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E. M. Friedlander
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(Editors)

Handbook of *K*-Theory



Springer

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Eric M. Friedlander

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Editors

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Library of Congress Control Number: 2005925753

ISBN-10 3-540-23019-X Springer Berlin Heidelberg New York

ISBN-13 978-3-540-23019-9 Springer Berlin Heidelberg New York

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Printed in Germany

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Typesetting and Production: LE-T_EX Jelonek, Schmidt & Vöckler GbR, Leipzig

Cover design: deblik, Berlin

Printed on acid-free paper 41/3142/YL 5 4 3 2 1 0

Preface

This volume is a collection of chapters reflecting the status of much of the current research in K -theory. As editors, our goal has been to provide an entry and an overview to K -theory in many of its guises. Thus, each chapter provides its author an opportunity to summarize, reflect upon, and simplify a given topic which has typically been presented only in research articles. We have grouped these chapters into five parts, and within each part the chapters are arranged alphabetically.

Informally, K -theory is a tool for probing the structure of a mathematical object such as a ring or a topological space in terms of suitably parameterized vector spaces. Thus, in some sense, K -theory can be viewed as a form of higher order linear algebra that has incorporated sophisticated techniques from algebraic geometry and algebraic topology in its formulation. As can be seen from the various branches of mathematics discussed in the succeeding chapters, K -theory gives intrinsic invariants which are useful in the study of algebraic and geometric questions. In low degrees, there are explicit algebraic definitions of K -groups, beginning with the Grothendieck group of vector bundles as K_0 , continuing with H. Bass's definition of K_1 motivated in part by questions in geometric topology, and including J. Milnor's definition of K_2 arising from considerations in algebraic number theory. On the other hand, even when working in a purely algebraic context, one requires techniques from homotopy theory to construct the higher K -groups K_i and to achieve computations. The resulting interplay of algebra, functional analysis, geometry, and topology in K -theory provides a fascinating glimpse of the unity of mathematics.

K -theory has its origins in A. Grothendieck's formulation and proof of his celebrated Riemann-Roch Theorem [5] in the mid-1950's. While K -theory now plays a significant role in many diverse branches of mathematics, Grothendieck's original focus on the interplay of algebraic vector bundles and algebraic cycles on algebraic varieties is much reflected in current research, as can be seen in the chapters of Part II. The applicability of the Grothendieck construction to algebraic topology was quickly perceived by M. Atiyah and F. Hirzebruch [1], who developed

topological K -theory into the first and most important example of a “generalized cohomology theory”. Also in the 1960’s, work of H. Bass and others resulted in the formulation and systematic investigation of constructions in geometric topology (e.g., that of the Whitehead group and the Swan finiteness obstruction) involving the K -theory of non-commutative rings such as the group ring of the fundamental group of a manifold. Others soon saw the relevance of K -theoretic techniques to number theory, for example in the solution by H. Bass, J. Milnor, and J.-P. Serre [2] of the congruence subgroup problem and the conjectures of S. Lichtenbaum [6] concerning the values of zeta functions.

In the early 1970’s, D. Quillen [8] provided the now accepted definition of higher algebraic K -theory and established remarkable properties of “Quillen’s K -groups”, thereby advancing the formalism of the algebraic side of K -theory and enabling various computations. An important application of Quillen’s theory is the identification by A. Merkurjev and A. Suslin [7] of $K_2 \otimes \mathbb{Z}/n$ of a field with n -torsion in the Brauer group. Others soon recognized that many of Quillen’s techniques could be applied to rings with additional structure, leading to the study of operator algebras and to L -theory in geometric topology. Conjectures by S. Bloch [4] and A. Beilinson [3] concerning algebraic K -theory and arithmetical algebraic geometry were also formulated during the 1970’s; these conjectures prepared the way for many current developments.

We now briefly mention the subject matter of the individual chapters, which typically present mathematics developed in the past twenty years.

Part I consists of five chapters, beginning with Gunnar Carlsson’s exposition of the formalism of infinite loop spaces and their role in K -theory. This is followed by the chapter by Daniel Grayson which discusses the many efforts, recently fully successful, to construct a spectral sequence converging to K -theory analogous to the very useful Atiyah-Hirzebruch spectral sequence for topological K -theory. Max Karoubi’s chapter is dedicated to the exposition of Bott periodicity in various forms of K -theory: topological K -theory of spaces and Banach algebras, algebraic and Hermitian K -theory of discrete rings. The chapters by Lars Hesselholt and Charles Weibel present two of the most successful computations of algebraic K -groups, namely that of truncated polynomial algebras over regular noetherian rings over a field and of rings of integers in local and global fields. These computations are far from elementary and have required the development of many new techniques, some of which are explained in these (and other) chapters.

Some of the important recent developments in arithmetic and algebraic geometry and their relationship to K -theory are explored in Part II. In addition to a discussion of much recent progress, the reader will find in these chapters considerable discussion of conjectures and their consequences. The chapter by Thomas Geisser gives an exposition of Bloch’s higher Chow groups, then discusses algebraic K -theory, étale K -theory, and topological cyclic homology. Henri Gillet explains how algebraic K -theory provides a useful tool in the study of intersection theory of cycles on algebraic varieties. Various constructions of regulator maps are presented in the chapter by Alexander Goncharov in order to investigate special values of L -functions of algebraic varieties. Bruno Kahn discusses the interplay of alge-

braic K -theory, arithmetic algebraic geometry, motives and motivic cohomology, describing fundamental conjectures as well as some progress on these conjectures. Marc Levine's chapter consists of an overview of mixed motives, including various constructions and their conjectural role in providing a fundamental understanding of many geometric questions.

Part III is a collection of three articles dedicated to constructions relating algebraic K -theory (including the K -theory of quadratic spaces) to "geometric topology" (i.e., the study of manifolds). In the first chapter, Paul Balmer gives a modern and general survey of Witt groups constructed in a fashion analogous to the construction of algebraic K -groups. Jonathan Rosenberg's chapter surveys a great range of topics in geometric topology, reviewing recent as well as classical applications of K -theory to geometry. Bruce Williams' chapter emphasizes the role of the K -theory of quadratic forms in the study of moduli spaces of manifolds.

In Part IV are grouped three chapters whose focus is on the (topological) K -theory of C^* -algebras and other topological algebras which arise in the study of differential geometry. Joachim Cuntz presents in his chapter an investigation of the K -theory, K -homology and bivariant K -theory of topological algebras and their relationship with cyclic homology theories via Chern character transformations. In their long survey, Wolfgang Lueck and Holger Reich discuss the significant progress made towards the complete solution of important conjectures which would identify the K -theory or L -theory of group rings and C^* -algebras with appropriate equivariant homology groups. In the chapter by Jonathan Rosenberg, the relationship between operator algebras and K -theory is motivated, investigated, and explained through applications.

The fifth and final part presents other forms and approaches to K -theory not found in earlier chapters. Eric Friedlander and Mark Walker survey recent work on semi-topological K -theory that interpolates between algebraic K -theory of varieties and topological K -theory of associated analytic spaces. Alexander Merkurjev develops the K -theory of G -vector bundles over an algebraic variety equipped with an action of a group G and presents some applications of this theory. Stephen Mitchell's chapter demonstrates how algebraic K -theory provides an important link between techniques in algebraic number theory and sophisticated constructions in homotopy theory. The final chapter by Amnon Neeman provides a historical overview and through investigation of the challenge of recovering K -theory from the structure of a triangulated category.

Finally, two Bourbaki articles (by Eric Friedlander and Bruno Kahn) are reprinted in the appendix. The first summarizes some of the important work of A. Suslin and V. Voevodsky on motivic cohomology, whereas the second outlines the celebrated theorem of Voevodsky establishing the validity of a conjecture by J. Milnor relating $K(-) \otimes \mathbb{Z}/2$, Galois cohomology, and quadratic forms.

Some readers will be disappointed to find no chapter dedicated specifically to low-degree (i.e., classical) algebraic K -groups and insufficient discussion of the role of algebraic K -theory to algebraic number. We fully acknowledge the many

limitations of this handbook, but hope that readers will appreciate the expository effort and skills of the authors.

April, 2005

Eric M. Friedlander
Daniel R. Grayson

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Table of Contents – Volume 1

I. Foundations and Computations

I.1 Deloopings in Algebraic K -Theory	
<i>Gunnar Carlsson</i>	3
I.2 The Motivic Spectral Sequence	
<i>Daniel R. Grayson</i>	39
I.3 K -Theory of Truncated Polynomial Algebras	
<i>Lars Hesselholt</i>	71
I.4 Bott Periodicity in Topological, Algebraic and Hermitian K -Theory	
<i>Max Karoubi</i>	111
I.5 Algebraic K -Theory of Rings of Integers in Local and Global Fields	
<i>Charles Weibel</i>	139

II. K -Theory and Algebraic Geometry

II.1 Motivic Cohomology, K -Theory and Topological Cyclic Homology	
<i>Thomas Geisser</i>	193
II.2 K -Theory and Intersection Theory	
<i>Henri Gillet</i>	235
II.3 Regulators	
<i>Alexander B. Goncharov</i>	295
II.4 Algebraic K -Theory, Algebraic Cycles and Arithmetic Geometry	
<i>Bruno Kahn</i>	351
II.5 Mixed Motives	
<i>Marc Levine</i>	429
Index	523

Table of Contents – Volume 2

III. K-Theory and Geometric Topology

III.1 Witt Groups	
<i>Paul Balmer</i>	539
III.2 K-Theory and Geometric Topology	
<i>Jonathan Rosenberg</i>	577
III.3 Quadratic K-Theory and Geometric Topology	
<i>Bruce Williams</i>	611

IV. K-Theory and Operator Algebras

IV.1 Bivariant K- and Cyclic Theories	
<i>Joachim Cuntz</i>	655
IV.2 The Baum–Connes and the Farrell–Jones Conjectures in K- and L-Theory	
<i>Wolfgang Lück, Holger Reich</i>	703
IV.3 Comparison Between Algebraic and Topological K-Theory for Banach Algebras and C^* -Algebras	
<i>Jonathan Rosenberg</i>	843

V. Other Forms of K-Theory

V.1 Semi-topological K-Theory	
<i>Eric M. Friedlander, Mark E. Walker</i>	877
V.2 Equivariant K-Theory	
<i>Alexander S. Merkurjev</i>	925

V.3 $K(1)$ -Local Homotopy, Iwasawa Theory and Algebraic K -Theory
Stephen A. Mitchell 955

V.4 The K -Theory of Triangulated Categories
Amnon Neeman 1011

Appendix: Bourbaki Articles on the Milnor Conjecture

A Motivic Complexes of Suslin and Voevodsky
Eric M. Friedlander 1081

B La conjecture de Milnor (d'après V. Voevodsky)
Bruno Kahn 1105

Index 1151

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Part I

Foundations and Computations

Deloopings in Algebraic K -Theory*

Gunnar Carlsson

1.1	<i>Introduction</i>	4
1.2	<i>Generic Deloopings Using Infinite Loop Space Machines</i>	5
1.3	<i>The Q-Construction and Its Higher Dimensional Generalization ..</i>	11
1.4	<i>Waldhausen's S_*-Construction</i>	15
1.5	<i>The Gersten–Wagoner Delooping</i>	22
1.6	<i>Deloopings Based on Karoubi's Derived Functors</i>	24
1.7	<i>The Pedersen–Weibel Delooping and Bounded K-Theory</i>	26
	<i>References</i>	36

* Supported in part by NSF DMS 0104162

Introduction

A crucial observation in Quillen's definition of higher algebraic K -theory was that the right way to proceed is to define the higher K -groups as the homotopy groups of a space ([23]). Quillen gave two different space level models, one via the plus construction and the other via the Q -construction. The Q -construction version allowed Quillen to prove a number of important formal properties of the K -theory construction, namely localization, devissage, reduction by resolution, and the homotopy property. It was quickly realized that although the theory initially revolved around a functor K from the category of rings (or schemes) to the category $\underline{Top}$ of topological spaces, K in fact took its values in the category of infinite loop spaces and infinite loop maps ([1]). In fact, K is best thought of as a functor not to topological spaces, but to the category of *spectra* ([2, 11]). Recall that a spectrum is a family of based topological spaces $\{X_i\}_{i \geq 0}$, together with bonding maps $\sigma_i : X_i \rightarrow \Omega X_{i+1}$, which can be taken to be homeomorphisms. There is a great deal of value to this refinement of the functor K . Here are some reasons.

- Homotopy colimits in the category of spectra play a crucial role in applications of algebraic K -theory. For example, the assembly map for the algebraic K -theory of group rings, which is the central object of study in work on the Novikov conjecture ([13, 24]), is defined on a spectrum obtained as a homotopy colimit of the trivial group action on the K -theory spectrum of the coefficient ring. This spectrum homotopy colimit is definitely not the same thing as the homotopy colimit computed in the category $\underline{Top}$, and indeed it is clear that no construction defined purely on the space level would give this construction.
- The lower K -groups of Bass [5] can only be defined as homotopy groups in the category of spectra, since there are no negative homotopy groups defined on the category $\underline{Top}$. These groups play a key role in geometric topology [3, 4], and to define them in a way which is consistent with the definition of the higher groups (i.e. as homotopy groups) is very desirable.
- When computing with topological spaces, there is a great deal of value in being able to study the homology (or generalized homology) of a space, rather than just its homotopy groups. A reason for this is that homology is relatively easy to compute, when compared with homotopy. One has the notion of *spectrum homology*, which can only be defined as a construction on spectra, and which is also often a relatively simple object to study. To simply study the homology of the zero-th space of a spectrum is not a useful thing to do, since the homology of these spaces can be extremely complicated.
- The category of spectra has the convenient property that given a map f of spectra, the fibre of f is equivalent to the loop spectrum of the cofibre. This linearity property is quite useful, and simplifies many constructions.

In this paper, we will give an overview of a number of different constructions of spectra attached to rings. Constructing spectra amounts to constructing “deloopings” of the K -theory space of a ring. We will begin with a “generic” construction,

which applies to any category with an appropriate notion of direct sum. Because it is so generic, this construction does not permit one to prove the special formal properties mentioned above for the K -theory construction. We will then outline Quillen's Q -construction, as well as iterations of it defined by Waldhausen [32], Gillet–Grayson [15], Jardine [17], and Shimakawa [27]. We then describe Waldhausen's S -construction, which is a kind of mix of the generic construction with the Q -construction, and which has been very useful in extending the range of applicability of the K -theoretic methods beyond categories of modules over rings or schemes to categories of spectra, which has been the central tool in studying pseudo-isotopy theory ([33]). Finally, we will discuss three distinct constructions of non-connective deloopings due to Gersten–Wagoner, M. Karoubi, and Pedersen–Weibel. These constructions give interpretations of Bass's lower K -groups as homotopy groups. The Pedersen–Weibel construction can be extended beyond just a delooping construction to a construction, for any metric space, of a K -theory spectrum which is intimately related to the *locally finite homology* of the metric space. This last extension has been very useful in work on the Novikov conjecture (see [19]).

We will assume the reader is familiar with the technology of simplicial sets and multisimplicial sets, the properties of the nerve construction on categories, and the definition of algebraic K -theory via the plus construction ([16]). We will also refer him/her to [2] or [11] for material on the category of spectra.

Generic Deloopings Using Infinite Loop Space Machines

1.2

To motivate this discussion, we recall how to construct Eilenberg–MacLane spaces for abelian groups. Let A be a group. We construct a simplicial set $B.A$ by setting $B_k A = A^k$, with face maps given by

$$\begin{aligned} d_0(a_0, a_1, \dots, a_{k-1}) &= (a_1, a_2, \dots, a_{k-1}) \\ d_i(a_0, a_1, \dots, a_{k-1}) &= (a_0, a_1, \dots, a_{i-2}, a_{i-1} + a_i, a_{i+1}, \dots, a_{k-1}) \text{ for } 0 < i < k \\ d_k(a_0, a_1, \dots, a_{k-1}) &= (a_0, a_1, \dots, a_{k-2}) . \end{aligned}$$

We note that due to the fact that A is abelian, the multiplication map $A \times A \rightarrow A$ is a homomorphism of abelian groups, so $B.A$ is actually a simplicial abelian group. Moreover, the construction is functorial for homomorphisms of abelian groups, and so we may apply the construction to a simplicial abelian group to obtain a bisimplicial abelian group. Proceeding in this way, we may start with an abelian group A , and obtain a collection of multisimplicial sets $B^n A$, where $B^n A$ is an n -simplicial set. Each n -simplicial abelian group can be viewed as a simplicial abelian group by restricting to the diagonal $\Delta^{op} \subseteq (\Delta^{op})^n$, and we obtain a family

of simplicial sets, which we also denote by $B^n A$. It is easy to see that we have exact sequences of simplicial abelian groups

$$B^{n-1} A \rightarrow E^n A \rightarrow B^n A$$

where $E^n A$ is a contractible simplicial group. Exact sequences of simplicial abelian groups realize to Serre fibrations, which shows that

$$|B^{n-1} A| \cong \Omega |B^n A|$$

and that therefore the family $\{B^n A\}_n$ forms a spectrum. The idea of the infinite loop space machine construction is now to generalize this construction a bit, so that we can use combinatorial data to produce spectra.

We first describe Segal's notion of Γ -spaces, as presented in [26]. We define a category Γ as having objects the finite sets, and where a morphism from X to Y is given by a function $\theta : X \rightarrow \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ denotes the power set of Y , such that if $x, x' \in X$, $x \neq x'$, then $\theta(x) \cap \theta(x') = \emptyset$. Composition of $\theta : X \rightarrow \mathcal{P}(Y)$ and $\eta : Y \rightarrow \mathcal{P}(Z)$ is given by $x \mapsto \bigcup_{y \in \theta(x)} \eta(y)$. There is a functor from the category Δ of finite totally ordered sets and order preserving maps to Γ , given on objects by sending a totally ordered set X to its set of non-minimal elements X^- , and sending an order-preserving map from $f : X \rightarrow Y$ to the function θ_f , defined by letting $\theta_f(x)$ be the intersection of the "half-open interval" $(f(x-1), f(x)]$ with Y^- . ($x-1$ denotes the immediate predecessor of x in the total ordering on X if there is one, and if there is not, the interval $[x-1, x)$ will mean the empty set.) There is an obvious identification of the category Γ^{op} with the category of finite based sets, which sends an object X in Γ^{op} to X_+ , X "with a disjoint base point added", and which sends a morphism $\theta : X \rightarrow \mathcal{P}(Y)$ to the morphism $f_\theta : Y_+ \rightarrow X_+$ given by $f_\theta(y) = x$ if $y \in \theta(x)$ and $f_\theta(y) = *$ if $y \notin \bigcup_x \theta(x)$. Let $\underline{n}$ denote the based set $\{1, 2, \dots, n\}_+$. We have the morphism $p_i : \underline{n} \rightarrow \underline{1}$ given by $p_i(i) = 1$ and $p_i(j) = *$ when $i \neq j$.

1

Definition 1 A Γ -space is a functor from Γ^{op} to the category of simplicial sets, so that

- $F(\emptyset_+)$ is weakly contractible.
- $\Pi_1^n F(p_i) : F(\underline{n}) \rightarrow \Pi_1^n F(\underline{1})$ is a weak equivalence of simplicial sets.

Note that we have a functor $\Delta^{op} \rightarrow \Gamma^{op}$, and therefore every Γ -space can be viewed as a simplicial simplicial set, i.e. a bisimplicial set.

We will now show how to use category theoretic data to construct Γ -spaces. Suppose that C denotes a category which contains a zero objects, i.e. an object which is both initial and terminal, in which every pair of objects $X, Y \in C$ admits a *categorical sum*, i.e an object $X \oplus Y \in C$, together with a diagram

$$X \rightarrow X \oplus Y \leftarrow Y$$

so that given any pair of morphisms $f : X \rightarrow Z$ and $g : Y \rightarrow Z$, there is a unique morphism $f \oplus g : X \oplus Y \rightarrow Z$ making the diagram

$$\begin{array}{ccccc} X & \longrightarrow & X \oplus Y & \longleftarrow & Y \\ & \searrow f & \downarrow f \oplus g & \swarrow g & \\ & & Z & & \end{array}$$

commute. We will now define a functor F_C from Γ^{op} to simplicial sets. For each finite based set X , we define $\Pi(X)$ to be the category of finite based subsets of X and inclusions of sets. Consider any functor $\varphi(X) : \Pi(X) \rightarrow C$. For any pair of based subsets $S, T \subseteq X$, we obtain morphisms $\varphi(S) \rightarrow \varphi(S \cup T)$ and $\varphi(T) \rightarrow \varphi(S \cup T)$, and therefore a well defined morphism $\varphi(S) \oplus \varphi(T) \rightarrow \varphi(S \cup T)$ for any choice of sum $\varphi(S) \oplus \varphi(T)$. We say the functor $\varphi : \Pi(X) \rightarrow C$ is *summing* if it satisfies two conditions.

- $\varphi(\emptyset)$ is a zero object in C
- For any based subsets $S, T \subseteq X$, with $S \cap T = \{*\}$, we have that the natural morphism $\varphi(S) \oplus \varphi(T) \rightarrow \varphi(S \cup T)$ is an isomorphism.

Let $Sum_C(X)$ denote the category whose objects are all summing functors from $\Pi(X)$ to C , and whose morphisms are all natural transformations which are isomorphisms at all objects of $\Pi(X)$.

We next observe that if we have a morphism $f : X \rightarrow Y$ of based sets, we may define a functor $Sum_C(X) \xrightarrow{Sum_C(f)} Sum_C(Y)$ by

$$Sum_C(f)(\varphi)(S) = \varphi(f^{-1}(S))$$

for any based subset $S \subseteq Y$. One verifies that this makes $Sum_C(-)$ into a functor from Γ^{op} to the category $\underline{CAT}$ of small categories. By composing with the nerve functor N_* , we obtain a functor $Sp_1(C) : \Gamma^{op} \rightarrow s.sets$. Segal [26] now proves

Proposition 2 The functor $Sp_1(C)$ is a Γ -space.

2

The category $Sum_C(\emptyset)$ is just the subcategory of zero objects in C , which has contractible nerve since it has an initial object. The map $\prod_{i=1}^n Sp_1(p_i) : Sp_1(\underline{n}) \rightarrow \prod_{i=1}^n Sp_1(C)(\underline{1})$ is obtained by applying the nerve functor to the functor $\prod_{i=1}^n Sum_C(p_i) : Sum_C(\underline{n}) \rightarrow \prod_{i=1}^n Sum_C(\underline{1})$. But this functor is an equivalence of categories, since we may define a functor $\theta : \prod_{i=1}^n Sum_C(\underline{1}) \rightarrow Sum_C(\underline{n})$ by

$$\theta(\varphi_1, \varphi_2, \dots, \varphi_n)(\{i_1, i_2, \dots, i_s\}) = \varphi_{i_1}(\underline{1}) \oplus \varphi_{i_2}(\underline{1}) \oplus \dots \oplus \varphi_{i_s}(\underline{1})$$

Here the sum denotes any choice of categorical sum for the objects in question. Any choices will produce a functor, any two of which are isomorphic, and it is easy

to verify that $\prod_{i=1}^n \text{Sum}_C(p_i) \circ \theta$ is equal to the identity, and that $\theta \circ \prod_{i=1}^n \text{Sum}_C(p_i)$ is canonically isomorphic to the identity functor.

We observe that this construction is also functorial in C , for functors which preserve zero objects and categorical sums. Moreover, when C possesses zero objects and categorical sums, the categories $\text{Sum}_C(X)$ are themselves easily verified to possess zero objects and categorical sums, and the functors $\text{Sum}_C(f)$ preserve them. This means that we can iterate the construction to obtain functors $Sp_n(C)$ from $(\Gamma^{op})^n$ to the category of $(n+1)$ -fold simplicial sets, and by restricting to the diagonal to the category of simplicial sets we obtain a family of simplicial sets we also denote by $Sp_n(C)$. We also note that the category $\text{Sum}_C(\underline{1})$ is canonically equivalent to the category C itself, and therefore that we have a canonical map from $N.C$ to $N.\text{Sum}_C(\underline{1})$. Since $\text{Sum}_C(\underline{1})$ occurs in dimension 1 of $Sp_1(C)$, and since $\text{Sum}_C(\emptyset)$ has contractible nerve, we obtain a map from $\Sigma N.C$ to $Sp_1(C)$. Iterating the Sp_1 -construction, we obtain maps $\Sigma Sp_n(C) \rightarrow Sp_{n+1}(C)$, and hence adjoints $\sigma_n : Sp_n(C) \rightarrow \Omega Sp_{n+1}(C)$. Segal proves

3 Theorem 3 The maps σ_n are weak equivalences for $n > 1$, and for $n = 0$, σ_0 can be described as a group completion. Taken together, the functors Sp_n yield a functor Sp from the category whose objects are categories containing zero objects and admitting categorical sums and whose morphisms are functors preserving zero objects and categorical sums to the category of spectra.

Example 4. For C the category of finite sets, $Sp(C)$ is the sphere spectrum.

Example 5. For the category of finitely generated projective modules over a ring A , this spectrum is the K -theory spectrum of A .

6 Remark 6 The relationship between this construction and the iterated delooping for abelian groups discussed above is as follows. When C admits zero objects and categorical sums, we obtain a functor $C \times C$ by choosing a categorical sum $a \oplus b$ for every pair of objects a and b in C . Applying the nerve functor yields a simplicial map $\mu : N.C \times N.C \rightarrow N.C$. The map μ behaves like the multiplication map in a simplicial monoid, except that the identities are only identities up to simplicial homotopy, and the associativity conditions only hold up to homotopy. Moreover, μ has a form of homotopy commutativity, in that the maps μT and μ are simplicially homotopic, where T denotes the evident twist map on $N.C \times N.C$. So $N.C$ behaves like a commutative monoid up to homotopy. On the other hand, in verifying the Γ -space properties for $Sp_1(C)$, we showed that $Sp_1(C)(\underline{n})$ is weakly equivalent to $\prod_{i=1}^n Sp_1(C)(\underline{1})$. By definition of the classifying spaces for abelian groups, the set in the n -th level is the product of n copies of G , which is the set in the first level. So, the construction $Sp_1(C)$ also behaves *up to homotopy equivalence* like the classifying space construction for abelian groups.

The construction we have given is restricted to categories with categorical sums, and functors which preserve those. This turns out to be unnecessarily restrictive. For example, an abelian group A can be regarded as a category $Cat(A)$ whose objects are the elements of the A , and whose morphisms consist only of identity morphisms. The multiplication in A gives a functor $Cat(A) \times Cat(A) \rightarrow Cat(A)$, and hence a map $N.Cat(A) \times N.Cat(A) \rightarrow N.Cat(A)$, which is in fact associative and commutative. One can apply the classifying space construction to $N.Cat(A)$ to obtain the Eilenberg–MacLane spectrum for A . However, this operation is not induced from a categorical sum. It is desirable to have the flexibility to include examples such as this one into the families of categories to which one can apply the construction Sp . This kind of extension has been carried out by *May* [20] and *Thomason* [28]. We will give a description of the kind of categories to which the construction can be extended. See *Thomason* [28] for a complete treatment.

