(X \wedge_R Y)
\end{tikzcd}$$

commute.

I.3.2 A cocommutative Hopf algebra

Let us introduce the Hopf algebra that we will work with through the remainder of this paper. The right R-action on $R[G] = R \wedge G_+$ is given by the composite map

$$R \wedge G_+ \wedge R \xrightarrow{1\wedge\tau} R \wedge R \wedge G_+ \xrightarrow{\mu\wedge 1} R \wedge G_+ \,.$$

Lemma I.3.2. *If $R[G]_* = \pi_*(R \wedge G_+)$ is flat as a (right) R_*-module, then $R[G]_*$ is naturally a cocommutative Hopf algebra over $R_* = \pi_*(R)$.*

Proof. We have a pairing

$$
\begin{aligned}
R[G]_* \otimes_{R_*} R[G]_* = \pi_*(R \wedge G_+) \otimes_{\pi_*(R)} \pi_*(R \wedge G_+) \\
\stackrel{\cdot}{\longrightarrow} \pi_*((R \wedge G_+) \wedge_R (R \wedge G_+)) \cong \pi_*(R \wedge G_+ \wedge G_+)\,,
\end{aligned}
$$

where the left R_*-action on the right hand copy of $R[G]_*$ is equal to that obtained by twisting the right R_*-action. When $R[G]_*$ is flat as a right R_*-module, it follows by a well-known induction over the cells of a CW structure on the right hand copy of G that the pairing above is an isomorphism of R_*-modules.

The unit inclusion $\{e\} \to G$, group multiplication $G \times G \to G$, collapse $G \to \{e\}$, diagonal $G \to G \times G$ and group inverse $G \to G$ give us R-module maps $R \to R \wedge G_+$, $R \wedge G_+ \wedge_R R \wedge G_+ \to R \wedge G_+$, $R \wedge G_+ \to R$, $R \wedge G_+ \to R \wedge G_+ \wedge_R R \wedge G_+$ and $R \wedge G_+ \to R \wedge G_+$ that induce R_*-module homomorphisms

$$
\begin{aligned}
\eta &\colon R_* \longrightarrow R[G]_* \\
\phi &\colon R[G]_* \otimes_{R_*} R[G]_* \longrightarrow R[G]_* \\
\epsilon &\colon R[G]_* \longrightarrow R_* \\
\psi &\colon R[G]_* \longrightarrow R[G]_* \otimes_{R_*} R[G]_* \\
\chi &\colon R[G]_* \longrightarrow R[G]_*
\end{aligned}
$$

which make $R[G]_*$ a Hopf algebra over R_*. The cocommutativity of the diagonal implies that ψ is cocommutative. □

By the discussion in Section I.2.1, the category of modules over $R[G]_*$ is closed symmetric monoidal. Note that if X is an R-module in orthogonal G-spectra, then the commuting right R- and G-actions combine to define an action

$$\gamma \colon X \wedge_R R[G] \cong X \wedge G_+ \longrightarrow X$$

which makes the underlying (non-equivariant) orthogonal spectrum of X into a right $R[G]$-module in the category of (non-equivariant) R-modules. The induced pairing

$$\gamma_* \colon \pi_*(X) \otimes_{R_*} R[G]_* \longrightarrow \pi_*(X)$$

then gives the (non-equivariant) homotopy groups $\pi_*(X)$ the structure of a right $R[G]_*$-module. If Y is a second R-module in orthogonal G-spectra, the pairing

$$\pi_*(X) \otimes_{R_*} \pi_*(Y) \stackrel{\cdot}{\longrightarrow} \pi_*(X \wedge_R Y)$$

is a homomorphism of $R[G]_*$-modules, where the Hopf algebra $R[G]_*$ acts diagonally on the left hand side. Likewise,

$$\pi_* F_R(X, Y) \longrightarrow \mathrm{Hom}_{R_*}(\pi_*(X), \pi_*(Y))$$

is a homomorphism of $R[G]_*$-modules, where the Hopf algebra $R[G]_*$ acts by conjugation on the right hand side.

The case we are the most interested in is when the Lie group is the circle, so let us compute the homotopy groups of the spherical group ring of this specific group.

Proposition I.3.3. *When $G = \mathbb{T} = U(1)$ is the circle group,*

$$R[\mathbb{T}]_* = R_*[s]/(s^2 = \eta s)$$

with $|s| = |\eta| = 1$. Here s generates the augmentation ideal

$$\overline{R[\mathbb{T}]}_* = \ker(\epsilon\colon R[\mathbb{T}]_* \longrightarrow R_*) = R_*\{s\}\,,$$

and η is the image of the complex Hopf map in $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$. The generator s is primitive, so the coproduct and involution are given by $\psi(s) = s \otimes 1 + 1 \otimes s$ and $\chi(s) = -s$.

Proof. It suffices to prove the result for $R = \mathbb{S}$. Proving this we would know that $\mathbb{S}[\mathbb{T}]_*$ is free over $\mathbb{S}_*$, so that the case of a general ground ring spectrum R follows immediately from the isomorphism $R[\mathbb{T}]_* \cong \mathbb{S}[\mathbb{T}]_* \otimes_{\mathbb{S}_*} R_*$.

To prove the result for the sphere spectrum, we start by noting that the cofibre sequence

$$S^0 \cong 1_+ \longrightarrow \mathbb{T}_+ \longrightarrow \mathbb{T} \cong S^1$$

admits a retraction $\mathbb{T}_+ \to 1_+$. Hence the induced stable cofibre sequence

$$\mathbb{S} \xrightarrow{i} \mathbb{S}[\mathbb{T}] \xrightarrow{p} \Sigma\mathbb{S}$$

admits a retraction $c\colon \mathbb{S}[\mathbb{T}] \to \mathbb{S}$ and a section $s\colon \Sigma\mathbb{S} \to \mathbb{S}[\mathbb{T}]$ with $ps \simeq 1$ and $cs \simeq 0$. The maps i and s represent classes in $\mathbb{S}[\mathbb{T}]_*$ of total (and internal) degree 0 and 1, respectively, and induce an isomorphism $\mathbb{S}_*\{i, s\} \cong \mathbb{S}[\mathbb{T}]_*$. Here, i is the multiplicative unit and s generates the augmentation ideal $\overline{\mathbb{S}[\mathbb{T}]}_* = \mathbb{S}_*\{s\}$. It only remains to prove that we have the relation $s^2 = \eta s$ in $\mathbb{S}[\mathbb{T}]_2$. This is the content of formula (1.4.4) in [Hes96]. We give the following direct argument using the bar construction [Seg68, §3], [May75, §7] and the bar spectral sequence [Seg68, §5], [May72, §11]. We shall discuss these tools at greater length in Section I.5.1.

The bar construction of $\mathbb{T}$ is the geometric realization $B\mathbb{T} = |B_\bullet\mathbb{T}| \simeq \mathbb{C}P^\infty$ of the simplicial space

$$[q] \mapsto B_q\mathbb{T} = \mathbb{T}^q\,,$$

with the usual face and degeneracy maps. There is a standard filtration of $B\mathbb{T}$ by simplicial skeleta. The associated spectral sequence in (reduced) stable homotopy has E^1-page given as the normalised bar complex $NB_*(\mathbb{S}_*, \mathbb{S}[\mathbb{T}]_*, \mathbb{S}_*)$

$$0 \leftarrow 0 \xleftarrow{d^1_1} \overline{\mathbb{S}[\mathbb{T}]}_* \xleftarrow{d^1_2} \overline{\mathbb{S}[\mathbb{T}]}_* \otimes_{\mathbb{S}_*} \overline{\mathbb{S}[\mathbb{T}]}_* \xleftarrow{d^1_3} \cdots\,,$$

which we reviewed in Construction I.2.24. This spectral sequence converges (strongly) to $\pi_*\Sigma^\infty(B\mathbb{T}) \cong \pi_*\Sigma^\infty(\mathbb{C}P^\infty)$. The part of the E^1-page that will be relevant to us is pictured below, with the origin in the lower left hand corner:

$$
\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \overline{\mathbb{S}[\mathbb{T}]}_2 & \overline{\mathbb{S}[\mathbb{T}]}_1 \otimes \overline{\mathbb{S}[\mathbb{T}]}_1 \arrow[l] & \cdots & \cdots \\
0 & \overline{\mathbb{S}[\mathbb{T}]}_1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

Firstly,

$$d^1_2(x \otimes y) = \epsilon(x)y - xy + x\epsilon(y) = -xy$$

for $x, y \in \overline{\mathbb{S}[\mathbb{T}]}_*$, since x and y both augment to zero. With prior understanding of the stable homotopy groups of $\mathbb{C}P^\infty$ we can now figure out the displayed differential. The stable class of the inclusion $S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty$ is well-known to generate $\pi_2 \Sigma^\infty \mathbb{C}P^\infty \cong \mathbb{Z}$. For degree reasons this must be detected by $\pm s$ in $E^\infty_{1,1} = E^1_{1,1} = \overline{\mathbb{S}[\mathbb{T}]}_1 = \mathbb{Z}\{s\}$. The 4-cell in $\mathbb{C}P^2$ is attached by the Hopf fibration η to $\mathbb{C}P^1$, so that $\pi_3 \Sigma^\infty \mathbb{C}P^\infty = 0$. This forces $\eta s \in E^1_{1,2} = \overline{\mathbb{S}[\mathbb{T}]}_2 = \mathbb{Z}/2\{\eta s\}$ to be a boundary in the spectral sequence. For degree reasons, the only possibility is that

$$\eta s = d^1_2(s \otimes s)\,,$$

so that $s^2 = \eta s$.

Note that the coproduct $\psi(s)$ must contain the terms $s \otimes 1$ and $1 \otimes s$ by counitality, and cannot contain other terms since $\mathbb{S}[\mathbb{T}]_*$ is connected and $|s| = 1$. Hence s is a primitive element of our Hopf algebra. □

Proposition I.3.4. *When $G = \mathbb{U} = Sp(1)$ is the 3-sphere group,*

$$R[\mathbb{U}]_* = R_*[t]/(t^2 = \dot{\nu} t)$$

with $|t| = |\dot{\nu}| = 3$. Here t generates the augmentation ideal

$$\overline{R[\mathbb{U}]}_* = \ker(\epsilon\colon R[\mathbb{U}]_* \longrightarrow R_*) = R_*\{t\}\,,$$

and $\dot{\nu}$ is the image of a generator of $\pi_3(\mathbb{S}) \cong \mathbb{Z}/24$. The coproduct is given by $\psi(t) = t \otimes 1 + 1 \otimes t$.

Proof. Similar to the circle case. □

Note that we cannot assert from this line of argument that $\dot{\nu}$ is the image of the quaternionic Hopf map. The bar spectral sequence argument for $R = \mathbb{S}$ only shows that $t^2 = \dot{\nu} t$ with $\dot{\nu}$ some generator of $\pi_3(\mathbb{S}) \cong \mathbb{Z}/24$. A more geometric argument might link $\dot{\nu}$ to the standard generator ν of $\pi_3(\mathbb{S})$, but we will not pursue this here.

I.3.3 A restriction homomorphism

Any G-spectrum can be viewed as a non-equivariant spectrum via the inclusion homomorphism $1 \to G$. This gives rise to a map of graded abelian groups

$$\mathrm{res}^G_1 \colon \pi^G_*(X) \longrightarrow \pi_*(X)$$

taking the homotopy class of a G-map $f \colon \Sigma^V S^q \to X(V)$ to the homotopy class of the underlying non-equivariant map, and similarly for G-maps $\Sigma^{V-\mathbb{R}^q} S^0 \to X(V)$. Here 1 denotes the trivial group, and we write $\pi_*(X)$ in place of $\pi^1_*(X)$, as they are simply the ordinary non-equivariant homotopy groups of X viewed as a non-equivariant orthogonal spectrum X. Tracing through definitions shows that res^G_1 is R_*-linear if X is an R-module in orthogonal G-spectra. We are interested in the following refined restriction homomorphism.

Lemma I.3.5. *There is a natural R_*-module homomorphism*

$$\omega_X \colon \pi^G_*(X) \longrightarrow \mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X))$$

making the diagram

$$\begin{tikzcd}
\pi^G_*(X) \arrow[d, "\omega_X"'] \arrow[dr, "\mathrm{res}^G_1"] & & \\
\mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X)) \arrow[r, tail] & \pi_*(X) \arrow[r, shift left, "\tilde{\gamma}"] \arrow[r, shift right, "\tilde{\epsilon}"'] & \mathrm{Hom}_{R_*}(R[G]_*, \pi_*(X))
\end{tikzcd}$$

commute. Here $\tilde{\epsilon}$ denotes the adjoint of the trivial $R[G]_*$-action ϵ_* on $\pi_*(X)$, which equals the composite

$$\epsilon_* \colon \pi_*(X) \otimes_{R_*} R[G]_* \xrightarrow{1\otimes\epsilon} \pi_*(X) \otimes_{R_*} R_* \cong \pi_*(X)\,.$$

Similarly $\tilde{\gamma}$ denotes the adjoint to the $R[G]_*$-action

$$\gamma_* \colon \pi_*(X) \otimes_{R_*} R[G]_* \longrightarrow \pi_*(X)$$

on $\pi_*(X)$.

Proof. We claim that the two composite homomorphisms

$$\begin{tikzcd}
\pi^G_*(X) \otimes_{R_*} R[G]_* \arrow[r, "\mathrm{res}^G_1 \otimes 1"] & \pi_*(X) \otimes_{R_*} R[G]_* \arrow[r, shift left, "\gamma_*"] \arrow[r, shift right, "\epsilon_*"'] & \pi_*(X)
\end{tikzcd}$$

are equal. This implies that the two adjoint homomorphisms

$$\begin{tikzcd}
\pi^G_*(X) \arrow[r, "\mathrm{res}^G_1"] & \pi_*(X) \arrow[r, shift left, "\tilde{\gamma}"] \arrow[r, shift right, "\tilde{\epsilon}"'] & \mathrm{Hom}_{R_*}(R[G]_*, \pi_*(X))
\end{tikzcd}$$

are equal, so that res^G_1 factors uniquely through the equalizer $\mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X))$ of the two right-hand arrows.

By fibrant replacement we may assume that X is an Ω-G-spectrum [MM02, Def. III.3.1], meaning that each adjoint structure map $X(V) \to \Omega^{W-V} X(W)$ is a weak G-equivalence, where $V \subset W$ lie in our fixed complete G-universe $\mathscr{U}$. Then each element x in $\pi^G_*(X)$ is represented by the homotopy class $[f]$ of a G-map $f \colon S^m \to X(\mathbb{R}^n)$, for suitable non-negative integers m and n. Here G acts trivially on S^m, so f factors through the fixed points $X(\mathbb{R}^n)^G$, where the G-action γ is trivial. It follows that $\tilde{\gamma}$ and $\tilde{\epsilon}$ agree on $\mathrm{res}^G_1(x) \otimes y$ for any $y \in R[G]_*$, as claimed. $\square$

Proposition I.3.6. *If $R[G]_*$ is projective as an R_*-module, and $X \simeq F(G_+, Y)$ for some R-module Y in orthogonal G-spectra, then the natural homomorphism*

$$\omega_X \colon \pi^G_*(X) \xrightarrow{\cong} \mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X))$$

is an isomorphism.

Proof. By fibrant replacement, we may assume that Y is an Ω-G-spectrum. As usual we give $F(G_+, Y) \cong F_R(R[G], Y)$ the conjugate G-action. By naturality of ω_X we may assume that $X = F(G_+, Y)$, in which case X is also an Ω-G-spectrum.

Let us consider the commutative diagram

$$\begin{tikzcd}
\pi^G_*(X) \arrow[r, "\mathrm{res}^G_1"] \arrow[d, "\cong"] & \pi_*(X) \arrow[d, "\cong"] \\
\pi_*(Y) \arrow[r, "\tilde{\gamma}"] & \mathrm{Hom}_{R_*}(R[G]_*, \pi_*(Y))
\end{tikzcd} .$$

We make the maps involved a bit more explicit. The vertical isomorphisms are given as follows. The left hand vertical isomorphism $\pi^G_*(X) = \pi^G_*(F(G_+, Y)) \to \pi_*(Y)$ takes the homotopy class of a G-map $f \colon S^m \to X(\mathbb{R}^n) = F(G_+, Y(\mathbb{R}^n))$ bijectively to the homotopy class of $f' \colon S^m \to Y(\mathbb{R}^n)$ given by $f'(s) = f(s)(e)$, where $e \in G$ is the unit element of our group. The right hand vertical isomorphism is the special case $Z = R[G]$ of the natural $R[G]_*$-module homomorphism

$$\pi_* F_R(Z, Y) \longrightarrow \mathrm{Hom}_{R_*}(\pi_*(Z), \pi_*(Y)) .$$

Indeed, this is an isomorphism whenever $\pi_*(Z)$ is projective as an R_*-module. The top horizontal map is the restriction homomorphism we described at the beginning of this section, and the lower horizontal homomorphism $\tilde{\gamma}$ is adjoint to the $R[G]_*$-module action on $\pi_*(Y)$. Note that the diagram does indeed commute, since the lower and upper compositions both send the homotopy class of the G-map $f \colon S^m \to F(G_+, Y(\mathbb{R}^n))$ to the homomorphism $R[G]_* \to \pi_*(Y)$ induced by the left adjoint $S^m \wedge G_+ \to Y(\mathbb{R}^n)$ of f.

The homomorphism $\tilde{\gamma}$ identifies $\pi_*(Y)$ with the R_*-submodule

$$\mathrm{Hom}_{R[G]_*}(R_*, \mathrm{Hom}_{R_*}(R[G]_*, \pi_*(Y))) \cong \mathrm{Hom}_{R[G]_*}(R[G]_*, \pi_*(Y))$$

of its target, while res^G_1 factors through $\mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X))$ by the previous lemma. Hence we can apply $\mathrm{Hom}_{R[G]_*}(R_*, -)$ to the right hand vertical isomorphism in the diagram above to obtain another isomorphism, and a commutative square

$$\begin{tikzcd}
\pi^G_*(X) \arrow[r, "\omega_X"] \arrow[d, "\cong"'] & \mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X)) \arrow[d, "\cong"] \\
\pi_*(Y) \arrow[r, "\cong"] & \mathrm{Hom}_{R[G]_*}(R_*, \mathrm{Hom}_{R_*}(R[G]_*, \pi_*(Y)))\,.
\end{tikzcd}$$

It follows that ω_X is an isomorphism, as asserted. □

To handle multiplicative structure, we need the following observation.

Lemma I.3.7. *The natural transformation ω is monoidal, in the sense that the diagram*

$$\begin{tikzcd}
\pi^G_*(X) \otimes_{R_*} \pi^G_*(Y) \arrow[rr, "\cdot"] \arrow[d, "\omega_X \otimes \omega_Y"'] & & \pi^G_*(X \wedge_R Y) \arrow[dd, "\omega_{X \wedge_R Y}"] \\
\mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X)) \otimes_{R_*} \mathrm{Hom}_{R[G]_*}(R_*, \pi_*(Y)) \arrow[d, "\alpha"'] & & \\
\mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X) \otimes_{R_*} \pi_*(Y)) \arrow[rr, "\cdot"] & & \mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X \wedge_R Y))
\end{tikzcd}$$

commutes.

Proof. Since $\mathrm{Hom}_{R[G]_*}(R_*, \pi_*(X \wedge_R Y)) \to \pi_*(X \wedge_R Y)$ is a monomorphism it suffices to show that

$$\begin{tikzcd}
\pi^G_*(X) \otimes_{R_*} \pi^G_*(Y) \arrow[r, "\cdot"] \arrow[d, "\mathrm{res}^G_1 \otimes \mathrm{res}^G_1"'] & \pi^G_*(X \wedge_R Y) \arrow[d, "\mathrm{res}^G_1"] \\
\pi_*(X) \otimes_{R_*} \pi_*(Y) \arrow[r, "\cdot"] & \pi_*(X \wedge_R Y)
\end{tikzcd}$$

commutes, which is clear. □

I.4 Sequences of spectra and spectral sequences

In this section we associate a Cartan–Eilenberg system, an exact couple, and a spectral sequence to any sequence of orthogonal G-spectra. We identify certain well-behaved sequences called filtrations, and use these to show how pairings of sequences induce pairings of Cartan–Eilenberg systems and spectral sequences. This is essentially the content of Section I.4.5 and Section I.4.6, culminating in Theorem I.4.26 and Theorem I.4.27. We shall rely on the classical telescope construction to approximate general sequences by equivalent filtrations. Finally we discuss how pairings can be internalized in terms of the convolution product of two sequences.

I.4.1 Cartan–Eilenberg systems

A Cartan–Eilenberg system is an algebraic structure, introduced in [CE56], which determines two exact couples [Mas52] and a spectral sequence. This structure has the advantage that one can give a useful definition of a pairing of Cartan–Eilenberg systems, which determines a pairing of the corresponding spectral sequences. These definitions were reviewed by Douady in [Dou59a] and [Dou59b]. As opposed to these sources, which use cohomological indexing, we adopt homological indexing for our Cartan–Eilenberg systems, as in [HR19].

We start with some preliminary definitions. We will in particular make use of the posets $[1] = \{0 \to 1\}$ and $[2] = \{0 \to 1 \to 2\}$ regarded as categories. Note that we have three functors

$$\delta_0, \delta_1, \delta_2 \colon [1] \longrightarrow [2]\,,$$

with subscript indicating which object of the target is skipped. In addition, we have natural transformations

$$i \colon \delta_2 \longrightarrow \delta_1 \quad \text{and} \quad p \colon \delta_1 \longrightarrow \delta_0\,.$$

Let $(\mathcal{I}, \leq)$ be a linearly ordered set. The following definitions can be found in [HR19, Def. 4.1].

Definition I.4.1.

- Let $\mathcal{I}^{[1]} = \mathrm{Fun}([1], \mathcal{I})$. The objects in this category are pairs (i, j) in $\mathcal{I}$ with $i \leq j$, and we have a single morphism $(i, j) \to (i', j')$ precisely when $i \leq i'$ and $j \leq j'$.
- Let $\mathcal{I}^{[2]} = \mathrm{Fun}([2], \mathcal{I})$. The objects in this category are triples (i, j, k) in $\mathcal{I}$ with $i \leq j \leq k$, and we have a single morphism $(i, j, k) \to (i', j', k')$ precisely when $i \leq i'$, $j \leq j'$ and $k \leq k'$.

The functors δ_0, δ_1, and δ_2 defined above induce functors

$$d_0, d_1, d_2 \colon \mathcal{I}^{[2]} \longrightarrow \mathcal{I}^{[1]}\,.$$

These map (i, j, k) to (j, k), (i, k) and (i, j), respectively. The natural transformations i and p induce natural transformations

$$\iota \colon d_2 \longrightarrow d_1 \quad \text{and} \quad \pi \colon d_1 \longrightarrow d_0$$

with components $\iota \colon (i, j) \to (i, k)$ and $\pi \colon (i, k) \to (j, k)$, respectively.

Let k be a graded ring and let $\mathscr{A}$ be the graded abelian category of k-modules. The grading $\|x\|$ of a homogeneous element $x \in M$ in an object M of $\mathscr{A}$ will be referred to as its **total degree**.

Definition I.4.2 ([HR19, Def. 4.2, Def. 6.1])**.** An **$\mathcal{I}$-system** in $\mathscr{A}$ is a pair (H, ∂) where $H : \mathcal{I}^{[1]} \to \mathscr{A}$ is a functor and $\partial \colon Hd_0 \to Hd_2$ is a natural transformation of functors $\mathcal{I}^{[2]} \to \mathscr{A}$, such that the triangle

$$\begin{tikzcd} Hd_2 \arrow[r, "H\iota"] & Hd_1 \arrow[d, "H\pi"] \\ & Hd_0 \arrow[ul, "\partial"] \end{tikzcd}$$

is exact. We assume that $H\iota$ and $H\pi$ have total degree 0, while ∂ has total degree -1. We generically write $\eta \colon H(i,j) \to H(i',j')$ for the total degree 0 morphisms in $\mathscr{A}$ induced by morphisms in $\mathcal{I}^{[1]}$.

Definition I.4.3.

- A **finite Cartan–Eilenberg system** is a $\mathbb{Z}$-system (H, ∂), where $\mathbb{Z}$ denotes the integers with its usual linear ordering.
- An **extended Cartan–Eilenberg system** is a $\mathcal{I}$-system (H, ∂) for $\mathcal{I} = \mathbb{Z} \cup \{\pm\infty\}$, with the extended linear ordering where $-\infty$ is initial and $+\infty$ is terminal.
- A **Cartan–Eilenberg system** is an extended Cartan–Eilenberg system (H, ∂) such that the following condition, called (SP.5), is satisfied: The canonical homomorphism
$$\operatorname*{colim}_j H(i,j) \xrightarrow{\cong} H(i,\infty)$$
is an isomorphism for all $i \in \mathbb{Z}$.

An extended Cartan–Eilenberg system thus associates to each pair (i,j) with $-\infty \le i \le j \le \infty$ a module $H(i,j)$, in a functorial way. Furthermore, it associates to each triple (i,j,k) with $-\infty \le i \le j \le k \le \infty$ a long exact sequence

$$\ldots \longrightarrow H(i,j) \longrightarrow H(i,k) \longrightarrow H(j,k) \xrightarrow{\partial} H(i,j) \longrightarrow \ldots ,$$

where ∂ is a natural transformation of total degree -1. If the homomorphism in condition (SP.5) is an isomorphism for one $-\infty \le i < \infty$, then it is an isomorphism for every such i. This follows by using the 5-Lemma twice in the following map of exact sequences:

$$\begin{tikzcd} \ldots \arrow[r] & H(-\infty,i) \arrow[r] \arrow[d, "="] & \operatorname{colim}_j H(-\infty,j) \arrow[r] \arrow[d] & \operatorname{colim}_j H(i,j) \arrow[r, "\partial"] \arrow[d] & H(-\infty,i) \arrow[r] \arrow[d, "="] & \ldots \\ \ldots \arrow[r] & H(-\infty,i) \arrow[r] & H(-\infty,\infty) \arrow[r] & H(i,\infty) \arrow[r, "\partial"] & H(-\infty,i) \arrow[r] & \ldots . \end{tikzcd}$$

An extended Cartan–Eilenberg system determines a finite Cartan–Eilenberg system by restriction to (i,j) with $-\infty < i \le j < \infty$.

Remark I.4.4. Apart from the switch in variance, the definition given by Cartan and Eilenberg in [CE56, §XV.7] corresponds to our Cartan–Eilenberg systems. This is also the definition recalled in [McC01, Ex. 2.2]. In [Dou59a, § II C], Douady works with data defining an Adams spectral sequence, which is concentrated in non-negative cohomological (so: non-positive homological) filtration degrees. He therefore assumes that $H(i,0) = H(i,j)$ for all $i \leq 0 \leq j \leq \infty$, so that condition (SP.5) is trivially satisfied.

Definition I.4.5 ([HR19, Def. 7.1])**.** Let (H, ∂) be a Cartan–Eilenberg system. Let the **left couple** (A, E^1) be the exact couple given by

$$A_s = H(-\infty, s) \quad \text{and} \quad E^1_s = H(s-1, s)$$

fitting together in the exact triangle

$$\begin{tikzcd} A_{s-1} \arrow[r] & A_s \arrow[d] \\ & E^1_s \arrow[ul, "\partial"] \end{tikzcd}$$

associated to the triple $(-\infty, s-1, s)$. The **abutment** of this exact couple is

$$A_\infty = \operatorname*{colim}_s A_s \cong H(-\infty, \infty)\,.$$

This abutment is exhaustively filtered by the images

$$F_s A_\infty = \operatorname{im}(A_s \longrightarrow A_\infty)\,.$$

The Cartan–Eilenberg system and the left couple give rise to the same spectral sequence (E^r, d^r). The pages of this spectral sequence are given by

$$E^r_s = Z^r_s / B^r_s$$

and the differentials $d^r_s \colon E^r_s \to E^r_{s-r}$ are of total degree -1. Here

$$\begin{aligned} Z^r_s &= \ker(\partial \colon E^1_s \longrightarrow H(s-r, s-1)) \\ B^r_s &= \operatorname{im}(\partial \colon H(s, s+r-1) \longrightarrow E^1_s) \end{aligned}$$

define the r-cycles and r-boundaries in **filtration degree** s, respectively, and

$$d^r_s([x]) = [\partial(\tilde{x})]\,,$$

where $x \in Z^r_s$ and $\tilde{x} \in H(s-r, s)$ satisfies $\eta(\tilde{x}) = x$. There are preferred isomorphisms $H(E^r, d^r) \cong E^{r+1}$. We let

$$Z^\infty_s = \lim_r Z^r_s\,, \quad B^\infty_s = \operatorname*{colim}_r B^r_s\,, \quad \text{and} \quad E^\infty_s = Z^\infty_s / B^\infty_s\,.$$

We refer to [CE56, §XV.1] or [HR19, Prop. 4.9] for the verification that (E^r, d^r) is indeed a spectral sequence. Note in particular that it only depends on the finite part of the Cartan–Eilenberg system (H, ∂). The abutment and E^∞-page are related as follows.

Lemma I.4.6. *There is a natural monomorphism*

$$\beta\colon \frac{F_s A_\infty}{F_{s-1} A_\infty} \longrightarrow E_s^\infty$$

in each filtration degree s.

Proof. See [Boa99, Lem. 5.6] or [HR19, Lem. 3.15(a)]. □

The main purpose of reviewing the above definitions is to let us record the following definitions of pairings of (finite and classical) Cartan–Eilenberg systems. We assume that k is graded commutative, and write $\otimes$ in place of $\otimes_k$.

Definition I.4.7. Let (H', ∂), (H'', ∂) and (H, ∂) be finite Cartan–Eilenberg systems in $\mathscr{A}$. A **pairing** $\phi : (H', H'') \to H$ of such systems is a collection of k-module homomorphisms

$$\phi_r\colon H'(i-r, i) \otimes H''(j-r, j) \longrightarrow H(i+j-r, i+j)$$

of total degree 0, for all $i, j \in \mathbb{Z}$ and $r \geq 1$. These are required to satisfy the following two conditions:

SSP I Each square

$$\begin{tikzcd}
H'(i-r,i) \otimes H''(j-r,j) \arrow[r, "\phi_r"] \arrow[d, "\eta\otimes\eta"'] & H(i+j-r,i+j) \arrow[d, "\eta"] \\
H'(i'-r',i') \otimes H''(j'-r',j') \arrow[r, "\phi_{r'}"] & H(i'+j'-r',i'+j')
\end{tikzcd}$$

commutes, for all integers i, j, i', j' and $r, r' \geq 1$ with $i \leq i'$, $i-r \leq i'-r'$, $j \leq j'$ and $j-r \leq j'-r'$.

SSP II In the (non-commutative) diagram

$$\begin{tikzcd}
H'(i-r,i) \otimes H''(j-r,j) \arrow[rr, "\eta\otimes\partial"] \arrow[dr, "\phi_r"] \arrow[dd, "\partial\otimes\eta"'] & & H'(i-1,i) \otimes H''(j-r-1,j-r) \arrow[dd, "\phi_1"] \\
& H(i+j-r,i+j) \arrow[dr, "\partial"] & \\
H'(i-r-1,i-r) \otimes H''(j-1,j) \arrow[rr, "\phi_1"] & & H(i+j-r-1,i+j-r)
\end{tikzcd}$$

the inner composition is the sum of the two outer ones:

$$\partial\phi_r = \phi_1(\partial \otimes \eta) + \phi_1(\eta \otimes \partial)\,.$$

In terms of elements, this identity in $H(i+j-r-1, i+j-r)$ can be written

$$\partial\phi_r(x \otimes y) = \phi_1(\partial x \otimes \eta y) + (-1)^{\|x\|}\phi_1(\eta x \otimes \partial y)$$

for $x \in H'(i-r, i)$ of total degree $\|x\|$ and $y \in H''(j-r, j)$.

Remark I.4.8. Apart from the switch in variance, this definition agrees with that of Douady [Dou59b, § II A], except for the fact that we ignore $r = 0$, since ϕ_0 carries no information, and Douady omits the cases $i > 0$ and $j > 0$, due to his focus on Adams spectral sequences. In the definition given in [McC01, Ex. 2.3], the homomorphism φ_1 is missing from the right hand term in his equation (2), and the conditions $n \geq 0$ and $q \geq 0$ should be omitted.

Definition I.4.9. Let $('E^r, d^r)$, $(''E^r, d^r)$ and (E^r, d^r) be k-module spectral sequences. A **pairing** $\phi\colon ('E^*, ''E^*) \to E^*$ of such spectral sequences consists of a collection of k-module homomorphisms

$$\phi^r : 'E^r \otimes ''E^r \longrightarrow E^r$$

for all $r \geq 1$, such that:

1. The Leibniz rule

$$d^r \phi^r = \phi^r(d^r \otimes 1) + \phi^r(1 \otimes d^r)$$

holds as an equality of homomorphisms $'E^r_i \otimes ''E^r_j \longrightarrow E^r_{i+j-r}$ for all $i, j \in \mathbb{Z}$ and $r \geq 1$.

2. The diagram

$$\begin{tikzcd} 'E^{r+1} \otimes ''E^{r+1} \arrow[r, "\phi^{r+1}"] \arrow[d] & E^{r+1} \arrow[d, "\cong"] \\ H('E^r \otimes ''E^r) \arrow[r, "H(\phi^r)"] & H(E^r) \end{tikzcd}$$

commutes for all $r \geq 1$.

By a **multiplicative spectral sequence**, we mean a spectral sequence (E^r, d^r) equipped with a pairing

$$\phi\colon (E^*, E^*) \longrightarrow E^* .$$

If $\phi^a : E^a \otimes E^a \to E^a$ (usually with $a = 1$ or $a = 2$) is associative and unital, then each pairing ϕ^r for $r \geq a$ is also associative and unital, and we call $(E^r, d^r)_{r\geq a}$ an **algebra spectral sequence**.

Theorem I.4.10 ([Dou59b, Thm. II A])**.** *Let (H', ∂), (H'', ∂) and (H, ∂) be finite Cartan–Eilenberg systems, with associated spectral sequences $('E^r, d^r)$, $(''E^r, d^r)$ and (E^r, d^r). Let $\phi : (H', H'') \to H$ be a pairing of finite Cartan–Eilenberg systems. Then there is a pairing $\phi\colon ('E^*, ''E^*) \to E^*$ of spectral sequences, uniquely defined by the condition $\phi^1 = \phi_1$.*

Proof. Douady leaves the proof to the reader ("s'il existe"). Starting with setting $\phi^1 : 'E^1_i \otimes ''E^1_j \to E^1_{i+j}$ equal to

$$\phi_1 : 'H(i-1, i) \otimes ''H(j-1, j) \longrightarrow H(i+j-1, i+j)\,,$$

the point is to inductively show that d^r satisfies the Leibniz rule with respect to the pairing ϕ^r of E^r-pages, so that ϕ^{r+1} can be defined to be equal to the induced pairing in homology with respect to d^r. A full proof can be found in [Hel17, Prop. 3.4.2]. □

We now move from finite Cartan–Eilenberg systems to classical ones.

Definition I.4.11. Let (H', ∂), (H'', ∂) and (H, ∂) be Cartan–Eilenberg systems. A **pairing** $\phi\colon (H', H'') \to H$ of such systems consists of a pairing $(\phi_r)_{r\geq 1}$ of the restricted finite Cartan–Eilenberg systems, together with k-module homomorphisms

$$\phi_\infty \colon H'(-\infty, i) \otimes H''(-\infty, j) \longrightarrow H(-\infty, i+j)$$

of total degree 0, for all $i, j \in \mathbb{Z}$. These are required to satisfy the following additional condition:

SPP III Each square

$$\begin{tikzcd}
H'(-\infty, i) \otimes H''(-\infty, j) \arrow[r, "\phi_\infty"] \arrow[d, "\eta\otimes\eta"'] & H(-\infty, i+j) \arrow[d, "\eta"] \\
H'(i-r, i) \otimes H''(j-r, j) \arrow[r, "\phi_r"] & H(i+j-r, i+j)
\end{tikzcd}$$

commutes, for all integers i, j and $r \geq 1$.

We emphasize that the ϕ_r in the definition above must satisfy (SPP I) and (SPP II), by virtue of defining a pairing of finite Cartan–Eilenberg systems. The new condition (SPP III) is an analogue of (SPP I) for $r = \infty$.

With notation as in Definition I.4.5, we can rewrite ϕ_∞ as compatible pairings

$$\phi_{i,j} \colon A'_i \otimes A''_j \longrightarrow A_{i+j}$$

in the corresponding left couples, for all $i, j \in \mathbb{Z}$. Passing to colimits, we obtain a pairing of abutments

$$\phi_* \colon A'_\infty \otimes A''_\infty \longrightarrow A_\infty\,.$$

This is filtration-preserving in the sense that it sends $F_iA'_\infty \otimes F_jA''_\infty$ to $F_{i+j}A_\infty$, by virtue of the commutative diagram

$$\begin{tikzcd}
A'_i \otimes A''_j \arrow[r, "\phi_{i,j}"] \arrow[d, two heads] & A_{i+j} \arrow[d, two heads] \\
F_iA'_\infty \otimes F_jA''_\infty \arrow[r] \arrow[d] & F_{i+j}A_\infty \arrow[d, tail] \\
A'_\infty \otimes A''_\infty \arrow[r, "\phi_*"] & A_\infty\,.
\end{tikzcd}$$

Being filtration-preserving, the pairing ϕ_* then induces pairings of filtration subquotients

$$\bar{\phi}_* \colon \frac{F_i A'_\infty}{F_{i-1} A'_\infty} \otimes \frac{F_j A''_\infty}{F_{j-1} A''_\infty} \longrightarrow \frac{F_{i+j} A_\infty}{F_{i+j-1} A_\infty}$$

for all $i, j \in \mathbb{Z}$.

In a similar way, the spectral sequence pairing $\phi = (\phi^r)_{r\geq 1}$, induced by the pairing $(\phi_r)_{r\geq 1}$ per Theorem I.4.10, maps

$$\begin{aligned} {}'Z^r \otimes {}''Z^r &\longrightarrow Z^r \,, \\ {}'B^r \otimes {}''Z^r &\longrightarrow B^r \,, \\ {}'Z^r \otimes {}''B^r &\longrightarrow B^r \,, \end{aligned}$$

hence also maps

$$\begin{aligned} {}'Z^\infty \otimes {}''Z^\infty &\longrightarrow Z^\infty \,, \\ {}'B^\infty \otimes {}''Z^\infty &\longrightarrow B^\infty \,, \\ {}'Z^\infty \otimes {}''B^\infty &\longrightarrow B^\infty \,. \end{aligned}$$

It follows that ϕ also induces k-module homomorphisms

$$\phi^\infty \colon {}'E^\infty_i \otimes {}''E^\infty_j \longrightarrow E^\infty_{i+j} \,, \tag{I.2}$$

sending $[x] \otimes [y]$ to $[\phi^1(x \otimes y)]$ for any pair of infinite cycles x and y. Condition (SPP III) ensures that we have the following compatibility.

Proposition I.4.12. *Let $\phi = (\phi_r) \colon (H', H'') \to H$ be a pairing of Cartan–Eilenberg systems, with induced pairing $\phi = (\phi^r) \colon ({}'E^*, {}''E^*) \to E^*$ of spectral sequences, per Theorem I.4.10. Then the pairing ϕ_* of filtered abutments is compatible with the pairing ϕ^∞ of E^∞-pages, in the sense that the diagram*

$$\begin{tikzcd} \dfrac{F_i A'_\infty}{F_{i-1} A'_\infty} \otimes \dfrac{F_j A''_\infty}{F_{j-1} A''_\infty} \arrow[r, "\bar{\phi}_*"] \arrow[d, "\beta\otimes\beta"'] & \dfrac{F_{i+j} A_\infty}{F_{i+j-1} A_\infty} \arrow[d, tail, "\beta"] \\ {}'E^\infty_i \otimes {}''E^\infty_j \arrow[r, "\phi^\infty"] & E^\infty_{i+j} \end{tikzcd}$$

commutes, for all $i, j \in \mathbb{Z}$.

Proof. A detailed proof is given in [Hel17, Prop. 3.4.4]. □

Remark I.4.13. As a consequence of Theorem I.4.10, if (H, ∂) is a multiplicative Cartan–Eilenberg system, meaning that it is equipped with a pairing $\phi : (H, H) \to H$, then the associated spectral sequence (E^r, d^r) is also multiplicative. Moreover, Proposition I.4.12 tells us that the induced pairing on the filtered abutment A_∞ is compatible with the induced pairing on the E^∞-page of the spectral sequence. In this situation, we say that A_∞ is a **multiplicative abutment**. When (E^r, d^r) converges weakly to A_∞, meaning that the filtration $(F_s A_\infty)_s$ is exhaustive and β is an isomorphism, multiplicativity of the abutment means that we can reconstruct the product ϕ_* on A_∞ from the product ϕ^∞ on E^∞, up to the usual ambiguity created by extensions.

I.4.2 Sequences

Our Cartan–Eilenberg systems will in practice be obtained from filtrations and sequences. Let us first set up some terminology, so that it is clear what we are discussing. Again, G denotes a compact Lie group.

Definition I.4.14. A sequential diagram $X_\star$ of orthogonal G-spectra and G-maps on the form

$$\cdots \longrightarrow X_{i-1} \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots ,$$

indexed over $i \in \mathbb{Z}$, is called a **sequence**.

We can extend the sequence to be indexed over $\mathbb{Z} \cup \{\pm\infty\}$ by setting

$$X_{-\infty} = * \quad \text{and} \quad X_\infty = \mathrm{Tel}(X_\star)\,,$$

where

$$\mathrm{Tel}(X_\star) = \bigvee_{i\in\mathbb{Z}} [i, i+1]_+ \wedge X_i/{\sim}$$

is the classical **telescope construction**. Here, the equivalence relation $\sim$ is given by identifying $\{i\}_+ \wedge X_{i-1}$ with $\{i\}_+ \wedge X_i$ using the G-map $X_{i-1} \to X_i$. There are standard inclusions $X_i \cong \{i\}_+ \wedge X_i \subset \mathrm{Tel}(X_\star)$ for all $i \in \mathbb{Z}$, and each diagram

$$\begin{tikzcd}
X_{i-1} \arrow[r] \arrow[dr] & X_i \arrow[d] \\
 & \mathrm{Tel}(X_\star)
\end{tikzcd}$$

commutes up to preferred homotopy.

Definition I.4.15. The Cartan–Eilenberg system $(H = H(X_\star), \partial)$ associated to the sequence $X_\star$ of orthogonal G-spectra is given by

$$H(i,j) = \pi_*^G(X_i \longrightarrow X_j)$$

for all $-\infty \le i \le j \le \infty$, and

$$\partial\colon \pi_*^G(X_j \to X_k) \longrightarrow \pi_{*-1}^G(X_i \to X_j)$$

for all $-\infty \le i \le j \le k \le \infty$.

Let us elaborate on the definition above. In the $q \geq 0$ case,

$$H(i,j) = \pi_q^G(X \to Y)$$

denotes the colimit, over the partially ordered set of finite-dimensional G-subrepresentations V of the complete G-universe $\mathscr{U}$, of the groups of homotopy classes $[f', f]$ of pairs (f', f) of G-maps $f' \colon \Sigma^V S^{q-1} \to X(V)$ and $f \colon \Sigma^V D^q \to Y(V)$ making the square

$$\begin{tikzcd}
\Sigma^V S^{q-1} \arrow[r] \arrow[d, "f'"'] & \Sigma^V D^q \arrow[d, "f"] \\
X(V) \arrow[r] & Y(V)
\end{tikzcd}$$

commute. Similar definitions can be made for $q \leq 0$, but will be left to the reader. By the stability of the homotopy category of orthogonal G-spectra there is a natural isomorphism

$$\pi_q^G(X \to Y) \cong \pi_q^G(Y \cup CX)\,,$$

where $Y \cup CX$ is the mapping cone of $X \to Y$. This isomorphism takes the homotopy class $[f', f]$ to (the image in the colimit over V of) the homotopy class of the composite map

$$\Sigma^V S^q \xrightarrow{\simeq} \Sigma^V (D^q \cup CS^{q-1}) \xrightarrow{f \cup Cf'} (Y \cup CX)(V)\,,$$

where the first map is a (V-suspended) homotopy inverse to the collapse map $D^q \cup CS^{q-1} \to D^q/S^{q-1} \cong S^q$.

The connecting homomorphism $\partial \colon \pi_q^G(Y \to Z) \to \pi_{q-1}^G(X \to Y)$ mentioned in the definition takes the homotopy class $[g', g]$ of a pair of G-maps $g' \colon \Sigma^V S^{q-1} \to Y(V)$ and $g \colon \Sigma^V D^q \to Z(V)$ to the homotopy class $[*, g'\pi]$ of the maps

$$* \colon \Sigma^V S^{q-2} \longrightarrow * \longrightarrow X(V) \quad \text{and} \quad g'\pi \colon \Sigma^V D^{q-1} \xrightarrow{\pi} \Sigma^V S^{q-1} \longrightarrow Y(V)\,,$$

where $\pi \colon D^{q-1} \to S^{q-1}$ identifies D^{q-1}/S^{q-2} with S^{q-1}. The diagram

$$\begin{tikzcd}
\Sigma^V S^{q-2} \arrow[r] \arrow[d] & \Sigma^V D^{q-1} \arrow[d, "\pi"] \\
* \arrow[r] \arrow[d] & \Sigma^V S^{q-1} \arrow[d, "g'"] \\
X(V) \arrow[r] & Y(V)
\end{tikzcd}$$

evidently commutes. Under the isomorphism $\pi_{q-1}^G(X \to Y) \cong \pi_{q-1}^G(Y \cup CX)$ the homotopy class $[*, g'\pi]$ corresponds to the homotopy class of the composite map

$$\Sigma^V S^{q-1} \xrightarrow{g'} Y(V) \longrightarrow (Y \cup CX)(V)\,.$$

Note that the graded abelian group $\pi_*^G(X_i \to X_j)$ is functorial in $i \le j$, the homomorphism ∂ is natural in $i \le j \le k$, and the sequence

$$\cdots \longrightarrow \pi_q^G(X_i \to X_j) \longrightarrow \pi_q^G(X_i \to X_k) \longrightarrow \pi_q^G(X_j \to X_k) \xrightarrow{\partial} \pi_{q-1}^G(X_i \to X_j) \longrightarrow \cdots$$

is exact for all $i \le j \le k$ and $q \in \mathbb{Z}$. The canonical homomorphism

$$\operatorname*{colim}_j \pi_*^G(X_j) \xrightarrow{\cong} \pi_*^G \operatorname{Tel}(X_\star)$$

is an isomorphism, which implies that condition (SP.5) is satisfied. Hence (H, ∂) is indeed a Cartan–Eilenberg system in the sense of Definition I.4.3.

We can extract two different exact couples [Mas52] from $(H(X_\star), \partial)$, but shall only be concerned with the 'left' couple of Definition I.4.5. Explicitly, the exact couple $(A, E^1) = (A(X_\star), E^1(X_\star))$ associated to $X_\star$ is given by

$$A_s = \pi_*^G(X_s) \quad \text{and} \quad E_s^1 = \pi_*^G(X_{s-1} \to X_s)\,,$$

fitting together in the exact triangle

$$\begin{tikzcd}
\pi_*^G(X_{s-1}) \arrow[rr] & & \pi_*^G(X_s) \arrow[d] \\
& & \pi_*^G(X_{s-1} \to X_s) \arrow[llu, "\partial"]
\end{tikzcd}$$

where ∂ has total degree -1.

Recall from [Boa99, Def. 5.10] that the spectral sequence associated to the unrolled exact couple (A, E^1) is said to be **conditionally convergent** to the abutment

$$A_\infty = \operatorname*{colim}_s \pi_*^G(X_s) \cong \pi_*^G \operatorname{Tel}(X_\star) \tag{I.3}$$

if and only if

$$A_{-\infty} = \lim_s A_s = 0 \quad \text{and} \quad RA_{-\infty} = \operatorname*{Rlim}_s A_s = 0\,.$$

Here $\operatorname{Rlim} = \lim^1$ denotes the (first right) derived limit of a sequence. In view of the short exact sequence

$$0 \longrightarrow \operatorname*{Rlim}_s \pi_{*+1}^G(X_s) \longrightarrow \pi_*^G(\operatorname*{holim}_s X_s) \longrightarrow \lim_s \pi_*^G(X_s) \longrightarrow 0$$

this is equivalent to the condition that

$$\pi_*^G(\operatorname*{holim}_s X_s) = 0\,.$$

In particular, conditional convergence holds if $\operatorname{holim}_s X_s \simeq_G *$.

The **spectral sequence** $(E^r = E^r(X_\star), d^r)_{r \ge 1}$ associated to the sequence $X_\star$ (and the Cartan–Eilenberg system $(H(X_\star), \partial)$, and the exact couple $(A(X_\star), E^1(X_\star))$) has

$$E_{s,t}^1 = \pi_{s+t}^G(X_{s-1} \to X_s)$$

and $d^1 \colon E^1_{s,t} \to E^1_{s-1,t}$ is equal to the composite homomorphism

$$\pi^G_{s+t}(X_{s-1} \to X_s) \xrightarrow{\partial} \pi^G_{s+t-1}(X_{s-1}) \longrightarrow \pi^G_{s+t-1}(X_{s-2} \to X_{s-1}) .$$

Here $s+t$ is the **total degree**, s is the **filtration degree**, and t will be called the **internal degree**. The d^r-differentials have the form

$$d^r \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$$

and there are preferred isomorphisms $H(E^r, d^r) \cong E^{r+1}$ for all $r \geq 1$.

I.4.3 Filtrations

The category of orthogonal G-spectra is based topological, meaning that it is enriched in the closed symmetric monoidal category of compactly generated weak Hausdorff spaces with base point.

Definition I.4.16. Let $I = [0, 1]$, with boundary $\partial I = \{0, 1\}$.

- A G-map $i \colon A \to X$ of orthogonal G-spectra is an ***h*-cofibration** (= Hurewicz cofibration) if it has the homotopy extension property with respect to any target Z:

$$\begin{tikzcd} X \cup_A A \wedge I_+ \arrow[r] \arrow[d] & Z \\ X \wedge I_+ \arrow[ur, dashed] & \end{tikzcd} .$$

- A G-map $p \colon E \to B$ of orthogonal G-spectra is an ***h*-fibration** (= Hurewicz fibration) if it has the homotopy lifting property with respect to any source X:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \wedge I_+ \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} .$$

- Adapting [SV02, Def. 2.4], we say that $i \colon A \to X$ is a **strong *h*-cofibration** if the G-map $X \cup_A A \wedge I_+ \to X \wedge I_+$ has the left lifting property with respect to any h-fibration:

$$\begin{tikzcd} X \cup_A A \wedge I_+ \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \wedge I_+ \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} .$$

Strong h-cofibrations are closed under cobase change, retracts, arbitrary sums, and sequential colimits [SV02, Lem. 2.6]. For each map $f\colon X \to Y$ the inclusion $i_0\colon X \to Y \cup_X X \wedge I_+$ is a strong h-cofibration [SV02, Rmk. 3.3(2)]. It follows that each q-cofibration (= Quillen cofibration, [MM02, Def. III.2.3]) is a strong h-cofibration. Each strong h-cofibration is evidently an h-cofibration. Our main reason for working with strong h-cofibrations is the availability of the following theorem.

Theorem I.4.17 ([SV02, Theorem 2.7])**.** *If $i\colon A \to X$ and $j\colon B \to Y$ are strong h-cofibrations, then the pushout-product map*

$$i \wedge 1 \cup 1 \wedge j\colon A \wedge Y \cup_{A \wedge B} X \wedge B \longrightarrow X \wedge Y$$

is a strong h-cofibration.

We can now specify well-behaved sequences, called filtrations, for which we can directly prove that pairings of sequences induce pairings of Cartan–Eilenberg systems and of spectral sequences.

Definition I.4.18. Let $X_\star$ be a sequence of orthogonal G-spectra. We say that $X_\star$ is a **filtration** if each G-map $X_{i-1} \to X_i$ for $i \in \mathbb{Z}$ is a strong h-cofibration.

In particular, if $X_\star$ is a filtration, then the G-maps are all h-cofibrations, so the canonical maps

$$X_j \cup CX_i \longrightarrow X_j/X_i \quad \text{and} \quad \mathrm{Tel}(X_\star) \longrightarrow \operatorname*{colim}_i X_i = \bigcup_i X_i$$

are G-equivalences, so that

$$H(i,j) \cong \pi_*^G(X_j/X_i) \quad \text{and} \quad A_\infty \cong \pi_*^G\left(\bigcup_i X_i\right)$$

in the associated Cartan–Eilenberg system.

We can always approximate a sequence $X_\star$ with an equivalent filtration $T_\star(X)$. To do this, we proceed as follows. For each integer j we let

$$T_j(X) = \{j\}_+ \wedge X_j \vee \bigvee_{i<j} [i, i+1]_+ \wedge X_i/\sim$$

be the subspectrum of $\mathrm{Tel}(X_\star)$ with telescope coordinate in the interval $(-\infty, j]$ within the real line $(-\infty, \infty) = \bigcup_i [i, i+1]$. The sequence $T_\star(X)$ given by the inclusions

$$\dots \longrightarrow T_{j-1}(X) \longrightarrow T_j(X) \longrightarrow T_{j+1}(X) \longrightarrow \dots$$

is then a filtration.

For each integer j there is a deformation retraction

$$\epsilon_j\colon T_j(X) \longrightarrow X_j$$

identifying $\{j\}_+ \wedge X_j$ with X_j and mapping $[i, i+1]_+ \wedge X_i$ to X_j by the evident composition $[i, i+1]_+ \wedge X_i \to X_i \to X_j$, for each $i < j$. The resulting diagram

$$\begin{tikzcd}
\dots \arrow[r] & T_{j-1}(X) \arrow[r] \arrow[d, "\simeq_G"'] & T_j(X) \arrow[r] \arrow[d, "\simeq_G"'] & T_{j+1}(X) \arrow[r] \arrow[d, "\simeq_G"'] & \dots \\
\dots \arrow[r] & X_{j-1} \arrow[r] & X_j \arrow[r] & X_{j+1} \arrow[r] & \dots
\end{tikzcd}$$

commutes, and defines a G-equivalence of sequences $\epsilon \colon T_\star(X) \to X_\star$. It follows that the associated maps

$$* = T_{-\infty}(X) \longrightarrow X_{-\infty} = * \quad \text{and} \quad T_\infty(X) \longrightarrow X_\infty = \mathrm{Tel}(X_\star)$$

are both G-equivalences. Hence the induced maps of Cartan–Eilenberg systems

$$H(T_\star(X))(i,j) = \pi_*^G(T_i(X) \longrightarrow T_j(X)) \xrightarrow{\cong} \pi_*^G(X_i \to X_j) = H(X_\star)(i,j)\,,$$

and of spectral sequences

$$(E^r(T_\star(X)), d^r)_{r\geq 1} \xrightarrow{\cong} (E^r(X_\star), d^r)_{r \geq 1}\,,$$

are both isomorphisms. Their common abutment for convergence to the colimit is $A_\infty(X_\star) \cong \pi_*^G \mathrm{Tel}(X_\star)$, filtered by the image subsequence

$$F_s A_\infty(X_\star) \cong F_s \pi_*^G \mathrm{Tel}(X_\star) = \mathrm{im}(\pi_*^G(X_s) \to \pi_*^G \mathrm{Tel}(X_\star))\,.$$

Remark I.4.19. Some form of cofibrant replacement of maps is necessary to convert general sequences to filtrations. We have chosen to use mapping cylinders and telescopes, which have convenient monoidal properties. For finite groups, Hesselholt and Madsen [HM03, §4.3] instead use a functorial G-CW replacement to convert G-spectra to G-CW spectra. There exists a functorial G-CW replacement also for compact Lie groups [Sey83], but its construction is comparatively intricate, and the monoidal properties are less clear, which may partially justify our choice.

I.4.4 Pairings of sequences

We now turn to discussing pairings of sequences and how these behave under passage to mapping telescopes.

Definition I.4.20. Let $X_\star$, $Y_\star$ and $Z_\star$ be sequences of orthogonal G-spectra. A **pairing** $\phi \colon (X_\star, Y_\star) \to Z_\star$ is a collection of G-maps

$$\phi_{i,j} \colon X_i \wedge Y_j \longrightarrow Z_{i+j}$$

for all integers i and j, making the squares

$$\begin{tikzcd}
X_{i-1} \wedge Y_j \arrow[r, "\phi_{i-1,j}"] \arrow[d] & Z_{i+j-1} \arrow[d] & X_i \wedge Y_{j-1} \arrow[l, "\phi_{i,j-1}"'] \arrow[d] \\
X_i \wedge Y_j \arrow[r, "\phi_{i,j}"] & Z_{i,j} & X_i \wedge Y_j \arrow[l, "\phi_{i,j}"']
\end{tikzcd} \tag{I.4}$$

commute. We say that a sequence $X_\star$ is **multiplicative** if it comes equipped with a pairing $\phi\colon (X_\star, X_\star) \to X_\star$.

We note that, from [MM02, §II.3] and [Sch18, §3.5], the smash product $X_i \wedge Y_j$ of orthogonal G-spectra is defined in such a way that $\phi_{i,j}$ associates to each pair of G-representations $U$and V a G-map of based G-spaces

$$\phi_{i,j}(U,V)\colon X_i(U) \wedge Y_j(V) \longrightarrow Z_{i+j}(U \oplus V)\,,$$

subject to bilinearity relations for varying U and V.

Lemma I.4.21. *A pairing of sequences $\phi\colon (X_\star, Y_\star) \to Z_\star$ induces a pairing of sequences $T(\phi)\colon (T_\star(X), T_\star(Y)) \to T_\star(Z)$, such that the diagram*

$$\begin{tikzcd}
T_i(X) \wedge T_j(Y) \arrow[r, "T(\phi)_{i,j}"] \arrow[d, "\simeq_G"'] & T_{i+j}(Z) \arrow[d, "\simeq_G"] \\
X_i \wedge Y_j \arrow[r, "\phi_{i,j}"] & Z_{i+j}
\end{tikzcd}$$

commutes for all integers i and j.

Proof. Given a pairing ϕ of sequences, note that we can form G-maps

$$[i, i+1]_+ \wedge X_i \wedge [j, j+1]_+ \wedge Y_j \longrightarrow [k, k+2]_+ \wedge Z_k \longrightarrow \mathrm{Tel}(Z_\star) \qquad \text{(I.5)}$$

for any integers i and j, with $k = i + j$. Here $[i, i+1] \times [j, j+1] \to [k, k+2]$ sends (x, y) to $x + y$, while $X_i \wedge Y_j \to Z_k$ is given by $\phi_{i,j}$. The second map factors through

$$[k, k+1]_+ \wedge Z_k \cup [k+1, k+2]_+ \wedge Z_{k+1}\,,$$

and is given by $Z_k \to Z_{k+1}$ on $[k+1, k+2] \subset [k, k+2]$. The maps (I.5) for varying i and j are compatible with the identifications defining $\mathrm{Tel}(X_\star)$ and $\mathrm{Tel}(Y_\star)$, hence combine to define a G-map

$$\mathrm{Tel}(\phi)\colon\ \mathrm{Tel}(X_\star) \wedge \mathrm{Tel}(Y_\star) \longrightarrow \mathrm{Tel}(Z_\star)\,.$$

By construction, it restricts to compatible G-maps

$$T(\phi)_{i,j}\colon T_i(X) \wedge T_j(Y) \longrightarrow T_{i+j}(Z)$$

for all integers i and j, defining the pairing of sequences $T(\phi)\colon (T_\star(X), T_\star(Y)) \to T_\star(Z))$. It is then clear that the square in the lemma commutes, and that the vertical maps are G-equivariant deformation retractions. □

Corollary I.4.22. *If $(X_\star, \phi)$ is a multiplicative sequence, then so is $(T_\star(X), T(\phi))$. Moreover, the equivalence $\epsilon\colon T_\star(X) \to X_\star$ respects the multiplicative structures.*

I.4.5 Pairings of Cartan–Eilenberg systems, I

The goal of the following two sections is to show that a pairing of sequences gives rise to a pairing of the resulting Cartan–Eilenberg systems. By Theorem I.4.10 and Proposition I.4.12 this is enough to guarantee that we have a pairing of the associated spectral sequences in such a way that the induced pairing on filtered abutments is compatible with the pairing on E^∞-pages. Referring back to Definition I.4.7 and Definition I.4.11, we note that there are three things to check. In this section we deal with (SPP I) and (SPP III).

Let $\phi\colon (X_\star, Y_\star) \to Z_\star$ be a pairing of sequences. For integers i, j and r, with $r \geq 1$, we define induced pairings

$$\phi_r \colon H(X_\star)(i-r, i) \otimes H(Y_\star)(j-r, j) \longrightarrow H(Z_\star)(i+j-r, i+j)$$

as homomorphisms

$$\phi_r \colon \pi_p^G(X_{i-r} \to X_i) \otimes \pi_q^G(Y_{j-r} \to Y_j) \longrightarrow \pi_{p+q}^G(Z_{i+j-r} \to Z_{i+j}) \,.$$

Here p and q range over all integers, but for (relative) brevity we concentrate on the case when $p \geq 0$ and $q \geq 0$. Given two pairs of vertical G-maps

$$\begin{tikzcd} \Sigma^U S^{p-1} \arrow[r] \arrow[d, "f'"] & \Sigma^U D^p \arrow[d, "f"] \\ X_{i-r}(U) \arrow[r] & X_i(U) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Sigma^V S^{q-1} \arrow[r] \arrow[d, "g'"] & \Sigma^V D^q \arrow[d, "g"] \\ Y_{j-r}(V) \arrow[r] & Y_j(V) \end{tikzcd}$$

we first form the commutative diagram

$$\begin{tikzcd}
\Sigma^{U\oplus V} S^{p-1} \wedge S^{q-1} \arrow[rr] \arrow[dr] \arrow[dd, "\cong"'] & & \Sigma^{U\oplus V} D^p \wedge S^{q-1} \arrow[dr] \arrow[dd, "\cong"] & \\
& \Sigma^{U\oplus V} S^{p-1} \wedge D^q \arrow[rr] \arrow[dd, "\cong"'] & & \Sigma^{U\oplus V} D^p \wedge D^q \arrow[dd, "\cong"] \\
\Sigma^U S^{p-1} \wedge \Sigma^V S^{q-1} \arrow[rr] \arrow[dr] \arrow[dd, "f' \wedge g'"'] & & \Sigma^U D^p \wedge \Sigma^V S^{q-1} \arrow[dr] \arrow[dd, "f \wedge g'"] & \\
& \Sigma^U S^{p-1} \wedge \Sigma^V D^q \arrow[rr] \arrow[dd, "f' \wedge g"'] & & \Sigma^U D^p \wedge \Sigma^V D^q \arrow[dd, "f \wedge g"] \\
X_{i-r}(U) \wedge Y_{j-r}(V) \arrow[rr] \arrow[dr] \arrow[dd, "\phi_{i-r,j-r}(U,V)"'] & & X_i(U) \wedge Y_{j-r}(V) \arrow[dr] \arrow[dd, "\phi_{i,j-r}(U,V)"] & \\
& X_{i-r}(U) \wedge Y_j(V) \arrow[rr] \arrow[dr, "\phi_{i-r,j}(U,V)"'] & & X_i(U) \wedge Y_j(V) \arrow[d, "\phi_{i,j}(U,V)"] \\
Z_{i+j-2r}(U \oplus V) \arrow[rr] & & Z_{i+j-r}(U \oplus V) \arrow[r] & Z_{i+j}(U \oplus V) \,.
\end{tikzcd}$$

For typographical reasons, we will often suppress the stabilising G-representations U and V and simply display this diagram as

$$\begin{tikzcd}
S^{p-1}\wedge S^{q-1} \arrow[rr] \arrow[dr] \arrow[dd, "f'\wedge g'"'] & & D^p\wedge S^{q-1} \arrow[dr] \arrow[dd, "f\wedge g'"] & \\
& S^{p-1}\wedge D^q \arrow[rr] \arrow[dd, "f'\wedge g"'] & & D^p\wedge D^q \arrow[dd, "f\wedge g"] \\
X_{i-r}\wedge Y_{j-r} \arrow[rr] \arrow[dr] \arrow[dd, "\phi_{i-r,j-r}"'] & & X_i\wedge Y_{j-r} \arrow[dr] \arrow[dd, "\phi_{i,j-r}"] & \\
& X_{i-r}\wedge Y_j \arrow[rr] \arrow[dr, "\phi_{i-r,j}"'] & & X_i\wedge Y_j \arrow[d, "\phi_{i,j}"] \\
Z_{i+j-2r} \arrow[rr] & & Z_{i+j-r} \arrow[r] & Z_{i+j}\,.
\end{tikzcd}$$

Let

$$S^{p+q-1} = S^{p-1}\wedge D^q \cup_{S^{p-1}\wedge S^{q-1}} D^p\wedge S^{q-1} \quad \text{and} \quad W = X_{i-r}\wedge Y_j \cup_{X_{i-r}\wedge Y_{j-r}} X_i\wedge Y_{j-r}$$

denote the pushouts in the squares of the upper and middle layer of the diagram, respectively. In particular, S^{p+q-1} is the boundary of $D^p \wedge D^q \cong D^{p+q}$. We then have an induced commutative diagram

$$\begin{tikzcd}
S^{p-1}\wedge S^{q-1} \arrow[r] \arrow[d, "f'\wedge g'"] & S^{p+q-1} \arrow[r] \arrow[d, "(f\wedge g)'"] & D^{p+q} \arrow[d, "f\wedge g"] \\
X_{i-r}\wedge Y_{j-r} \arrow[r] \arrow[d, "\phi_{i-r,j-r}"] & W \arrow[r] \arrow[d, "\phi_W"] & X_i\wedge Y_j \arrow[d, "\phi_{i,j}"] \\
Z_{i+j-2r} \arrow[r] & Z_{i+j-r} \arrow[r] & Z_{i+j}\,.
\end{tikzcd} \tag{I.6}$$

Here, $(f\wedge g)' : S^{p+q-1} \to W$ is the induced map between the pushouts in the top and bottom square of the boxed-shaped diagram appearing above, and ϕ_W is the induced map in the diagram

$$\begin{tikzcd}
X_{i-r}\wedge Y_{j-r} \arrow[r] \arrow[d] & X_i\wedge Y_{j-r} \arrow[d] \arrow[ddr, bend left, "\phi_{i,j-r}"] & \\
X_{i-r}\wedge Y_j \arrow[r] \arrow[drr, bend right, "\phi_{i-r,j}"'] & W \arrow[dr, "\phi_W"] & \\
& & Z_{i+j-r}\,.
\end{tikzcd}$$

We define the homomorphism

$$\phi_r \colon \pi_p^G(X_{i-r} \to X_i) \otimes \pi_q^G(Y_{j-r} \to Y_j) \to \pi_{p+q}^G(Z_{i+j-r} \to Z_{i+j})$$

as sending $[f', f] \otimes [g', g]$ to the homotopy class of the pair

$$\phi_W(f \wedge g)' \colon S^{p+q-1} \to Z_{i+j-r} \quad \text{and} \quad \phi_{i,j}(f \wedge g) \colon D^{p+q} \to Z_{i+j} \,,$$

which is an element of $\pi_{p+q}^G(Z_{i+j-r} \to Z_{i+j})$. As a diagram, this pair is visualised as the commutative square

$$\begin{tikzcd}
S^{p+q-1} \arrow[r] \arrow[d, "\phi_W(f\wedge g)'"'] & D^{p+q} \arrow[d, "\phi_{i,j}(f\wedge g)"] \\
Z_{i+j-r} \arrow[r] & Z_{i+j} \,.
\end{tikzcd}$$

In symbols:

$$\phi_r \colon [f', f] \otimes [g', g] \longmapsto [\phi_W(f \wedge g)', \phi_{i,j}(f \wedge g)] \,.$$

Spelled out with the stabilising G-representations U and V, this diagram should be interpreted as the commutative diagram

$$\begin{tikzcd}
\Sigma^{U\oplus V} S^{p+q-1} \arrow[r] \arrow[d, "\cong"'] & \Sigma^{U\oplus V} D^{p+q} \arrow[d, "\cong"] \\
\Sigma^U S^{p-1} \wedge \Sigma^V D^q \cup \Sigma^U D^p \wedge \Sigma^V S^{q-1} \arrow[r] \arrow[d, "(f\wedge g)'"'] & \Sigma^U D^p \wedge \Sigma^V D^q \arrow[d, "f\wedge g"] \\
X_{i-r}(U) \wedge Y_j(V) \cup X_i(U) \wedge Y_{j-r}(V) \arrow[r] \arrow[d, "\phi_W(U,V)"'] & X_i(U) \wedge Y_j(V) \arrow[d, "\phi_{i,j}(U,V)"] \\
Z_{i+j-r}(U \oplus V) \arrow[r] & Z_{i+j}(U \oplus V)
\end{tikzcd}$$

The pushouts on the left hand side are formed along $\Sigma^U S^{p-1} \wedge \Sigma^V S^{q-1}$ and $X_{i-r}(U) \wedge Y_{j-r}(V)$, respectively.

Remark I.4.23. We note that the pushout W is not generally equivalent to the corresponding homotopy pushout, but this will hold if $X_\star$ and $Y_\star$ are filtrations.

We also note that if one only has a weak pairing, in the sense that the squares (I.4) commute up to homotopy, then there is in general no preferred commuting homotopy in the diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\phi_W"'] & X_i \wedge Y_j \arrow[d, "\phi_{i,j}"] \\
Z_{i+j-r} \arrow[r] & Z_{i+j} \,,
\end{tikzcd}$$

and therefore no well-defined pairing ϕ_r. Any construction of spectral sequence pairings that only assumes such compatibility at the level of the (stable) homotopy category is therefore likely to contain a logical gap.

The pairing ϕ_r is evidently natural in i, j and r, in the sense that the square

$$\begin{array}{ccc} H(X_\star)(i-r,i) \otimes H(Y_\star)(j-r,j) & \xrightarrow{\phi_r} & H(Z_\star)(i+j-r,i+j) \\ \downarrow & & \downarrow \\ H(X_\star)(i'-r',i') \otimes H(Y_\star)(j'-r',j') & \xrightarrow{\phi_{r'}} & H(Z_\star)(i'+j'-r',i'+j') \end{array} \tag{I.7}$$

commutes for all integers i, j, $r \ge 1$, i', j' and $r' \ge 1$ with $i \le i'$, $i-r \le i'-r'$, $j \le j'$ and $j-r \le j'-r'$. As we recalled from [Dou59b, § II A] in Definition I.4.7, this is the first (SPP I) of two conditions for $(\phi_r)_{r\ge1}$ to define a pairing of (finite) Cartan–Eilenberg systems.

We now check condition (SPP III). The pairings ϕ_r can be extended to the case $r = \infty$ by letting

$$\phi_\infty\colon H(X_\star)(-\infty,i) \otimes H(Y_\star)(-\infty,j) \longrightarrow H(Z_\star)(-\infty,i+j)$$

be defined by homomorphisms

$$\phi_\infty\colon \pi_p^G(X_i) \otimes \pi_q^G(Y_j) \longrightarrow \pi_{p+q}^G(Z_{i+j})\,.$$

Given G-maps $f\colon \Sigma^U S^p \to X_i(U)$ and $g\colon \Sigma^V S^q \to Y_j(V)$, the homomorphism ϕ_∞ sends the homotopy classes $[f]$ and $[g]$ to the homotopy class of the composite

$$\Sigma^{U\oplus V} S^{p+q} \cong \Sigma^U S^p \wedge \Sigma^V S^q \xrightarrow{f\wedge g} X_i(U) \wedge Y_j(V) \xrightarrow{\phi_{i,j}(U,V)} Z_{i+j}(U\oplus V)\,.$$

In symbols, suppressing U and V:

$$\phi_\infty\colon [f]\otimes[g] \longmapsto [\phi_{i,j}(f\wedge g)]\,.$$

Recalling that our convention is such that $X_{-\infty} = Y_{-\infty} = Z_{-\infty} = *$, we note that the isomorphism $\pi_p^G(X_i) \cong \pi_p^G(X_{-\infty} \to X_i)$ takes the homotopy class of f to the homotopy class $[*, f\pi]$ of the pair $*\colon \Sigma^U S^{p-1} \to X_{-\infty}(U)$ and $f\pi\colon \Sigma^U D^p \to X_i(U)$. Here $\pi\colon D^p \to S^p$ identifies D^p/S^{p-1} with S^p. The pairing ϕ_∞ then corresponds to the pairing ϕ_r as defined in the paragraph above, for $r = \infty$, with every reference to X_{i-r}, Y_{j-r}, Z_{i+j-r} and Z_{i+j-2r} replaced by $*$. By the discussion above, it then also follows that the extended naturality condition

$$\begin{array}{ccc} H(X_\star)(-\infty,i) \otimes H(Y_\star)(-\infty,j) & \xrightarrow{\phi_\infty} & H(Z_\star)(-\infty,i+j) \\ \downarrow & & \downarrow \\ H(X_\star)(i-r,i) \otimes H(Y_\star)(j-r,j) & \xrightarrow{\phi_r} & H(Z_\star)(i+j-r,i+j) \end{array} \tag{I.8}$$

holds for the pairings ϕ_r with $1 \le r \le \infty$. This is condition (SPP III) from Definition I.4.11.

I.4.6 Pairings of Cartan–Eilenberg systems, II

Having proved (SPP I) and (SPP III), we now turn to the second condition (SPP II) from [Dou59b, § II A]. Recall that it says that, for $(\phi_r)_{r\geq 1}$ to define a pairing of Cartan–Eilenberg systems, we want the Leibniz rule

$$\partial\phi_r = \phi_1(\partial\otimes\eta) + \phi_1(\eta\otimes\partial) \tag{I.9}$$

to hold. That is, we want the composite

$$\begin{aligned} H(X_\star)(i-r,i)\otimes H(Y_\star)(j-r,j) &\xrightarrow{\phi_r} H(Z_\star)(i+j-r,i+j) \\ &\xrightarrow{\partial} H(Z_\star)(i+j-r-1,i+j-r) \end{aligned}$$

to be equal to the sum of the composite homomorphisms

$$\begin{aligned} H(X_\star)(i-r,i)\otimes H(Y_\star)(j-r,j) &\xrightarrow{\partial\otimes\eta} H(X_\star)(i-r-1,i-r)\otimes H(Y_\star)(j-1,j) \\ &\xrightarrow{\phi_1} H(Z_\star)(i+j-r-1,i+j-r) \end{aligned}$$

and

$$\begin{aligned} H(X_\star)(i-r,i)\otimes H(Y_\star)(j-r,j) &\xrightarrow{\eta\otimes\partial} H(X_\star)(i-1,i)\otimes H(Y_\star)(j-r-1,j-r) \\ &\xrightarrow{\phi_1} H(Z_\star)(i+j-r-1,i+j-r)\,. \end{aligned}$$

Here

$$\begin{aligned} \eta\colon H(X_\star)(i-r,i) &\longrightarrow H(X_\star)(i-1,i) \\ \eta\colon H(Y_\star)(j-r,j) &\longrightarrow H(Y_\star)(j-1,j) \end{aligned}$$

denote the natural maps. Regarding signs in the Leibniz rule, we recall the convention that

$$(\partial\otimes 1)(x\otimes y) = \partial x\otimes y \quad\text{and}\quad (1\otimes\partial)(x\otimes y) = (-1)^p x\otimes\partial y\,,$$

for $x\in\pi_p^G(X_{i-r}\to X_i)$ in total degree p of $H(X_\star)(i-r,i)$.

To verify condition (I.9) for a given pairing $\phi\colon (X_\star, Y_\star)\to Z_\star$, it follows from the naturality of the boundary homomorphisms ∂, and the case $i=i'$, $j=j'$, $r\geq r'=1$ of (I.7), that it suffices to establish the rule

$$\partial\phi_r = \phi_r(\partial\otimes 1) + \phi_r(1\otimes\partial)\,. \tag{I.10}$$

Here the left hand side is the composite

$$\begin{aligned} H(X_\star)(i-r,i)\otimes H(Y_\star)(j-r,j) &\xrightarrow{\phi_r} H(Z_\star)(i+j-r,i+j) \\ &\xrightarrow{\partial} H(Z_\star)(i+j-2r,i+j-r)\,, \end{aligned}$$

and the right hand side is the sum of the two composite homomorphisms

$$\begin{aligned} H(X_\star)(i-r,i) \otimes H(Y_\star)(j-r,j) &\xrightarrow{\partial\otimes 1} H(X_\star)(i-2r,i-r) \otimes H(Y_\star)(j-r,j) \\ &\xrightarrow{\phi_r} H(Z_\star)(i+j-2r,i+j-r) \end{aligned}$$

and

$$\begin{aligned} H(X_\star)(i-r,i) \otimes H(Y_\star)(j-r,j) &\xrightarrow{1\otimes\partial} H(X_\star)(i-r,i) \otimes H(Y_\star)(j-2r,j-r) \\ &\xrightarrow{\phi_r} H(Z_\star)(i+j-2r,i+j-r)\,. \end{aligned}$$

We shall now show that the identity (I.10) holds for pairings of filtrations of orthogonal G-spectra. Thereafter we use approximation by mapping telescopes to deduce that the identity holds for pairings of arbitrary sequences, as well.

Proposition I.4.24. *If $\phi\colon (X_\star, Y_\star) \to Z_\star$ is a pairing of sequences of orthogonal G-spectra, and $X_\star$ and $Y_\star$ are filtrations, then*

$$\partial\phi_r = \phi_r(\partial\otimes 1) + \phi_r(1\otimes\partial)$$

as homomorphisms

$$H(X_\star)(i-r,i) \otimes H(Y_\star)(j-r,j) \longrightarrow H(Z_\star)(i+j-2r,i+j-r)$$

for all integers i, j, r with $r \ge 1$.

Proof. In this proof we will, for the same typographical reasons as in Section I.4.5, suppress the stabilising representations U and V implicit in the presentation of elements of $\pi_p^G(X_{i-r} \to X_i)$ and $\pi_q^G(Y_{j-r} \to Y_j)$ by homotopy classes of pairs (f', f) and (g', g) of G-maps. The reader can reconstruct how the diagrams could be embellished with these suspensions and shifts.

For each map $A \to B$ we have natural maps $B \to B \cup CA \to B/A$ to the homotopy cofibre and cofibre. The right hand map is an equivalence when $A \to B$ is an h-cofibration. Applied to the left hand maps in diagram (I.6), this gives us a commutative diagram

$$\begin{tikzcd} S^{p+q-1} \arrow[r] \arrow[d, "(f\wedge g)'"'] & S^{p+q-1} \cup C(S^{p-1}\wedge S^{q-1}) \arrow[r, "\simeq"] \arrow[d, "(f\wedge g)'\cup C(f'\wedge g')"] & S^{p-1}\wedge S^q \vee S^p \wedge S^{q-1} \arrow[d, "f'\wedge g''\vee f''\wedge g'"] \\ W \arrow[r] \arrow[d, "\phi_W"'] & W \cup C(X_{i-r}\wedge Y_{j-r}) \arrow[r, "(\simeq)", "\Theta"'] \arrow[d, "\Phi"] & X_{i-r}\wedge Y_j/Y_{j-r} \vee X_i/X_{i-r}\wedge Y_{j-r} \\ Z_{i+j-r} \arrow[r] & Z_{i+j-r} \cup CZ_{i+j-2r}\,. & \end{tikzcd}$$

Here

$$f''\colon S^p \longrightarrow X_i/X_{i-r} \quad\text{and}\quad g''\colon S^q \longrightarrow Y_j/Y_{j-r}$$

are the quotient maps induced by (f', f) and (g', g), respectively, and we write

$$\Phi = \phi_W \cup C\phi_{i-r,j-r}$$

for brevity. If $X_\star$ and $Y_\star$ are filtrations, as we assume, then

$$X_{i-r} \wedge Y_{j-r} \longrightarrow W$$

is a h-cofibration, so the collapse map

$$\Theta \colon W \cup C(X_{i-r} \wedge Y_{j-r}) \longrightarrow X_{i-r} \wedge Y_j/Y_{j-r} \vee X_i/X_{i-r} \wedge Y_{j-r}$$

is an equivalence.

The left hand side of Equation (I.10) applied to $[f', f] \otimes [g', g]$ is

$$\begin{aligned} \partial\phi_r([f', f] \otimes [g', g]) &= \partial[\phi_W(f \wedge g)', \phi_{i,j}(f \wedge g)] \\ &= [*, \phi_W(f \wedge g)'\pi]\,. \end{aligned} \tag{I.11}$$

Under the isomorphism $\pi^G_{p+q-1}(Z_{i+j-2r} \to Z_{i+j-r}) \cong \pi^G_{p+q-1}(Z_{i+j-r} \cup CZ_{i+j-2r})$ this corresponds to the homotopy class of the composite map

$$S^{p+q-1} \longrightarrow Z_{i+j-r} \longrightarrow Z_{i+j-r} \cup CZ_{i+j-2r}$$

in the diagram above. Equivalently, by the commutativity of the diagram, we can describe it as the homotopy class of the composite map

$$\begin{aligned} S^{p+q-1} &\longrightarrow S^{p+q-1} \cup C(S^{p-1} \wedge S^{q-1}) \\ &\xrightarrow{(f\wedge g)' \cup C(f' \wedge g')} W \cup C(X_{i-r} \wedge Y_{j-r}) \\ &\xrightarrow{\Phi} Z_{i+j-r} \cup C(Z_{i+j-2r})\,. \end{aligned}$$

Alternatively, we can describe it as the image $\Phi_*([a])$ of the homotopy class $[a]$ of the composite map

$$\begin{aligned} a \colon S^{p+q-1} &\longrightarrow S^{p+q-1} \cup C(S^{p-1} \wedge S^{q-1}) \\ &\xrightarrow{(f\wedge g)' \cup C(f' \wedge g')} W \cup C(X_{i-r} \wedge Y_{j-r}) \end{aligned}$$

under the homomorphism

$$\Phi_* \colon \pi^G_{p+q-1}(W \cup C(X_{i-r} \wedge Y_{j-r})) \longrightarrow \pi^G_{p+q-1}(Z_{i+j-r} \cup CZ_{i+j-2r})\,.$$

We shall confirm that Equation (I.10) holds by writing it in the form

$$\Phi_*([a]) = \Phi_*([b]) + (-1)^p \Phi_*([c])$$

for some specific classes $[b]$ and $[c]$, to be defined later, and showing that

$$[a] = [b] + (-1)^p [c]$$

in $\pi^G_{p+q-1}(W \cup C(X_{i-r} \wedge Y_{j-r}))$. The latter identity will be confirmed by showing that

$$\Theta_*([a]) = \Theta_*([b]) + (-1)^p \Theta_*([c]) \,,$$

where Θ_* is the isomorphism

$$\Theta_* \colon \pi^G_{p+q-1}(W \cup C(X_{i-r} \wedge Y_{j-r})) \xrightarrow{(\cong)} \pi^G_{p+q-1}(X_{i-r} \wedge Y_j / Y_{j-r} \vee X_i / X_{i-r} \wedge Y_{j-r})$$

induced by the map Θ on homotopy. With this aim in mind, we first note that $\Theta_*([a])$ is the homotopy class of the composite

$$\begin{aligned} S^{p+q-1} &\longrightarrow S^{p+q-1} \cup C(S^{p-1} \wedge S^{q-1}) \\ &\xrightarrow{\simeq} (S^{p-1} \wedge S^q) \vee (S^p \wedge S^{q-1}) \\ &\xrightarrow{(f' \wedge g'') \vee (f'' \wedge g')} X_{i-r} \wedge Y_j / Y_{j-r} \vee X_i / X_{i-r} \wedge Y_{j-r} \,, \end{aligned}$$

again by commutativity of the above diagram. Checking orientations in the boundary of $D^p \wedge D^q$, the composition of all but the last map in the displayed map has degree $+1$ when projected to $S^{p-1} \wedge S^q$, and degree $(-1)^p$ when projected to $S^p \wedge S^{q-1}$. Hence $\Theta_*([a])$ is the sum of the homotopy class of the composite

$$\begin{aligned} S^{p-1} \wedge S^q &\xrightarrow{f' \wedge g''} X_{i-r} \wedge Y_j / Y_{j-r} \\ &\xrightarrow{\mathrm{in}_1} X_{i-r} \wedge Y_j / Y_{j-r} \vee X_i / X_{i-r} \wedge Y_{j-r} \end{aligned} \tag{I.12}$$

and $(-1)^p$ times the homotopy class of the composite

$$\begin{aligned} S^p \wedge S^{q-1} &\xrightarrow{f'' \wedge g'} X_i / X_{i-r} \wedge Y_{j-r} \\ &\xrightarrow{\mathrm{in}_2} X_{i-r} \wedge Y_j / Y_{j-r} \vee X_i / X_{i-r} \wedge Y_{j-r} \,. \end{aligned} \tag{I.13}$$

The first term of the right hand side of Equation (I.10) applied to $[f', f] \otimes [g', g]$ is

$$\phi_r(\partial \otimes 1)([f', f] \otimes [g', g]) = \phi_r([*, f'\pi] \otimes [g', g]) \,. \tag{I.14}$$

Unravelling the definition of ϕ_r, see the discussion in Section I.4.5 for more details, we form the commutative diagram

$$
\begin{tikzcd}
S^{p-2} \wedge S^{q-1} \arrow[rr] \arrow[dr] \arrow[dd] & & D^{p-1} \wedge S^{q-1} \arrow[dr] \arrow[dd, "f'\pi \wedge g'"] & \\
& S^{p-2} \wedge D^{q} \arrow[rr, crossing over] & & D^{p-1} \wedge D^{q} \arrow[dd, "f'\pi \wedge g"] \\
\ast \arrow[rr] \arrow[dr] \arrow[dd] & & X_{i-r} \wedge Y_{j-r} \arrow[dr] \arrow[dd, "="] & \\
& \ast \arrow[from=uu, crossing over] \arrow[rr, crossing over] & & X_{i-r} \wedge Y_{j} \arrow[dd, "="] \\
X_{i-2r} \wedge Y_{j-r} \arrow[rr] \arrow[dr] \arrow[dd, "\phi_{i-2r,j-r}"'] & & X_{i-r} \wedge Y_{j-r} \arrow[dr] \arrow[dd, "\phi_{i-r,j-r}"] & \\
& X_{i-2r} \wedge Y_{j} \arrow[from=uu, crossing over] \arrow[rr, crossing over] \arrow[dr, "\phi_{i-2r,j}"'] & & X_{i-r} \wedge Y_{j} \arrow[d, "\phi_{i-r,j}"] \\
Z_{i+j-3r} \arrow[rr] & & Z_{i+j-2r} \arrow[r] & Z_{i+j-r} \,.
\end{tikzcd}
$$

We also introduce the pushouts

$$
S^{p+q-2} = S^{p-2} \wedge D^{q} \cup_{S^{p-2} \wedge S^{q-1}} D^{p-1} \wedge S^{q-1}
$$

and

$$
U = X_{i-2r} \wedge Y_{j} \cup_{X_{i-2r} \wedge Y_{j-r}} X_{i-r} \wedge Y_{j-r}
$$

mapping to $D^{p-1} \wedge D^{q} \cong D^{p+q-1}$ and $X_{i-r} \wedge Y_{j}$, respectively. This leads to the commutative diagram

$$
\begin{tikzcd}
S^{p+q-2} \arrow[r] \arrow[d, "\ast \cup f'\pi \wedge g'"'] & D^{p+q-1} \arrow[d, "f'\pi \wedge g"] & \\
X_{i-r} \wedge Y_{j-r} \arrow[r] \arrow[d] \arrow[ddd, bend right=40, "\phi_{i-r,j-r}"'] & X_{i-r} \wedge Y_{j} \arrow[d, "="] & \\
U \arrow[r] \arrow[dd, "\phi_{U}"] & X_{i-r} \wedge Y_{j} \arrow[dd, "\phi_{i-r,j}"] \arrow[dr] & \\
& & W \arrow[dl, "\phi_{W}"] \\
Z_{i+j-2r} \arrow[r] & Z_{i+j-r} \,. &
\end{tikzcd}
$$

The class in $\pi^G_{p+q-1}(Z_{i+j-2r} \to Z_{i+j-r})$ described in Equation (I.14) is represented visually as the big rectangle in the diagram, that is, by the pair of maps

$$\begin{aligned} \phi_{i-r,j-r}(* \cup f'\pi \wedge g') \colon S^{p+q-2} &\to Z_{i+j-2r} \\ \phi_{i-r,j}(f'\pi \wedge g) \colon D^{p+q-1} &\to Z_{i+j-r} \,. \end{aligned}$$

We can extend the diagram to the right, as follows,

$$\begin{tikzcd}
D^{p+q-1} \arrow[r] \arrow[d, "f'\pi\wedge g"'] & D^{p+q-1} \cup CS^{p+q-2} \arrow[r, "\simeq"] \arrow[d, "f'\pi\wedge g \cup C(*\cup f'\pi\wedge g')"] & S^{p+q-1} \arrow[d, "f'\wedge g''"] \\
X_{i-r} \wedge Y_j \arrow[r] \arrow[d] & X_{i-r} \wedge (Y_j \cup CY_{j-r}) \arrow[r, "(\simeq)"] \arrow[d] & X_{i-r} \wedge Y_j/Y_{j-r} \arrow[d, "\mathrm{in}_1"] \\
W \arrow[r] \arrow[d, "\phi_W"'] & W \cup C(X_{i-r} \wedge Y_{j-r}) \arrow[r, "(\simeq)", "\Theta"'] \arrow[d, "\Phi"] & X_{i-r} \wedge Y_j/Y_{j-r} \vee X_i/X_{i-r} \wedge Y_{j-r} \\
Z_{i+j-r} \arrow[r] & Z_{i+j-r} \cup CZ_{i+j-2r} &
\end{tikzcd}$$

where the maps marked $(\simeq)$ are equivalences by our assumption that $X_{i-r} \to X_i$ and $Y_{j-r} \to Y_j$ are h-cofibrations. Under the isomorphism $\pi^G_{p+q-1}(Z_{i+j-2r} \to Z_{i+j-r}) \cong \pi^G_{p+q-1}(Z_{i+j-r} \cup CZ_{i+j-2r})$ the class described in Equation (I.14) is given by the composite

$$\begin{aligned} S^{p+q-1} &\xrightarrow{\simeq} D^{p+q-1} \cup CS^{p+q-2} \\ &\longrightarrow X_{i-r} \wedge (Y_j \cup CY_{j-r}) \\ &\longrightarrow W \cup C(X_{i-r} \wedge Y_{j-r}) \\ &\xrightarrow{\Phi} Z_{i+j-r} \cup CZ_{i+j-2r} \,, \end{aligned}$$

where the first map is a homotopy inverse to the collapse map. This is the image $\Phi_*([b])$ of the homotopy class $[b]$ of the composite map

$$b : S^{p+q-1} \xrightarrow{\simeq} D^{p+q-1} \cup CS^{p+q-2} \longrightarrow W \cup C(X_{i-r} \wedge Y_{j-r}) \,.$$

Since the composite

$$S^{p+q-1} \xrightarrow{\simeq} D^{p+q-1} \cup CS^{p+q-2} \xrightarrow{\simeq} S^{p+q-1} \cong S^{p-1} \wedge S^q$$

is homotopic to the identity, $\Theta_*([b])$ is the homotopy class of the map (I.12). That is, it is the image of $[f' \wedge g'']$ under the inclusion $(\mathrm{in}_1)_*$.

The second term of the right hand side of (I.10) applied to $[f', f] \otimes [g', g]$ is

$$\phi_r(1 \otimes \partial)([f', f] \otimes [g', g]) = (-1)^p \phi_r([f', f] \otimes [*, g'\pi]) \,. \tag{I.15}$$

By a similar analysis as for the first term of the sum, the class $\phi_r([f', f] \otimes [*, g'\pi])$ in $\pi^G_{p+q-1}(Z_{i+j-2r} \to Z_{i+j-r})$ is represented by a pair of maps

$$\phi_{i-r,j-r}(f' \wedge g'\pi \cup *) \colon S^{p+q-2} \longrightarrow Z_{i+j-2r}$$
$$\phi_{i,j-r}(f \wedge g'\pi) \colon D^{p+q-1} \longrightarrow Z_{i+j-r}\,,$$

where S^{p+q-2} is the boundary of $D^{p+q-1} \cong D^p \wedge D^{q-1}$. The corresponding class in $\pi^G_{p+q-1}(Z_{i+j-r} \cup CZ_{i+j-2r})$ is the image $\Phi_*([c])$ under Φ_* of the homotopy class $[c]$ of the composite map

$$c \colon S^{p+q-1} \xrightarrow{\simeq} D^{p+q-1} \cup CS^{p+q-2} \longrightarrow W \cup C(X_{i-r} \wedge Y_{j-r})\,.$$

The class $\Theta_*([c])$ is then the homotopy class of the map (I.13). That is, it is the image of $[f'' \wedge g']$ under $(\mathrm{in}_2)_*$.

Summarising, we have now defined classes $[a]$, $[b]$, and $[c]$ in $\pi^G_{p+q-1}(W \cup C(X_{i-r} \wedge Y_{j-r}))$ satisfying

$$\Theta_*([a]) = \Theta_*([b]) + (-1)^p \Theta_*([c])\,.$$

Since Θ_* is an isomorphism, we deduce that

$$[a] = [b] + (-1)^p [c] \quad \text{and} \quad \Phi_*([a]) = \Phi_*([b]) + (-1)^p \Phi_*([c])\,.$$

Since $\Phi_*([a])$, $\Phi_*([b])$ and $(-1)^p \Phi_*([c])$ are the three parts in Equation (I.10) evaluated at $[f', f] \otimes [g', g]$, and $[f', f]$ and $[g', g]$ were arbitrarily chosen, it follows that Equation (I.10) holds whenever $X_\star$ and $Y_\star$ are filtrations. □

We now extend the result above to all pairings of sequences.

Proposition I.4.25. *If $\phi \colon (X_\star, Y_\star) \to Z_\star$ is a pairing of sequences of orthogonal G-spectra, then*

$$\partial\phi_r = \phi_r(\partial \otimes 1) + \phi_r(1 \otimes \partial)$$

as homomorphisms

$$H(X_\star)(i-r, i) \otimes H(Y_\star)(j-r, j) \longrightarrow H(Z_\star)(i+j-2r, i+j-r)$$

for all integers i, j, r with $r \geq 1$.

Proof. Let $T(\phi) \colon (T_\star(X), T_\star(Y)) \to T_\star(Z)$ be the pairing of filtrations defined as in the proof of Lemma I.4.21. The equivalence $\epsilon \colon T_\star(X) \to X_\star$ and its analogues for $Y_\star$ and $Z_\star$ are compatible with the pairings. Hence we have commutative diagrams with vertical isomorphisms

$$\begin{tikzcd}
H(T_\star(X))(i-r, i) \arrow[r, "\partial"] \arrow[d, "\epsilon"', "\cong"] & H(T_\star(X))(i-2r, i-r) \arrow[d, "\cong"', "\epsilon"] \\
H(X_\star)(i-r, i) \arrow[r, "\partial"] & H(X_\star)(i-2r, i-r)\,,
\end{tikzcd}$$

together with its analogues for $Y_\star$ and $Z_\star$, and

$$\begin{tikzcd}
H(T_\star(X))(i-r,i) \otimes H(T_\star(Y))(j-r,j) \arrow[r, "T(\phi)_r"] \arrow[d, "\epsilon\otimes\epsilon"', "\cong"] & H(T_\star(Z))(i+j-r,i+j) \arrow[d, "\cong"', "\epsilon"] \\
H(X_\star)(i-r,i) \otimes H(Y_\star)(j-r,j) \arrow[r, "\phi_r"] & H(Z_\star)(i+j-r,i+j)\,,
\end{tikzcd}$$

for all $r \geq 1$. By Proposition I.4.24 applied to the pairing of filtrations $T(\phi)$ we know that

$$\partial T(\phi)_r = T(\phi)_r(\partial \otimes 1) + T(\phi)_r(1 \otimes \partial)$$

as homomorphisms

$$H(T_\star(X))(i-r,i) \otimes H(T_\star(Y))(j-r,j) \longrightarrow H(T_\star(Z))(i+j-2r,i+j-r)\,.$$

In view of the vertical isomorphisms ϵ, this implies that $\partial\phi_r = \phi_r(\partial \otimes 1) + \phi_r(1 \otimes \partial)$ as homomorphisms $H(X_\star)(i-r,i) \otimes H(Y_\star)(j-r,j) \to H(Z_\star)(i+j-2r,i+j-r)$. □

This finishes the goal we set out for ourselves at the start of Section I.4.5, namely to prove that a pairing of sequences gives rise to a pairing of the resulting Cartan–Eilenberg systems. Let us phrase this conclusion in a theorem, so that we can refer back to it when needed.

Theorem I.4.26. *A pairing $\phi\colon (X_\star, Y_\star) \to Z_\star$ of sequences gives rise to a pairing $\phi : (H(X_\star), H(Y_\star)) \to H(Z_\star)$ of the associated Cartan–Eilenberg systems, in the sense of Definition I.4.11.*

Proof. The proof of (SPP I) and (SPP III) is the content of Section I.4.5 and the proof of (SPP II) is the content of the present section. □

This directly gives us the following consequence for the associated spectral sequences.

Theorem I.4.27. *A pairing $\phi\colon (X_\star, Y_\star) \to Z_\star$ of sequences of orthogonal G-spectra gives rise to a pairing $\phi : (E^*(X_\star), E^*(Y_\star)) \to E^*(Z_\star)$, in the sense of Definition I.4.9. Explicitly, we have access to a collection of homomorphisms*

$$\phi^r : E^r(X_\star) \otimes E^r(Y_\star) \longrightarrow E^r(Z_\star)$$

for all $r \geq 1$, such that:

1. *The Leibniz rule*
$$d^r\phi^r = \phi^r(d^r \otimes 1) + \phi^r(1 \otimes d^r)$$
holds as an equality of homomorphisms $E^r_i(X_\star) \otimes E^r_j(Y_\star) \longrightarrow E^r_{i+j-r}(Z_\star)$ for all $i, j \in \mathbb{Z}$ and $r \geq 1$.

2. The diagram

$$\begin{tikzcd} E^{r+1}(X_\star) \otimes E^{r+1}(Y_\star) \arrow[r, "\phi^r"] \arrow[d] & E^{r+1}(Z_\star) \arrow[d, "\cong"] \\ H(E^r(X_\star) \otimes E^r(Y_\star)) \arrow[r, "H(\phi^{r+1})"] & H(E^r(Z_\star)) \end{tikzcd}$$

commutes for all $r \geq 1$.

Moreover, the induced pairing ϕ_* on filtered abutments is compatible with the pairing ϕ^∞ of E^∞-pages in the sense of Proposition I.4.12. Explicitly, the diagram

$$\begin{tikzcd} \dfrac{F_i A_\infty(X_\star)}{F_{i-1} A_\infty(X_\star)} \otimes \dfrac{F_j A_\infty(Y_\star)}{F_{j-1} A_\infty(Y_\star)} \arrow[r, "\bar{\phi}_*"] \arrow[d, "\beta\otimes\beta"'] & \dfrac{F_{i+j} A_\infty(Z_\star)}{F_{i+j-1} A_\infty(Z_\star)} \arrow[d, tail, "\beta"] \\ E_i^\infty(X_\star) \otimes E_j^\infty(Y_\star) \arrow[r, "\phi^\infty"] & E_{i+j}^\infty(Z_\star) \end{tikzcd}$$

commutes, for all $i, j \in \mathbb{Z}$. Here the abutments are given as

$$A_\infty(X_\star) \cong \pi_*^G \operatorname{Tel}(X_\star)$$
$$A_\infty(Y_\star) \cong \pi_*^G \operatorname{Tel}(Y_\star)$$
$$A_\infty(Z_\star) \cong \pi_*^G \operatorname{Tel}(Z_\star)\,,$$

and are filtered by the images

$$F_i A_\infty(X_\star) = \operatorname{im}(\pi_*^G(X_i) \longrightarrow A_\infty(X_\star))$$
$$F_j A_\infty(Y_\star) = \operatorname{im}(\pi_*^G(Y_j) \longrightarrow A_\infty(Y_\star))$$
$$F_k A_\infty(Z_\star) = \operatorname{im}(\pi_*^G(Z_k) \longrightarrow A_\infty(Z_\star))\,,$$

respectively.

Proof. This follows from combining Theorem I.4.26 with Theorem I.4.10 and Proposition I.4.12. □

Corollary I.4.28. *If $(X_\star, \phi)$ is a multiplicative sequence of orthogonal G-spectra, then the associated spectral sequence $(E(X_\star), d)$ is multiplicative with multiplicative abutment.*

I.4.7 The convolution product

Given two sequences $X_\star$ and $Y_\star$ of orthogonal G-spectra there is an initial pairing $\iota\colon (X_\star, Y_\star) \to Z_\star$, where the sequence $Z_\star$ is given at each level by

$$Z_k = \operatorname*{colim}_{i+j \leq k} X_i \wedge Y_j\,,$$

with the canonical G-maps $Z_{k-1} \to Z_k$ between them. We call this sequence $Z_\star$ the **(Day) convolution product** of $X_\star$ and $Y_\star$, and write

$$Z_\star = (X \wedge Y)_\star \,.$$

The universal pairing $\iota\colon (X_\star, Y_\star) \to (X \wedge Y)_\star$ has components

$$\iota_{i,j}\colon X_i \wedge Y_j \longrightarrow (X \wedge Y)_{i+j} \,,$$

each given by a structure map to the colimit. Per the discussion of Section I.4.5, the universal pairing $\iota : (X_\star, Y_\star) \to (X \wedge Y)_\star$ induces homomorphisms

$$\iota_r\colon \pi_p^G(X_{i-r} \to X_i) \otimes \pi_q^G(Y_{j-r} \to Y_j) \longrightarrow \pi_{p+q}^G((X \wedge Y)_{i+j-r} \to (X \wedge Y)_{i+j})$$

for $r \geq 1$, and Theorem I.4.27 shows that the pairing ι extends to a pairing $\iota\colon E^*(X_\star) \otimes E^*(Y_\star) \to E^*((X \wedge Y)_\star)$ of spectral sequences, in such a way that the induced pairing on filtered abutments

$$\bar\iota_*\colon \frac{F_i \pi_*^G \operatorname{Tel}(X_\star)}{F_{i-1}\pi_*^G \operatorname{Tel}(X_\star)} \otimes \frac{F_j \pi_*^G \operatorname{Tel}(Y_\star)}{F_{j-1}\pi_*^G \operatorname{Tel}(Y_\star)} \longrightarrow \frac{F_{i+j} \pi_*^G \operatorname{Tel}((X \wedge Y)_\star)}{F_{i+j-1}\pi_*^G \operatorname{Tel}((X \wedge Y)_\star)}$$

is compatible with the induced pairing on E^∞-pages.

Remark I.4.29. The colimit defining Z_k can equally well be calculated over the cofinal subcategory of pairs $(i, j) \in \mathbb{Z}^2$ with $k - 1 \leq i + j \leq k$, i.e., as the colimit of the zig-zag diagram:

$$\begin{tikzcd}
\dots \arrow[r] & X_{i-1} \wedge Y_{k-i+1} & & \\
& X_{i-1} \wedge Y_{k-i} \arrow[u] \arrow[r] & X_i \wedge Y_{k-i} & \\
& & X_i \wedge Y_{k-i-1} \arrow[u] \arrow[r] & \dots
\end{tikzcd}$$

If $(X_\star, \phi)$ and $(Y_\star, \psi)$ are multiplicative sequences of orthogonal G-spectra, then the convolution product $((X \wedge Y)_\star, \phi \wedge \psi)$ is a multiplicative sequence as well. Here, the component

$$(\phi \wedge \psi)_{i,j}\colon (X \wedge Y)_i \wedge (X \wedge Y)_j \longrightarrow (X \wedge Y)_{i+j}$$

is defined as the colimit over $i_1 + i_2 \leq i$ and $j_1 + j_2 \leq j$ of the composite maps

$$\begin{aligned} X_{i_1} \wedge Y_{i_2} \wedge X_{j_1} \wedge Y_{j_2} &\xrightarrow{1\wedge\tau\wedge 1} X_{i_1} \wedge X_{j_1} \wedge Y_{i_2} \wedge Y_{j_2} \xrightarrow{\phi_{i_1,j_1}\wedge\psi_{i_2,j_2}} X_{i_1+j_1} \wedge Y_{i_2+j_2} \\ &\xrightarrow{\iota_{i_1+j_1,i_2+j_2}} (X \wedge Y)_{i_1+j_1+i_2+j_2} \longrightarrow (X \wedge Y)_{i+j} \,. \end{aligned}$$

Lemma I.4.30. *If $(X_\star, \phi)$ and $(Y_\star, \psi)$ are multiplicative sequences, then the homomorphism $\iota^1\colon E^1(X_\star) \otimes E^1(Y_\star) \to E^1((X \wedge Y)_\star)$ is multiplicative, in the sense that the diagram*

$$\begin{tikzcd}
E^1(X_\star) \otimes E^1(Y_\star) \otimes E^1(X_\star) \otimes E^1(Y_\star) \arrow[r, "\iota^1 \otimes \iota^1"] \arrow[d, "1\otimes\tau\otimes 1"', "\cong"] & E^1((X \wedge Y)_\star) \otimes E^1((X \wedge Y)_\star) \arrow[dd, "(\phi\wedge\psi)^1"] \\
E^1(X_\star) \otimes E^1(X_\star) \otimes E^1(Y_\star) \otimes E^1(Y_\star) \arrow[d, "\phi^1\otimes\psi^1"'] & \\
E^1(X_\star) \otimes E^1(Y_\star) \arrow[r, "\iota^1"] & E^1((X \wedge Y)_\star)
\end{tikzcd}$$

commutes.

Proof. Let us write $\theta = \phi \wedge \psi$ for brevity. The diagrams

$$\begin{tikzcd}
X_{i_1} \wedge Y_{i_2} \wedge X_{j_1} \wedge Y_{j_2} \arrow[r, "\iota_{i_1,i_2} \wedge \iota_{j_1,j_2}"] \arrow[d, "1\wedge\tau\wedge 1"', "\cong"] & (X \wedge Y)_{i_1+i_2} \wedge (X \wedge Y)_{j_1+j_2} \arrow[dd, "\theta_{i_1+i_2,j_1+j_2}"] \\
X_{i_1} \wedge X_{j_1} \wedge Y_{i_2} \wedge Y_{j_2} \arrow[d, "\phi_{i_1,j_1}\wedge\psi_{i_2,j_2}"'] & \\
X_{i_1+j_1} \wedge Y_{i_2+j_2} \arrow[r, "\iota_{i_1+j_1,i_2+j_2}"] & (X \wedge Y)_{i_1+j_1+i_2+j_2}
\end{tikzcd}$$

commute, and are compatible, for all i_1, i_2, j_1 and j_2. This implies that the composite homomorphism

$$\begin{aligned}
&H(X_\star)(i_1 - r, i_1) \otimes H(Y_\star)(i_2 - r, i_2) \otimes H(X_\star)(j_1 - r, j_1) \otimes H(Y_\star)(j_2 - r, j_2) \\
&\qquad \xrightarrow{\iota_r \otimes \iota_r} H((X \wedge Y)_\star)(i_1 + i_2 - r, i_1 + i_2) \otimes H((X \wedge Y)_\star)(j_1 + j_2 - r, j_1 + j_2) \\
&\qquad \xrightarrow{\theta_r} H((X \wedge Y)_\star)(i_1 + i_2 + j_1 + j_2 - r, i_1 + i_2 + j_1 + j_2)
\end{aligned}$$

is equal to the composite homomorphism

$$\begin{aligned}
&H(X_\star)(i_1 - r, i_1) \otimes H(Y_\star)(i_2 - r, i_2) \otimes H(X_\star)(j_1 - r, j_1) \otimes H(Y_\star)(j_2 - r, j_2) \\
&\xrightarrow{\cong} H(X_\star)(i_1 - r, i_1) \otimes H(X_\star)(j_1 - r, j_1) \otimes H(Y_\star)(i_2 - r, i_2) \otimes H(Y_\star)(j_2 - r, j_2) \\
&\xrightarrow{\phi_r \otimes \psi_r} H(X_\star)(i_1 + j_1 - r, i_1 + j_1) \otimes H(Y_\star)(i_2 + j_2 - r, i_2 + j_2) \\
&\xrightarrow{\iota_r} H((X \wedge Y)_\star)(i_1 + j_1 + i_2 + j_2 - r, i_1 + j_1 + i_2 + j_2)
\end{aligned}$$

for each $r \geq 1$, where the homomorphisms ι_r, ϕ_r, ψ_r and θ_r are defined as in Section I.4.5. For $r = 1$, this gives the claim of the lemma. □

Note that for general sequences $X_\star$ and $Y_\star$ we typically have no homotopical control of their convolution product. However, if both $X_\star$ and $Y_\star$ are filtrations, then we can view each $X_i \wedge Y_j$ as a subspectrum of

$$\operatorname*{colim}_i X_i \wedge \operatorname*{colim}_j Y_j = \bigcup_i X_i \wedge \bigcup_j Y_j$$

and their colimit for $i + j \le k$ can be formed as the union

$$(X \wedge Y)_k = \bigcup_{i+j=k} X_i \wedge Y_j \,.$$

Proposition I.4.31. *If the sequences $X_\star$ and $Y_\star$ are filtrations, then their convolution product $(X \wedge Y)_\star$ is a filtration.*

Proof. We must show that each map

$$(X \wedge Y)_{k-1} \longrightarrow (X \wedge Y)_k$$

is a strong h-cofibration. This is the colimit of a sequence of maps, each of which is the cobase change of a pushout-product map

$$X_{i-1} \wedge Y_j \cup X_i \wedge Y_{j-1} \longrightarrow X_i \wedge Y_j$$

with $i + j = k$, where the pushout is formed over $X_{i-1} \wedge Y_{j-1}$. By assumption $X_{i-1} \to X_i$ and $Y_{j-1} \to Y_j$ are strong h-cofibrations, so the conclusion follows immediately from Theorem I.4.17. $\square$

In the special case when two arbitrary sequences $X_\star$ and $Y_\star$ are first replaced with equivalent filtrations $T_\star(X)$ and $T_\star(Y)$, we can give the following alternative, more explicit, argument for why the resulting convolution product is always a filtration.