Definition 7 A symmetric monoidal category is a small category $\mathbf{S}$ together with a functor $\oplus : \mathbf{S} \times \mathbf{S} \rightarrow \mathbf{S}$ and an object 0 , together with three natural isomorphisms of functors

$$\alpha : (S_1 \oplus S_2) \oplus S_3 \xrightarrow{\sim} S_1 \oplus (S_2 \oplus S_3)$$

$$\lambda : 0 \oplus S_1 \xrightarrow{\sim} S_1$$

and

$$\gamma : S_1 \oplus S_2 \xrightarrow{\sim} S_2 \oplus S_1$$

satisfying the condition that $\gamma^2 = Id$ and so that the following three diagrams commute.

$$\begin{array}{ccc} ((S_1 \oplus S_2) \oplus S_3) \oplus S_4 & \xrightarrow{\alpha \oplus S_4} & (S_1 \oplus (S_2 \oplus S_3)) \oplus S_4 \\ \downarrow \alpha & & \downarrow \alpha \\ (S_1 \oplus S_2) \oplus (S_3 \oplus S_4) & & \\ \downarrow \alpha & & \\ S_1 \oplus (S_2 \oplus (S_3 \oplus S_4)) & \xleftarrow{S_1 \oplus \alpha} & S_1 \oplus ((S_2 \oplus S_3) \oplus S_4) \end{array}$$

$$\begin{array}{ccccc} (S_1 \oplus S_2) \oplus S_3 & \xrightarrow{\alpha} & S_1 \oplus (S_2 \oplus S_3) & \xrightarrow{\gamma} & (S_2 \oplus S_3) \oplus S_1 \\ \downarrow \gamma \oplus S_3 & & & & \downarrow \alpha \\ (S_2 \oplus S_1) \oplus S_3 & \xrightarrow{\alpha} & S_2 \oplus (S_1 \oplus S_3) & \xrightarrow{S_2 \oplus \gamma} & S_2 \oplus (S_3 \oplus S_1) \end{array}$$

$$\begin{array}{ccccc}
 (0 \oplus S_1) \oplus S_2 & \xrightarrow{\gamma \oplus S_2} & (S_1 \oplus 0) \oplus S_2 & \xrightarrow{\alpha} & S_1 \oplus (0 \oplus S_2) \\
 & \searrow \lambda \oplus S_2 & & \swarrow S_1 \oplus \lambda & \\
 & & S_1 \oplus S_2 & &
 \end{array}$$

Given two symmetric monoidal categories $\mathbf{S}$ and $\mathbf{T}$, a *symmetric monoidal functor* from $\mathbf{S}$ to $\mathbf{T}$ is a triple $(F, f, \bar{f})$, where $F : \mathbf{S} \rightarrow \mathbf{T}$ is a functor, and where

$$f : FS_1 \oplus FS_2 \rightarrow F(S_1 \oplus S_2) \text{ and } \bar{f} : 0 \rightarrow F0$$

are natural transformations of functors so that the diagrams

$$\begin{array}{ccccccc}
 (FS_1 \oplus FS_2) \oplus FS_3 & \xrightarrow{f \oplus FS_2} & F(S_1 \oplus S_2) \oplus FS_3 & \xrightarrow{f} & F((S_1 \oplus S_2) \oplus S_3) \\
 \downarrow \alpha & & & & \downarrow F\alpha \\
 FS_1 \oplus (FS_2 \oplus FS_3) & \xrightarrow{FS_3 \oplus f} & FS_1 \oplus F(S_2 \oplus S_3) & \xrightarrow{f} & F(S_1 \oplus (S_2 \oplus S_3))
 \end{array}$$

and

$$\begin{array}{ccc}
 FS_1 \oplus FS_2 & \xrightarrow{f} & F(S_1 \oplus S_2) \\
 \downarrow \gamma & & \downarrow F\gamma \\
 FS_2 \oplus FS_1 & \xrightarrow{f} & F(S_2 \oplus S_1)
 \end{array}
 \qquad
 \begin{array}{ccc}
 F0 \oplus FS & \xrightarrow{f} & F(0 \oplus S) \\
 \bar{f} \oplus FS \uparrow & & \downarrow F\lambda \\
 0 \oplus FS & \xrightarrow{\lambda} & FS
 \end{array}$$

all commute.

It is easy to see that if we are given a category C with zero objects and which admits categorical sums, then one can produce the isomorphisms in question by making arbitrary choices of zero objects and categorical sums for each pair of objects of C , making C into a symmetric monoidal category. In [28], Thomason now shows that it is possible by a construction based on the one given by Segal to produce a Γ -space $Sp_1(\mathbf{S})$ for any symmetric monoidal category, and more generally Γ^n -spaces, i.e. functors $(\Gamma^{op})^n \rightarrow s.sets$ which fit together into a spectrum, and that these constructions agree with those given by Segal in the case where the symmetric monoidal sum is given by a categorical sum. May [20] has also given a construction for *permutative categories*, i.e. symmetric monoidal categories where the associativity isomorphism α is actually the identity. He uses his theory of *operads* instead of Segal's Γ -spaces. It should be pointed out that the restriction to permutative categories is no real restriction, since every symmetric monoidal category is symmetric monoidally equivalent to a permutative category.

The Q-Construction and Its Higher Dimensional Generalization

1.3

It was Quillen's crucial insight that the higher algebraic K -groups of a ring A could be defined as the homotopy groups of the nerve of a certain category Q constructed from the category of finitely generated projective modules over A . He had previously defined the K -groups as the homotopy groups of the space $BGL^+(A) \times K_0(A)$. The homotopy groups and the K -groups are related via a dimension shift of one, i.e.

$$K_i(A) \cong \pi_{i+1} N.Q$$

This suggests that the loop space $\Omega N.Q$ should be viewed as the “right” space for K -theory, and indeed Quillen ([16, 23]) showed that $\Omega N.Q$ could be identified as a group completion of the nerve of the category of finitely generated projective A -modules and their isomorphisms. From this point of view, the space $N.Q$ can be viewed as a delooping of $BGL^+(A) \times K_0(A)$, and it suggests that one should look for ways to construct higher deloopings which would agree with $N.Q$ in the case $n = 1$. This was carried out by Waldhausen in [32], and developed in various forms by Gillet [15], Jardine [17], and Shimakawa [27]. We will outline Shimakawa's version of the construction.

We must first review Quillen's Q -construction. We first recall that its input is considerably more general than the category of finitely generated projective modules over a ring. In fact, the input is an *exact category*, a concept which we now recall.

Definition 8 A category is *additive* if it admits sums and products and if every *Hom*-set is given an abelian group structure, such that the composition pairings are bilinear. An *exact category* is an additive category C equipped with a family E of diagrams of the form

$$C' \xrightarrow{i} C \xrightarrow{p} C''$$

which we call the *exact sequences*, satisfying certain conditions to be specified below. Morphisms which occur as “ i ” in an exact sequence are called *admissible monomorphisms* and morphisms which occur as “ p ” are called *admissible epimorphisms*. The family E is now required to satisfy the following five conditions.

- Any diagram in C which is isomorphic to one in E is itself in E .
- The set of admissible monomorphisms is closed under composition, and the cobase change exists for an arbitrary morphism. The last statement says that the pushout of any diagram of the form

8

$$\begin{array}{ccc}
 C & \xrightarrow{f} & D \\
 i \downarrow & & \downarrow \hat{i} \\
 C' & \cdots \rightarrow & C' \times_C D
 \end{array}$$

exists, and the morphism $\hat{i}$ is also an admissible monomorphism.

- The set of admissible epimorphisms is closed under composition, and the base change exists for an arbitrary morphism. This corresponds to the evident dual diagram to the preceding condition.
- Any sequence of the form

$$C \rightarrow C \oplus C' \rightarrow C'$$

is in $\mathbf{E}$.

- In any element of $\mathbf{E}$ as above, i is a kernel for p and p is a cokernel for i .

We also define an *exact functor* as a functor between exact categories which preserves the class of exact sequences, in an obvious sense, as well as base and cobase changes.

Exact categories of course include the categories of finitely generated projective modules over rings, but they also contain many other categories. For example, any abelian category is an exact category. For any exact category $(\mathbf{C}, \mathbf{E})$, Quillen now constructs a new category $Q(\mathbf{C}, \mathbf{E})$ as follows. Objects of $Q(\mathbf{C}, \mathbf{E})$ are the same as the objects of $\mathbf{C}$, and a morphism from C to C' in $Q(\mathbf{C}, \mathbf{E})$ is a diagram of the form

$$\begin{array}{ccc}
 D & \xrightarrow{i} & C' \\
 p \downarrow & & \\
 C & &
 \end{array}$$

where i is an admissible monomorphism and p is an admissible epimorphism. The diagrams are composed using a pullback construction, so the composition of the two diagrams

$$\begin{array}{ccc}
 D & \xrightarrow{i} & C' \\
 p \downarrow & & \\
 C & &
 \end{array}
 \quad
 \begin{array}{ccc}
 D' & \xrightarrow{i'} & C'' \\
 p' \downarrow & & \\
 C' & &
 \end{array}$$

is the diagram

$$\begin{array}{ccc} D \times_{C'} D' & \longrightarrow & C'' \\ \downarrow & & \\ C & & \end{array}$$

Quillen now defines the higher K -groups for the exact category (C, E) by $K_{i-1}(C, E) = \pi_i N.Q(C, E)$. The problem before us is now how to construct higher deloopings, i.e. spaces X_n so that $K_{i-n}(C, E) = \pi_i X_n$. Shimakawa [27] proceeds as follows.

We will first need the definition of a *multicategory*. To make this definition, we must first observe that a category C is uniquely determined by

- The set A_C of all the morphisms in C , between any pairs of objects.
- A subset O_C of A_C , called the objects, identified with the set of identity morphisms in A_C .
- The source and target maps $S : A_C \rightarrow O_C$ and $T : A_C \rightarrow O_C$.
- The composition pairing is a map $\circ$ from the pullback

$$\begin{array}{ccc} A_C \times_{O_C} A_C & \longrightarrow & A_C \\ \downarrow & & \downarrow T \\ A_C & \xrightarrow{S} & O_C \end{array}$$

to A_C .

Definition 9 An n -multicategory is a set A equipped with n different category structures $(S_j, T_j, \circ_j)$ for $j = 1, \dots, n$ satisfying the following compatibility conditions for all pairs of distinct integers j and k , with $1 \leq j, k \leq n$.

- $S_j S_k x = S_k S_j x$, $S_j T_k x = T_k S_j x$, and $T_j T_k x = T_k T_j x$
- $S_j(x \circ_k y) = S_j x \circ_k S_j y$ and $T_j(x \circ_k y) = T_j x \circ_k T_j y$
- $(x \circ_k y) \circ_j (z \circ_k w) = (x \circ_j z) \circ_k (y \circ_j w)$

The notion of an n -multifunctor is the obvious one.

It is clear that one can define the notion of an n -multicategory object in any category which admits finite limits (although the only limits which are actually needed are the pullbacks $A \times_{O_j} A$). In particular, one may speak of an n -multicategory object in the category $\underline{CAT}$, and it is readily verified that such objects can be identified with $(n+1)$ -multicategories.

There is a particularly useful way to construct n -multicategories from ordinary categories. Let I denote the category associated with the totally ordered set $\{0, 1\}$,

$0 < 1$, and let λ equal the unique morphism $0 \rightarrow 1$ in I . For any category $\mathbf{C}$, define an n -multicategory structure on the set of all functors $I^n \rightarrow \mathbf{C}$ as follows. The j -th source and target functions are given on any functor f and any vector $u = (u_1, \dots, u_n) \in I^n$ by

$$(S_j f)(u) = f(u_1, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_n)$$

and

$$(T_j f)(u) = f(u_1, \dots, u_{j-1}, 1, u_{j+1}, \dots, u_n)$$

The j -th composition pairing is defined by

$$(g \circ_j f) = \begin{cases} fu & : \text{ if } u_j = 0 \\ gu & : \text{ if } u_j = 1 \\ gu \circ fu & : \text{ if } u_j = \lambda \text{ and } u_k \in \{0, 1\} \text{ for all } k \neq j \end{cases}$$

We will write $\mathbf{C}^{[n]}$ for this n -multicategory, for any $\mathbf{C}$.

We will now define an analogue of the usual nerve construction on categories. The construction applied to an n -multicategory will yield an n -multisimplicial set. To see how to proceed, we note that for an ordinary category $\mathbf{C}$, regarded as a set A with S , T , and $\circ$ operators, and O the set of objects, the set $N_k \mathbf{C}$ can be identified with the pullback

$$\underbrace{A \times_O A \times_O A \cdots \times_O A}_{k \text{ factors}}$$

i.e the set of vectors $(a_1, a_2, \dots, a_k)$ so that $S(a_j) = T(a_{j-1})$ for $2 \leq j \leq k$. Note that $N_0 \mathbf{C}$ conventionally denotes O . In the case of an n -multicategory $\mathbf{C}$, we can therefore construct this pullback for any one of the n category structures. Moreover, because of the commutation relations among the operators S_j , T_j , and $\circ_j$ for the various values of j , the nerve construction in one of the directions respects the operators in the other directions. This means that if we let N_{s,k_s} denote the k -dimensional nerve operator attached to the s -th category structure, we may define an n -multisimplicial set $NC : (\Delta^{op})^n \rightarrow \mathbf{Sets}$ by the formula

$$NC(i_1, i_2, \dots, i_n) = N_{1,i_1} N_{2,i_2} \cdots N_{n,i_n} \mathbf{C}$$

The idea for constructing deloopings of exact categories is to define a notion of an n -multiexact category, and to note that it admits a Q -construction which is an n -multicategory. whose nerve will become the n -th delooping.

Definition 10 A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ of small categories is called *strongly good* if for any object $x \in \mathbf{C}$ and any isomorphism $f : Fx \rightarrow y$ in $\mathbf{D}$, there is a unique isomorphism $f' : x \rightarrow y'$ in $\mathbf{C}$ such that $Ff' = f$.

Definition 11 Let $P \subseteq \{1, \dots, n\}$. Then by P -exact category, we mean an n -fold category $\mathbf{C}$ so that every $\mathbf{C}_p$, $p \in P$, is equipped with the structure of an exact category, so that the following conditions hold for every pair p, j , with $p \in P$ and $j \neq p$.

- $o_j \mathbf{C}_p$ is an exact subcategory of $\mathbf{C}_p$.
- $(S_j, T_j) : \mathbf{C}_p \rightarrow o_j \mathbf{C}_p \times o_j \mathbf{C}_p$ is strongly good and exact.
- $\circ_j : \mathbf{C}_p \times_{o_j \mathbf{C}_p} \mathbf{C}_p \rightarrow \mathbf{C}_p$ is exact.
- If j also belongs to P , the class E_j of exact sequences of $\mathbf{C}_j$ becomes an exact subcategory of $\mathbf{C}_p \times_{o_j \mathbf{C}_p} \mathbf{C}_p$

One direct consequence of the definition is that if we regard a P -exact category as an $(n-1)$ -multicategory object in $\underline{\mathbf{CAT}}$, with the arguments in the $(n-1)$ -multicategory taking their values in $\mathbf{C}_p$, with $p \in P$, we find that we actually obtain an $(n-1)$ -multicategory object in the category $\underline{\mathbf{EXCAT}}$ of exact categories and exact functors. The usual Q -construction gives a functor from $\underline{\mathbf{EXCAT}}$ to $\underline{\mathbf{CAT}}$, which preserves the limits used to define n -multicategory objects, so we may apply Q in the p -th coordinate to obtain an $(n-1)$ -multicategory object in $\underline{\mathbf{CAT}}$, which we will denote by $Q_p(\mathbf{C})$. We note that $Q_p(\mathbf{C})$ is now an n -multicategory, and Shimakawa shows that there is a natural structure of a $(P - \{p\})$ -exact multicategory on $Q_p(\mathbf{C})$. One can therefore begin with an $\{1, \dots, n\}$ -exact multicategory $\mathbf{C}$, and construct an n -multicategory $Q_n Q_{n-1} \cdots Q_1 \mathbf{C}$. It can further be shown that the result is independent of the order in which one applies the operators Q_i . The nerves of these constructions provide us with n -multisimplicial sets, and these can be proved to yield a compatible system of deloopings and therefore of spectra.

Waldhausen's $S_\bullet$ -Construction

In this section, we describe a family of deloopings constructed by F. Waldhausen in [33] which combine the best features of the generic deloopings with the important special properties of Quillen's Q -construction delooping. The input to the construction is a *category with cofibrations and weak equivalences*, a notion defined by Waldhausen, and which is much more general than Quillen's exact categories. For example, it will include categories of spaces with various special conditions, or spaces over a fixed space as input. These cannot be regarded as exact categories in any way, since they are not additive. On the other hand, the construction takes into account a notion of exact sequence, which permits one to prove versions of the localization and additivity theorems for it. It has permitted Waldhausen to construct spectra $A(X)$ for spaces X , so that for X a manifold, $A(X)$ contains the stable pseudo-isotopy space as a factor. See [33] for details.

We begin with a definition.

Definition 12 A category $\mathcal{C}$ is said to be *pointed* if it is equipped with a distinguished object $*$ which is both an initial and terminal object. A *category with cofibrations*

is a pointed category $\mathcal{C}$ together with a subcategory $co\mathcal{C}$. The morphisms of $co\mathcal{C}$ are called the cofibrations. The subcategory $co\mathcal{C}$ satisfies the following properties.

1. Any isomorphism in $\mathcal{C}$ is in $co\mathcal{C}$. In particular, any object of $\mathcal{C}$ is in $co\mathcal{C}$.
2. For every object $X \in \mathcal{C}$, the unique arrow $* \rightarrow X$ is a cofibration.
3. For any cofibration $i : X \hookrightarrow Y$ and any morphism $f : X \rightarrow Z$ in $\mathcal{C}$, there is a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ i \downarrow & & \downarrow \hat{i} \\ Y & \longrightarrow & W \end{array}$$

in $\mathcal{C}$, and the natural map $\hat{i}$ is also a cofibration.

13

Definition 13 Let $\mathcal{C}$ be a category with cofibrations. A *category of weak equivalences* in $\mathcal{C}$ is a subcategory $w\mathcal{C}$ of $\mathcal{C}$, called the *weak equivalences*, which satisfy two axioms.

1. All isomorphisms in $\mathcal{C}$ are in $w\mathcal{C}$.
2. For any commutative diagram

$$\begin{array}{ccccc} Y & \xleftarrow{i} & X & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \xleftarrow{i'} & X' & \longrightarrow & Z' \end{array}$$

in $\mathcal{C}$, where i and i' are cofibrations, and all the vertical arrows are weak equivalences, the induced map on pushouts is also a weak equivalence.

If $\mathcal{C}$ and $\mathcal{D}$ are both categories with cofibrations and weak equivalences, we say a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is *exact* if it preserves pushouts, $co\mathcal{C}$, and $w\mathcal{C}$, and $f* = *$.

Here are some examples.

Example 14. The category of based finite sets, with $*$ a single point space, the cofibrations the based inclusions, and the weak equivalences being the bijections.

Example 15. The category of based simplicial sets with finitely many cells (i.e. non-degenerate simplices), the one point based set as $*$, the level-wise inclusions as the cofibrations, and the usual weak equivalences as $w\mathcal{C}$.

Example 16. Any exact category $\mathcal{E}$ in the sense of Quillen [23] can be regarded as a category with cofibrations and weak equivalences as follows. $*$ is chosen to be any zero object, the cofibrations are the admissible monomorphisms, and the weak equivalences are the isomorphisms.

Example 17. Let A be any ring, and let $\mathcal{C}$ denote the category of chain complexes of finitely generated projective A -modules, which are bounded above and below. The zero complex is $*$, the cofibrations will be the levelwise split monomorphisms, and the weak equivalences are the chain equivalences, i.e. maps inducing isomorphisms on homology. A variant would be to consider the homologically finite complexes, i.e. complexes which are not necessarily bounded above but which have the property that there exists an N so that $H_n = 0$ for $n > N$.

We will now outline how Waldhausen constructs a spectrum out of a category with cofibrations and weak equivalences. For each n , we define a new category $\mathcal{S}_n\mathcal{C}$ as follows. Let $\underline{n}$ denote the totally ordered set $\{0, 1, \dots, n\}$ with the usual ordering. Let $Ar[n] \subseteq \underline{n} \times \underline{n}$ be the subset of all (i, j) such that $i \leq j$. The category $Ar[n]$ is a partially ordered set, and as such may be regarded as a category. We define the objects of $\mathcal{S}_n\mathcal{C}$ as the collection of all functors $\theta : Ar[n] \rightarrow \mathcal{C}$ satisfying the following conditions.

- $\theta(i, i) = *$ for all $0 \leq i \leq n$.
- $\theta((i, j) \leq (i, j'))$ is a cofibration.
- For all triples i, j, k , with $i \leq j \leq k$, the diagram

$$\begin{array}{ccc} \theta(i, j) & \longrightarrow & \theta(i, k) \\ \downarrow & & \downarrow \\ \theta(j, j) = * & \longrightarrow & \theta(j, k) \end{array}$$

is a pushout diagram.

Remark 18 Note that each object in $\mathcal{S}_n\mathcal{C}$ consists of a composable sequence of cofibrations

$$* = \theta(0, 0) \hookrightarrow \theta(0, 1) \hookrightarrow \theta(0, 2) \hookrightarrow \dots \hookrightarrow \theta(0, n-1) \hookrightarrow \theta(0, n)$$

together with choices of quotients for each cofibration $\theta((0, i) \leq (0, j))$, when $i \leq j$.

$\mathcal{S}_n\mathcal{C}$ becomes a category by letting the morphisms be the natural transformations of functors. We can define a category of cofibrations on $\mathcal{S}_n\mathcal{C}$ as follows. A morphism $\Phi : \theta \rightarrow \theta'$ determines morphisms $\Phi_{ij} : \theta(ij) \rightarrow \theta'(ij)$. In order for Φ to be a cofibration in $\mathcal{S}_n\mathcal{C}$, we must first require that Φ_{ij} is a cofibration in $\mathcal{C}$ for every

i and j . In addition, we require that for every triple i, j, k , with $i \leq j \leq k$, the commutative diagram

$$\begin{array}{ccc} \theta(i, j) & \longrightarrow & \theta(i, k) \\ \downarrow & & \downarrow \\ \theta'(i, j) & \longrightarrow & \theta'(j, k) \end{array}$$

is a pushout diagram in $\mathcal{C}$. We further define a category $w\mathcal{S}_n\mathcal{C}$ of weak equivalences on $\mathcal{S}_n\mathcal{C}$ by $\Phi \in w\mathcal{S}_n\mathcal{C}$ if and only if $\Phi_{ij} \in w\mathcal{C}$ for all $i \leq j$. One can now check that $\mathcal{S}_n\mathcal{C}$ is a category with cofibrations and weak equivalences.

We now further observe that we actually have a functor from $\Delta \rightarrow \underline{CAT}$ given by $\underline{n} \rightarrow Ar[n]$, where Δ as usual denotes the category of finite totally ordered sets and order preserving maps of such. Consequently, if we denote by $\mathcal{F}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$ (the morphisms are natural transformations), we obtain a simplicial category $n \rightarrow \mathcal{F}(Ar[n], \mathcal{C})$ for any category $\mathcal{C}$. One checks that if $\mathcal{C}$ is a category with cofibrations and weak equivalences, then the subcategories $\mathcal{S}_n\mathcal{C} \subseteq \mathcal{F}(Ar[n], \mathcal{C})$ are preserved under the face and degeneracy maps, so that we actually have a functor $\mathcal{S}_\bullet$ from the category of categories with cofibrations and weak equivalences, and exact functors, to the category of simplicial categories with cofibrations and weak equivalences and levelwise exact functors. This construction can now be iterated to obtain functors $\mathcal{S}_\bullet^k$ which assign to a category with cofibrations and weak equivalences a k -simplicial category with cofibrations and weak equivalences. To obtain the desired simplicial sets, we first apply the w levelwise, to obtain a simplicial category, and then apply the nerve construction levelwise, to obtain a $(k+1)$ -simplicial set $N.w\mathcal{S}_\bullet^k\mathcal{C}$. We can restrict to the diagonal simplicial set $\Delta N.w\mathcal{S}_\bullet^k\mathcal{C}$ to obtain a family of simplicial sets, which by abuse of notation we also write as $\mathcal{S}_\bullet^k\mathcal{C}$.

Waldhausen's next observation is that for any category with cofibrations and weak equivalences $\mathcal{C}$, there is a natural inclusion

$$\Sigma N.w\mathcal{C} \rightarrow \mathcal{S}_\bullet\mathcal{C}$$

The suspension is the reduced suspension using $*$ as the base point in $N.w\mathcal{C}$. This map exists because by definition, $\mathcal{S}_0\mathcal{C} = *$, and $\mathcal{S}_1\mathcal{C} = N.w\mathcal{C}$, so we obtain a map $\sigma : \Delta[1] \times N.w\mathcal{C} \rightarrow \mathcal{S}_\bullet\mathcal{C}$, and it is easy to see that the subspace

$$\partial\Delta[1] \times N.w\mathcal{C} \cup \Delta[1] \times *$$

maps to $*$ under σ , inducing the desired map which we also denote by σ . The map σ is natural for exact functors, and we therefore obtain maps $\mathcal{S}_\bullet^k(\sigma) : \Sigma\mathcal{S}_\bullet^k\mathcal{C} \rightarrow \mathcal{S}_\bullet^{k+1}\mathcal{C}$ for each k . Waldhausen now proves

Theorem 19 The adjoint to $\mathcal{S}^k(\sigma)$ is a weak equivalence of simplicial sets from $\mathcal{S}^k\mathcal{C}$ to $\Omega\mathcal{S}^{k+1}\mathcal{C}$ for $k \geq 1$, so the spaces $|\mathcal{S}^k\mathcal{C}|$ form a spectrum except in dimension $k = 0$, when it can be described as a homotopy theoretic group completion. These deloopings agree with Segal's generic deloopings when $\mathcal{C}$ is a category with sums and zero object, and the cofibrations are chosen to be only the sums of the form $X \rightarrow X \vee T \rightarrow Y$.

We will write $\mathcal{S}\mathcal{C}$ for this spectrum. The point of Waldhausen's construction is that it produces a spectrum from the category $\mathcal{C}$ in such a way that many of the useful formal properties of Quillen's construction hold in this much more general context. We will discuss the analogues of the localization and additivity theorems, from the work of Quillen [23].

We will first consider localization. Recall that Quillen proved a localization theorem in the context of quotients of abelian categories. The context in which Waldhausen proves his localization result is the following. Given a category with cofibrations and weak equivalences $\mathcal{C}$, we define $Ar\mathcal{C}$ to be the category whose objects are morphisms $f : X \rightarrow Y$ in $\mathcal{C}$, and where a morphism from $f : X \rightarrow Y$ to $f' : X' \rightarrow Y'$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

It is easy to check that $Ar\mathcal{C}$ becomes a category with cofibrations and weak equivalences if we declare that the cofibrations (respectively weak equivalences) are diagrams such as the ones above in which both vertical arrows are cofibrations (respectively weak equivalences). If $\mathcal{C}$ is the category of based topological spaces, then the *mapping cylinder* construction can be viewed as a functor from $Ar\mathcal{C}$ to spaces, satisfying certain conditions. In order to construct a localization sequence, Waldhausen requires an analogue of the mapping cylinder construction in the category $\mathcal{C}$.

Definition 20 A *cylinder functor* on a category with cofibrations and weak equivalences $\mathcal{C}$ is a functor T which takes objects $f : X \rightarrow Y$ in $Ar\mathcal{C}$ to diagrams of shape

$$\begin{array}{ccccc} X & \xrightarrow{j} & T(f) & \xleftarrow{k} & Y \\ & \searrow f & \downarrow p & \swarrow id & \\ & & Y & & \end{array}$$

satisfying the following two conditions.

- For any two objects X and Y in $\mathcal{C}$, we denote by $X \vee Y$ pushout of the diagram

$$X \leftarrow * \rightarrow Y$$

We require that the canonical map $X \vee Y \rightarrow T(f)$ coming from the diagram above be a cofibration, and further that if we have any morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

in $Ar\mathcal{C}$, then the associated diagram

$$\begin{array}{ccc} X \vee Y & \longrightarrow & T(f) \\ \downarrow & & \downarrow \\ X' \vee Y' & \longrightarrow & T(f') \end{array}$$

is a pushout.

- $T(* \rightarrow X) = X$ for every $X \in \mathcal{C}$, and k and p are the identity map in this case.

(The collection of diagrams of this shape form a category with natural transformations as morphisms, and T should be a functor to this category.) We say the cylinder functor satisfies the *cylinder condition* if p is in $w\mathcal{C}$ for every object in $Ar\mathcal{C}$.

We will also need two axioms which apply to categories with cofibrations and weak equivalences.

21 **Axiom 21:** *Saturation axiom:* If f and g are composable morphisms in $\mathcal{C}$, and if two of f , g , and gf are in $w\mathcal{C}$, then so is the third.

22 **Axiom 22:** *Extension Axiom:* If we have a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow{i'} & Y' & \longrightarrow & Z' \end{array}$$

where i and i' are cofibrations, and Z and Z' are pushouts of the diagrams $* \leftarrow X \rightarrow Y$ and $* \leftarrow X' \rightarrow Y'$ respectively, and if the arrows $X \rightarrow X'$ and $Z \rightarrow Z'$ are in $w\mathcal{C}$, then it follows that $Y \rightarrow Y'$ is in $w\mathcal{C}$ also.

The setup for the localization theorem is now as follows. We let $\mathcal{C}$ be a category equipped with a category of cofibrations, and two different categories of weak equivalences v and w , so that $v\mathcal{C} \subseteq w\mathcal{C}$. Let $\mathcal{C}^w$ denote the subcategory with cofibrations on $\mathcal{C}$ given by the objects $X \in \mathcal{C}$ having the property that the map $* \rightarrow X$ is in $w\mathcal{C}$. It will inherit categories of weak equivalences $v\mathcal{C}^w = \mathcal{C}^w \cap v\mathcal{C}$ and $w\mathcal{C}^w = \mathcal{C}^w \cap w\mathcal{C}$. Waldhausen's theorem is now as follows.

Theorem 23 If $\mathcal{C}$ has a cylinder functor, and the category of weak equivalences $w\mathcal{C}$ satisfies the cylinder axiom, saturation axiom, and extension axiom, then the square

23

$$\begin{array}{ccc} v\mathcal{S}.\mathcal{C}^w & \longrightarrow & w\mathcal{S}.\mathcal{C}^w \\ \downarrow & & \downarrow \\ v\mathcal{S}.\mathcal{C} & \longrightarrow & w\mathcal{S}.\mathcal{C} \end{array}$$

is homotopy Cartesian, and $w\mathcal{S}.\mathcal{C}^w$ is contractible. In other words, we have up to homotopy a fibration sequence

$$v\mathcal{S}.\mathcal{C}^w \rightarrow v\mathcal{S}.\mathcal{C} \rightarrow w\mathcal{S}.\mathcal{C}$$

The theorem extends to the deloopings by applying $\mathcal{S}$. levelwise, and we obtain a fibration sequence of spectra.