Lemma I.4.32. *For any two sequences $X_\star$ and $Y_\star$ of orthogonal G-spectra, the convolution product $(T(X) \wedge T(Y))_\star$ is a filtration.*

Proof. In degree k,

$$(T(X) \wedge T(Y))_k = \bigcup_{i+j=k} T_i(X) \wedge T_j(Y) \,.$$

This is the subspectrum of $\mathrm{Tel}(X_\star) \wedge \mathrm{Tel}(Y_\star)$ with telescope coordinates x and y satisfying $\lceil x \rceil + \lceil y \rceil \le k$. Here $\lceil x \rceil$ denotes the least integer i with $x \le i$. The inclusion $(T(X) \wedge T(Y))_{k-1} \to (T(X) \wedge T(Y))_k$ is then the composite of a sequence of cobase changes of maps of the form

$$i_0 \colon A \longrightarrow B \cup_A A \wedge I_+ \,,$$

with A the double mapping cylinder of the diagram

$$X_{i-1} \wedge Y_j \longleftarrow X_{i-1} \wedge Y_{j-1} \longrightarrow X_i \wedge Y_{j-1}$$

and $B = X_i \wedge Y_j$, for $i + j = k$. Since each such map i_0 is a strong h-cofibration, so is the structure map in $(T(X) \wedge T(Y))_\star$, as claimed. $\square$

As a consequence of Proposition I.4.31, we can write first page of the spectral sequence associated to the convolution product of two filtrations $X_\star$ and $Y_\star$ as

$$\begin{aligned} E^1_k((X\wedge Y)_\star) &= \pi^G_*((X\wedge Y)_{k-1}\to (X\wedge Y)_k) \\ &\cong \pi^G_*((X\wedge Y)_k/(X\wedge Y)_{k-1}) \\ &\cong \bigoplus_{i+j=k} \pi^G_*(X_i/X_{i-1}\wedge Y_j/Y_{j-1}) \end{aligned}$$

since

$$\frac{(X\wedge Y)_k}{(X\wedge Y)_{k-1}} \cong \bigvee_{i+j=k} X_i/X_{i-1}\wedge Y_j/Y_{j-1}\,.$$

Furthermore, the diagram

$$\begin{tikzcd} E^1(X_\star)\otimes E^1(Y_\star) \arrow[r, "\iota^1"] \arrow[d, "d^1\otimes 1+1\otimes d^1"'] & E^1((X\wedge Y)_\star) \arrow[d, "d^1"] \\ E^1(X_\star)\otimes E^1(Y_\star) \arrow[r, "\iota^1"] & E^1((X\wedge Y)_\star) \end{tikzcd} \tag{I.16}$$

commutes. To proceed we usually need more explicit control of the d^1-differential for $(X\wedge Y)_\star$, e.g., by use of (I.16) in situations where ι^1 is surjective.

Suppose now that $(X_\star,\phi)$ and $(Y_\star,\psi)$ are multiplicative sequences, and also assume that the former is a filtration. This will be the situation when we filter the G-Tate construction in Section I.6. By Corollary I.4.22, the telescopic replacement $(T_\star(Y),T(\psi))$ is a multiplicative filtration, and by Proposition I.4.31 the convolution product $(X\wedge T(Y))_\star,\phi\wedge T(\psi))$ is then also a multiplicative filtration. Lemma I.4.30 shows that $\iota^1\colon E^1(X_\star)\otimes E^1(Y_\star)\to E^1((X\wedge Y)_\star)$ is multiplicative, in the sense that the diagram

$$\begin{tikzcd} \begin{matrix} E^1(X_\star)\otimes E^1(T_\star(Y)) \\ \otimes \\ E^1(X_\star)\otimes E^1(T_\star(Y)) \end{matrix} \arrow[r, "\iota^1\otimes\iota^1"] \arrow[d, "1\otimes\tau\otimes 1"', "\cong"] & E^1((X\wedge T(Y))_\star)\otimes E^1((X\wedge T(Y))_\star) \arrow[dd, "(\phi\wedge T(\psi))^1"] \\ \begin{matrix} E^1(X_\star)\otimes E^1(X_\star) \\ \otimes \\ E^1(T_\star(Y))\otimes E^1(T_\star(Y)) \end{matrix} \arrow[d, "\phi^1\otimes T(\psi)^1"'] & \\ E^1(X_\star)\otimes E^1(T_\star(Y)) \arrow[r, "\iota^1"] & E^1((X\wedge T(Y))_\star) \end{tikzcd}$$

commutes for all $i,j\in\mathbb{Z}$. In situations where ι^1 is surjective, this gives us algebraic control of the product on $E^1((X\wedge T(Y))_\star)$ in terms of the products on $E^1(X_\star)$ and $E^1(T_\star(Y))\cong E^1(Y_\star)$.

Remark I.4.33. The results of this section readily generalise to the case of sequences of R-modules in orthogonal G-spectra, for any fixed commutative orthogonal ring spectrum R. Letting $X_\star$ denote a sequence

$$\cdots \longrightarrow X_{i-1} \longrightarrow X_i \longrightarrow X_{i+1} \to \cdots$$

of R-module G-spectra and R-module G-maps, the telescope $\mathrm{Tel}(X_\star)$ is an R-module G-spectrum, and the Cartan–Eilenberg system (H, ∂), the exact couple (A, E^1), the filtered abutment $A_\infty \cong \pi_*^G \mathrm{Tel}(X_\star)$ and the spectral sequence (E^r, d^r) all live in the category of R_*-modules. The telescope filtration and equivalence

$$\epsilon\colon T_\star(X) \longrightarrow X_\star$$

also live in the category of R-modules.

Given sequences $X_\star$, $Y_\star$ and $Z_\star$ of R-modules in orthogonal G-spectra, an R-bilinear pairing

$$\phi\colon (X_\star, Y_\star) \longrightarrow Z_\star$$

consists of compatible R-linear G-maps

$$\phi\colon X_i \wedge_R Y_j \longrightarrow Z_{i+j}\,,$$

where the usual smash product has been replaced with the smash product over R. Such pairings induce R_*-module homomorphisms

$$\phi_r\colon H(X_\star)(i-r, i) \otimes_{R_*} H(Y_\star)(j-r, j) \longrightarrow H(Z_\star)(i+j-r, i+j)\,,$$

where the usual tensor product has been replaced with the tensor product over R_*. The Leibniz rule holds for ϕ_r, so that ϕ induces an R_*-linear pairing of R_*-module spectral sequences

$$\phi^r\colon E^r(X_\star) \otimes_{R_*} E^r(Y_\star) \longrightarrow E^r(Z_\star)\,.$$

The corresponding R_*-linear pairings $\bar\phi_*$ and ϕ^∞ of the filtration subquotients and E^∞-pages are compatible under the R_*-module monomorphism

$$\beta\colon \frac{F_i A_\infty}{F_{i-1} A_\infty} \longrightarrow E_i^\infty\,.$$

The universal R-bilinear pairing $\iota\colon (X_\star, Y_\star) \to Z_\star$ is given by the R-module convolution product $Z_\star = (X \wedge_R Y)_\star$, with

$$(X \wedge_R Y)_k = \operatorname*{colim}_{i+j \le k} X_i \wedge_R Y_j\,.$$

If $X_\star$ and $Y_\star$ are R-module filtrations, then $(X \wedge_R Y)_\star$ is an R-module filtration. For general R-module sequences $X_\star$ and $Y_\star$ the diagram (I.16) commutes after replacing $\otimes$ and $\wedge$ by $\otimes_{R_*}$ and $\wedge_R$, respectively. Finally, if $(X_\star, \phi)$ and $(Y_\star, \psi)$ are multiplicative R-module sequences then $\iota^1\colon E^1(X_\star) \otimes_{R_*} E^1(Y_\star) \to E^1((X \wedge_R Y)_\star)$ will be multiplicative. This depends on the existence of R-module maps

$$(23) = 1 \wedge \tau \wedge 1\colon X_{i_1} \wedge_R Y_{i_2} \wedge_R X_{j_1} \wedge_R Y_{j_2} \longrightarrow X_{i_1} \wedge_R X_{j_1} \wedge_R Y_{i_2} \wedge_R Y_{j_2}$$

that are strictly compatible for varying i_1, j_1, i_2 and j_2. This is a point where we use the assumption that R is strictly commutative, not just homotopy commutative, as an orthogonal ring spectrum.

I.5 The G-homotopy fixed point spectral sequence

Given an R-module X of orthogonal G-spectra we can define the G-homotopy fixed points as the genuine fixed points

$$X^{hG} = F(EG_+, X)^G = F_R(R \wedge EG_+, X)^G \,.$$

In this section we construct a spectral sequence

$$E^r_{*,*}(X) \Longrightarrow \pi_*(X^{hG})$$

with abutment being the homotopy groups of the G-homotopy fixed points of X, for any compact Lie group G. This spectral sequence will be induced by the filtration, covered in Section 5.1, on the free and contractible G-space EG coming from the simplicial bar construction. In Section 5.2, we show that this spectral sequence is multiplicative with multiplicative abutment. See Theorem I.5.6. Under the assumption that $R[G]_*$ is finitely generated and projective over R_* we can algebraically identify the E^2-page of the G-homotopy fixed point spectral sequence as

$$E^2_{*,*}(X) \cong \mathrm{Ext}^{-*}_{R[G]_*}(R_*, \pi_*(X))\,,$$

with the multiplicative structure being identified with the cup product on the right-hand side. See Theorem I.5.14. Lastly, in Section 5.4 we discuss the relationship between the simplicial skeletal filtration on $E\mathbb{T}$ and the often used filtration coming from odd-dimensional spheres.

I.5.1 The filtered G-space EG

As always, G is a compact Lie group. We let

$$EG = B(*, G, G)$$

be the free and contractible (right) G-space obtained by taking the geometric realization of the simplicial space

$$[q] \mapsto B_q(*, G, G) = G^q \times G\,,$$

with the usual face and degeneracy maps [May75, §7]. There is a simplicial contraction of $B_\bullet(*, G, G)$, so EG is indeed contractible [May72, Prop. 9.8]. We let F_iEG be the image of the structure map $\Delta^i \times B_i(*, G, G) \to EG$ to the geometric realization, yielding the following exhaustive filtration [May72, Def. 11.1]:

$$\emptyset = F_{-1}EG \subset F_0EG \subset \cdots \subset F_{i-1}EG \subset F_iEG \subset \cdots \subset EG\,.$$

Here, the group G acts freely from the right in each simplicial degree, hence also on each term in this filtration. The structure map induces a G-equivariant homeomorphism

$$\Sigma^i(G^{\wedge i} \wedge G_+) \cong \Delta^i/\partial\Delta^i \wedge G^{\wedge i} \wedge G_+ \cong F_iEG/F_{i-1}EG$$

for each $i \geq 0$. Each smash power $G^{\wedge i} = G \wedge \cdots \wedge G$ (with i copies of G) is formed with respect to the base point $e \in G$ given by the unit element.

Remark I.5.1. When G is finite, F_iEG gives the i-skeleton of a free G-CW-structure on EG. When $G = \mathbb{T} = U(1)$ is the circle group, F_iEG gives the $2i$- and $2i+1$-skeleta of a G-CW structure. Similarly, when $G = \mathbb{U} = Sp(1)$ is the 3-sphere, F_iEG gives the $4i$-, $4i+1$-, $4i+2$- and $4i+3$-skeleta of a G-CW structure. For other Lie groups the relationship is more complicated. Hence our filtration will agree with that used by Greenlees and May in [GM95, §9] when G is finite, be a doubly accelerated version when $G = \mathbb{T}$, and be a quadruply accelerated version when $G = \mathbb{U}$. The two filtrations might be quite different for other compact Lie groups G, though.

We give the cartesian product $EG \times EG$ the product filtration:

$$F_k(EG \times EG) = \bigcup_{i+j=k} F_iEG \times F_jEG\,.$$

Note that the diagonal G-map $\Delta\colon EG \to EG \times EG$, sending x to $\Delta(x) = (x, x)$, is not filtration-preserving. However, by [Seg68, Lem. 5.4] or [May72, Lem. 11.15] it is naturally homotopic to a filtration-preserving map $D\colon EG \to EG \times EG$, which we call a **diagonal approximation** for EG. By inspection, both D and the natural homotopy $\Delta \simeq D$ are G-equivariant. Subject to this condition, the precise choice of diagonal approximation will not be important, only its existence.

Lemma I.5.2. *Any diagonal approximation D induces a commutative diagram of based G-spaces and G-maps*

$$\begin{tikzcd}
\dfrac{F_kEG}{F_{k-1}EG} \arrow[r, "D'_k"] \arrow[d, tail] \arrow[rr, bend left, "D'_{i,j}"] & \dfrac{F_k(EG \times EG)}{F_{k-1}(EG \times EG)} \arrow[r, "\mathrm{pr}_{i,j}"] \arrow[d, tail] & \dfrac{F_iEG}{F_{i-1}EG} \wedge \dfrac{F_jEG}{F_{j-1}EG} \arrow[d, tail] \\
\dfrac{EG}{F_{k-1}EG} \arrow[r, "D_k"] \arrow[rr, bend right, "D_{i,j}"'] & \dfrac{EG \times EG}{F_{k-1}(EG \times EG)} \arrow[r] & \dfrac{EG}{F_{i-1}EG} \wedge \dfrac{EG}{F_{j-1}EG}
\end{tikzcd}$$

for all $i + j = k$. The G-maps $D_{i,j}$ are compatible for varying i and j, in the sense that the squares

$$\begin{tikzcd}
\dfrac{EG}{F_{i-1}EG} \wedge \dfrac{EG}{F_{j-1}EG} \arrow[d] & \dfrac{EG}{F_{k-1}EG} \arrow[l, "D_{i,j}"'] \arrow[r, "D_{i,j}"] \arrow[d] & \dfrac{EG}{F_{i-1}EG} \wedge \dfrac{EG}{F_{j-1}EG} \arrow[d] \\
\dfrac{EG}{F_iEG} \wedge \dfrac{EG}{F_{j-1}EG} & \dfrac{EG}{F_kEG} \arrow[l, "D_{i+1,j}"'] \arrow[r, "D_{i,j+1}"] & \dfrac{EG}{F_{i-1}EG} \wedge \dfrac{EG}{F_jEG}\,.
\end{tikzcd}$$

commute.

Proof. This follows from the inclusions

$$D(F_{k-1}EG) \subset F_{k-1}(EG \times EG) \subset (F_{i-1}EG \times EG) \cup (EG \times F_{j-1}EG)$$

and the splitting

$$\frac{F_k EG}{F_{k-1}EG} \cong \bigvee_{i+j=k} \frac{F_i EG}{F_{i-1}EG} \wedge \frac{F_j EG}{F_{j-1}EG}\,.$$

□

I.5.2 G-homotopy fixed points

Let X be an orthogonal G-spectrum. In this section we will construct a spectral sequence computing the homotopy groups of the G-homotopy fixed points of X, that is, the G-fixed points of a fibrant replacement of the function spectrum $F(EG_+, X)$:

$$X^{hG} = F(EG_+, X)^G\,.$$

To this end, note that the sequence of based G-spaces

$$EG_+ = \frac{EG}{F_{-1}EG} \to \frac{EG}{F_0 EG} \to \cdots \to \frac{EG}{F_{i-1}EG} \to \frac{EG}{F_i EG} \to \cdots \to *$$

induces a sequence

$$M_\star(X) = F(EG/EG_{-\star-1}, X)$$

of orthogonal G-spectra. Explicitly, $M_\star(X)$ is the sequence

$$\cdots \to F\left(\frac{EG}{F_i EG}, X\right) \to F\left(\frac{EG}{F_{i-1}EG}, X\right) \to \ldots$$
$$\cdots \to F\left(\frac{EG}{F_0 EG}, X\right) \to F(EG_+, X) = F(EG_+, X) = \ldots\,.$$

Definition I.5.3. The spectral sequence $(E^r(X), d^r) = (E^r(M_\star(X)), d^r)$ associated to the sequence $M_\star(X)$ above is called the **G-homotopy fixed point spectral sequence** of X.

Each map of function spectra $F(EG/F_iEG, X) \to F(EG/F_{i-1}EG, X)$ is a monomorphism of orthogonal G-spectra, but it is unlikely in general that these maps are h-cofibrations, so $M_\star(X)$ need not be a filtration. Since the sequence $M_\star(X)$ is eventually constant, there is a natural G-equivalence

$$M_\infty(X) = \mathrm{Tel}(M_\star(X)) \simeq_G F(EG_+, X)\,.$$

There is also a G-equivalence

$$\operatorname*{holim}_s M_s(X) = \operatorname*{holim}_s F\left(\frac{EG}{F_{-s-1}EG}, X\right) \cong F\left(\operatorname*{hocolim}_i \frac{EG}{F_{i-1}EG}, X\right) \simeq_G *\,,$$

since $\operatorname{hocolim}_i F_{i-1}EG \simeq_G EG$. We conclude that the G-homotopy fixed point spectral sequence is always conditionally convergent to the abutment

$$A_\infty(M_\star(X)) \cong \pi_*^G \operatorname{Tel}(M_\star(X)) \cong \pi_*^G F(EG_+, X) = \pi_*(X^{hG}) .$$

Let us now explicitly compute the E^1-page of this spectral sequence.

Lemma I.5.4. *The E^1-page of the G-homotopy fixed point spectral sequence of X is given by*

$$E^1_{-i,*}(M_\star(X)) \cong \pi^G_{-i+*} F\left(\frac{F_iEG}{F_{i-1}EG}, X\right) \cong \pi_*^G F(G_+, F(G^{\wedge i}, X))$$

and the differential

$$d^1_{-i,*} \colon E^1_{-i,*}(X) \to E^1_{-i-1,*}(X)$$

is contravariantly induced by the composite G-map

$$\frac{F_{i+1}EG}{F_iEG} \longrightarrow \frac{EG}{F_iEG} \simeq \frac{EG}{F_{i-1}EG} \cup C\Big(\frac{F_iEG}{F_{i-1}EG}\Big) \longrightarrow \Sigma \frac{F_iEG}{F_{i-1}EG} .$$

Proof. The cofibre sequence

$$\frac{F_iEG}{F_{i-1}EG} \longrightarrow \frac{EG}{F_{i-1}EG} \longrightarrow \frac{EG}{F_iEG}$$

of based G-spaces is a homotopy cofibre sequence, hence induces a homotopy fibre sequence

$$M_{-i-1}(X) \longrightarrow M_{-i}(X) \longrightarrow F\left(\frac{F_iEG}{F_{i-1}EG}, X\right)$$

of orthogonal G-spectra. It follows that

$$\begin{aligned} E^1_{-i,*}(X) &\cong \pi^G_{-i+*} F\left(\frac{F_iEG}{F_{i-1}EG}, X\right) \\ &\cong \pi^G_{-i+*} F(\Sigma^i(G^{\wedge i} \wedge G_+), X) \\ &\cong \pi^G_* F(G_+, F(G^{\wedge i}, X)) \end{aligned}$$

for $i \geq 0$. The d^1-differential is the composite of the connecting homomorphism

$$\pi^G_{-i+*}(M_{-i-1}(X) \to M_{-i}(X)) \xrightarrow{\partial} \pi^G_{-i-1+*}(M_{-i-1}(X)) \cong \pi^G_{-i-1+*} F\left(\frac{EG}{F_iEG}, X\right)$$

induced by

$$\frac{EG}{F_iEG} \simeq \frac{EG}{F_{i-1}EG} \cup C\left(\frac{F_iEG}{F_{i-1}EG}\right) \longrightarrow \Sigma \frac{F_iEG}{F_{i-1}EG} ,$$

and the homomorphism

$$\pi^G_{-i-1+*}(M_{-i-1}(X)) \longrightarrow \pi^G_{-i-1+*}(M_{-i-2}(X) \to M_{-i-1}(X))$$

induced by

$$\frac{F_{i+1}EG}{F_iEG} \longrightarrow \frac{EG}{F_iEG}\,.$$

□

Remark I.5.5. When G is finite, the spectral sequence $E^r(M_\star(X))$ agrees with the G-homotopy fixed point spectral sequence associated to a G-equivariant Whitehead (or Postnikov) tower for X. When $G = \mathbb{T}$ or $\mathbb{U}$ it is an accelerated version of the latter spectral sequence. In Theorem I.5.14 we will give an algebraic description of $E^2(M_\star(X))$ when X is an R-module and $R[G]_*$ is finitely generated and projective over R_*.

The homotopy fixed point construction is a lax symmetric monoidal functor. To see this, let $\mu\colon X\wedge Y \to Z$ be a pairing of orthogonal G-spectra, and recall the diagonal map $\Delta : EG \to EG \times EG$. The associated pairing $X^{hG} \wedge Y^{hG} \to Z^{hG}$ is given by the composite

$$\begin{aligned} F(EG_+, X)^G \wedge F(EG_+, Y)^G &\xrightarrow{\alpha} F(EG_+ \wedge EG_+, X \wedge Y)^G \\ &\xrightarrow{(\Delta_+)^*} F(EG_+, X \wedge Y)^G \\ &\xrightarrow{\mu_*} F(EG_+, Z)^G\,. \end{aligned}$$

From this point of view it is hence relevant to understand how the homotopy fixed point spectral sequence interacts with multiplicative structures. First note that the maps $D_{i,j}$ from Lemma I.5.2 induce G-maps

$$\begin{aligned} F\left(\frac{EG}{F_{i-1}EG}, X\right) \wedge F\left(\frac{EG}{F_{j-1}EG}, Y\right) &\xrightarrow{\alpha} F\left(\frac{EG}{F_{i-1}EG} \wedge \frac{EG}{F_{j-1}EG}\,,\, X \wedge Y\right) \\ &\xrightarrow{D^*_{i,j}} F\left(\frac{EG}{F_{k-1}EG}\,,\, X \wedge Y\right) \\ &\xrightarrow{\mu_*} F\left(\frac{EG}{F_{k-1}EG}\,,\, Z\right) \end{aligned}$$

for $k = i + j$. These are compatible for varying i and j, in the sense of Definition I.4.20, and so define the components $\bar\mu_{-i,-j}$ of a pairing

$$\bar\mu\colon (M_\star(X), M_\star(Y)) \to M_\star(Z)$$

of sequences of orthogonal G-spectra.

Theorem I.5.6. *Let $\mu\colon X\wedge Y \to Z$ be a pairing of orthogonal G-spectra. There is then a pairing*

$$\bar\mu^r : E^r(M_\star(X)) \otimes E^r(M_\star(Y)) \longrightarrow E^r(M_\star(Z))$$

of the associated G-homotopy fixed point spectral sequences, and the induced pairing $\bar\mu_*$ on filtered abutments is compatible with the induced pairing

$$\bar\mu^\infty : E^\infty(M_\star(X)) \otimes E^\infty(M_\star(Y)) \to E^\infty(M_\star(Z))$$

of E^∞-pages.

Moreover, the pairing $\bar\mu^1$ of E^1-pages is contravariantly induced by

$$D'_{i,j}\colon \frac{F_kEG}{F_{k-1}EG} \longrightarrow \frac{F_iEG}{F_{i-1}EG} \wedge \frac{F_jEG}{F_{j-1}EG}$$

under the isomorphism of Lemma I.5.4, and the pairing

$$\bar\mu_*\colon \pi_*(X^{hG}) \otimes \pi_*(Y^{hG}) \longrightarrow \pi_*(Z^{hG})$$

equals the pairing induced by $X^{hG} \wedge Y^{hG} \to Z^{hG}$.

Proof. In the paragraph before this theorem we noted that a map $\mu : X \wedge Y \to Z$ of orthogonal G-spectra gives rise to a pairing $\bar\mu : (M_\star(X), M_\star(Y)) \to M_\star(Z)$ of sequences. By Theorem I.4.27 it follows that we have an induced pairing between the associated spectral sequences, and that the induced pairing $\bar\mu_*$ on filtered abutments is compatible with the pairing $\bar\mu^\infty$ of E^∞-pages.

Tracing through the definitions shows that the pairing $\bar\mu^1_{-i,-j}$ of E^1-pages is compatible with the pairing induced by $D'_{i,j}$ under the isomorphism

$$\begin{aligned} E^1_{-i,*}(M_\star(X)) &= \pi^G_{-i+*}(M_{-i-1}(X) \to M_{-i}(X)) \\ &\cong \pi^G_{-i+*}F\left(\frac{F_iEG}{F_{i-1}EG}, X\right) \end{aligned}$$

and its analogues for Y and Z.

The abutment $A_\infty(M_\star(X)) \cong \pi^G_* F(EG_+, X)$ is filtered by the images

$$F_s\pi^G_* F(EG_+, X) = \operatorname{im}(\pi^G_* F(EG/EG_{-s-1}, X) \longrightarrow \pi^G_* F(EG_+, X))\,.$$

Note that this exhaustive filtration is constant for $s \ge 0$. The pairing $\bar\mu_*$ is induced by the composite map

$$\begin{aligned} \bar\mu_{0,0}\colon F(EG_+, X) \wedge F(EG_+, Y) &\xrightarrow{\alpha} F(EG_+ \wedge EG_+, X \wedge Y) \\ &\xrightarrow{D^*_{0,0}} F(EG_+, X \wedge Y) \\ &\xrightarrow{\mu_*} F(EG_+, Z)\,. \end{aligned}$$

In view of the based G-homotopy $\Delta_+ \simeq D_+ = D_{0,0}$, it is also induced by the composite map

$$\begin{aligned} F(EG_+, X) \wedge F(EG_+, Y) &\xrightarrow{\alpha} F(EG_+ \wedge EG_+, X \wedge Y) \\ &\xrightarrow{\Delta^*_+} F(EG_+, X \wedge Y) \\ &\xrightarrow{\mu_*} F(EG_+, Z)\,, \end{aligned}$$

where $\Delta_+\colon EG_+ \to (EG \times EG)_+ \cong EG_+ \wedge EG_+$. □

Corollary I.5.7. *If X is a multiplicative orthogonal G-spectrum, then the G-homotopy fixed point spectral sequence $(E^r(M_\star(X)), d^r)$ is a conditionally convergent and multiplicative spectral sequence, with multiplicative abutment $\pi_*(X^{hG})$.*

I.5.3 Algebraic description of the E^1- and E^2-pages

Under suitable flatness hypotheses there is an algebraic description of the first two pages of the homotopy fixed point spectral sequence. Recall from Section I.3 that R is our 'ground' commutative orthogonal ring spectrum. We write

$$R_*(X) = \pi_*(R \wedge X)$$

for the associated (reduced) homology theory. We will assume that $R[G]_*$ is flat over R_*, so that $R[G]_*$ is a cocommutative Hopf algebra over R_*, per Lemma I.3.2. Let us write

$$E = R \wedge EG_+ \quad \text{and} \quad E_i = R \wedge (F_i EG)_+ \,.$$

Each map $E_{i-1} \to E_i$ is a q-cofibration, hence a strong h-cofibration, so that $E_\star$ is a filtration

$$\cdots \longrightarrow E_{i-1} \longrightarrow E_i \longrightarrow E_{i+1} \longrightarrow \cdots$$

of R-modules in orthogonal G-spectra. Here $E_i = *$ for $i < 0$, and

$$E_\infty = \mathrm{Tel}(E_\star) \simeq_G E \,.$$

The R- and G-equivariant collapse map $1 \wedge c \colon E = R \wedge EG_+ \to R \wedge S^0 = R$ is a non-equivariant R-equivalence, inducing an $R[G]_*$-module isomorphism $\pi_*(E) \cong R_*$.

Definition I.5.8. Let $(P_{*,*}, \partial) = NB_*(R_*, R[G]_*, R[G]_*)$ denote the normalised bar resolution, as defined in Construction I.2.24.

Explicitly, the normalised bar resolution of the $R[G]_*$-module R_* is a non-negative chain complex given in homological degree $n \geq 0$ as

$$P_{n,*} = NB_n(R_*, R[G]_*, R[G]_*) = \overline{R[G]}_*^{\otimes n} \otimes_{R_*} R[G]_* \,,$$

where

$$\overline{R[G]}_* = \mathrm{coker}(\eta \colon R_* \to R[G]_*) \cong \ker(\epsilon \colon R[G]_* \to R_*)$$

denotes the augmentation (co-)ideal, and $\overline{R[G]}_*^{\otimes n}$ is its n-th tensor power over R_*. The boundary $\partial_n \colon P_{n,*} \to P_{n-1,*}$ is induced by the alternating sum of face operators

$$\sum_{i=0}^{n} (-1)^i d_i$$

for $n \geq 1$, with

$$d_i = \begin{cases} \epsilon \otimes 1^{\otimes n} & \text{for } i = 0, \\ 1^{\otimes i-1} \otimes \phi \otimes 1^{\otimes n-i} & \text{for } 0 < i \leq n. \end{cases}$$

Note that the simplicial contraction [May72, Prop. 9.8] of $B_\bullet(R_*, R[G]_*, R[G]_*)$ shows that the augmentation $\epsilon\colon P_{0,*} = R[G]_* \to R_*$ admits an R_*-linear chain homotopy inverse, so that the augmented chain complex

$$\dots \longrightarrow P_{q,*} \xrightarrow{\partial_q} P_{q-1,*} \to \cdots \to P_{1,*} \xrightarrow{\partial_1} P_{0,*} \xrightarrow{\epsilon} R_* \longrightarrow 0$$

is exact. Hence $(P_{*,*}, \partial)$ is a flat $R[G]_*$-module resolution of R_*.

Lemma I.5.9. *If $R[G]_*$ is flat over R_*, then the (E^1, d^1)-page of the non-equivariant homotopy spectral sequence*

$$E^1_{i,*} = \pi_{i+*}(E_{i-1} \to E_i)$$

associated to $E_\star$ is isomorphic to $(P_{*,*}, \partial)$. The edge homomorphism $P_{0,*} \to \pi_*(E) \cong R_*$ is equal to the augmentation $\epsilon\colon R[G]_* \to R_*$, and makes $(P_{*,*}, \partial)$ a flat $R[G]_*$-module resolution of R_*. In particular, the spectral sequence collapses at the E^2-page, where is it given by

$$E^2 = E^\infty \cong R_*$$

concentrated in filtration degree $i = 0$.

Proof. The R-module filtration $E_\star$ has an associated $R[G]_*$-module spectral sequence (for non-equivariant homotopy groups) with E^1-page

$$E^1_{i,*} = \pi_{i+*}(E_{i-1} \to E_i) \cong R_{i+*}\left(\frac{F_i EG}{F_{i-1} EG}\right)$$

and d^1-differential equal to the composite

$$R_{i+*}\left(\frac{F_i EG}{F_{i-1} EG}\right) \xrightarrow{\partial} R_{i-1+*}(F_{i-1}EG_+) \longrightarrow R_{i-1+*}\left(\frac{F_{i-1} EG}{F_{i-2} EG}\right).$$

By the proof of [Seg68, Prop. 5.1] or [May72, Thm. 11.14], (E^1, d^1) is the normalized chain complex associated to the simplicial $R[G]_*$-module

$$[q] \mapsto R_*(B_q(*, G, G)_+) = R_*((G^q \times G)_+).$$

The products

$$\begin{aligned} &R[G]_* \otimes_{R_*} R[G]_* \otimes_{R_*} \cdots \otimes_{R_*} R[G]_* \\ &\qquad \overset{\cdot}{\longrightarrow} \pi_*(R \wedge G_+ \wedge_R R \wedge G_+ \wedge_R \cdots \wedge_R R \wedge G_+) \\ &\qquad\qquad \cong \pi_*(R \wedge G_+ \wedge G_+ \wedge \cdots \wedge G_+) \end{aligned}$$

induce a homomorphism of simplicial $R[G]_*$-modules

$$B_\bullet(R_*, R[G]_*, R[G]_*) \longrightarrow R_*(B_\bullet(*, G, G))_+).$$

Since $R[G]_*$ is assumed to be flat over R_* the products are isomorphisms, so that (E^1, d^1) is indeed isomorphic to the normalized chain complex associated to the simplicial $R[G]_*$-module $B_\bullet(R_*, R[G]_*, R[G]_*)$. □

Remark I.5.10. If $R[G]_*$ is projective over R_*, then $\overline{R[G]}_*$ is also R_*-projective, and each $P_{q,*}$ is $R[G]_*$-projective by Lemma I.2.2. It follows that the chain complex $(P_{*,*}, \partial)$ is a projective $R[G]_*$-module resolution of R_*. Moreover, if $R[G]_*$ is finitely generated over R_*, then so is $\overline{R[G]}_*$, and each $P_{q,*}$ is finitely generated as an $R[G]_*$-module. We conclude that $(P_{*,*}, \partial)$ is a projective resolution of finite type, in this case.

To deal with the multiplicative structure of the spectral sequence we introduce the convolution product $(E \wedge_R E)_\star$. Explicitly, this is given by

$$(E \wedge_R E)_k = R \wedge F_k(EG \times EG)_+ \,,$$

with filtration subquotients

$$\frac{E \wedge_R E)_k}{(E \wedge_R E)_{k-1}} \cong \bigvee_{i+j=k} \frac{E_i}{E_{i-1}} \wedge_R \frac{E_j}{E_{j-1}} \,.$$

Let $\mathrm{in}_{i,j}$ denote the inclusion of the (i,j)-th summand in this splitting.

Lemma I.5.11. *The (E^1, d^1)-page of the homotopy spectral sequence associated to $(E \wedge_R E)_\star$ is isomorphic to the tensor product*

$$(P_{*,*} \otimes_{R_*} P_{*,*}, \partial \otimes 1 + 1 \otimes \partial) \,,$$

with the same signs occurring in the boundary as specified in Section I.2.2. In particular, this spectral sequence collapses at the E^2-page, where it is given by

$$E^2 = E^\infty \cong R_* \otimes_{R_*} R_* \cong R_*$$

concentrated in filtration degree 0.

Proof. Theorem I.4.27 applied to the initial pairing $\iota\colon (E_\star, E_\star) \to (E \wedge_R E)_\star$ gives us a pairing

$$\iota^r : E^r(E_\star) \otimes_{R_*} E^r(E_\star) \longrightarrow E^r((E \wedge_R E)_\star)$$

of $R[G]_*$-module spectral sequences. Since each copy of $E_\star$ is a filtration, the pairing

$$\iota^1_{i,j} : P_{i,*} \otimes_{R_*} P_{j,*} = E^1_{i,*}(E_\star) \otimes_{R_*} E^1_{j,*}(E_\star) \longrightarrow E^1_{k,*}((E \wedge_R E)_\star) \,,$$

for $r = 1$ and $i + j = k$, is induced by the product

$$P_{i,*} \otimes_{R_*} P_{j,*} \overset{\cdot}{\longrightarrow} \pi_* \left(\frac{E_i}{E_{i-1}} \wedge_R \frac{E_j}{E_{j-1}} \right)$$

and the inclusion

$$\mathrm{in}_{i,j} : \frac{E_i}{E_{i-1}} \wedge_R \frac{E_j}{E_{j-1}} \longrightarrow \frac{(E \wedge_R E)_k}{(E \wedge_R E)_{k-1}} \,.$$

Since $R[G]_*$ is flat over R_*, so that each $P_{i,*}$ is flat over R_*, the product is an isomorphism. Adding these together for $i+j=k$ we obtain the degree k part of an isomorphism of $R[G]_*$-module chain complexes

$$\iota^1 \colon P_{*,*} \otimes_{R_*} P_{*,*} \xrightarrow{\cong} E^1_{*,*}((E \wedge_R E)_\star) .$$

In particular, Theorem I.4.27 ensures that the tensor product boundary operator $\partial \otimes 1 + 1 \otimes \partial$ on the left hand side corresponds to the d^1-differential on the right hand side. The calculation of the E^2-page then follows as in the proof of Proposition I.2.28. □

Lemma I.5.12. *The diagonal approximation $D \colon EG \to EG \times EG$ induces a map of filtrations $1 \wedge D_+ \colon E_\star \to (E \wedge_R E)_\star$ and a chain map*

$$(1 \wedge D_+)^1 \colon E^1(E_\star) \longrightarrow E^1((E \wedge_R E)_\star) ,$$

which corresponds, under the isomorphisms of Lemma I.5.9 and Lemma I.5.11, to an $R[G]_*$-module chain map

$$\Psi \colon P_{*,*} \longrightarrow P_{*,*} \otimes_{R_*} P_{*,*} .$$

In particular, the component

$$\Psi_{i,j} = \mathrm{pr}_{i,j} \circ \Psi_k \colon P_{k,*} \to P_{i,*} \otimes_{R_*} P_{j,*}$$

of Ψ_k, for $k = i + j$, is induced by the G-map $D'_{i,j}$ of Lemma I.5.2 and Theorem I.5.6.

The chain map Ψ is characterised, uniquely up to chain homotopy equivalence, by the commutative square

$$\begin{tikzcd}
P_{*,*} \arrow[r, "\Psi"] \arrow[d, "\epsilon"'] & P_{*,*} \otimes_{R_*} P_{*,*} \arrow[d, "\epsilon\otimes\epsilon"] \\
R_* \arrow[r, "\cong"] & R_* \otimes_{R_*} R_*
\end{tikzcd}$$

of $R[G]_*$-module complexes.

Proof. The map of E^1-pages induced by the diagonal approximation is induced by $1 \wedge D'_k$, and the (i,j)-th component in the direct sum splitting of its target can be recovered by projecting to that summand, which is therefore induced by $1 \wedge D'_{i,j}$.

By naturality of the edge homomorphism, we have a commutative square of $R[G]_*$-modules

$$\begin{tikzcd}
P_{0,*} \arrow[r, "\Psi_0"] \arrow[d, "\epsilon"'] & P_{0,*} \otimes_{R_*} P_{0,*} \arrow[d, "\epsilon\otimes\epsilon"] \\
R_* \arrow[r, "\cong"] & R_* \otimes_{R_*} R_* .
\end{tikzcd}$$

Hence the $R[G]_*$-module chain map $\Psi\colon P_{*,*} \to P_{*,*} \otimes_{R_*} P_{*,*}$ is a lift of the isomorphism $R_* \cong R_* \otimes_{R_*} R_*$. Since $\epsilon\colon P_{*,*} \to R_*$ is an $R[G]_*$-projective complex over R_*, and $\epsilon \otimes \epsilon\colon P_{*,*} \otimes_{R_*} P_{*,*} \to R_* \otimes_{R_*} R_*$ is a resolution, it follows from [Mac95, Thm. III.6.1] that such a chain map Ψ exists and is unique up to chain homotopy. □

We now suppose that X is an R-module in orthogonal G-spectra. There are then compatible adjunction equivalences

$$F_R(E/E_{i-1}, X) \cong F(EG/F_{i-1}EG, X) = M_{-i}(X)$$

for all i. The left hand side exhibits $M_\star(X)$ as a sequence of R-modules in orthogonal G-spectra, so that the G-homotopy fixed point spectral sequence $E^r(M_\star(X))$ is a spectral sequence of R_*-modules. Theorem I.5.6 readily generalizes: If Y and Z are also R-modules in orthogonal G-spectra, and $\mu\colon X \wedge_R Y \to Z$ is a map in this category, then we obtain a pairing of R_*-module spectral sequences

$$\bar{\mu}^r\colon E^r(M_\star(X)) \otimes_{R_*} E^r(M_\star(Y)) \longrightarrow E^r(M_\star(Z))$$

such that the resulting pairing of E^∞-pages is compatible with the R_*-linear pairing

$$\bar{\mu}_*\colon \pi_*^G F(EG_+, X) \otimes_{R_*} \pi_*^G F(EG_+, Y) \longrightarrow \pi_*^G F(EG_+, Z)$$

of abutments. We can now give algebraic descriptions of the (E^1, d^1)-pages and the pairing $\bar{\mu}^1$, for $R[G]_*$ projective over R_*.

Proposition I.5.13. *Assume that $R[G]_*$ is projective as an R_*-module. There is then a natural isomorphism*

$$E^1_{-i,*}(M_\star(X)) \cong \mathrm{Hom}_{R[G]_*}(P_{i,*}, \pi_*(X))$$

of R_*-modules. Under this isomorphism, the d^1-differential

$$d^1_{-i,*}\colon E^1_{-i,*}(M_\star(X)) \longrightarrow E^1_{-i-1,*}(M_\star(X))$$

corresponds to the boundary in the chain complex, with signs as specified in Section I.2.2. The pairing

$$\bar{\mu}^1\colon E^1_{-i,*}(M_\star(X)) \otimes_{R_*} E^1_{-j,*}(M_\star(Y)) \longrightarrow E^1_{-k,*}(M_\star(Z))$$

with $i + j = k$ is contravariantly induced by the component

$$\Psi_{i,j}\colon P_{k,*} \longrightarrow P_{i,*} \otimes_{R_*} P_{j,*}$$

of the chain map Ψ.

Proof. By Lemma I.5.4 and adjunction isomorphisms

$$E^1_{-i,*}(M_\star(X)) \cong \pi^G_{-i+*} F(F_iEG/F_{i-1}EG, X) \cong \pi^G_{-i+*} F_R(E_i/E_{i-1}, X)\,.$$

Note that the spectrum appearing in the last term can be written $F_R(E_i/E_{i-1}, X) \cong F(G_+, X')$ with

$$X' = F_R(R \wedge G^{\wedge i}, X) \cong F(G^{\wedge i}, X)\,.$$

Under our assumption that $R[G]_*$ is projective, it follows from Proposition I.3.6 that the natural R_*-module homomorphism

$$\omega\colon \pi^G_{-i+*} F_R(E_i/E_{i-1}, X) \xrightarrow{\cong} \mathrm{Hom}_{R[G]_*}(R_*, \pi_{-i+*} F_R(E_i/E_{i-1}, X))$$

is an isomorphism. Moreover, since $P_{i,*} = \pi_{i+*}(E_i/E_{i-1})$ is projective over $R[G]_*$ and hence also over R_*, it follows that the natural $R[G]_*$-module homomorphism

$$\pi_{-i+*} F_R(E_i/E_{i-1}, X) \xrightarrow{\cong} \mathrm{Hom}_{R_*}(\pi_{i+*}(E_i/E_{i-1}), \pi_*(X)) = \mathrm{Hom}_{R_*}(P_{i,*}, \pi_*(X))$$

is an isomorphism. Applying the functor $\mathrm{Hom}_{R[G]_*}(R_*, -)$ this yields an isomorphism

$$\begin{aligned}&\mathrm{Hom}_{R[G]_*}(R_*, \pi_{-i+*} F_R(E_i/E_{i-1}, X)) \\ &\qquad \xrightarrow{\cong} \mathrm{Hom}_{R[G]_*}(R_*, \mathrm{Hom}_{R_*}(P_{i,*}, \pi_*(X))) \cong \mathrm{Hom}_{R[G]_*}(P_{i,*}, \pi_*(X))\,.\end{aligned}$$

Composing this chain of R_*-module isomorphisms gives the asserted natural isomorphism.

We now identify the d^1-differential. By Lemma I.5.4 again, we have a commutative diagram

$$\begin{tikzcd}
E^1_{-i,*}(M_\star(X)) \arrow[r, "\cong"] \arrow[dd, "d^1_{-i,*}"'] & \pi^G_{-i+*} F_R(E_i/E_{i-1}, X) \arrow[d] \\
 & \pi^G_{-i-1+*} F_R(E/E_i, X) \arrow[d] \\
E^1_{-i-1,*}(M_\star(X)) \arrow[r, "\cong"] & \pi^G_{-i-1+*} F_R(E_{i+1}/E_i, X)
\end{tikzcd}$$

of R_*-modules. By the naturality of ω in Lemma I.3.5 the diagram

$$\begin{tikzcd}
\pi^G_{-i+*} F_R(E_i/E_{i-1}, X) \arrow[r, "\omega", "\cong"'] \arrow[d] & \mathrm{Hom}_{R[G]_*}(R_*, \pi_{-i+*} F_R(E_i/E_{i-1}, X)) \arrow[d] \\
\pi^G_{-i-1+*} F_R(E/E_i, X) \arrow[r, "\omega"] \arrow[d] & \mathrm{Hom}_{R[G]_*}(R_*, \pi_{-i-1+*} F_R(E/E_i, X)) \arrow[d] \\
\pi^G_{-i-1+*} F_R(E_{i+1}/E_i, X) \arrow[r, "\omega", "\cong"'] & \mathrm{Hom}_{R[G]_*}(R_*, \pi_{-i-1+*} F_R(E_{i+1}/E_i, X))
\end{tikzcd}$$

commutes. Note that these two diagrams fit together along one edge. We also have a commutative diagram of $R[G]_*$-modules

$$\begin{tikzcd}
\pi_{-i+*}F_R(E_i/E_{i-1}, X) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{R_*}(P_{i,*}, \pi_*(X)) \arrow[d] \arrow[dd, bend left, "\mathrm{Hom}(\partial_{i+1},1)"] \\
\pi_{-i-1+*}F_R(E/E_i, X) \arrow[r] \arrow[d] & \mathrm{Hom}_{R_*}(\pi_{i+1+*}(E/E_i), \pi_*(X)) \arrow[d] \\
\pi_{-i-1+*}F_R(E_{i+1}/E_i, X) \arrow[r, "\cong"] & \mathrm{Hom}_{R_*}(P_{i+1,*}, \pi_*(X))
\end{tikzcd}$$

since $\partial_{i+1} : P_{i+1,*} \to P_{i,*}$ can be calculated by either composite from the left to the right in the diagram

$$\begin{tikzcd}
\pi_{i+1+*}(E_{i+1}/E_i) \arrow[r] \arrow[dr, "\partial"'] & \pi_{i+1+*}(E/E_i) \arrow[d, "\partial"] \arrow[dr, "\partial"] \\
& \pi_{i+*}(E_i) \arrow[r] & \pi_{i+*}(E_i/E_{i-1}) \,.
\end{tikzcd}$$

Applying $\mathrm{Hom}_{R[G]_*}(R_*, -)$ we obtain a commutative diagram of R_*-modules, which fits together with the previous one. Hence the square

$$\begin{tikzcd}
E^1_{-i,*}(M_\star(X)) \arrow[r, "\cong"] \arrow[d, "d^1_{-i,*}"'] & \mathrm{Hom}_{R[G]_*}(P_{i,*}, \pi_*(X)) \arrow[d, "\mathrm{Hom}(\partial_{i+1},1)"] \\
E^1_{-i-1,*}(M_\star(X)) \arrow[r, "\cong"] & \mathrm{Hom}_{R[G]_*}(P_{i+1,*}, \pi_*(X))
\end{tikzcd}$$

commutes, as asserted.

We now identify the multiplicative structure on the E^1-page. By Theorem I.5.6, the diagram

$$\begin{tikzcd}
E^1_{-i,*}(M_\star(X)) \otimes_{R_*} E^1_{-j,*}(M_\star(Y)) \arrow[r, "\bar{\mu}^1"] \arrow[d, "\cong"'] & E^1_{-k,*}(M_\star(Z)) \arrow[d, "\cong"] \\
\pi^G_{-i+*}F_R(E_i/E_{i-1}, X) \otimes_{R_*} \pi^G_{-j+*}F_R(E_j/E_{j-1}, Y) \arrow[r] & \pi^G_{-k+*}F_R(E_k/E_{k-1}, Z)
\end{tikzcd}$$

commutes, where the lower arrow is induced by

$$1 \wedge D'_{i,j} : E_k/E_{k-1} \to E_i/E_{i-1} \wedge_R E_j/E_{j-1} \,.$$

Since the natural homomorphism ω is monoidal, per Lemma I.3.7, the composite

$$\pi^G_{-i+*}F_R(E_i/E_{i-1}, X) \otimes_{R_*} \pi^G_{-j+*}F_R(E_j/E_{j-1}, Y) \longrightarrow \pi^G_{-k+*}F_R(E_k/E_{k-1}, Z)$$
$$\xrightarrow{\omega} \mathrm{Hom}_{R[G]_*}(R_*, \pi_{-k+*}F_R(E_k/E_{k-1}, Z))$$

is equal to the composite

$$\begin{aligned}
&\pi^G_{-i+*}F_R(E_i/E_{i-1},X)\otimes_{R_*}\pi^G_{-j+*}F_R(E_j/E_{j-1},Y)\xrightarrow{\omega\otimes\omega}\\
&\mathrm{Hom}_{R[G]_*}(R_*,\pi_{-i+*}F_R(E_i/E_{i-1},X))\otimes_{R_*}\mathrm{Hom}_{R[G]_*}(R_*,\pi_{-j+*}F_R(E_j/E_{j-1},Y))\\
&\quad\xrightarrow{\alpha}\mathrm{Hom}_{R[G]_*}(R_*,\pi_{-i+*}F_R(E_i/E_{i-1},X)\otimes_{R_*}\pi_{-j+*}F_R(E_j/E_{j-1},Y))\\
&\qquad\xrightarrow{\mu_*}\mathrm{Hom}_{R[G]_*}(R_*,\pi_{-k+*}F_R(E_k/E_{k-1},Z))\,.
\end{aligned}$$

Note that the final arrow is also induced by $1\wedge D'_{i,j}\colon E_k/E_{k-1}\to E_i/E_{i-1}\wedge_R E_j/E_{j-1}$. Next, we use the commutative diagram

$$\begin{tikzcd}
\pi_{-i+*}F_R(E_i/E_{i-1},X)\otimes_{R_*}\pi_{-j+*}F_R(E_j/E_{j-1},Y) \arrow[r] \arrow[d, "\cong"'] & \pi_{-k+*}F_R(E_k/E_{k-1},Z) \arrow[d, "\cong"] \\
\mathrm{Hom}_{R_*}(P_{i,*},\pi_*(X))\otimes_{R_*}\mathrm{Hom}_{R_*}(P_{j,*},\pi_*(Y)) \arrow[r] & \mathrm{Hom}_{R_*}(P_{k,*},\pi_*(Z))
\end{tikzcd}$$

of $R[G]_*$-modules, where the lower homomorphism is induced by $1\wedge D'_{i,j}$. In view of the isomorphism $P_{i,*}\otimes_{R_*}P_{j,*}\cong\pi_{i+j}(E_i/E_{i-1}\wedge_R E_j/E_{j-1})$ from the proof of Lemma I.5.11, this is the same homomorphism as that induced by $\Psi_{i,j}$, as defined in Lemma I.5.12.

Applying the monoidal functor $\mathrm{Hom}_{R[G]_*}(R_*,-)$, we obtain a commutative square of R_*-modules. Combining these results we have a commutative square

$$\begin{tikzcd}
E^1_{-i,*}(M_\star(X))\otimes_{R_*}E^1_{-j,*}(M_\star(Y)) \arrow[r, "\bar\mu^1"] \arrow[d, "\cong"'] & E^1_{-k,*}(M_\star(Z)) \arrow[d, "\cong"] \\
\mathrm{Hom}_{R[G]_*}(P_{i,*},\pi_*(X))\otimes_{R_*}\mathrm{Hom}_{R[G]_*}(P_{j,*},\pi_*(Y)) \arrow[r] & \mathrm{Hom}_{R[G]_*}(P_{k,*},\pi_*(Z))
\end{tikzcd}$$

where the lower homomorphism is induced by $\Psi_{i,j}$, meaning that it is equal to the composite

$$\begin{aligned}
&\mathrm{Hom}_{R[G]_*}(P_{i,*},\pi_*(X))\otimes_{R_*}\mathrm{Hom}_{R[G]_*}(P_{j,*},\pi_*(Y))\\
&\quad\xrightarrow{\alpha}\mathrm{Hom}_{R[G]_*}(P_{i,*}\otimes_{R_*}P_{j,*},\pi_*(X)\otimes_{R_*}\pi_*(Y))\\
&\quad\xrightarrow{\Psi^*_{i,j}}\mathrm{Hom}_{R[G]_*}(P_{k,*},\pi_*(X)\otimes_{R_*}\pi_*(Y))\\
&\quad\xrightarrow{\mu_*}\mathrm{Hom}_{R[G]_*}(P_{k,*},\pi_*(Z))\,.
\end{aligned}$$

This is the same as the (i,j)-component of the chain map

$$\begin{aligned}
&\mathrm{Hom}_{R[G]_*}(P_{*,*},\pi_*(X))\otimes_{R_*}\mathrm{Hom}_{R[G]_*}(P_{*,*},\pi_*(Y))\\
&\quad\xrightarrow{\alpha}\mathrm{Hom}_{R[G]_*}(P_{*,*}\otimes_{R_*}P_{*,*},\pi_*(X)\otimes_{R_*}\pi_*(Y))\\
&\quad\xrightarrow{\Psi^*}\mathrm{Hom}_{R[G]_*}(P_{*,*},\pi_*(X)\otimes_{R_*}\pi_*(Y))\\
&\quad\xrightarrow{\mu_*}\mathrm{Hom}_{R[G]_*}(P_{*,*},\pi_*(Z))
\end{aligned}$$

induced by Ψ. □

As a direct consequence, we get a description of the E^2-page of the homotopy fixed point spectral sequence.

Theorem I.5.14. *Let G be a compact Lie group and let R be a commutative orthogonal ring spectrum. Moreover, let $\mu\colon X \wedge_R Y \to Z$ be a pairing of R-modules in orthogonal G-spectra. Assume that $R[G]_*$ is projective as an R_*-module. Then there is a natural isomorphism*

$$E^2_{-i,*}(M_\star(X)) \cong \operatorname{Ext}^i_{R[G]_*}(R_*, \pi_*(X))$$

of R_*-modules, for each integer i. The pairing

$$\bar{\mu}^2\colon E^2_{-i,*}(M_\star(X)) \otimes_{R_*} E^2_{-j,*}(M_\star(Y)) \longrightarrow E^2_{-i-j,*}(M_\star(Z))$$

is given by the cup product

$$\smile\colon \operatorname{Ext}^i_{R[G]_*}(R_*, \pi_*(X)) \otimes_{R_*} \operatorname{Ext}^j_{R[G]_*}(R_*, \pi_*(Y)) \longrightarrow \operatorname{Ext}^{i+j}_{R[G]_*}(R_*, \pi_*(Z))$$

associated to the $R[G]_*$-module pairing $\mu_*\colon \pi_*(X) \otimes_{R_*} \pi_*(Y) \to \pi_*(Z)$, in Ext *over the Hopf algebra $R[G]_*$.*

Proof. By Proposition I.5.13, the first page of the spectral sequence, together with its d^1-differential, is identified with the R_*-chain complex $\operatorname{Hom}_{R[G]_*}(P_{*,*}, \pi_*(X))$ where $(P_{*,*}, \partial)$ is a projective resolution of R_*. It follows that the E^2-page is given by the homology of this chain complex, which by definition is the graded R_*-module

$$E^2_{*,*}(X) \cong \operatorname{Ext}^*_{R[G]_*}(R_*, \pi_*(X))\,.$$

Let us now identify the multiplication on the E^2-page with the cup product. Let $f\colon P_{i,*} \to \pi_*(X)$ and $g\colon P_{j,*} \to \pi_*(Y)$ be (graded) $R[G]_*$-module homomorphisms with $f\partial_{i+1} = 0$ and $g\partial_{j+1} = 0$. They correspond to i- and j-cycles in $\operatorname{Hom}_{R[G]_*}(P_{*,*}, \pi_*(X))$ and $\operatorname{Hom}_{R[G]_*}(P_{*,*}, \pi_*(Y))$, respectively, with homology classes $[f] \in E^2_{-i,*}(X)$ and $[g] \in E^2_{-j,*}(Y)$. The pairing of E^2-pages sends $[f]\otimes[g]$ to the homology class in $E^2_{-k,*}(Z)$ of the k-cycle given by the composite (graded) $R[G]_*$-module homomorphism

$$P_{k,*} \xrightarrow{\Psi_{i,j}} P_{i,*} \otimes_{R_*} P_{j,*} \xrightarrow{f\otimes g} \pi_*(X) \otimes_{R_*} \pi_*(Y) \xrightarrow{\mu_*} \pi_*(Z)\,.$$

The verification that $\mu_*(f \otimes g)\Psi_{i,j}$ is a k-cycle uses the fact that $\Psi_{i,j}$ is a component of an $R[G]_*$-module chain map $\Psi\colon P_{*,*} \to P_{*,*} \otimes_{R_*} P_{*,*}$, so that

$$\Psi_{i,j}\partial_{k+1} = (\partial_{i+1} \otimes 1)\Psi_{i+1,j} + (1 \otimes \partial_{j+1})\Psi_{i,j+1}\,.$$

This is the definition of the cup product

$$\smile\colon \operatorname{Ext}^*_{R[G]_*}(R_*, \pi_*(X)) \otimes_{R_*} \operatorname{Ext}^*_{R[G]_*}(R_*, \pi_*(Y)) \longrightarrow \operatorname{Ext}^*_{R[G]_*}(R_*, \pi_*(Z))$$

associated to the pairing μ_*. See Section I.2.5. $\square$

Remark I.5.15. A well-known consequence of the comparison theorem [Mac95, Thm. III.6.1] is that

$$\mathrm{Ext}^i_{R[G]_*}(R_*, \pi_*(X)) = H^i(\mathrm{Hom}_{R[G]_*}(P_{*,*}, \pi_*(X)))$$

can be calculated with any projective $R[G]_*$-module resolution $P_{*,*}$ of R_*, not necessarily the one introduced in Definition I.5.8. Likewise, by Proposition I.2.28, the cup product can be calculated with any $R[G]_*$-module chain map

$$\Psi\colon P_{*,*} \longrightarrow P_{*,*} \otimes_{R_*} P_{*,*}$$

lifting $R_* \cong R_* \otimes_{R_*} R_*$, not necessarily the one induced by a given diagonal approximation D.