Remark 24 The reader may wonder what the relationship between this sequence and Quillen's localization sequence is. One can see that the category of finitely generated projective modules over a Noetherian commutative ring A , although it is a category with cofibrations and weak equivalences, does not admit a cylinder functor with the cylinder axiom, and it seems that this theorem does not apply. However, one can consider the category of chain complexes $Comp(A)$ of finitely generated projective chain complexes over A , which are bounded above and below. The category $Comp(A)$ does admit a cylinder functor satisfying the cylinder axiom (just use the usual algebraic mapping cylinder construction). It is shown in [29] that $Proj(A) \hookrightarrow Comp(A)$ of categories with cofibrations and weak equivalences induces an equivalences on spectra $\mathcal{S}(Proj(A)) \rightarrow \mathcal{S}(Comp(A))$ for any ring. Moreover, if S is a multiplicative subset in the regular ring A , then the category $Comp(A)^S$ of objects $C_* \in Comp(A)$ which have the property that $S^{-1}C_*$ is acyclic, i.e. has vanishing homology, has the property that $\mathcal{S}.Comp(A)^S$ is weakly equivalent to the $\mathcal{S}$. spectrum of the exact category $Mod(A)^S$ of finitely generated

24

A -modules M for which $S^{-1}M = 0$. In this case, the localization theorem above applies, with ν being the usual category of weak equivalences, and where w is the class of chain maps whose algebraic mapping cone has homology annihilated by S^{-1} . Finally, if we let $\mathcal{D}$ denote the category with cofibrations and weak equivalences consisting with $\mathcal{C}$ as underlying category, and with w as the weak equivalences, then $\mathcal{S}\mathcal{D} \cong \mathcal{S}\text{Comp}(S^{-1}A)$. Putting these results together shows that we obtain Quillen's localization sequence in this case.

The key result in proving 23 is Waldhausen's version of the Additivity Theorem. Suppose we have a category with cofibrations and weak equivalences $\mathcal{C}$, and two subcategories with cofibrations and weak equivalences $\mathcal{A}$ and $\mathcal{B}$. This means that $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathcal{C}$, each given a structure of a category with cofibrations and weak equivalences, so that the inclusions are exact. Then we define a new category $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ to have objects cofibration sequences

$$A \hookrightarrow C \rightarrow B$$

with $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $C \in \mathcal{C}$. This means that we are given a specific isomorphism from a pushout of the diagram $* \leftarrow A \rightarrow C$ to B . The morphisms in $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ are maps of diagrams. We define a category of cofibrations on $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ to consist of those maps of diagrams which are cofibrations at each point in the diagram. We similarly define the weak equivalences to be the pointwise weak equivalences. We have an exact functor $(s, q) : E(\mathcal{A}, \mathcal{C}, \mathcal{B}) \rightarrow \mathcal{A} \times \mathcal{B}$, given by $s(A \hookrightarrow C \rightarrow B) = A$ and $q(A \hookrightarrow C \rightarrow B) = B$.

25 **Theorem 25** The exact functor (s, q) induces a homotopy equivalence from $\mathcal{S}E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ to $\mathcal{S}\mathcal{A} \times \mathcal{S}\mathcal{B}$.

Finally, Waldhausen proves a comparison result between his delooping and the nerve of Quillen's Q -construction.

26 **Theorem 26** There is a natural weak equivalence of spaces from $|w\mathcal{S}E|$ to $|N.Q(E)|$ for any exact category E . (Recall that E can be viewed as a category with cofibrations and weak equivalences, and hence $w\mathcal{S}$ can be evaluated on it).

1.5 The Gersten–Wagoner Delooping

All the constructions we have seen so far have constructed simplicial and category theoretic models for deloopings of algebraic K -theory spaces. It turns out that there is a way to construct the deloopings directly on the level of rings, i.e. for any ring R there is a ring μR whose K -theory space actually deloops the K -theory

space of R . Two different versions of this idea were developed by S. Gersten [14] and J. Wagoner [30]. The model we will describe was motivated by problems in high dimensional geometric topology [12], and by the observation that the space of Fredholm operators on an infinite dimensional complex Hilbert space provides a delooping of the infinite unitary group U . This delooping, and the Pedersen–Weibel delooping which follows in the last section, are *non-connective*, i.e. we can have $\pi_i X_n \neq 0$ for $i < n$, where X_n denotes the n -th delooping in the spectrum. The homotopy group $\pi_i X_{i+n}$ are equal to Bass’s lower K -group $K_{-n}(R)$ ([5]), and these lower K -groups have played a significant role in geometric topology, notably in the study of stratified spaces ([3, 4]).

Definition 27 Let R be a ring, and let lR denote the ring of infinite matrices over R in which each row and column contains only finitely many non-zero elements. The subring $mR \subseteq lR$ will be the set of all matrices with only finitely many non-zero entries; mR is a two-sided ideal in lR , and we define $\mu R = lR/mR$.

27

Remark 28 μR is a ring of a somewhat unfamiliar character. For example, it does not admit a rank function on projective modules.

28

Wagoner shows that $BGL^+(\mu R)$ is a delooping of $BGL^+(R)$. He first observes that the construction of the $n \times n$ matrices $M_n(R)$ for a ring R does not require that R has a unit. Of course, if R doesn’t have a unit, then neither will $M_n(R)$. Next, for a ring R (possibly without unit), he defines $GL_n(R)$ be the set of $n \times n$ matrices P so that there is an $n \times n$ matrix Q with $P + Q + PQ = 0$, and equips $GL_n(R)$ with the multiplication $P \circ Q = P + Q + PQ$. (Note that for a ring with unit, this corresponds to the usual definition via the correspondence $P \rightarrow I + P$.) GL is now defined as the union of the groups GL_n under the usual inclusions. We similarly define $E_n(R)$ to be the subgroup generated by the elementary matrices $e_{ij}(r)$, for $i \neq j$, whose ij -th entry is r and for which the kl -th entry is zero for $(k, l) \neq (i, j)$. The group $E(R)$ is defined as the union of the groups $E_n(R)$. By definition, $GL(R)/E(R) \cong K_1 R$. The group $E(R)$ is a perfect group, so we may perform the plus construction to the classifying space $BE(R)$. Wagoner proves that there is a fibration sequence up to homotopy

$$BGL^+(R) \times K_0(R) \rightarrow E \rightarrow BGL^+(\mu R) \quad (1.1)$$

where E is a contractible space. This clearly shows that $BGL^+(\mu R)$ deloops $BGL^+(R) \times K_0(R)$. The steps in Wagoner’s argument are as follows.

- There is an equivalence $BGL^+(R) \cong BGL^+(mR)$, coming from a straightforward isomorphism of rings (without unit) $M_\infty(mR) \cong mR$, where $M_\infty(R)$ denotes the union of the rings $M_n(R)$ under the evident inclusions.

- There is an exact sequence of groups $GL(mR) \rightarrow E(IR) \rightarrow E(\mu R)$. This follows directly from the definition of the rings mR , IR , and μR , together with the fact that $E(IR) = GL(IR)$. It yields a fibration sequence of classifying spaces

$$BGL(mR) \rightarrow BE(IR) \rightarrow BE(\mu R) \quad (1.2)$$

- The space $BE(IR)$ has trivial homology, and therefore the space $BE^+(IR)$ is contractible.
- The action of $E(\mu(R)) = \pi_1 BE^+(\mu R)$ on the homology of the fiber $BGL^+(mR)$ in the fibration 1.2 above is trivial. Wagoner makes a technical argument which shows that this implies that the sequence

$$BGL^+(mR) \rightarrow BGL^+(IR) \rightarrow BGL^+(\mu R)$$

is a fibration up to homotopy.

- $K_1(\mu(R)) \cong K_0(R)$.

Wagoner assembles these facts into a proof that we have a fibration of the form 1.1. Iterating the μ construction and applying $BGL^+(-)$ now yields the required family of deloopings.

Deloopings Based on Karoubi's Derived Functors

Max Karoubi ([18]) developed a method for defining the lower algebraic K -groups which resembles the construction of derived functors in algebra. The method permits the definition of these lower K -groups in a very general setting. As we have seen, the lower K -groups can be defined as the homotopy groups of non-connective deloopings of the zeroth space of the K -theory spectrum. Karoubi observed that his techniques could be refined to produce deloopings rather than just lower K -groups, and this was carried out by Pedersen and Weibel in [22].

Karoubi considers an additive category $\mathcal{A}$, i.e. a category so that every morphism set is equipped with the structure of an abelian group, so that the composition pairings are bilinear, and so that every finite set of objects admits a sum which is simultaneously a product. He supposes further that $\mathcal{A}$ is embedded as a full subcategory of another additive category $\mathcal{U}$. He then makes the following definition.

Definition 29 $\mathcal{U}$ is said to be $\mathcal{A}$ -filtered if every object $U \in \mathcal{U}$ is equipped with a family of direct sum decompositions $\varphi_i: U \xrightarrow{\sim} E_i \oplus U_i$, $i \in I_U$, where I_U is an indexing set depending on U , with each $E_i \in \mathcal{A}$, satisfying the following axioms.

- For each U , the collection of decompositions form a filtered poset, when we equip it with the partial order $\{\varphi_i: U \xrightarrow{\sim} E_i \oplus U_i\} \leq \{\varphi_j: U \xrightarrow{\sim} E_j \oplus U_j\}$ if and

- only if the composite $U_j \hookrightarrow E_j \oplus U_j \xrightarrow{\varphi_j} U$ factors as $U_j \rightarrow U_i \hookrightarrow E_i \oplus U_i \xrightarrow{\varphi_i} U$ and the composite $E_i \hookrightarrow E_i \oplus U_i \xrightarrow{\varphi_i} U$ factors as $E_i \rightarrow E_j \hookrightarrow E_j \oplus U_j \xrightarrow{\varphi_j} U$.
- For any objects $A \in \mathcal{A}$ and $U \in \mathcal{U}$, and any morphism $f: A \rightarrow U$ in $\mathcal{U}$, f factors as $A \rightarrow E_i \hookrightarrow E_i \oplus U_i \xrightarrow{\varphi_i} U$ for some i .
 - For any objects $A \in \mathcal{A}$ and $U \in \mathcal{U}$, and any morphism $f: U \rightarrow A$ in $\mathcal{U}$, f factors as $U \xrightarrow{\varphi_i^{-1}} E_i \oplus U_i \xrightarrow{\pi} E_i \rightarrow A$ for some i .
 - For each $U, V \in \mathcal{U}$, the given partially ordered set of filtrations on $U \oplus V$ is equivalent to the product of the partially ordered sets of filtrations on U and V . That is to say, if the decompositions for U , V , and $U \oplus V$ are given by $\{E_i \oplus U_i\}_{i \in I_U}$, $\{E_j \oplus V_j\}_{j \in I_V}$, and $\{E_k \oplus W_k\}_{k \in I_{U \oplus V}}$, then the union of the collections of decompositions $\{(E_i \oplus E_j) \oplus (U_i \oplus V_j)\}_{(i,j) \in I_U \times I_V}$ and $\{E_k \oplus W_k\}_{k \in I_{U \oplus V}}$ also form a filtered partially ordered set under the partial ordering specified above.
 - If $\varphi_i: U \rightarrow E_i \oplus U_i$ is one of the decompositions for U , and E_i can be decomposed as $E_i \cong A \oplus B$ in $\mathcal{A}$, then the decomposition $U \cong A \oplus (B \oplus U_i)$ is also one of the given family of decompositions for U .

Karoubi also defines an additive category $\mathcal{U}$ to be *flasque* if there a functor $e: \mathcal{U} \rightarrow \mathcal{U}$ and a natural isomorphism from e to $e \oplus id_{\mathcal{U}}$. Given an inclusion $\mathcal{A} \rightarrow \mathcal{U}$ as above, he also defines the quotient category $\mathcal{U}/\mathcal{A}$ to be the category with the same objects as $\mathcal{U}$, but with $Hom_{\mathcal{U}/\mathcal{A}}(U, V) \cong Hom_{\mathcal{U}}(U, V)/K$, where K is the subgroup of all morphisms from U to V which factor through an object of $\mathcal{A}$. The quotient category $\mathcal{U}/\mathcal{A}$ is also additive.

In [22], the following results are shown.

- Any additive category $\mathcal{A}$ admits an embedding in an $\mathcal{A}$ -filtered flasque additive category.
- For any flasque additive category $\mathcal{U}$, $K\mathcal{U}$ is contractible, where $K\mathcal{U}$ denotes the Quillen K -theory space of $\mathcal{U}$.
- For any semisimple, idempotent complete additive category $\mathcal{A}$ and any embedding of $\mathcal{A}$ into an $\mathcal{A}$ -filtered additive category $\mathcal{U}$, we obtain a homotopy fibration sequence

$$K\mathcal{A} \rightarrow K\mathcal{U} \rightarrow K\mathcal{U}/\mathcal{A}$$

It now follows that if we have an embedding $\mathcal{A} \rightarrow \mathcal{U}$ of additive categories, where $\mathcal{A}$ is semisimple and idempotent complete, and $\mathcal{U}$ is flasque, then $K\mathcal{U}/\mathcal{A}$ is a delooping of the K -theory space $K\mathcal{A}$. By applying the idempotent completion construction to $\mathcal{U}/\mathcal{A}$, one can iterate this construction to obtain a non-connective family of deloopings and therefore a non-connective spectrum. This delooping is equivalent to the Gersten–Wagoner deloopings of the last section and to the Pedersen–Weibel deloopings to be described in the next section. Finally, we note that M. Schlichting (see [25]) has constructed a version of the deloopings discussed in this section which applies to any idempotent complete exact category.

The Pedersen–Weibel Delooping and Bounded K -Theory

1.7

In this final section we will discuss a family of deloopings which were constructed by Pedersen and Weibel in [22] using the ideas of “bounded topology”. This work is based on much earlier work in high-dimensional geometric topology, notably by E. Connell [10]. The idea is to consider categories of possibly infinitely generated free modules over a ring A , equipped with a basis, and to suppose further that elements in the basis lie in a metric space X . One puts restrictions on both the objects and the morphisms, i.e. the modules have only finitely many basis elements in any given ball, and morphisms have the property that they send basis elements to linear combinations of “nearby elements”. When one applies this construction to the metric spaces $\mathbb{R}^n$, one obtains a family of deloopings of the K -theory spectrum of A . The construction has seen application in other problems as well, when applied to other metric spaces, such as the universal cover of a $K(\pi, 1)$ -manifold, or a finitely generated group Γ with word length metric attached to a generating set for Γ . In that context, the method has been applied to prove the so-called Novikov conjecture and its algebraic K -theoretic analogue in a number of interesting cases ([6, 7], and [8]). See [19] for a complete account of the status of this conjecture. This family of deloopings is in general non-connective, like the Gersten–Wagoner delooping, and produces a homotopy equivalent spectrum.

We begin with the construction of the categories in question.

30

Definition 30 Let A denote a ring, and let X be a metric space. We define a category $\mathcal{C}_X(A)$ as follows.

- The objects of $\mathcal{C}_X(A)$ are triples (F, B, φ) , where F is a free left A -module (not necessarily finitely generated), B is a basis for F , and $\varphi : B \rightarrow X$ is a function so that for every $x \in X$ and $R \in [0, +\infty)$, the set $\varphi^{-1}(B_R(x))$ is finite, where $B_R(x)$ denotes the ball of radius R centered at x .
- Let $d \in [0, +\infty)$, and let (F, B, φ) and (F', B', φ') denote objects of $\mathcal{C}_X(A)$. Let $f : F \rightarrow F'$ be a homomorphism of A -modules. We say f is bounded with bound d if for every $\beta \in B$, $f\beta$ lies in the span of $\varphi'^{-1}B_d(\varphi(x)) = \{\beta' \mid d(\varphi(\beta), \varphi'(\beta')) \leq d\}$. The morphisms in $\mathcal{C}_X(A)$ from (F, B, φ) to (F', B', φ') are the A -linear homomorphisms which are bounded with some bound d .

It is now easy to observe that $i\mathcal{C}_X(A)$, the category of isomorphisms in $\mathcal{C}_X(A)$, is a symmetric monoidal category, and so the construction of section 1.2 allows us to construct a spectrum $Sp(i\mathcal{C}_X(A))$. Another observation is that for metric spaces X and Y , we obtain a tensor product pairing $i\mathcal{C}_X(A) \times i\mathcal{C}_Y(B) \rightarrow i\mathcal{C}_{X \times Y}(A \otimes B)$, and a corresponding pairing of spectra $Sp(i\mathcal{C}_X(A)) \wedge Sp(i\mathcal{C}_Y(B)) \rightarrow Sp(i\mathcal{C}_{X \times Y}(A \otimes B))$ (see [21]). We recall from section 1.2 that for any symmetric monoidal category $\mathcal{C}$, there is a canonical map

$$N.C \rightarrow Sp(C)_0$$

where $Sp(C)_0$ denotes the zero-th space of the spectrum $Sp(C)$. In particular, if f is an endomorphism of any object in C , f determines an element in $\pi_1(Sp(C))$. Now consider the case $X = \mathbb{R}$. For any ring A , let M_A denote the object $(F_A(\mathbb{Z}), \mathbb{Z}, i)$, where $i \hookrightarrow \mathbb{R}$ is the inclusion. Let σ_A denote the automorphism of M_A given on basis elements by $\sigma_A([n]) = [n + 1]$. σ_A determines an element in $\pi_1 Sp(iC_{\mathbb{R}}(A))$. Therefore we have maps of spectra

$$\begin{aligned} \Sigma Sp(iC_{\mathbb{R}^n}(A)) &\cong S^1 \wedge Sp(iC_{\mathbb{R}^n}(A)) \rightarrow Sp(iC_{\mathbb{R}}(\mathbb{Z})) \wedge Sp(iC_{\mathbb{R}^n}(A)) \rightarrow \\ &\rightarrow Sp(iC_{\mathbb{R} \times \mathbb{R}^n}(\mathbb{Z} \otimes A)) \rightarrow Sp(iC_{\mathbb{R}^{n+1}}(\mathbb{Z} \otimes A)) \cong Sp(iC_{\mathbb{R}^{n+1}}(A)) \end{aligned}$$

and therefore adjoint maps of spaces

$$Sp(iC_{\mathbb{R}^n}(A))_0 \rightarrow \Omega(Sp(iC_{\mathbb{R}^{n+1}}(A)))_0$$

Assembling these maps together gives the Pedersen–Weibel spectrum attached to the ring A , which we denote by $\mathcal{K}(A)$. Note that we may also include a metric space X as a factor, we obtain similar maps

$$Sp(iC_{X \times \mathbb{R}^n}(A))_0 \rightarrow \Omega(Sp(iC_{X \times \mathbb{R}^{n+1}}(A)))_0$$

We will denote this spectrum by $\mathcal{K}(X; A)$, and refer to it as the *bounded K-theory spectrum of X with coefficients in the ring A* . A key result concerning $\mathcal{K}(X; A)$ is the following excision result (see [8]).

Proposition 31 Suppose that the metric space X is decomposed as a union $X = Y \cup Z$. For any subset $U \subseteq X$, and any $r \in [0, +\infty)$, we let $N_r U$ denote r -neighborhood of U in X . We consider the diagram of spectra

31

$$\operatorname{colim}_r Sp(iC_{N_r Y}(A)) \leftarrow \operatorname{colim}_r Sp(iC_{N_r Y \cap N_r Z}(A)) \rightarrow \operatorname{colim}_r Sp(iC_{N_r Z}(A))$$

and let $\mathbf{P}$ denote its pushout. Then the evident map $\mathbf{P} \rightarrow Sp(iC_X(A))$ induces an isomorphism on π_i for $i > 0$. It now follows that if we denote by $\mathcal{P}$ the pushout of the diagram of spectra

$$\operatorname{colim}_r \mathcal{K}(N_r Y; A) \leftarrow \operatorname{colim}_r \mathcal{K}(N_r Y \cap N_r Z; A) \rightarrow \operatorname{colim}_r \mathcal{K}(N_r Z; A)$$

then the evident map $\mathcal{P} \rightarrow \mathcal{K}(X; A)$ is an equivalence of spectra.

Remark 32 The spectra $\operatorname{colim}_r \mathcal{K}(N_r Y; A)$ and $\operatorname{colim}_r \mathcal{K}(N_r Z; A)$ are in fact equivalent to the spectra $\mathcal{K}(Y; A)$ and $\mathcal{K}(Z; A)$ as a consequence of the coarse invariance property for the functor $\mathcal{K}(-; A)$ described below.

32

Using the first part of 31, Pedersen and Weibel now prove the following properties of their construction.

- The homotopy groups $\pi_i \mathcal{K}(A)$ agree with Quillen's groups for $i \geq 0$.
- For $i < 0$ the groups $\pi_i \mathcal{K}(A)$ agree with Bass's lower K -groups. In particular, they vanish for regular rings.
- $\mathcal{K}(A)$ is equivalent to the Gersten–Wagoner spectrum.

The Pedersen–Weibel spectrum is particularly interesting because of the existence of the spectra $\mathcal{K}(X; A)$ for metric spaces X other than $\mathbb{R}^n$. This construction is quite useful for studying problems in high dimensional geometric topology. The spectrum $\mathcal{K}(X; A)$ has the following properties.

- **(Functoriality)** $\mathcal{K}(-; A)$ is functorial for *proper eventually continuous* map of metric spaces. A map of $f : X \rightarrow Y$ is said to be proper if for any bounded set $U \in Y$, $f^{-1}U$ is a bounded set in X . The map f is said to be eventually continuous if for every $R \in [0, +\infty)$, there is a number $\delta(R)$ so that $d_X(x_1, x_2) \leq R \Rightarrow d_Y(fx_1, fx_2) \leq \delta(R)$.
- **(Homotopy Invariance)** If $f, g : X \rightarrow Y$ are proper eventually continuous maps between metric spaces, and so that $d(f(x), g(x))$ is bounded for all x , then the maps $\mathcal{K}(f; A)$ and $\mathcal{K}(g; A)$ are homotopic.
- **(Coarse invariance)** $\mathcal{K}(X; A)$ depends only on the *coarse type* of X , i.e. if $Z \subseteq X$ is such that there is an $R \in [0, +\infty)$ so that $N_R Z = X$, then the map $\mathcal{K}(Z; A) \rightarrow \mathcal{K}(X; A)$ is an equivalence of spectra. For example, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces an equivalence on $\mathcal{K}(-; A)$. The spectrum $\mathcal{K}(-; A)$ does not “see” any local topology, only “topology at infinity”.
- **(Triviality on bounded spaces)** If X is a bounded metric space, then $\mathcal{K}(X; A) \cong \mathcal{K}(A)$.

To show the reader how this $\mathcal{K}(X; A)$ behaves, we first remind him/her about locally finite homology. Recall that the singular homology of a space X is defined to be the homology of the singular complex, i.e. the chain complex $C_* X$, with $C_k X$ denoting the free abelian group on the set of singular k -simplices, i.e. continuous maps from the standard k -simplex $\Delta[k]$ into X . This means that we are considering *finite* formal linear combinations of singular k -simplices.

33

Definition 33 Let X denote a locally compact topological space. We define $C_k^{lf} X$ to be the infinite formal linear combinations of singular k -simplices $\sum n_\sigma \sigma$, which have the property that for any compact set K in X , there are only finitely many σ with $im(\sigma) \cap K \neq \emptyset$, and $n_\sigma \neq 0$. The groups $C_k^{lf} X$ fit together into a chain complex, whose homology is denoted by $H_*^{lf} X$. The construction H_*^{lf} is functorial with respect to proper continuous maps, and is proper homotopy invariant.

34

Remark 34 H_*^{lf} is formally dual to cohomology with compact supports.

Example 35. $H_*^{lf} \mathbb{R}^n$ vanishes for $* \neq n$, and $H_n^{lf} \mathbb{R}^n \cong \mathbb{Z}$.

Example 36. If X is compact, $H_*^{lf} X \cong H_* X$.

Example 37. Suppose that X is the universal cover of a bouquet of two circles, so it is an infinite tree. It is possible to compactify X by adding a Cantor set onto X . The Cantor set can be viewed as an inverse system of spaces $\underline{C} = \dots C_n \rightarrow C_{n-1} \rightarrow \dots$, and we have $H_*^{lf} X \cong 0$ for $* \neq 1$, and $H_1^{lf} X \cong \varprojlim \mathbb{Z}[C_n]$.

Example 38. For any manifold with boundary $(X, \partial X)$, $H_*^{lf}(X) \cong H_*(X, \partial X)$.

A variant of this construction occurs when X is a metric space.

Definition 39 Suppose that X is a *proper metric space*, i.e. that all closed balls are compact. We now define a subcomplex ${}^s C_*^{lf} X \subseteq C_*^{lf} X$ by letting ${}^s C_k^{lf} X$ denote the infinite linear combinations $\sum_{\sigma} n_{\sigma} \sigma \in C_k^{lf} X$ so that the set $\{diam(im(\sigma)) | n_{\sigma} \neq 0\}$ is bounded above. Informally, it consists of linear combinations of singular simplices which have images of uniformly bounded diameter. We denote the corresponding homology theory by ${}^s H_*^{lf} X$. There is an evident map ${}^s H_*^{lf} X \rightarrow H_*^{lf} X$, which is an isomorphism in this situation, i.e. when X is proper.

39

In order to describe the relationship between locally finite homology and bounded K -theory, we recall that spectra give rise to *generalize homology theories* as follows. For any spectrum $\mathcal{S}$ and any based space X , one can construct a new spectrum $X \wedge \mathcal{S}$, which we write as $h(X, \mathcal{S})$. Applying homotopy groups, we define the *generalized homology groups* of the space X with coefficients in $\mathcal{S}$, $h_i(X, \mathcal{S}) = \pi_i h(X, \mathcal{S})$. The graded group $h_*(X, \mathcal{S})$ is a generalized homology theory in X , in that it satisfies all of the Eilenberg–Steenrod axioms for a homology theory except the dimension hypothesis, which asserts that $h_i(S^0, \mathcal{S}) = 0$ for $i \neq 0$ and $h_0(X, \mathcal{S}) = \mathbb{Z}$. In this situation, when we take coefficients in the Eilenberg–MacLane spectrum for an abelian group A , we obtain ordinary singular homology with coefficients in A . It is possible to adapt this idea for the theories H_*^{lf} and ${}^s H_*^{lf}$.

Proposition 40 (See [8]) Let $\mathcal{S}$ be any spectrum. Then there are spectrum valued functors $h^{lf}(-, \mathcal{S})$ and ${}^s h^{lf}(-, \mathcal{S})$, so that the graded abelian group valued functors $\pi_* h^{lf}(-, \mathcal{S})$ and $\pi_* {}^s h^{lf}(-, \mathcal{S})$ agree with the functors $H_*^{lf}(-, A)$ and ${}^s H_*^{lf}(-, A)$ defined above in the case where $\mathcal{S}$ denotes the Eilenberg–MacLane spectrum for A .

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The relationship with bounded K -theory is now given as follows.

41 Proposition 41 There is a natural transformation of spectrum valued functors

$$\alpha^R(-) : {}^s h^{\text{lf}}(-, \mathcal{K}(R)) \rightarrow \mathcal{K}(-; R)$$

which is an equivalence for discrete metric spaces. (The constructions above extend to metric spaces where the distance function is allowed to take the value $+\infty$. A metric space X is said to be *discrete* if $x_1 \neq x_2 \Rightarrow d(x_1, x_2) = +\infty$.)

The value of this construction is in its relationship to the K -theoretic form of the Novikov conjecture. We recall ([8]) that for any group Γ , we have the assembly map $A_\Gamma^R : h(B\Gamma_+, \mathcal{K}(R)) \rightarrow \mathcal{K}(R[\Gamma])$, and the following conjecture.

42 Conjecture 42: (*Integral K -theoretic Novikov conjecture for Γ*) A_Γ induces a split injection on homotopy groups.

43 Remark 43 This conjecture has attracted a great deal of interest due to its relationship with the original Novikov conjecture, which makes the same assertion after tensoring with the rational numbers, and using the analogous statement for L -theory. Recall that L -theory is a quadratic analogue of K -theory, made periodic, which represents the obstruction to completing non simply connected surgery. The L -theoretic version is also closely related to the Borel conjecture, which asserts that two homotopy equivalent closed $K(\Gamma, 1)$ -manifolds are homeomorphic. This geometric consequence would require that we prove an isomorphism statement for A_Γ rather than just an injectivity statement.

We now describe the relationship between the locally finite homology, bounded K -theory, and Conjecture 42. We recall that if X is any metric space, and $d \geq 0$, then the *Rips complex for X with parameter d* , $R[d](X)$, is the simplicial complex whose vertex set is the underlying set of X , and where $\{x_0, x_1, \dots, x_k\}$ spans a k -simplex if and only if $d(x_i, x_j) \leq d$ for all $0 \leq i, j \leq k$. Note that we obtain a directed system of simplicial complexes, since $R[d] \subseteq R[d']$ when $d \leq d'$. We say that a metric space is *uniformly finite* if for every $R \geq 0$, there is an N so that for every $x \in X$, $\#B_R(x) \leq N$. We note that if X is uniformly finite, then each of the complexes $R[d](X)$ is locally finite and finite dimensional. If X is a finitely generated discrete group, with word length metric associated to a finite generating set, then X is uniformly finite. Also, again if $X = \Gamma$, with Γ finitely generated, Γ acts on the right of $R[d](X)$, and the orbit space is homeomorphic to a finite simplicial complex. It may be necessary to subdivide $R[d](X)$ for the orbit space to be a simplicial complex. This Γ -action is free if Γ is torsion free. Further, $R[\infty](\Gamma) = \bigcup_d R[d](\Gamma)$ is contractible, so when Γ is torsion free, $R[\infty](\Gamma)/\Gamma$ is a model for the classifying space $B\Gamma$. The complex $R[d](X)$ is itself

equipped with a metric, namely the path length metric. For a uniformly finite metric space X and spectrum $\mathcal{S}$, we now define a new spectrum valued functor $\mathcal{E}(X, \mathcal{S})$ by

$$\mathcal{E}(X, \mathcal{S}) = \operatorname{colim}_d {}^s h^{lf}(R[d](X), \mathcal{S})$$

Note that without the uniform finiteness hypothesis, the spaces $R[d](X)$ would not be locally compact. We may now apply the assembly map $\alpha^R(-)$ to obtain a commutative diagram

$$\begin{array}{ccc} \begin{array}{c} \vdots \\ \downarrow \\ {}^s h^{lf}(R[d](X), \mathcal{K}(R)) \\ \downarrow \\ {}^s h^{lf}(R[d+1](X), \mathcal{K}(R)) \\ \downarrow \\ \vdots \end{array} & \xrightarrow{\alpha^R(R[d](X)/)} & \begin{array}{c} \vdots \\ \downarrow \\ \mathcal{K}(R[d](X), R) \\ \downarrow \\ \mathcal{K}(R[d+1](X), R) \\ \downarrow \\ \vdots \end{array} \end{array}$$

which yields a natural transformation

$$\alpha_{\mathcal{E}}^R(X) : \mathcal{E}(X, \mathcal{K}(R)) \rightarrow \operatorname{colim}_d \mathcal{K}(R[d](X), R)$$

It follows directly from the coarse invariance property of $\mathcal{K}(-, R)$ that the natural inclusion $\mathcal{K}(X, R) \rightarrow \operatorname{colim}_d \mathcal{K}(R[d](X), R)$ is an equivalence of spectra, and by abuse of notation we regard $\alpha_{\mathcal{E}}^R(X)$ as a natural transformation from $\mathcal{E}(X, \mathcal{K}(R))$ to $\mathcal{K}(X, R)$.

Theorem 44 ([8]) Let Γ be a finitely generated group, with finite classifying space, and let Γ be regarded as a metric space via the word length metric associated to any finite generating set. If $\alpha_{\mathcal{E}}^R(\Gamma)$ is an equivalence, then the K -theoretic Novikov conjecture holds for the group Γ and the coefficient ring R .

44

The value of a theorem of this type is that $\pi_* {}^s h^{lf}(X, \mathcal{K}(R))$ has many good properties, including an excision property. It does not involve the intricacies present in the (complicated) group ring $R[\Gamma]$, which makes it difficult to deal with the algebraic K -theory of this group ring directly. Two important advantages of this method are as follows.

- Experience shows that it generally works equally well for the case of L -theory, which is the case with direct geometric consequences.
- This method produces integral results. As such, it has the potential to contribute directly to the solution of the Borel conjecture.

We give an outline of the proof. A first observation is that there is an equivariant version of the construction $\mathcal{K}(-, R)$, which when applied to the metric space Γ with left action by the group Γ produces a spectrum $\mathcal{K}^\Gamma(\Gamma, R)$ with Γ -action, which is equivalent to the usual (non-equivariant) spectrum $\mathcal{K}(\Gamma, R)$, and whose fixed point set is the K -theory spectrum $\mathcal{K}(R[\Gamma])$. Next recall that for any space (or spectrum) X which is acted on by a group Γ , we may define the homotopy fixed point space (or spectrum) $X^{h\Gamma}$ to be the space (or spectrum) $F^\Gamma(E\Gamma, X)$ of equivariant maps from $E\Gamma$ to X , where $E\Gamma$ denotes a contractible space on which Γ acts trivially. The homotopy fixed set $X^{h\Gamma}$ has the following properties.

- The construction $X \rightarrow X^{h\Gamma}$ is functorial for maps of Γ -spaces (spectra).
- There is a map $X^\Gamma \rightarrow X^{h\Gamma}$, which is natural for Γ -equivariant maps, where X^Γ denotes the fixed point space (spectrum).
- Suppose that $f : X \rightarrow Y$ is an equivariant map of Γ -spaces (spectra), which is a weak equivalence as a non-equivariant map. Then the natural map $X^{h\Gamma} \rightarrow Y^{h\Gamma}$ is also a weak equivalence.
- For groups with finite classifying spaces, the functor $(-)^{h\Gamma}$ commutes with arbitrary filtering colimits.

In order to apply these facts, we will also need to construct an equivariant version of $\mathcal{E}(X, \mathcal{S})$. The facts concerning this construction are as follows.

- It is possible to construct an equivariant version of the functor on proper metric spaces $X \rightarrow {}^s h^{lf}(X, \mathcal{S})$, whose fixed point spectrum is equivalent to ${}^s h^{lf}(X/\Gamma, \mathcal{S})$. Such a construction yields naturally an equivariant version of the functor $\mathcal{E}(X, \mathcal{S})$.
- When X is a locally finite, finite dimensional simplicial complex with free simplicial Γ action, then the fixed point spectrum of the action of Γ on the equivariant model is ${}^s h^{lf}(X/\Gamma, \mathcal{S})$. In particular, we find that for a uniformly finite metric space X , $\mathcal{E}(X, \mathcal{S})^\Gamma$ is equivalent to

$$\operatorname{colim}_d h^{lf}(R[d](X)/\Gamma, \mathcal{S})$$

When $X = \Gamma$, equipped with a word length metric, and Γ is torsion free, we find that since $R[d](\Gamma)/\Gamma$ is a finite simplicial complex, we have

$$\mathcal{E}(\Gamma, \mathcal{S})^\Gamma \cong \operatorname{colim}_d h^{lf}(R[d](\Gamma)/\Gamma, \mathcal{S}) \cong \operatorname{colim}_d h(R[d](X)/\Gamma, \mathcal{S})$$

$$\cong h\left(\operatorname{colim}_d R[d](X)/\Gamma, \mathcal{S}\right) \cong h(B\Gamma, \mathcal{S})$$

- The assembly map A_Γ^R is the map obtained by restricting $\alpha_\mathcal{E}^R(\Gamma)$ to fixed point sets.
- Suppose Γ is a discrete group, and X is a finite dimensional simplicial complex equipped with a free simplicial Γ -action, with only a finite number of orbits in each simplicial dimension. Then $h^{lf}(X, \mathcal{S})^\Gamma \simeq h^{lf}(X, \mathcal{S})^{h\Gamma} \cong h(X/\Gamma, \mathcal{S})$, and

similarly for ${}^sh^f$. Therefore, when Γ is torsion free, so that $R[d](\Gamma)$ is a filtering direct system of such complexes, we have

$$\mathcal{E}(\Gamma, \mathcal{S})^\Gamma \cong \mathcal{E}(\Gamma, \mathcal{S})^{h\Gamma}$$

In order to prove Theorem 44 from these facts, we consider the following diagram.

$$\begin{array}{ccc} h(B\Gamma, \mathcal{K}R) \cong \mathcal{E}(\Gamma, \mathcal{K}R)^\Gamma & \xrightarrow{(\alpha_\mathcal{E}^R)^\Gamma = A_\Gamma^R} & \mathcal{K}(R[\Gamma]) \cong \mathcal{K}(\Gamma, R)^\Gamma \\ \downarrow & & \downarrow \\ \mathcal{E}(\Gamma, \mathcal{K}R)^{h\Gamma} & \xrightarrow{\quad\quad\quad} & \mathcal{K}(\Gamma, R)^{h\Gamma} \end{array}$$

We wish to prove that the upper horizontal arrow is the inclusion of a spectrum summand. To prove this, it will suffice that the composite to the lower right hand corner is an equivalence. From the discussion above, it follows that the left hand vertical arrow is an equivalence. the hypothesis of Theorem 44 shows that the map $\mathcal{E}(\Gamma, \mathcal{K}R) \rightarrow \mathcal{K}(\Gamma, R)$ is a weak equivalence. It now follow from the properties of homotopy fixed points enumerated above the composite $h(B\Gamma, \mathcal{K}R) \rightarrow \mathcal{K}(\Gamma, R)^{h\Gamma}$ is a weak equivalence. It now follows that the map $h(B\Gamma, \mathcal{K}R) \rightarrow \mathcal{K}(R[\Gamma])$ is the inclusion on a wedge product of spectra.

Finally, we wish to give an indication about how one can prove that the hypothesis of Theorem 44 holds for some particular groups. In order to do this, we need to formulate a reasonable excision property for bounded K -theory. By a covering of a metric space X , we will mean a family of subsets $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of X so that $X = \bigcup_\alpha U_\alpha$. We will say that the covering has covering dimension $\leq d$ if whenever $\alpha_0, \alpha_1, \dots, \alpha_k$ are distinct elements of A , with $k > d$, then $U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_k} = \emptyset$. For $R \geq 0$, we say that a covering $\mathcal{U}$ is R -lax d -dimensional if the covering $N_R \mathcal{U} = \{N_R U\}_{U \in \mathcal{U}}$ is d -dimensional. We also say that a covering $\mathcal{V}$ refines $\mathcal{U}$ if and only if every element of $\mathcal{V}$ is contained in an element of $\mathcal{U}$.

Definition 45 An *asymptotic covering* of dimension $\leq d$ of a metric space X is a family of coverings $\mathcal{U}_n$ of X satisfying the following properties.

45

- $\mathcal{U}_i$ refines $\mathcal{U}_{i+1}$ for all i .
- $\mathcal{U}_i$ is R_i -lax d -dimensional, where $R_i \rightarrow +\infty$.

Bounded K -theory has an excision property for one-dimensional asymptotic coverings. We first recall that the bounded K -theory construction can accept as input metric spaces in which the value $+\infty$ is an allowed value of the metric, where points x, y so that $d(x, y) = +\infty$ are understood to be “infinitely far apart”. Suppose that we have a family of metric spaces X_α . Then we define $\coprod_\alpha X_\alpha$ to be the metric space

whose underlying set is the disjoint union of the X_α 's, and where the metric is given by $d(x, y) = d_\alpha(x, y)$ when $x, y \in X_\alpha$, and where $d(x, y) = +\infty$ when $x \in X_\alpha$, $y \in X_\beta$, and $\alpha \neq \beta$. Consider a one-dimensional asymptotic covering of X , and for each i and each set $U \in \mathcal{U}_i$, select a set $\Theta(U) \in \mathcal{U}_{i+1}$ so that $U \subseteq \Theta(U)$. For each i , we now construct the two metric spaces

$$N_0(i) = \coprod_{U \in \mathcal{U}_i} U$$

and

$$N_1(i) = \coprod_{(U, V) \in \mathcal{U}_i \times \mathcal{U}_i, U \cap V \neq \emptyset} U \cap V$$

There are now two maps of metric spaces $d_0^i, d_1^i : N_1(i) \rightarrow N_0(i)$, one induced by the inclusions $U \cap V \hookrightarrow U$ and the other induced by the inclusions $U \cap V \hookrightarrow V$. Recall that for any pair of maps $f, g : X \rightarrow Y$, we may construct the *double mapping cylinder* $Dcyl(f, g)$ as the quotient

$$X \times [0, 1] \coprod Y / \simeq$$

where $\simeq$ is generated by the relations $(x, 0) \simeq f(x)$ and $(x, 1) \simeq g(x)$. This construction has an obvious extension to a spectrum level construction, and so we can construct $Dcyl(d_0^i, d_1^i)$ for each i . Moreover, the choices $\Theta(U)$ give us maps

$$Dcyl(d_0^i, d_1^i) \rightarrow Dcyl(d_0^{i+1}, d_1^{i+1})$$

for each i . Furthermore, for each i , we obtain a map

$$\lambda_i : Dcyl(d_0^i, d_1^i) \rightarrow \mathcal{K}(X, R)$$

which on the metric spaces $N_0(i)$ and $N_1(i)$ is given by inclusions on the factors U and $U \cap V$ respectively. The excision result for bounded K -theory which we require is now the following.

46 Theorem 46 The maps λ_i determine a map of spectra

$$\Lambda : \operatorname{colim}_i Dcyl(d_0^i, d_1^i) \rightarrow \mathcal{K}(X, R)$$

which is a weak equivalence of spectra.

Instead of applying $\mathcal{K}(-, R)$ to the diagram

$$\begin{array}{ccccccc} & \longrightarrow & N_1(i) & \longrightarrow & N_1(i+1) & \longrightarrow & \\ & & \downarrow d_0^i, d_1^i & & \downarrow d_0^{i+1}, d_1^{i+1} & & \\ & \longrightarrow & N_0(i) & \longrightarrow & N_0(i+1) & \longrightarrow & \end{array}$$

we apply $\mathcal{E}(\mathcal{K}R)$ to it. We obtain double mapping cylinders $D\text{cyl}_{\mathcal{E}}(d_0^i, d_1^i)$, maps $\lambda_{i,\mathcal{E}} : D\text{cyl}_{\mathcal{E}}(d_0^i, d_1^i) \rightarrow \mathcal{E}(X, \mathcal{K}R)$, and finally a map

$$\Lambda_{\mathcal{E}} : \text{colim}_i D\text{cyl}_{\mathcal{E}}(d_0^i, d_1^i) \rightarrow \mathcal{E}(X, \mathcal{K}R).$$

Proposition 47 The map $\Lambda_{\mathcal{E}}$ is a weak equivalence of spectra.

47

Due to the naturality of the constructions, we now conclude the following.

Corollary 48 Suppose that we have an asymptotic covering of a metric space X of dimension d , and suppose that the maps

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$$\alpha_{\mathcal{E}}^R(N_0(i)) \text{ and } \alpha_{\mathcal{E}}^R(N_i(i))$$

are weak equivalences of spectra for all i . Then $\alpha_{\mathcal{E}}^R(X)$ is a weak equivalence of spectra.

This result is a useful induction result. We need an absolute statement for some family of metric spaces, though. We say that a metric space X is *almost discrete* if there is a number R so that for all $x, y \in X$, $d(x, y) \geq R \Rightarrow d(x, y) = +\infty$.

Theorem 49 If X is almost discrete, then $\alpha_{\mathcal{E}}^R(X)$ is a weak equivalence of spectra.

49

The proof of this theorem relies on an analysis of the K -theory of infinite products of categories with cofibrations and weak equivalences, which is given in [9]. An iterated application of Corollary 48 now gives the following result.

Theorem 50 Suppose that X is a metric space, and that we have a finite family of asymptotic coverings $\mathcal{U}^j = \{\mathcal{U}_k^j\}_{k \in A_i^j}$ of dimension 1, with $1 \leq j \leq N$. Suppose further that for each i , there is an R_i so that for any family of elements $V_j \in \mathcal{U}_i^j$, the intersection $\cap_j V_j$ has diameter bounded by R_i . Then $\alpha_{\mathcal{E}}^R(X)$ is a weak equivalence of spectra.

50

Remark 51 The existence of asymptotic coverings of this form can be verified in many cases. For instance, in [8], it is shown that such coverings exist for the homogeneous space G/K , where G is a Lie group, and K is a maximal compact subgroup. This gives the result for torsion free, cocompact subgroups of Lie groups. In the case of the real line, such a family of coverings can be given by the family of coverings $\mathcal{U}_i = \{[2^i k, 2^i(k+1)]\}_{k \in \mathbb{Z}}$. By taking product hypercubes, one obtains similar asymptotic coverings for Euclidean space. It can be shown that similar coverings exist for trees, and therefore it follows that one can construct a finite

51

family of asymptotic coverings satisfying the hypotheses of Theorem 50 for any finite product of trees, and therefore for subspaces of products of trees.

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The Motivic Spectral Sequence*

I.2

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2.1	<i>Introduction</i>	40
2.2	<i>Algebraic K-Theory and Cohomology</i>	40
2.3	<i>Topological K-Theory and Cohomology</i>	42
2.4	<i>The Motivic Spectral Sequence</i>	44
2.5	<i>Filtrations as a Source of Spectral Sequences</i>	45
2.6	<i>Commuting Automorphisms</i>	47
2.7	<i>Cancellation and Comparison with Motivic Cohomology</i>	53
2.8	<i>Higher Chow Groups and a Motivic Spectral Sequence</i>	58
2.9	<i>Extension to the Global Case</i>	61
2.10	<i>The Slice Filtration</i>	63
2.11	<i>Filtrations for General Cohomology Theories</i>	64
	<i>References</i>	66

* This work has been supported by NSF grant DMS 99-70085.

Summary. We give an overview of the search for a motivic spectral sequence: a spectral sequence connecting algebraic K -theory to motivic cohomology that is analogous to the Atiyah–Hirzebruch spectral sequence that connects topological K -theory to singular cohomology.

2.1

Introduction

In this chapter we explain the Atiyah–Hirzebruch spectral sequence that relates topological K -theory to singular cohomology and try to motivate the search for a motivic version. In the time since [18] appeared, which concerns motivation for such a motivic spectral sequence, many authors have produced results in this direction. We describe the Bloch–Lichtenbaum spectral sequence [8] for the spectrum of a field together with the Friedlander–Suslin and Levine extensions [12, 28] to the global case for a smooth variety over a field. We explain the Goodwillie–Lichtenbaum idea involving tuples of commuting automorphisms and the theorem [19] that uses it to produce a motivic spectral sequence for an affine regular noetherian scheme, unfortunately involving certain non-standard motivic cohomology groups. We present Suslin’s result [41], that, for smooth varieties over a field, these non-standard motivic cohomology groups are isomorphic to the standard groups. We sketch Voevodsky’s approach via the slice filtration [43, 47, 49], much of which remains conjectural. Finally, we sketch Levine’s recent preprint [29], which gives a novel approach that yields a spectral sequence for smooth varieties over a field and makes it extremely clear which formal properties of K -theory are used in the proof. At this point we refer the reader also to [11] where a similar spectral sequence is developed for semi-topological K -theory.

The importance of the motivic spectral sequence lies in its applications. Important work of Voevodsky [42, 44, 48] makes motivic cohomology amenable to computation, and the motivic spectral sequence is the route by which those computations can be used to compute algebraic K -groups. For such applications, see, for example, [22, 24, 34–36, 39, 40, 51].

The main open question now seems to be how to handle a general noetherian regular scheme, such as those that arise in number theory. In all the papers cited, except for [18], the strongest results are true only for smooth varieties over a field.

Many fine papers have been written on this topic – any difference in the depths to which I manage to expose them are due more to personal limitations of time and ability than to any judgment of their relative importance.

2.2

Algebraic K -Theory and Cohomology

The Riemann–Roch theorem for a complete nonsingular algebraic curve X over a field k relates the degree of a line bundle $\mathcal{L}$ to the dimension of its space $\Gamma(X, \mathcal{L})$ of global sections. The functor $\Gamma(X, -)$ is not an exact functor, so what really

enters into the theorem (on one side of an equation) is the Euler characteristic, defined by $\chi(\mathcal{L}) := \sum_i (-1)^i \dim H^i(X, \mathcal{L})$. The Euler characteristic is *additive* in the sense that $\chi(\mathcal{E}) = \chi(\mathcal{E}') + \chi(\mathcal{E}'')$ whenever $0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$ is an exact sequence of (locally free) coherent sheaves on X . The natural way to prove formulas relating one additive function on coherent sheaves to another is to work with the universal target for such additive functions, so Grothendieck defined $K_0(X)$ to be the abelian group generated by the isomorphism classes $[\mathcal{E}]$ of locally free coherent sheaves on X , with relations $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ for each exact sequence as before. Alternatively, one defines $K'_0(X)$ by using all coherent sheaves, not just the locally free ones. Tensor product of coherent sheaves makes $K_0(X)$ into a ring and $K'_0(X)$ into a module over it. For a nonsingular quasi-projective algebraic variety X , the natural map $K_0(X) \rightarrow K'_0(X)$ is an isomorphism, because a coherent sheaf has a resolution of finite length by locally free coherent sheaves.

The group $K'_0(X)$ has a filtration whose i -th member $F^i K'_0(X)$ is the subgroup generated by the classes of coherent sheaves whose support has codimension at least i . The ring $K_0(X)$ has a more complicated filtration with members $F^i_\gamma K_0(X)$ called the γ -filtration (formerly, the λ -filtration) [14, III, § 1], arising from a detailed consideration of the way exterior powers of vector bundles behave with respect to short exact sequences of bundles. Let $Gr^i_\gamma K_0(X)$ and $Gr^i K'_0(X)$ denote the associated graded groups. When X is a nonsingular quasi-projective algebraic variety the map $K_0(X) \rightarrow K'_0(X)$ respects the filtration [21, X 1.3.2] [14, VI 5.5], and the induced map $Gr^i_\gamma K_0(X)_\mathbb{Q} \rightarrow Gr^i K'_0(X)_\mathbb{Q}$ is an isomorphism [21, VII 4.11, X 1.3.2], where $(-)_{\mathbb{Q}}$ denotes tensoring with the field of rational numbers.

The Grothendieck group $K_0(X)$ was an essential tool in Grothendieck's proof of the Grothendieck–Riemann–Roch theorem [21], which extended the Riemann–Roch theorem for curves to nonsingular varieties of any dimension. The other important ingredient was the Chow ring. An *algebraic cycle* on X of codimension i is a formal linear combination of closed subvarieties Z (reduced and irreducible) of X of codimension i . The group of such cycles is denoted by $Z^i(X)$. Such cycles arise naturally when intersecting two subvarieties as a way of keeping track of the multiplicities with which the components of the intersection should be counted. Two algebraic cycles are called *linearly equivalent* if they are members of the same family parametrized algebraically by the points of the affine line $\mathbb{A}^1$. The automorphism group of a projective space contains plenty of straight lines, so linear equivalence of algebraic cycles allows pairs of cycles whose intersection doesn't have the maximal possible codimension to be moved to achieve that condition. Let $CH^i(X)$ denote the group of codimension i algebraic cycles on X modulo linear equivalence; it is the degree i component of a graded ring $CH(X)$ whose multiplication comes from intersection of cycles.

One consequence of the Grothendieck–Riemann–Roch theorem is that when X is a nonsingular quasi-projective algebraic variety, the algebraic cycles of codimension at least i account for all the classes of coherent sheaves of codimension at least i , up to torsion. More precisely, given an algebraic cycle, each component of it is a subvariety $Z \subseteq X$; the coherent sheaf $\mathcal{O}_Z$ on X gives a class $[\mathcal{O}_Z]$ in $K'_0(X)$. The resulting well-defined map $CH^i(X) \rightarrow Gr^i K'_0(X)$ induces an isomorphism

$CH^i(X)_{\mathbb{Q}} \xrightarrow{\cong} Gr^i K'_0(X)_{\mathbb{Q}}$ [21, XIV 4.2, IV 2.9]. The proof involves the use of Chern classes to construct an inverse map, and indeed, the Chern character, as defined by Grothendieck, gives the following isomorphism of rings [13, 15.2.16(b)].

$$\text{ch}: K_0(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_i CH^i(X)_{\mathbb{Q}} \quad (2.1)$$

In [37] Quillen defined, for $n \geq 0$, the higher algebraic K -group $K_n(X)$ for a variety X as the homotopy group $\pi_n K(X)$ of a certain topological space $K(X)$ constructed from the category of locally free coherent sheaves on X , and he defined $K'_n(X)$ for a variety X as the homotopy group $\pi_n K'(X)$ of a certain topological space $K'(X)$ constructed in the same way from the category of coherent sheaves on X . For a nonsingular quasi-projective algebraic variety X , the natural map $K_n(X) \rightarrow K'_n(X)$ is an isomorphism and both groups have filtrations analogous to those for K_0 and K'_0 . It follows from the main result in [16] that the map respects the filtrations in the sense that $F_j^i K_n(X)$ lands in $F^{i-n} K'_n(X)$ but the filtrations may disagree rationally. We may suspect such behavior by considering the spectrum of a field: it has topological dimension 0, but its étale cohomological dimension can be greater than 0, and the higher K -groups harbor elements whose Chern characters involve the higher cohomology groups.

An immediate question is whether there is an analogue of (2.1) for the higher K -groups that reflects the γ -filtration on $K_n(X)$.

$$\text{ch}: K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_i (?)_{\mathbb{Q}} \quad (2.2)$$

The abelian groups replacing the question mark should have an interesting structure in the sense that cognate groups should exist which handle torsion coefficient groups.

2.3

Topological K -Theory and Cohomology

In the late 1950's Atiyah and Hirzebruch combined Grothendieck's formalism with Bott's periodicity theorem to invent a generalized cohomology theory called topological K -theory. In this section we sketch the definition of topological K -theory and the relationship between it and singular cohomology provided by the Atiyah–Hirzebruch spectral sequence. The basic objects of study are finite cell complexes, and the spectral sequence arises from the skeletal filtration of a cell complex. We follow the discussion in [18]; see also [3].

Let X be a finite cell complex, and let $\mathbb{C}(X^{\text{top}})$ denote the topological ring of continuous functions $X \rightarrow \mathbb{C}$. Although this isn't the way it was originally envisioned, it turns out that there is a way [33] to take the topology of a ring into account when defining the algebraic K -groups, yielding the topological K -groups $K_n(X^{\text{top}}) := K_n(\mathbb{C}(X^{\text{top}}))$. Let $K(X^{\text{top}})$ denote the space obtained, so that

$K_n(X^{\text{top}}) = \pi_n K(X^{\text{top}})$. The space $K(X^{\text{top}})$ is naturally an infinite loop space, with the deloopings getting more and more connected.

Let $*$ denote the one point space. Bott computed the homotopy groups of $K(*^{\text{top}})$.

$$K_{2i}(*^{\text{top}}) = \pi_{2i} K(*^{\text{top}}) = \mathbb{Z}(i) \quad (2.3)$$

$$K_{2i-1}(*^{\text{top}}) = \pi_{2i-1} K(*^{\text{top}}) = 0 \quad (2.4)$$

We write $\mathbb{Z}(i)$ above to mean simply the group $\mathbb{Z}$; the annotation (i) is there simply to inform us that its elements are destined for the i -th stage in the weight filtration. The identification $\mathbb{Z}(1) = K_2(*^{\text{top}})$ can be decomposed as the following sequence of isomorphisms.

$$K_2(*^{\text{top}}) = \pi_2 BGL(\mathbb{C}^{\text{top}}) \cong \pi_2 BGL_1(\mathbb{C}^{\text{top}}) \cong \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}(1) \quad (2.5)$$

A generator for $\pi_1(\mathbb{C}^\times)$ gives a generator β for $K_2(*^{\text{top}})$. Bott's theorem includes the additional statement that multiplication by β gives a homotopy equivalence of spaces $K(*^{\text{top}}) \rightarrow \Omega^2 K(*^{\text{top}})$. This homotopy equivalence gives us a non-connected delooping $\Omega^{-2} K(*^{\text{top}})$ of $K(*^{\text{top}})$, which is $K(*^{\text{top}})$ itself. These deloopings can be composed to give deloopings of every order, and hence yields an Ω -spectrum called BU that has $K(*^{\text{top}})$ as its underlying infinite loop space $\Omega^\infty BU$, and whose homotopy group in dimension $2i$ is $\mathbb{Z}(i)$, for every integer i .

There is a homotopy equivalence of the mapping space $K(*^{\text{top}})^X$ with $K(X^{\text{top}})$ and from this it follows that $K_n(X^{\text{top}}) = [X_+, \Omega^\infty \Omega^n BU]$, where X_+ denotes X with a disjoint base point adjoined. When $n < 0$ there might be a bit of ambiguity about what we might mean when we write $K_n(X^{\text{top}})$; we let it always denote $[X_+, \Omega^\infty \Omega^n BU]$, so that $K_n(X^{\text{top}}) = K_{n+2}(X^{\text{top}})$ for all $n \in \mathbb{Z}$.

It is a theorem of Atiyah and Hirzebruch that the Chern character for topological vector bundles gives an the following isomorphism.

$$ch: K_n(X^{\text{top}})_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_i H^{2i-n}(X, \mathbb{Q}) \quad (2.6)$$

For $n = 0$ comparison of this formula with (2.1) shows us that $CH^i(X)_{\mathbb{Q}}$ (defined for a variety X) is a good algebraic analogue of $H^{2i}(X, \mathbb{Q})$ (defined for a topological space X).