Example I.5.16. When G is finite,

$$R[G]_* = R_*[G] \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} R_*\,,$$

any projective $\mathbb{Z}[G]$-module resolution Q_* of $\mathbb{Z}$ induces up to a projective $R_*[G]$-module resolution $P_{*,*} = Q_* \otimes_{\mathbb{Z}} R_*$ of R_*, and any $\mathbb{Z}[G]$-module diagonal approximation $\Psi\colon Q_* \to Q_* \otimes_{\mathbb{Z}} Q_*$ induces up to an $R_*[G]$-module diagonal approximation $\Psi \otimes 1\colon P_{*,*} = Q_* \otimes_{\mathbb{Z}} R_* \to Q_* \otimes_{\mathbb{Z}} Q_* \otimes_{\mathbb{Z}} R_* \cong P_{*,*} \otimes_{R_*} P_{*,*}$. Hence there is a natural isomorphism

$$\mathrm{Ext}^i_{R_*[G]}(R_*, \pi_*(X)) \cong \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, \pi_*(X)) = H^i(G, \pi_*(X))$$

identifying the E^2-page of the G-homotopy fixed point spectral sequence with the group cohomology of the G-module $\pi_*(X)$, and this identification is compatible with the cup product structure on both sides.

Example I.5.17. When $G = \mathbb{T}$ is the circle group, we showed in Proposition I.3.3 that

$$R[\mathbb{T}]_* = R_*[s]/(s^2 = \eta s) \quad \text{and} \quad \overline{R[\mathbb{T}]}_* = R_*\{s\}\,.$$

As we discussed in Definition I.5.8, the normalized bar resolution gives a (minimal) resolution $P_{*,*} = NB_*(R_*, R[\mathbb{T}]_*, R[\mathbb{T}]_*)$ of R_*, with

$$P_{i,*} = \overline{R[\mathbb{T}]}_*^{\otimes i} \otimes_{R_*} R[\mathbb{T}]_* \cong R[\mathbb{T}]_*\{\bar{p}_i\}\,.$$

Here $\bar{p}_i = s \otimes \dots \otimes s \otimes 1 = [s|\dots|s]1$ has homological degree i, internal degree $|\bar{p}_i| = i$ and total degree $\|p_i\| = 2i$, for $i \geq 0$. The differential is given by

$$\partial_i(\bar{p}_i) = \bar{p}_{i-1}((i-1)\eta + (-1)^i s)$$

for $i \geq 1$. This means that $P_{*,*}$ is not strictly equal to the resolution P_* specified at the beginning of Section I.2.6, with $P_i = R[\mathbb{T}]_*\{p_i\}$ and $\partial_i(p_i) = p_{i-1}(s + (i-1)\eta)$, due to the sign $(-1)^i$ before the contribution from the last face operator. However, the two resolutions are isomorphic, by way of the chain map sending $\bar{p}_i$ to $(-1)^{i(i+1)/2}p_i$ for each $i \geq 0$. Even without this isomorphism, we are free to use P_* to calculate $\operatorname{Ext}^*_{R[\mathbb{T}]_*}(R_*, \pi_*(X))$ as the homology of $\operatorname{Hom}_{R[\mathbb{T}]_*}(P_*, \pi_*(X))$, and that calculation was essentially done in Section I.2.6. For each $b \geq 0$ the rule

$$x \longmapsto f_b \cdot x := \begin{pmatrix} p_b \mapsto x \\ p_b s \mapsto xs \end{pmatrix}$$

defines a bijection $\Sigma^{-b}\pi_*(X) \cong \operatorname{Hom}_{R_*[G]}(\operatorname{Hom}(P_b, \pi_*(X)))$, and the boundary on such an element is given by

$$\partial^v(f_b \cdot x) = \begin{cases} -(-1)^{|x|} f_{b+1} \cdot xs & \text{for } b \geq 0 \text{ even,} \\ -(-1)^{|x|} f_{b+1} \cdot x(s+\eta) & \text{for } b \geq 1 \text{ odd.} \end{cases}$$

Hence we can compute the homology as

$$\operatorname{Ext}^b_{R[\mathbb{T}]_*}(R_*, \pi_*(X)) \cong \begin{cases} f_0 \cdot \ker(s\colon \pi_*(X) \to \pi_{*+1}(X)) & \text{for } b = 0, \\[2ex] f_b \cdot \dfrac{\ker(s+\eta\colon \pi_*(X) \to \pi_{*+1}(X))}{\operatorname{im}(s\colon \pi_{*-1}(X) \to \pi_*(X))} & \text{for } b \geq 1 \text{ odd,} \\[2ex] f_b \cdot \dfrac{\ker(s\colon \pi_*(X) \to \pi_{*+1}(X))}{\operatorname{im}(s+\eta\colon \pi_{*-1}(X) \to \pi_*(X))} & \text{for } b \geq 2 \text{ even.} \end{cases}$$

Please compare with Proposition I.2.36, Lemma I.2.37, and Proposition I.2.42.

For a description of the cup product, we can use any chain map $\Psi : P_* \to P_* \otimes_{R_*} P_*$ lifting the identity on R_*. Such a map is given in Lemma I.2.44, so that we can compute the cup product as

$$f_{b_1} \cdot x \smile f_{b_2} \cdot y = f_{b_1+b_2} \cdot x \otimes y\,.$$

Please compare with Lemma I.2.46. Formally writing the class of $f_b \cdot x$ as $t^b \cdot x$, we can then express $\operatorname{Ext}_{R[\mathbb{T}]_*}(R_*, \pi_*(X))$ as the homology of the differential graded $R[\mathbb{T}]_*$-module

$$\pi_*(X)[t]$$

with differential given by $d(x) = txs$ and $d(t) = t^2\eta$, for $x \in \pi_*(X)$. Here, t has homological degree -1, internal degree $|t| = -1$ and total degree $\|t\| = -2$.

I.5.4 The odd spheres filtration

In the important case $G = \mathbb{T}$, the circle action on odd-dimensional spheres provides a pleasant alternative model for EG. For each $i \geq 0$ let $S(i\mathbb{C}) = S^{2i-1}$ be the unit sphere in $i\mathbb{C} = \mathbb{C}^i$, with the standard, free $\mathbb{T}$-action. We obtain an exhaustive filtration

$$\emptyset \subset S(\mathbb{C}) \subset \cdots \subset S(i\mathbb{C}) \subset S((i+1)\mathbb{C}) \subset \cdots \subset S(\infty\mathbb{C})$$

of free $\mathbb{T}$-spaces. Here $S((i+1)\mathbb{C})$ is obtained from $S(i\mathbb{C})$ by attaching a free $\mathbb{T}$-equivariant $2i$-cell $D^{2i} \times \mathbb{T}$ along the group action map

$$S^{2i-1} \times \mathbb{T} \cong S(i\mathbb{C}) \times \mathbb{T} \to S(i\mathbb{C})\,,$$

so that $S((i+1)\mathbb{C})$ is the $2i$-skeleton in a free $\mathbb{T}$-CW structure on $S(\infty\mathbb{C})$. This filtered model for a free, contractible $\mathbb{T}$-CW complex was used in [BR05, §2] to discuss the $\mathbb{T}$-homotopy fixed point spectral sequence.

There are well-known $\mathbb{T}$-equivariant homeomorphisms

$$S((i+1)\mathbb{C}) \cong \mathbb{T} * \cdots * \mathbb{T} * \mathbb{T}$$

with $(i+1)$ copies of $\mathbb{T}$, where $*$ denotes the join of spaces. These homeomorphisms are compatible for varying $i \geq 0$, and $S(\infty\mathbb{C})$ is isomorphic as a filtered space to Milnor's infinite join construction from [Mil56], for $G = \mathbb{T}$, which we denote by

$$\mathscr{E}G = G * G * G * \dots .$$

The identifications made in the iterated join are included among those made in geometric realization. Hence the structure map $\Delta^i \times G^i \times G \to EG$ factors through a G-map

$$q_i \colon G * \cdots * G * G \longrightarrow F_i EG$$

with $(i+1)$ copies of G, collapsing degenerate simplices. These are compatible for varying i, yielding a G-map $q \colon \mathscr{E}G \to EG$. As explained in [Seg68, §3], the Milnor join construction is a special case $\mathscr{E}G \cong EG_{\mathbb{N}}$ of the two-sided bar construction for a topological category $G_{\mathbb{N}}$, and there is a continuous functor $G_{\mathbb{N}} \to G$ inducing the G-maps q_i and q. It follows that the filtration-preserving diagonal approximation $D_{\mathbb{N}} \colon EG_{\mathbb{N}} \to EG_{\mathbb{N}} \times EG_{\mathbb{N}}$ constructed in [Seg68, Lem. 5.4] is compatible with the diagonal approximation $D \colon EG \to EG \times EG$ that we have used in the present paper. In particular, the $\mathbb{T}$-map

$$q^* \colon F(E\mathbb{T}_+, X) \longrightarrow F(\mathscr{E}\mathbb{T}_+, X) \cong F(S(\infty\mathbb{C})_+, X)$$

maps our multiplicative sequence $M_\star(X)$ to the multiplicative tower used in [BR05, §4]. Furthermore, for $G = \mathbb{T}$ the G-maps q_i and q are equivalences, so that the two multiplicative towers of orthogonal G-spectra are equivalent. Hence they give isomorphic $\mathbb{T}$-homotopy fixed point spectral sequences, converging to the same multiplicative filtration on the abutment.

A similar discussion applies for the 3-sphere $G = \mathbb{U} = Sp(1)$ acting on the unit spheres in $i\mathbb{H} = \mathbb{H}^i$, showing that $S(\infty\mathbb{H}) \cong \mathscr{E}\mathbb{U}$ is a perfectly good alternative filtered model for $E\mathbb{U}$.

I.6 The G-Tate spectral sequence

Given an R-module X in orthogonal G-spectra we can define its G-Tate construction as the genuine fixed points

$$X^{tG} = (\widetilde{EG} \wedge F(EG_+, X))^G\,,$$

where $\widetilde{EG}$ is the mapping cone of the collapse map $EG_+ \to S^0$. In this section we construct an R_*-module spectral sequence

$$\hat{E}^r_{*,*} \Longrightarrow \pi_*(X^{tG})$$

with abutment the G-equivariant homotopy groups of $\widetilde{EG} \wedge F(EG_+, X)$, for any compact Lie group G. We do this by letting the filtration $E_\star$ induce a filtration $\widetilde{E}_\star$ of $R \wedge \widetilde{EG}$ and consider the so-called Hesselholt–Madsen filtration

$$HM_\star(X) = (\widetilde{E} \wedge_R T(M(X)))_\star$$

obtained by forming a convolution product. Under the assumption that $R[G]_*$ is finitely generated and projective over R_* we show that the resulting spectral sequence $\hat{E}^r_{*,*}(X) = E^r_{*,*}(HM_\star(X))$ is multiplicative, as a functor of X, with multiplicative abutment. With the same assumptions we also algebraically identify the E^2-page as

$$\hat{E}^2_{*,*}(X) \cong \widehat{\operatorname{Ext}}^{-*}_{R[G]_*}(R_*, \pi_*(X)) \,,$$

with the multiplicative structure given by cup product on the right-hand side. See Theorem I.6.18. To say something about the convergence of this spectral sequence we compare the Hesselholt–Madsen filtration to another filtration $GM_\star(X)$ of $\widetilde{EG} \wedge F(EG_+, X)$, dubbed the Greenlees–May filtration. While the multiplicative properties of the Greenlees–May G-Tate spectral sequence are less clear, it is easy to obtain convergence results for the latter spectral sequence. By the comparison we can then also obtain convergence results for the Hesselholt–Madsen G-Tate spectral sequence. See Section I.6.6, and in particular Theorem I.6.44.

I.6.1 The filtered G-space $\widetilde{EG}$

As always, let G be any compact Lie group. Let $c\colon EG_+ \to S^0$ denote the based and G-equivariant collapse map, and define

$$\widetilde{EG} = S^0 \cup C(EG_+)$$

to be its reduced mapping cone, as in [Car84, p. 198] and [GM95, p. 2]. Non-equivariantly, c is an equivalence, so $\widetilde{EG}$ is (non-equivariantly) contractible. For $i \geq 0$ we let

$$F_i\widetilde{EG} = S^0 \cup C(F_{i-1}EG_+)$$

be the mapping cone of c restricted to $F_{i-1}EG_+$, where $F_{i-1}EG$ is defined as in Section I.5.1. For $i < 0$, we set $F_i\widetilde{EG} = *$. This defines an exhaustive filtration

$$* = F_{-1}\widetilde{EG} \subset S^0 = F_0\widetilde{EG} \subset \cdots \subset F_{i-1}\widetilde{EG} \subset F_i\widetilde{EG} \subset \cdots \subset \widetilde{EG} \qquad \text{(I.17)}$$

of based G-spaces. Each map $F_{i-1}\widetilde{EG} \to F_i\widetilde{EG}$ is a strong h-cofibration, so this is indeed a filtration, as opposed to simply a sequence. Moreover, there are homeomorphisms

$$\frac{F_i\widetilde{EG}}{F_{i-1}\widetilde{EG}} \cong \Sigma\frac{F_{i-1}EG}{F_{i-2}EG}$$

for $i \geq 1$. Per Theorem I.4.17, each pushout-product map

$$F_{i-1}\widetilde{EG} \wedge F_j\widetilde{EG} \cup F_i\widetilde{EG} \wedge F_{j-1}\widetilde{EG} \longrightarrow F_i\widetilde{EG} \wedge F_j\widetilde{EG}$$

is a strong h-cofibration, with cofibre

$$\frac{F_i\widetilde{EG}}{F_{i-1}\widetilde{EG}} \wedge \frac{F_j\widetilde{EG}}{F_{j-1}\widetilde{EG}}\,.$$

Remark I.6.1. When G is finite, $F_i\widetilde{EG}$ gives the i-skeleton of a based and non-free G-CW structure on $\widetilde{EG}$. When $G = \mathbb{T} = U(1)$, $F_0\widetilde{EG} = S^0$ is the 0-skeleton, while $F_i\widetilde{EG}$ for $i \geq 1$ is the $2i-1$-and $2i$-skeleton of a G-CW structure on $\widetilde{EG}$. Similarly, when $G = \mathbb{U} = Sp(1)$, $F_i\widetilde{EG}$ gives the $4i-3$-, $4i-2$-, $4i-1$- and $4i$-skeleta of a G-CW structure.

Remark I.6.2. For $G = \mathbb{T}$, the G-equivalences $q_{i-1} \colon S(i\mathbb{C}) \to F_{i-1}EG$ from Section I.5.4 induce G-equivalences $\tilde{q}_i \colon S^{i\mathbb{C}} \to F_i\widetilde{EG}$, where we identify the one-point compactification $S^{i\mathbb{C}}$ with the mapping cone $S^0 \cup C(S(i\mathbb{C})_+)$. Hence we have a G-equivalence from the exhaustive filtration

$$* \to S^0 \to \cdots \to S^{(i-1)\mathbb{C}} \to S^{i\mathbb{C}} \to \cdots \to S^{\infty\mathbb{C}}$$

to (I.17), showing that we may use $S^{\infty\mathbb{C}}$ as a filtered replacement for $\widetilde{EG}$, if desired.

We give $\widetilde{EG} \wedge \widetilde{EG}$ the (convolved) smash product filtration, with

$$F_k(\widetilde{EG} \wedge \widetilde{EG}) = \bigcup_{i+j=k} F_i\widetilde{EG} \wedge F_j\widetilde{EG}\,.$$

The identifications $S^0 \wedge \widetilde{EG} \cong \widetilde{EG} \cong \widetilde{EG} \wedge S^0$ agree on $S^0 \wedge S^0 \cong S^0$, hence combine to a fold map

$$\nabla \colon \widetilde{EG} \cup_{S^0} \widetilde{EG} \cong \widetilde{EG} \wedge S^0 \cup S^0 \wedge \widetilde{EG} \longrightarrow \widetilde{EG}\,.$$

We seek a G-map $N \colon \widetilde{EG} \wedge \widetilde{EG} \to \widetilde{EG}$ extending ∇, so that the diagram

$$\begin{tikzcd}
\widetilde{EG} \wedge S^0 \cup S^0 \wedge \widetilde{EG} \arrow[r] \arrow[dr, "\nabla"'] & \widetilde{EG} \wedge \widetilde{EG} \arrow[d, "N"] \\
& \widetilde{EG}
\end{tikzcd}$$

commutes. For these pairings to induce pairing of spectral sequences, we must arrange that N is filtration-preserving. We do not know how to give a direct definition of such an extension $N\colon \widetilde{EG}\wedge\widetilde{EG} \to \widetilde{EG}$, in analogy with the explicit diagonal approximation $D\colon EG \to EG \times EG$. Instead we will use obstruction theory to show that such a filtration-preserving extension N of ∇ exists after base change to our ground ring spectrum R, assuming that $R[G]_*$ is projective over R_*. See Proposition I.6.9.

Definition I.6.3. Let

$$\widetilde{E} = R \wedge \widetilde{EG} \quad \text{and} \quad \widetilde{E}_i = R \wedge F_i\widetilde{EG}\,.$$

Each map $\widetilde{E}_{i-1} \to \widetilde{E}_i$ is a strong h-cofibration, so that $\widetilde{E}_\star$ is a filtration

$$\dots \longrightarrow \widetilde{E}_{i-1} \longrightarrow \widetilde{E}_i \longrightarrow \widetilde{E}_{i+1} \longrightarrow \dots$$

of R-modules in orthogonal G-spectra. Here $\widetilde{E}_i = *$ for $i < 0$, $\widetilde{E}_0 = R$, and

$$\widetilde{E}_\infty = \mathrm{Tel}(\widetilde{E}_\star) \simeq_G \widetilde{E}\,.$$

Since $\widetilde{EG}$ is non-equivariantly contractible, $\pi_*(\widetilde{E}) = 0$.

Applying non-equivariant homotopy we obtain the following unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*(\widetilde{E}_{i-1}) \arrow[r, "\alpha"] & \pi_*(\widetilde{E}_i) \arrow[r] \arrow[d, "\beta"] & \cdots \\
& & \pi_*(\widetilde{E}_{i-1} \to \widetilde{E}_i) \arrow[ul, "\partial"] &
\end{tikzcd} \tag{I.18}$$

with ∂ of total degree -1. Recall the $R[G]_*$-module resolution $(P_{*,*}, \partial)$ of R_*, introduced in Definition I.5.8.

Definition I.6.4. Let $(\widetilde{P}_{*,*}, \widetilde{\partial})$ be the mapping cone of the augmentation $\epsilon\colon P_{*,*} \to R_*$, in the sense of Definition I.2.11.

Explicitly, we have

$$\widetilde{P}_{i,*} \cong \begin{cases} R_* & \text{for } i = 0 \\ P_{i-1,*} & \text{for } i \geq 1 \end{cases}$$

with boundary $\tilde{\partial} : \widetilde{P}_{i,*} \to \widetilde{P}_{i-1,*}$ given as

$$\tilde{\partial} = \begin{cases} \epsilon(x) & \text{for } i = 1 \\ -\partial(x) & \text{for } i \geq 2. \end{cases}$$

We note that $\widetilde{P}_{*,*}$ is an exact complex of flat R_*-modules, by our standing assumption that $R[G]_*$ is flat. If, furthermore, $R[G]_*$ is finitely generated projective over R_*, then so is each $\widetilde{P}_{i,*}$.

Lemma I.6.5. *If $R[G]_*$ is flat over R_*, then the (E^1, d^1)-page of the non-equivariant homotopy spectral sequence*

$$\widetilde{E}^1_{i,*} = \pi_{i+*}(\widetilde{E}_{i-1} \to \widetilde{E}_i)$$

associated to $\widetilde{E}_\star$ is isomorphic to $(\widetilde{P}_{*,*}, \tilde{\partial})$. In particular, the spectral sequence collapses of the E^2-page, where it is given by

$$\widetilde{E}^2 = \widetilde{E}^\infty = 0\,.$$

Proof. Note that $\widetilde{E}$ is the mapping cone of the collapse map $1 \wedge c\colon E \to R$ and can be viewed as the pushout

$$\begin{tikzcd} E \arrow[r] \arrow[d, "1\wedge c"'] & I \wedge E \arrow[d] \\ R \arrow[r] & \widetilde{E}\,. \end{tikzcd}$$

Let $I_\star$ be the filtration

$$* \longrightarrow \{0,1\} \longrightarrow I \xrightarrow{=} I \xrightarrow{=} I \xrightarrow{=} \cdots$$

of the unit interval $I = [0,1]$, where $\partial I = \{0,1\}$ sits in filtration degree 0. Let $L_\star(R)$ be the non-negative filtration consisting of R's and identity maps between them. We then have a pushout of filtrations

$$\begin{tikzcd} E_\star \arrow[r] \arrow[d, "1\wedge c"] & (I \wedge E)_\star \arrow[d] \\ L_\star(R) \arrow[r] & \widetilde{E}_\star \end{tikzcd} \tag{I.19}$$

with colimit being the pushout square above. That this is indeed a pushout of filtrations can be checked in each filtration degree separately, noting that

$$(I \wedge E)_k = \partial I \wedge E_k \cup I \wedge E_{k-1} \cong E_k \cup CE_{k-1}\,.$$

It follows as in Lemma I.5.9 that we have a commutative square of associated chain complexes

$$\begin{tikzcd} P_{*,*} \arrow[r, "\cong"] \arrow[d, "\epsilon"'] & \partial I_* \otimes P_{*,*} \arrow[r, tail] & I_* \otimes P_{*,*} \arrow[d] \\ R_* \arrow[rr, tail] & & \widetilde{E}^1_{*,*}\,. \end{tikzcd} \tag{I.20}$$

Here R_* is the chain complex consisting of R_* concentrated in homological degree 0, and I_* is the reduced cellular chain complex

$$0 \longrightarrow \mathbb{Z}\{i_1\} \xrightarrow{\partial_1} \mathbb{Z}\{i_0\} \longrightarrow 0$$

of I, with $\partial_1(i_1) = i_0$. Both i_0 and i_1 have internal degree 0, and lie in homological degree as indicated by their subscript. The chain complex ∂I_* is the subcomplex given by $\mathbb{Z}\{i_0\}$ concentrated in homological degree 0. Since the map

$$P_{*,*} \cong \partial I_* \otimes P_{*,*} \longrightarrow I_* \otimes P_{*,*}$$

is injective, a Mayer–Vietoris argument for the filtration subquotients of (I.19) shows that (I.20) is in fact a pushout of chain complexes. This proves that $\widetilde{E}^1_{*,*}$ is indeed the algebraic mapping cone of $\epsilon\colon P_{*,*} \to R_*$, by the definition of the latter chain complex. □

Lemma I.6.6. *The (E^1, d^1)-page of the non-equivariant homotopy spectral sequence associated to $(\widetilde{E} \wedge_R \widetilde{E})_\star$ is isomorphic to $(\widetilde{P}_{*,*} \otimes_{R_*} \widetilde{P}_{*,*}, \tilde{\partial} \otimes 1 + 1 \otimes \tilde{\partial})$.*

Proof. This is very similar to Lemma I.5.11. □

Lemma I.6.7. *The homomorphism $\pi_*(\widetilde{E}_{i-1}) \to \pi_*(\widetilde{E}_i)$ is zero, for each i.*

Proof. This follows from the exactness of $(\widetilde{E}^1_{*,*}, \widetilde{d}^1) \cong (\widetilde{P}_{*,*}, \tilde{\partial})$, by an induction on i in the unrolled exact couple (I.18). The claim is clear for $i \leq 0$. Assume by induction that $\alpha\colon \pi_*(\widetilde{E}_{i-1}) \to \pi_*(\widetilde{E}_i)$ is zero, for some $i \geq 0$. Then $\beta\colon \pi_{i+*}(\widetilde{E}_i) \to \widetilde{P}_{i,*}$ is injective. Consider any class $x \in \pi_{i+*}(\widetilde{E}_i)$. Since $\tilde{\partial}_i(\beta(x)) = \beta\partial\beta(x) = 0$, exactness at $\widetilde{P}_{i,*}$ implies that $\beta(x) = \tilde{\partial}_{i+1}(y) = \beta\partial(y)$ for some $y \in \widetilde{P}_{i+1,*}$. By injectiveness of β it follows that $x = \partial(y)$. Since x was arbitrary, $\partial\colon \widetilde{P}_{i+1,*} \to \pi_{i+*}(\widetilde{E}_i)$ is surjective, so $\alpha\colon \pi_*(\widetilde{E}_i) \to \pi_*(\widetilde{E}_{i+1})$ is zero. □

Lemma I.6.8. *There always exists an R-module map of orthogonal G-spectra*

$$N\colon \widetilde{E} \wedge_R \widetilde{E} \longrightarrow \widetilde{E}$$

extending $\nabla\colon \widetilde{E} \cup_R \widetilde{E} \to \widetilde{E}$, and any two choices are homotopic.

Proof. This follows by obstruction theory, since

$$\widetilde{EG} \cup \widetilde{EG} \cong \widetilde{EG} \wedge S^0 \cup S^0 \wedge \widetilde{EG} \subset \widetilde{EG} \wedge \widetilde{EG}$$

can be given the structure of a free relative G-CW complex, and $\pi_*(\widetilde{E}) = 0$. □

The above lemma, together with the map $\Delta_+ : EG_+ \to EG_+ \wedge EG_+$, makes sure that the Tate construction is multiplicative, in the sense that each G-equivariant R-module pairing $X \wedge_R Y \to Z$ induces an R-module pairing $X^{tG} \wedge_R Y^{tG} \to Z^{tG}$. See Section I.6.2. To arrange that the Tate spectral sequence preserves this structure we need to make sure that we can find a filtration-preserving approximation of N, in the same way as we could find the filtration-preserving approximation of D. The following proposition addresses difficulties raised in Problem 11.8 and Problem 14.8 of [GM95].

Proposition I.6.9. *Suppose that $R[G]_*$ is projective over R_*. Then there exists a filtration-preserving map*

$$N\colon (\widetilde{E} \wedge_R \widetilde{E})_\star \longrightarrow \widetilde{E}_\star$$

of R-modules in orthogonal G-spectra, extending the fold map

$$\nabla\colon \widetilde{E}_\star \cup_R \widetilde{E}_\star \cong (\widetilde{E}_\star \wedge_R R) \cup (R \wedge_R \widetilde{E}_\star) \longrightarrow \widetilde{E}_\star \,.$$

Proof. We inductively assume that ∇ has been extended to a filtration-preserving map $N_{k-1}\colon (\widetilde{E} \wedge_R \widetilde{E})_{k-1} \to \widetilde{E}_{k-1}$, and show that N_{k-1} can be further extended to a filtration-preserving map $N_k\colon (\widetilde{E} \wedge_R \widetilde{E})_k \to \widetilde{E}_k$. It suffices to extend N_{k-1} over $\widetilde{E}_i \wedge_R \widetilde{E}_j$ for $i, j \geq 1$ with $i + j = k$. In particular, there is only something to prove for $k \geq 2$. Let us consider the diagram

$$\begin{tikzcd}
\widetilde{E}_{i-1} \wedge_R \widetilde{E}_j \cup \widetilde{E}_i \wedge_R \widetilde{E}_{j-1} \arrow[r, "N_{k-1}"] \arrow[d] & \widetilde{E}_{k-1} \arrow[d, "\alpha"] \\
\widetilde{E}_i \wedge_R \widetilde{E}_j \arrow[r, dashed, "N_{i,j}"] \arrow[d] & \widetilde{E}_k \\
\widetilde{E}_i/\widetilde{E}_{i-1} \wedge_R \widetilde{E}_j/\widetilde{E}_{j-1} &
\end{tikzcd}$$

where the left hand column is a (Hurewicz) cofibre sequence. By the homotopy extension property, in order to find a dashed map $N_{i,j}$ making the diagram commute, it suffices to find an extension up to homotopy of $\alpha \circ N_{k-1}$. Let

$$W = E_{i-1}/E_{i-2} \wedge_R E_{j-1}/E_{j-2} \cong R \wedge G^{\wedge i-1} \wedge G_+ \wedge G^{\wedge j-1} \wedge G_+$$

so that $\Sigma^2 W \cong \widetilde{E}_i/\widetilde{E}_{i-1} \wedge_R \widetilde{E}_j/\widetilde{E}_{j-1}$. There is then a (stably defined) homotopy cofibre sequence

$$\Sigma W \xrightarrow{\partial} \widetilde{E}_{i-1} \wedge_R \widetilde{E}_j \cup \widetilde{E}_i \wedge_R \widetilde{E}_{j-1} \longrightarrow \widetilde{E}_i \wedge_R \widetilde{E}_j \longrightarrow \Sigma^2 W$$

and it suffices to prove that $\alpha \circ N_{k-1} \circ \partial\colon \Sigma W \to \widetilde{E}_k$ is null-homotopic. We confirm this by showing that α induces the trivial homomorphism

$$\alpha_*\colon [\Sigma W, \widetilde{E}_{k-1}]^G_R \longrightarrow [\Sigma W, \widetilde{E}_k]^G_R \,,$$

where $[-,-]^G_R$ denotes homotopy classes of G-maps of R-modules in orthogonal G-spectra. Note that G acts diagonally on the two copies of G_+ in W, so that there is an untwisting isomorphism $W \cong V \wedge G_+$ where

$$V = R \wedge G^{\wedge i-1} \wedge G^{\wedge j-1} \wedge G_+$$

has trivial G-action. By adjunction we can therefore rewrite the homomorphism above as

$$\alpha_*\colon [\Sigma V, \widetilde{E}_{k-1}]_R \longrightarrow [\Sigma V, \widetilde{E}_k]_R$$

where $[-,-]_R$ denotes homotopy classes of maps of (non-equivariant) R-modules. By our assumption that $R[G]_*$ is R_*-projective, it follows that

$$\pi_*(V) \cong \overline{R[G]}_*^{\otimes i-1} \otimes_{R_*} \overline{R[G]}_*^{\otimes j-1} \otimes_{R_*} R[G]_*$$

is R_*-projective. Hence we can rewrite α_* as the homomorphism

$$\mathrm{Hom}_{R_*}(\Sigma\pi_*(V), \pi_*(\widetilde{E}_{k-1})) \longrightarrow \mathrm{Hom}_{R_*}(\Sigma\pi_*(V), \pi_*(\widetilde{E}_k))$$

given by composition with $\alpha\colon \pi_*(\widetilde{E}_{k-1}) \to \pi_*(\widetilde{E}_k)$. By Lemma I.6.7 that homomorphism is zero, which completes the proof. □

Definition I.6.10. Suppose that $R[G]_*$ is projective over R_*. Let

$$\Phi\colon \widetilde{P}_{*,*} \otimes_{R_*} \widetilde{P}_{*,*} \longrightarrow \widetilde{P}_{*,*}$$

be the $R[G]_*$-module chain map that corresponds, under the isomorphism of Lemma I.6.5 and Lemma I.6.6, to the pairing N^1 of (E^1, d^1)-pages induced by the filtration-preserving map $N\colon (\widetilde{E} \wedge_R \widetilde{E})_\star \to \widetilde{E}_\star$ of Proposition I.6.9.

Lemma I.6.11. *Suppose that $R[G]_*$ is projective over R_*. Then the map*

$$\Phi\colon \widetilde{P}_{*,*} \otimes_{R_*} \widetilde{P}_{*,*} \to \widetilde{P}_{*,*}$$

is uniquely characterized, up to $R[G]_*$-module chain homotopy, by being an $R[G]_*$-module chain map that extends the fold map ∇.

Proof. By construction, Φ extends the fold map, and it follows that this map is unique up chain homotopy equivalence by Proposition I.2.30. □

I.6.2 The G-Tate construction

Let X be an R-module in orthogonal G-spectra[8]. In this section, we discuss the Tate construction and its multiplicative properties.

Definition I.6.12. The **G-Tate construction** X^{tG} is the G-fixed point spectrum of (a fibrant replacement of) $\widetilde{EG} \wedge F(EG_+, X)$:

$$X^{tG} = \left(\widetilde{EG} \wedge F(EG_+, X)\right)^G$$

Note that the homotopy groups

$$\pi_*(X^{tG}) \cong \pi_*^G(\widetilde{EG} \wedge F(EG_+, X))$$

naturally form an R_*-module, and that we can write

$$\widetilde{EG} \wedge F(EG_+, X) \cong \widetilde{E} \wedge_R F_R(E, X)\,.$$

[8]If X is an orthogonal G-spectrum without R-action, the discussion in this section applies to $R \wedge X$ in place of X.

The inclusion $S^0 \to \widetilde{EG}$ induces a G-map

$$F(EG_+, X) \cong S^0 \wedge F(EG_+, X) \longrightarrow \widetilde{EG} \wedge F(EG_+, X)$$

and a map of their corresponding G-fixed points $X^{hG} \longrightarrow X^{tG}$. We can write these as maps of R-modules, using the inclusion $R \to \widetilde{E}$, to obtain a G-map

$$F_R(E, X) \cong R \wedge_R F_R(E, X) \longrightarrow \widetilde{E} \wedge_R F_R(E, X)$$

and a canonical map

$$\gamma\colon X^{hG} = F_R(E, X)^G \longrightarrow \left(\widetilde{E} \wedge_R F_R(E, X)\right)^G = X^{tG},$$

inducing a homomorphism $\pi_*(X^{hG}) \to \pi_*(X^{tG})$ of R_*-modules.

The Tate construction interacts well with the multiplicative structure on homotopy fixed points we described in the paragraph preceding Remark I.5.5. Note first that given a pairing $\mu : X \wedge_R Y \to Z$ of R-modules in orthogonal G-spectra, the R-module pairing $X^{hG} \wedge_R Y^{hG} \to Z^{hG}$ extends to R-module pairings $X^{tG} \wedge_R Y^{hG} \to Z^{tG}$ and $X^{hG} \wedge_R Y^{tG} \to Z^{tG}$. The first is given by a composite

$$\begin{aligned}
\left(\widetilde{E} \wedge_R F_R(E, X)\right)^G \wedge_R F_R(E, Y)^G &\xrightarrow{\wedge} \left(\widetilde{E} \wedge_R F_R(E, X) \wedge_R F_R(E, Y)\right)^G \\
&\xrightarrow{1\wedge\alpha} \left(\widetilde{E} \wedge_R F_R(E \wedge_R E, X \wedge_R Y)\right)^G \\
&\xrightarrow{1\wedge(1\wedge\Delta_+)^*} \left(\widetilde{E} \wedge_R F_R(E, X \wedge_R Y)\right)^G \\
&\xrightarrow{1\wedge\mu_*} \left(\widetilde{E} \wedge_R F_R(E, Z)\right)^G .
\end{aligned}$$

The second one is similar, and left to the reader. The two pairings induce R_*-module pairings $\pi_*(X^{tG}) \otimes_{R_*} \pi_*(Y^{hG}) \to \pi_*(Z^{tG})$ and $\pi_*(X^{hG}) \otimes_{R_*} \pi_*(Y^{tG}) \to \pi_*(Z^{tG})$. These pairings are all compatible via the canonical map, meaning that the R-module diagram

$$\begin{tikzcd}
X^{tG} \wedge_R Y^{hG} \arrow[d] & X^{hG} \wedge_R Y^{hG} \arrow[l, "\gamma\wedge 1"'] \arrow[r, "1\wedge\gamma"] \arrow[d] & X^{hG} \wedge_R Y^{tG} \arrow[d] \\
Z^{tG} & Z^{hG} \arrow[l, "\gamma"'] \arrow[r, "\gamma"] & Z^{tG} ,
\end{tikzcd}$$

and the induced R_*-module diagram both commute. Per Lemma I.6.8 we can choose a unique (up to homotopy) extension $N\colon \widetilde{E} \wedge_R \widetilde{E} \to \widetilde{E}$ of the fold map $\nabla\colon \widetilde{E} \cup_R \widetilde{E} \to \widetilde{E}$, in the category of R-modules in orthogonal G-spectra. We can then promote the two R-module pairings to an R-module pairing $X^{tG} \wedge_R Y^{tG} \to$

Z^{tG}, given by the composite

$$\begin{aligned}
&\Big(\widetilde{E} \wedge_R F_R(E,X)\Big)^G \wedge_R \Big(\widetilde{E} \wedge_R F_R(E,Y)\Big)^G \\
&\qquad \xrightarrow{\wedge} \Big(\widetilde{E} \wedge_R F_R(E,X) \wedge_R \widetilde{E} \wedge_R F_R(E,Y)\Big)^G \\
&\qquad \xrightarrow{1\wedge\tau\wedge 1} \Big(\widetilde{E} \wedge_R \widetilde{E} \wedge_R F_R(E,X) \wedge_R F_R(E,Y)\Big)^G \\
&\qquad \xrightarrow{1\wedge 1\wedge\alpha} \Big(\widetilde{E} \wedge_R \widetilde{E} \wedge_R F_R(E \wedge_R E, X \wedge_R Y)\Big)^G \\
&\qquad \xrightarrow{N\wedge(1\wedge\Delta_+)^*} \Big(\widetilde{E} \wedge_R F_R(E, X \wedge_R Y)\Big)^G \\
&\qquad \xrightarrow{1\wedge\mu_*} \Big(\widetilde{E} \wedge_R F_R(E,Z)\Big)^G .
\end{aligned}$$

These pairings are also compatible via the canonical map, meaning that the R-module diagram

$$\begin{tikzcd}
X^{tG} \wedge_R Y^{hG} \arrow[r, "1\wedge\gamma"] \arrow[dr] & X^{tG} \wedge_R Y^{tG} \arrow[d] & X^{hG} \wedge_R Y^{tG} \arrow[l, "\gamma\wedge 1"'] \arrow[dl] \\
& Z^{tG} &
\end{tikzcd}$$

and the induced R_*-module diagram both commute. Taken together, these diagrams show that

$$\gamma\colon X^{hG} \to X^{tG} \quad \text{and} \quad \gamma_*\colon \pi_*(X^{hG}) \to \pi_*(X^{tG})$$

are multiplicative. We would now like to access $\pi_*(X^{tG})$ and the pairings above through filtrations and their associated spectral sequences.

I.6.3 The Hesselholt–Madsen filtration

We can now generalize the filtration of X^{tG} from [HM03, §4.3] to the case of compact Lie groups G.

Definition I.6.13. Let

$$HM_\star(X) = (\widetilde{E} \wedge_R T(M(X)))_\star$$

be the filtration

$$\cdots \to HM_{k-1}(X) \to HM_k(X) \to HM_{k+1}(X) \to \dots$$

of R-modules in orthogonal G-spectra given by the Day convolution product of the filtrations $\widetilde{E}_\star$ and $T_\star(M(X))$.

Recall that we introduced the filtration $\widetilde{E}_\star$ in Definition I.6.3, the sequence $M_\star(X)$ in Section I.5.2, and its telescopic approximation $T_\star(M(X))$ in Section I.4.3. The convolution product of $\widetilde{E}_\star$ and $T_\star(M(X))$ was defined in Section I.4.7, and is a filtration by Proposition I.4.31. We can realize

$$HM_k(X) = \bigcup_{i+j=k} \widetilde{E}_i \wedge_R T_j(M(X))$$

as a subspectrum of $\widetilde{E} \wedge_R \mathrm{Tel}(M_\star(X))$. The structure maps $HM_{k-1}(X) \to HM_k(X)$ are then inclusions of subspectra. These are (strong) h-cofibrations, so the canonical map

$$\mathrm{Tel}(HM_\star(X)) \longrightarrow \operatorname*{colim}_k HM_k(X) = \widetilde{E} \wedge_R \mathrm{Tel}(M_\star(X))$$

is an equivalence. Since $M_j(X) = F_R(E, X)$ for all $j \geq 0$ there is a deformation retraction

$$\mathrm{Tel}(M_\star(X)) \xrightarrow{\simeq_G} F_R(E, X)$$

and a further equivalence

$$\widetilde{E} \wedge_R \mathrm{Tel}(M_\star(X)) \xrightarrow{\simeq_G} \widetilde{E} \wedge_R F_R(E, X) \cong \widetilde{EG} \wedge F(EG_+, X)\,.$$

Definition I.6.14. Let X be an R-module in orthogonal G-spectra. We define the **G-Tate spectral sequence** for X to be the R_*-module spectral sequence $(\hat{E}^r(X), d^r)$ associated to the filtration $HM_\star(X)$ with

$$\hat{E}^r(X) = E^r(HM_\star(X))$$

for each $r \geq 1$.

The abutment of the G-Tate spectral sequence for X is the colimit

$$A_\infty(HM_\star(X)) \cong \pi_*^G \,\mathrm{Tel}(HM_\star(X)) \cong \pi_*^G(\widetilde{E} \wedge_R F_R(E, X)) \cong \pi_*(X^{tG})\,,$$

filtered by the image submodules

$$F_k\pi_*(X^{tG}) = \mathrm{im}\left(\pi_*^G(HM_k(X)) \to \pi_*^G \,\mathrm{Tel}(HM_\star(X)) \cong \pi_*(X^{tG})\right).$$

Remark I.6.15. In general, we do not claim that the G-Tate spectral sequence converges to the stated abutment, neither in the conditional nor in the weak sense. As we recalled in Section I.4.2, conditional convergence to the colimit holds if $\operatorname{holim}_k HM_k(X) \simeq_G *$. The latter condition would follow from an interchange of homotopy colimits and homotopy limits. More precisely, for each $a \geq 0$ and integer k, consider the subspectrum

$$S_{a,k} = \bigcup_{\substack{i+j=k \\ i \leq a}} \widetilde{E}_i \wedge_R T_j(M(X))$$

of $\widetilde{E} \wedge_R \mathrm{Tel}(M_\star(X))$. Then $\mathrm{hocolim}_a S_{a,k} \simeq_G HM_k(X)$, and the sufficient condition $\mathrm{holim}_k HM_k(X) \simeq_G *$ for conditional convergence is equivalent to

$$\operatorname*{holim}_k \operatorname*{hocolim}_a S_{a,k} \simeq_G * \,. \tag{I.21}$$

On the other hand, $\mathrm{holim}_j \widetilde{E}_i \wedge_R T_j(M(X)) \simeq_G *$ for each i, since $F_i\widetilde{EG}$ is a finite G-CW space. It follows by induction that $\mathrm{holim}_k S_{a,k} \simeq_G *$ for each finite a, which implies that

$$\operatorname*{hocolim}_a \operatorname*{holim}_k S_{a,k} \simeq_G * \,. \tag{I.22}$$

Without further hypotheses we do not see how to deduce (I.21) from (I.22).

I.6.4 Algebraic description of $\hat{E}^1$ and $\hat{E}^2$

Under the assumption that $R[G]_*$ is finitely generated projective over R_*, we can algebraically describe the E^1- and E^2-pages of the G-Tate spectral sequence, in the same way as we did for the G-homotopy fixed points spectral sequence in Section I.5.3.

Proposition I.6.16. *Suppose that $R[G]_*$ is R_*-projective. There is then a natural isomorphism of R_*-module chain complexes*

$$E^1_{*,*}(HM_\star(X)) \cong \mathrm{Hom}_{R[G]_*}(R_*, \widetilde{P}_{*,*} \otimes_{R_*} \mathrm{Hom}_{R_*}(P_{*,*}, \pi_*(X)))\,.$$

In the notation of Definition I.2.12 and Definition I.6.14, we have

$$\hat{E}^1_{*,*}(X) \cong \mathrm{Hom}_{R[G]_*}(R_*, \mathrm{hm}_*(\pi_*(X)))\,,$$

where the d^1-differential on the left hand side corresponds to the boundary $\mathrm{Hom}(1, \partial_{\mathrm{hm}})$ *on the right hand side.*

Proof. We first check that the natural restriction homomorphism

$$\omega\colon E^1_{*,*}(HM_\star(X)) \xrightarrow{\cong} \mathrm{Hom}_{R[G]_*}(R_*, E^1_{*,*}((\widetilde{E} \wedge_R T(M(X)))_\star)) \tag{I.23}$$

from Lemma I.3.5 is an isomorphism of R_*-module chain complexes, where $HM_\star(X)$ at the left hand side is treated as an R-module filtration in orthogonal G-spectra, while $(\widetilde{E}\wedge_R T(M(X)))_\star$ at the right hand side refers to the underlying R-module filtration in non-equivariant orthogonal spectra, with the residual $R[G]$-module action. We first note that we have

$$\begin{aligned} E^1_{k,*}(HM_\star(X)) &= \pi^G_{k+*}(HM_{k-1}(X) \to HM_k(X)) \\ &\cong \pi^G_{k+*}(HM_k(X)/HM_{k-1}(X)) \end{aligned}$$

while

$$\begin{aligned} E^1_{k,*}((\widetilde{E} \wedge_R T(M(X)))_\star) &= \pi_{k+*}(HM_{k-1}(X) \to HM_k(X)) \\ &\cong \pi_{k+*}(HM_k(X)/HM_{k-1}(X))\,. \end{aligned}$$

Secondly, we note that

$$\begin{aligned} HM_k(X)/HM_{k-1}(X) &\cong \bigvee_{i+j=k} \widetilde{E}_i/\widetilde{E}_{i-1} \wedge_R T_j(M(X))/T_{j-1}(M(X)) \\ &\cong \bigvee_{i+j=k} \widetilde{E}_i/\widetilde{E}_{i-1} \wedge_R (M_j(X) \cup CM_{j-1}(X)) \\ &\simeq_G \bigvee_{i+j=k} \widetilde{E}_i/\widetilde{E}_{i-1} \wedge_R F_R(E_{-j}/E_{-j-1}, X)\,, \end{aligned}$$

which is moreover G-equivalent to $F(G_+, X')$ for

$$X' = \bigvee_{i+j=k} \widetilde{E}_i/\widetilde{E}_{i-1} \wedge \Sigma^j F(G^{\wedge -j}, X)\,.$$

This uses that $R[G]$ is dualisable (see Definition I.6.28). Proposition I.3.6 therefore implies that the natural restriction homomorphism (I.23) is an isomorphism in every homological degree. We check that it is also an isomorphism of chain complexes. The d^1-differential in the spectral sequence appearing in the left hand side corresponds to the composition

$$\begin{aligned} \pi^G_{k+*}(HM_k(X)/HM_{k-1}(X)) &\xrightarrow{\partial} \pi^G_{k-1+*}(HM_{k-1}(X)) \\ &\longrightarrow \pi^G_{k-1+*}(HM_{k-1}/HM_{k-2}(X)))\,, \end{aligned}$$

which by the naturality of ω corresponds to $\mathrm{Hom}(1, d^1_{k,*})$, where $d^1_{k,*}$ is the d^1-differential in the spectral sequence appearing in the right hand side. This is given by the composite

$$\begin{aligned} \pi_{k+*}(HM_k(X)/HM_{k-1}(X)) &\xrightarrow{\partial} \pi_{k-1+*}(HM_{k-1}(X)) \\ &\longrightarrow \pi_{k-1+*}(HM_{k-1}/HM_{k-2}(X)))\,. \end{aligned}$$

Hence (I.23) is indeed an isomorphism of chain complexes.

We now want to identify $E^1_{*,*}((\widetilde{E}\wedge_R T(M(X)))_\star)$ with the Tate complex $\mathrm{hm}_*(\pi_*(X))$. For this aim, we can use the canonical pairing

$$\iota\colon (\widetilde{E}_\star, T_\star(M(X))) \longrightarrow (\widetilde{E} \wedge_R T(M(X)))_\star$$

of R-module filtrations to obtain an $R[G]_*$-module chain map of the associated E^1-pages

$$\iota^1\colon E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(T_\star(M(X))) \longrightarrow E^1((\widetilde{E} \wedge_R T(M(X)))_\star)$$

as in Theorem I.4.27, but in the non-equivariant setting. This map is the direct sum of the maps

$$\begin{aligned} &\pi_{i+*}(\widetilde{E}_i/\widetilde{E}_{i-1}) \otimes_{R_*} \pi_{j+*}(T_j(M(X))/T_{j-1}(M(X))) \\ &\qquad\qquad \stackrel{\cdot}{\longrightarrow} \pi_{k+*}(\widetilde{E}_i/\widetilde{E}_{i-1} \wedge_R T_j(M(X))/T_{j-1}(M(X))) \end{aligned}$$

for $i + j = k$. Note that each one of these maps is an isomorphism, because $\widetilde{P}_{i,*} = \pi_{i+*}(\widetilde{E}_i/\widetilde{E}_{i-1})$ is projective, hence flat, over R_*. We conclude that ι^1 is an isomorphism of $R[G]_*$-chain complexes, and thus induces an isomorphism

$$\begin{aligned}\mathrm{Hom}(1, \iota^1)\colon\ &\mathrm{Hom}_{R[G]_*}(R_*, E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(T_\star(M(X)))) \\ &\xrightarrow{\cong} \mathrm{Hom}_{R[G]_*}(R_*, E^1((\widetilde{E} \wedge_R T(M(X)))_\star)) \qquad \text{(I.24)}\end{aligned}$$

of R_*-module complexes.

The equivalence $\epsilon\colon T_\star(M(X)) \to M_\star(X)$ induces an isomorphism

$$\epsilon\colon E^1(T_\star(M(X))) \xrightarrow{\cong} E^1(M_\star(X))$$

of $R[G]_*$-module chain complexes, which in turn induces an isomorphism

$$\begin{aligned}\mathrm{Hom}(1, 1 \otimes \epsilon)\colon\ &\mathrm{Hom}_{R[G]_*}(R_*, E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(T_\star(M(X)))) \\ &\xrightarrow{\cong} \mathrm{Hom}_{R[G]_*}(R_*, E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(M_\star(X)))\,, \qquad \text{(I.25)}\end{aligned}$$

of R_*-module chain complexes. Finally, we have

$$E^1_{j,*}(M_\star(X)) \cong \pi_{j+*}(F_R(E_{-j}/E_{-j-1}, X)) \cong \mathrm{Hom}_{R_*}(P_{-j,*}, \pi_*(X))$$

as $R[G]_*$-modules, because $\pi_{-j+*}(E_{-j}/E_{-j-1}) \cong P_{-j,*}$ is R_*-projective. The d^1-differentials correspond to ∂ and $\mathrm{Hom}(\partial, 1)$ by the argument in the proof of Proposition I.5.13. Hence we have an isomorphism

$$\begin{aligned}&\mathrm{Hom}_{R[G]_*}(R_*, E^1_{*,*}(\widetilde{E}_\star) \otimes_{R_*} E^1_{*,*}(M_\star(X))) \\ &\cong \mathrm{Hom}_{R[G]_*}(R_*, \widetilde{P}_{*,*} \otimes_{R_*} \mathrm{Hom}_{R_*}(P_{*,*}, \pi_*(X))) \qquad \text{(I.26)}\end{aligned}$$

of R_*-module complexes.

When strung together, the numbered isomorphisms (I.23) through (I.26) establish the asserted identification of the G-Tate spectral sequence (E^1, d^1)-page for the orthogonal G-spectrum R-module X with the Tate complex for the $R[G]_*$-module $\pi_*(X)$. □

Theorem I.6.17. *Let X be an R-module in orthogonal G-spectra, and suppose that $R[G]_*$ is R_*-projective. Then there is a natural R_*-module isomorphism*

$$\hat{E}^2_{i,*}(X) = E^2_{i,*}(HM_\star(X)) \cong \widehat{\mathrm{Ext}}^{-i}_{R[G]_*}(R_*, \pi_*(X))\,,$$

for each integer i.

Proof. This is immediate by passage to homology from Proposition I.6.16. Here we are using the definition of Hopf algebra Tate cohomology given in Definition I.2.13. □

We now go on to discuss the multiplicative structure of the Tate spectral sequence. Let $\mu\colon X \wedge_R Y \to Z$ be a pairing of R-modules in orthogonal G-spectra. As discussed in the paragraph before Theorem I.5.6, the diagonal approximation D and μ combine to define a pairing $\bar{\mu}\colon (M_\star(X), M_\star(Y)) \to M_\star(Z)$ of sequences of R-modules in orthogonal G-spectra. By Lemma I.4.21 we have an induced pairing

$$T(\bar{\mu})\colon (T_\star(M(X)), T_\star(M(Y))) \longrightarrow T_\star(M(Z))$$

of filtrations. By Proposition I.6.9 there is also a pairing of filtrations

$$N\colon (\widetilde{E}_\star, \widetilde{E}_\star) \longrightarrow \widetilde{E}_\star$$

which extends the fold map. Hence $(\widetilde{E}_\star, N)$ is a multiplicative R-module filtration in orthogonal G-spectra. We can now form the induced pairing of convolution filtrations

$$\theta = N \wedge T(\bar{\mu})\colon (HM_\star(X), HM_\star(Y)) \longrightarrow HM_\star(Z)\,.$$

This has components

$$\theta_{i,j}\colon HM_i(X) \wedge_R HM_j(Y) \longrightarrow HM_{i+j}(Z)$$

given by the union over $i_1 + i_2 = i$ and $j_1 + j_2 = j$ of the composite maps

$$\begin{aligned}
&\widetilde{E}_{i_1} \wedge_R T_{i_2}(M(X)) \wedge_R \widetilde{E}_{j_1} \wedge_R T_{j_2}(M(Y)) \\
&\qquad\xrightarrow{1\wedge\tau\wedge 1} \widetilde{E}_{i_1} \wedge_R \widetilde{E}_{j_1} \wedge_R T_{i_2}(M(X)) \wedge_R T_{j_2}(M(Y)) \\
&\qquad\xrightarrow{N_{i_1,j_1}\wedge T(\bar{\mu})_{i_2,j_2}} \widetilde{E}_{i_1+j_1} \wedge_R T_{i_2+j_2}(M(Z)) \\
&\qquad\xrightarrow{\iota_{i_1+j_1,i_2+j_2}} HM_{i+j}(Z)\,.
\end{aligned}$$

Viewing $HM_i(X)$ as a subspectrum of $\widetilde{E} \wedge_R \mathrm{Tel}(M_\star(X))$ (and similarly for Y and Z in place of X) the maps $\theta_{i,j}$ are compatible with the composite map

$$\begin{aligned}
\widetilde{E} \wedge_R F_R(E, X) \wedge_R \widetilde{E} \wedge_R F_R(E, Y) &\xrightarrow{1\wedge\tau\wedge 1} \widetilde{E} \wedge_R \widetilde{E} \wedge_R F_R(E, X) \wedge_R F_R(E, Y) \\
&\xrightarrow{1\wedge\alpha} \widetilde{E} \wedge_R \widetilde{E} \wedge_R F_R(E \wedge_R E, X \wedge_R Y) \\
&\xrightarrow{N\wedge 1} \widetilde{E} \wedge_R F_R(E \wedge_R E, X \wedge_R Y) \\
&\xrightarrow{(1\wedge D_+)^*} \widetilde{E} \wedge_R F_R(E, X \wedge_R Y) \\
&\xrightarrow{1\wedge\mu_*} \widetilde{E} \wedge_R F_R(E, Z)\,.
\end{aligned}$$

This is G-homotopic to the corresponding map with Δ in place of D, which defines the product

$$\theta_*\colon \pi_*(X^{tG}) \otimes_{R_*} \pi_*(Y^{tG}) \to \pi_*(Z^{tG})$$

that we introduced in Section I.6.2. Hence this product is filtration-preserving, taking $F_i\pi_*(X^{tG}) \otimes_{R_*} F_j\pi_*(Y^{tG})$ to $F_{i+j}\pi_*(Z^{tG})$ for all i and j. We write $\bar{\theta}_*$ for the induced pairing of filtration subquotients.

Theorem I.6.18. *Let $\mu\colon X \wedge_R Y \to Z$ be a pairing of R-modules in orthogonal G-spectra, and assume that $R[G]_*$ is projective over R_*. The pairing*

$$\theta = N \wedge T(\bar{\mu})\colon (HM_\star(X), HM_\star(Y)) \to HM_\star(Z)$$

of filtrations induces a pairing of G-Tate spectral sequences

$$\theta\colon \hat{E}^*(X) \otimes_{R_*} \hat{E}^*(Y) \longrightarrow \hat{E}^*(Z)\,.$$

in the sense of Definition I.4.9. Moreover, the induced pairing θ^∞ of E^∞-pages is compatible with the pairing $\bar{\theta}_*$ of filtration subquotients, in the sense of Proposition I.4.12.

Proof. This is a direct consequence of Theorem I.4.27. □

Corollary I.6.19. *If (X, μ) a multiplicative R-module in orthogonal G-spectra, then $(HM_\star(X), N \wedge T(\bar{\mu}))$ is a multiplicative filtration, and the G-Tate spectral sequence*

$$(\hat{E}^r(X), d^r) = (E^r(HM_\star(X)), d^r)$$

is a multiplicative spectral sequence with multiplicative abutment $\pi_*(X^{tG})$.

Proof. This follows from Corollary I.4.28. □

Proposition I.6.20. *Let $\mu\colon X \wedge_R Y \to Z$ be a pairing of R-modules in orthogonal G-spectra and assume that $R[G]_*$ is R_*-projective. Under the isomorphism of Proposition I.6.16, the pairing*

$$\theta^1\colon \hat{E}^1(X) \otimes_{R_*} \hat{E}^1(Y) \longrightarrow \hat{E}^1(Z)$$

corresponds to the pairing covariantly induced by $\Phi\colon \widetilde{P}_{*,*} \otimes_{R_*} \widetilde{P}_{*,*} \longrightarrow \widetilde{P}_{*,*}$ and contravariantly induced by $\Psi\colon P_{*,*} \longrightarrow P_{*,*} \otimes_{R_*} P_{*,*}$, as in Section I.2.5.

Proof. For typographical reasons we will use the abbreviation

$$M^{R[G]_*} = \mathrm{Hom}_{R[G]_*}(R_*, M)$$

in what follows, for various $R[G]_*$-modules M. In the same way as in the proof of Proposition I.6.16, the notation $E^1(HM_\star(X))$ will refer to the E^1-page of the associated spectral sequence on equivariant homotopy groups, while $E^1((\widetilde{E} \wedge_R T(M(X)))_\star)$ will refer to the E^1-page of the associated non-equivariant spectral sequence.

We first note some results regarding multiplicative compatibility. Firstly, the natural homomorphism ω is monoidal by Lemma I.3.7, so the diagram

$$\begin{tikzcd}
E^1(HM_\star(X)) \otimes_{R_*} E^1(HM_\star(Y)) \arrow[rr, "\theta^1"] \arrow[d, "\omega\otimes\omega"', "\cong"] & & E^1(HM_\star(Z)) \arrow[dd, "\cong"', "\omega"] \\
\begin{matrix} E^1((\widetilde{E} \wedge_R T(M(X)))_\star)^{R[G]_*} \\ \otimes_{R_*} \\ E^1((\widetilde{E} \wedge_R T(M(Y)))_\star)^{R[G]_*} \end{matrix} \arrow[d, "\alpha"'] & & \\
\begin{pmatrix} E^1((\widetilde{E} \wedge_R T(M(X)))_\star) \\ \otimes_{R_*} \\ E^1((\widetilde{E} \wedge_R T(M(X)))_\star) \end{pmatrix}^{R[G]_*} \arrow[rr, "(\theta^1)^{R[G]_*}"] & & E^1((\widetilde{E} \wedge_R T(M(Z)))_\star)^{R[G]_*}
\end{tikzcd}$$

commutes. Secondly, by a slight generalization of Lemma I.4.30, the pairing

$$\iota^1 \colon E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(T_\star(M(X))) \longrightarrow E^1((\tilde{E} \wedge_R T(M(X)))_\star)$$

and its variants for Y and Z in place of X are multiplicatively compatible in the sense that the diagram

$$\begin{tikzcd}
\begin{matrix} E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(T_\star(M(X))) \\ \otimes_{R_*} \\ E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(T_\star(M(Y))) \end{matrix} \arrow[r, "1\otimes\tau\otimes 1"] \arrow[dd, "\iota^1\otimes\iota^1"', "\cong"] & \begin{matrix} E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(\widetilde{E}_\star) \\ \otimes_{R_*} \\ E^1(T_\star(M(X))) \otimes_{R_*} E^1(T_\star(M(Y))) \end{matrix} \arrow[d, "N^1\otimes T(\bar{\mu})^1"] \\
 & E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(T_\star(M(Z))) \arrow[d, "\cong"', "\iota^1"] \\
\begin{matrix} E^1((\widetilde{E} \wedge_R T(M(X)))_\star) \\ \otimes_{R_*} \\ E^1((\widetilde{E} \wedge_R T(M(Y)))_\star) \end{matrix} \arrow[r, "\theta^1"] & E^1((\widetilde{E} \wedge_R T(M(Z)))_\star)
\end{tikzcd}$$

commutes. Note that, by Definition I.6.10, the map

$$N^1 \colon E^1(\widetilde{E}_\star) \otimes_{R_*} E^1(\widetilde{E}_\star) \longrightarrow E^1(\widetilde{E}_\star)$$

corresponds to $\Phi \colon \widetilde{P}_{*,*} \otimes_{R_*} \widetilde{P}_{*,*} \to \widetilde{P}_{*,*}$ under the isomorphism $E^1_{*,*}(\widetilde{E}_\star) \cong \widetilde{P}_{*,*}$. As discussed in the proofs of Proposition I.4.25, Theorem I.5.6 and (the non-equivariant version of) Proposition I.5.13,

$$T(\bar{\mu})^1 \colon E^1(T_\star(M(X))) \otimes_{R_*} E^1(T_\star(M(Y))) \longrightarrow E^1(T_\star(M(Z)))$$

corresponds to the composite homomorphism

$$\begin{aligned}\operatorname{Hom}_{R_*}(P_{*,*},\pi_*(X)) \otimes_{R_*} \operatorname{Hom}_{R_*}(P_{*,*},\pi_*(Y))& \\ &\xrightarrow{\alpha} \operatorname{Hom}_{R_*}(P_{*,*}\otimes_{R_*} P_{*,*},\pi_*(X)\otimes_{R_*}\pi_*(Y)) \\ &\xrightarrow{\Psi^*} \operatorname{Hom}_{R_*}(P_{*,*},\pi_*(X)\otimes_{R_*}\pi_*(Y)) \\ &\xrightarrow{\mu_*} \operatorname{Hom}_{R_*}(P_{*,*},\pi_*(Z))\end{aligned}$$

under the isomorphisms

$$E^1(T_\star(M(X))) \cong E^1(M_\star(X)) \cong \operatorname{Hom}_{R_*}(P_{*,*},\pi_*(X))$$

and their variants with Y and Z in place of X.

Combining all of these results, we have shown that θ^1 corresponds to the composite

$$\begin{aligned}&\operatorname{hm}(\pi_*(X))^{R[G]_*} \otimes_{R_*} \operatorname{hm}(\pi_*(Y))^{R[G]_*} \xrightarrow{\alpha} (\operatorname{hm}_*(\pi_*(X)) \otimes_{R_*} \operatorname{hm}(\pi_*(Y)))^{R[G]_*} \\ &\quad\xrightarrow{1\otimes\tau\otimes 1} (\widetilde{P}_{*,*} \otimes_{R_*} \widetilde{P}_{*,*} \otimes_{R_*} \operatorname{Hom}_{R_*}(P_{*,*},\pi_*(X)) \otimes_{R_*} \operatorname{Hom}_{R_*}(P_{*,*},\pi_*(Y)))^{R[G]_*} \\ &\quad\xrightarrow{1\otimes 1\otimes\alpha} (\widetilde{P}_{*,*} \otimes_{R_*} \widetilde{P}_{*,*} \otimes_{R_*} \operatorname{Hom}_{R_*}(P_{*,*}\otimes_{R_*} P_{*,*},\pi_*(X)\otimes_{R_*}\pi_*(Y)))^{R[G]_*} \\ &\quad\xrightarrow{\Phi\otimes 1} (\widetilde{P}_{*,*} \otimes_{R_*} \operatorname{Hom}_{R_*}(P_{*,*}\otimes_{R_*} P_{*,*},\pi_*(X)\otimes_{R_*}\pi_*(Y)))^{R[G]_*} \\ &\quad\xrightarrow{1\otimes\Psi^*} \operatorname{hm}_*(\pi_*(X)\otimes_{R_*}\pi_*(Y))^{R[G]_*} \\ &\quad\xrightarrow{1\otimes\mu_*} \operatorname{hm}_*(\pi_*(Z))^{R[G]_*}\,,\end{aligned}$$

where we have abbreviated

$$\operatorname{hm}_*(\pi_*(X)) = \widetilde{P}_{*,*} \otimes_{R_*} \operatorname{Hom}_{R_*}(P_{*,*},\pi_*(X))\,.$$

Note that this is the pairing that induces the cup product, as in Section I.2.5. □

Theorem I.6.21. *Let $\mu\colon X \wedge_R Y \to Z$ be a pairing of R-modules in orthogonal G-spectra, and assume that $R[G]_*$ is R_*-projective. Then the pairing*

$$\theta^2\colon E^2_{i,*}(HM_\star(X)) \otimes_{R_*} E^2_{j,*}(HM_\star(Y)) \longrightarrow E^2_{i+j,*}(HM_\star(Z))$$

of G-Tate spectral sequence E^2-pages corresponds, under the isomorphism of Theorem I.6.17, to the cup product

$$\smile\colon \widehat{\operatorname{Ext}}^{-i,*}_{R[G]_*}(R_*,\pi_*(X)) \otimes_{R_*} \widehat{\operatorname{Ext}}^{-j,*}_{R[G]_*}(R_*,\pi_*(Y)) \longrightarrow \widehat{\operatorname{Ext}}^{-i-j,*}_{R[G]_*}(R_*,\pi_*(Z))$$

associated to $\mu_*\colon \pi_*(X) \otimes_{R_*} \pi_*(Y) \to \pi_*(Z)$.

Proof. This is immediate by passage to homology from Proposition I.6.20. See Section I.2.5 for the definition of the cup product in Hopf algebra Tate cohomology. □

Corollary I.6.22. *If (X, μ) is a multiplicative R-module in orthogonal G-spectra, then the product in $\hat{E}^2(X) = E^2(HM_\star(X))$ corresponds to the cup product in $\widehat{\mathrm{Ext}}^*_{R[G]_*}(R_*, \pi_*(X))$ that is associated to the product μ_* in $\pi_*(X)$.*

Note independence of the particular choices of maps D and N, since the resulting chain homomorphisms Ψ and Φ are unique up to homotopy, per Proposition I.2.28 and Proposition I.2.30.

I.6.5 The Greenlees–May filtration

In [Gre87, §1], Greenlees spliced the filtration $F_\star\widetilde{EG}$ with its Spanier–Whitehead dual to obtain a sequence of G-spectra

$$\cdots \longrightarrow D(F_2\widetilde{EG}) \longrightarrow D(F_1\widetilde{EG}) \longrightarrow \mathbb{S} \longrightarrow \Sigma^\infty F_1\widetilde{EG} \longrightarrow \Sigma^\infty F_2\widetilde{EG} \longrightarrow \cdots$$

with mapping telescope equivalent to $\widetilde{EG}$. The induced sequence

$$\ldots \longrightarrow D(F_1\widetilde{EG})\wedge F(EG_+, X) \longrightarrow \Sigma^\infty F(EG_+, X) \longrightarrow F_1\widetilde{EG}\wedge F(EG_+, X) \longrightarrow \cdots$$

was used in [GM95, (9.5), Thm. 10.3] to define a spectral sequence with abutment being the homotopy groups of the G-Tate construction on X. In this section, we will define a spliced filtration $GM_\star(X)$ with a map to the Hesselholt–Madsen filtration $HM_\star(X)$, and show that the induced map of G-homotopy spectral sequences

$$\check{E}^r(X) = E^r(GM_\star(X)) \longrightarrow E^r(HM_\star(X)) = \hat{E}^r(X)$$

is an isomorphism for $r \geq 2$. Thereafter we show that $GM_\star(X)$ is equivalent to the spliced sequence of Greenlees and May, at least for finite groups G. For other compact Lie groups the sequences will differ in the same way that our filtration $F_\star\widetilde{EG}$ differs from the G-CW skeletal filtration. See Remark I.6.1.

Definition I.6.23. Recall the filtration $\widetilde{E}_\star$ from Definition I.6.3 and let $GM_\star(X)$ be the filtration of orthogonal G-spectra defined as

$$GM_k(X) = \begin{cases} \widetilde{E}_k \wedge_R T_0(M(X)) & \text{for } k \geq 0, \\ \widetilde{E}_0 \wedge_R T_k(M(X)) & \text{for } k \leq 0. \end{cases}$$

The structure maps $GM_{k-1}(X) \to GM_k(X)$ for $k \geq 1$ are induced by the maps $\widetilde{E}_{k-1} \to \widetilde{E}_k$ in the filtration $\widetilde{E}_\star$, while the maps for $k \leq 0$ are those of $T_\star(M(X))$. We refer to the filtration $GM_\star(X)$ as the **Greenlees–May filtration**.

Notation I.6.24. Let

$$\check{E}^r(X) = E^r(GM_\star(X))$$

denote the G-homotopy spectral sequence associated to the filtration $GM_\star(X)$.

We now discuss the map of filtrations between the Greenlees–May filtration and the Hesselholt–Madsen filtration.

Lemma I.6.25. *The inclusions* $\widetilde{E}_k \wedge_R T_0(M(X)) \to (\widetilde{E} \wedge_R T(M(X)))_k$ *for* $k \geq 0$ *and* $\widetilde{E}_0 \wedge_R T_k(M(X)) \to (\widetilde{E} \wedge_R T(M(X)))_k$ *for* $k \leq 0$ *define a map of filtrations*

$$\alpha \colon GM_\star(X) \longrightarrow HM_\star(X)$$

of R*-modules in orthogonal* G*-spectra. The induced maps of mapping telescopes and colimits*

$$\begin{tikzcd}
\mathrm{Tel}(GM_\star(X)) \arrow[r, "\simeq_G"] \arrow[d, "\simeq_G"'] & \mathrm{Tel}(HM_\star(X)) \arrow[d, "\simeq_G"] \\
\widetilde{E} \wedge_R T_0(M(X)) \arrow[r, "\simeq_G"] & \widetilde{E} \wedge_R \mathrm{Tel}(M_\star(X))
\end{tikzcd}$$

are all equivalences.

Proof. Recall from Section I.6.3 that

$$HM_k(X) = \bigcup_{i+j=k} \widetilde{E}_i \wedge_R T_j(M(X))$$

as a subspectrum of $\widetilde{E} \wedge_R \mathrm{Tel}(M(X))$. The existence of the filtered map α is then clear. The vertical maps from mapping telescopes to colimits are equivalences, since $GM_\star(X)$ and $HM_\star(X)$ are both filtrations. The lower horizontal map is also an equivalence, since the sequence $M_\star(X)$ is constant for $\star \geq 0$. $\square$

As a consequence of the above lemma, there is a map

$$\alpha \colon \check{E}^*(X) \to \hat{E}^*(X)$$

of R_*-module spectral sequences.

Remark I.6.26. Recall from Proposition I.6.9 that, under the assumption that $R[G]_*$ is R_*-projective, we have a filtration-preserving pairing $N : (\widetilde{E}_\star, \widetilde{E}_\star) \to \widetilde{E}_\star$. However, when (X, μ) is multiplicative, the induced pairing

$$N \wedge T(\bar{\mu}) : (HM_\star(X), HM_\star(X)) \longrightarrow HM_\star(X)$$

does usually not restrict to a multiplication on $GM_\star(X)$. For instance, $GM_a(X) \wedge_R GM_{-b}(X)$ with $a > 0$ and $b > 0$ maps to $HM_a(X) \wedge_R HM_{-b}(X)$ and $\widetilde{E}_a \wedge_R T_{-b}(M(X))$ in $HM_{a-b}(X)$, which is hardly ever in $GM_{a-b}(X)$. Hence $GM_\star(X)$ is not a multiplicative filtration, and $\check{E}^r(X)$ is not evidently a multiplicative spectral sequence. Nonetheless, we will show that $\check{E}^r(X)$ is isomorphic to the G-Tate spectral sequence $\hat{E}^r(X)$ for $r \geq 2$, which we showed to be multiplicative in Theorem I.6.18. This will then show that $(\check{E}^r(X), d^r)$ is also multiplicative, at least for $r \geq 2$.

Thinking only about the additive properties of the spectral sequence $\check{E}^r(X)$, we can safely replace the filtration $GM_\star(X)$ with a simpler, but equivalent, sequence.

Lemma I.6.27. *There is an equivalence from $GM_\star(X)$ to the sequence $GM'_\star(X)$ with*

$$GM'_k(X) = \begin{cases} \widetilde{E}_k \wedge_R F_R(E,X) & \text{for } k \geq 0, \\ F_R(E/E_{-k-1}, X) & \text{for } k \leq 0. \end{cases}$$

Proof. The equivalences $\epsilon\colon T_k(M(X)) \to M_k(X)$ induce the following commutative diagram.

$$\begin{tikzcd}
\ldots \arrow[r] & \widetilde{E}_0 \wedge_R T_{-1}(M(X)) \arrow[r] \arrow[d, "\epsilon"', "\simeq"] & \widetilde{E}_0 \wedge_R T_0(M(X)) \arrow[r] \arrow[d, "\epsilon"', "\simeq"] & \widetilde{E}_1 \wedge_R T_0(M(X)) \arrow[r] \arrow[d, "\epsilon"', "\simeq"] & \ldots \\
\ldots \arrow[r] & M_{-1}(X) \arrow[r] & F_R(E,X) \arrow[r] & \widetilde{E}_1 \wedge_R F_R(E,X) \arrow[r] & \ldots
\end{tikzcd}$$

Here $M_k(X) = F_R(E/E_{-k-1}, X)$ for $k \leq 0$. □

We refer to [LMSM86, §III.1] for the basic Spanier–Whitehead duality theory in a closed symmetric monoidal category. In the case of (the homotopy category of) R-modules in orthogonal G-spectra, we refer to the objects called 'finite' by Lewis and May as 'dualisable'.

Definition I.6.28. For an R-module X in orthogonal G-spectra, let

$$D(X) = F_R(X, R)$$

be its **functional dual**. For dualisable X we refer to $D(X)$ as the **Spanier–Whitehead dual** of X. There are natural maps

$$\rho\colon X \to D(D(X)) \quad \text{and} \quad \nu\colon DX \wedge_R Y \to F_R(X,Y)\,,$$

which are equivalences when X is dualisable, essentially by definition.

Lemma I.6.29. *Each term in the filtration $\widetilde{E}_\star$ is dualisable.*

Proof. We can give G a finite CW structure, with e as a 0-cell. It follows that the bar construction is a finite G-CW space in each simplicial degree $B_q(*, G, G) = G^q \times G$, so that $G^{\wedge q} \wedge G_+$ is a finite G-CW space and $R \wedge G^{\wedge q} \wedge G_+$ is a dualisable R-module in orthogonal G-spectra. By induction, this implies that E_{i-1} is dualisable, and therefore the mapping cone $\widetilde{E}_i$ is also dualisable, for each $i \geq 0$. □

Lemma I.6.30. *The E^1-page of the spectral sequence associated to the sequence $GM'_\star(X)$ is the R_*-module chain complex with*

$$\check{E}^1_{*,*}(X) \cong \mathrm{Hom}_{R[G]_*}(R_*, \mathrm{gm}_*(\pi_*(X)))$$

where we use the notation of Definition I.2.14.

Proof. For $\star \le 0$ the sequence $GM'_\star(X)$ agrees with the sequence $M_\star(X)$ from Section I.5.2, so $(\check{E}^1_*(X), d^1)$ for $* \le 0$ agrees with $\mathrm{Hom}_{R[G]_*}(P_{-*,*}, \pi_*(X))$ by Proposition I.5.13.

For $\star \ge 0$ the sequence $GM'_\star(X)$ agrees with the filtration $\widetilde{E}_\star \wedge_R F_R(E, X)$. Its subquotients for $i \ge 1$ are of the form $(\widetilde{E}_i/\widetilde{E}_{i-1}) \wedge_R F_R(E, X)$ with

$$\widetilde{E}_i/\widetilde{E}_{i-1} \cong \Sigma(E_{i-1}/E_{i-2}) \cong R \wedge \Sigma^i(G^{\wedge i-1} \wedge G_+)\,.$$

Let d be the dimension of G. Since G_+ is stably dualisable, with Spanier–Whitehead dual $D(G_+) \simeq_G \Sigma^{-d}G_+$, each subquotient above is equivalent to $F(G_+, X')$ for some R-module X' in orthogonal G-spectra. It follows from Proposition I.3.6 that

$$\check{E}^1_*(X) \cong \mathrm{Hom}_{R[G]_*}(R_*, E^1_*(\widetilde{E}_\star \wedge_R F_R(E, X)))$$

for $* \ge 1$. Here

$$\begin{aligned} E^1_i(\widetilde{E}_\star \wedge_R F_R(E,X)) &= \pi_{i+*}(\widetilde{E}_{i-1} \wedge_R F_R(E,X) \to \widetilde{E}_i \wedge_R F_R(E,X)) \\ &\cong \pi_{i+*}((\widetilde{E}_i/\widetilde{E}_{i-1} \wedge_R F_R(E,X))) \\ &\cong \pi_{i+*}(\widetilde{E}_i/\widetilde{E}_{i-1}) \otimes_{R_*} \pi_* F_R(E,X) \\ &\cong \widetilde{P}_{i,*} \otimes_{R_*} \pi_*(X) \end{aligned}$$

for $i \ge 1$, since $\widetilde{P}_{i,*} = \pi_{i+*}(\widetilde{E}_i/\widetilde{E}_{i-1})$ is projective, hence flat, over R_*, and $c\colon E \to R$ induces an isomorphism $\pi_*(X) \cong \pi_* F_R(E, X)$ of $R[G]_*$-modules. This shows that $(\check{E}^1_*(X), d^1)$ for $* \ge 1$ agrees with $\mathrm{Hom}_{R[G]_*}(R_*, \widetilde{P}_{*,*} \otimes_{R_*} \pi_*(X))$.

It remains to verify that $d^1\colon \check{E}^1_1(X) \to \check{E}^1_0(X)$ is as asserted. By definition, it is given by the left-to-right composite in the following diagram.

$$\begin{tikzcd} \pi^G_{1+*}((\widetilde{E}_1/\widetilde{E}_0) \wedge_R X) \arrow[r, "\partial"] \arrow[d, "1\wedge c^*"', "\cong"] & \pi^G_*(X) \arrow[d, "c^*"'] & \\ \pi^G_{1+*}((\widetilde{E}_1/\widetilde{E}_0) \wedge_R F_R(E,X)) \arrow[r, "\partial"] & \pi^G_*(F_R(E,X)) \arrow[r] & \pi^G_*(F_R(E_0,X)) \end{tikzcd}$$

By naturality of ω, as in Lemma I.3.5, this is obtained from the left-to-right composite

$$\begin{tikzcd} \pi_{1+*}((\widetilde{E}_1/\widetilde{E}_0) \wedge_R X) \arrow[r, "\partial"] & \pi_*(X) \arrow[d, "c^*"', "\cong"] & \\ & \pi_*(F_R(E,X)) \arrow[r] & \pi_*(F_R(E_0,X)) \end{tikzcd}$$

by applying $\mathrm{Hom}_{R[G]_*}(R_*, -)$. Under the isomorphisms above, this is the composition

$$\begin{aligned} &\widetilde{P}_{1,*} \otimes_{R_*} \pi_*(X) \xrightarrow{\tilde{\partial}_1} \widetilde{P}_{0,*} \otimes_{R_*} \pi_*(X) \\ &\qquad \cong \pi_*(X) \cong \mathrm{Hom}_{R_*}(R_*, \pi_*(X)) \xrightarrow{\epsilon^*} \mathrm{Hom}_{R_*}(P_0, \pi_*(X))\,. \end{aligned}$$

As we made explicit in Proposition I.2.15, this equals the boundary $\mathrm{gm}_1(\pi_*(X)) \to \mathrm{gm}_0(\pi_*(X))$. □

Proposition I.6.31. *The filtration-preserving map $\alpha\colon GM_\star(X) \to HM_\star(X)$ induces an isomorphism of spectral sequences*

$$\alpha^r\colon \check{E}^r(X) \xrightarrow{\cong} \hat{E}^r(X)$$

for $r \geq 2$.

Proof. Comparing Proposition I.6.16 and Lemma I.6.30 shows that $\alpha^1\colon \check{E}^1(X) \to \hat{E}^1(X)$ is the chain map $\mathrm{Hom}(1, \alpha)$ shown to be a quasi-isomorphism in Proposition I.2.16. Hence $\alpha^2 = H(\alpha^1, d^1)$ is an isomorphism, which implies that α^r is an isomorphism for each $r \geq 2$. □

Following [Gre87, §1], we can splice the filtration

$$R \cong \widetilde{E}_0 \longrightarrow \widetilde{E}_1 \longrightarrow \widetilde{E}_2 \longrightarrow \dots$$

with the Spanier–Whitehead dual sequence

$$\dots \longrightarrow D(\widetilde{E}_2) \longrightarrow D(\widetilde{E}_1) \longrightarrow D(\widetilde{E}_0) \cong R$$

to obtain a bi-infinite sequence

$$\dots \longrightarrow D(\widetilde{E}_2) \longrightarrow D(\widetilde{E}_1) \longrightarrow R \longrightarrow \widetilde{E}_1 \longrightarrow \widetilde{E}_2 \longrightarrow \dots \tag{I.27}$$

of dualisable R-modules in orthogonal G-spectra. This is the sequence [GM95, (9.5)] used by Greenlees and May to define their Tate spectral sequence, at least for finite G. For $G = \mathbb{T} = U(1)$ or $\mathbb{U} = Sp(1)$ they instead repeat each term in this sequence two or four times, respectively. For other compact Lie groups, the connection is less direct.

Proposition I.6.32. *There is a zig-zag of equivalences from $GM'_\star(X)$ to the sequence $GM''_\star(X)$ with*

$$GM''_k(X) = \begin{cases} \widetilde{E}_k \wedge_R F_R(E, X) & \text{for } k \geq 0, \\ D(\widetilde{E}_{-k}) \wedge_R F_R(E, X) & \text{for } k \leq 0. \end{cases}$$

Hence the spectral sequence $\check{E}^r(X)$ is isomorphic to the Greenlees–May Tate spectral sequence [GM95, Thm. 10.3] for π_*^G applied to the sequence $GM''_\star(X)$.

Proof. The zig-zag of equivalences connecting $GM'_\star(X)$ to $GM''_\star(X)$ consists of identity maps for $\star \ge 0$. For $\star \le 0$ it takes the following form:

$$\begin{array}{ccccccc}
\dots \longrightarrow & F_R(E/E_1, X) & \longrightarrow & F_R(E/E_0, X) & \longrightarrow & F_R(E, X) \\
& \big\downarrow \simeq_G & & \big\downarrow \simeq_G & & \big\downarrow = \\
\dots \longrightarrow & F_R(E \cup CE_1, X) & \longrightarrow & F_R(E \cup CE_0, X) & \longrightarrow & F_R(E, X) \\
& \tilde\Delta^* \big\uparrow \simeq_G & & \tilde\Delta^* \big\uparrow \simeq_G & & \tilde\Delta^* \big\uparrow \cong \\
\dots \longrightarrow & F_R(\widetilde{E}_2 \wedge_R E, X) & \longrightarrow & F_R(\widetilde{E}_1 \wedge_R E, X) & \longrightarrow & F_R(\widetilde{E}_0 \wedge_R E, X) \\
& \nu \big\uparrow \simeq_G & & \nu \big\uparrow \simeq_G & & \nu \big\uparrow \cong \\
\dots \to & D(\widetilde{E}_2) \wedge_R F_R(E, X) & \to & D(\widetilde{E}_1) \wedge_R F_R(E, X) & \to & D(\widetilde{E}_0) \wedge_R F_R(E, X)
\end{array}$$

The two top rows are equivalent because each quotient map $E \cup CE_{i-1} \to E/E_{i-1}$ is an equivalence, since $E_{i-1} \to E$ is a (strong) h-cofibration. The equivalence between the middle two rows is induced by the map $\widetilde{\Delta}$ of mapping cones associated to the diagonal equivalence $\Delta\colon E_{i-1} \to E_{i-1} \wedge_R E$:

$$\begin{array}{ccccc}
E_{i-1} & \longrightarrow & E & \longrightarrow & E \cup CE_{i-1} \\
\Delta \big\downarrow \simeq_G & & \big\downarrow = & & \widetilde{\Delta} \big\downarrow \simeq_G \\
E_{i-1} \wedge_R E & \xrightarrow{c \wedge 1} & R \wedge_R E & \longrightarrow & \widetilde{E}_i \wedge_R E \,.
\end{array}$$

The lower two rows are equivalent because each $\widetilde{E}_i$ is dualisable by Lemma I.6.29. □

Remark I.6.33. Our comparison of the Hesselholt–Madsen Tate spectral sequence

$$\hat{E}^r(X) = E^r(HM_\star(X))$$

and the Greenlees–May Tate spectral sequence

$$\check{E}^r(X) = E^r(GM_\star(X)) \cong E^r(GM'_\star(X)) \cong E^r(GM''_\star(X))$$

is a little different from that of [HM03, Rmk. 4.3.6], since we obtain the Greenlees sequence $GM''_\star(X)$ by splicing the two perpendicular edges $(i = 0, j \le 0)$ and $(i \ge 0, j = 0)$ of the bifiltration

$$HM_{i,j}(X) = \widetilde{E}_i \wedge_R T_j(M(X)) \simeq_G \widetilde{E}_i \wedge_R F_R(E/E_{-j-1}, X)\,,$$

while Hesselholt and Madsen first invert a quasi-isomorphism, so as to position both halves of the Greenlees sequence on the line $j = 0$.

Remark I.6.34. In the case of the circle group $G = \mathbb{T}$ we can work over $R = \mathbb{S}$ and use the odd spheres $S((k+1)\mathbb{C})$ to filter $E\mathbb{T} = S(\infty\mathbb{C})$, so that

$\widetilde{E}_k = S^{k\mathbb{C}}$ equals (the suspension spectrum of) a representation sphere. Then $D(\widetilde{E}_{-k}) = D(S^{-k\mathbb{C}}) = S^{k\mathbb{C}}$ is a virtual representation sphere for each $k < 0$. For brevity, let us also write $\widetilde{E}_k$ for the latter $\mathbb{T}$-spectra, so that $\{\widetilde{E}_k\}_{k\in\mathbb{Z}}$ is the bi-infinite sequence (I.27), with cofibre sequences $\widetilde{E}_{k-1} \to \widetilde{E}_k \to \Sigma^{2k-1}\mathbb{T}_+$. The Greenlees–May spectral sequence associated to the sequence

$$\widetilde{E}_\star \wedge F(E\mathbb{T}_+, X)$$

of $\mathbb{T}$-spectra has E^1-page

$$\check{E}^1_{k,*}(X) = \pi^{\mathbb{T}}_{k+*}(\Sigma^{2k-1}\mathbb{T}_+ \wedge F(E\mathbb{T}_+, X)) \cong \pi_{*-k}(X)$$

for each $k \in \mathbb{Z}$, which we can formally write as $t^{-k} \cdot \pi_*(X)$ with t in bidegree $(-1,-1)$. As remarked earlier, we may reindex the filtration and spectral sequence so as to put t in bidegree $(-2,0)$, in which case $E^2_{2k,*}(X) \cong t^{-k} \cdot \pi_*(X)$ and $E^2_{2k-1,*}(X) = 0$. Greenlees and May [GM95, Thm. B.8] prove that the latter spectral sequence is isomorphic to one associated to the tower of $\mathbb{T}$-spectra

$$\dots \longrightarrow \widetilde{E\mathbb{T}} \wedge F(E\mathbb{T}_+, X^{\ell+1}) \longrightarrow \widetilde{E\mathbb{T}} \wedge F(E\mathbb{T}_+, X^{\ell}) \longrightarrow \dots .$$

Here $\{X^\ell\}_\ell$ denotes a $\mathbb{T}$-equivariant Whitehead tower for X, with homotopy fibre sequences $X^{\ell+1} \to X^\ell \to \Sigma^\ell H\pi_\ell(X)$. The latter spectral sequence is indexed so that

$$E^2_{*,\ell}(X) = \pi^{\mathbb{T}}_{*+\ell}(\widetilde{E\mathbb{T}} \wedge F(E\mathbb{T}_+, \Sigma^\ell H\pi_\ell(X))) \cong \pi_*(H\pi_\ell(X)^{t\mathbb{T}})$$

for each integer ℓ. In particular, $\pi_{2k}(H\pi_\ell(X)^{t\mathbb{T}}) \cong t^{-k}\cdot\pi_\ell(X)$ and $\pi_{2k-1}(H\pi_\ell(X)^{t\mathbb{T}}) = 0$, so that, formally, $\pi_*(H\pi_\ell(X)^{t\mathbb{T}}) \cong \pi_\ell(X)[t, t^{-1}]$. Furthermore, Greenlees and May argue that the latter spectral sequence is multiplicative, with respect to some topologically defined pairings of the form

$$\pi_*(H\pi_i(X)^{t\mathbb{T}}) \otimes \pi_*(H\pi_j(Y)^{t\mathbb{T}}) \longrightarrow \pi_*(H\pi_{i+j}(Z)^{t\mathbb{T}}) .$$

However, as is implicit in [GM95, Prob. 14.8], they do not establish that these topological pairings agree with the evident algebraic pairings

$$\pi_i(X)[t, t^{-1}] \otimes \pi_j(Y)[t, t^{-1}] \longrightarrow \pi_{i+j}(Z)[t, t^{-1}] .$$

Hence they do not assert that the isomorphism $E^2_{*,*}(X) \cong \pi_*(X)[t, t^{-1}]$ takes the topological product to the algebraic product. In particular, the higher differentials in this spectral sequence are known to obey a Leibniz rule, but conceivably not with respect to the most evident algebraic product.

Nonetheless, we can confirm directly that the first differential in each of these spectral sequences is a derivation with respect to the algebraic product. To express this, we return to the indexing used elsewhere in the paper, i.e., to the Greenlees–May spectral sequence $\check{E}^r_{*,*}(X)$. Up to the technical issue we have pointed out about compatibility of product structures, the following result is due to Hesselholt [Hes96, Lem. 1.4.2].

Proposition I.6.35. *Let X be any orthogonal $\mathbb{T}$-spectrum, so that $\pi_*(X)$ is a right $\mathbb{S}[\mathbb{T}]_*$-module. There is a natural isomorphism*

$$\check{E}^1_{*,*}(X) \cong \pi_*(X)[t, t^{-1}]$$

with t in bidegree $(-1,-1)$, such that $d^1 : \check{E}^1_{k,*}(X) \to \check{E}^1_{k-1,*}(X)$ corresponds to the differential $d\colon t^{-k} \cdot \pi_*(X) \to t^{-k+1} \cdot \pi_*(X)$ given by

$$d(t^{-k} \cdot x) = \begin{cases} t^{-k+1} \cdot xs & \text{for } k \text{ even,} \\ t^{-k+1} \cdot x(s+\eta) & \text{for } k \text{ odd.} \end{cases}$$

Proof. By naturality of the Greenlees–May spectral sequence with respect to $\mathbb{T}$-maps $x\colon \Sigma^\ell \mathbb{S}[\mathbb{T}] \to X$, corresponding to homotopy classes $x \in \pi_\ell(X)$, it suffices to prove the result in the case $X = \mathbb{S}[\mathbb{T}]$ and $x = 1 \in \pi_0(\mathbb{S}[\mathbb{T}])$.

Consider the case $X = H\mathbb{Z}[\mathbb{T}]$. We have $\pi_*(H\mathbb{Z}) = \mathbb{Z}\{1, \sigma\}$ and $H\mathbb{Z}[\mathbb{T}]^{t\mathbb{T}} \simeq *$ since $H\mathbb{Z}$, as a $\mathbb{T}$-spectrum, is induced up from $H\mathbb{Z}$. For bidegree reasons the $\mathbb{T}$-Tate spectral sequence must collapse to zero at the E^2-page, which forces

$$d(t^{-k} \cdot 1) = \pm t^{-k+1} \cdot \sigma\,.$$

Here, we can iteratively fix the sign of t^{-k} implicit in the identification $\check{E}^1_{k,*}(X) \cong t^{-k} \cdot \pi_*(X)$ so that each of these signs is a plus. By naturality with respect to the Hurewicz homomorphism $\mathbb{S}[\mathbb{T}] \to H\mathbb{Z}[\mathbb{T}]$ it follows that

$$d(t^{-k} \cdot 1) \equiv t^{-k+1} \cdot s \mod t^{-k+1} \cdot \eta$$

in the $\mathbb{T}$-Tate spectral sequence for $\mathbb{S}[\mathbb{T}]$, since $\pi_1(\mathbb{S}[\mathbb{T}]) = \mathbb{Z}\{s\} \oplus \mathbb{Z}/2\{\eta\}$ with the Hurewicz homomorphism mapping s to σ.

Now consider the case $X = \mathbb{S}$ with trivial $\mathbb{T}$-action. The part $k \ge 1$ of the Greenlees–May spectral sequence maps to the Atiyah–Hirzebruch spectral sequence for $\Sigma^2 \mathbb{C}P^\infty_+$. Since the $2k$-cell in $\Sigma^2\mathbb{C}P^\infty$ is stably attached to the $2k-2$-cell by $k\eta$, it follows that

$$d(t^{-k} \cdot 1) = t^{-k+1} \cdot k\eta$$

for $k \ge 2$. Similarly, the part $k \le 0$ receives a map from the Atiyah–Hirzebruch spectral sequence for $D(\mathbb{C}P^\infty_+)$, where the $-2k$-cell is attached to the $-2k-2$-cell by $k\eta$, so that

$$d(t^{-k} \cdot 1) = t^{-k+1} \cdot k\eta$$

for $k \le 0$, as well. Finally, for $k = 1$ the differential is induced by the composite $\mathbb{T}$-map

$$\Sigma^{-1} \widetilde{E}_1/\widetilde{E}_0 \wedge F(E\mathbb{T}_+, \mathbb{S}) \longrightarrow \widetilde{E}_0 \wedge F(E\mathbb{T}_+, \mathbb{S}) \longrightarrow \widetilde{E}_0/\widetilde{E}_{-1} \wedge F(E\mathbb{T}_+, \mathbb{S})\,,$$

which we can rewrite in terms of the counit $\epsilon\colon \mathbb{T}_+ \to S^0$ and its Spanier–Whitehead dual $D(\epsilon)\colon \mathbb{S} \to D(\mathbb{T}_+)$ as

$$F(E\mathbb{T}_+, \mathbb{T}_+) \xrightarrow{\epsilon_*} F(E\mathbb{T}_+, \mathbb{S}) \xrightarrow{D(\epsilon)_*} F(E\mathbb{T}_+, D(\mathbb{T}_+))\,.$$

Passing to $\mathbb{T}$-fixed points, this is a composite

$$\Sigma\mathbb{S} \simeq \mathbb{S}[\mathbb{T}]^{h\mathbb{T}} \longrightarrow \mathbb{S}^{h\mathbb{T}} \longrightarrow D(\mathbb{T}_+)^{h\mathbb{T}} \simeq \mathbb{S}\,,$$

which we claim equals $\eta \in \pi_1(\mathbb{S})$. This can be seen using the Pontryagin–Thom collapse $S^{\mathbb{C}} \to S^{\mathbb{C}}/S^0 \cong \Sigma(\mathbb{T}_+)$ associated to the embedding $\mathbb{T} \subset \mathbb{C}$, and the untwisting isomorphism $\zeta\colon \Sigma^2(\mathbb{T}_+) \cong \Sigma^{\mathbb{C}}(\mathbb{T}_+)$. Then $\zeta(1 \wedge t)\colon \Sigma S^{\mathbb{C}} \to \Sigma^{\mathbb{C}}(\mathbb{T}_+)$ defines a stable $\mathbb{T}$-map $S^1 \to \mathbb{T}_+$. The composite $\epsilon\zeta(1 \wedge t)\colon \Sigma S^{\mathbb{C}} \to S^{\mathbb{C}}$ has (non-equivariant) Hopf invariant ± 1, since the preimages of two generic points in $S^{\mathbb{C}}$ are circles in $\Sigma S^{\mathbb{C}}$ with that linking number.

This proves $d(t^{-1} \cdot 1) = t^0 \cdot \eta$ in the $\mathbb{T}$-Tate spectral sequence for $\mathbb{S}$ and, by naturality with respect to $\mathbb{S}[\mathbb{T}] \to \mathbb{S}$, the asserted formulas follow. □

I.6.6 Convergence

In this section we use Proposition I.6.31 to deduce convergence results for the Hesselholt–Madsen spectral sequence $\hat{E}^r(X)$ from corresponding results for the Greenlees–May spectral sequence $\check{E}^r(X)$.

Lemma I.6.36. *The map of abutments*

$$\alpha_\infty\colon A_\infty(GM_\star(X)) \xrightarrow{\cong} A_\infty(HM_\star(X)) \cong \pi_*(X^{tG})$$

is an isomorphism. Hence the homomorphism

$$\alpha_s\colon F_sA_\infty(GM_\star(X)) \longrightarrow F_sA_\infty(HM_\star(X))$$

is injective, for each $s \in \mathbb{Z}$.

Proof. The first assertion follows from Lemma I.6.25. The commutative diagram

$$\begin{tikzcd}
A_s(GM_\star(X)) \arrow[r, two heads] \arrow[d] & F_sA_\infty(GM_\star(X)) \arrow[r, tail] \arrow[d, "\alpha_s"'] & A_\infty(GM_\star(X)) \arrow[d, "\alpha_\infty"', "\cong"] \\
A_s(HM_\star(X)) \arrow[r, two heads] & F_sA_\infty(HM_\star(X)) \arrow[r, tail] & A_\infty(HM_\star(X))
\end{tikzcd}$$

implies the injectivity assertion. □

Lemma I.6.37. *The spectral sequence*

$$\check{E}^r(X) = E^r(GM_\star(X)) \Longrightarrow \pi_*(X^{tG})$$

is conditionally convergent.

Proof. The spectral sequence associated to $GM_\star(X)$ is conditionally convergent to the colimit, in the sense of [Boa99, Def. 5.10], whenever $\operatorname{holim}_s GM_s(X) \simeq_G *$. Since the sequences $GM_\star(X)$ and $GM'_\star(X)$ are equivalent by Lemma I.6.27, we may equally well verify that $\operatorname{holim}_s GM'_s(X) \simeq_G *$. But

$$GM'_s(X) = F_R(E/E_{s-1}, X) = M_s(X)$$

for $s \le 0$, and we saw in Section I.5.2 that $\operatorname{holim}_s M_s(X) \simeq_G *$. Hence the Greenlees–May G-Tate spectral sequence $\check{E}^r(X)$ is always conditionally convergent. $\square$

Lemma I.6.38. *The maps of E^∞- and RE^∞-pages*

$$E^\infty(GM_\star(X)) \xrightarrow{\cong} E^\infty(HM_\star(X))$$
$$RE^\infty(GM_\star(X)) \xrightarrow{\cong} RE^\infty(HM_\star(X))$$

are isomorphisms.

Proof. Recall from [Boa99, (5.1)] that for each spectral sequence (E^r, d^r) there are filtrations

$$0 = B^1 \subset B^2 \subset B^3 \subset \cdots \subset Z^3 \subset Z^2 \subset Z^1 = E^1$$

with $E^r \cong Z^r/B^r$ for $r \ge 1$. We set

$$B^\infty = \operatorname*{colim}_r B^r\,, \quad Z^\infty = \lim_r Z_r\,, \quad E^\infty = Z^\infty/B^\infty\,, \quad RE^\infty = \operatorname*{Rlim}_r Z_r\,.$$

Letting $\bar{B}^r = B^r/B^2$ and $\bar{Z}^r = Z^r/B^2$ for $r \ge 2$, we obtain a filtration

$$0 = \bar{B}^2 \subset \bar{B}^3 \subset \cdots \subset \bar{Z}^3 \subset \bar{Z}^2 \cong E^2$$

with $E^r \cong \bar{Z}^r/\bar{B}^r$ for $r \ge 2$. Let

$$\bar{B}^\infty = \operatorname*{colim}_r \bar{B}^r \quad \text{and} \quad \bar{Z}^\infty = \lim_r \bar{Z}_r\,.$$

Then $\bar{B}^\infty \cong B^\infty/B^2$, while $\bar{Z}^\infty \cong Z^\infty/B^2$ and $\operatorname{Rlim}_r Z^r \cong \operatorname{Rlim}_r \bar{Z}^r$ by the lim-Rlim exact sequence. Hence $E^\infty \cong \bar{Z}^\infty/\bar{B}^\infty$ by the Noether isomorphism, and $RE^\infty \cong \operatorname{Rlim}_r \bar{Z}^r$.

A map of spectral sequences inducing an isomorphism of E^2-pages will by induction induce isomorphisms of $\bar{B}^r$- and $\bar{Z}^r$-pages for all $r \ge 2$, and therefore also of $\bar{B}^\infty$-, $\bar{Z}^\infty$-, E^∞- and RE^∞-pages. $\square$

When X is bounded below, and G is finite or equal to $\mathbb{T} = U(1)$ or $\mathbb{U} = Sp(1)$, the E^1-pages $\check{E}^1(X)$ and $\hat{E}^1(X)$ are both concentrated in half-planes with entering differentials [Boa99, §7]. However, for more general groups G the E^1-page $\check{E}^1(X)$ occupies a region that is only bounded by a broken line, and $\hat{E}^1(X)$ may not be bounded in any ordinary sense. We therefore need to discuss convergence for the spectral sequences $\check{E}^r(X)$ and $\hat{E}^r(X)$ in the generality of whole-plane spectral sequences [Boa99, §8].

Definition I.6.39. Let (A, E^1) be the exact couple associated to a Cartan–Eilenberg system (H, ∂). Boardman's **whole-plane obstruction group** W is defined in [Boa99, Lem. 8.5] by an expression

$$W = \operatorname*{colim}_s \operatorname*{Rlim}_r K_\infty \operatorname{im}^r A_s\,.$$

We refer to Boardman's paper for an explanation of the notation. By [HR19, Thm. 7.5] there is an isomorphism

$$W \cong \ker(\kappa)$$

where

$$\kappa\colon \operatorname*{colim}_j \lim_i H(i,j) \longrightarrow \lim_i \operatorname*{colim}_j H(i,j)$$

is the interchange morphism, which is always surjective.

While W is defined in terms of the underlying exact couple (or Cartan–Eilenberg system), Boardman gives the following criterion for the vanishing of W, which is internal to the spectral sequence.

Lemma I.6.40 ([Boa99, Lem. 8.1])**.** *Suppose that for each m, there exist numbers $u(m)$ and $v(m)$ such that for all $u \geq u(m)$ and $v \geq v(m)$, the differential*

$$d^{u+v}\colon E^{u+v}_{u,m-u} \longrightarrow E^{u+v}_{-v,m+v-1}$$

vanishes. Then $W = 0$.

Remark I.6.41. If for some fixed r the E^r-page of the spectral sequence is bounded from the side of entering differentials, in the sense that for each m there is a number $u(m)$ such that $E^r_{u,m-u} = 0$ for all $u \geq u(m)$, then Boardman's vanishing criterion is satisfied with $v(m) = r - u(m)$. Hence $W = 0$ in these cases.

Alternatively, if the spectral sequence collapses at the E^r-page, so that d^r and all later differentials are zero, then Boardman's vanishing criterion is satisfied with $u(m) = r$ and $v(m) = 0$. Thus $W = 0$ also in these cases.

Theorem I.6.42. *The spectral sequence $\check{E}^r(X) = E^r(GM_\star(X))$ converges strongly to $A_\infty(GM_\star(X)) \cong \pi_*(X^{tG})$ if and only if $RE^\infty = 0$ and $W = 0$ for this spectral sequence.*

Proof. We saw that $\check{E}^r(X)$ is conditionally convergent in Lemma I.6.37. Hence the statements '$RE^\infty = 0$ and $W = 0$' and 'the spectral sequence is strongly convergent' are equivalent by [Boa99, Thm. 8.10]. □

Theorem I.6.43. *If the Greenlees–May spectral sequence*

$$\check{E}^r(X) = E^r(GM_\star(X)) \Longrightarrow \pi_*(X^{tG})$$

is strongly convergent, then the Hesselholt–Madsen spectral sequence

$$\hat{E}^r(X) = E^r(HM_\star(X)) \Longrightarrow \pi_*(X^{tG})$$

is strongly convergent, as well. Moreover, $F_s A_\infty(GM_\star(X)) = F_s A_\infty(HM_\star(X))$ for all integers s.

Proof. We assume $\check{E}^r(X)$ is strongly convergent. Explicitly, this means that the exhaustive filtration $(F_sA_\infty(GM_\star(X)))_s$ of $A_\infty(GM_\star(X)) \cong \pi_*(X^{tG})$ is complete Hausdorff, and the left hand monomorphism β in the commutative square

$$\begin{tikzcd}
\dfrac{F_sA_\infty(GM_\star(X))}{F_{s-1}A_\infty(GM_\star(X))} \arrow[r, "\bar{\alpha}_s"] \arrow[d, "\beta"', "\cong"] & \dfrac{F_sA_\infty(HM_\star(X))}{F_{s-1}A_\infty(HM_\star(X))} \arrow[d, tail, "\beta"'] \\
E_s^\infty(GM_\star(X)) \arrow[r, "\alpha_s^\infty", "\cong"'] & E_s^\infty(HM_\star(X))
\end{tikzcd}$$

is an isomorphism. It follows that the right hand monomorphism β is also an isomorphism. Since the filtration $(F_sA_\infty(HM_\star(X)))_s$ is exhaustive, this means that $\hat{E}^r(X)$ converges weakly to $\pi_*(X^{tG})$. It also follows that the upper homomorphism $\bar{\alpha}_s$ is an isomorphism. By induction, this implies that the map of filtration quotients

$$\frac{F_tA_\infty(GM_\star(X))}{F_sA_\infty(GM_\star(X))} \xrightarrow{\cong} \frac{F_tA_\infty(HM_\star(X))}{F_sA_\infty(HM_\star(X))}$$

is an isomorphism for all integers $s \le t$.

Passing to colimits over t, and using the fact that

$$\alpha_\infty \colon A_\infty(GM_\star(X)) \xrightarrow{\cong} A_\infty(HM_\star(X))$$

is an isomorphism by Lemma I.6.36, we deduce that

$$\alpha_s \colon F_sA_\infty(GM_\star(X)) \to F_sA_\infty(HM_\star(X))$$

is an isomorphism, for each $s \in \mathbb{Z}$. The filtration $(F_sA_\infty(HM_\star(X)))_s$ is therefore complete and Hausdorff, meaning that $\hat{E}^r(X)$ converges strongly to $\pi_*(X^{tG})$. □

Combining these results we obtain the following theorem, which often compensates for the problem that we do not *a priori* know when $\hat{E}^r(X) = E^r(HM_\star(X))$ is conditionally convergent, cf. Remark I.6.15.

Theorem I.6.44. *If $RE^\infty = 0$ and Boardman's vanishing criterion for W from Lemma I.6.40 is satisfied for the G-Tate spectral sequence $\hat{E}^r(X) = E^r(HM_\star(X))$, then this spectral sequence converges strongly and conditionally to $A_\infty(HM_\star(X)) \cong \pi_*(X^{tG})$.*

Note that we are not just assuming that $W = 0$ for $\hat{E}^r(X)$, but that this group vanishes for the reason given by Boardman's criterion.

Proof. Since $\check{E}^r(X) \cong \hat{E}^r(X)$ for $r \ge 2$, the vanishing of RE^∞ for $\hat{E}^r(X)$ implies the vanishing of RE^∞ for $\check{E}^r(X)$. Furthermore, the hypothesis of Boardman's criterion for $\hat{E}^r(X)$ implies the same hypothesis for $\check{E}^r(X)$. Hence $\check{E}^r(X)$ converges strongly by Theorem I.6.42, which implies that $\hat{E}^r(X)$ converges strongly by Theorem I.6.43. By [Boa99, Thm. 8.10] strong convergence and the vanishing of RE^∞ and W imply conditional convergence. □

I.6.7 Summary : The $\mathbb{T}$-Tate spectral sequence

The main example we had in mind when writing this paper was $G = \mathbb{T}$. Note that when discussing **the $\mathbb{T}$-Tate spectral sequence for a $\mathbb{T}$-spectrum** X one could really refer to at least[9] two different spectral sequences: one arising from the Greenlees–May filtration and one from the Hesselholt–Madsen filtration. The first has better convergence properties, while the latter has better multiplicative properties. Fortunately there are quite good comparison results between the two, as covered in Section I.6.6.

Let us start by summarizing the additive results regarding the Greenlees–May and Hesselholt–Madsen versions of the $\mathbb{T}$-Tate spectral sequence. We work over $R = \mathbb{S}$, and write $\otimes$ for $\otimes_{\mathbb{S}_*}$. We first note that by virtue of X being a $\mathbb{T}$-spectrum, there is an action

$$\gamma : X \wedge \mathbb{T}_+ \cong X \wedge \mathbb{S}[\mathbb{T}] \longrightarrow X$$

which makes X into a right module over the spherical group ring $\mathbb{S}[\mathbb{T}]$. The induced pairing

$$\gamma_* : \pi_*(X) \otimes \mathbb{S}[\mathbb{T}]_* \longrightarrow \pi_*(X)$$

on homotopy groups then gives $\pi_*(X)$ the structure of a right module over the Hopf algebra

$$\mathbb{S}[\mathbb{T}]_* \cong \mathbb{S}_*[s]/(s^2 = \eta s)\,, \quad |s| = 1\,.$$

Here η is the image of the complex Hopf map in $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$. Note that this Hopf algebra is finitely generated and projective over $\mathbb{S}_*$. We denote the image $\gamma_*(x \otimes s)$ by xs.

There is a minimal projective $\mathbb{S}[\mathbb{T}]_*$-module resolution P_* of $\mathbb{S}_*$, with $P_k = \mathbb{S}[\mathbb{T}]\{p_k\}$ and $\partial(p_k) = p_{k-1}(s + (k-1)\eta)$. Let $\widetilde{P}_*$ be the mapping cone of the augmentation $\epsilon\colon P_* \to \mathbb{S}_*$. Let the complete resolution $\hat{P}_*$ be the fibre product $P_* \times_{\mathbb{S}_*} D(\widetilde{P}_*)$, which is obtained by splicing P_* with its dual.

Theorem I.6.45 (Greenlees–May–Tate spectral sequence)**.** *Given an orthogonal $\mathbb{T}$-spectrum X, there is a filtration $GM_\star(X)$ of orthogonal $\mathbb{T}$-spectra, and an associated $\mathbb{S}_*$-module spectral sequence*

$$\check{E}^r(X) = E^r(GM_\star(X))$$

with abutment

$$A_\infty(GM_\star(X)) \cong \pi_*(X^{t\mathbb{T}})$$

filtered by the images $\operatorname{im}(\pi_* GM(X_s) \to \pi_*(X^{t\mathbb{T}}))$. *We refer to this spectral sequence as the Greenlees–May $\mathbb{T}$-Tate spectral sequence for X. The following hold:*

[9] There is also at least one more Tate spectral sequence, namely the one arising from a Postnikov or Whitehead tower of X; see Remark I.5.5 and Remark I.6.34.

E^1-page *The E^1-page of the Greenlees–May $\mathbb{T}$-Tate spectral sequence can be written*

$$E^1_{*,*}(GM_\star(X)) \cong \mathrm{Hom}_{\mathbb{S}[\mathbb{T}]_*}(\hat{P}_*, \pi_*(X))$$

where $\hat{P}_*$ is a complete resolution of $\mathbb{S}_*$ as a trivial $\mathbb{S}[\mathbb{T}]_*$-module. For the minimal such resolution we can write

$$E^1_{*,*}(GM_\star(X)) \cong \pi_*(X)[t, t^{-1}]$$

with t in bidegree $(-1,-1)$, and then $d^1(t^c \cdot x) = t^{c+1} \cdot x(s + c\eta)$ for all $c \in \mathbb{Z}$ and $x \in \pi_*(X)$.