The isomorphism (2.6) was obtained in [2] from a spectral sequence known as the Atiyah-Hirzebruch spectral sequence. One construction of the spectral sequence uses the skeletal filtration $\text{sk}_p X$ of X as follows. (Another one maps X into the terms of the Postnikov tower of BU , as we'll see in Sect. 2.5.)

A cofibration sequence $A \subseteq B \rightarrow B/A$ of pointed spaces and an Ω -spectrum E give rise to a long exact sequence $\cdots \rightarrow [A, \Omega^\infty \Omega^1 E] \rightarrow [B/A, \Omega^\infty E] \rightarrow [B, \Omega^\infty E] \rightarrow [A, \Omega^\infty E] \rightarrow [B/A, \Omega^\infty \Omega^{-1} E] \rightarrow \cdots$.

We introduce the following groups.

$$E_1^{pq} := [\text{sk}_p X / \text{sk}_{p-1} X, \Omega^\infty \Omega^{-p-q} BU] \quad (2.7)$$

$$D_1^{pq} := [(\text{sk}_p X)_+, \Omega^\infty \Omega^{-p-q} BU] \quad (2.8)$$

The long exact sequence provides an exact couple $\cdots \rightarrow D_1^{p-1,q} \rightarrow E_1^{pq} \rightarrow D_1^{pq} \rightarrow D_1^{p-1,q+1} \rightarrow \cdots$. The explicit computation of the homotopy groups of BU presented above, together with fact that the space $\mathrm{sk}_p X / \mathrm{sk}_{p-1} X$ is a bouquet of the p -cells from X , leads to the computation that

$$E_1^{pq} = \begin{cases} C^p(X, \mathbb{Z}(-q/2)) & \text{if } q \text{ is even} \\ 0 & \text{if } q \text{ is odd,} \end{cases} \quad (2.9)$$

where C^p denotes the group of cellular cochains. We will abbreviate this conclusion by regarding $\mathbb{Z}(-q/2)$ as zero when q is odd. The differential $d_1: E_1^{pq} \rightarrow E_1^{p+1,q}$ is seen to be the usual differential for cochains, so that $E_2^{pq} = H^p(X, \mathbb{Z}(-q/2))$. The exact couple gives rise to a convergent spectral sequence because X is a finite dimensional cell complex. The abutment is $[X_+, \Omega^\infty \Omega^{-p-q} BU] = K_{-p-q}(X^{\mathrm{top}})$, so the resulting spectral sequence may be displayed as follows.

$$E_2^{pq} = H^p(X, \mathbb{Z}(-q/2)) \Rightarrow K_{-p-q}(X^{\mathrm{top}}) \quad (2.10)$$

This spectral sequence is concentrated in quadrants I and IV, is nonzero only in the rows where q is even, and is periodic with respect to the translation $(p, q) \mapsto (p, q - 2)$. Using the Chern character Atiyah and Hirzebruch show that the differentials in this spectral sequence vanish modulo torsion, and obtain the canonical isomorphism (2.6).

The odd-numbered rows in the spectral sequence (2.10) are zero, so the even-numbered differentials are, also. The spectral sequence can be reindexed to progress at double speed, in which case it will be indexed as follows.

$$E_2^{pq} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X^{\mathrm{top}}) \quad (2.11)$$

2.4

The Motivic Spectral Sequence

Now let X be a nonsingular algebraic variety, or more generally, a regular scheme.

Consider a finitely generated regular ring A and a prime number ℓ . In [38], motivated by the evident success of étale cohomology as an algebraic analogue for varieties of singular cohomology with finite coefficients and by conjectures of Lichtenbaum relating K -theory of number rings to étale cohomology, Quillen asked whether there is a spectral sequence analogous to (2.10) of the following form, converging at least in degrees $-p - q > \dim(A) + 1$.

$$E_2^{pq} = H_{\mathrm{et}}^p(\mathrm{Spec}(A[\ell^{-1}]), \mathbb{Z}_\ell(-q/2)) \Rightarrow K_{-p-q}(A) \otimes \mathbb{Z}_\ell \quad (2.12)$$

The spectral sequence would degenerate in case A is the ring of integers in a number field and either ℓ is odd or A is totally imaginary.

Beilinson asked ([4], see also [5, p. 182]) whether there is an integral version of (2.12) serving as a totally algebraic analogue of (2.10) that would look like this:

$$E_2^{pq} = H^p(X, \mathbb{Z}(-q/2)) \Rightarrow K_{-p-q}(X). \quad (2.13)$$

The groups $H^p(X, \mathbb{Z}(-q/2))$ would be called *motivic cohomology* groups, and comparison with (2.1) suggests we demand that $H^{2i}(X, \mathbb{Z}(i)) = CH^i(X)$. In Beilinson's formulation, $\mathbb{Z}(i)$ would be a cohomological complex of sheaves of abelian groups in the Zariski topology on X concentrated in degrees $1, 2, \dots, i$ (except for $i = 0$, where $\mathbb{Z}(0) = \mathbb{Z}$) and $H^p(X, \mathbb{Z}(-q/2))$ would be the hypercohomology of the complex. In an alternative formulation [30] advanced by Lichtenbaum the complex $\mathbb{Z}(i)$ is derived from such a complex in the étale topology on X .

An alternative indexing scheme for the spectral sequence eliminates the odd-numbered rows whose groups are zero anyway, and looks like this.

$$E_2^{pq} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X) \quad (2.14)$$

Filtrations as a Source of Spectral Sequences

2.5

The most basic way to make a (convergent) spectral sequence is to start with a homological bicomplex [31, XI.6], but for more generality one can also start with a filtered chain complex [31, XI.3], or even just with an exact couple [31, XI.5]. An exact couple is basically an exact triangle of bigraded abelian groups, where two of the three terms are the same, as in the following diagram.

$$\begin{array}{ccc} D & \xrightarrow{\quad} & D \\ & \swarrow \quad \searrow & \\ & E & \end{array}$$

Such exact couples can arise from long exact sequences where the terms are indexed by a pair of integers, and, aside from a difference of indices, two of the terms look the same. If C is a chain complex, and $C = F^0 C \supseteq F^1 C \supseteq F^2 C \supseteq \dots$ is a descending filtration by subcomplexes, then the long exact sequences

$$\dots \rightarrow H_q(F^{p+1}C) \rightarrow H_q(F^p C) \rightarrow H_q(F^p C / F^{p+1}C) \rightarrow H_{q-1}(F^{p+1}C) \rightarrow \dots$$

provide the exact couple that in turn provides the spectral sequence associated to the filtration.

More generally, we could start simply with a sequence of maps $C = F^0 C \leftarrow F^1 C \leftarrow F^2 C \leftarrow \dots$, for we may replace the quotient chain complex $F^p C / F^{p+1} C$

with the mapping cone of the map $F^p C \leftarrow F^{p+1} C$, preserving the basic shape of the long exact sequences above, which is all that is needed to make an exact couple.

Another source of long exact sequences is homotopy theory, where a fibration sequence $F \rightarrow X \rightarrow Y$ of pointed spaces gives rise to a long exact sequence $\cdots \rightarrow \pi_n F \rightarrow \pi_n X \rightarrow \pi_n Y \rightarrow \pi_{n-1} F \rightarrow \cdots \rightarrow \pi_0 X \rightarrow \pi_0 Y$. If $X \rightarrow Y$ is any map of pointed spaces, then letting F be the homotopy fiber of the map provides the desired fibration sequence. If we insist that X and Y be homotopy commutative group-like H -spaces (Abelian groups up to homotopy) and that the map $X \rightarrow Y$ be compatible with the H -space structure, then F will be an H -space, too, the terms in the long exact sequence will be abelian groups, and the maps in it will be homomorphisms. Finally, if we assume that $\pi_0 X \rightarrow \pi_0 Y$ is surjective, and define the $\pi_n = 0$ for $n < 0$, the long exact sequence will be exact also at $\pi_0 Y$ and thus will extend infinitely far in both directions. Ultimately, we may assume that $X \rightarrow Y$ is a map of spectra, in which case a homotopy cofiber for the map $X \rightarrow Y$ exists, serving as a complete analogue of the mapping cone for a map of chain complexes.

The long exact sequences of homotopy theory can produce spectral sequences, too. For example, let Y be a connected space with abelian fundamental group, and let $Y = F^0 Y \leftarrow F^1 Y \leftarrow F^2 Y \leftarrow \cdots$ be a sequence of maps between connected spaces $F^p Y$ with abelian fundamental group; call such a thing a *filtration* of Y or also a *tower*. By a slight abuse of notation let $\Omega^{F^{p|p+1}} Y$ denote the homotopy fiber of the map $F^{p+1} Y \rightarrow F^p Y$. The long exact sequences $\cdots \rightarrow \pi_q F^{p+1} Y \rightarrow \pi_q F^p Y \rightarrow \pi_{q-1} \Omega^{F^{p|p+1}} Y \rightarrow \pi_{q-1} F^{p+1} Y \rightarrow \cdots$ form an exact couple. (Observe also that $\pi_0 \Omega^{F^{p|p+1}} Y$ appears in the long exact sequence as the cokernel of a homomorphism between abelian groups, so naturally is one as well.) The corresponding spectral sequence will converge to $\pi_n Y$ if for every q and for every sufficiently large p , $\pi_q F^p Y = 0$ [12, A.6].

If Y is a space, then the Postnikov tower of Y is a filtration $Y = F^0 Y \leftarrow F^1 Y \leftarrow F^2 Y \leftarrow \cdots$ that comes equipped with spaces $F^{p|p+1} Y$ fitting into fibration sequences $F^{p+1} Y \rightarrow F^p Y \rightarrow F^{p|p+1} Y$, and $F^{p|p+1} Y$ is an Eilenberg–MacLane space $K(\pi_p Y, p)$. The corresponding spectral sequence is uninteresting, because it gives no new information about the homotopy groups $\pi_p Y$ or the space Y .

More generally, let Y be a spectrum, let $Y = F^0 Y \leftarrow F^1 Y \leftarrow F^2 Y \leftarrow \cdots$ is a sequence of maps of spectra; call such a thing a filtration of Y . Let $F^{p|p+1} Y$ denote the homotopy cofiber of the map $F^{p+1} Y \rightarrow F^p Y$; we call it the p -th *layer* of the filtration. The long exact sequences $\cdots \rightarrow \pi_q F^{p+1} Y \rightarrow \pi_q F^p Y \rightarrow \pi_q F^{p|p+1} Y \rightarrow \pi_{q-1} F^{p+1} Y \rightarrow \cdots$ form an exact couple. The corresponding spectral sequence will converge to $\pi_n Y$ if for every q and for every sufficiently large p , $\pi_q F^p Y = 0$.

If Y is a spectrum, possibly with negative homotopy groups, then the Postnikov filtration has terms $F^p Y$ with $p < 0$.

The Postnikov filtration of the spectrum BU involves Eilenberg–MacLane spaces $F^{2i|2i+1} BU \cong K(\mathbb{Z}, 2i)$; the other steps in the filtration are trivial. Taking a finite cell complex X , the mapping spectra $(F^p BU)^X$ provide a filtration of $Y := BU^X \cong K(X^{\text{top}}, \mathbb{Z})$ and fit into fibration sequences $(F^{2i+1} BU)^X \rightarrow (F^{2i} BU)^X \rightarrow K(\mathbb{Z}, 2i)^X$. The homotopy groups of $K(\mathbb{Z}, 2i)^X$ turn out to be cohomology groups of X ,

for $\pi_p K(\mathbb{Z}, 2i)^X \cong [X_+ \wedge S^p, K(\mathbb{Z}, 2i)] \cong [X_+, \Omega^p K(\mathbb{Z}, 2i)] \cong [X_+, K(\mathbb{Z}, 2i - p)] \cong H^{2i-p}(X, \mathbb{Z})$. Indeed, the spectrum $K(\mathbb{Z}, 2i)^X$ is the generalized Eilenberg–MacLane spectrum corresponding to the singular cochain complex of X , shifted in degree by $2i$ [10, IV.2.4-5], and the spectral sequence resulting from this filtration can be identified with the Atiyah–Hirzebruch spectral sequence of X , defined as above using the skeletal filtration of X ; see [20, theorem B.8].

In general, given a filtration of a spectrum Y , we may ask that the layers $F^{p/p+1} Y$ be generalized Eilenberg–MacLane spectra, i.e., should come from chain complexes of abelian groups. Intuitively, such a spectral sequence describes something complicated (homotopy groups) in terms of something simpler and hopefully more computable (homology or cohomology groups). Ideally, those chain complexes would be explicitly constructible without using any higher homotopy groups.

The motivic spectral sequences turn out to be of the type just described, and it's not surprising, because constructing a filtration of a spectrum is a natural way to proceed, postponing as long as possible the study of the homotopy groups themselves. That is the hope expressed in [18].

Here is a further important motivational remark of Goodwillie. If R is a commutative ring and Y is the K -theory spectrum $K(R)$ derived from projective finitely generated R -modules, then tensor product over R makes Y into a ring spectrum. If the proposed filtration is compatible with products, then $F^{0/1} Y$ is also a ring spectrum and each cofiber $F^{p/p+1} Y$ is a module over it. If, moreover, $F^{0/1} Y$ is the Eilenberg–MacLane spectrum $H_{\mathbb{Z}}$ that classifies ordinary homology, then it follows that $F^{p/p+1} Y$ must be an Eilenberg–MacLane spectrum, too, i.e., come from a chain complex of abelian groups. Further work may be required to make the chain complex explicit, but at least our search for a suitable filtration can be limited to those compatible with products.

Commuting Automorphisms

2.6

In this section we explain the Goodwillie–Lichtenbaum idea involving tuples of commuting automorphisms and the theorem [19] that gives a spectral sequence relating K -theory to chain complexes constructed from direct-sum Grothendieck groups of tuples of commuting automorphisms. (The intrusion of of the direct-sum Grothendieck groups during the construction of the spectral sequence was unwelcome.) One aspect of the proof I want to emphasize is the “cancellation” theorem for the space $\text{Stab}(P, Q)$ of stable isomorphisms between projective modules P and Q . It says that $\text{Stab}(P, Q)$ and $\text{Stab}(P \oplus X, Q \oplus X)$ are homotopy equivalent, and requires the ground ring to be a connected simplicial ring. The proof is sort of similar to Voevodsky's proof of his cancellation theorem, which we'll cover in Sect. 2.7.

Let's consider an affine regular noetherian scheme X . The Fundamental Theorem [37] says that the map $K(X) \rightarrow K(X \times \mathbb{A}^1)$ is a homotopy equivalence; this property of the functor K is called *homotopy invariance*.

There is a standard way of converting a functor into one that satisfies homotopy invariance, first used¹ by Gersten in [15] to describe the higher K -theory of rings developed by Karoubi and Villamayor in [23], an attempt which turned out to give the right answer for regular noetherian commutative rings. The standard topological simplices Δ^n form a cosimplicial space $\Delta: n \mapsto \Delta^n$ in which the transition maps are affine maps that send vertices to vertices. For example, some of the transition maps are the inclusion maps $\Delta^{n-1} \hookrightarrow \Delta^n$ whose images are the faces of codimension 1. Gersten considered the analogous cosimplicial affine space $\mathbb{A}: n \mapsto \mathbb{A}^n$ whose transition maps are given by the same formulas. We regard $\mathbb{A}^n$ as a *simplex*, the set of affine linear combinations of $n + 1$ distinguished points which are called its *vertices*; its *faces* are the subaffine spaces spanned by subsets of the vertices. It is an elementary fact (see, for example, [19]) that if F is a contravariant functor to spaces from a category of smooth varieties that includes the affine spaces, the $G(X) := |n \mapsto F(X \times \mathbb{A}^n)|$ is homotopy invariant, and the map $F \rightarrow G$ is, in some up-to-homotopy sense, the universal map to a homotopy invariant functor.

Let $K(X \times \mathbb{A}^n)$ denote the geometric realization of simplicial space $n \mapsto K(X \times \mathbb{A}^n)$. It follows that the map $K(X) \rightarrow K(X \times \mathbb{A}^n)$ is a homotopy equivalence. The simplicial space $n \mapsto K(X \times \mathbb{A}^n)$ is analogous to a bicomplex, with the spaces $K(X \times \mathbb{A}^n)$ playing the role of the columns. This particular simplicial space isn't interesting, because the face and degeneracy maps are homotopy equivalences, but we can relate it to some simplicial spaces that are.

Suppose X and Y are separated noetherian schemes. Then the union of two subschemes of $X \times Y$ that are finite over X will also be finite over X , and thus an extension of two coherent sheaves on $X \times Y$ whose supports are finite over X will also be finite over X . Thus we may define the exact category $\mathcal{P}(X, Y)$ consisting of those coherent sheaves on $X \times Y$ that are flat over X and whose support is finite over X . If $X' \rightarrow X$ is a map, then there is an exact base-change functor $\mathcal{P}(X, Y) \rightarrow \mathcal{P}(X', Y)$.

For example, the category $\mathcal{P}(X, \text{Spec } \mathbb{Z})$ is equivalent to the category $\mathcal{P}(X)$ of locally free coherent sheaves on X . When $X = \text{Spec}(R)$ is affine, $\mathcal{P}(X)$ is equivalent to the category $\mathcal{P}(R)$ of finitely generated projective R -modules. When $X = \text{Spec}(R)$ and $Y = \text{Spec}(S)$ are both affine, the category $\mathcal{P}(X, Y)$ is equivalent to the category $\mathcal{P}(R, S)$ of R - S -bimodules that are finitely generated and projective as R -modules.

The category $\mathcal{P}(X, \mathbb{A}^1)$ is equivalent to the category of pairs (M, f) where M is a locally free coherent sheaf on X and f is an endomorphism of M . If R is a commutative ring and X is the affine scheme $\text{Spec } R$, then M is essentially the same as a finitely generated projective R -module. Similarly, the category $\mathcal{P}(X, \mathbb{A}^n)$ is equivalent to the category of tuples $(M, f_1, \dots, f_n)$ where M is a locally free coherent sheaf on X and $f_1, \dots, f_n$ are commuting endomorphisms of M .

Let $\mathbb{G}_m = \text{Spec } \mathbb{Z}[u, u^{-1}]$. (It is a group scheme, but we won't use its multiplication operation.) The category $\mathcal{P}(X, \mathbb{G}_m)$ is equivalent to the category of pairs (M, f) where M is a locally free coherent sheaf on X and f is an automorphism of M . The

¹ according to [1, p. 78]

category $\mathcal{P}(X, \mathbb{G}_m^n)$ is equivalent to the category of tuples $(M, f_1, \dots, f_n)$ where M is a locally free coherent sheaf on X and $f_1, \dots, f_n$ are commuting automorphisms of M .

The identity section $*$: $\mathrm{Spec} \mathbb{Z} \rightarrow \mathbb{G}_m$ of $\mathbb{G}_m$ plays the role of the base point, so let $*$ denote it. Following Voevodsky, more or less, define the algebraic circle $\mathbb{S}^1$ to be the pair $(\mathbb{G}_m, *)$. Consider the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Spec} \mathbb{Z} & \xrightarrow{*} & \mathbb{G}_m \\ & \searrow & \downarrow \\ & 1 & \mathrm{Spec} \mathbb{Z} \end{array}$$

From it we see that if F is any contravariant (or covariant) functor from schemes to abelian groups, we can define $F(\mathbb{S}^1)$ as the complementary summand in the decomposition $F(\mathbb{G}_m) \cong F(\mathrm{Spec} \mathbb{Z}) \oplus F(\mathbb{S}^1)$ derived from the following diagram.

$$\begin{array}{ccc} F(\mathrm{Spec} \mathbb{Z}) & \xrightarrow{*} & F(\mathbb{G}_m) \\ & \searrow & \downarrow \\ & 1 & F(\mathrm{Spec} \mathbb{Z}) \end{array}$$

The result will be natural in F . We can iterate this: considering $F(X \times Y)$ first as a functor of X we can give a meaning to $F(\mathbb{S}^1 \times Y)$, and then considering $F(\mathbb{S}^1 \times Y)$ as a functor of Y we can give a meaning to $F(\mathbb{S}^1 \times \mathbb{S}^1)$. Usually that would be written as $F(\mathbb{S}^1 \wedge \mathbb{S}^1)$ if $\mathbb{S}^1$ is regarded as a pointed object rather than as a pair. We may also define $\mathbb{S}^t := \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ as a “product” of t copies of $\mathbb{S}^1$, and then $F(\mathbb{S}^t)$, interpreted as above, is the summand of $F(\mathbb{G}_m^t)$ that is new in the sense that it doesn’t come from $F(\mathbb{G}_m^{t-1})$ via any of the standard inclusions.

We define $\mathbb{Z}^{\mathrm{ex}}(t)(X)$ to be the chain complex associated to the simplicial abelian group $n \mapsto K_0(\mathcal{P}(X \times \mathbb{A}^n, \mathbb{S}^t))$, regarded as a cohomological chain complex, and shifted so that the group $K_0(\mathcal{P}(X \times \mathbb{A}^t, \mathbb{S}^t))$ is in degree 0. In a context where a complex $\mathbb{Z}^{\mathrm{ex}}(t)$ of sheaves is required, we sheafify the presheaf $U \mapsto \mathbb{Z}^{\mathrm{ex}}(t)(U)$, where U ranges over open subsets of X .

Recall the direct sum Grothendieck group $K_0^{\oplus}(\mathcal{M})$, where $\mathcal{M}$ is a small additive category. It is defined to be the abelian group given by generators $[M]$, one for each object M of $\mathcal{M}$, and by relations $[M] = [M'] + [M'']$, one for each isomorphism of the form $M \cong M' \oplus M''$. If $\mathcal{M}$ is not small, but equivalent to an additive category $\mathcal{M}'$ that is, we define $K_0^{\oplus}(\mathcal{M}) := K_0^{\oplus}(\mathcal{M}')$ and observe that up to isomorphism it is independent of the choice of $\mathcal{M}'$. It is an easy exercise to check that two classes $[M]$ and $[N]$ are equal in $K_0^{\oplus}(\mathcal{M})$ if and only if M and N are *stably isomorphic*, i.e., there is another object C such that $M \oplus C \cong N \oplus C$. There is an evident natural surjection $K_0^{\oplus}(\mathcal{M}) \twoheadrightarrow K_0(\mathcal{M})$.

Now we present the motivic cohomology complex encountered in [19]. We define $\mathbb{Z}^\oplus(t)(X)$ to be the chain complex associated to the simplicial abelian group $n \mapsto K_0^\oplus(\mathcal{P}(X \times \mathbb{A}^n, \mathbb{S}^t))$, reversed and shifted as for $\mathbb{Z}^{\text{ex}}(t)(X)$, and we define a complex of sheaves $\mathbb{Z}^\oplus(t)$ as before. There is a natural map $\mathbb{Z}^\oplus(t) \rightarrow \mathbb{Z}^{\text{ex}}(t)$.

If $X = \text{Spec}(R)$ is the spectrum of a regular noetherian ring R , then from [19, 9.7] one derives a spectral sequence of the following form.

$$E_2^{pq} = H^{p-q}(\mathbb{Z}^\oplus(-q)(X)) \Rightarrow K_{-p-q}(X) \quad (2.15)$$

The rest of this section will be devoted to sketching some of the details of the proof.

The coordinate rings $n \mapsto R[\mathbb{A}^n]$ form a contractible simplicial ring, where $R[\mathbb{A}^n] = R[T_1, \dots, T_n]$. Let's examine the part in degrees 0 and 1, where we have the ring homomorphism $R \hookrightarrow R[T]$ and the evaluation maps $e_0, e_1: R[T] \rightrightarrows R$ defined by $f \mapsto f(0)$ and $f \mapsto f(1)$. The two evaluation maps allow us to regard a polynomial in $R[T]$, a matrix over $R[T]$, or an $R[T]$ -module, as a sort of homotopy connecting the two specializations obtained using the two evaluation maps. There are two simple remarks about such *algebraic homotopies* that play a role in the proofs.

1

Remark 1 Firstly, short exact sequences always split up to homotopy, so working with direct sum K -theory might not be so bad. Here is the homotopy. Start with a short exact sequence $E: 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of R -modules, and define an $R[T]$ -module $\tilde{M}$ as the pull back in the following diagram.

$$\begin{array}{ccccccccc} \tilde{E}: & 0 & \longrightarrow & M'[T] & \longrightarrow & \tilde{M} & \longrightarrow & M''[T] & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow^T & & \\ E: & 0 & \longrightarrow & M'[T] & \longrightarrow & M[T] & \longrightarrow & M''[T] & \longrightarrow & 0 \end{array}$$

The short exact sequence $\tilde{E}$ specializes to E when $T = 1$ and to $0 \rightarrow M' \rightarrow M' \oplus M'' \rightarrow M'' \rightarrow 0$ when $T = 0$, and provides the desired homotopy.

2

Remark 2 Secondly, signed permutation matrices of determinant 1 don't matter, as they are homotopic to the identity. Here is an example of such a homotopy: the invertible matrix

$$\begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} \begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix}$$

specializes to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ at $T = 0$ and to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ at $T = 1$.

Here is an application of the second remark that played a role in the proof in [19], abstracted for examination. Suppose $\mathcal{M}$ is an additive category. Given two objects M and N of $\mathcal{M}$ let's define a new category $\text{Stab}(M, N)$, the category of *stable equivalences* between M and N . A stable equivalence is a pair (C, θ) where $C \in \mathcal{M}$ and $\theta: M \oplus C \xrightarrow{\cong} N \oplus C$. An object of $\text{Stab}(M, N)$ is a stable equivalence. An arrow $(C, \theta) \rightarrow (C', \theta')$ of $\text{Stab}(M, N)$ is an isomorphism class of pairs (D, ψ) with $D \in \mathcal{M}$ and $\psi: C \oplus D \xrightarrow{\cong} C'$, such that $\theta' = (1_N \oplus \psi)(\theta \oplus 1_D)(1_M \oplus \psi)^{-1}$. In effect, an arrow connects a stable isomorphism to one obtained from it by direct sum with an identity isomorphism, underscoring the point that stable isomorphisms related in that way are not much different from each other.

Given $P \in \mathcal{M}$ there are natural functors $\text{Stab}(M, N) \rightleftarrows \text{Stab}(M \oplus P, N \oplus P)$. The rightward map μ adds the identity isomorphism 1_P to a stable isomorphism of M with N , getting a stable isomorphism of $M \oplus P$ with $N \oplus P$. The leftward map φ sends an object (D, ψ) to $(P \oplus D, \psi)$.

The composite functor $\text{Stab}(M, N) \xrightarrow{\mu} \text{Stab}(M \oplus P, N \oplus P) \xrightarrow{\varphi} \text{Stab}(M, N)$ is the target of a natural transformation whose source is the identity functor. The arrows in the transformation connect (C, θ) to something equivalent to $(C \oplus P, \theta \oplus 1_P)$. After geometric realization the natural transformation gives a homotopy from the identity to $\varphi \circ \mu$.

Our goal is to show the functors φ and μ are inverse homotopy equivalences. It turns out there is a switching swindle that produces a homotopy from $\mu \circ \varphi \sim 1$ for free from the homotopy $1 \sim \varphi \circ \mu$, which we describe now.

The composite functor $\text{Stab}(M \oplus P, N \oplus P) \xrightarrow{\varphi} \text{Stab}(M, N) \xrightarrow{\mu} \text{Stab}(M \oplus P, N \oplus P)$ sends an object (D, ψ) , where $\psi: (M \oplus P) \oplus D \xrightarrow{\cong} (N \oplus P) \oplus D$, to an object of the form $(P \oplus D, \beta)$. The isomorphism $\beta: (M \oplus P) \oplus P \oplus D \xrightarrow{\cong} (N \oplus P) \oplus P \oplus D$ has 1_P as a direct summand, provided by μ , but the identity map is on the first P , which is the wrong one! If only it were on the second one, we could get a natural transformation as before. An equivalent way to visualize the situation is to embed the composite map into the following diagram.

$$\begin{array}{ccc}
 \text{Stab}(M \oplus P, N \oplus P) & \xrightarrow{\varphi} & \text{Stab}(M, N) \\
 \mu_2 \downarrow & & \downarrow \mu \\
 \text{Stab}(M \oplus P \oplus P, N \oplus P \oplus P) & \xrightleftharpoons[\varphi_1]{\varphi_2} & \text{Stab}(M \oplus P, N \oplus P)
 \end{array} \tag{2.16}$$

Here φ_1 and φ_2 are analogues of φ that deal with the first and second P , respectively, and μ_2 adds in an identity map on the second P . The square commutes up to homotopy if φ_2 is used, so we need a homotopy $\varphi_1 \sim \varphi_2$. The trick used is to put ourselves in a world where remark 2 applies and incorporate a homotopy from the identity map to a signed permutation that switches one copy of P with the other.

That world is the world where we are working over the simplicial coordinate ring $R[\mathbb{A}^1]$ of $\mathbb{A}^1$, i.e., $\mathcal{M}$ is replaced by a simplicial additive category $n \mapsto \mathcal{M}_n$ over $R[\mathbb{A}^1]$, so such homotopies are available. The details of that homotopy are the main technical point of [19, 8.3]. Following Voevodsky, we may refer to the homotopy equivalence $\text{Stab}(M, N) \sim \text{Stab}(M \oplus P, N \oplus P)$ as a cancellation theorem. Now we explain briefly how it gets used in the construction of the spectral sequence (2.15).

We recall $S^{-1}S(\mathcal{M})$, the category constructed by Quillen in [17] whose homotopy groups are the higher direct sum K -groups of the additive category $\mathcal{M}$. Its objects are pairs (M, N) of objects in $\mathcal{M}$, and an arrow $(M, N) \rightarrow (M', N')$ is an isomorphism class of triples (C, α, β) where $C \in \mathcal{M}$ and $\alpha: M \oplus C \xrightarrow{\cong} M'$ and $\beta: N \oplus C \xrightarrow{\cong} N'$ are isomorphisms. The construction is designed so $\pi_0 S^{-1}S(\mathcal{M}) \cong K_0^\oplus(\mathcal{M})$.

By techniques due to Quillen [37, theorem B] homotopy equivalences between naive approximations to the homotopy fibers of a functor yield homotopy fibration sequences. The categories $\text{Stab}(M, N)$ appear as naive approximations to the homotopy fibers of the functor from the (contractible) path space of $S^{-1}S(\mathcal{M})$ to $S^{-1}S(\mathcal{M})$. That follows from two observations: a stable isomorphism of M with N is essentially a diagram of the following type in $S^{-1}S(\mathcal{M})$;

$$(0, 0) \longrightarrow (M \oplus C, N \oplus C) \longleftarrow (M, N)$$

and an arrow in the category $\text{Stab}(M, N)$ is essentially a diagram of the following type in $S^{-1}S(\mathcal{M})$.

$$\begin{array}{ccccc} (0, 0) & \longrightarrow & (M \oplus C, N \oplus C) & \longleftarrow & (M, N) \\ & \searrow & \downarrow & \swarrow & \\ & & (M \oplus C \oplus D, N \oplus C \oplus D) & & \end{array}$$

Since all the fibers are the same up to homotopy equivalence, a variant of Quillen's Theorem B adapted to the simplicial world tells us that any of the fibers, say $\text{Stab}(0, 0)$, is almost the loop space of $S^{-1}S(\mathcal{M})$. We have to say "almost" because not every object (M, N) of $S^{-1}S(\mathcal{M})$ has components M and N that are stably isomorphic. For an additive category $\mathcal{M}$ the obstruction to stable isomorphism is captured precisely by the group $K_0^\oplus(\mathcal{M})$. In our situation, $\mathcal{M}$ is a simplicial additive category, which means that the space associated to the simplicial abelian group $n \mapsto K_0^\oplus(\mathcal{M}_n)$ enters into a fibration with the other two spaces. An object of $\text{Stab}(0, 0)$ is a stable isomorphism of 0 with 0, and is essentially a pair (C, θ) where C is an object of $\mathcal{M}$ and θ is an automorphism of C . That's how automorphisms enter into the picture. Proceeding inductively, one encounters the additive category whose objects are such pairs (C, θ) . An automorphism of such an object is an automorphism of C that commutes with θ . That explains why, at subsequent

stages, tuples of *commuting* automorphisms are involved. Starting the game off with $\mathcal{M} := (n \mapsto \mathcal{P}(R[\mathbb{A}^n]))$ one can see now how $\mathbb{Z}^\oplus(t)(X)$ arises.

We discuss the extent to which the map $\mathbb{Z}^\oplus(t)(X) \rightarrow \mathbb{Z}(t)(X)$ is a quasi-isomorphism in Sect. 2.7.

Cancellation and Comparison with Motivic Cohomology

2.7

In this section we describe the cancellation theorem of Voevodsky [45] in the form presented by Suslin in [41, §4]. It is one of the tools used by Suslin to prove that $\mathbb{Z}^\oplus(t)(X) \rightarrow \mathbb{Z}(t)(X)$ is a quasi-isomorphism locally on a smooth variety X , and we'll also discuss how that goes. We take some liberties with Suslin's presentation for the sake of motivation and for the sake of hiding technicalities. In this section $X = \text{Spec}(R)$ and $Y = \text{Spec}(S)$ are affine regular noetherian schemes.

There is a functor $\mathcal{P}(X, Y) \rightarrow \mathcal{P}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ arising from tensor product with the structure sheaf of the graph of the identity map $\mathbb{G}_m \rightarrow \mathbb{G}_m$.

Let $K_0^\oplus(\mathcal{P}(X \times \mathbb{A}^1, Y))$ denote the simplicial abelian group $n \mapsto K_0^\oplus(\mathcal{P}(X \times \mathbb{A}^n, Y))$, or when needed, the chain complex associated to it. The cancellation theorem states that the induced map

$$K_0^\oplus(\mathcal{P}(X \times \mathbb{A}^1, Y)) \rightarrow K_0^\oplus(\mathcal{P}(X \times \mathbb{S}^1 \times \mathbb{A}^1, Y \times \mathbb{S}^1)) \quad (2.17)$$

is a quasi-isomorphism. Roughly speaking, the idea is to use the same switching swindle as in the previous section. One could imagine trying to construct the following diagram, analogous to (2.16), with $\mathbb{S}^1$ here playing the role P played there.

$$\begin{array}{ccc} K_0^\oplus(\mathcal{P}(X \times \mathbb{S}^1 \times \mathbb{A}^1, Y \times \mathbb{S}^1)) & \xrightarrow{\varphi} & K_0^\oplus(\mathcal{P}(X \times \mathbb{A}^1, Y)) \\ \mu_2 \downarrow & & \downarrow \mu \\ K_0^\oplus(\mathcal{P}(X \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{A}^1, Y \times \mathbb{S}^1 \times \mathbb{S}^1)) & \begin{array}{c} \xrightarrow{\varphi_2} \\ \xleftarrow{\varphi_1} \end{array} & K_0^\oplus(\mathcal{P}(X \times \mathbb{S}^1 \times \mathbb{A}^1, Y \times \mathbb{S}^1)) \end{array} \quad (2.18)$$

Here the map μ would be external product with the identity map on $\mathbb{S}^1$, and hopefully, one could find a map φ for which $\varphi \circ \mu \sim 1$. The maps φ_1 and φ_2 would be instances of φ , but based on different $\mathbb{S}^1$ factors. Then one could hope to use the switching swindle to get the homotopy $\mu \circ \varphi \sim 1$ for free.

We may regard a class $[M]$ of $K_0^\oplus(\mathcal{P}(X, Y))$ as a *direct sum Grothendieck group correspondence* from X to Y . Intuitively, a *correspondence* is a relation that relates

some points of Y to each point of X . Geometrically, we imagine that the points (x, y) in the support of M lying over a point $x \in X$ give the values y of the correspondence. Since the support of M is finite over X , each x corresponds to only finitely many y . Since M is flat over X , the number of points y corresponding to x , counted with multiplicity, is the rank of M as a locally free $\mathcal{O}_X$ -module near x , and is a locally constant function of x .

For example, an element of $K_0^\oplus(\mathcal{P}(X, \mathbb{A}^1))$ is essentially a square matrix θ of regular functions on X , and to a point x it associates the eigenvalues of $\theta(x)$. An element of $K_0^\oplus(\mathcal{P}(X, \mathbb{G}_m))$ is essentially an invertible square matrix θ of regular functions on X , and to a point x it associates the eigenvalues of $\theta(x)$, which are nonzero numbers.

We define the additive category of correspondences; its objects are the symbols $[X]$, one for each noetherian separated scheme X . An arrow $[X] \rightarrow [Y]$ is an element of $\text{Hom}([X], [Y]) := K_0^\oplus(\mathcal{P}(X, Y))$. The composition $[N] \circ [M]$ of correspondences is defined to be $[M \otimes_{\mathcal{O}_Y} N]$. The direct sum $[X] \oplus [X']$ of two objects is represented by the disjoint union $[X \sqcup X']$ of schemes, because $K_0^\oplus(\mathcal{P}(X \sqcup X', Y)) \cong K_0^\oplus(\mathcal{P}(X, Y)) \oplus K_0^\oplus(\mathcal{P}(X', Y))$ and $K_0^\oplus(\mathcal{P}(X, Y \sqcup Y')) \cong K_0^\oplus(\mathcal{P}(X, Y)) \oplus K_0^\oplus(\mathcal{P}(X, Y'))$. We define $[X] \otimes [X'] := [X \times X']$ on objects, and extend it to a bilinear function on arrows.

There is a function $\text{Hom}(X, Y) \rightarrow \text{Hom}([X], [Y])$ which sends a map f to the class $[f]$ of the structure sheaf of its graph. It is compatible with composition and defines a functor $X \mapsto [X]$ from the category of separated schemes to the category of correspondences.

A *homotopy* of correspondences from X to Y will be an element of $\text{Hom}([X \times \mathbb{A}^1], [Y])$. If two correspondences f and g are homotopic, we'll write $f \sim g$. Composition preserves homotopies.

We are particularly interested in correspondences $[X] \rightarrow [\mathbb{G}_m]$. We can use companion matrices to construct them. Let $f = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in R[T]$ be a polynomial with unit constant term $a_0 \in R^\times$. Since f is monic, as an R -module P is free with rank equal to n . Since T acts invertibly on $P := R[T]/f$, we can regard P as a finitely generated $R[T, T^{-1}]$ -module, hence as an object of $\mathcal{P}(X, \mathbb{G}_m)$. The eigenvalues of T acting on P are the roots of f , so let's use the notation $[f = 0]$ for the correspondence $[P]$.

3 Lemma 3 If f and g are monic polynomials with unit constant term, then $[f = 0] + [g = 0] \sim [fg = 0]: [X] \rightarrow [\mathbb{G}_m]$.

Proof The exact sequence

$$0 \rightarrow R[T, T^{-1}]/f \rightarrow R[T, T^{-1}]/fg \rightarrow R[T, T^{-1}]/g \rightarrow 0$$

splits up to homotopy, according to Remark 1.

If $a \in R^\times$ is a unit, it may be regarded as a map $a: X \rightarrow \mathbb{G}_m$, and then $[a] = [T - a = 0]$.

Lemma 4 If $a, b \in R^\times$, then $[ab] \sim [a] + [b] - [1]: [X] \rightarrow [\mathbb{G}_m]$.

4

Proof (See [41, 4.6.1].) We compute $[a] + [b] = [T - a = 0] + [T - b = 0] \sim [(T - a)(T - b) = 0] = [T^2 - (a + b)T + ab = 0] \sim [T^2 - (1 + ab)T + ab = 0] = [(T - 1)(T - ab) = 0] \sim [(T - 1)] + [(T - ab) = 0] = [1] + [ab]$. The homotopy in the middle arises from adjoining a new variable V and using the homotopy $[T^2 - (V(a + b) + (1 - V)(1 + ab))T + ab = 0]$, which is valid because its constant term ab is a unit.

Now consider correspondences of the form $[X] \rightarrow [\mathbb{G}_m \times \mathbb{G}_m]$. The simplest ones are obtained from maps $X \rightarrow \mathbb{G}_m \times \mathbb{G}_m$, i.e., from pairs (b, c) of units in R . Let $[b, c]$ denote such a correspondence.

Lemma 5 If $a, b, c \in R^\times$, then $[ab, c] \sim [a, c] + [b, c] - [1, c]: [X] \rightarrow [\mathbb{G}_m \times \mathbb{G}_m]$.

5

Proof (See [41, 4.6.2].) The function $\text{Hom}([X], [\mathbb{G}_m]) \rightarrow \text{Hom}([X], [\mathbb{G}_m \times \mathbb{G}_m])$ defined by $f \mapsto (f \otimes [c]) \circ [D]$, where D is the diagonal embedding $X \rightarrow X \times X$, sends $[a]$ to $[a, c]$ and preserves homotopies.

Since $\text{Hom}([X], [Y])$ is an abelian group, the convention introduced above for applying functors to $\mathbb{S}^1$ or to $\mathbb{S}^f$ applies, and we can attach a meaning to $\text{Hom}([X], [\mathbb{S}^1])$ or to $\text{Hom}([\mathbb{S}^1], [Y])$. If we enlarge the category of correspondences slightly by taking its *idempotent completion* we may even interpret $[\mathbb{S}^1]$ as an object of the category of correspondences. To do so we introduce new objects denoted by $p[X]$ whenever X is a scheme and $p \in \text{Hom}([X], [X])$ is an idempotent, i.e., satisfies the equation $p^2 = p$. We define $\text{Hom}(p[X], q[Y]) := q \text{Hom}([X], [Y]) p \subseteq \text{Hom}([X], [Y])$, and with this definition composition is nothing new. We define $p[X] \otimes q[Y] := (p \otimes q)[X \times Y]$. A homotopy between maps $p[X] \rightrightarrows q[Y]$ will be a map $p[X] \otimes [\mathbb{A}^1] \rightarrow q[Y]$.

A map $f: [X] \rightarrow [Y]$ may be said to induce the map $qfp: p[X] \rightarrow q[Y]$, but that procedure is not necessarily compatible with composition, for in terms of matrices it is the function $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$. Nevertheless, we will abuse notation slightly and denote that induced map by $f: p[X] \rightarrow q[Y]$, leaving it to the reader to understand the necessity of composing f with p and q . Similarly, if we have an equation $f = g: p[X] \rightarrow q[Y]$ or a homotopy $f \sim g: p[X] \rightarrow q[Y]$, we'll understand that both f and g are to be treated that way.

We identify $[X]$ with $1[X]$, and prove easily that $[X] \cong p[X] \oplus \bar{p}[X]$, where $\bar{p} := 1 - p$. Any functor F from the old category of correspondences to the category of abelian groups can be extended to the new category by defining $F(p[X]) := F(p)F([X])$. For example, with this notation, $F([\mathbb{S}^1]) = F(\bar{e}[\mathbb{G}_m])$, where e is the composite map $\mathbb{G}_m \rightarrow \operatorname{Spec} \mathbb{Z} \xrightarrow{*} \mathbb{G}_m$. Thus we may as well identify $[\mathbb{S}^1]$ with $\bar{e}[\mathbb{G}_m]$, and similarly, $[\mathbb{S}^2]$ with $[\mathbb{S}^1] \otimes [\mathbb{S}^1] = (\bar{e} \otimes \bar{e})[\mathbb{G}_m \times \mathbb{G}_m]$.

6 **Lemma 6** Suppose $a, b, c \in R^\times$. Then $[ab] \sim [a] + [b]: [X] \rightarrow [\mathbb{S}^1]$ and $[ab, c] \sim [a, c] + [b, c]: [X] \rightarrow [\mathbb{S}^2]$

Proof Recall our convention about composing with idempotents when necessary. Use Lemma 4 to get $\bar{e}[ab] \sim \bar{e}([a] + [b] - [1]) = \bar{e}([a] + [b]) - \bar{e}[1]$ and then compute $\bar{e}[1] = [1] - [1] = 0$. Use Lemma 5 to get $(\bar{e} \otimes \bar{e})[ab, c] \sim (\bar{e} \otimes \bar{e})([a, c] + [b, c] - [1, c])$ and then compute $(\bar{e} \otimes \bar{e})[1, c] = [1, c] - [1, c] - [1, 1] + [1, 1] = 0$.

7 **Lemma 7** Let T, U be the standard coordinates on $\mathbb{G}_m \times \mathbb{G}_m$. Then $[TU] \sim 0: [\mathbb{S}^2] \rightarrow [\mathbb{G}_m]$.

Proof We compute $[TU](\bar{e} \otimes \bar{e}) = [TU] - [TU](1 \otimes e) - [TU](e \otimes 1) + [TU](e \otimes e) = [TU] - [T] - [U] + [1] \sim 0$, applying Lemma 4.

8 **Corollary 8** $[TU, -TU] \sim 0: [\mathbb{S}^2] \rightarrow [\mathbb{G}_m^2]$.

Proof Compose the result of the lemma with the map $\mathbb{G}_m \rightarrow \mathbb{G}_m^2$ defined by $v \mapsto (v, -v)$.

9 **Corollary 9** $[T, U] + [U, T] \sim 0: [\mathbb{S}^2] \rightarrow [\mathbb{G}_m^2]$.

Proof Compute $0 \sim [TU, -TU] \sim [T, -TU] + [U, -TU] \sim [T, -T] + [T, U] + [U, T] + [U, -U]: [\mathbb{S}^2] \rightarrow [\mathbb{G}_m^2]$ and then observe that $[T, -T] \sim 0 \sim [U, -U]: [\mathbb{S}^2] \rightarrow [\mathbb{G}_m^2]$, because, for example $[T, -T](\bar{e} \otimes \bar{e}) = [T, -T] - [T, -T] - [1, -1] + [1, -1] = 0$.

Corollary 10 $[T, U] \sim [U^{-1}, T]: [\mathbb{S}^2] \rightarrow [\mathbb{G}_m^2]$.

10

Proof Compute $0 = [1, T] = [UU^{-1}, T] \sim [U, T] + [U^{-1}, T]: [\mathbb{S}^2] \rightarrow [\mathbb{G}_m^2]$, and then apply Corollary 9.

Corollary 11 $[U^{-1}, T] \sim 1: [\mathbb{S}^2] \rightarrow [\mathbb{S}^2]$.

11

Proof Use Corollary 10 and observe that $[T, U]: [\mathbb{S}^2] \rightarrow [\mathbb{S}^2]$ is the identity map.

The map $[U^{-1}, T]: [\mathbb{S}^2] \rightarrow [\mathbb{S}^2]$ is an analogue of the signed permutation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ considered in Remark 2, and the homotopy found in Corollary 11 gives us a switching swindle that can be used to prove the cancellation theorem.

Having done that, the next task is to construct the map φ used in (2.18): it doesn't quite exist (!), but one can proceed as follows. Suppose $P \in \mathcal{P}(X \times \mathbb{S}^1, Y \times \mathbb{S}^1)$, so P is an $R[T, T^{-1}]$ - $S[V, V^{-1}]$ -module that is finitely generated and projective as an $R[T, T^{-1}]$ -module. We try to define $\varphi[P] := [P/(T^n - 1)] - [P/(T^n - V)] \in K_0^\oplus(\mathcal{P}(X, Y))$. It's easy to check that $P/(T^n - 1)$ is a projective R -module. It turns out that for n sufficiently large, multiplication by $T^n - V$ on P is injective, and the bound is independent of the ring, so injectivity is preserved by base change, implying that $P/(T^n - V)$ is flat, hence projective as an S -module, since it's finitely generated. To show that φ is a left inverse for μ in (2.18), assume that $Q \in \mathcal{P}(X, Y)$ and let $P = Q \otimes_R R[T, T^{-1}]$ with V acting on P the same way T does, and compute $\varphi[P] = [Q^n] - [Q^{n-1}] = [Q]$. One handles the problem of choosing n in the definition of φ by explicit computations with cycle classes.

The usual definition of motivic cohomology is this: the chain complex $\mathbb{Z}^{\text{cor}}(t)$, usually called $\mathbb{Z}(t)$, is defined as $\mathbb{Z}^{\text{ex}}(t)$ was, but $K_0(\mathcal{P}(X, Y))$ is replaced by the group $\text{Cor}(X, Y)$ of *finite correspondences*, those algebraic cycles on $X \times Y$ each component of which maps finitely and surjectively onto some irreducible component of Y . See [46] for the equivalence between $\mathbb{Z}^{\text{cor}}(t)$ and $\mathbb{Z}^{\text{Bl}}(i)$. There is a natural map $\mathbb{Z}^{\text{ex}}(t) \rightarrow \mathbb{Z}^{\text{cor}}(t)$, which sends a coherent sheaf to its support cycle; according to [50, corollary 6.32] the map is an equivalence. Thus in order for the spectral sequence (2.15) to be useful, the main remaining problem is to show that the map $\mathbb{Z}^\oplus(t) \rightarrow \mathbb{Z}^{\text{ex}}(t)$ is an equivalence, or equivalently, that $\mathbb{Z}^\oplus(t) \rightarrow \mathbb{Z}^{\text{cor}}(t)$ is an equivalence.

Now let's sketch the rest of the proof of Suslin's theorem in [41] that, locally, for smooth varieties over a field, that the map $\mathbb{Z}^\oplus(t) \rightarrow \mathbb{Z}^{\text{cor}}(t)$ is an equivalence.

The basic strategy is to prove that the groups $H^*(X, \mathbb{Z}^\oplus(t))$ share enough properties with motivic cohomology $H^*(X, \mathbb{Z}^{\text{cor}}(t))$ that they must be equal. First comes the statement that $H^k(X, \mathbb{Z}^\oplus(t)) \cong H^{k+1}(X \times \mathbb{S}^1, \mathbb{Z}^\oplus(t+1))$, which follows from cancellation. (It corresponds to the part of the Fundamental Theorem of algebraic K -theory that says that $K_n(R) \cong K_{n+1}(\text{Spec } R \times \mathbb{S}^1)$ for a regular noetherian ring R .) Relating $\mathbb{S}^1$ to $\mathbb{G}_m$, which is an open subset of $\mathbb{A}^1$, yields a statement about cohomology with supports, $H^k(X, \mathbb{Z}^\oplus(t)) \cong H_{X \times 0}^{k+2}(X \times \mathbb{A}^1, \mathbb{Z}^\oplus(t+1))$, for $k \geq 0$. By induction one gets $H^k(X, \mathbb{Z}^\oplus(t)) \cong H_{X \times 0}^{k+2m}(X \times \mathbb{A}^m, \mathbb{Z}^\oplus(t+m))$ for $m \geq 0$. This works also for any vector bundle E of rank m over X , yielding $H^k(X, \mathbb{Z}^\oplus(t)) \cong H_{s(X)}^{k+2m}(E, \mathbb{Z}^\oplus(t+m))$, where $s: X \rightarrow E$ is the zero section. Now take a smooth closed subscheme Z of X of codimension m , and use deformation to the normal bundle to get $H^k(Z, \mathbb{Z}^\oplus(t)) \cong H_Z^{k+2m}(X, \mathbb{Z}^\oplus(t+m))$.

Now it's time to show that the map $H_Z^*(X, \mathbb{Z}^\oplus(t)) \rightarrow H_Z^*(X, \mathbb{Z}^{\text{cor}}(t))$ is an isomorphism, but it's still not easy. The multi-relative cohomology groups with supports, $H_Z^*(\mathbb{A}^m, \partial \mathbb{A}^m; \mathbb{Z}^\oplus(t))$ arise (analogous to the multi-relative K -groups described in Sect. 2.8), and Suslin develops a notion he calls “rationally contractible presheaves” to handle the rest of the proof.

Higher Chow Groups and a Motivic Spectral Sequence

2.8

In this section we describe Bloch's approach to motivic cohomology via higher Chow groups and then describe the argument of Bloch and Lichtenbaum for a motivic spectral sequence.

The definition of linear equivalence of algebraic cycles amounts to saying that there is the following exact sequence, in which the two arrows are derived from the inclusion maps $\mathbb{A}^0 \rightrightarrows \mathbb{A}^1$ corresponding to the points 0 and 1.

$$CH^i(X) \leftarrow \mathbb{Z}^i(X) \rightrightarrows \mathbb{Z}^i(X \times \mathbb{A}^1) \quad (2.19)$$

The first step in developing motivic cohomology groups to serve in (2.2) or (2.13) is to bring homological algebra to bear: evidently $CH^i(X)$ is a cokernel, and in homological algebra one can't consider a cokernel without also considering the kernel. One way to handle that is to try to continue the sequence above forever, forming a chain complex like the one in (2.20).

$$\mathbb{Z}^i(X) \leftarrow \mathbb{Z}^i(X \times \mathbb{A}^1) \leftarrow \mathbb{Z}^i(X \times \mathbb{A}^2) \leftarrow \cdots \quad (2.20)$$

For this purpose, Bloch used the cosimplicial affine space $\mathbb{A}^\cdot$, introduced earlier, and algebraic cycles on the corresponding cosimplicial variety $X \times \mathbb{A}^\cdot$. He defined $\mathbb{Z}^i(X, n) \subseteq \mathbb{Z}^i(X \times \mathbb{A}^n)$ to be the group of algebraic cycles of codimension i on $X \times \mathbb{A}^n$ meeting all the faces properly. (Two subvarieties are said to *meet properly* if the codimension of the intersection is the sum of the codimensions of

the subvarieties.) The result is a simplicial abelian group $n \mapsto \mathbb{Z}^i(X, n)$. Its face maps involve intersecting a cycle with a hyperplane, keeping track of intersection multiplicities. Bloch defined the higher Chow group $CH^i(X, n)$ as its homology group in degree n , and we'll introduce the auxiliary notation $H^{2i-n}(X, \mathbb{Z}^{\text{Bl}}(i))$ for it here to make the anticipated use as motivic cohomology more apparent. A useful abuse of notation is to write $\mathbb{Z}^{\text{Bl}}(i) = (n \mapsto \mathbb{Z}^i(X, 2i-n))$, leaving it to the reader to remember to replace the simplicial abelian group by its associated chain complex. We let $H^{2i-n}(X, \mathbb{Q}^{\text{Bl}}(i)) := H^{2i-n}(X, \mathbb{Z}^{\text{Bl}}(i)) \otimes \mathbb{Q}$.

A good survey of results about Bloch's higher Chow groups is available in [27, II, §2.1]; for the original papers see [6, 7, 26].

The main results are that $H^n(X, \mathbb{Z}^{\text{Bl}}(i))$ is homotopy invariant, fits into a localization sequence for an open subscheme and its complement, can be made into a contravariant functor (on nonsingular varieties), and can be compared rationally with K -theory.

For example, in [7] Bloch proves a moving lemma which implies a localization theorem for $\mathbb{Z}^{\text{Bl}}$. Suppose X is quasiprojective, let $U \subseteq X$ be an open subset, let $Z = X - U$, and assume Z has codimension p in X . The localization theorem provides the following long exact sequence.

$$\begin{aligned} \dots \rightarrow H^{j-1}(U, \mathbb{Z}^{\text{Bl}}(i)) &\rightarrow H^{j-2p}(Z, \mathbb{Z}^{\text{Bl}}(i-p)) \rightarrow H^j(X, \mathbb{Z}^{\text{Bl}}(i)) \\ &\rightarrow H^j(U, \mathbb{Z}^{\text{Bl}}(i)) \rightarrow \dots \end{aligned}$$

In [6, 9.1] Bloch presents a comparison isomorphism

$$\tau: K'_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_d H^{2i-n}(X, \mathbb{Q}^{\text{Bl}}(i))$$

for X a quasiprojective variety over a field k . The proof (with some flaws corrected later) proceeds by using the localization theorem to reduce to the case where X is affine and smooth over k , then uses relative K -theory to reduce to the case where $m = 0$, and finally appeals to Grothendieck's result (2.1). In the case where X is nonsingular the isomorphism τ differs from the higher Chern character map ch by multiplication by the Todd class of X , which is a unit in the Chow ring, so Bloch's result implies that the map $\text{ch}: K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_d H^{2i-n}(X, \mathbb{Q}^{\text{Bl}}(i))$ is an isomorphism, too. In [26, 3.1] Levine proves the same result, but avoids Chern classes by using a more detailed computation of the relative K -groups. Finally, a detailed summary of the complete proof when X is nonsingular is available in [27, III, §3.6].

Now we sketch some details of the argument from the unpublished preprint [8] for a motivic spectral sequence for the spectrum of a field k .

A *cube* of dimension n is a diagram (in some category) indexed by the partially ordered set of subsets of the set $\{1, 2, \dots, n\}$. The *homotopy fiber* $\text{hofib } C$ of a cube C of dimension n of spaces or spectra can be constructed inductively as follows. When $n = 0$, $\text{hofib } C$ is the single space in the diagram. For $n > 0$ consider

the two cubes C' and C'' of dimension $n - 1$ appearing as faces perpendicular to a chosen direction and take $\text{hofib } C$ to be the homotopy fiber of the map $\text{hofib } C' \rightarrow \text{hofib } C''$.