Convergence *The Greenlees–May spectral sequence converges conditionally to the abutment. It converges strongly to the abutment if and only if the derived E^∞-page RE^∞ and Boardman's whole plane obstruction group W are both trivial.*

Proof. The first statement is Lemma I.6.30 combined with Proposition I.2.22 and Proposition I.6.35. The second statement is Lemma I.6.37 combined with Theorem I.6.42. □

Theorem I.6.46 (Hesselholt–Madsen–Tate spectral sequence)**.** *Given an orthogonal $\mathbb{T}$-spectrum X, there is a filtration $HM_\star(X)$ of orthogonal $\mathbb{T}$-spectra, and an associated $\mathbb{S}_*$-module spectral sequence*

$$\hat{E}^r(X) = E^r(HM_\star(X))$$

with abutment

$$A_\infty(HM_\star(X)) \cong \pi_*(X^{t\mathbb{T}})$$

filtered by the images $\mathrm{im}(\pi_* HM(X_s) \to \pi_*(X^{t\mathbb{T}}))$. We refer to this spectral sequence as the Hesselholt–Madsen $\mathbb{T}$-Tate spectral sequence for X. The following hold:

E^1-page *The E^1-page of the Hesselholt–Madsen $\mathbb{T}$-Tate spectral sequence can be written*

$$E^1_{*,*}(HM_\star(X)) \cong \mathrm{Hom}_{\mathbb{S}[\mathbb{T}]_*}(\mathbb{S}_*, \widetilde{P}_* \otimes \mathrm{Hom}(P_*, \pi_*(X)))$$

where P_* is a projective resolution of $\mathbb{S}_*$ as a trivial $\mathbb{S}[\mathbb{T}]_*$-module and $\widetilde{P}_*$ denotes the mapping cone of the augmentation $\epsilon : P_* \to \mathbb{S}_*$.

E^2-page *The E^2-page is given in terms of Hopf algebra Tate cohomology, alias complete* Ext*, as*

$$E^2_{*,*}(HM_\star(X)) \cong \widehat{\mathrm{Ext}}^{-*}_{\mathbb{S}[\mathbb{T}]_*}(\mathbb{S}_*, \pi_*(X)) \,.$$

Convergence *If the Greenlees–May $\mathbb{T}$-Tate spectral sequence for X converges strongly, then the Hesselholt–Madsen $\mathbb{T}$-Tate spectral sequence for X converges strongly, and the two associated filtrations of $\pi_*(X^{t\mathbb{T}})$ agree.*

Proof. The first statement is Proposition I.6.16, the second is Theorem I.6.17, and the third statement is Theorem I.6.43. □

Worth pointing out is that the Greenlees–May and the Hesselholt–Madsen versions of the $\mathbb{T}$-Tate spectral sequence are isomorphic from the E^2-page and on, per Proposition I.6.31. In particular, the E^2-page of both spectral sequences is given by

$$\begin{aligned}\hat{E}^2_{-c,*}(X) \cong \check{E}^2_{-c,*}(X) &\cong \widehat{\operatorname{Ext}}^c_{\mathbb{S}[\mathbb{T}]_*}(\mathbb{S}_*, \pi_*(X)) \\ &\cong \begin{cases} \dfrac{\ker(s : \pi_*(X) \to \pi_{*+1}(X))}{\operatorname{im}(s+\eta : \pi_{*-1}(X) \to \pi_*(X))} & \text{for } c \text{ even,} \\[2ex] \dfrac{\ker(s+\eta : \pi_*(X) \to \pi_{*+1}(X))}{\operatorname{im}(s : \pi_{*-1}(X) \to \pi_*(X))} & \text{for } c \text{ odd,}\end{cases}\end{aligned}$$

where the last isomorphism is the result of the computation of Section I.2.6.

Regarding convergence, we note that Lemma I.6.40 gives a criterion, internal to the spectral sequence itself, for when Boardman's whole-plane obstruction vanishes. In particular, if X is bounded below, either version of the Tate spectral sequence is a half-plane spectral sequence with entering differentials (at least from the E^2-page), which guarantees this. In the applications we have in mind, we are in this situation if we consider topological Hochschild homology $X = \mathrm{THH}(B)$ for some connective orthogonal ring spectrum B.

Let us now summarise the multiplicative structure of the two spectral sequences discussed.

Theorem I.6.47.

Multiplicativity *The Hesselholt–Madsen $\mathbb{T}$-Tate spectral sequence is multiplicative in the sense that a pairing $\phi : X \wedge Y \to Z$ of orthogonal $\mathbb{T}$-spectra gives rise to a pairing of the associated spectral sequences $\phi : (\hat{E}^*(X), \hat{E}^*(Y)) \to \hat{E}^*(Z)$. Explicitly, ϕ gives rise to homomorphisms*

$$\phi^r : \hat{E}^r(X_\star) \otimes \hat{E}^r(Y_\star) \longrightarrow \hat{E}^r(Z_\star)$$

for all $r \geq 1$, such that:

1. *The Leibniz rule*

$$d^r \phi^r = \phi^r(d^r \otimes 1) + \phi^r(1 \otimes d^r)$$

holds as an equality of homomorphisms $\hat{E}^r_i(X) \otimes \hat{E}^r_j(Y) \longrightarrow \hat{E}^r_{i+j-r}(Z)$ for all $i, j \in \mathbb{Z}$ and $r \geq 1$.

2. *The diagram*

$$\begin{tikzcd} \hat{E}^{r+1}(X) \otimes \hat{E}^{r+1}(Y) \arrow[r, "\phi^{r+1}"] \arrow[d] & \hat{E}^{r+1}(Z) \arrow[d, "\cong"] \\ H(\hat{E}^r(X) \otimes \hat{E}^r(Y)) \arrow[r, "H(\phi^r)"] & H(\hat{E}^r(Z)) \end{tikzcd}$$

commutes for all $r \geq 1$.

3. *The pairing*

$$\phi^2 : \hat{E}^2(X_\star) \otimes \hat{E}^2(Y_\star) \longrightarrow \hat{E}^2(Z_\star)$$

agrees with the cup product

$$\smile : \widehat{\mathrm{Ext}}^{-i,*}_{\mathbb{S}[\mathbb{T}]_*}(\mathbb{S}_*, \pi_*(X)) \otimes \widehat{\mathrm{Ext}}^{-j,*}_{\mathbb{S}[\mathbb{T}]_*}(\mathbb{S}_*, \pi_*(Y)) \longrightarrow \widehat{\mathrm{Ext}}^{-i-j,*}_{\mathbb{S}[\mathbb{T}]_*}(\mathbb{S}_*, \pi_*(Z)) .$$

For $r \geq 2$ the same statements hold for the Greenlees–May spectral sequence, with $\check{E}^r$ in place of $\hat{E}^r$.

Multiplicative abutment *We have an induced pairing*

$$\phi_* : \pi_*(X^{t\mathbb{T}}) \otimes \pi_*(Y^{t\mathbb{T}}) \longrightarrow \pi_*(Z^{t\mathbb{T}})$$

of abutments with the Hesselholt–Madsen filtrations, which is compatible with the pairing ϕ^∞ of E^∞-pages. Explicitly, the diagram

$$\begin{array}{ccc} \dfrac{F_i\pi_*(X^{t\mathbb{T}})}{F_{i-1}\pi_*(X^{t\mathbb{T}})} \otimes \dfrac{F_j\pi_*(Y^{t\mathbb{T}})}{F_{j-1}\pi_*(Y^{t\mathbb{T}})} & \xrightarrow{\bar{\phi}_*} & \dfrac{F_{i+j}\pi_*(Z^{t\mathbb{T}})}{F_{i+j-1}\pi_*(Z^{t\mathbb{T}})} \\ \downarrow{\scriptstyle \beta\otimes\beta} & & \downarrow{\scriptstyle \beta} \\ \hat{E}^\infty_i(X) \otimes \hat{E}^\infty_j(Y) & \xrightarrow{\phi^\infty} & \hat{E}^\infty_{i+j}(Z) \end{array}$$

commutes, for all $i, j \in \mathbb{Z}$.

If the Greenlees–May spectral sequence is strongly convergent, then the same statements hold for the Greenlees–May filtrations and $\check{E}^\infty_{*,*}$.

Proof. For the Hesselholt–Madsen $\mathbb{T}$-Tate spectral sequence this is Theorem I.6.18 and Theorem I.6.21. The statements about multiplicativity of the E^r-pages and d^r-differentials can be transported to the Greenlees–May spectral sequence for $r \geq 2$ by way of the isomorphism of Proposition I.6.31. The statements about multiplicativity of filtered abutments carry over to the Greenlees–May spectral sequence when $GM_\star(X)$ and $HM_\star(X)$ induce the same filtration on $\pi_*(X^{t\mathbb{T}})$, which holds under the hypothesis of strong convergence by Theorem I.6.43. □

Recalling the discussion of Remark I.2.54, in the context of the circle group, the Hopf algebra Tate cohomology can also be described as the homology of the differential graded $\mathbb{S}[\mathbb{T}]_*$-module

$$\pi_*(X)[t, t^{-1}] , \quad |t| = -1$$

with boundary characterised by

$$d(x) = txs \quad \text{and} \quad d(t) = t^2\eta .$$

Moreover, given a pairing $X \wedge Y \to Z$ we have an induced pairing $\pi_*(X) \otimes_{\mathbb{S}_*} \pi_*(Y) \to \pi_*(Z)$ on homotopy groups, and the cup product on Tate cohomology is precisely the one induced by the obvious map

$$\pi_*(X)[t, t^{-1}] \otimes \pi_*(Y)[t, t^{-1}] \longrightarrow \pi_*(Z)[t, t^{-1}]$$

on homology. By Theorem I.6.21, the multiplicative structure on the second page of (both versions of) the $\mathbb{T}$-Tate spectral sequence corresponds to this cup product. By Proposition I.6.35 we can formally impose this algebra structure on the Greenlees–May E^1-page, in which case the d^1-differential is a derivation, and this lets us extend the multiplicativity statement for the Greenlees–May $\mathbb{T}$-Tate spectral sequence to the range $r \geq 1$, in place of $r \geq 2$, even if there is no underlying topological source of the pairing of E^1-pages.

References

[AR12] C. Ausoni and J. Rognes. "Algebraic K-theory of the first Morava K-theory". In: *J. Eur. Math. Soc.* 14.4 (2012), pp. 1041–1079.

[BM94] M. Bökstedt and I. Madsen. "Topological cyclic homology of the integers". In: *Astérisque* 226 (1994). K-theory (Strasbourg, 1992), pp. 7–8, 57–143.

[BMS19] B. Bhatt, M. Morrow, and P. Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310.

[Boa99] J. M. Boardman. "Conditionally convergent spectral sequences". In: *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*. Vol. 239. Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 49–84.

[Bök85] M. Bökstedt. "Topological Hochschild homology". Preprint. 1985.

[BR05] R. R. Bruner and John Rognes. "Differentials in the homological homotopy fixed point spectral sequence". In: *Algebr. Geom. Topol.* 5 (2005), pp. 653–690.

[Bro82] K. S. Brown. *Cohomology of groups.* Vol. 87. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982.

[Car84] G. Carlsson. "Equivariant stable homotopy and Segal's Burnside ring conjecture". In: *Ann. of Math. (2)* 120.2 (1984), pp. 189–224.

[CE56] H. Cartan and S. Eilenberg. *Homological algebra.* Princeton University Press, Princeton, N. J., 1956.

[CK97] J. Cornick and P. H. Kropholler. "On complete resolutions". In: *Topology Appl.* 78.3 (1997), pp. 235–250.

[Dou59a] A. Douady. "La suite spectrale d'Adams". In: *Séminaire Henri Cartan* 11.2 (1959).

[Dou59b] A. Douady. “La suite spectrale d’Adams : structure multiplicative”. In: *Séminaire Henri Cartan* 11.2 (1959).

[GM74] V. K. A. M. Gugenheim and J. P. May. *On the theory and applications of differential torsion products*. Memoirs of the American Mathematical Society, No. 142. American Mathematical Society, Providence, R.I., 1974.

[GM95] J. P. C. Greenlees and J. P. May. “Generalized Tate cohomology”. In: *Mem. Amer. Math. Soc.* 113.543 (1995).

[Gre87] J. P. C. Greenlees. “Representing Tate cohomology of G-spaces”. In: *Proc. Edinburgh Math. Soc. (2)* 30.3 (1987), pp. 435–443.

[Gre95] J. P. C. Greenlees. “Commutative algebra in group cohomology”. In: *J. Pure Appl. Algebra* 98.2 (1995), pp. 151–162.

[Hel17] G. O. Helle. “Pairings and convergence of spectral sequences”. In: (2017). Master’s Thesis, Department of Mathematics, University of Oslo.

[Hes18] L. Hesselholt. “Topological Hochschild homology and the Hasse–Weil zeta function”. In: (2018), pp. 157–180.

[Hes96] L. Hesselholt. “On the p-typical curves in Quillen’s K-theory”. In: *Acta Math.* 177.1 (1996), pp. 1–53.

[HM03] L. Hesselholt and I. Madsen. “On the K-theory of local fields”. In: *Ann. of Math. (2)* 158.1 (2003), pp. 1–113.

[HR19] G. O. Helle and J. Rognes. “Boardman’s whole-plane obstruction group for Cartan–Eilenberg systems”. In: *Doc. Math.* 24 (2019).

[LMSM86] L. G. Lewis Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*. Vol. 1213. Lecture Notes in Mathematics. With contributions by J. E. McClure. Springer-Verlag, Berlin, 1986.

[Mac95] S. Mac Lane. *Homology*. Classics in Mathematics. Reprint of the 1975 edition. Springer-Verlag, Berlin, 1995.

[Mas52] W. S. Massey. “Exact couples in algebraic topology. I, II”. In: *Ann. of Math. (2)* 56 (1952), pp. 363–396.

[May72] J. P. May. *The geometry of iterated loop spaces*. Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972.

[May75] J. P. May. “Classifying spaces and fibrations”. In: *Mem. Amer. Math. Soc.* 1.1, 155 (1975).

[McC01] J. McCleary. *A user’s guide to spectral sequences*. 2nd ed. Vol. 58. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2001.

[Mil56] J. Milnor. “Construction of universal bundles. II”. In: *Ann. of Math. (2)* 63 (1956), pp. 430–436.

[MM02] M. A. Mandell and J. P. May. “Equivariant orthogonal spectra and S-modules”. In: *Mem. Amer. Math. Soc.* 159.755 (2002).

[NS18] T. Nikolaus and P. Scholze. “On topological cyclic homology”. In: *Acta Math.* 221.2 (2018), pp. 203–409.

[Par71] B. Pareigis. “When Hopf algebras are Frobenius algebras”. In: *J. Algebra* 18 (1971), pp. 588–596.

[Sch18] S. Schwede. *Global homotopy theory*. Vol. 34. New Mathematical Monographs. Cambridge University Press, Cambridge, 2018.

[Seg68] G. Segal. “Classifying spaces and spectral sequences”. In: *Inst. Hautes Études Sci. Publ. Math.* 34 (1968), pp. 105–112.

[Sey83] R. M. Seymour. “Some functional constructions on G-spaces”. In: *Bull. London Math. Soc.* 15.4 (1983), pp. 353–359.

[SV02] R. Schwänzl and R. M. Vogt. “Strong cofibrations and fibrations in enriched categories”. In: *Arch. Math. (Basel)* 79.6 (2002), pp. 449–462.

[Tat52] J. Tate. “The higher dimensional cohomology groups of class field theory”. In: *Ann. of Math. (2)* 56 (1952), pp. 294–297.

Paper II

Multiplicative spectral sequences via décalage

Alice Hedenlund

Abstract

We give a definition of décalage for filtrations of spectra, using the Beilinson t-structure, and use this to construct spectral sequences. In particular, we make sure that all the constructions are compatible with the multiplicative structures involved. We use this to prove that the functor from the ∞-category of filtrations of spectra, equipped with the symmetric monoidal structure coming from Day convolution, to the multicategory of spectral sequences of abelian groups, equipped with its standard notion of multilinear pairings, can be endowed with the structure of a map of ∞-operads.

Introduction

Spectral sequences have proven to be immensely powerful tools in modern mathematics. Unfortunately, they have gathered a, not completely unfounded, reputation of being "difficult" and "complicated". The aim of this article is to give a clear account of the subject of multiplicative spectral sequences using the modern language of ∞-categories.

Background and aim

Recall that a spectral sequence (of abelian groups) is a collection of bigraded abelian groups $E^r_{*,*}$, usually called pages, together with differentials on each page, and identifications of the $(r+1)$th page with the homology of the rth page. Defining morphisms in the appropriate way as morphisms of bigraded abelian groups compatible with the extra structure, we obtain a category of spectral sequences, which we will denote as SSEQ. While spectral sequences can certainly be viewed as mathematical objects in their own right, they are more often used as machines to analyse other mathematical objects. As such, it is perhaps a good idea to think of spectral sequences as book-keeping tools; convenient ways to store and process large amounts of data associated to mathematical objects. A common situation where spectral sequences arise and are heavily used is when one studies filtrations. It is well-known that each filtration of chain complexes has an associated spectral sequence of abelian groups. This spectral sequence can then be used to extract information about the chain complex available in the filtration in question and process it in the standard fashion of homological algebra.

The main focus of this paper is multiplicative structures on spectral sequences. Whenever the mathematical object we want to study has some extra structure, say the structure of an $\mathbb{E}_n$-algebra, it is often useful to encode this structure in the spectral sequence used to study the object, in some way.

In this paper, we approach spectral sequences from the point of view of décalage. Décalage was originally introduced by Deligne and one can roughly describe the process as providing us with a way to encode "turning the page of a spectral sequence" on the level of filtrations [Del71]. More specifically, given a filtration (C, X) on a chain complex C, the decalée $(C, \mathrm{Déc}(X))$ is a new filtration of C constructed in such a way that spectral sequence associated to $\mathrm{Déc}(X)$ is isomorphic, after reindexing, to the spectral sequence associated to X shifted forward one page:

$$E^r_{n,s}(\mathrm{Déc}(X)) \cong E^{r+1}_{n,s-n}(X)$$

Although not originally phrased in this way, décalage can be made sense in terms of taking covers of a filtration in a certain t-structure on the category of filtered complexes.

Main results

In this paper, we show that the language of the Beilinson t-structure and décalage provide access to highly structured results on filtered spectra and their associated spectral sequences[1]. Here, a filtered spectrum is an object of the stable ∞-category $\mathrm{Tow(Sp)} = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp})$, where Sp denotes the ∞-category of spectra. To shorten notation, we will write

$$\mathrm{Gr}^q(X) = X(q)/X(q+1) = \mathrm{cofib}(X(q+1) \to X(n))$$

for a given filtration X. In Section II.2, we recall from [BMS19, Section 5.1] that this stable ∞-category can be endowed with a t-structure, the Beilinson t-structure, by declaring Beilinson n-connective filtrations to be objects in the subcategory

$$\mathrm{Tow(Sp)}^{\mathrm{Bei}}_{\geq n} = \{X \in \mathrm{Tow(Sp)} \mid \mathrm{Gr}^q(X) \in \mathrm{Sp}_{\geq n-q} \text{ for all } n\}.$$

[1] For pedagogical reasons, the paper is written as treating spectral sequences of abelian groups coming from filtrations of spectra. It is of course possible to talk about spectral sequences in some other abelian category, and in general, given a filtration in some stable ∞-category $\mathscr{C}$ equipped with a t-structure, there is a completely analogous way in which every filtration of objects in $\mathscr{C}$ gives rise to a spectral sequence in the abelian category $\mathscr{C}^{\heartsuit}$. Of course, some care must be taken when translating the results of this paper to some other choice of $\mathscr{C}$. However, it is worth pointing out that all results in this paper translate directly and hold true for $\mathscr{C} = \mathcal{D}\mathbb{Z}$ (denoting the derived ∞-category of abelian groups equipped with its standard t-structure and symmetric monoidal structure), which we expect to be the most relevant stable ∞-category other than Sp.

We write $\tau^{\mathrm{Bei}}_{\geq n}$, $\tau^{\mathrm{Bei}}_{\leq n}$, and π^{Bei}_n for the cover, truncation, and homotopy functors in the Beilinson t-structure. We go on to note, in Theorem II.2.10, that the heart of the Beilinson t-structure is equivalent to the category of chain complexes of abelian groups, and that the nth Beilinson homotopy group $\pi^{\mathrm{Bei}}_n(X)$, regarded as a chain complex via this equivalence, is given as

$$\cdots \longrightarrow \pi_{n+1}(\mathrm{Gr}^{-1}(X)) \longrightarrow \pi_n(\mathrm{Gr}^0(X)) \longrightarrow \pi_{n-1}(\mathrm{Gr}^1(X)) \longrightarrow \cdots$$

where $\pi_n(\mathrm{Gr}^0(X))$ is placed in homological degree 0. Here, the boundaries in the chain complex are induced by the connecting homomorphisms in the pullback squares

$$\begin{tikzcd}
\mathrm{Gr}^{q+1}(X) \arrow[r] \arrow[d] & X(q)/X(q+2) \arrow[d] \\
0 \arrow[r] & \mathrm{Gr}^q(X)\,.
\end{tikzcd}$$

It is an observation by Antieau that décalage can be phrased using the Beilinson cover functors, and that this extends the notion of décalage to the setting of filtered spectra.

Definition 1. The **décalée** of a filtration X is defined as the filtration $\mathrm{Déc}(X)$ given by

$$\cdots \longrightarrow \operatorname*{colim}_q(\tau^{\mathrm{Bei}}_{\geq n+1}X)(q) \longrightarrow \operatorname*{colim}_q(\tau^{\mathrm{Bei}}_{\geq n}X)(q) \longrightarrow \operatorname*{colim}_q(\tau^{\mathrm{Bei}}_{\geq n-1}X)(q) \longrightarrow \cdots,$$

where the middle term is placed in filtration degree n. The above construction gives us a functor

$$\mathrm{Déc} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \mathrm{Tow}(\mathrm{Sp})$$

that we refer to as **décalage**.

At the heart of the paper, and referred to as Theorem II.2.20, lies the computation of the associated graded of the décalée as the Eilenberg–Mac Lane spectrum of a Beilinson homotopy group:

$$\mathrm{Gr}^n\,\mathrm{Déc}(X) \simeq H\pi^{\mathrm{Bei}}_n(X)[n]\,.$$

We note that the following theorem, which in the text is numbered as Theorem II.3.2, is essentially trivial once we have this expression for the associated graded.

Theorem A. *Given a filtration X, the assignment*

$$E^r_{n,s}(X) = \pi_n(\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)))$$

determines a spectral sequence. Here, the differential $d^r_{n,s} : E^r_{n,s} \to E^r_{n-1,s+r}$ *is induced by the connecting homomorphism in the pushout square*

$$\begin{tikzcd}
\mathrm{Gr}^{(r-1)n+s+1}(\mathrm{D\acute{e}c}^{r-1}(X)) \arrow[r] \arrow[d] & \dfrac{\mathrm{D\acute{e}c}^{r-1}(X)((r-1)n+s)}{\mathrm{D\acute{e}c}^{r-1}(X)((r-1)n+s+2)} \arrow[d] \\
0 \arrow[r] & \mathrm{Gr}^{(r-1)n+s}(\mathrm{D\acute{e}c}^{r-1}(X)) .
\end{tikzcd}$$

In particular, the above theorem tells us that we have a functor

$$E^{\star}_{*,*} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \mathrm{SSEQ}$$

from the ∞-category of spectra to the 1-category of spectral sequences of abelian groups. A significant part of this paper concerns establishing multiplicative properties of this functor. Explicitly, we prove the following theorem, which in the text is Theorem II.3.5.

Theorem B. *The functor* $E^{\star}_{*,*} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{SSEQ}$ *admits the structure of a map of* ∞*-operads.*

Let us elaborate on this statement. The source Tow(Sp) can be endowed with the structure of a symmetric monoidal ∞-category through the use of Day convolution. Degree-wise this is given as

$$(X \otimes Y)(n) \simeq \operatorname*{colim}_{i+j \geq n} X(i) \otimes Y(j) .$$

In contrast, note that the target SSEQ is not a monoidal category. However, although there is no tensor product of spectral sequences *per se*, it is possible to talk about multilinear maps of spectral sequences. This makes the category of spectral sequences into a multicategory, or a coloured operad, if you so will. This puts both Tow(Sp) and SSEQ into the context of ∞-operads, and the above statement makes sense. The proof of the above theorem essentially relies on two facts. The first important fact is that the equivalence between the heart of the Beilinson t-structure and the category of chain complexes of abelian groups that we mentioned earlier is an equivalence of symmetric monoidal categories. This is a 1-categorical statement that we show in Theorem II.2.7. The second essential part is that the computation of the associated graded of the décalée as an Eilenberg–Mac Lane spectrum of a Beilinson homotopy group is compatible with all symmetric monoidal structures involved. This is the content of Theorem II.2.22.

We end the paper with an appendix on the Tate spectral sequence, which is the author's personal reason for studying multiplicative spectral sequences in general. In particular, we give a short proof that the Tate spectral sequence is a multiplicative spectral sequence, using the results covered in the main article, and use multiplicativity to deduce the d^1-differential in the $\mathbb{T}$-Tate spectral sequence, where $\mathbb{T}$ is the circle group.

Notation

As we will use ∞-categories extensively throughout the paper, we will follow common notation and terminology for these, following [Lur09; Lur17]. In particular, we will use the symbol $\otimes$ to denote the smash product of spectra. We will typically not differentiate between a 1-category and its corresponding ∞-category obtained via the nerve construction.

What we mean by a colimit and limit will depend on the context; if we are working in an ∞-category this is what classically might have been referred to as homotopy colimits and limits. We hope that this will not lead to confusion.

To differentiate between different mapping objects we will use the following notation:

$\mathrm{Hom}_{\mathscr{C}}(X,Y)$	a mapping *abelian group*
$\mathrm{Map}_{\mathscr{C}}(X,Y)$	a mapping *space*
$\mathrm{map}_{\mathscr{C}}(X,Y)$	a mapping *spectrum*

We will exclusively use homological indexing throughout the paper. In particular, we use homological Adams gradings for spectral sequences.

Acknowledgements

This paper grew out of a joint project with Achim Krause and Thomas Nikolaus on setting up a modern framework for dealing with spectral sequences. I am thankful to both of them for many interesting and enlightening discussions on the topic. The idea of using décalage is due to Ben Antieau, and came to me via Achim and Thomas, and I am very grateful for him pointing us in the right direction. Finally, my PhD advisor John Rognes was the one who set me on the path of studying multiplicative structures in spectral sequences in the first place, and I wish to thank him for this, as well as for his constructive criticism and support in this project and other mathematical endeavours.

II.1 Filtered objects

In this section, we introduce the necessary terminology and some basic results regarding filtrations. For pedagogical reasons, we will focus on filtered objects in the ∞-category of spectra. We start with the basic definitions in Section II.1.1, and go on to discuss complete filtrations more specifically in Section II,1.2. In Section 1.3, we discuss the symmetric monoidal structure on filtrations coming from Day convolution, and elaborate in particular on the monoidal properties of the associated graded functor. Finally, in Section II.1.4, we discuss the so-called canonical t-structure on filtrations and how to use it to define the Whitehead filtration of a spectrum.

II.1.1 Preliminary definitions

We start with defining the ∞-category of filtered objects in some stable ∞-category $\mathscr{C}$. To make sure no confusion arises regarding the direction of arrows: $\mathbb{Z}$ is the 1-category whose objects are integers, and where the mapping set $\mathbb{Z}(n, m)$ has a single element if $n \geq m$, and is empty otherwise.

Definition II.1.1. The **stable ∞-category of filtered objects in $\mathscr{C}$** is the functor category

$$\mathrm{Tow}(\mathscr{C}) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathscr{C}) .$$

Note that $\mathrm{Tow}(\mathscr{C})$ is indeed stable since colimits and limits are computed object-wise in functor categories. Throughout the paper, we will use the notation

$$X(-\infty) = \operatorname{colim} X \quad \text{and} \quad X(\infty) = \lim X$$

for the colimit and the limit of a filtration X, respectively, should they exist. We will regard the colimit as the "underlying object" of our filtration. For brevity, we will also write

$$X(m)/X(n) = \mathrm{cofib}(X(n) \to X(m)) , \quad -\infty \leq m \leq n \leq \infty .$$

For each integer n, there is an evaluation functor $\mathrm{ev}_p : \mathrm{Tow}(\mathscr{C}) \to \mathscr{C}$. This functor commutes with all limits and colimits, and has left and right adjoints. We will be most concerned with its left adjoint, which we denote as

$$L_p : \mathscr{C} \to \mathrm{Tow}(\mathscr{C}) .$$

This is the functor which assigns to an object A the filtration given in each filtration degree as

$$L_p(A)(s) = \begin{cases} A & \text{if } s \leq p \\ 0 & \text{otherwise,} \end{cases}$$

with identity maps between the copies of A. We refer to the subcategory of $\mathrm{Tow}(\mathscr{C})$ consisting of filtrations of this form as $\mathrm{Tow}^{\mathrm{Gen}}(\mathscr{C})$. These act as "generators" for the ∞-category of filtrations, in the sense of the following proposition.

Proposition II.1.2. *The relative Yoneda embedding*

$$Y^{\mathrm{Gen}} : \mathrm{Tow}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathrm{Tow}^{\mathrm{Gen}}(\mathscr{C})) , \quad X \mapsto \mathrm{Map}_{\mathrm{Tow}(\mathscr{C})}(-, X)$$

is fully faithful.

Proof. The Yoneda embedding $\mathscr{C} \longrightarrow \mathrm{PSh}(\mathscr{C})$ is fully faithful, and it follows that the induced functor

$$\mathrm{Tow}(\mathscr{C}) \longrightarrow \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{PSh}(\mathscr{C})) , \quad X \mapsto (p \mapsto \mathrm{Map}_{\mathscr{C}}(-, X(p)))$$

is fully faithful, as well. Note that we have a natural equivalence

$$\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}} \times \mathscr{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{PSh}(\mathscr{C}))$$

and that the functor $p \mapsto \mathrm{Map}_{\mathscr{C}}(-, X(p))$, and element in the right hand side, corresponds to the functor

$$(p, A) \mapsto \mathrm{Map}_{\mathscr{C}}(A, X(p)) \simeq \mathrm{Map}_{\mathrm{Tow}(\mathscr{C})}(L_p(A), X)$$

on the left hand side, where we have used the L_p-ev_p-adjunction. Since $\mathbb{Z} \times \mathscr{C} \simeq \mathrm{Tow}^{\mathrm{Gen}}(\mathscr{C})$, the total functor

$$\mathrm{Tow}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathrm{Tow}(\mathscr{C})^{\mathrm{Gen}}), \quad X \mapsto \mathrm{Map}_{\mathrm{Tow}(\mathscr{C})}(-, X)$$

is precisely the relative Yoneda embedding, which finishes the proof. □

Corollary II.1.3. *Let $\mathscr{C}$ and $\mathscr{D}$ be a cocomplete ∞-categories. Then*

$$\mathrm{Fun}^{\mathrm{Lex}}(\mathrm{Tow}(\mathscr{C}), \mathscr{D}) \simeq \mathrm{Fun}(\mathrm{Tow}^{\mathrm{Gen}}(\mathscr{C}), \mathscr{D}),$$

where the left hand side refers to functors preserving all colimits.

Remark II.1.4. Other accounts of the subject of spectral sequences in ∞-categories, like [Lur17, Section 1.2.2] and [GP18] employ "sequences" (that is, objects in $\mathrm{Fun}(\mathbb{Z}, \mathscr{C})$), rather than "towers" (that is, objects in $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathscr{C})$), as we have opted to do. Since the Whitehead filtration is most easily phrased in the language of towers, rather than sequences, and since it has better monoidal properties than the Postnikov filtration, we have chosen to work with the "tower convention". Many of the results given in Section II.1 can be found in mentioned references; we have merely rewritten everything to the dualised setting here.

II.1.2 Complete towers and graded equivalences

In this section, we introduce the notion of a complete filtration and discuss how it relates to graded equivalence. We will assume that $\mathscr{C}$ has sequential limits in this section.

Definition II.1.5. If X is a filtration such that $X(\infty) \simeq 0$, we will refer to X as a **complete filtration**.

Let us denote the subcategory of $\mathrm{Tow}(\mathscr{C})$ consisting of those filtrations that are complete by $\widehat{\mathrm{Tow}}(\mathscr{C})$. Note that every filtration X can be made into a complete one by modding out by the limit $X(\infty)$ in every filtration degree. The resulting functor

$$\widehat{(-)} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \widehat{\mathrm{Tow}(\mathrm{Sp})}, \quad X \mapsto (n \mapsto X(n)/X(\infty))$$

is referred to as **completion**.

Proposition II.1.6. *Completion is left adjoint to the inclusion $\widehat{\mathrm{Tow}}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$.*

Proof. Let X and Y be filtrations, where Y is assumed to be complete. A map $\hat{X} \to Y$ corresponds to a map $X \to Y$ by pre-composing with the morphism $X \to \hat{X}$. Conversely, suppose that we are given a map $X \to Y$, and consider the commutative diagrams

$$\begin{tikzcd} X(\infty) \arrow[r] \arrow[d] & X(n) \arrow[r] \arrow[d] & \hat{X}(n) \arrow[d] \\ 0 \arrow[r] & Y(n) \arrow[r, equal] & Y(n) \end{tikzcd}$$

in which the upper and lower rows are cofibre sequences. From this, we see that the maps $X(n) \to Y(n)$ can be factored through $\hat{X}(n)$, and it follows that $X \to Y$ can be factored as

$$X \longrightarrow \hat{X} \longrightarrow Y\,,$$

which provides us with the wanted map $\hat{X} \to Y$. $\square$

Corollary II.1.7. *The subcategory of complete towers is stable.*

Proof. Follows from [Lur17, Lemma 1.4.4.7]. $\square$

Complete filtrations are related to so-called graded equivalence. We will not cover this in detail, as we will not need it, but instead refer the interested reader to [GP18]. We will need the most basic definition, though.

Definition II.1.8. The functor

$$\mathrm{Gr}^q : \mathrm{Tow(Sp)} \longrightarrow \mathrm{Sp}\,, \quad X \mapsto X(q)/X(q+1)$$

is referred to as the q**th associated graded functor**. The collection of all of these forms a functor

$$\mathrm{Gr} : \mathrm{Tow(Sp)} \longrightarrow \prod_{\mathbb{Z}} \mathrm{Sp}\,, \quad X \mapsto (X(q)/X(q+1))_{q\in\mathbb{Z}}$$

referred to as the **total associated graded functor**.

We say that a map $f : X \to Y$ of filtrations is a **graded equivalence** if $\mathrm{Gr}(f)$ is an equivalence. By [GP18, Remark 2.16] localisation with respect to graded equivalences is equivalent to localisation with respect to completion. The following result follows.

Proposition II.1.9. *Let f be a map of filtrations. If $\mathrm{Gr}(f)$ and $f(\infty)$ are equivalences, then f is an equivalence.*

II.1.3 Monoidal properties of the associated graded

Assume that $(\mathscr{C}, \otimes, 1)$ is a symmetric monoidal ∞-category. We can then endow the ∞-category of filtered objects in $\mathscr{C}$ with a symmetric monoidal structure via Day convolution, by noting that $\mathbb{Z}^{\mathrm{op}}$ has a symmetric monoidal structure given by addition of integers. See [Gla16] for a lift of Day convolution to the ∞-categorical level, later extended in [Nik16]. Explicitly, the convolution product of two filtrations X and Y is given in each filtration degree as

$$(X \otimes Y)(n) \simeq \operatorname*{colim}_{i+j \geq n} X(i) \otimes Y(j)\,.$$

Note that the unit of the symmetric monoidal structure on $\mathrm{Tow}(\mathrm{Sp})$ is the filtration $\mathbb{1} = L_0(1)$, where 1 is the unit for the symmetric monoidal structure on $\mathscr{C}$. Note also that an $\mathbb{E}_1$-algebra (resp. $\mathbb{E}_\infty$-algebra) in $\mathrm{Tow}(\mathscr{C})$ is nothing more than a lax (symmetric) monoidal functor $\mathbb{Z}^{\mathrm{op}} \to \mathscr{C}$.

Definition II.1.10. A map $X \otimes Y \to Z$ of filtered objects is called a **pairing of filtered objects**.

To settle questions about how the symmetric monoidal structure on filtered spectra interacts with the associated spectral sequences, it is of interest to know how the symmetric monoidal structure interacts with the associated graded functors introduced in Definition II.1.8.

Lemma II.1.11. *The subcategory* $\mathrm{Tow}^{\mathrm{Gen}}(\mathscr{C})$ *is closed under the tensor products.*

Proof. The unit $L_0(1)$ belongs to the specified subcategory and one readily checks that

$$L_p(A) \otimes L_p(B) \simeq L_{p+q}(A \otimes B)$$

for arbitrary p, $q \in \mathbb{Z}$ and A, $B \in \mathrm{Sp}$. $\square$

Proposition II.1.12. *Given a cocontinuous functor* $F : \mathrm{Tow}(\mathscr{C}) \longrightarrow \mathscr{D}$*, where* $\mathscr{D}$ *is an arbitrary cocomplete symmetric monoidal* ∞*-category, a lax symmetric monoidal structure on* F *can be constructed for the restricted functor* $F : \mathrm{Tow}^{\mathrm{Gen}}(\mathscr{C}) \to \mathscr{D}$ *and then canonically extended to* $\mathrm{Tow}(\mathscr{C})$ *by cocontinuity.*

Proof. Follows from Corollary II.1.3 and the relative Yoneda embedding being strong symmetric monoidal. $\square$

Proposition II.1.13. *The total associated graded functor*

$$\mathrm{Gr} : \mathrm{Tow}(\mathscr{C}) \longrightarrow \prod_{\mathbb{Z}} \mathscr{C}\,, \quad X \mapsto (X(q)/X(q+1))_{q \in \mathbb{Z}}$$

is (strong) symmetric monoidal. In particular, there is an equivalence

$$\mathrm{Gr}^q(X \otimes Y) \simeq \bigoplus_{i+j=q} \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y)$$

for all integers q*.*

Proof. Since Gr is a cocontinuous functor it suffices to check this on the filtrations of $\mathrm{Tow}^{\mathrm{Gen}}(\mathscr{C})$, by Remark II.1.12. This is straight forward:

$$\mathrm{Gr}(L_p(A) \otimes L_q(B)) \simeq \mathrm{Gr}(L_{p+q}(A \otimes B)) \simeq \mathrm{Gr}(L_p(A)) \otimes \mathrm{Gr}(L_q(B)) \,.$$

□

Given a map $\phi : X \otimes Y \to Z$ of filtrations, let us describe the induced map $\mathrm{Gr}(\phi) : \mathrm{Gr}(X) \otimes \mathrm{Gr}(Y) \to \mathrm{Gr}(Z)$, explicitly. By virtue of $X \otimes Y \to Z$ being a pairing of filtrations, we are provided maps

$$\phi_{i,j} : X(i) \otimes Y(j) \longrightarrow Z(i+j) \,,$$

compatible in the sense that we have commutative diagrams

$$\begin{tikzcd}
X(i+1) \otimes Y(j) \arrow[d, "\phi_{i+1,j}"] & X(i) \otimes Y(j) \arrow[l] \arrow[r] \arrow[d, "\phi_{i,j}"] & X(i) \otimes Y(j+1) \arrow[d, "\phi_{i.j+1}"] \\
Z(i+j+1) & Z(i+j) \arrow[l] \arrow[r] & Z(i+j+1)
\end{tikzcd}$$

for all i and j. Consider the commutative diagram

$$\begin{tikzcd}
X(i+1) \otimes Y(j+1) \arrow[rr] \arrow[dr] \arrow[dd, "\phi_{i+1,j+1}"'] & & X(i) \otimes Y(j+1) \arrow[dr] \arrow[dd, "\phi_{i,j+1}"] & \\
& X(i+1) \otimes Y(j) \arrow[rr] \arrow[dr, "\phi_{i+1,j}"'] & & X(i) \otimes Y(j) \arrow[d, "\phi_{i,j}"] \\
Z(i+j+2) \arrow[rr] & & Z(i+j+1) \arrow[r] & Z(i+j) \,.
\end{tikzcd}$$

For brevity, let us refer to the pushout of the upper square by

$$W_{i,j} = X(i) \otimes Y(j+1) \oplus_{X(i+1) \otimes Y(j+1)} X(i+1) \otimes Y(j) \,.$$

By the universal property of pushouts, we have a induced map $\psi_{i,j}$ in the diagram

$$\begin{tikzcd}
X(i+1) \otimes Y(j+1) \arrow[r] \arrow[d] & X(i) \otimes Y(j+1) \arrow[d] \arrow[ddr, bend left, "\phi_{i,j+1}"] & \\
X(i+1) \otimes Y(j) \arrow[r] \arrow[drr, bend right, "\phi_{i+1,j}"'] & W_{i,j} \arrow[dr, dotted, "\psi_{i,j}"] & \\
& & Z(i+j+1) \,,
\end{tikzcd}$$

as well as the induced map $\alpha_{i,j}$ is the diagram

$$\begin{tikzcd}
X(i+1) \otimes Y(j+1) \arrow[r] \arrow[d] & X(i) \otimes Y(j+1) \arrow[d] \arrow[ddr, bend left] & \\
X(i+1) \otimes Y(j) \arrow[r] \arrow[drr, bend right] & W_{i,j} \arrow[dr, dotted, "\alpha_{i,j}"] & \\
 & & X(i) \otimes Y(j) \,.
\end{tikzcd}$$

We conclude that we have a commutative diagram

$$\begin{tikzcd}
W_{i,j} \arrow[r, "\alpha_{i,j}"] \arrow[d, "\psi_{i,j}"] & X(i) \otimes Y(j) \arrow[r, dotted, "\iota_i^X \otimes \iota_j^Y"] \arrow[d, "\phi_{i,j}"] & \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \arrow[d, dotted, "\mathrm{Gr}^{i,j}(\phi)"] \\
Z(i+j+1) \arrow[r] & Z(i+j) \arrow[r, dotted, "\iota_{i+j}^Z"] & \mathrm{Gr}^{i+j}(Z) \,,
\end{tikzcd} \tag{II.1}$$

where we have denoted the inclusion $X(i) \to \mathrm{Gr}^i(X)$ by ι_i^X, and correspondingly for the filtrations Y and Z. Here, the dotted horizontal maps are those induced by taking cofibres, and the vertical dotted map

$$\mathrm{Gr}^{i,j}(\phi) : \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \longrightarrow \mathrm{Gr}^{i+j}(Z)$$

is the map induced on cofibres. The collection

$$\mathrm{Gr}^q(\phi) = (\mathrm{Gr}^{i,j}(\phi))_{i+j=q} : \mathrm{Gr}^q(X \otimes Y) \simeq \bigoplus_{i+j=q} \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \longrightarrow \mathrm{Gr}^q(Z) \,.$$

describes the qth summand of the induced map

$$\mathrm{Gr}(\phi) : \mathrm{Gr}(X) \otimes \mathrm{Gr}(Y) \longrightarrow \mathrm{Gr}(Z) \,.$$

Later on in the paper we need to understand how connecting homomorphisms of various relevant pushout squares related to the associated graded functor interact, especially with respect to symmetric monoidal structures. To more easily keep track of the connecting homomorphisms, let us introduce the following notations.

Notation II.1.14. Given a filtration X, let us denote the connecting homomorphism of the pushout

$$\begin{tikzcd}
\mathrm{Gr}^{i+1}(X) \arrow[r] \arrow[d] & X(i)/X(i+2) \arrow[d] \\
0 \arrow[r] & \mathrm{Gr}^i(X) \,,
\end{tikzcd}$$

by

$$\delta_i^X : \mathrm{Gr}^i(X) \longrightarrow \mathrm{Gr}^{i+1}(X)[1] \,,$$

Notation II.1.15. Given a filtration X, let us denote the connecting homomorphism of the pushout

$$\begin{tikzcd} X(i+1) \arrow[r] \arrow[d] & X(i) \arrow[d, "\iota^X_{i,j}"] \\ 0 \arrow[r] & \mathrm{Gr}^i(X)\,, \end{tikzcd}$$

by

$$\partial^X_i : \mathrm{Gr}^i(X) \longrightarrow X(i+1)[1]\,.$$

Given two filtrations X and Y, let us denote the "total connecting homomorphism" by

$$\partial^{\mathrm{Tot}}_{i,j} = \partial^X_i \otimes 1 + 1 \otimes \partial^Y_j\,.$$

The following result follows directly from the above notational definitions.

Lemma II.1.16. *Given a filtration X and integer i, there is a commutative diagram*

$$\begin{tikzcd} X(i+2) \arrow[r] \arrow[d] & X(i+1) \arrow[r] \arrow[d, "\iota^X_{i+1}"] & X(i) \arrow[d] \\ 0 \arrow[r] & \mathrm{Gr}^{i+1}(X) \arrow[r] \arrow[d] & X(i)/X(i+2) \arrow[d] \\ & 0 \arrow[r] & \mathrm{Gr}^i(X)\,, \end{tikzcd}$$

where all the squares are pushouts. In particular, the connecting homomorphism $\delta^X_i : \mathrm{Gr}^i(X) \to \mathrm{Gr}^{i+1}(X)[1]$ can be identified with the composition

$$\mathrm{Gr}^i(X) \xrightarrow{\partial^X_i} X(i+1)[1] \xrightarrow{\iota^X_{i+1}} \mathrm{Gr}^{i+1}(X)[1]\,.$$

We also need to keep track of the connecting homomorphism associated to the top row of Diagram (II.1).

Notation II.1.17. Given two filtrations X and Y, let us denote the connecting homomorphism of the pushout

$$\begin{tikzcd} W_{i,j} \arrow[r, "\alpha_{i,j}"] \arrow[d] & X(i) \otimes Y(j) \arrow[d, "\iota^X_i \otimes \iota^Y_j"] \\ 0 \arrow[r] & \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y)\,, \end{tikzcd}$$

by

$$\partial^W_{i,j} : \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \longrightarrow W_{i,j}[1]\,,$$

Proposition II.1.18. *There is a commutative square*

$$\begin{tikzcd}
\mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \arrow[r, "\partial^W_{i,j}"] \arrow[d, "\mathrm{Gr}^{i,j}(\phi)"'] & W_{i,j}[1] \arrow[d, "\psi_{i,j}"] \\
\mathrm{Gr}^{i+j}(Z) \arrow[r, "\partial^Z_{i+j}"] & Z(i+j+1)[1]
\end{tikzcd}$$

Proof. Follow directly by considering Diagram (II.1). ☐

Finally, we need to understand how the connecting homomorphism $\partial^{i,j}_W$ relates to the "total connecting homomorphism" we defined in Notation II.1.15. Let us start with a more general result.

Lemma II.1.19. *Suppose we are given a commutative diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D\,,
\end{tikzcd} \tag{☐}$$

of spectra, and let P denote the pushout in the diagram. The following hold:

1. *The sequence $A \to P \to D$ gives rise to cofibre sequence*

   $$P/A \longrightarrow D/A \longrightarrow D/P\,,$$

   in a natural way. Let us refer to the connecting homomorphism of this cofibre sequence by

   $$\partial^{\mathrm{Tot}} : D/P \longrightarrow P/A[1]\,.$$

2. *There are cofibre sequences*

   $$C/A \longrightarrow D/B \longrightarrow D/P \quad \text{and} \quad B/A \longrightarrow D/C \longrightarrow D/P\,,$$

   obtained by taking vertical and horizontal cofibres in the diagram (☐), respectively. Let us denote the connecting homomorphisms of these two cofibre sequences by

   $$\partial^h : D/P \longrightarrow C/A[1] \quad \text{and} \quad \partial^v : D/P \longrightarrow B/A[1]\,,$$

   respectively.

3. *There is a natural equivalence*

   $$P/A \simeq C/A \oplus B/A\,.$$

   Moreover, the total connecting homomorphism can be written as the sum of the vertical and horizontal connecting homomorphisms:

   $$\partial^{\mathrm{Tot}} = \partial^h + \partial^v : D/P \longrightarrow (C/A \oplus B/A)[1]\,.$$

Proof.

(i) Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & D \arrow[d] \\ 0 \arrow[r] & P/A \arrow[r] \arrow[d] & D/A \arrow[d] \\ & 0 \arrow[r] & D/P\,, \end{tikzcd}$$

in which every square is a pushout.

(ii) Recall that the so-called "total cofibre" of the square (□) can be computed in three ways:

a) By taking the cofibre of the map $P \to D$.
b) By taking vertical cofibres and then the cofibre of the induced map $C/A \to D/B$.
c) By taking horizontal cofibres and then the cofibre of the induced map $B/A \to D/C$.

We leave it to the reader to check that these three ways do indeed compute equivalent spectra; it is essentially an exercise in commuting colimits. In particular, we have

$$\mathrm{cofib}(C/A \to D/B) \simeq \mathrm{cofib}(B/A \to D/C) \simeq D/P\,,$$

which is what we wanted to show.

(iii) Establishing the natural equivalence

$$P/A \simeq C/A \oplus B/A$$

is again an exercise in commuting colimits, and we leave this to the reader. To show that the "total connecting homomorphism" is the sum of the horizontal and vertical connecting homomorphisms we need to show that the compositions

$$D/P \xrightarrow{\partial^{\mathrm{Tot}}} P/A[1] \xrightarrow{p_{C/A}} C/A[1] \quad \text{and} \quad D/P \xrightarrow{\partial^{\mathrm{Tot}}} P/A[1] \xrightarrow{p_{B/A}} B/A[1]$$

are ∂^h and ∂^v, respectively. Here, $p_{C/A}$ and $p_{B/A}$ denotes the canonical projections to the two summands, which occur as the cofibres of the inclusions $i_{B/A} : B/A \to P/A$ and $i_{C/A} : C/A \to P/A$, respectively, since Sp is a stable ∞-category, so that finite sums and finite products agree. Let us show that the first of the compositions is equal to ∂^h; the other one is shown analogously. First note that the map $D/A \to D/P$ factors through D/B, and then consider the commutative diagram

$$\begin{tikzcd}
P/A \arrow[r] \arrow[d, "p_{C/A}"'] & D/A \arrow[d] \\
C/A \arrow[r] \arrow[d] & D/B \arrow[d] \\
0 \arrow[r] & D/P
\end{tikzcd}$$

in which the lower square is a pushout square since $C/A \to D/B \to D/P$ is a cofibre sequence, and the big rectangle is a pushout square since $P/A \to D/A \to D/P$ is a cofibre sequence. The result now follows.

$\square$

Proposition II.1.20. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \arrow[rr, "\partial^W_{i,j}"] \arrow[dr, "\partial^{\mathrm{Tot}}_{i,j}"'] & & W_{i,j}[1] \arrow[dl] \\
& \big(X(i+1) \otimes \mathrm{Gr}^j(Y) \oplus \mathrm{Gr}^i(X) \otimes Y(j+1)\big)[1] &
\end{tikzcd},$$

Proof. For brevity, let us write

$$\begin{aligned}
U_{i,j} &= \mathrm{cofib}\,(X(i+1) \otimes Y(j+1) \longrightarrow W_{i,j}) \\
&\simeq X(i+1) \otimes \mathrm{Gr}^j(Y) \oplus \mathrm{Gr}^i(X) \otimes Y(j+1)
\end{aligned}$$

and

$$V_{i,j} = \mathrm{cofib}\,((X(i+1) \otimes Y(i+1) \to X(i) \otimes Y(j)) \ .$$

Applying Lemma II.1.19 to the commutative diagram

$$\begin{tikzcd}
X(i+1) \otimes Y(j+1) \arrow[r] \arrow[d] & X(i) \otimes Y(j+1) \arrow[d] \\
X(i+1) \otimes Y(j) \arrow[r] & X(i) \otimes Y(j)
\end{tikzcd}$$

tells us that the connecting homomorphism of the cofibre sequence

$$U_{i,j} \longrightarrow V_{i,j} \longrightarrow \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y)$$

is the sum

$$\partial^{\mathrm{Tot}}_{i,j} = \partial^X_i \otimes 1 + 1 \otimes \partial^Y_j \ .$$

The result now follows from considering the commutative diagram

$$\begin{tikzcd}
X(i+1)\otimes Y(j+1) \arrow[r] \arrow[d] & W_{i,j} \arrow[r] \arrow[d] & X(i)\otimes Y(j) \arrow[d] \\
0 \arrow[r] & U_{i,j} \arrow[r] \arrow[d] & V_{i,j} \arrow[d] \\
 & 0 \arrow[r] & \mathrm{Gr}^i(X)\otimes \mathrm{Gr}^j(Y)\,,
\end{tikzcd}$$

in which every square is a pushout. □

With all of these lemmas in place, let us now phrase and prove the following result.

Theorem II.1.21 (The Leibniz rule)**.** *Given a pairing $\phi : X\otimes Y \to Z$ of filtrations, the diagram*

$$\begin{tikzcd}
\mathrm{Gr}^i(X)\otimes \mathrm{Gr}^j(Y) \arrow[r, "\delta_i^X\otimes 1+1\otimes\delta_j^Y"] \arrow[d, "\mathrm{Gr}^{i,j}(\phi)"'] & \big(\mathrm{Gr}^{i+1}(X)\otimes \mathrm{Gr}^j(Y)\oplus \mathrm{Gr}^i(X)\otimes \mathrm{Gr}^{j+1}(Y)\big)[1] \arrow[d, "\mathrm{Gr}^{i+1,j}(\phi)\oplus\mathrm{Gr}^{i,j+1}(\phi)"] \\
\mathrm{Gr}^{i+j}(Z) \arrow[r, "\delta_{i+j}^Z"] & \mathrm{Gr}^{i+j+1}(Z)[1]\,.
\end{tikzcd}$$

commutes for all integers i and j.

Proof. By Proposition II.1.16 and Proposition II.1.20, we can rewrite the upper composition as the composition

$$\begin{aligned}
\mathrm{Gr}^i(X)\otimes \mathrm{Gr}^j(Y) \xrightarrow{\partial_{i,j}^W} W_{i,j}[1] &\longrightarrow U_{i,j}[1] \\
&\longrightarrow \big(\mathrm{Gr}^{i+1}(X)\otimes \mathrm{Gr}^j(Y)\oplus \mathrm{Gr}^i(X)\otimes \mathrm{Gr}^{j+1}(Y)\big)[1] \\
&\longrightarrow \mathrm{Gr}^{i+j+1}(Z)[1]\,,
\end{aligned}$$

where we are using the notation introduced in the proof of Proposition II.1.19. We claim that we can identify the part $W_{i,j}[1]\to \mathrm{Gr}^{i+j+1}(Z)[1]$ of the above composition with

$$W_{i,j}[1] \xrightarrow{\psi_{i,j}} Z(i+j+1)[1] \xrightarrow{\iota_{i+j+1}^Z} \mathrm{Gr}^{i+j+1}(Z)[1]\,.$$

This can be shown on the unsuspended versions of all the maps, so let us do this, as it will make our notation slightly less involved. Consider the commutative diagram

$$\begin{tikzcd}
X(i+1)\otimes Y(j+1) \arrow[r] \arrow[d, "\phi_{i+1,j+1}"'] & W_{i,j} \arrow[r] \arrow[d, "\psi_{i,j}"'] & U_{i,j} \arrow[d] \\
Z(i+j+2) \arrow[r] & Z(i+j+1) \arrow[r, "\iota_{i+j+1}^Z"] & \mathrm{Gr}^{i+j+1}(Z)\,,
\end{tikzcd}$$

in which the right-most terms are obtained by taking cofibres of the left-most horizontal maps. The commutativity of the right hand square gives a proof of our claim since

$$U_{i,j} \longrightarrow \mathrm{Gr}^{i+1}(X) \otimes \mathrm{Gr}^{j}(Y) \oplus \mathrm{Gr}^{i}(X) \otimes \mathrm{Gr}^{j+1}(Y) \longrightarrow \mathrm{Gr}^{i+j+1}(Z)$$

can be identified with the right-most vertical map in the diagram. Using the claim, the upper composition can be further rewritten as

$$\begin{aligned} \mathrm{Gr}^{i}(X) \otimes \mathrm{Gr}^{j}(Y) &\xrightarrow{\mathrm{Gr}^{i,j}(\phi)} \mathrm{Gr}^{i+j}(Z) \\ &\xrightarrow{\partial^Z_{i+j}} Z(i+j+1)[1] \\ &\longrightarrow \mathrm{Gr}^{i+j+1}(Z)[1]\,, \end{aligned}$$

using Proposition II.1.18. The result now follows by appealing to Proposition II.1.16. □

II.1.4 The canonical t-structure and the Whitehead filtration

Given a spectrum A, we can construct two filtrations of A by killing off homotopy groups either over or under a certain degree. These two filtrations are called the Postnikov and Whitehead filtration, respectively. In a wider generality, these constructions can be made in any stable ∞-category $\mathscr{C}$ with a t-structure. We will focus on the Whitehead filtration, as it has better multiplicative properties than the Postnikov one. In order to deduced these multiplicative properties, it is useful to describe the Whitehead filtration as the connective cover of a constant filtration. In this section, we describe how to equip $\mathrm{Tow}(\mathscr{C})$ with a suitable t-structure which makes this description valid.

Consider the functor $\tau^{\mathrm{can}}_{\geq 0} : \mathrm{Tow}(\mathscr{C}) \to \mathrm{Tow}(\mathscr{C})$ which sends the filtration X to the filtration

$$(\tau^{\mathrm{can}}_{\geq 0} X)(n) = \tau_{\geq n} X(n)\,.$$

Note that the essential image of this functor is the subcategory

$$\mathrm{Tow}(\mathrm{Sp})^{\mathrm{can}}_{\geq 0} = \{X \in \mathrm{Tow}(\mathscr{C}) \mid X(n) \in \mathscr{C}_{\geq n}\}\,.$$

We show that this does indeed determine a t-structure on the category of filtered objects in $\mathscr{C}$.

Proposition II.1.22. *This does indeed determine a t-structure on the category of filtered objects in $\mathscr{C}$.*

Proof. This is an application of the dual of [Lur17, Proposition 1.2.1.16]. Observe that $\tau^{\mathrm{can}}_{\geq 0}$ is an idempotent functor, so it indeed constitutes a colocalization functor. We only need to show that $\mathrm{Tow}(\mathscr{C})^{\mathrm{can}}_{\geq 0}$ is closed under extensions. Let X and Z be connective filtrations and consider a cofibre sequence

$$X \to Y \to Z$$

in $\mathrm{Tow}(\mathscr{C})$. This data gives us cofibre sequences

$$X(n) \to Y(n) \to Z(n)$$

in $\mathscr{C}$ for every integer n. Since the subcategories $\mathscr{C}_{\geq n}$ are closed under extensions for every n, we conclude that $Y(n) \in \mathscr{C}_{\geq n}$ for every n, which is what we wanted to show. $\square$

We shall refer to this t-structure as the **canonical**[2] **t-structure** on $\mathrm{Tow}(\mathscr{C})$. Let us write $\tau_{\geq n}^{\mathrm{can}}$, $\tau_{\leq n}^{\mathrm{can}}$, and π_n^{can} for the cover, truncation, and homotopy functors in this t-structure. From this point and on, we will make the standing assumption that $\mathscr{C}$ is endowed with a symmetric monoidal structure that is compatible with the t-structure on $\mathscr{C}$, in the sense of Definition II.A.6.

Proposition II.1.23. *The canonical t-structure on* $\mathrm{Tow}(\mathscr{C})$ *is compatible with the symmetric monoidal structure on* $\mathrm{Tow}(\mathscr{C})$ *given via Day convolution.*

Proof. We need to check the two conditions of Definition II.A.6. Recall that the unit for the tensor product on $\mathrm{Tow}(\mathscr{C})$ is the tower $L_0(1)$, where 1 is the tensor unit of $\mathscr{C}$. This is an object of $\mathrm{Tow}(\mathscr{C})_{\geq 0}$ since 1 is an object of $\mathscr{C}_{\geq 0}$ by assumption, and hence also an object of $\mathscr{C}_{\geq n}$ for all negative n. Let X and Y be connective filtrations. We want to show that $X \otimes Y$ is connective as well. By definition of Day convolution we have

$$(X \otimes Y)(n) \simeq \operatorname*{colim}_{i+j \geq n} X(i) \otimes Y(j)\,.$$

The object $X(i) \otimes Y(j)$ lies in $\mathscr{C}_{\geq i+j}$ by Lemma II.A.8, and $\mathscr{C}_{\geq i+j} \subseteq \mathscr{C}_{\geq n}$ whenever $i + j \geq n$. We conclude that $(X \otimes Y)(n)$ lies in $\mathscr{C}_{\geq n}$ since $\mathscr{C}_{\geq n}$ is stable under all colimits in $\mathscr{C}$. $\square$

Proposition II.1.24. *There is an isomorphism*

$$\mathrm{Tow}(\mathscr{C})^{\mathrm{can},\heartsuit} \simeq \prod_{\mathbb{Z}} \mathscr{C}^{\heartsuit}$$

of symmetric monoidal categories. Here, the symmetric monoidal structure on the canonical heart is given by

$$X \otimes_{\mathrm{can},\heartsuit} Y = \tau_{\leq 0}^{\mathrm{can}}(X \otimes Y)$$

and the symmetric monoidal structure on graded objects is the sign-introducing one.

Proof. It is not hard to see that the functor

$$F : \mathrm{Tow}(\mathscr{C})^{\mathrm{can},\heartsuit} \longrightarrow \prod_{\mathbb{Z}} \mathscr{C}^{\heartsuit}\,, \quad X \mapsto (\pi_n(X(n)))_{n\in\mathbb{Z}}$$

[2] The word "canonical" is mainly there to differentiate it from another, more subtle, t-structure on filtered objects that will be relevant later on. See Section II.2.

constitutes an isomorphism of categories. Note that the canonical maps $X(i) \otimes Y(j) \to (X \otimes Y)(i+j)$ and lax symmetric monoidality of the functor

$$\pi_* : \mathscr{C} \longrightarrow \prod_{\mathbb{Z}} \mathscr{C}^{\heartsuit},$$

ensure that we have a map

$$F(X) \otimes F(Y) \longrightarrow F(X \otimes_{\mathrm{can},\heartsuit} Y).$$

of graded objects in $\mathscr{C}^{\heartsuit}$. □

As mentioned, we wanted to use the canonical t-structure primarily for defining the so-called Whitehead filtration.

Definition II.1.25. We define the **Whitehead filtration of** A as the canonical connective cover of the constant filtration of A:

$$\tau_{\geq \bullet}(A) = \tau_{\geq 0}^{\mathrm{can}}(\mathrm{const}(A))$$

Explicitly, the Whitehead filtration of A is given in each filtration degree as the covers of A in the relevant t-structure:

$$(\tau_{\geq \bullet} A)(n) \simeq \tau_{\geq n} A.$$

Note that that if the t-structure on $\mathscr{C}$ is right- and left complete, then we have

$$(\tau_{\geq \bullet} A)(-\infty) \simeq A \quad \text{and} \quad (\tau_{\geq \bullet} A)(\infty) \simeq 0,$$

respectively. An important example where this holds is the ∞-category of spectra with its usual Postnikov t-structure.

Proposition II.1.26. *The Whitehead filtration construction*

$$\tau_{\geq \bullet} : \mathscr{C} \longrightarrow \mathrm{Tow}(\mathscr{C})$$

is lax symmetric monoidal.

Proof. The functor $\tau_{\geq \bullet} : \mathscr{C} \to \mathrm{Tow}(\mathscr{C})_{\geq 0}$ is the right adjoint to the composition

$$\mathrm{Tow}(\mathscr{C})_{\geq 0} \xrightarrow{I} \mathrm{Tow}(\mathscr{C}) \xrightarrow{\mathrm{colim}} \mathscr{C},$$

and hence lax symmetric monoidal since both I and colim are strong symmetric monoidal. □

II.2 The Beilinson t-structure and décalage

In the last section, we saw that the stable ∞-category of filtrations can be equipped with the so-called canonical t-structure. However, there is also another t-structure we can put on filtered spectra, called the Beilinson t-structure, which is strongly connected to spectral sequences and Deligne's notion of décalage. In Section II.2.1, we recall the Beilinson t-structure and some basic results regarding it. In Section II.2.2, we show that the Beilinson t-structure is compatible with the symmetric monoidal structure on Tow(Sp) we defined earlier, and in Section II.2.3 we identify the heart of the Beilinson t-structure with the category of chain complexes of abelian groups equipped with its standard (non-derived) symmetric monoidal structure. We describe how to use the Beilinson t-structure to define the décalée of a filtration in Section II.2.4, and express the associated graded of the décalée as an Eilenberg-Mac Lane spectrum in II.2.5.

II.2.1 The Beilinson t-structure

We start by recalling the Beilinson t-structure. This is essentially due to Beilinson, with the original source being [Beĭ87]. For a modern approach using stable ∞-categories, see [BMS19, Section 5.1] which contains many of the results listed here. The **Beilinson t-structure** is determined by defining the Beilinson n-connective filtrations to be

$$\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei}}_{\geq n} = \left\{ X \in \mathrm{Tow}(\mathrm{Sp}) \mid \mathrm{Gr}^q(X) \in \mathrm{Sp}_{\geq n-q} \text{ for all } q \right\}.$$

Dually, we define the Beilinson n-coconnective filtrations as

$$\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei}}_{\leq n} = \left\{ X \in \mathrm{Tow}(\mathrm{Sp}) \mid X(q) \in \mathrm{Sp}_{\leq n-q} \text{ for all } q \right\},$$

Worth noting is that the definition is not symmetric.

Proposition II.2.1. *This does indeed determine a t-structure on the category of filtered spectra.*

Proof. Note that the subcategory $\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei}}_{\geq 0}$ is closed under colimits, since the associated graded functor commutes with colimits and each subcategory $\mathrm{Sp}_{\geq q}$ is closed under colimits. By presentability of Tow(Sp), there is hence a functor

$$\tau^{\mathrm{Bei}}_{\geq 0} : \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei}}_{\geq 0} \to \mathrm{Tow}(\mathrm{Sp})$$

which is right adjoint to the corresponding inclusion functor. We can now define a functor $\tau^{\mathrm{Bei}}_{\leq -1} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$ characterised by the fact that we want cofibre sequences

$$\tau^{\mathrm{Bei}}_{\geq 0} X \longrightarrow X \longrightarrow \tau^{\mathrm{Bei}}_{\leq -1} X$$

for all filtrations X. To make sure that $\tau^{\mathrm{Bei}}_{\leq -1}$ actually deserves its name, we need to check that $\tau^{\mathrm{Bei}}_{\leq -1} X$ is indeed an object of $\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei}}_{\leq -1}$ for all filtrations X.

Referring back to the definition of this subcategory: we need to check that $(\tau^{\mathrm{Bei}}_{\leq -1}X)(q) \in \mathrm{Sp}_{\leq -1-q}$ for all q. This can be checked by proving that

$$\mathrm{Map}_{\mathrm{Tow(Sp)}}(L_q(Y), \tau^{\mathrm{Bei}}_{\leq -1}X) \simeq \mathrm{Map}_{\mathrm{Sp}}(Y, (\tau^{\mathrm{Bei}}_{\leq -1}X)(q)) \simeq 0$$

for all $Y \in \mathrm{Sp}_{\geq -q}$. Here, we have used the L_q-ev_q-adjunction for the left hand equivalence. We conclude that it is enough to show that $Y \in \mathrm{Sp}_{\geq -q}$ guarantees that an arbitrary map $L_q(Y) \to \tau^{\mathrm{Bei}}_{\leq -1}X$ is zero. To show this, consider the commutative diagram

$$\begin{tikzcd}
\tau^{\mathrm{Bei}}_{\geq 0}X \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[d] \\
0 \arrow[r] & L_q(Y) \arrow[r] & \tau^{\mathrm{Bei}}_{\leq -1}X
\end{tikzcd}$$

where P simply denotes the pullback of the right hand square. Since the big rectangle is a pullback, by definition of the functor $\tau^{\mathrm{Bei}}_{\leq -1} : \mathrm{Tow(Sp)} \to \mathrm{Tow(Sp)}$, so is the left hand square. Note that $\tau^{\mathrm{Bei}}_{\geq 0}X$ is an object of $\mathrm{Tow(Sp)}^{\mathrm{Bei}}_{\geq 0}$, by construction, and it is straight-forward to check that $L_q(Y)$ is an object of $\mathrm{Tow(Sp)}^{\mathrm{Bei}}_{\geq 0}$, as well. By stability under extensions, we conclude that P is an object of $\mathrm{Tow(Sp)}^{\mathrm{Bei}}_{\geq 0}$, as well. By the universal property of the map, we now know that the map $P \to X$ factors uniquely over $\tau^{\mathrm{Bei}}_{\geq 0}X \to X$. It follows that we have a diagram

$$\begin{tikzcd}
\tau^{\mathrm{Bei}}_{\geq 0}X \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & \tau^{\mathrm{Bei}}_{\geq 0}X \arrow[r] \arrow[d] & X \arrow[d] \\
0 \arrow[r] & L_q(Y) \arrow[r] & 0 \arrow[r] & \tau^{\mathrm{Bei}}_{\leq -1}X
\end{tikzcd}$$

in which every square is a pullback. In particular, note that the map $L_q(Y) \to \tau^{\mathrm{Bei}}_{\leq -1}X$ factors through zero, which is what we wanted to show. $\square$

Let us write $\tau^{\mathrm{Bei}}_{\geq i}$, $\tau^{\mathrm{Bei}}_{\leq i}$, and π^{Bei}_i for the cover, truncation, and homotopy functors in the Beilinson t-structure.

Proposition II.2.2. *The Beilinson t-structure is right complete, but not left complete.*

Proof. By definition, the subcategory

$$\mathrm{Tow(Sp)}^{\mathrm{Bei}}_{\geq \infty} = \bigcap_n \mathrm{Tow(Sp)}^{\mathrm{Bei}}_{\geq n}$$

of ∞-connective objects consists of those towers such that

$$\mathrm{Gr}^q(X) \in \mathrm{Sp}_{\geq n-q}$$

for all q and n. This only happens if

$$\mathrm{Gr}^q(X) \simeq 0$$

for all q, in other words, only for constant towers. In particular, the category of ∞-connected objects is not contractible, so by [Lur17, Proposition 1.2.1.19], the Beilinson t-structure is not left complete. We can show that the t-structure is right complete in a similar fashion, by using the dual of [Lur17, Proposition 1.2.1.19], and we leave this to the reader. $\square$

In particular, note that the above result means that the colimit of the Beilinson–Whitehead filtration $\tau^{\mathrm{Bei}}_{\geq\bullet} X$ is equivalent to X, but that its limit does not vanish, generally. We can still say something about how the Beilinson t-structure interacts with complete towers, though.

Proposition II.2.3.

1. *The Beilinson truncations* $\tau^{\mathrm{Bei}}_{\leq n} X$ *are complete filtrations for all* n.
2. *If* X *is a complete filtration, then so are the Beilinson covers* $\tau^{\mathrm{Bei}}_{\geq n} X$ *for all* n.

Proof.

1. Since $(\tau^{\mathrm{Bei}}_{\leq n} X)(q) \in \mathrm{Sp}_{\leq n-q}$, essentially by definition, it follows that

$$(\tau^{\mathrm{Bei}}_{\leq n} X)(\infty) = \lim_q (\tau^{\mathrm{Bei}}_{\leq n} X)(q) \simeq 0$$

2. Consider the cofibre sequence

$$\tau^{\mathrm{Bei}}_{\geq n} X \longrightarrow X \longrightarrow \tau^{\mathrm{Bei}}_{\leq n-1} X\,.$$

Since the two right-most terms are complete, the first by assumption and the second one by (1), it follows that the fibre $\tau^{\mathrm{Bei}}_{\geq n} X$ is complete, as well.

$\square$

The following result is often useful when studying associated graded in the Beilinson covers.

Proposition II.2.4. *There are equivalences of functors* $\mathrm{Tow}(\mathrm{Sp}) \longrightarrow \mathrm{Sp}$*:*

$$\begin{aligned}
\mathrm{Gr}^q \circ \tau^{\mathrm{Bei}}_{\geq n} &\simeq \tau_{\geq n-q} \circ \mathrm{Gr}^q \\
\mathrm{Gr}^q \circ \tau^{\mathrm{Bei}}_{\leq n} &\simeq \tau_{\leq n-q} \circ \mathrm{Gr}^q \\
\mathrm{Gr}^q \circ \pi^{\mathrm{Bei}}_{n} &\simeq \Sigma^{-q} \circ \pi_{n-q} \circ \mathrm{Gr}^q
\end{aligned}$$

Proof. Note that the functor Gr^q is exact, and that

$$\mathrm{Gr}^q\left(\mathrm{Tow}(X)^{\mathrm{Bei}}_{\geq n}\right) \subseteq \mathrm{Sp}_{\geq n-q} \quad \text{and} \quad \mathrm{Gr}^q\left(\mathrm{Tow}(X)^{\mathrm{Bei}}_{\leq n}\right) \subseteq \mathrm{Sp}_{\leq n-q}\,,$$

so that the functor Gr^q is t-exact with respect to a q-shifted version of the standard t-structure on Sp. Any exact and t-exact functor between stable ∞-categories equipped with t-structures commutes with the cover/truncation functors associated to the t-structures, from which the first two results follow.

For the result concerning the Beilinson homotopy groups, we note that

$$\pi_n^{\mathrm{Bei}} = \tau_{\geq n}^{\mathrm{Bei}} \circ \tau_{\leq n}^{\mathrm{Bei}} \circ \Sigma^{-n}\,,$$

by definition. Here $\Sigma^{-n} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$ refers to the functor that suspends every filtration degree by $-n$. By the above results on the Beilinson covers and truncations, we have

$$\begin{aligned}
\mathrm{Gr}^q \circ \pi_n^{\mathrm{Bei}} &= \mathrm{Gr}^q \circ \tau_{\geq n}^{\mathrm{Bei}} \circ \tau_{\leq n}^{\mathrm{Bei}} \circ \Sigma^{-n} \\
&\simeq \tau_{\geq n-q} \circ \tau_{\leq n-q} \circ \mathrm{Gr}^q \circ \Sigma^{-n} \\
&\simeq \tau_{\geq n-q} \circ \tau_{\leq n-q} \circ \Sigma^{-n} \circ \mathrm{Gr}^q \\
&\simeq \tau_{\geq n-q} \circ \tau_{\leq n-q} \circ \Sigma^{-q} \circ \Sigma^{q-n} \circ \mathrm{Gr}^q \\
&\simeq \Sigma^{-q} \circ \tau_{\geq n-q} \circ \tau_{\leq n-q} \circ \Sigma^{q-n} \circ \mathrm{Gr}^q \\
&\simeq \Sigma^{-q} \circ \pi_{n-q} \circ \mathrm{Gr}^q
\end{aligned}$$

□

Using the above result, we compute the Beilinson covers of some standard filtrations.

Lemma II.2.5. *Consider a trivial filtration $L_p(A)$ of some fixed spectrum A and integer p. The Beilinson covers, truncations, and homotopy groups*[3] *of this filtration are given by*

$$\begin{aligned}
\tau_{\geq n}^{\mathrm{Bei}} L_p(A) &\simeq L_p(\tau_{\geq n-p} A) \\
\tau_{\leq n}^{\mathrm{Bei}} L_p(A) &\simeq L_p(\tau_{\leq n-p} A) \\
\pi_n^{\mathrm{Bei}} L_p(A) &\simeq L_p(H\pi_{n-p}(A)[-p])
\end{aligned}$$

Proof. Since we have maps $\tau_{\geq n-p}A \to A$, we are provided with a map $L_p(\tau_{\geq n-p}K) \to X$. Note that the source of this map is Beilinson n-connective, so by universal property, we are provided with a map

$$\gamma : L_p(\tau_{\geq n-p}K) \to \tau_{\geq n}^{\mathrm{Bei}} X$$

[3] In the chain complex point of view on the Beilinson homotopy group, see Remark II.2.14, this would correspond to the chain complex

$$\pi_n^{\mathrm{Bei}} L_p(A) \simeq \pi_{n-p}(A)[-p]\,.$$

of filtrations. Since the total associated graded functor is conservative on complete filtrations, by Proposition II.1.9, it suffices to show that the map induces equivalences

$$\bar{\gamma}_q : \mathrm{Gr}^q(L_p(\tau_{\geq n-p}K)) \to \mathrm{Gr}^q(\tau^{\mathrm{Bei}}_{\geq n}X)$$

on all associated graded. This is clear, both source and target is zero unless $q = p$ when the map is the identity map. The Beilinson truncations follows from the cofibre sequences

$$\tau^{\mathrm{Bei}}_{\geq n}X \longrightarrow X \longrightarrow \tau^{\mathrm{Bei}}_{\leq n-1}X\ ,$$

and the Beilinson homotopy groups readily follows. □

Lemma II.2.6. *Consider a Whitehead filtration $\tau_{\geq \bullet}A$ of some spectrum A. The Beilinson covers of this filtration are given degree-wise by*

$$\left(\tau^{\mathrm{Bei}}_{\geq n}\tau_{\geq \bullet}A\right)(q) \simeq \begin{cases} \tau_{\geq q}A & \textit{if } 2q \geq n \\ \tau_{\geq \lceil n/2 \rceil}A & \textit{otherwise.} \end{cases}$$

Proof. Let us denote the tower specified in the statement of the lemma by X. Note that we have a map $X \to \tau_{\geq \bullet}(A)$. The associated graded of the tower X is

$$\mathrm{Gr}^q(X) \simeq \begin{cases} H\pi_q A[q] & \text{if } 2q \geq n \\ 0 & \text{otherwise.} \end{cases}$$

The inequality $2q \geq n$ is equivalent to the inequality $q \geq n - q$, so that

$$H\pi_q A[q] \in \mathrm{Sp}_{\geq q} \subseteq \mathrm{Sp}_{n-q}\ .$$

This proves that X is Beilinson n-connective, so by universal property we are provided with a map $X \to \tau^{\mathrm{Bei}}_{\geq n}\tau_{\geq \bullet}(A)$. Left to show is that this map is an equivalence. By Proposition II.1.9, is suffices to show that the map is an equivalence on all associated graded, which is readily checked by using Lemma II.2.4. □

II.2.2 Compatibility with multiplicative structures

In this section, we discuss the interaction between the Beilinson t-structure and the symmetric monoidal structure on filtered spectra given by Day convolution and introduced in Section II.1.3.

Proposition II.2.7. *The Beilinson t-structure on* Tow(Sp) *is compatible with the symmetric monoidal structure on* Tow(Sp) *given by Day convolution.*

Proof. According to Definition II.A.6, we need to check two things, namely that the unit $L_0(\mathbb{S})$ is Beilinson connective, and that if X and Y are Beilinson connective filtrations, then so is their tensor product. The associated graded of $L_0(\mathbb{S})$ is

$$\mathrm{Gr}^q(L_0(\mathbb{S})) \simeq \begin{cases} \mathbb{S} & \text{if } q = 0 \\ 0 & \text{otherwise .} \end{cases}$$

Since $\mathbb{S}$ lies in $\mathrm{Sp}_{\geq 0}$, the unit $L_0(\mathbb{S})$ is Beilinson connective. Next we check that $X \otimes Y$ is Beilinson connective, assuming that X and Y are. The total associated graded functor is strong symmetric monoidal by Proposition II.1.13, so that

$$\mathrm{Gr}^q(X \otimes Y) \simeq \bigoplus_{i+j=q} \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y)\,.$$

By assumption, $\mathrm{Gr}^i(X) \in \mathrm{Sp}_{\geq -i}$ and $\mathrm{Gr}^j(Y) \in \mathrm{Sp}_{\geq -j}$, which implies that

$$\mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \in \mathrm{Sp}_{\geq -i-j} = \mathrm{Sp}_{\geq -q}\,.$$

Since this subcategory is stable under colimits, we conclude that $\mathrm{Gr}^n(X \otimes Y) \in \mathrm{Sp}_{\geq -q}$, which is what we wanted to prove. □

Corollary II.2.8. *The Beilinson–Whitehead filtration construction*

$$\tau^{\mathrm{Bei}}_{\geq \bullet} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \mathrm{Tow}(\mathrm{Tow}(\mathrm{Sp}))$$

is lax symmetric monoidal.

Proof. Use Proposition II.2.7 combined with Proposition II.1.26. □

II.2.3 The heart of the Beilinson t-structure

Let us now study the heart of the Beilinson t-structure. The goal is to show that it is equivalent to the category of chain complexes of abelian groups, as abelian symmetric monoidal categories. In particular, this shows that the symmetric monoidal structure on the Beilinson heart is compatible with the ordinary (non-derived) tensor product of chain complexes. We start with characterising the filtrations that lie in the Beilinson heart.

Proposition II.2.9. *A filtration X lies in the heart of the Beilinson t-structure if and only if the two following properties hold:*

1. *The qth associated graded $\mathrm{Gr}^q(X)$ is concentrated in homological degree $-q$ for all q. That is, we have*

$$\pi_n(\mathrm{Gr}^q(X)) \simeq 0$$

   unless $n = -q$.

2. *The filtration X is complete. That is, we have*

$$X(\infty) \simeq 0\,.$$

Proof. Assume that X is a filtration in the heart of the Beilinson t-structure, and let us show that the two properties in the statement of the proposition are satisfied.

1. Note that $X(q) \in \mathrm{Sp}_{\leq -q}$ by virtue of $X \in \mathrm{Tow}(\mathscr{C})_{\leq 0}$. Since $\mathrm{Sp}_{\leq -q}$ is closed under extensions it follows that $\mathrm{Gr}^q(X) \in \mathrm{Sp}_{\leq -q}$, as well. We also have that $\mathrm{Gr}^q(X) \in \mathrm{Sp}_{\geq -q}$ by virtue of $X \in \mathrm{Tow}(\mathscr{C})_{\geq 0}$. Together, these two facts imply that $\mathrm{Gr}^q(X)$ is concentrated in homological degree $-q$ for all q.

2. Since $X(q) \in \mathrm{Sp}_{\leq -q}$ and the subcategories $\mathrm{Sp}_{\leq -q}$ are closed under limits, we have that $X(\infty) \simeq 0$.

Conversely, assume that X satisfies the two properties in the statement of the proposition. We want to show that X lies in the heart of the Beilinson t-structure. Property (1) directly implies that $X \in \mathrm{Tow}(\mathrm{Sp})_{\geq 0}$. To show that we also have $X \in \mathrm{Tow}(\mathrm{Sp})_{\leq 0}$, note that

$$X(q) \simeq \lim_{m \geq q} X(q)/X(m)$$

by virtue of property (2). The result now follows from (1), and from $\mathrm{Sp}_{\leq -q}$ being closed under limits. □

We now prove that we can identify the heart of the Beilinson t-structure with the category of chain complexes of abelian groups.

Theorem II.2.10. *There is an equivalence of abelian categories*

$$\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit} \simeq \mathrm{Ch}(\mathrm{Ab})\,.$$

More explicitly:

- *The direction* $\Phi : \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit} \longrightarrow \mathrm{Ch}(\mathrm{Ab})$ *is explicitly given as the functor that sends a filtration* X *in the Beilinson heart to the chain complex*

$$\cdots \longrightarrow \pi_1(\mathrm{Gr}^{-1}(X)) \longrightarrow \pi_0(\mathrm{Gr}^0(X)) \longrightarrow \pi_{-1}(\mathrm{Gr}^1(X)) \longrightarrow \cdots$$

  where $\pi_0(\mathrm{Gr}^0(X))$ *is placed in homological degree 0. Here, the boundaries in the chain complex are induced by the connecting homomorphisms in the pushout squares*

$$\begin{tikzcd} \mathrm{Gr}^{q+1}(X) \arrow[r] \arrow[d] & X(q)/X(q+2) \arrow[d] \\ 0 \arrow[r] & \mathrm{Gr}^q(X)\,. \end{tikzcd}$$

- *The direction* $\Psi : \mathrm{Ch}(\mathrm{Ab}) \longrightarrow \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit}$ *is explicitly given as the functor that sends a complex* C *to the filtration given degree-wise by*

$$(\Psi C)(n) = H(\sigma_{\leq -n} C)$$

  where the functors $\sigma_{\leq -n} : \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Ch}(\mathrm{Ab})$ *denotes the "stupid" truncations of a chain complex.*

Proof. We start by showing that the functor Ψ is well-defined. That is, we need to check that the filtration $\Psi(C)$ actually lies in the Beilinson heart, as claimed. By Proposition II.2.9, is suffices to check two conditions, the second of which is clear. The associated graded of the tower is given by

$$\mathrm{Gr}^q(\Psi C) \simeq H(\sigma_{\leq -q}C/\sigma_{\leq -q-1}C) \simeq (HC_{-q})[-q]$$

which is indeed concentrated in homological degree $-q$, which also shows the first condition. Note that the functor $\Psi : \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit}$ preserves colimits, so by the adjoint functor theorem it admits a right adjoint. We claim that that this right adjoint is (naturally isomorphic to) Φ. To see this, consider the chain complex Λ_n given as

$$0 \longrightarrow \mathbb{Z}\{x_n\} \xrightarrow{\partial} \mathbb{Z}\{x_{n-1}\} \to 0\,, \quad \partial(x_n) = x_{n-1}$$

where subindicies on the generators indicate in which homological degree they are located. Note that Λ_n detects the abelian group in homological degree n, in the sense that we have natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Ab})}(\Lambda_n, C) \cong \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, C_n) \cong C_n$$

of abelian groups. If we write $\phi : \Lambda_{n-1} \to \Lambda_n$ for the obvious chain map

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & \mathbb{Z}\{x_{n-1}\} \arrow[r] \arrow[d, equal] & \mathbb{Z}\{x_{n-2}\} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{x_n\} \arrow[r] & \mathbb{Z}\{x_{n-1}\} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

between these two chain complexes, then the contravariantly induced map

$$\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Ab})}(\phi, C) : \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Ab})}(\Lambda_n, C) \longrightarrow \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Ab})}(\Lambda_{n-1}, C)$$

is precisely the boundary $\partial_n : C_{n-1} \to C_n$ under the natural isomorphisms above. We can use the above facts to figure out what the right adjoint of Ψ should look like. Writing Φ for the right adjoint, which *a priori* has nothing to do with the Φ we specified in the statement of the theorem, consider the natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Tow}(\mathrm{Sp})}(\Psi(\Lambda_n), X) \cong \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Ab})}(\Lambda_n, \Phi(X)) \cong \Phi(X)_n$$

of abelian groups. Note that the "stupid inverted Postnikov tower" of the chain complex Λ_n is the filtration of chain complexes visualised as

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}\{x_{n-1}\} \longrightarrow \Lambda_n = \Lambda_n = \cdots$$

where the term $\mathbb{Z}\{x_{n-1}\}$ appears in filtration degree $1-n$. The chain complex Λ_n is exact, so that they vanish when we apply the Eilenberg–Mac Lane functor, and we see that $\Psi(\Lambda_n)$ is a filtration of spectra given by $H\mathbb{Z}[n-1]$ concentrated in filtration degree $1-n$. We conclude that the spectrum $\mathrm{map}_{\mathrm{Tow(Sp)}}(\Psi\Lambda_n, X_{1-n})$ is equivalent to the spectrum of maps $H\mathbb{Z}[n-1] \to X_{1-n}$ making the diagram

$$\begin{tikzcd} H\mathbb{Z}[n-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ X_{1-n} \arrow[r] & X_{-n} \end{tikzcd}$$

commute, which is in turn equivalent to the space of maps between the horizontal cofibres in this diagram. In other words:

$$\mathrm{map}_{\mathrm{Tow(Sp)}}(\Psi\Lambda_n, X) \simeq \mathrm{map}_{\mathrm{Sp}}(H\mathbb{Z}[n], \mathrm{Gr}^{-n}(X)) .$$

Since X lies in the Beilinson heart, by assumption, we know that $\mathrm{Gr}^{-n}(X)$ is concentrated in homological degree n, by Proposition II.2.9. We hence have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Tow(Sp)}}(\Psi\Lambda_n, X) &\cong \pi_0\mathrm{map}_{\mathrm{Sp}}(H\mathbb{Z}[n], \mathrm{Gr}^{-n}(X)) \\ &\cong \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \pi_n(\mathrm{Gr}^{-n}(X))) \\ &\cong \pi_n(\mathrm{Gr}^{-n}(X)) \end{aligned}$$

of abelian groups. This shows that

$$\Phi(X)_n \cong \pi_n(\mathrm{Gr}^{-n}(X)) ,$$

as wanted. Although this is technically enough to show that we have an equivalence of categories, we also show that the boundaries in the chain complex $\Phi(X)$ are given as in the statement of the theorem. For this, let $\phi : \Lambda_{n-1} \to \Lambda_n$ be as above. The "stupid truncated Postnikov tower" of the map ϕ is the map of filtrations of chain complexes visualised as

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}\{x_{n-2}\} \arrow[r] \arrow[d] & \Lambda_{n-1} \arrow[r, equal] \arrow[d] & \Lambda_{n-1} \arrow[r, equal] \arrow[d, "\phi"] & \Lambda_{n-1} \arrow[r, equal] \arrow[d, "\phi"] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{x_{n-1}\} \arrow[r] & \Lambda_n \arrow[r, equal] & \Lambda_n \arrow[r, equal] & \cdots \end{tikzcd}$$

where the map $\Lambda_{n-1} \to \mathbb{Z}\{x_{n-2}\}$ occurs in filtration degree $1-n$. From this, we conclude that we have a fibre sequence

$$\Xi_n \longrightarrow \Psi(\Lambda_{n-1}) \longrightarrow \Psi(\Lambda_n)$$

in Tow(Sp), where Ξ_n is the filtration of spectra given by

$$\cdots \longrightarrow 0 \longrightarrow H\mathbb{Z}[n-2] = H\mathbb{Z}[n-2] \longrightarrow 0 \longrightarrow \cdots .$$

Note that the spectrum $\mathrm{map}_{\mathrm{Tow(Sp)}}(\Xi_n, X)$ is equivalent to the spectrum of maps $H\mathbb{Z}[n-2] \to X_{2-n}$ making the diagram

$$\begin{tikzcd} H\mathbb{Z}[n-2] \arrow[r] \arrow[d] & 0 \arrow[d] \\ X_{2-n} \arrow[r] & X_{-n} \end{tikzcd}$$

commute. In other words, we have an equivalence

$$\mathrm{map}_{\mathrm{Tow(Sp)}}(\Xi_n, X) \simeq \mathrm{map}_{\mathrm{Sp}}(H\mathbb{Z}[n-1], X(-n)/X(2-n))\,.$$

Per what we have done so far, we have an equivalence of fibre sequences

$$\begin{tikzcd}
\mathrm{map}(\Psi(\Lambda_n), X) \arrow[r, "\partial"] \arrow[d, "\simeq"] & \mathrm{map}(\Psi(\Lambda_{n-1}), X) \arrow[r] \arrow[d, "\simeq"] & \mathrm{map}(\Xi_n, X) \arrow[d, "\simeq"] \\
\mathrm{map}(H\mathbb{Z}[n], \mathrm{Gr}^{-n}(X)) \arrow[r, "\partial"] & \mathrm{map}(H\mathbb{Z}[n-1], \mathrm{Gr}^{1-n}(X)) \arrow[r] & \mathrm{map}\left(H\mathbb{Z}[n-1], \dfrac{X(-n)}{X(2-n)}\right)
\end{tikzcd}$$

in Sp, where the upper row refers to the mapping spectra in Tow(Sp) and the lower row refers to the mapping spectra in Sp. We remind the reader that boundary map in the chain complex $\Phi(X)$ is determined by what happens to the either of the maps marked as ∂ on π_0. Note that the ∂ on the lower row is precisely induced by the connecting homomorphism in the pushout square

$$\begin{tikzcd}
\mathrm{Gr}^{1-n}(X) \arrow[r] \arrow[d] & X(-n)/X(2-n) \arrow[d] \\
0 \arrow[r] & \mathrm{Gr}^{-n}(X)\,,
\end{tikzcd}$$

which finishes the proof that Φ, as given in the statement of the theorem, is indeed the right adjoint to the functor Ψ.

What is left is showing that we have an equivalence of categories. By virtue of having an adjunction, we are provided with natural transformations

$$\eta : 1 \longrightarrow \Phi \circ \Psi \quad \text{and} \quad \epsilon : \Psi \circ \Phi \longrightarrow 1\,.$$

In order to show that we have an equivalence of categories, we need to show that

$$\eta_C : C \longrightarrow (\Phi \circ \Psi)(C) \quad \text{and} \quad \epsilon_X : (\Psi \circ \Phi)(X) \longrightarrow X$$

are isomorphisms for all chain complexes C and filtrations X in the Beilinson heart. The first statement is a straight-forward check in every homological degree. For the second statement, it is enough to check that we have an equivalence on the associated graded functor, by Proposition II.1.9. That is, we need to show that the map

$$\mathrm{Gr}^q(\epsilon_X) : (\mathrm{Gr}^q \circ \Psi \circ \Phi)(X) \longrightarrow \mathrm{Gr}^q X$$

is an equivalence. This is also straight-forward. □

Theorem II.2.11. *The equivalence*

$$\mathrm{Tow(Sp)}^{\mathrm{Bei},\heartsuit} \simeq \mathrm{Ch(Ab)}$$

is symmetric monoidal. Here, Ch(Ab) *is equipped with its standard (non-derived) tensor product and* $\mathrm{Tow(Sp)}^{\mathrm{Bei},\heartsuit}$ *is equipped with the symmetric monoidal structure given by*

$$X \otimes_{\mathrm{Bei},\heartsuit} Y = \tau^{\mathrm{Bei}}_{\leq 0}(X \otimes Y)\,.$$

Proof. It suffices to show that the functor $\Phi : \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit} \to \mathrm{Ch}(\mathrm{Ab})$ is (strong) symmetric monoidal. By the definition of the symmetric monoidal structure on the category of chain complexes, we have

$$\begin{aligned}(\Phi(X) \otimes \Phi(Y))_n &= \bigoplus_{i+j=n} \Phi(X)_i \otimes \Phi(Y)_j \\ &\cong \bigoplus_{i+j=n} \pi_{-i}(\mathrm{Gr}^i(X)) \otimes \pi_{-j}(\mathrm{Gr}^j(Y)) .\end{aligned}$$

This maps to

$$\begin{aligned}\Phi(X \otimes_{\mathrm{Bei},\heartsuit} Y)_n &= \pi_{-n}(\mathrm{Gr}^n(\tau^{\mathrm{Bei}}_{\leq 0}(X \otimes Y))) \\ &\cong \pi_{-n}(\tau_{\leq -n} \mathrm{Gr}^n(X \otimes Y))) \\ &\cong \pi_{-n}\left(\bigoplus_{i+j=n} \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y)\right) \\ &\cong \bigoplus_{i+j=n} \pi_{-n}\left(\mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y)\right)\end{aligned}$$

by lax symmetric monoidality of the functor

$$\pi_* : \mathrm{Sp} \longrightarrow \prod_{\mathbb{Z}} \mathrm{Ab}\,, \quad A \mapsto (\pi_n A)_{n\in\mathbb{Z}} \,.$$

This specifies the wanted map $\Phi(X) \otimes \Phi(Y) \longrightarrow \Phi(Z)$ in every homological degree. Note in particular that this map is an equivalence; this follows from the fact that the associated graded of a filtration in the Beilinson heart is concentrated in a single homological degree by Proposition II.2.9. It follows that the restricted functor $\Phi : \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit} \to \prod_{\mathbb{Z}} \mathrm{Ab}$ is (strong) symmetric monoidal. We also need to show that the maps $(\Phi(X) \otimes \Phi(Y))_n \to \Phi(X \otimes_{\mathrm{Bei},\heartsuit} Y)_n$ we have defined are compatible with the boundary maps of source and the target, so that we do indeed have a chain map. This is essentially the content of Theorem II.1.21 for the identity map $\mathrm{id} : X \otimes Y \to X \otimes Y$.

□

Before we end this section, let us give a description of the Beilinson homotopy groups viewed as chain complexes.

Proposition II.2.12. *The nth Beilinson homotopy group $\pi^{\mathrm{Bei}}_n X$, regarded as a chain complex of abelian groups via the equivalence of Theorem II.2.10, is the chain complex*

$$\cdots \longrightarrow \pi_{n+1}(\mathrm{Gr}^{-1}(X)) \longrightarrow \pi_n(\mathrm{Gr}^0(X)) \longrightarrow \pi_{n-1}(\mathrm{Gr}^1(X)) \longrightarrow \cdots$$

where $\pi_n(\mathrm{Gr}^0(X))$ is placed in homological degree 0. The boundaries in the chain complex are induced by the connecting homomorphisms in the pullback *squares*

$$\begin{tikzcd} \mathrm{Gr}^{q+1}(X) \arrow[r] \arrow[d] & X(q)/X(q+2) \arrow[d] \\ 0 \arrow[r] & \mathrm{Gr}^{q}(X)\,. \end{tikzcd}$$

Proof. We are looking for an explicit description for the composition

$$\mathrm{Tow}(\mathrm{Sp}) \xrightarrow{\pi_n^{\mathrm{Bei}}} \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit} \xrightarrow{\Phi} \mathrm{Ch}(\mathrm{Ab})\,.$$

From the explicit description of the functor Φ from Theorem II.2.10, the composition evaluates a tower X to the chain complex

$$\cdots \longrightarrow \pi_1(\mathrm{Gr}^{-1}(\pi_n^{\mathrm{Bei}}X)) \longrightarrow \pi_0(\mathrm{Gr}^{0}(\pi_n^{\mathrm{Bei}}X)) \longrightarrow \pi_{-1}(\mathrm{Gr}^{1}(\pi_n^{\mathrm{Bei}}X)) \longrightarrow \cdots$$

from which the result follows by applying Proposition II.2.4. □

Theorem II.2.13. *The functor*

$$\pi_*^{\mathrm{Bei}} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab})\,, \quad X \mapsto (\pi_n^{\mathrm{Bei}}(X))_{n\in\mathbb{Z}}$$

is lax symmetric monoidal with respect to the standard (non-derived) symmetric monoidal structure on the category of chain complexes.

Proof. The functor

$$\pi_*^{\mathrm{Bei}} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \prod_{\mathbb{Z}} \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit}$$

being lax symmetric monoidal follows from the Beilinson t-structure being compatible with the symmetric monoidal structure on Tow(Sp), per Proposition II.2.7. The result now follows from the functor

$$\Phi : \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit} \longrightarrow \mathrm{Ch}(\mathrm{Ab})$$

being (strong) symmetric monoidal, per Theorem II.2.11. □

Remark II.2.14. Occasionally, we will want to switch between the perspective of the Beilinson homotopy group $\pi_n^{\mathrm{Bei}}(X)$ *as a chain complex* and the Beilinson homotopy group *as a filtration*. To avoid confusion, the notation $\pi_n^{\mathrm{Bei}}(X)$ will, from now on, always refer to the chain complex perspective. If we want to view the Beilinson homotopy group as a filtration, we will do so via the equivalence $\Psi : \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit}$, and write $\Psi\pi_n^{\mathrm{Bei}}(X)$.

II.2.4 Décalage

Given a filtration, there is a way of constructing a new filtration with the same associated spectral sequence, but shifted forward one page. This is referred to as **décalage**, and was originally introduced by Deligne in [Del71] in the setting of filtered chain complexes. It is an observation of Antieau that décalage can be expressed in terms of the Beilinson t-structure, and that this provides a way to define décalage in the setting of filtrations of spectra, as well. Given a filtration X, consider the Beilinson Whitehead tower of X. That is, consider the bifiltered spectrum

$$\cdots \longrightarrow \tau^{\mathrm{Bei}}_{\geq n+1} X \longrightarrow \tau^{\mathrm{Bei}}_{\geq n} X \longrightarrow \tau^{\mathrm{Bei}}_{\geq n-1} X \longrightarrow \cdots,$$

briefly denoted $\tau^{\mathrm{Bei}}_{\geq \bullet}(X)$. Taking the colimit in the other filtration direction gives us a new filtered spectrum

$$\cdots \longrightarrow (\tau^{\mathrm{Bei}}_{\geq n+1} X)(-\infty) \longrightarrow (\tau^{\mathrm{Bei}}_{\geq n} X)(-\infty) \longrightarrow (\tau^{\mathrm{Bei}}_{\geq n-1} X)(-\infty) \longrightarrow \cdots.$$

This construction is important enough that it deserves a name.

Definition II.2.15. We refer to the filtration above as the **décalée of** X, and denote it as $\mathrm{Déc}(X)$.

The above recipe provides us with an endofunctor

$$\mathrm{Déc} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp}),$$

on filtered spectra. We note that the functor is not idempotent, so it makes sense to iterate the process. We will denote the functor expressing n applications of décalage by

$$\mathrm{Déc}^n : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp}).$$

Lemma II.2.16. *Consider the filtration $L_p(A)$ for a fixed spectrum A and integer p. The décalée of this filtration is a shift of the Whitehead tower of A:*

$$\mathrm{Déc}(L_p(A)) \simeq \mathrm{sh}^{-p}\, \tau_{\geq \bullet} A.$$

Proof. Follows directly from Lemma II.2.5. □

Lemma II.2.17. *The décalage functor on a Whitehead filtration is given by*

$$\mathrm{Déc}(\tau_{\geq \bullet} A)(n) \simeq \tau_{\geq \lceil n/2 \rceil} A.$$

Proof. Follows directly from Lemma II.2.6. □

Note that in the above example, the décalée of our original filtration of A gave us another filtration of A. The following lemma tells us that this is true, in general.

Lemma II.2.18. *If X is a complete filtration of A, then $\mathrm{Déc}(X)$ is a complete filtration of A, as well.*

Proof. By Lemma II.2.3.(2), each Beilinson cover $\tau^{\mathrm{Bei}}_{\geq n} X$ is complete whenever X is complete. Since complete towers are closed under all limits, it follows that $(\tau^{\mathrm{Bei}}_{\geq n} X)(\infty)$ is a complete filtration. The associated graded of this filtration is

$$\begin{aligned} \mathrm{Gr}^q \left(\lim_n \tau^{\mathrm{Bei}}_{\geq n} X \right) &\simeq \lim_n \mathrm{Gr}^q \left(\tau^{\mathrm{Bei}}_{\geq n} X \right) \\ &\simeq \lim_n \tau_{\geq n-q} \mathrm{Gr}^q(X) \\ &\simeq 0 \,, \end{aligned}$$

which implies that $(\tau^{\mathrm{Bei}}_{\geq n} X)(\infty)$ is a constant tower, and hence

$$(\tau^{\mathrm{Bei}}_{\geq n} X)(\infty) \simeq 0$$

by completeness. It follows that the filtration $(\tau^{\mathrm{Bei}}_{\geq \bullet} X)(q)$ is complete for any fixed integer q. Since complete filtrations are closed under colimits by Corollary II.1.7, it follows that

$$\lim_n \mathrm{Déc}(X)(n) \simeq \lim_n \operatorname*{colim}_q (\tau^{\mathrm{Bei}}_{\geq n} X)(q) \simeq 0 \,,$$

which proves that $\mathrm{Déc}(X)$ is a complete filtration, as well. Left to show is that the colimit of $\mathrm{Déc}(X)$ is equivalent to A, if the colimit of X is equivalent to A. For a fixed q, consider the cofibre sequence

$$(\tau^{\mathrm{Bei}}_{\geq n} X)(q) \longrightarrow X(q) \longrightarrow (\tau^{\mathrm{Bei}}_{\leq n-1} X)(q) \,.$$

Since $(\tau^{\mathrm{Bei}}_{\leq n-1} X)(q) \in \mathrm{Sp}_{\leq n-1-q}$ it follows that the map $(\tau^{\mathrm{Bei}}_{\geq n} X)(q) \longrightarrow X(q)$ is an equivalence after applying $\tau_{\geq n-q} : \mathrm{Sp} \longrightarrow \mathrm{Sp}$. We conclude that the map

$$\operatorname*{colim}_n (\tau^{\mathrm{Bei}}_{\geq n} X)(q) \longrightarrow X(q)$$

is an equivalence for all q. It follows that

$$\begin{aligned} \operatorname*{colim}_n \mathrm{Déc}(X)(n) &= \operatorname*{colim}_n \operatorname*{colim}_q (\tau^{\mathrm{Bei}}_{\geq n} X)(q) \\ &\simeq \operatorname*{colim}_q \operatorname*{colim}_n (\tau^{\mathrm{Bei}}_{\geq n} X)(q) \\ &\simeq \operatorname*{colim}_q X(q) \\ &\simeq A \,. \end{aligned}$$

□

Proposition II.2.19. *The functor* $\mathrm{Déc} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$ *is lax symmetric monoidal.*

Proof. The Beilinson–Whitehead tower construction is lax symmetric monoidal by Corollary II.2.8. Taking level-wise colimits is symmetric monoidal by commuting colimits. □

II.2.5 Associated graded of the décalée

To be able to understand how décalage interacts with spectral sequences we need to first understand the associated graded of the décalée of a filtration.

Theorem II.2.20. *The diagram*

$$\begin{tikzcd}
\mathrm{Tow}(\mathrm{Sp}) \arrow[rr, "\mathrm{Déc}"] \arrow[d, "\pi_*^{\mathrm{Bei}}"'] & & \mathrm{Tow}(\mathrm{Sp}) \arrow[d, "\mathrm{Gr}"] \\
\prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \arrow[r, "\Sigma^{\mathrm{Tot}}"] & \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \arrow[r, "H"] & \prod_{\mathbb{Z}} \mathrm{Sp}
\end{tikzcd}$$

commutes[4]*. Here, the functor*

$$H : \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \longrightarrow \prod_{\mathbb{Z}} \mathrm{Sp}$$

is the Eilenberg-Mac Lane functor applied levelwise, and

$$\Sigma^{\mathrm{Tot}} : \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \longrightarrow \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}), \quad (C_n)_{n\in\mathbb{Z}} \mapsto (C_n[n])_{n\in\mathbb{Z}}$$

is the functor that suspends the chain complex in degree n by n. In particular, the qth associated graded of $\mathrm{Déc}(X)$ is the Eilenberg-Mac Lane spectrum of the chain complex

$$\cdots \longrightarrow \pi_1(\mathrm{Gr}^{q-1}(X)) \longrightarrow \pi_0(\mathrm{Gr}^q(X)) \longrightarrow \pi_{-1}(\mathrm{Gr}^{q+1}(X)) \longrightarrow \cdots .$$

where $\pi_0(\mathrm{Gr}^q(X))$ lies in homological degree 0.

Proof. Let us start by understanding the top composition in the diagram degree-wise. By commuting colimits we have:

$$\begin{aligned}
\mathrm{Gr}^q(\mathrm{Déc}(X)) &= \operatorname*{colim}_n(\tau^{\mathrm{Bei}}_{\geq q}X)(n)/\operatorname*{colim}_n(\tau^{\mathrm{Bei}}_{\geq q+1})(n) \\
&\simeq \operatorname*{colim}_n(\tau^{\mathrm{Bei}}_{\geq q}X/\tau^{\mathrm{Bei}}_{\geq q+1}X)(n) \\
&\simeq \left(\operatorname*{colim}_n \Psi(\pi^{\mathrm{Bei}}_q(X))(n)\right)[q] .
\end{aligned}$$

See Remark II.2.14 for a reminder of what we mean by the notation $\Psi(\pi^{\mathrm{Bei}}_q(X))$. We now have

$$\begin{aligned}
\operatorname*{colim}_n \Psi(\pi^{\mathrm{Bei}}_q(X))(n) &= \operatorname*{colim}_n H\left(\sigma_{\leq -n}\pi^{\mathrm{Bei}}_q(X)\right) \\
&\simeq H\left(\operatorname*{colim}_n \sigma_{\leq -n}\pi^{\mathrm{Bei}}_q(X)\right) \\
&\simeq H\pi^{\mathrm{Bei}}_q(X) .
\end{aligned}$$

[4]Degree-wise, this means that there is an equivalence

$$\mathrm{Gr}^q\,\mathrm{Déc}(X) \simeq H\pi^{\mathrm{Bei}}_q(X)[q] .$$

However, the statement regarding the monoidal properties of this equivalence is phrased more succinctly by using the "total graded functors", which is why we have phrased the theorem in these terms.

Note that in the above, we are implicitly switching between homotopy colimits and colimits, which is allowed since the category Ch(Ab) is AB5, so that sequential colimits are already exact. □

Corollary II.2.21. *We have*

$$\pi_n(\mathrm{Gr}^q(\mathrm{Déc}(X))) \cong \frac{\ker\left(\partial : \pi_n(\mathrm{Gr}^{q-n}(X)) \longrightarrow \pi_{n-1}(\mathrm{Gr}^{q-n+1}(X))\right)}{\mathrm{im}\left(\partial : \pi_{n+1}(\mathrm{Gr}^{q-n-1}(X)) \longrightarrow \pi_n(\mathrm{Gr}^{q-n}(X))\right)}.$$

Theorem II.2.22. *The natural equivalence*

$$\mathrm{Gr} \circ \mathrm{Déc} \simeq H \circ \Sigma^{\mathrm{Tot}} \circ \pi_*^{\mathrm{Bei}}$$

is symmetric monoidal.

Proof. The proof of Theorem II.2.20 can be visualised as the commutative diagram

$$\begin{tikzcd}
\mathrm{Tow}(\mathrm{Sp}) \arrow[r, "\tau^{\mathrm{Bei}}_{\geq \bullet}"] \arrow[dd, "\pi_*^{\mathrm{Bei}}"'] & \mathrm{Tow}(\mathrm{Tow}(\mathrm{Sp})) \arrow[r, "\mathrm{colim}"] \arrow[d, "\mathrm{Gr}"] & \mathrm{Tow}(\mathrm{Sp}) \arrow[dd, "\mathrm{Gr}"] \\
& \prod_{\mathbb{Z}} \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit} \arrow[dr, "\mathrm{colim}"] & \\
\prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \arrow[r, "\Sigma^{\mathrm{Tot}}"] & \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \arrow[r, "H"] \arrow[u, "\Psi"] & \prod_{\mathbb{Z}} \mathrm{Sp}\,.
\end{tikzcd}$$

of functors. In order to prove the statement of the theorem, it suffices to show that each one of the parts of the diagram represents a symmetric monoidal equivalence.

Tetragon in the right upper corner of the diagram This indicates the commuting colimits which occurs at the very start of the proof of Theorem II.2.20. Commuting colimits is symmetric monoidal, so here there is nothing to show.

Pentagon in the left part of the diagram This represents the computation of the associated graded in the Whitehead–Beilinson filtration which also occurs early in the proof. That this is a symmetric monoidal equivalence follows from the Beilinson heart being equivalent to the category of chain complexes as symmetric monoidal categories. This is Theorem II.2.11.

Triangle at the lower right of the diagram This represents the last step of the proof. Let us write this out a little bit, and consider the diagram

$$\begin{tikzcd}
\mathrm{Ch}(\mathrm{Ab}) \arrow[r, "\sigma_{-\bullet}"] \arrow[dr, equal] & \mathrm{Tow}(\mathrm{Ch}(\mathrm{Ab})) \arrow[r, "H"] \arrow[d, "\mathrm{colim}"] & \mathrm{Tow}(\mathrm{Sp}) \arrow[d, "\mathrm{colim}"] \\
& \mathrm{Ch}(\mathrm{Ab}) \arrow[r, "H"] & \mathrm{Sp}
\end{tikzcd}$$

where the upper composition is the map Ψ, essentially by definition. Here, commutativity of the left hand square comes down to the fact that sequential colimits in $\mathrm{Ch}(\mathrm{Ab})$ are already exact. This is symmetric monoidal essentially by definition.