If $Y_1, \dots, Y_n$ are open or closed subschemes of a scheme X , their intersections $Y_{i_1} \cap \dots \cap Y_{i_k}$ form a cube of schemes. Applying the K -theory functor (which is contravariant) gives a cube of spectra. When $Y_1, \dots, Y_n$ are closed subschemes, the homotopy fiber of the cube is denoted by $K(X; Y_1, \dots, Y_n)$ and is called *multi-relative K -theory*. If $Y_1 = X - V$ is an open subscheme of X , the homotopy fiber of the cube is denoted by $K^V(X; Y_2, \dots, Y_n)$, and is called *multi-relative K -theory with supports* in V . If $\mathcal{V}$ is a family of closed subsets of X that is closed under finite unions, then the inductive limit $\text{colim}_{V \in \mathcal{V}} K^V(X; Y_2, \dots, Y_n)$ is denoted by $K^\mathcal{V}(X; Y_2, \dots, Y_n)$, and is called *multi-relative K -theory with supports* in $\mathcal{V}$. If $\mathcal{W}$ is another such family of closed subsets of X , then $\text{colim}_{W \in \mathcal{W}} \text{colim}_{V \in \mathcal{V}} K^{V-W}(X - W; Y_2 - W, \dots, Y_n - W)$ is denoted by $K^{\mathcal{V}-\mathcal{W}}(X; Y_2, \dots, Y_n)$, which one might call *multi-relative K -theory with supports in $\mathcal{V}$ away from $\mathcal{W}$* . Finally, if $p \in \mathbb{Z}$, we let $K_p^{\mathcal{V}-\mathcal{W}}(X; Y_2, \dots, Y_n)$ be the homotopy group $\pi_p K^{\mathcal{V}-\mathcal{W}}(X; Y_2, \dots, Y_n)$, and similarly for the other notations.

The notation for multi-relative K -theory is applied to the simplices $\mathbb{A}^p = \mathbb{A}_k^p$ as follows. For a variety X and for any $p \geq 0$ we let $\mathcal{V}^n$ denote the family of closed subsets of $X \times \mathbb{A}^p$ which are finite unions of subvarieties of codimension n meeting $X \times \mathbb{A}^m$ properly, for each face $\mathbb{A}^m \subseteq \mathbb{A}^p$ of the simplex $\mathbb{A}^p$. In this section we are interested in the case where $X = \text{Spec } k$.

When K -theory (or multi-relative K -theory) with supports in $\mathcal{V}^n$ is intended, we'll write $K^{(n)}$ in place of $K^{\mathcal{V}^n}$. The spectrum $K^{(n)}(X \times \mathbb{A}^p)$ is functorial in the sense that if $\mathbb{A}^q \rightarrow \mathbb{A}^p$ is an affine map sending vertices to vertices, there is an induced map $K^{(n)}(X \times \mathbb{A}^p) \rightarrow K^{(n)}(X \times \mathbb{A}^q)$. These maps can be assembled to form a simplicial space $p \mapsto K^{(n)}(X \times \mathbb{A}^p)$ which we'll call $K^{(n)}(X \times \mathbb{A}^\bullet)$.

Let $H_0, \dots, H_p$ be the codimension 1 faces of $\mathbb{A}^p$. Let $\partial \mathbb{A}^p$ be an abbreviation for the sequence $H_0, \dots, H_p$, and let Σ be an abbreviation for the sequence $H_0, \dots, H_{p-1}$.

The main result is [8, theorem 1.3.3], which states that the following sequence of multi-relative K_0 groups is exact.

$$\begin{aligned} \dots &\rightarrow K_0^{(n+1)}(\mathbb{A}^p; \partial \mathbb{A}^p) \xrightarrow{i} K_0^{(n)}(\mathbb{A}^p; \partial \mathbb{A}^p) \rightarrow K_0^{(n)}(\mathbb{A}^p; \Sigma) \\ &\rightarrow K_0^{(n)}(\mathbb{A}^{p-1}; \partial \mathbb{A}^{p-1}) \rightarrow K_0^{(n-1)}(\mathbb{A}^{p-1}; \partial \mathbb{A}^{p-1}) \rightarrow \dots \end{aligned}$$

The proof of exactness hinges on showing that the related map

$$K_0^{\mathcal{V}^n - \mathcal{V}^{n+1}}(\mathbb{A}^p; \partial \mathbb{A}^p) \rightarrow K_0^{\mathcal{V}^n - \mathcal{V}^{n+1}}(\mathbb{A}^p; \Sigma)$$

is injective, which in turn depends on a *moving lemma* that occupies the bulk of the paper. Two of every three terms in the long exact sequence look like $K_0^{(n)}(\mathbb{A}^p; \partial \mathbb{A}^p)$ with a change of index, so it's actually an exact couple. The abutment of the corresponding spectral sequence is the colimit of the chain of maps like the one labelled i above, and thus is $K_0(\mathbb{A}^p; \partial \mathbb{A}^p)$, which, by an easy computation, turns out to be isomorphic to $K_p(k)$. The E_1 term $K_0^{\mathcal{V}^n}(\mathbb{A}^p; \Sigma)$ is isomorphic to $K_0^{\mathcal{V}^n - \mathcal{V}^{n+1}}(\mathbb{A}^p; \Sigma)$,

which in turn is isomorphic to the subgroup of $\mathbb{Z}^n(X, p)$ consisting of cycles which pull back to 0 in each face mentioned in Σ ; it follows that the E_2 term is $CH^n(X, p)$. The final result of Bloch and Lichtenbaum is the following theorem.

Theorem 12 If k is a field and $X = \operatorname{Spec} k$, then there is a motivic spectral sequence of the following form.

12

$$E_2^{pq} = H^{p-q}(X, \mathbb{Z}^{\operatorname{Bl}}(-q)) \Rightarrow K_{-p-q}(X) \quad (2.21)$$

We also mention [16, §7], which shows that the filtration of the abutment provided by this spectral sequence is the γ -filtration and that the spectral sequence degenerates rationally.

We continue this line of development in Sect. 2.9.

Extension to the Global Case

2.9

In this section we sketch the ideas of Friedlander and Suslin [12] for generalizing the Bloch–Lichtenbaum spectral sequence, Theorem 12, to the global case, in other words, to establish it for any nonsingular variety X . The first step is to show that the Bloch–Lichtenbaum spectral sequence arises from a filtration of the K -theory spectrum, and that the successive quotients are Eilenberg–MacLane spectra. For the subsequent steps, Levine gives an alternate approach in [28], which we also sketch briefly.

The paper [12] is a long one, so it will be hard to summarize it here, but it is carefully written, with many foundational matters spelled out in detail.

Let Δ be the category of finite nonempty ordered sets of the form $[p] := \{0, 1, 2, \dots, p\}$ for some p , so that a simplicial object is a contravariant functor from Δ to some other category. Since any finite nonempty ordered set is isomorphic in a unique way to some object of Δ , we can think of a simplicial object as a functor on the category of all finite nonempty ordered sets.

Recall that the relative K -group $K_0^{(n)}(\mathbb{A}^p; \partial\mathbb{A}^p)$ is constructed from a cube whose vertices are indexed by the intersections of the faces of the simplex $\mathbb{A}^p$. Any such intersection is a face itself, unless it's empty. The faces are indexed by the nonempty subsets of the ordered set $[p]$, so the intersections of faces are indexed by the subsets of $[p]$ (including the empty subset), and those subsets correspond naturally to the vertices of a $p+1$ -dimensional cube. The K -theory space of the empty scheme is the K -theory space of the zero exact category, so is a one point space. If Z is a pointed simplicial space (or spectrum), then by defining $Z(\emptyset)$ to be the one point space, we can apply it to the cube of subsets of $[p]$, yielding a $p+1$ -dimensional cube of spaces (or spectra) we'll call $\operatorname{cube}_{p+1} Z$. As in [12, A.1] we see that this procedure, when applied to $K^{(n)}(\mathbb{A}^\cdot) = (q \mapsto K^{(n)}(\mathbb{A}^q))$, yields a cube whose homotopy fiber is $K^{(n)}(\mathbb{A}^p; \partial\mathbb{A}^p)$.

It is well known that homotopy cofibers (mapping cones) and homotopy fibers amount to the same thing (with a degree shift of 1) for spectra. Given an m -dimensional cube Y of pointed spaces form the *homotopy cofiber* $\text{hocofib } Y$: the construction is inductive, as for the homotopy fiber of a cube described above, but at each stage the mapping cone replaces the homotopy fiber of a map. Friedlander and Suslin relate the homotopy fiber to the homotopy cofiber by proving that the natural map $\text{hofib } Y \rightarrow \Omega^m \text{hocofib } Y$ is a $2m - N + 1$ -equivalence if each space in the cube Y is N -connected [12, 3.4]. The proof uses the Freudenthal Suspension theorem and the Blakers–Massey Excision theorem.

A cube of the form $\text{cube}_m Z$ arising from a simplicial space Z comes with an interesting natural map

$$\text{hocofib cube}_m Z \rightarrow \Sigma[Z] \quad (2.22)$$

(see [12, 2.6]), which for $N \geq 1$ is $N + m + 1$ -connected if each space Z_i is N -connected [12, 2.11]. The proof goes by using homology to reduce to the case where Z is a simplicial abelian group.

Combining the two remarks above and passing to spectra, we get a map $K_i^{(n)}(\mathbb{A}^P; \partial\mathbb{A}^P) \rightarrow \pi_{i+p} K^{(n)}(\mathbb{A}^\cdot)$ which is an isomorphism for $i \leq -1$. It turns out that $i = 0$ (the value occurring in the construction of Bloch and Lichtenbaum) is close enough to $i = -1$ so that further diagram chasing [12, §5–6] allows the exact couple derived from the exact couple of Bloch and Lichtenbaum to be identified with the exact couple arising from the filtration of spectra defined by $F^n := |K^{(n)}(\mathbb{A}^\cdot)|$, and for the successive quotients in the filtration to be identified with the Eilenberg–MacLane spectrum arising from the chain complex defining $\mathbb{Z}^{\text{Bl}}$.

So much for the first step, which was crucial, since it brings topology into the game. Next the authors show that $K^{(n)}(X \times \mathbb{A}^\cdot)$ is homotopy invariant in the sense that it doesn't change if X is replaced by $X \times \mathbb{A}^1$; by induction, the same is true for the product with $\mathbb{A}^m$. Combining that with a result of Landsburg [25] establishes the motivic spectral sequence for $X = \mathbb{A}^m$. To go further, we take $X = \mathbb{A}^m$ and $m = n$ and examine $K^{(n)}(\mathbb{A}^n \times \mathbb{A}^\cdot)$. Cycles on $\mathbb{A}^n \times \mathbb{A}^m$ that are quasifinite over $\mathbb{A}^m$ have codimension n . A moving lemma of Suslin asserts that such cycles are general enough, allowing us to replace $K^{(n)}(\mathbb{A}^n \times \mathbb{A}^\cdot)$ by $K^{\mathcal{Q}}(\mathbb{A}^n \times \mathbb{A}^\cdot)$, where $\mathcal{Q}$ denotes the family of support varieties that are quasifinite over the base. The advantage here is increased functoriality, since quasifiniteness is preserved by base change. Transfer maps can also be defined, allowing the globalization theorem of Voevodsky for “pretheories” to be adapted for the current situation, thereby establishing the spectral sequence when X is a smooth affine semilocal variety over a field. Finally, the globalization techniques of Brown and Gersten [9] involving hypercohomology of sheaves of spaces are used to establish the spectral sequence when X is a smooth affine variety.

Levine's approach [28] to globalizing the spectral sequence is somewhat different. Instead of developing transfer maps as Friedlander and Suslin do, he replaces the K -theory of locally free sheaves by the K -theory of coherent sheaves (which is

called G -theory) so singular varieties can be handled. Then he develops a localization theorem [28, corollary 7.10] that provides, when U is an open subvariety of a variety X , a fibration sequence $G^{(n)}((X-U) \times \mathbb{A}^1) \rightarrow G^{(n)}(X \times \mathbb{A}^1) \rightarrow G^{(n)}(U \times \mathbb{A}^1)$. The proof of the localization theorem involves a very general moving lemma [28, theorem 0.9] that, roughly speaking, takes a cycle on $U \times \mathbb{A}^N$ meeting the faces properly and moves it to a cycle whose closure in X still meets the faces properly. But what's really "moving" is the ambient affine space, which is blown up repeatedly along faces – the blow-ups are then "trianglated" by more affine spaces.

Levine's final result is more general than the Friedlander–Suslin version, since it provides a motivic spectral sequence for any smooth scheme X over a regular noetherian scheme of dimension 1.

The Slice Filtration

2.10

In this section we sketch some of Voevodsky's ideas [43, 47, 49] aimed at producing a motivic spectral sequence. The setting is Voevodsky's $\mathbb{A}^1$ -homotopy theory for schemes [32], which can be briefly described as follows. Modern algebraic topology is set in the world of simplicial sets or their geometric realizations; the spaces to be studied can be viewed as being obtained from colimits of diagrams of standard simplices Δ^n , where the arrows in the diagram are affine maps that send vertices to vertices and preserve the ordering of the vertices. Voevodsky enlarges the notion of "space" by considering a field k (or more generally, a noetherian finite dimensional base scheme), replacing the simplices Δ^n by affine spaces $\mathbb{A}_k^n$ over k , and throwing in all smooth varieties over k , as well as all colimits of diagrams involving such varieties. The colimits may be realized in a universal way as presheaves on the category of smooth varieties over k . These presheaves are then sheafified in the Nisnevich topology, a topology intermediate between the Zariski and étale topologies. Finally, the affine "simplices" $\mathbb{A}_k^n$ are forced to be contractible spaces. The result is the $\mathbb{A}^1$ -homotopy category, a world where analogues of the techniques of algebraic topology have been developed. Spectra in this world are called *motivic spectra*. In this world there are two types of circles, the usual topological circle S^1_s and the algebraic circle $S^1_t := \mathrm{Spec}(k[U, U^{-1}])$, so the spheres $S^{n,i} := S^{n-i}_s \wedge S^i_t$ have an extra index, as do the motivic spectra and the generalized cohomology theories they represent. The sphere $T := S^{2,1}$, half topological and half algebraic, turns out to be equivalent to the projective line $\mathbb{P}^1$, so the Fundamental Theorem of algebraic K -theory, which identifies $K(\mathbb{P}^1_A)$ with $K(A) \times K(A)$ for a ring A , shows that the motivic spectrum KGL representing K -theory is $(2, 1)$ -periodic.

Voevodsky's idea for constructing a motivic spectral sequence is to build a filtration of the motivic spectrum KGL , in which each term $F^p KGL$ is to be a motivic spectrum. As in Goodwillie's remark in Sect. 2.5, the filtration is to be compatible with products, the cofiber $F^{0/1} KGL$ is to be equivalent to the motivic analogue $H_{\mathbb{Z}}$ of the Eilenberg–MacLane ring spectrum associated to the ring $\mathbb{Z}$, and hence each

quotient $F^{p/p+1}KGL$ will be an $H_{\mathbb{Z}}$ -module. Modulo problems with convergence, filtrations of motivic spectra yield spectral sequences as before.

The *slice filtration* of any motivic spectrum Y is introduced in [43, §2]. The spectrum $F^q Y$ is obtained from Y as that part of it that can be constructed from $(2q, q)$ -fold suspensions of suspension spectra of smooth varieties. Smashing two motivic spheres amounts to adding the indices, so the filtration is compatible with any multiplication on Y . The layer $s_p(Y) := F^{p/p+1}Y$ is called the p -th *slice* of Y .

Voevodsky states a number of interlocking conjectures about slice filtrations of various standard spectra [43]. For example, Conjecture 1 states that the slice filtration of $H_{\mathbb{Z}}$ is trivial, i.e., the slice $s_0(H_{\mathbb{Z}})$ includes the whole thing.

Conjecture 10 (the main conjecture) says $s_0(\mathbf{1}) = H_{\mathbb{Z}}$, where $\mathbf{1}$ denotes the sphere spectrum. By compatibility with multiplication, a corollary would be that the slices of any motivic spectrum are modules over $H_{\mathbb{Z}}$.

Conjecture 7 says that $s_0(KGL) = H_{\mathbb{Z}}$. Since KGL is $(2, 1)$ -periodic, Conjecture 7 implies that $s_q(KGL) = \Sigma^{2q,q}H_{\mathbb{Z}}$, thereby identifying the E_2 term of the spectral sequence, and providing a motivic spectral sequence of the desired form. In [47] Voevodsky shows Conjecture 7 is implied by Conjecture 10 and a seemingly simpler conjecture that doesn't refer to K -theory or the spectrum representing it. In [49] Voevodsky proves Conjecture 10 over fields of characteristic 0, providing good evidence for the conjecture in general.

Filtrations for General Cohomology Theories

2.11

We'll sketch briefly some of Levine's ideas from [29] that lead to a new replacement for the spectral sequence construction of Bloch–Lichtenbaum [8].

Levine's *homotopy coniveau filtration* is defined for a contravariant functor E from the category of smooth schemes over a noetherian separated scheme S of finite dimension to the category of spectra, but for simplicity, since some of his results require it, we'll assume S is the spectrum of an infinite field k .

In Sect. 2.8 we defined K -theory with supports. The same definitions can be applied to the functor E as follows. If V is a closed subset of X , we let $E^V(X)$ denote the homotopy fiber of the map $E(X) \rightarrow E(X - V)$. If $\mathcal{V}$ is a family of closed subsets of X that is closed under finite unions, then we let $E^{\mathcal{V}}(X)$ denote the inductive limit $\operatorname{colim}_{V \in \mathcal{V}} E^V(X)$.

For each smooth variety X Levine provides a natural filtration $E(X) = F^0 E(X) \leftarrow F^1 E(X) \leftarrow F^2 E(X) \leftarrow \dots$ of $E(X)$ as follows.

In Sect. 2.8 we introduced the family of supports $\mathcal{V}^n$ on $X \times \mathbb{A}^p$. Levine modifies the definition of it slightly, considering it instead to be the family of closed subsets of $X \times \mathbb{A}^p$ of codimension at least n meeting each face in a subset of codimension at least n in that face; when X is a quasiprojective variety it amounts to the same thing. We define $F^n E(X) = |p \mapsto E^{\mathcal{V}^n}(X \times \mathbb{A}^p)|$.

We say that E is *homotopy invariant* if for all smooth varieties X , the map $E(X) \rightarrow E(X \times \mathbb{A}^1)$ is a weak homotopy equivalence. Homotopy invariance of E ensures that the natural map $E(X) \rightarrow F^0 E(X)$ is an equivalence, so the filtration above can be regarded as a filtration of $E(X)$. Taking homotopy groups and exact couples leads to a spectral sequence, as before, but the homotopy groups of E should be bounded below if the spectral sequence is to converge. Levine's *axiom 1* is homotopy invariance of E .

The terms $F^n E$ and the layers $F^{n|n+1} E$ of the filtration are contravariant functors from the category of smooth varieties (but with just the equidimensional maps) to the category of spectra, so the procedure can be iterated. In particular, we can consider $(F^n E)^V(X)$, $(F^n E)^{\vee}(X)$, and $F^m F^n E(X)$.

We say that E satisfies *Nisnevich excision* if for any étale map $f: X' \rightarrow X$ and for any closed subset $V \subseteq X$ for which the map f restricts to an isomorphism from $V' := f^{-1}(V)$ to V , it follows that the map $E^V(X) \rightarrow E^{V'}(X')$ is a weak homotopy equivalence. (A special case is where f is the inclusion of an open subset X' of X into X , so that $V' = V \subseteq X' \subseteq X$.) Levine's *axiom 2* is Nisnevich excision for E .

The first main consequence of assuming that E satisfies Nisnevich excision is the localization theorem [29, 2.2.1], which, for a closed subset Z of X , identifies $(F^n E)^Z(X)$ with $|p \mapsto E^{\mathcal{W}}(X \times \mathbb{A}^p)|$, where $\mathcal{W}$ is the family of closed subsets of $X \times \mathbb{A}^p$ that meet each face in a subset of codimension at least n and are contained in $Z \times \mathbb{A}^p$. There is an analogous statement for the layers.

Define the p -fold T -loop space $\Omega_T^p E$ of E by the formula $(\Omega_T E)(X) := E^{X \times 0}(X \times \mathbb{A}^p)$. Levine's *axiom 3* is that there is a functor E' satisfying axioms 1 and 2 and there is a natural weak equivalence $E \xrightarrow{\sim} \Omega_T^2 E'$.

The first main consequence of axiom 3 (which implies axioms 1 and 2) is the *moving lemma*, which is phrased as follows. Let $f: Y \rightarrow X$ be a map of smooth varieties over k . Let $\mathcal{U}^n$ be the family of closed subsets $V \subseteq X \times \mathbb{A}^p$ of codimension at least n meeting the faces in sets of codimension at least n , whose pullbacks $f^{-1}(V)$ have the same property, and let $F_f^n E(X) = |p \mapsto E^{\mathcal{U}^n}(X \times \mathbb{A}^p)|$. Levine's moving lemma states that, provided X is a smooth variety of dimension d admitting a closed embedding into $\mathbb{A}^{d+2}$ with trivial normal bundle², the map $F_f^n E(X) \rightarrow F^n E(X)$ is a weak homotopy equivalence.

Finally, we say that E is *well-connected* if it satisfies axioms 1, 2, and 3, and: (1) for every smooth variety X and every closed subset $W \subseteq X$, the spectrum $E^W(X)$ is -1 -connected; and (2) $\pi_n(F^{0|1}(\Omega_T^d E)(F)) = 0$ for every finitely generated field extension F of k , for every $d \geq 0$, and for every $n \neq 0$.

The main virtue of a well-connected functor E is that it allows computations in terms of cycles. Indeed, Levine defines a generalization of Bloch's higher Chow groups that is based on E , which enters into computations of the layers in the filtration.

The objects $E(X)$ are already spectra, but the additional T -loop space functor Ω_T (or a related version called $\Omega_{\mathbb{P}^1}$) allows for the possibility of considering spectra formed with respect to it, whose terms are spectra in the usual sense.

² all small enough open subsets of X have such an embedding

The associated machinery allows Levine to compare his coniveau filtration with the slice filtration of Voevodsky in Sect. 2.10 and conclude they are equal. Finally, he is able to construct a *homotopy coniveau* spectral sequence, analogous to the Atiyah–Hirzebruch spectral sequence, converging to the homotopy groups of $E(X)$ (suitably completed to ensure convergence). In the case where $E = K$ is K -theory itself, he checks that K is well-connected and that the spectral sequence agrees with the Bloch–Lichtenbaum spectral sequence of Sect. 2.8 as globalized by Friedlander–Suslin in Sect. 2.9.

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K-Theory of Truncated Polynomial Algebras*

Lars Hesselholt

3.1	<i>Introduction</i>	72
3.2	<i>Topological Hochschild Homology</i>	74
3.3	<i>The Topological Hochschild Spectrum</i>	77
3.4	<i>Equivariant Homotopy Theory</i>	79
3.5	<i>Pointed Monoid Algebras</i>	81
3.6	<i>The Cyclic Bar-construction of Π_e</i>	86
3.7	<i>Topological Cyclic Homology</i>	87
3.8	<i>The de Rham–Witt Complex</i>	90
3.9	<i>Cyclic Homology of $A[x]/(x^e)$</i>	92
3.10	<i>Topological Cyclic Homology of $A[x]/(x^e)$</i>	94
3.11	<i>The Characteristic Zero Case</i>	98
3.12	<i>The Groups $\mathrm{TR}_{q-\lambda}^n(A; p)$</i>	99
3.13	<i>The Positive Characteristic Case</i>	103
3.14	<i>Miscellaneous</i>	105
	<i>References</i>	107

* Supported in part by a grant from the National Science Foundation

Introduction

In general, if A is a ring and $I \subset A$ a two-sided ideal, one defines the K -theory of A relative to I to be the mapping fiber of the map of K -theory spectra induced by the canonical projection from A to A/I . Hence, there is a natural exact triangle of spectra

$$K(A, I) \rightarrow K(A) \rightarrow K(A/I) \xrightarrow{\partial} K(A, I)[-1]$$

and an induced natural long-exact sequence of K -groups

$$\dots \rightarrow K_q(A, I) \rightarrow K_q(A) \rightarrow K_q(A/I) \xrightarrow{\partial} K_{q-1}(A, I) \rightarrow \dots$$

If the ideal $I \subset A$ is nilpotent, the relative K -theory $K(A, I)$ can be expressed completely in terms of the cyclic homology of Connes [14] and the topological cyclic homology of Bökstedt–Hsiang–Madsen [7]. Indeed, on the one hand, Goodwillie [21] has shown that rationally

$$K_q(A, I) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{HC}_q^-(A \otimes \mathbb{Q}, I \otimes \mathbb{Q}) \xleftarrow{\sim} \mathrm{HC}_{q-1}(A, I) \otimes \mathbb{Q},$$

and on the other hand, McCarthy [47] has shown that p -adically

$$K_q(A, I, \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{TC}_q(A, I; p, \mathbb{Z}_p).$$

In both cases, the argument uses the calculus of functors in the sense of Goodwillie [22, 23]. Thus, the problem of evaluating the relative K -theory is translated to the problem of evaluating the relative cyclic theories. The definitions of cyclic homology and of topological cyclic homology are given in Sect. 3.7 below.

Let A be a commutative algebra over a field k and suppose that A is a regular noetherian ring. By Popescu [50] (see also [54]), this is equivalent to A being a filtered colimit of smooth k -algebras. It is then possible by the above approach to completely evaluate the groups $K_q(A[x]/(x^e), (x))$ for truncated polynomial algebras over A relative to the ideal generated by the variable. The calculation of Connes' cyclic homology follows from Masuda and Natsume [46] and Kassel [38], but see also the Buenos Aires Cyclic Homology Group [24]. The topological cyclic homology was evaluated by Madsen and the author [28, 30].

If the field k has characteristic zero, the relative K -groups are expressed in terms of the (absolute) differential forms $\Omega_A^* = \Omega_{A/\mathbb{Z}}^*$ of the ring A . In short, Ω_A^* is the initial example of an anti-commutative differential graded ring E^* with a ring homomorphism $\lambda: A \rightarrow E^0$. The result, which we prove in Sect. 3.11 below, then is a natural (in A) isomorphism of abelian groups

$$K_{q-1}(A[x]/(x^e), (x)) \xleftarrow{\sim} \bigoplus_{m \geq 1} \left(\Omega_A^{q-2m} \right)^{e-1}.$$

Here the superscript $e-1$ on the right indicates product. In particular, the relative K -groups are uniquely divisible groups. For $e=2$ and $q=3$, this was first obtained by van der Kallen [56].

If the field k has positive characteristic p , the relative K -groups are expressed in terms of the (big) de Rham–Witt differential forms $W.\Omega_A^*$ of the ring A . One defines $W.\Omega_A^*$ as the initial example of a *big Witt complex* [30]. The result, which we prove in Sect. 3.13 below, then is a natural (in A) long-exact sequence of abelian groups

$$\dots \rightarrow \bigoplus_{m \geq 1} W_m \Omega_A^{q-2m} \xrightarrow{V_e} \bigoplus_{m \geq 1} W_{me} \Omega_A^{q-2m} \rightarrow K_{q-1}(A[x]/(x^e), (x)) \rightarrow \dots$$

In particular, the relative K -groups are p -primary torsion groups. The result for A a finite field, $e = 2$, and $q \leq 4$ was obtained first by Evens and Friedlander [15] and by Aisbett, Lluís-Puebla and Snaith [2]. The big de Rham–Witt groups and the map V_e can be described in terms of the more familiar p -typical de Rham–Witt groups $W.\Omega_A^*$ of Bloch–Deligne–Illusie [5, 35]. There is a canonical decomposition

$$W_m \Omega_A^* \xrightarrow{\sim} \prod_d W_{s,d} \Omega_A^*,$$

where on the right $1 \leq d \leq m$ and prime to p , and where $s = s(m, d)$ is given by $p^{s-1}d \leq m < p^s d$. Moreover, if we write $e = p^v e'$ with e' prime to p , then the map V_e takes the factor $W_s \Omega_A^*$ indexed by $1 \leq d \leq m$ to the factor $W_{s+v} \Omega_A^*$ indexed by $1 \leq de' \leq me$ by the map

$$e' V^v: W_s \Omega_A^* \rightarrow W_{s+v} \Omega_A^*.$$

If k is *perfect* and if A is smooth of relative dimension r over k , then the groups $W_m \Omega_A^q$ are concentrated in degrees $0 \leq q \leq r$. The de Rham–Witt complex is discussed in Sect. 3.8 below.

Finally, we remark that regardless of the characteristic of the field k , the spectrum $K(A[x]/(x^e), (x))$ is a product of Eilenberg–MacLane spectra, and hence, its homotopy type is completely determined by the homotopy groups.

All rings (resp. graded rings, resp. monoids) considered in this paper are assumed to be commutative (resp. graded-commutative, resp. commutative) and unital. We denote by $\mathbb{N}$ (resp. by $\mathbb{N}_0$, resp. by I_p) the set of positive integers (resp. non-negative integers, resp. positive integers prime to p). By a pro-object of a category $\mathcal{C}$ we mean a functor from $\mathbb{N}$, viewed as a category with one arrow from $n+1$ to n , to $\mathcal{C}$, and by a *strict* map between pro-objects we mean a natural transformation. A general map between pro-objects X and Y of $\mathcal{C}$ is an element of

$$\mathrm{Hom}_{\mathrm{pro-}\mathcal{C}}(X, Y) = \lim_n \mathrm{colim}_m \mathrm{Hom}_{\mathcal{C}}(X_m, Y_n).$$

We view objects of $\mathcal{C}$ as constant pro-objects of $\mathcal{C}$. We denote by $\mathbb{T}$ the multiplicative group of complex numbers of modulus one and by $C_r \subset \mathbb{T}$ the subgroup of order r . A map of $\mathbb{T}$ -spaces (resp. $\mathbb{T}$ -spectra) is an $\mathcal{F}$ -equivalence if the induced map of C_r -fixed points is a weak equivalence of spaces (resp. spectra), for all $C_r \subset \mathbb{T}$.