$\square$

II.3 Spectral sequences and décalage

The most common ways to encounter spectral sequences are via filtrations of objects in some category. In particular, the aim of this section is to describe how a filtration of spectra gives rise to spectral sequence of abelian groups. Roughly, passing from a filtration to its décalée is the filtration level version of "turning the page" in our spectral sequence. We use this point of view, as well as previous considerations of the multiplicative properties of the décalage functor, to show that a pairing of filtrations gives rise to a pairing of spectral sequences.

II.3.1 Spectral sequences and décalage

We start by recalling the definition of a spectral sequence. For ease of presentation, we consider the case of spectral sequences in abelian groups. We will use homological Adams grading throughout, but all of the results can of course be formulated in terms of any other grading convention the reader prefers.

Definition II.3.1. A **spectral sequence** (of abelian groups) consists of the following data:

1. for every $r \geq 1$, a bigraded abelian group $E^r = E^r_{n,s}$;

2. for every $r \geq 1$, a map $d^r : E^r \to E^r$ of bidegree $(-1, r)$ such that $d^r \circ d^r = 0$;

3. for every $r \geq 1$, an isomorphism $E^{r+1} \cong H(E^r, d^r)$ of bigraded abelian groups where H refers to taking homology.

A morphism of spectral sequences is a collection of morphisms of bigraded abelian groups compatible with the differentials, and the isomorphisms $E^{r+1} \cong H(E^r, d^r)$, in the obvious way. This makes spectral sequences into an ordinary 1-category. We denote this category by SSEQ.

Theorem II.3.2. *Given a filtration X, the assignment*

$$E^r_{n,s}(X) = \pi_n(\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)))$$

determines a spectral sequence. Here, the differential $d^r_{n,s} : E^r_{n,s} \to E^r_{n-1,s+r}$ *is induced by the connecting homomorphism in the pushout square*

$$\begin{tikzcd}
\mathrm{Gr}^{(r-1)n+s+1}(\mathrm{Déc}^{r-1}(X)) \arrow[r] \arrow[d] & \dfrac{\mathrm{Déc}^{r-1}(X)((r-1)n+s)}{\mathrm{Déc}^{r-1}(X)((r-1)n+s+2)} \arrow[d] \\
0 \arrow[r] & \mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)) \,.
\end{tikzcd}$$

Proof. We need to show that the recipe provided in the statement of the theorem actually determines a spectral sequence. That the differentials satisfy $d^r \circ d^r = 0$ follows from pasting pushout squares, similar to the argument that the functor $\Phi : \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit} \to \mathrm{Ch}(\mathrm{Ab})$ is well-defined provided in the proof of Theorem II.2.10. The preferred isomorphisms

$$E^{r+1}_{n,s}(X) \cong H(E^r_{n,s}(X), d^r)$$

follow from Corollary II.2.21 by noting that

$$\begin{aligned}
E^{r+1}_{n,s}(X) &= \pi_n(\mathrm{Gr}^{rn+s}(\mathrm{Déc}^r(X))) \\
&\cong \frac{\ker\left(\pi_n(\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X))) \to \pi_{n-1}(\mathrm{Gr}^{(r-1)(n-1)+s+1}(\mathrm{Déc}^{r-1}(X)))\right)}{\mathrm{im}\left(\pi_{n+1}(\mathrm{Gr}^{(r-1)(n+1)+s-1}(\mathrm{Déc}^{r-1}(X))) \to \pi_n(\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)))\right)} \\
&= \frac{\ker\left(d^r : E^r_{n,s}(X) \to E^r_{n-1,s+r}(X)\right)}{\mathrm{im}\left(d^r : E^r_{n+1,s-r}(X) \to E^r_{n,s}(X)\right)} \\
&= H(E^r_{n,s}(X), d^r)\,.
\end{aligned}$$

□

Remark II.3.3. It is worth noting that the Beilinson homotopy group $\pi_n^{\mathrm{Bei}}(\mathrm{Déc}^r(X))$ appears as the chain complex

$$\cdots \xrightarrow{d^{r+1}} E^{r+1}_{n+1,-nr-r-1}(X) \xrightarrow{d^{r+1}} E^{r+1}_{n,-nr}(X) \xrightarrow{d^{r+1}} E^{r+1}_{n-1,-nr+r+1}(X) \xrightarrow{d^{r+1}} \cdots$$

where $E_{n,-nr}(X)$ is placed in homological degree 0, essentially by definition. Conversely, the term $E^r_{n,s}(X)$, together with the d^r-differentials entering and exiting this group, appears as the term

$$E^r_{n,s}(X) = \left(\pi^{\mathrm{Bei}}_{nr+s}(\mathrm{Déc}^{r-1}(X))\right)_{n-nr-s}\,,$$

together with its the entering and exiting boundaries in the relevant Beilinson homotopy group.

II.3.2 Multiplicativity of spectral sequences

In this section, we specify what we mean when we say that a spectral sequence is multiplicative. We show that a pairing of filtrations gives rise to a pairing of spectral sequences. Note that category of spectral sequences is not a symmetric monoidal category; there is no tensor product of spectral sequences *per se*. However, one can still talk about multilinear maps of spectral sequences. This endows the category of spectral sequences with the structure of a coloured operad, or a symmetric multicategory, if you so prefer. The following definition specifies the bilinear maps; what is usually referred to as pairings of spectral sequences. Multilinear maps on more than two source objects are defined completely analogously.

Definition II.3.4. Given spectral sequences (C^r, d^r), (D^r, d^r), and (E^r, d^r), a **bilinear map** (or **pairing**)

$$\phi : (C^\star_{*,*}, D^\star_{*,*}) \longrightarrow E^\star_{*,*}$$

is a collection of morphisms

$$\phi^r : C^r_{n,s} \otimes D^r_{n',s'} \to E^r_{n+n',s+s'}$$

such that the following conditions hold:

1. We have
$$d^r \phi^r = \phi^r(d^r \otimes 1 + 1 \otimes d^r)$$
as morphisms $C^r_{n,s} \otimes D^r_{n',s'} \to E^r_{n+n'-1,s+s'+r}$.
2. The diagram
$$\begin{tikzcd}
C^{r+1}_{n,s} \otimes D^{r+1}_{n',s'} \arrow[r, "\phi^{r+1}"] \arrow[d] & E^{r+1}_{n+n',s+s'} \arrow[d, "\cong"] \\
H(C^r_{n,s} \otimes D^r_{n',s'}) \arrow[r, "H(\phi^r)"] & H(E^r_{n+n',s+s'}, d^r)
\end{tikzcd}$$
commutes.

Our goal of this section is to show that the functor

$$E^\star_{*,*} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \mathrm{SSEQ}$$

that we specified in the previous section can be endowed with the structure of a map of ∞-operads. This statement does indeed make sense; Tow(Sp) is an example of an ∞-operad by virtue of being a lax symmetric monoidal ∞-category, and SSEQ is an example of an ∞-operad by virtue of being a multicategory.

Theorem II.3.5. *The functor* $E^\star_{*,*} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{SSEQ}$ *admits the structure of a map of* ∞*-operads.*

Proof. Since SSEQ is a 1-operad, it suffices to check that multilinear maps of filtrations give rise to multilinear maps between the associated spectral sequences. That is, there are no higher coherences to check. Moreover, since Tow(Sp) is symmetric monoidal, every multilinear map

$$(X_1, \dots, X_n) = X_1 \otimes \cdots \otimes X_n \longrightarrow Y$$

in Tow(Sp) can be written as a composite of bilinear ones. This reduces the question to checking that a pairing $\phi : X \otimes Y \to Z$ of filtrations gives rise to a pairing $\phi : E^\star(X) \otimes E^\star(X) \to E^\star(Z)$ of the associated spectral sequences. Let us start by specifying the morphisms

$$\phi^r : E^r_{n,s}(X) \otimes E^r_{n',s'}(Y) \longrightarrow E^r_{n+n',s+s'}(Z)$$

Recall from Section II.1.3, that a pairing $X \otimes Y \to Z$ of filtrations induces pairings $\mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \longrightarrow \mathrm{Gr}^{i+j}(Z)$. On homotopy, this induces maps

$$\phi^1 : \pi_n(\mathrm{Gr}^i(X)) \otimes \pi_{n'}(\mathrm{Gr}^j(Y)) \longrightarrow \pi_{n+n'}(\mathrm{Gr}^{i+j}(Z))$$

on the associated spectral sequences, by virtue of the functor

$$\pi_* : \mathrm{Sp} \longrightarrow \prod_{\mathbb{Z}} \mathrm{Ab}$$

being lax symmetric monoidal. By replacing the pairing ϕ with the pairing

$$\mathrm{Déc}^{r-1}(\phi) : \mathrm{Déc}^{r-1}(X) \otimes \mathrm{Déc}^{r-1}(Y) \longrightarrow \mathrm{Déc}^{r-1}(Z)$$

induced on the $(r-1)$th décalées we obtain the wanted maps

$$\begin{aligned}\phi^r : \pi_n(\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X))) \otimes \pi_{n'}(\mathrm{Gr}^{(r-1)n'+s'}(\mathrm{Déc}^{r-1}(Y))) \\ \longrightarrow \pi_{n+n'}(\mathrm{Gr}^{(r-1)(n+n')+s+s'}(\mathrm{Déc}^{r-1}(Z)))\,.\end{aligned}$$

Here, we have used that the décalage functor is lax symmetric monoidal, by Proposition II.2.19. The fact that all the functors involved are lax symmetric monoidal guarantees that the restricted functor

$$E^r_{*,*} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \prod_{\mathbb{Z}\times\mathbb{Z}} \mathrm{Ab}$$

from filtered spectra to bigraded abelian groups is lax symmetric monoidal, as well. What remains is showing that the maps ϕ^r we have just defined are compatible with the additional data of a spectral sequence. More precisely, we need to check that the two conditions of Definition II.3.4 are satisfied.

Condition 1: Recall that the differential $d^r_{n,s} : E^r_{n,s}(X) \to E^r_{n-1,s+r}(X)$ in the spectral sequence associated to X is induced by the connecting homomorphism in the pushout

$$\begin{tikzcd}
\mathrm{Gr}^{(r-1)n+s+1}(\mathrm{Déc}^{r-1}(X)) \arrow[r] \arrow[d] & \dfrac{\mathrm{Déc}^{r-1}(X)((r-1)n+s)}{\mathrm{Déc}^{r-1}(X)((r-1)n+s+2)} \arrow[d] \\
0 \arrow[r] & \mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)) \,,
\end{tikzcd}$$

and analogously for the spectral sequence associated to Y and Z. Recalling and using Notation II.1.14, this is the connecting homomorphism we would denote

$$\delta^{\mathrm{Déc}^{r-1}(X)}_{(r-1)n+s} : \mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)) \longrightarrow \mathrm{Gr}^{(r-1)n+s+1}(\mathrm{Déc}^{r-1}(X))[1] \,.$$

Since the décalage functor is lax symmetric monoidal, the pairing $X \otimes Y \to Z$ induces pairings

$$\mathrm{Déc}^q(X) \otimes \mathrm{Déc}^q(Y) \longrightarrow \mathrm{Déc}^q(Z)$$

for all $q \geq 0$. By Theorem II.1.21, the diagrams

$$\begin{tikzcd}
\mathrm{Gr}^i(\mathrm{Déc}^q(X)) \otimes \mathrm{Gr}^j(\mathrm{Déc}^q(Y)) \arrow[r, "\delta\otimes 1 + 1 \otimes \delta"] \arrow[d, "\mathrm{Gr}^{i,j}(\mathrm{Déc}^q(\phi))"] & C^q_{i,j}[1] \arrow[d, "\mathrm{Gr}^{i+1,j}(\mathrm{Déc}^q(\phi)) \oplus \mathrm{Gr}^{i,j+1}(\mathrm{Déc}^q(\phi))"] \\
\mathrm{Gr}^{i+j}(\mathrm{Déc}^q(Z)) \arrow[r, "\delta"] & \mathrm{Gr}^{i+j+1}(\mathrm{Déc}^q(Z))[1] \,.
\end{tikzcd}$$

commute for all $q \geq 0$. Here, we have written

$$C^q_{i,j} = \left(\mathrm{Gr}^{i+1}(\mathrm{Déc}^q(X)) \otimes \mathrm{Gr}^j(\mathrm{Déc}^q(Y)) \oplus \mathrm{Gr}^i(\mathrm{Déc}^q(X)) \otimes \mathrm{Gr}^{j+1}(\mathrm{Déc}^q(Y)) \right) ,$$

as well as

$$i = (r-1)n + s \quad \text{and} \quad j = (r-1)n' + s' \,,$$

and refrained from using sub- and super-indices on the δ's to avoid excessively long expressions and cluttering. Condition 1 follows by passing to homotopy groups, since the homotopy groups functor is lax symmetric monoidal.

Condition 2: Recall that the isomorphism

$$\begin{aligned}
E^{r+1}_{n,s}(X) &= \pi_n(\mathrm{Gr}^{rn+s}(\mathrm{Déc}^r(X))) \\
&\cong H_{n-rn-s}(\pi^{\mathrm{Bei}}_{rn+s}(\mathrm{Déc}^{r-1}(X))) \\
&= H(E^r_{n,s}(X), d^r) \,,
\end{aligned}$$

is induced by the equivalence

$$\mathrm{Gr} \circ \mathrm{Déc} \simeq H \circ \Sigma^{\mathrm{Tot}} \circ \pi^{\mathrm{Bei}}_* \,.$$

This equivalence is symmetric monoidal by Theorem II.2.22. Monoidality of the equivalence implies that the induced map on homotopy is monoidal in the sense that the diagram

$$\begin{array}{ccc}
\pi_{i+j}(\mathrm{Gr}^p(\mathrm{Déc}(X)) \otimes \mathrm{Gr}^q(\mathrm{Déc}(Y))) & \longrightarrow & \pi_{i+j}(\mathrm{Gr}^{p+q}(\mathrm{Déc}(Z))) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\pi_{i+j}(H\pi_p^{\mathrm{Bei}}(X)[p] \otimes H\pi_q^{\mathrm{Bei}}(Y)[q]) & \longrightarrow & \pi_{i+j}(H\pi_{p+q}^{\mathrm{Bei}}(Z)[p+q])
\end{array}$$

commutes for all choices of i, j, p, and q, given a pairing of filtrations $X \otimes Y \to Z$. Lax monoidality of the functor

$$\pi_* : \mathrm{Sp} \longrightarrow \prod_{\mathbb{Z}} \mathrm{Sp}\,, \quad A \mapsto (\pi_n(A))_{n\in\mathbb{Z}}$$

implies that we have a commutative diagram

$$\begin{array}{ccc}
\pi_i(\mathrm{Gr}^p(\mathrm{Déc}(X)) \otimes \pi_j(\mathrm{Gr}^q(\mathrm{Déc}(Y))) & \longrightarrow & \pi_{i+j}(\mathrm{Gr}^{p+q}(\mathrm{Déc}(Z))) \\
\downarrow & & \| \\
\pi_{i+j}(\mathrm{Gr}^p(\mathrm{Déc}(X) \otimes \mathrm{Gr}^q(\mathrm{Déc}(Y))) & \longrightarrow & \pi_{i+j}(\mathrm{Gr}^{p+q}(\mathrm{Déc}(Z)))\,,
\end{array}$$

while lax monoidality of the Eilenberg-Mac Lane functor $H : \mathrm{Ch(Ab)} \longrightarrow \mathrm{Sp}$ implies that we have a commutative diagram

$$\begin{array}{ccc}
\pi_{i+j}(H\pi_p^{\mathrm{Bei}}(X)[p] \otimes H\pi_q^{\mathrm{Bei}}(Y)[q]) & \longrightarrow & \pi_{i+j}(H\pi_{p+q}^{\mathrm{Bei}}(Z)[p+q]) \\
\downarrow & & \| \\
\pi_{i+j}(H(\pi_p^{\mathrm{Bei}}(X) \otimes \pi_q^{\mathrm{Bei}}(Y))[p+q]) & \longrightarrow & \pi_{i+j}(H\pi_{p+q}^{\mathrm{Bei}}(Z)[p+q])\,.
\end{array}$$

Pasting these three commutative diagrams together we obtain the commutative diagram

$$\begin{array}{ccc}
\pi_i(\mathrm{Gr}^p(\mathrm{Déc}(X)) \otimes \pi_j(\mathrm{Gr}^q(\mathrm{Déc}(Y))) & \longrightarrow & \pi_{i+j}(\mathrm{Gr}^{p+q}(\mathrm{Déc}(Z))) \\
\downarrow & & \downarrow{\cong} \\
H_{i+j-p-q}(\pi_p^{\mathrm{Bei}}(X) \otimes \pi_q^{\mathrm{Bei}}(Y)) & \longrightarrow & H_{i+j-p-q}(\pi_{p+q}^{\mathrm{Bei}}(Z))\,.
\end{array}$$

Setting

$$i = n\,, \quad j = n'\,, \quad p = n + s\,, \quad q = n' + s'$$

gives us the commutative diagram we want in the case $r = 1$. The reader may want to use Remark II.3.3 to keep track of the indices. For the higher pages we replace $X \otimes Y \to Z$ with the pairing induced on the appropriate number of iterated décalage. $\square$

Appendix II.A t-structures

The notion of a t-structure first appears in [BBD82] in the setting of triangulated categories, and are a way to make sense of truncations, covers, and homotopy groups in such a category. As we use t-structure extensively in this paper, we here give a short appendix on this topic, in the context of stable ∞-categories. More details can be found in [Lur17, Section 1.2.1]. Note that we will use homological grading, as in [Lur17], rather than cohomological grading, as in [BBD82].

II.A.1 t-structures: Basics

Before we go into the definitions, let us first introduce some conventions. If X is an object of the stable ∞-category $\mathscr{C}$, we will write $X[n]$ for its n-fold suspension. Given a full subcategory $\mathscr{D} \subseteq \mathscr{C}$, we will write $\mathscr{D}[n]$ for the full subcategory of $\mathscr{C}$ spanned by objects on the form $X[n]$ where $X \in \mathscr{D}$.

Definition II.A.1. A **t-structure** on a stable ∞-category $\mathscr{C}$ consists of a pair $(\mathscr{C}_{\geq 0}, \mathscr{C}_{\leq 0})$ of full subcategories of $\mathscr{C}$ such that the following hold:

1. We have inclusions $\mathscr{C}_{\geq 0}[1] \subseteq \mathscr{C}_{\geq 0}$ and $\mathscr{C}_{\leq 0} \subseteq \mathscr{C}_{\leq 0}[1]$.
2. If $X \in \mathscr{C}_{\geq 0}$ and $Y \in \mathscr{C}_{\leq 0}$, then
$$\mathrm{Map}_{\mathscr{C}}(X, Y[-1]) \simeq 0\,.$$
3. Every $X \in \mathscr{C}$ fits into a cofibre sequence
$$\tau_{\geq 0}X \longrightarrow X \longrightarrow \tau_{\leq -1}X$$
where $\tau_{\geq 0}X \in \mathscr{C}_{\geq 0}$ and $\tau_{\leq -1}X \in \mathscr{C}_{\leq}[-1]$.

Remark II.A.2. Note that either one of the subcategories $\mathscr{C}_{\geq 0}$ and $\mathscr{C}_{\leq 0}$ determines the other; if we are given $\mathscr{C}_{\geq 0}$ we can simply define $\mathscr{C}_{\leq 0}$ as the subcategory spanned by those objects Y such that
$$\mathrm{Map}_{\mathscr{C}}(X, Y[-1]) \simeq 0$$
for all $X \in \mathscr{C}_{\geq 0}$.

It is common to write
$$\mathscr{C}_{\geq n} = \mathscr{C}_{\geq n}[n] \quad \text{and} \quad \mathscr{C}_{\leq n} = \mathscr{C}_{\leq n}[n]$$
and we will do so from now on. These are colocalisation and localisations of $\mathscr{C}$, respectively. We will write
$$\tau_{\geq n} : \mathscr{C} \longrightarrow \mathscr{C}_{\geq n} \quad \text{and} \quad \tau_{\leq n} : \mathscr{C} \longrightarrow \mathscr{C}_{\leq n}$$
for the right and left adjoints of the inclusions $\mathscr{C}_{\geq 0} \subseteq \mathscr{C}$ and $\mathscr{C}_{\leq 0} \subseteq \mathscr{C}$, respectively. These are referred to as cover and truncation functors, respectively.

Definition II.A.3. The intersection

$$\mathscr{C}^{\heartsuit} = \mathscr{C}_{\geq 0} \cap \mathscr{C}_{\leq 0}$$

is called the **heart of the t-structure**.

The homotopy category of the heart of a t-structure is an abelian category. Moreover, $\mathscr{C}^{\heartsuit}$ is equivalent to (the nerve of) its homotopy category. By abuse of notation and terminology, we will often identify the heart of a t-structure with this abelian category. We write

$$\pi_n = \tau_{\geq n}\tau_{\leq n}[-n] : \mathscr{C} \longrightarrow \mathscr{C}^{\heartsuit},$$

and refer this to the nth homotopy object functor on $\mathscr{C}$.

Definition II.A.4. An exact functor $F : \mathscr{C} \to \mathscr{D}$ of stable ∞-categories equipped with t-structures is called **left exact** (resp. **right exact**) if it maps $\mathscr{C}_{\leq 0}$ to $\mathscr{D}_{\leq 0}$ (resp. $\mathscr{C}_{\geq 0}$ to $\mathscr{D}_{\geq 0}$). If F is both left and right t-exact, we simply call it **t-exact**.

Note that a t-exact functor commutes with the cover/truncation functors in the t-structure on the source and target. We use this in the proof of Lemma II.2.4.

Definition II.A.5.

- A t-structure is called **left complete** if the natural map

  $$\mathscr{C} \longrightarrow \lim_n \left(\cdots \longrightarrow \mathscr{C}_{\leq n+1} \xrightarrow{\tau_{\leq n}} \mathscr{C}_{\leq n} \xrightarrow{\tau_{\leq n-1}} \mathscr{C}_{\leq n-1} \longrightarrow \cdots \right)$$

  is an equivalence.

- A t-structure is called **right complete** if the natural map

  $$\mathscr{C} \longrightarrow \lim_n \left(\cdots \longrightarrow \mathscr{C}_{\geq n-1} \xrightarrow{\tau_{\geq n}} \mathscr{C}_{\geq n} \xrightarrow{\tau_{\geq n+1}} \mathscr{C}_{\geq n+1} \longrightarrow \cdots \right)$$

  is an equivalence.

- A t-structure is called **complete** if it is both left and right complete.

Note that left completeness implies that the limit of a Whitehead filtration is contractible, while right completeness implies that the colimit of the Whitehead filtration of an object A is equivalent to A.

II.A.2 Compatibility of t-structures and symmetric monoidal structures

If $\mathscr{C}$ has the structure of a symmetric monoidal category we want to be able to talk about multiplicative spectral sequences in $\mathscr{C}^{\heartsuit}$. However, the construction of a spectral sequence from a filtration in some stable ∞-category involves the t-structure, and there is no *a priori* reason why this structure should interact well with the symmetric monoidal structure. We require some sort of compatibility between these structures so that the construction works out. The following definition and results can be found in [AN20, Appendix A.2].

Definition II.A.6. Let $\mathscr{C}$ be a stable ∞-category with a t-structure and a symmetric monoidal structure. We say that **the t-structure is compatible with the symmetric monoidal structure** if the following conditions hold:

1. The tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is exact in each variable;

2. The tensor unit is connective;

3. The tensor product of two connective objects is connective.

The ordinary Postnikov t-structure on the category of spectra is compatible with the smash product symmetric monoidal structure. The following two lemmas will be very useful to us.

Lemma II.A.7. *There is a unique symmetric monoidal structure on $\mathscr{C}^\heartsuit$ and a unique symmetric monoidal structure on the functor $\pi_0 : \mathscr{C}_{\geq 0} \to \mathscr{C}^\heartsuit$.*

Lemma II.A.8. *Let $\mathscr{C}$ be a stable ∞-category equipped with a symmetric monoidal structure and a compatible t-structure. Then, $X \in \mathscr{C}_{\geq m}$ and $Y \in \mathscr{C}_{\geq n}$ implies that $X \otimes Y \in \mathscr{C}_{\geq m+n}$.*

Appendix II.B The Tate spectral sequence

The author's personal reason for studying multiplicative structures on spectral sequences comes from her studying the Tate spectral sequence, which is a spectral sequence on which it should be possible to endow a multiplicative structure, but where technicalities makes it hard to do so classically. In this appendix, the aim is to give a straight-forward proof of the Tate spectral sequence being multiplicative using the results of the main paper.

II.B.1 Parametrised spectra

Before we introduce the Tate construction, we need to introduce the context in which it appears, namely parametrised spectra. In what follows, B will be a fixed Kan complex and Sp will denote the ∞-category of spectra.

Definition II.B.1. The ∞-category of **spectra parametrised over** B is the functor category

$$\mathrm{Sp}^B = \mathrm{Fun}(B, \mathrm{Sp})\,.$$

Remark II.B.2. If B is the classifying space BG of some topological group G, we often refer to the objects in the above category as spectra with G-action. Note that this is different to what we usually refer to G-spectra in genuine equivariant stable homotopy theory. However, the constructions that we are concerned with only depend on the naïve homotopy type of the G-spectra we want to consider, so this is a point that we will sweep under the rug.

Since colimit and limits in functor categories are constructed object-wise, the ∞-category of parametrised spectra is stable. There is also an alternative perspective on spectra parametrised over some space, which describes it as the module category via the Schwede-Shipley theorem.

Proposition II.B.3. *There is an equivalence of categories*

$$\mathrm{Sp}^B \simeq \prod_{b\in\pi_0 B} \mathrm{Mod}_{\mathbb{S}[\Omega_b B]} \ .$$

Proof. The category Sp^B is compactly generated by the objects $i_!\mathbb{S}$ where i ranges over the inclusions of all points $i : b \to X$. It suffices to consider one point in each connected component of B. Assuming now that B is connected, the Schwede-Shipley theorem gives us

$$\mathrm{Sp}^B \simeq \mathrm{Mod}_{\mathrm{End}_{\mathrm{Sp}^B}(i_!\mathbb{S})} \ .$$

We now want to understand spectrum of endomorphisms of $i_!\mathbb{S}$. By the $i_! - i^*$-adjunction we get

$$\mathrm{End}_{\mathrm{Sp}^B}(i_!\mathbb{S}) = \mathrm{map}_{\mathrm{Sp}^B}(i_!\mathbb{S}, i_!\mathbb{S}) \simeq \mathrm{map}_{\mathrm{Sp}}(\mathbb{S}, i^*i_!\mathbb{S}) \simeq i^*i_!\mathbb{S}\,.$$

Fibre-wise we have

$$(i_!\mathbb{S})_b = C_*(\mathrm{fib}(i); i^*\mathbb{S}) \simeq \mathbb{S}[\Omega_b B]\,.$$

which finishes the proof. $\square$

We can put a t-structure on the category of parametrised spectra by considering the underlying t-structure on the category of spectra:

$$\mathrm{Sp}^B_{\geq 0} = \mathrm{Fun}(B, \mathrm{Sp}_{\geq 0}) \quad \text{and} \quad \mathrm{Sp}^B_{\leq 0} = \mathrm{Fun}(B, \mathrm{Sp}_{\leq 0})\,.$$

The heart of this t-structure naturally equivalent to the functor category

$$\mathrm{Sp}^{B,\heartsuit} \simeq \mathrm{Fun}(\Pi_1(B), \mathrm{Ab})\,,$$

where $\Pi_1(B)$ denotes the fundamental groupoid of the space B. Alternatively, with the perspective of Proposition II.B.3 in mind, the heart of the t-structure is given as

$$\mathrm{Sp}^{B,\heartsuit} \simeq \prod_{b\in\pi_0 B} \mathrm{Mod}_{\pi_0(\mathbb{S}[\Omega_x B])} \simeq \prod_{b\in\pi_0 B} \mathrm{Mod}_{\mathbb{Z}[\pi_1(B,b)]} \ .$$

The symmetric monoidal structure on Sp^B is given by the pullback

$$(\mathrm{Sp}^B)^\otimes = \mathrm{Fun}(B, \mathrm{Sp}^\otimes) \times_{\mathrm{Fun}(B,\mathrm{Fin}_*)} \mathrm{Fin}_* \ .$$

In particular, if $B = BG$ is the classifying of a topological group, then the above results tell us that G-spectra are the same thing as modules over the spherical group ring $\mathbb{S}[G]$, and that the heart of t-structure on G-spectra is modules over $\mathbb{Z}[G]$. The t-structure and the symmetric monoidal structures are compatible by [AN20, Proposition A.17].

II.B.2 The dualising spectrum

Given a parametrised spectrum X there are two obvious things one can do to it, namely take the colimit and the limit. We will refer to these as the homology and the cohomology of the parametrised spectrum.

Definition II.B.4. The **homology** and the **cohomology** of the B-parametrised spectrum X is

$$p_!(X) = \operatorname*{colim}_B X \quad \text{and} \quad p_*(X) = \lim_B X\,,$$

respectively.

Note that if E is trivially parametrised over B, then the above definition recovers E-homology and E-cohomology of B, in the sense that

$$\pi_*(p_!(E)) \cong E_*(B) \quad \text{and} \quad \pi_*(p_*(E)) \cong E^{-*}(B)\,,$$

which justifies the terminology.

Remark II.B.5. If $B = BG$ for some topological group G, the above constructions are usually referred to as the **homotopy orbits** and **homotopy fixed points** of the G-spectrum X, and denoted

$$X_{hG} = p_!(X) = \operatorname*{colim}_{BG} X \quad \text{and} \quad X^{hG} = p_*(X) = \lim_{BG} X\,,$$

respectively. If G is a finite group, and M is a G-module, then HM is an example of a G-spectrum, and the homotopy orbits and homotopy fixed points recover group homology and group cohomology of G with coefficients in M, in the sense that

$$\pi_*(HM_{hG}) \cong H_*(G; M) \quad \text{and} \quad \pi_*(HM^{hG}) \cong H^{-*}(G; M)\,.$$

Given a Kan complex B, we will now introduce an important spectrum parametrised over B called the dualising spectrum of B. For this end, we consider spectra parametrised over the cartesian product $B \times B$. Let us introduce some notation. We write

$$p_1 : B \times B \to B \quad \text{and} \quad p_2 : B \times B \to B$$

for the projection onto the first and second factor, respectively. We can use the pullbacks

$$(p_1)^* : \mathrm{Sp}^B \to \mathrm{Sp}^{B\times B} \quad \text{and} \quad (p_2)^* : \mathrm{Sp}^B \to \mathrm{Sp}^{B\times B}$$

of these maps to view a spectrum parametrised over B as a spectrum parametrised over $B \times B$ by adding a trivial left or right action. We will also make use of the diagonal map $\delta : B \to B \times B$, and the functor

$$\delta_! : \mathrm{Sp}^B \to \mathrm{Sp}^{B\times B}$$

induced by it; the left adjoint to the pullback $\delta^* : \mathrm{Sp}^{B\times B} \to \mathrm{Sp}^B$.

Definition II.B.6. The **dualising spectrum of** B is the spectrum over B defined as

$$D_B = (p_2)_*(\delta_!(\mathbb{S}))\,,$$

where $\mathbb{S}$ denoted the trivially parametrised sphere spectrum.

Remark II.B.7. If $B = BG$ for some topological group G, then

$$\delta_!(\mathbb{S}) \simeq \mathbb{S}[G] = \Sigma^\infty_+ G$$

which its obvious left and right actions coming from multiplication of G on itself from the left and from the right.

The dualising spectrum is generally quite hard to understand explicitly, but there are a couple of situations where we have a good handle on it, essentially by work of Klein [Kle01; Kle02; Kle07].

Remark II.B.8. If $B = M$ is a closed manifold of finite dimension n, then the dualising spectrum is precisely

$$D_M = \mathbb{S}^{-TM}$$

where TM is the tangent bundle of M. More generally, the dualising spectrum is tensor invertible in Sp^B precisely if B is a Poincaré duality space. In this case, it agrees with the stabilisation of the Spivak normal bundle shifted in degree $-n$.

Remark II.B.9. If $B = BG$ for some compact Lie group G we can describe the dualising spectrum in terms of the adjoint representation of G. More detailed, if G is a compact Lie group, then there is an equivalence

$$D_{BG} \simeq \mathbb{S}^{\mathrm{Ad}(G)}$$

of G-spectra, where the right hand side refers to the suspension spectrum of the representation sphere of the adjoint representation of G. In particular, if G is a finite group, then the adjoint representation is trivial of dimension 0, which explains why the dualising spectrum does not show up in the classical case.

II.B.3 The norm map

Classically, Tate cohomology is obtained by splicing group homology and group cohomology into a single cohomology theory. This is usually done via the so-called norm map. In this section we define the analogue of the norm map in the setting of homotopical algebra. Let X be a spectrum parametrised over B. Our goal is to construct a natural map

$$\mathrm{Nm}_B : p_!(X \otimes D_B) \to p_*(X)\,.$$

We follow [Rog08, Section 5.2]. Consider the $B \times B$-spectrum $(p_2)^*(X) \otimes \delta_!(\mathbb{S}) : B \times B \longrightarrow \mathrm{Sp}$, or more specifically the canonical colimit-limit interchange map

$$\kappa : (p_! \circ (p_2)_*)\big((p_2)^*(X) \otimes \delta_!(\mathbb{S})\big) \to (p_* \circ (p_1)_!)\big((p_2)^*(X) \otimes \delta_!(\mathbb{S})\big)\,.$$

We unravel what the source and the target of this map are. For the source, observe that the right action on $p^*(X) \otimes \delta_!(\mathbb{S})$ only affects the second factor, so we have

$$\begin{aligned}(p_! \circ (p_2)_*)\big((p_2)^*(X) \otimes \delta_!(\mathbb{S})\big) &\simeq p_!\big(X \otimes (p_2)_*(\delta_!(\mathbb{S}))\big) \\ &= p_!(X \otimes D_B),\end{aligned}$$

In the target of the colimit-limit interchange map we have

$$\begin{aligned}(p_* \circ (p_1)_!)\big((p_2)^*(X) \otimes \delta_!(\mathbb{S})\big) &\simeq (p_* \circ (p_1)_!)\big(\delta_!(\delta^*((p_2)^*(X)) \otimes \mathbb{S})\big) \\ &\simeq p_*\big((p_1 \circ \delta)_!((p_2 \circ \delta)^*(X))\big) \\ &\simeq p_*(X),\end{aligned}$$

where we have used the projection formula in the first equivalence.

Definition II.B.10. The **norm map**

$$\mathrm{Nm}_B : p_!(X \otimes D_B) \to p_*(X)$$

is the colimit-limit exchange map under these equivalences.

In general, one can think of the norm map as a map which wants to compare twisted homology to cohomology, and thus as a generalised Poincaré duality map. In this logic, the Tate construction measures the deviation for B to satisfy some generalised form of Poincaré duality. Indeed, when B is a closed manifold of dimension n, then the norm map precisely induces the Poincaré duality map on homotopy groups, in which case this is map isomorphism.

II.B.4 The Tate construction

We now define the Tate construction for a parametrised spectrum, and list some important results.

Definition II.B.11. The **Tate construction** on a spectrum X parametrised over B is defined as the cofiber

$$p_*^{\mathrm{Tate}}(X) = \mathrm{cofib}\left(\mathrm{Nm}_B : p_!(X \otimes D_B) \to p_*(X)\right) .$$

Remark II.B.12. If X is a G-spectrum, then the Tate construction is often denoted

$$X^{tG} = p_*^{\mathrm{Tate}}(X) .$$

If G is a finite group and M is a G-module, then the Tate construction recovers Tate cohomology of G with coefficients in M in the sense that

$$\pi_*(HM^{tG}) \cong \hat{H}^{-*}(G; M) .$$

In light of the previous remark, if L is a local system of abelian groups on B, then the Eilenberg–Mac Lane spectrum HL is a spectrum parametrised over B, and we define Tate cohomology of B with coefficients in L as

$$\hat{H}^{-*}(B; L) = \pi_*(p_*^{\mathrm{Tate}}(HL)) .$$

In many cases, the Tate construction can be endowed with a lax symmetric monoidal structure. We recall [NS18, Theorem I.4.1].

Theorem II.B.13. *Assume that the Tate construction vanishes on all induced objects. Then space consisting of all pairs of a lax symmetric monoidal structure on the functor $p_*^{\mathrm{Tate}} : \mathrm{Sp}^B \to \mathrm{Sp}$ together with a symmetric monoidal refinement of the canonical functor $p_* \to p_*^{\mathrm{Tate}}$ is contractible.*

In practice, this theorem tells us that if the Tate construction vanishes on all induced objects, then the Tate construction can be endowed with a lax symmetric monoidal structure in such a way that the natural transformation $p_* \to p_*^{\mathrm{Tate}}$. Moreover, there is an essentially unique way this can be done.

II.B.5 The Tate spectral sequence

We now arrive at the construction of the Tate spectral sequence; which is a spectral sequence which computes the homotopy groups of the Tate construction. Consider the stable ∞-category of spectra parametrised over B with its t-structure and symmetric monoidal structure as in Section II.B.1. Given a parametrised spectrum X we can form the Whitehead filtration and postcompose with the Tate construction. This gives us a filtration

$$\cdots \longrightarrow p_*^{\mathrm{Tate}}(\tau_{\geq n+1}X) \longrightarrow p_*^{\mathrm{Tate}}(\tau_{\geq n}X) \longrightarrow p_*^{\mathrm{Tate}}(\tau_{\geq n-1}X) \longrightarrow \cdots ,$$

which we shall refer to as the **Tate filtration** on X. Since the Tate construction is an exact functor, the associated graded of this filtration is given by

$$\mathrm{Gr}^s\, p_*^{\mathrm{Tate}}(\tau_{\geq \bullet}X) \simeq p_*^{\mathrm{Tate}}(H\pi_s(X))[s] .$$

Note that if X is an $\mathbb{E}_n$-algebra in Sp^B, then the Tate tower is an $\mathbb{E}_n$-algebra in the ∞-category $\mathrm{Tow}(\mathrm{Sp})$; this follows directly from Proposition II.1.26 and Proposition II.B.13. By Theorem II.3.5 it follows that the associated spectral sequence

$$E^1_{n,s} \cong \hat{H}^{s-n}(B; \pi_s X) \Longrightarrow \pi_n(p_*^{\mathrm{Tate}}(X))$$

is multiplicative. We will make no claims about the convergence of this spectral sequence, but we provide the reader with the following result, which we presume is relevant to such questions.

Proposition II.B.14. *If the dualising spectrum D_B is bounded below, then*

$$p_*^{\mathrm{Tate}}(\tau_{\geq \bullet}(X))(-\infty) \simeq p_*^{\mathrm{Tate}}(X) \quad \textit{and} \quad p_*^{\mathrm{Tate}}(\tau_{\geq \bullet}(X))(\infty) \simeq 0 .$$

Proof. We first show that the filtrations $p_!((\tau_{\geq \bullet} X) \otimes D_B)$ and $p_*(\tau_{\geq \bullet} X)$ satisfy similar properties. We start by showing that

$$p_!((\tau_{\geq \bullet} X) \otimes D_B)(-\infty) \simeq p_!(X \otimes D_B) \quad \text{and} \quad p_!((\tau_{\geq \bullet} X) \otimes D_B)(\infty) \simeq 0\,.$$

Note that the first one of these follows directly from the functor $p_!(- \otimes D_B)$ commuting with colimits. For the second one, note that the functor $p_!(- \otimes D_B)$ sends n-connective parametrised spectra to $n+k$-connective parametrised spectra, where k is the connectivity of D_B. The result then follows. Proving that

$$p_*(\tau_{\geq \bullet} X)(-\infty) \simeq p_*(X) \quad \text{and} \quad p_*(\tau_{\geq \bullet} X)(\infty) \simeq 0$$

is similar, noting that p_* preserves limits and coconnectivity. Since p_*^{Tate} is defined as the cofibre of a natural transformation $p_!(- \otimes D_B) \to p_*$, it follows that

$$\begin{aligned} p_*^{\mathrm{Tate}}(\tau_{\geq \bullet}(X))(-\infty) &\simeq \operatorname{cofib}(p_!((\tau_{\geq \bullet} X) \otimes D_B)(-\infty) \to p_*(\tau_{\geq \bullet} X)(-\infty)) \\ &\simeq \operatorname{cofib}(p_!(X \otimes D_B) \to p_*(X)) \\ &\simeq p_*^{\mathrm{Tate}}(X) \end{aligned}$$

and

$$\begin{aligned} p_*^{\mathrm{Tate}}(\tau_{\geq \bullet}(X))(\infty) &\simeq \operatorname{cofib}(p_!((\tau_{\geq \bullet} X) \otimes D_B)(\infty) \to p_*(\tau_{\geq \bullet} X)(\infty)) \\ &\simeq 0\,. \end{aligned}$$

□

In particular, note that the condition of the result holds true in the case $B = BG$ for a compact Lie group G.

II.B.6 The $\mathbb{T}$-Tate spectral sequence and topological periodic homology

The author's main motivation behind studying the Tate spectral sequence for non-finite groups comes from the study of topological Hochschild homology $\mathrm{THH}(R)$ and its various refinements [AN20; NS18]. By virtue of being a cyclic object, topological Hochschild homology is a $\mathbb{T}$-spectrum, where $\mathbb{T}$ denotes the circle group. The $\mathbb{T}$-Tate construction on $\mathrm{THH}(R)$ is denoted

$$\mathrm{TP}(R) = \mathrm{THH}(R)^{t\mathbb{T}}$$

and referred to as **topological periodic homology**, and has important connections to crystalline cohomology [Hes18]. In this section, we study the $\mathbb{T}$-Tate spectral sequence closer, with a specific focus on the $\mathbb{T}$-Tate spectral sequence for topological Hochschild homology.

The E^1-page of the $\mathbb{T}$-Tate spectral sequence can be understood in terms of the stunted complex projective spectra

$$\mathbb{CP}^{\infty}_{k} = \mathrm{Th}(k\gamma^1)$$

obtained by taking the Thom spectrum of the k-fold sum of the tautological bundle on $\mathbb{CP}^\infty$. Explicitly, this is a spectrum with one cell in each even degree $\geq 2k$. These fit into a pro-object in the ∞-category of spectra that is usually referred to as $\mathbb{CP}^\infty_{-\infty}$. The trivial $\mathbb{T}$-Tate construction is pro-represented by this object in the sense that we have a natural equivalence

$$X^{t\mathbb{T}} \simeq \operatorname*{colim}_k \operatorname{map}(\mathbb{CP}^\infty_{-k}, X)\,;$$

see [GM95, Section 16]. We conclude that the first page of the $\mathbb{T}$-Tate spectral sequence for $\mathbb{S}$ with trivial action is given by

$$E^1_{n,s} = \pi_{s-n}(H\pi_s(\mathbb{S})^{t\mathbb{T}}) = \hat{H}^{s-n}(\mathbb{CP}^\infty; \pi_s(\mathbb{S})) \cong \pi_{s-n}(\mathbb{S})[t^{\pm 1}]\,.$$

Here, we have made a choice of generator

$$t \in E^1_{-2,0} \cong H^2(\mathbb{CP}^\infty, \mathbb{Z})\,.$$

In general, every $\mathbb{T}$-Tate spectral sequence is a module over the Tate spectral sequence for $\mathbb{S}$ with trivial action, and we can write

$$E^1_{*,*} \cong \pi_*(X)[t^\pm]\,, \quad |t| = (-2,0)$$

for the first page of the $\mathbb{T}$-Tate spectral sequence for the $\mathbb{T}$-spectrum X. Note that a spectrum with $\mathbb{T}$-action is the same thing as a module over the spherical group ring $\mathbb{S}[\mathbb{T}]$, per Proposition II.B.3. The homotopy groups of the latter are given as

$$\pi_*(\mathbb{S}[\mathbb{T}]) \cong \pi_*(\mathbb{S})[b]/(b^2 = \eta b)\,,$$

where η is the standard generator of $\pi_1(\mathbb{S})$; see for example [Mat15, Proposition 4.2]. Thus, if X is a $\mathbb{T}$-spectrum, we get an action of the above algebra on the homotopy groups of X. In particular, we have an operator

$$b : \pi_*(X) \to \pi_{*+1}(X)\,.$$

In the study of cyclic homology this is often referred to as **Connes' B-operator**. Our goal is to understand the d^1-differential on the E^1-page of the $\mathbb{T}$-Tate spectral sequence in terms of this operator b and the element η. Let us start with a preliminary result.

Lemma II.B.15. *The $\mathbb{T}$-Tate spectral sequence for $\mathbb{Z}[\mathbb{T}] = H\mathbb{Z} \otimes \Sigma^\infty_+ \mathbb{T}$ collapses on the E^2-page. The d^1-differential is given by*

$$d^1(t^k) = \pm b t^{k+1}\,.$$

Proof. The first page of the $\mathbb{T}$-Tate spectral sequence for $\mathbb{Z}[\mathbb{T}] = H\mathbb{Z} \otimes \Sigma^\infty_+ \mathbb{T}$ is given as

$$E^2_{*,*} \cong \mathbb{Z}[b, t^{\pm 1}]/(b^2)$$

where $|t| = (-2, 0)$ and $|b| = (1, 1)$. In particular, we note that this spectral sequence is concentrated on the lines $s = 0$ and $s = 1$. Since $\mathbb{Z}[\mathbb{T}]$, as an $\mathbb{T}$-spectrum, is induced up from $H\mathbb{Z}$, we know that $H\mathbb{Z}^{t\mathbb{T}} \simeq 0$. By degree reasons, the spectral sequence must collapse after the d^1-differential, and this forces:

$$d^1(t^k) = \pm bt^{k+1} .$$

□

Since we have a choice in what generator of $H^2(\mathbb{CP}^1, \mathbb{Z})$ to call t, we will choose t to be the one that satisfies $d^1(t) = bt^2$ in the lemma above. Next, we try to understand the $\mathbb{T}$-spectral sequence for the spherical group ring on $\mathbb{T}$.

Proposition II.B.16. *In the $\mathbb{T}$-Tate spectral sequence for $\mathbb{S}[\mathbb{T}]$, we have*

$$d^1(1) = bt .$$

Proof. The E^1 page of the $\mathbb{T}$-Tate spectral sequence for $\mathbb{S}[\mathbb{T}]$ is given as

$$E^1_{*,*} \cong \pi_*(\mathbb{S}[\mathbb{T}])[t^{\pm 1}] \cong \pi_*(\mathbb{S})[b, t^{\pm 1}]/(b^2 = \eta b) ,$$

with $|t| = (-2, 0)$ and $|b| = (1, 1)$. The relevant differential can therefore be written

$$d^1(1) = mbt + n\eta t \in E^1_{-1,1} \cong \mathbb{Z}\{bt\} \oplus \mathbb{Z}/2\{\eta t\} .$$

We claim that $n = 0$ and that $m = 1$.

The first claim can be shown by considering map $\mathbb{S} \to \mathbb{S}[\mathbb{T}]$. This induces an injective map of spectral sequences. On E^1-pages, this map is

$$\pi_*(\mathbb{S})[t^{\pm 1}] \to \pi_*(\mathbb{S})[b, t^{\pm 1}]/(b^2 = \eta b) ,$$

which in particular sends $t \mapsto t$ and $x \mapsto x$ for $x \in \pi_*(\mathbb{S})$. In the $\mathbb{T}$-Tate spectral sequence for $\mathbb{S}$ with trivial action, we have $d^2(1) = 0$, which forces $n = 0$.

The second claim can be shown by considering the Hurewicz map $\mathbb{S}[\mathbb{T}] \to \mathbb{Z}[\mathbb{T}]$. This induces a map of spectral sequences, which on E^1-pages is given by the surjective map

$$\pi_*(\mathbb{S})[b, t^{\pm 1}]/(b^2 = \eta b) \to \mathbb{Z}[b, t^{\pm 1}]/(b^2) .$$

By Lemma II.B.15, we know that $d^1(1) = bt^2$, with the correct choice of t, in this spectral sequence, so $m = 1$. □

Given a spectrum X with $\mathbb{T}$-action and an element $x \in \pi_n(X)$, there is a map $x : \mathbb{S}[\mathbb{T}] \to X$ that sends $1 \mapsto x$ and $b \mapsto b(x)$ on homotopy groups. This map induces a map of $\mathbb{T}$-Tate spectral sequences, and from the result above we can conclude that

$$d^1(x) = b(x)t$$

in the $\mathbb{T}$-Tate spectral sequence for X. What is left to do for a full computation of the d^2-differential on the $\mathbb{T}$-Tate spectral sequence, is to understand the d^1-differential on t. It suffices to understand this in the universal case; the $\mathbb{T}$-Tate spectral sequence for $\mathbb{S}$ with trivial $\mathbb{T}$-action.

Proposition II.B.17. *In the $\mathbb{T}$-Tate spectral sequence for $\mathbb{S}$ with trivial action we have*

$$d^1(t) = \eta t^2 \quad .$$

Proof. The target of the relevant d^1-differential is $\mathbb{Z}/2\{\eta t^2\}$, so its suffices to rule out the possibility that $d^1(t) = 0$. We proceed by proof by contradiction, so assume that $d^1(t) = 0$. Since every $\mathbb{T}$-Tate spectral sequence is a module over the $\mathbb{T}$-Tate spectral sequence for $\mathbb{S}$, this forces $d^1(t) = 0$ also in the general case. By the Leibniz rule, we would then have

$$d^1(xt) = d^1(x)t + xd^1(t) = d^1(x)t = b(x)t^2\,,$$

in general. This contradicts d^1 being a differential, since, again by the Leibniz rule,

$$d^1(d^1(x)) = d^1(b(x)t) = d^1(b(x))t + b(x)d^1(t) = b^2(x)t = \eta b(x)t \neq 0$$

in the general case. $\square$

We summarise all of the above in the following theorem.

Theorem II.B.18. *There is a choice of generator t, so that the d^1-differential of the $\mathbb{T}$-Tate spectral sequence for the $\mathbb{T}$-spectrum X is given by*

$$d^1 : \pi_*(X)[t^{\pm 1}] \to \pi_*(X)[t^{\pm}]\,, \quad d^1(xt^k) = \begin{cases} b(x)t^{k+1} & \text{if } k \text{ is even} \\ (b(x) + \eta x)t^{k+1} & \text{if } k \text{ is odd.} \end{cases}$$

We now return to the specific case of topological Hochschild homology. The most obvious Tate spectral sequence to consider when computing topological periodic homology is the one obtained by taking the Whitehead filtration on the $\mathbb{T}$-equivariant spectrum $\mathrm{THH}(R)$, per the previous section:

$$E^1_{*,*} = \mathrm{THH}_*(R)[t^{\pm 1}] \Longrightarrow \mathrm{TP}_*(R)\,.$$

However, in addition to the cyclic structure, topological Hochschild homology also has an additional structure not available in the classical case, namely the structure of a cyclotomic spectrum [NS18, Section II]. This allows for several additional refinements of $\mathrm{THH}(R)$, one of which is topological restriction homology $\mathrm{TR}(R, p)$, where p is a fixed prime. By [AN20, Corollary 10], there is a p-adic equivalence

$$\mathrm{TP}(R) \simeq \mathrm{TR}(R, p)^{t\mathbb{T}}$$

for every connective R. This is especially useful when R is (ind)-smooth over a perfect ring k of characteristic p; in this case, we have

$$\pi_*(\mathrm{TR}(R, p)) \cong W\Omega^*_{R/k}\,,$$

where the right hand side denotes the de Rham-Witt complex of R over k, by [Hes96]. The $\mathbb{T}$-Tate spectral sequence obtained from $\mathrm{TR}(R, p)$ then has the form

$$E^1_{*,*} = W\Omega^*_{R/k}[t^{\pm 1}] \Longrightarrow \mathrm{TP}_*(R)\,.$$

Note that the E^1-page of our spectral sequence only uses the individual de Rham-Witt sheaves, but not *a priori* the de Rham-Witt differential. Therefore, the E^2-page is *a priori* unrelated to the cohomology of that complex, namely crystalline cohomology. However, when R is a (ind)-smooth $\mathbb{F}_p$-algebra, then the element η acts trivially on $\mathrm{TR}_*(R,p) \cong W\Omega^*_{R/k}$ since everything is a module over $\mathrm{TR}_*(\mathbb{F}_p) \cong H\mathbb{Z}_p$. Thus, Connes' operator b is actually a differential on $\mathrm{TR}_*(R)$ in this case, and under the isomorphism with the de Rham–Witt complex it corresponds precisely to the de Rham–Witt differential. We thus conclude that the d^1-differential in the $\mathbb{T}$-Tate spectral sequence of $\mathrm{TR}(R,p)$ is indeed given by the de Rham–Witt differential, from which it follows that the E^2-page is given by 2-periodic crystalline cohomology of $\mathrm{Spec}(R)$:

$$E^2 = H^*(\mathrm{Spec}(R)/W)[t^{\pm}] \Longrightarrow \mathrm{TP}_*(R) \ .$$

This makes precise in which sense topological periodic homology is a form of 2-periodic crystalline cohomology.

References

[AN20] B. Antieau and T. Nikolaus. “Cartier modules and cyclotomic spectra”. `https://arxiv.org/abs/1809.01714`. Jan. 2020.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. “Faisceaux pervers”. In: *Analysis and topology on singular spaces, I (Luminy, 1981)*. Vol. 100. Astérisque. Soc. Math. France, Paris, 1982, pp. 5–171.

[Beĭ87] A. A. Beĭlinson. “On the derived category of perverse sheaves”. In: *K-theory, arithmetic and geometry (Moscow, 1984–1986)*. Vol. 1289. Lecture Notes in Math. Springer, Berlin, 1987, pp. 27–41.

[BMS19] B. Bhatt, M. Morrow, and P. Scholze. “Topological Hochschild homology and integral p-adic Hodge theory”. In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310.

[Del71] P. Deligne. “Théorie de Hodge. II”. In: *Inst. Hautes Études Sci. Publ. Math.* 40 (1971), pp. 5–57.

[Gla16] S. Glasman. “Day convolution for ∞-categories”. In: *Math. Res. Lett.* 23.5 (2016), pp. 1369–1385.

[GM95] J. P. C. Greenlees and J. P. May. “Generalized Tate cohomology”. In: *Mem. Amer. Math. Soc.* 113.543 (1995).

[GP18] O. Gwilliam and D. Pavlov. “Enhancing the filtered derived category”. In: *J. Pure Appl. Algebra* 222.11 (2018), pp. 3621–3674.

[Hes18] L. Hesselholt. “Topological Hochschild homology and the Hasse–Weil zeta function”. In: (2018), pp. 157–180.

[Hes96] L. Hesselholt. “On the p-typical curves in Quillen's K-theory”. In: *Acta Math.* 177.1 (1996), pp. 1–53.

[Kle01] J. R. Klein. "The dualizing spectrum of a topological group". In: *Math. Ann.* 319.3 (2001), pp. 421–456.

[Kle02] J. R. Klein. "Axioms for generalized Farrell-Tate cohomology". In: *J. Pure Appl. Algebra* 172.2-3 (2002), pp. 225–238.

[Kle07] J. R. Klein. "The dualizing spectrum. II". In: *Algebr. Geom. Topol.* 7 (2007), pp. 109–133.

[Lur09] J. Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.

[Lur17] J. Lurie. "Higher Algebra". `www.math.ias.edu/ lurie/papers/HA.pdf`. 2017.

[Mat15] A. Mathew. "Torus actions on stable module categories, Picard groups, and localizing subcategories". `https://arxiv.org/abs/1512.01716`. Dec. 2015.

[Nik16] T. Nikolaus. "Stable ∞-operads and the multiplicative Yoneda lemma". `https://arxiv.org/abs/1608.02901`. Aug. 2016.

[NS18] T. Nikolaus and P. Scholze. "On topological cyclic homology". In: *Acta Math.* 221.2 (2018), pp. 203–409.

[Rog08] J. Rognes. "Galois extensions of structured ring spectra. Stably dualizable groups". In: *Mem. Amer. Math. Soc.* 192.898 (2008).