This paper was written on visits to the Australian National University and the University of Tokyo. It is a pleasure to thank the two institutions for their hospitality and support.

Topological Hochschild Homology

We first recall the Hochschild complex associated with the ring A . This is the cyclic abelian group $\mathrm{HH}(A)$, with k -simplices

$$\mathrm{HH}(A)_k = A \otimes \dots \otimes A \quad (k+1 \text{ factors})$$

and with the cyclic structure maps

$$\begin{aligned} d_r(a_0 \otimes \dots \otimes a_k) &= a_0 \otimes \dots \otimes a_r a_{r+1} \otimes \dots \otimes a_k, & 0 \leq r < k, \\ &= a_k a_0 \otimes a_1 \otimes \dots \otimes a_{k-1}, & r = k, \\ s_r(a_0 \otimes \dots \otimes a_k) &= a_0 \otimes \dots \otimes a_r \otimes 1 \otimes a_{r+1} \otimes \dots \otimes a_k, & 0 \leq r \leq k, \\ t_k(a_0 \otimes \dots \otimes a_k) &= a_k \otimes a_0 \otimes a_1 \otimes \dots \otimes a_{k-1}. \end{aligned}$$

The Hochschild homology groups $\mathrm{HH}_*(A)$ are defined as the homology groups of the associated chain complex (with differential given by the alternating sum of the face maps d_r), or equivalently [58, theorem 8.4.1] as the homotopy groups of the geometric realization of the underlying simplicial set

$$\mathrm{HH}(A) = |[k] \mapsto \mathrm{HH}(A)_k|.$$

It was discovered by Connes that the action of the cyclic group of order $k+1$ on the set of k -simplices $\mathrm{HH}(A)_k$ gives rise to a continuous $\mathbb{T}$ -action on the space $\mathrm{HH}(A)$, see [41, 7.1.9] or [37].

We next recall the topological Hochschild space $\mathrm{THH}(A)$. The idea in the definition is to change the ground ring for the tensor product in the Hochschild complex from the ring of integers to the sphere spectrum. This was carried out by Bökstedt [6] and, as it turns out, before him by Breen [10]. To give the definition, we first associate a commutative symmetric ring spectrum $\tilde{A}$ in the sense of Hovey–Shipley–Smith [34] to the ring A . Let $S^1 = \Delta^1 / \partial \Delta^1$ be the standard simplicial circle, and let $S^i = S^1 \wedge \dots \wedge S^1$ be the i -fold smash product. Then

$$\tilde{A}_i = |[k] \mapsto A\{S_k^i\} / A\{s_{0,k}\}|$$

is an Eilenberg–MacLane space for A concentrated in degree i . Here $A\{S_k^i\}$ is the free A -module generated by the set of k -simplices S_k^n , and $A\{s_{0,k}\}$ is the sub- A -module generated by the base-point. The action of the symmetric group Σ_i by permutation of the i smash factors of S^i induces a Σ_i -action on $\tilde{A}_i$. In addition, there are natural multiplication and unit maps

$$\mu_{i,i'}: \tilde{A}_i \wedge \tilde{A}_{i'} \rightarrow \tilde{A}_{i+i'}, \quad \eta_i: S^i \rightarrow \tilde{A}_i,$$

which are $\Sigma_i \times \Sigma_{i'}$ -equivariant and $\Sigma_{i+i'}$ -equivariant, respectively.

Let I be the category with objects the finite sets

$$\underline{i} = \{1, 2, \dots, i\}, \quad i \geq 1,$$

and the empty set $\underline{0}$, and with morphisms all injective maps. We note that every morphism in I can be written (non-uniquely) as the standard inclusion followed by an automorphism. Concatenation of sets and maps defines a strict monoidal (but not symmetric monoidal) structure on I . There is a functor $G_k(A; X)$ from I^{k+1} to pointed spaces that on objects is given by

$$G_k(A)(\underline{i}_0, \dots, \underline{i}_k) = F(S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k}), \quad (3.1)$$

where the right-hand side is the space of continuous base-point preserving maps with the compact-open topology. Let $\iota: \underline{i}_r \rightarrow \underline{i}'_r$ be the standard inclusion and write $i'_r = i_r + j_r$. Then $G_k(A)(\underline{i}_0, \dots, \underline{i}, \dots, \underline{i}_k)$ takes the map

$$S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \xrightarrow{f} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \dots \wedge \tilde{A}_{i_k}$$

to the composite

$$\begin{aligned} S^{i_0} \wedge \dots \wedge S^{i'_r} \wedge \dots \wedge S^{i_k} &\xrightarrow{\sim} S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \wedge S^{j_r} \\ &\xrightarrow{f \wedge \eta_{j_r}} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \dots \wedge \tilde{A}_{i_k} \wedge \tilde{A}_{j_r} \\ &\xrightarrow{\sim} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \tilde{A}_{j_r} \wedge \dots \wedge \tilde{A}_{i_k} \\ &\xrightarrow{\mu_{i_r j_r}} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i'_r} \wedge \dots \wedge \tilde{A}_{i_k}, \end{aligned}$$

where the first and third maps are the canonical isomorphisms (in the symmetric monoidal category of pointed spaces and smash product [43]), and where we have suppressed identity maps. The symmetric group $\Sigma_{i'_r}$ acts on $S^{i'_r}$ and $\tilde{A}_{i'_r}$ and by conjugation on $G_k(\underline{i}_0, \dots, \underline{i}'_r, \dots, \underline{i}_k)$. This defines the functor G_k on morphisms.

One now defines a cyclic space $\mathrm{THH}(A)$, with k -simplices the homotopy colimit (see [9, §XII] for the definition of homotopy colimits)

$$\mathrm{THH}(A)_k = \mathrm{hocolim}_{I^{k+1}} G_k(A). \quad (3.2)$$

Although this is not a filtered homotopy colimit, it still has the desired homotopy type in that the canonical map

$$G_k(A)(\underline{i}_0, \dots, \underline{i}_k) \rightarrow \mathrm{THH}(A)_k \quad (3.3)$$

is $(i_0 + \dots + i_k - 1)$ -connected. The proof of this *approximation lemma* is similar to the proof of the lemma in the proof of Quillen's theorem B [51] and can be found in [44, lemma 2.3.7]. We define the face maps

$$d_r: \mathrm{THH}(A)_k \rightarrow \mathrm{THH}(A)_{k-1}, \quad 0 \leq r \leq k; \quad (3.4)$$

the degeneracies and the cyclic operator are defined in a similar manner. We let $\mathfrak{d}_r: I^{k+1} \rightarrow I^k$ be the functor given by

$$\begin{aligned} \mathfrak{d}_r(\underline{i}_0, \dots, \underline{i}_k) 0 &= (\underline{i}_0, \dots, \underline{i}_r \sqcup \underline{i}_{r+1}, \dots, \underline{i}_k), \quad 0 \leq r < k, \\ &= (\underline{i}_k \sqcup \underline{i}_0, \underline{i}_1, \dots, \underline{i}_{k-1}), \quad r = k, \end{aligned}$$

on objects and similarly on morphisms, and let

$$\delta_r: G_k(A) \rightarrow G_{k-1} \circ \mathfrak{d}_r$$

be the natural transformation that takes the map

$$S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \xrightarrow{f} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \dots \wedge \tilde{A}_{i_k}$$

to the composite

$$\begin{aligned} S^{i_0} \wedge \dots \wedge S^{i_r+i_{r+1}} \wedge \dots \wedge S^{i_k} &\xrightarrow{\sim} S^{i_0} \wedge \dots \wedge S^{i_r} \wedge S^{i_{r+1}} \wedge \dots \wedge S^{i_k} \\ &\xrightarrow{f} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r} \wedge \tilde{A}_{i_{r+1}} \wedge \dots \wedge \tilde{A}_{i_k} \\ &\xrightarrow{\mu_{i_r, i_{r+1}}} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_r+i_{r+1}} \wedge \dots \wedge \tilde{A}_{i_k}, \end{aligned}$$

if $0 \leq r < k$, and to the composite

$$\begin{aligned} S^{i_k+i_0} \wedge S^{i_1} \wedge \dots \wedge S^{i_{k-1}} &\xrightarrow{\sim} S^{i_0} \wedge \dots \wedge S^{i_{k-1}} \wedge S^{i_k} \\ &\xrightarrow{f} \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_{k-1}} \wedge \tilde{A}_{i_k} \\ &\xrightarrow{\sim} \tilde{A}_{i_k} \wedge \tilde{A}_{i_0} \wedge \tilde{A}_{i_1} \wedge \dots \wedge \tilde{A}_{i_{k-1}} \\ &\xrightarrow{\mu_{i_k, i_0}} \tilde{A}_{i_k+i_0} \wedge \tilde{A}_{i_1} \wedge \dots \wedge \tilde{A}_{i_{k-1}}, \end{aligned}$$

if $r = k$. Here again the unnamed isomorphisms are the canonical ones. The face map d_r in (3.4) is now defined to be the composite

$$\mathrm{hocolim}_{I^{k+1}} G_k(A) \xrightarrow{\delta_r} \mathrm{hocolim}_{I^{k+1}} G_{k-1}(A) \circ \mathfrak{d}_r \xrightarrow{\mathfrak{d}_r^*} \mathrm{hocolim}_{I^k} G_{k-1}(A).$$

Then Bökstedt's topological Hochschild space is the $\mathbb{T}$ -space

$$\mathrm{THH}(A) = |[k] \mapsto \mathrm{THH}(A)_k|. \quad (3.5)$$

The homotopy groups $\mathrm{THH}_*(A)$ can be defined in several other ways. Notably, Pirashvili–Waldhausen [49] have shown that these groups are canonically isomorphic to the homology groups of the category $\mathcal{P}(A)$ of finitely generated projective A -modules with coefficients in the bifunctor Hom , as defined by Jibladze–Pirashvili [36].

The Topological Hochschild Spectrum

It is essential for understanding the topological cyclic homology of truncated polynomial algebras that topological Hochschild homology be defined not only as a $\mathbb{T}$ -space $\mathrm{THH}(A)$ but as a $\mathbb{T}$ -spectrum $T(A)$.

In general, if G is a compact Lie group, the G -stable category is a triangulated category and a closed symmetric monoidal category, and the two structures are compatible [40, II.3.13]. The objects of the G -stable category are called G -spectra. A monoid for the smash product is called a ring G -spectrum. We denote the set of maps between two G -spectra T and T' by $[T, T']_G$. Associated with a pointed G -space X one has the suspension G -spectrum which we denote again by X . If λ is a finite dimensional orthogonal G -representation, we denote by S^λ its one-point compactification. Then the G -stable category is stable in the strong sense that the suspension homomorphism

$$[T, T']_G \xrightarrow{\sim} [T \wedge S^\lambda, T' \wedge S^\lambda]_G \quad (3.6)$$

is an isomorphism [40, I.6.1]. As a model for the G -stable category we use symmetric orthogonal G -spectra; see [45] and [33, theorem 5.10].

Let X be a pointed space. Then there is a functor $G_k(A; X)$ from I^{k+1} to pointed spaces that is defined on objects by

$$G_k(A; X) (\underline{i}_0, \dots, \underline{i}_k) = F(S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge X)$$

and on morphisms in a manner similar to (3.1). If n is a non-negative integer, we let (n) be the finite ordered set $\{1, 2, \dots, n\}$ and define $I^{(n)}$ to be the product category. (The category $I^{(0)}$ is the category with one object and one morphism.) The category $I^{(n)}$ is a strict monoidal category under component-wise concatenation of sets and maps. In addition, there is a functor

$$\sqcup_n: I^{(n)} \rightarrow I$$

given by the concatenation of sets and maps according to the ordering of (n) . (The functor $\sqcup_0$ takes the unique object to $\underline{0}$.) Let $G_k^{(n)}(A; X)$ be the functor from $(I^{(n)})^{k+1}$ to pointed spaces defined as the composition

$$G_k^{(n)}(A, X) = G_k(A, X) \circ (\sqcup_n)^{k+1}.$$

There is a cyclic space $\mathrm{THH}_k^{(n)}(A; X)$ with k -simplices the homotopy colimit

$$\mathrm{THH}_k^{(n)}(A; X) = \mathrm{hocolim}_{(I^{(n)})^{k+1}} G_k^{(n)}(A; X)$$

and with the cyclic structure maps defined by the same formulas as in (3.4), the only difference being that the concatenation functor $\sqcup$ in the formula for the functor ∂_r must be replaced by the component-wise concatenation functor $\sqcup^{(n)}$. Then we define the $\mathbb{T}$ -space

$$\mathrm{THH}^{(n)}(A; X) = |[k] \mapsto \mathrm{THH}_k^{(n)}(A; X)|.$$

An ordered inclusion $\iota: (n) \rightarrow (n')$ gives rise to a $\mathbb{T}$ -equivariant map

$$\iota_*: \mathrm{THH}^{(n)}(A; X) \rightarrow \mathrm{THH}^{(n')}(A; X),$$

which, by the approximation lemma (3.3), is an equivalence of $\mathbb{T}$ -spaces, provided that $n \geq 1$. For $n = 0$, there is canonical $\mathbb{T}$ -equivariant homeomorphism

$$N^{\mathrm{cy}}(A) \wedge X \xrightarrow{\sim} \mathrm{THH}^{(0)}(A; X),$$

where the first smash factor on the left is the cyclic bar-construction of the pointed monoid given by the ring A which we now recall.

We define a *pointed monoid* to be a monoid in the symmetric monoidal category of pointed spaces and smash product. The ring A determines a pointed monoid, which we also denote A , with A considered as a pointed set with basepoint 0 and with the multiplication and unit maps given by multiplication and unit maps from the ring structure. Let $N^{\mathrm{cy}}(\Pi)$ be the cyclic space with k -simplices

$$N_k^{\mathrm{cy}}(\Pi) = \Pi \wedge \dots \wedge \Pi \quad (k+1 \text{ times}) \quad (3.7)$$

and with the Hochschild-type cyclic structure maps

$$\begin{aligned} d_i(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_0 \wedge \dots \wedge \pi_i \pi_{i+1} \wedge \dots \wedge \pi_k, & 0 \leq i < k, \\ &= \pi_k \pi_0 \wedge \pi_1 \wedge \dots \wedge \pi_{k-1}, & i = k, \\ s_i(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_0 \wedge \dots \wedge \pi_i \wedge 1 \wedge \pi_{i+1} \wedge \dots \wedge \pi_k, & 0 \leq i \leq k, \\ t_k(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_k \wedge \pi_0 \wedge \pi_1 \wedge \dots \wedge \pi_{k-1}. \end{aligned}$$

Then $N^{\mathrm{cy}}(\Pi)$ is the geometric realization.

We define of the symmetric orthogonal $\mathbb{T}$ -spectrum $T(A)$. Let n be a non-negative integer, and let λ be a finite-dimensional orthogonal $\mathbb{T}$ -representation. Then the (n, λ) th space of $T(A)$ is defined to be the space

$$T(A)_{n, \lambda} = \mathrm{THH}^{(n)}(A; S^n \wedge S^\lambda) \quad (3.8)$$

with the diagonal $\mathbb{T}$ -action induced by the $\mathbb{T} \times \mathbb{T}$ -action, where the action by one factor is induced by the $\mathbb{T}$ -action on λ , and where the other is the canonical action by $\mathbb{T}$ on the realization of a cyclic space. Similarly, we give $T(A)_{n, \lambda}$ the diagonal Σ_n -action induced by the $\Sigma_n \times \Sigma_n$ -action, where the two factors act by permutation respectively on (n) and on the smash factors of $S^n = S^1 \wedge \dots \wedge S^1$. In particular, $T(A)_{0,0} = N^{\mathrm{cy}}(A)$. The definition of the $\Sigma_n \times \Sigma_{n'} \times \mathbb{T}$ -equivariant spectrum structure maps

$$\sigma_{n, n', \lambda, \lambda'}: T(A)_{n, \lambda} \wedge S^{n'} \wedge S^{\lambda'} \rightarrow T(A)_{n+n', \lambda \oplus \lambda'}$$

is straightforward, but can be found in [20, section 2.2]. We also refer to *op. cit.*, appendix, for the definition of the ring spectrum product maps

$$\mu_{n, \lambda, n', \lambda'}: T(A)_{n, \lambda} \wedge T(A)_{n', \lambda'} \rightarrow T(A)_{n+n', \lambda \oplus \lambda'},$$

which are $\Sigma_n \times \Sigma_{n'} \times \mathbb{T}$ -equivariant with $\mathbb{T}$ acting diagonally on the left. With this product $T(A)$, becomes a commutative $\mathbb{T}$ -ring spectrum. The following result is proved in [29, proposition 2.4].

Proposition 1 Suppose that $n \geq 1$. Then the adjoint of the structure map

1

$$\tilde{\sigma}_{n,n',\lambda,\lambda'}: T(A)_{n,\lambda} \rightarrow F\left(S^{n'} \wedge S^{\lambda'}, T(A)_{n+n',\lambda \oplus \lambda'}\right)$$

is an $\mathcal{F}$ -equivalence of pointed $\mathbb{T}$ -spaces.

As a corollary of the proposition, we have a canonical isomorphism

$$\mathrm{THH}_q(A) \xrightarrow{\sim} [S^q \wedge \mathbb{T}_+, T(A)]_{\mathbb{T}}.$$

We also define a Hochschild $\mathbb{T}$ -spectrum $H(A)$ and a map of $\mathbb{T}$ -spectra

$$\ell: T(A) \rightarrow H(A),$$

which is called the linearization map. The construction of $H(A)$ is completely analogous to that of $T(A)$ but with the functor $G_k(A)$ replaced by the functor $G'_k(A)$ that is defined on objects by

$$G'_k(A)(\underline{i}_0, \dots, \underline{i}_k) = F\left(S^{i_0} \wedge \dots \wedge S^{i_k}, (A \widetilde{\otimes} \dots \otimes A)_{i_0+\dots+i_k}\right)$$

and on morphisms in a manner similar to the formulas following (3.1). There are $k+1$ tensor factors on the right and the ground ring for the tensor products is the ring of rational integers.

Equivariant Homotopy Theory

3.4

Before we proceed, we discuss a few concepts and elementary results from the homotopy theory of G -spaces. We refer the reader to Adams [1] for an introduction to this material.

The homotopy category of pointed spaces is equivalent to the category of pointed CW-complexes and pointed homotopy classes of pointed cellular maps. Similarly, the homotopy category of pointed G -spaces is equivalent to the category of pointed G -CW-complexes and pointed G -homotopy classes of pointed cellular G -maps. Let G be a compact Lie group. Then a pointed G -CW-complex is a pointed (left) G -space X together with a sequence of pointed sub- G -spaces

$$* = \mathrm{sk}_{-1} X \subset \mathrm{sk}_0 X \subset \mathrm{sk}_1 X \subset \dots \subset \mathrm{sk}_n X \subset \dots \subset X,$$

and for all $n \geq 0$, a push-out square of (un-pointed) G -spaces

$$\begin{array}{ccc} \coprod_{\alpha} \partial D^n \times G/H_{\alpha} & \xrightarrow{\varphi_n|_{\partial}} & \mathrm{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \coprod_{\alpha} D^n \times G/H_{\alpha} & \xrightarrow{\varphi_n} & \mathrm{sk}_n X, \end{array}$$

with each $H_{\alpha} \subset G$ a closed subgroup, such that the canonical pointed G -map

$$\mathrm{colim}_n \mathrm{sk}_n X \rightarrow X$$

is a homeomorphism. The map φ_n restricts to an embedding on the interior of $D^n \times G/H_{\alpha}$. We say that the image of this embedding is a cell of dimension n and orbit-type G/H_{α} .

A pointed G -map $f: X \rightarrow X'$ between pointed G -CW-complexes is cellular if $f(\mathrm{sk}_n X) \subset \mathrm{sk}_n X'$, for all $n \geq -1$.

2

Lemma 2 Let G be a finite group, let X be a pointed G -CW-complex of finite dimension, and let $d(H)$ be the supremum of the dimension of the cells of X of orbit-type G/H . Let Y be a pointed G -space such that Y^H is $n(H)$ -connected. Then the equivariant mapping space $F(X, Y)^G$ is m -connected with

$$m = \inf\{n(H) - d(H) \mid H \in \mathcal{O}_G(X)\}.$$

Here $\mathcal{O}_G(X)$ denotes the set of subgroups $H \subset G$ for which X has a cell of orbit-type G/H .

Proof We show by induction on $n \geq -1$ that $F(\mathrm{sk}_n X, Y)^G$ is m -connected. The case $n = -1$ is trivial, so we assume the statement for $n - 1$. One shows as usual that the map $\mathrm{sk}_{n-1} X \hookrightarrow \mathrm{sk}_n X$ has the G -homotopy extension property, and that the induced map

$$F(\mathrm{sk}_n X, Y)^G \rightarrow F(\mathrm{sk}_{n-1} X, Y)^G$$

has the homotopy lifting property [52]. The fiber of the latter map over the base-point is canonically homeomorphic to

$$\begin{aligned} F(\mathrm{sk}_n X / \mathrm{sk}_{n-1} X, Y)^G &\xrightarrow{\sim} F\left(\bigvee_{\alpha} S^n \wedge G/H_{\alpha+}, Y\right)^G \\ &\xrightarrow{\sim} \prod_{\alpha} F(S^n \wedge G/H_{\alpha+}, Y)^G \xrightarrow{\sim} \prod_{\alpha} F(S^n, Y^{H_{\alpha}}), \end{aligned}$$

where the first map is induced by the map φ_n . But the space $F(S^n, Y^{H_{\alpha}})$ is $(n(H_{\alpha}) - n)$ -connected and $n(H_{\alpha}) - n \geq m$, so the induction step follows. Since $X = \mathrm{sk}_n X$, for some n , we are done.

Pointed Monoid Algebras

3.5

Let A be a ring, and let Π be a discrete pointed monoid, that is, a monoid in the symmetric monoidal category of pointed sets and smash product. Then the pointed monoid algebra $A(\Pi)$ is defined to be the quotient of the monoid algebra $A[\Pi]$ by the ideal generated by the base-point of Π . For example,

$$A[x]/(x^e) = A(\Pi_e) ,$$

where $\Pi_e = \{0, 1, x, \dots, x^{e-1}\}$ considered as a pointed monoid with base-point 0 and with the multiplication given by $x^e = 0$. There is a canonical maps $\phi: A \rightarrow A(\Pi)$ and $\iota: \Pi \rightarrow A(\Pi)$ of rings and pointed monoids, respectively, given by $\phi(a) = a \cdot 1$ and $\iota(\pi) = 1 \cdot \pi$. The following is [29, theorem 7.1].

Proposition 3 Let A be a ring and Π a pointed monoid. Then the composite

3

$$T(A) \wedge N^{\text{cy}}(\Pi) \xrightarrow{\phi \wedge \iota} T(A(\Pi)) \wedge N^{\text{cy}}(A(\Pi)) \xrightarrow{\mu} T(A(\Pi))$$

is an $\mathcal{F}$ -equivalence of $\mathbb{T}$ -spectra.

Before we give the proof, we mention the analogous result in the linear situation. The derived category of abelian groups is a triangulated category and a symmetric monoidal category. A monoid for the tensor product is called a differential graded ring. If $C_\bullet$ is a simplicial abelian group, we write $C_\bullet^*$ for the associated chain complex. If $R_\bullet$ is a simplicial ring, then $R_\bullet^*$ is a differential graded ring with product given by the composite

$$R_\bullet \otimes R_\bullet \xrightarrow{\theta} (R_\bullet \otimes R_\bullet)_\bullet \xrightarrow{\mu_*} R_\bullet ,$$

where the left-hand map is the Eilenberg–Zilber shuffle map [58, 8.5.4]. If A is a (commutative) ring, then $\text{HH}(A)_\bullet$ is a simplicial ring with the product

$$(a_0 \otimes \dots \otimes a_k) \cdot (a'_0 \otimes \dots \otimes a'_k) = a_0 a'_0 \otimes \dots \otimes a_k a'_k ,$$

and hence $\text{HH}(A)_\bullet^*$ is a differential graded ring. We claim that the composite

$$\text{HH}(A)_\bullet^* \otimes \mathbb{Z}(N_\bullet^{\text{cy}}(\Pi))_\bullet^* \rightarrow \text{HH}(A(\Pi))_\bullet^* \otimes \text{HH}(A(\Pi))_\bullet^* \xrightarrow{\mu_*} \text{HH}(A(\Pi))_\bullet^*$$

is a quasi-isomorphism. Indeed, this map is equal to the composite of the Eilenberg–Zilber map

$$\text{HH}(A)_\bullet^* \otimes \mathbb{Z}(N_\bullet^{\text{cy}}(\Pi))_\bullet^* \xrightarrow{\theta} (\text{HH}(A)_\bullet \otimes \mathbb{Z}(N_\bullet^{\text{cy}}(\Pi)))_\bullet^* ,$$

which is a quasi-isomorphism [58, theorem 8.5.1], and the map of simplicial abelian groups

$$\text{HH}(A)_\bullet \otimes \mathbb{Z}(N_\bullet^{\text{cy}}(\Pi)) \rightarrow \text{HH}(A(\Pi))_\bullet \otimes \text{HH}(A(\Pi))_\bullet \xrightarrow{\mu} \text{HH}(A(\Pi))_\bullet ,$$

which is an isomorphism, since the tensor product of simplicial abelian groups is formed degree-wise.

Proof (*Proof of proposition 3*) We shall use the following criterion for a map $f: X \rightarrow Y$ of symmetric orthogonal $\mathbb{T}$ -spectra to be an $\mathcal{F}$ -equivalence. Suppose that for all integers $n \geq 0$, all finite dimensional orthogonal $\mathbb{T}$ -representations λ , and all finite subgroups $C \subset \mathbb{T}$, the induced map

$$(f_{n,\lambda})^C: (X_{n,\lambda})^C \rightarrow (Y_{n,\lambda})^C$$

is $(n + \dim_{\mathbb{R}}(\lambda^C) + \varepsilon_C(n, \lambda))$ -connected, where $\varepsilon_C(n, \lambda)$ tends to infinity with n and λ . Then $f: X \rightarrow Y$ is an $\mathcal{F}$ -equivalence of $\mathbb{T}$ -spectra. This follows directly from the definition of $\mathcal{F}$ -equivalence [33, 45].

We note that a C -equivariant isometric isomorphism $\lambda \xrightarrow{\sim} \lambda'$ between two finite dimensional orthogonal $\mathbb{T}$ -representations induces a natural C -equivariant homeomorphism $T(A)_{n,\lambda} \xrightarrow{\sim} T(A)_{n,\lambda'}$. Hence, for a given finite subgroup $C \subset \mathbb{T}$, it suffices to consider the map

$$(f_{n,\lambda})^C: (T(A)_{n,\lambda})^C \wedge N^{\text{cy}}(\Pi)^C \rightarrow (T(A(\Pi))_{n,\lambda})^C$$

induced by the map of the statement in the case where λ is a direct sum of copies of the regular representation ϱ_C . To prove the proposition, we show that the map $(f_{n,m\varrho_C})^C$ is $(n + m + m - 1)$ -connected. We first unravel the definition of this map.

The composite of canonical maps

$$\begin{aligned} & F(S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge S^{n+m\varrho_C}) \wedge \Pi^{\wedge(k+1)} \\ & \rightarrow F(S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge \Pi^{\wedge(k+1)} \wedge S^{n+m\varrho_C}) \\ & \rightarrow F(S^{i_0} \wedge \dots \wedge S^{i_k}, \tilde{A}(\Pi)_{i_0} \wedge \dots \wedge \tilde{A}(\Pi)_{i_k} \wedge S^{n+m\varrho_C}) \end{aligned} \quad (3.9)$$

defines a natural transformation of functors from I^{k+1} to pointed spaces

$$G_k(A; S^{n+m\varrho_C}) \wedge N_k^{\text{cy}}(\Pi) \rightarrow G_k(A(\Pi); S^{n+m\varrho_C}).$$

If we pre-compose on both sides by the functor $(\sqcup_n)^{k+1}$, we get a similar natural transformation with $G_k^{(n)}$ in place of G_k . Taking homotopy colimits over $(I^{(n)})^{k+1}$, we obtain a map

$$\text{THH}_k^{(n)}(A; S^{n+m\varrho_C}) \wedge N_k^{\text{cy}}(\Pi) \rightarrow \text{THH}_k^{(n)}(A(\Pi); S^{n+m\varrho_C}).$$

Here we have used that taking homotopy colimits and smashing by a (fixed) pointed space commute up to canonical homeomorphism. Finally, as k varies, these maps constitute a map of cyclic spaces, and hence we get an induced map after geometric realization

$$\text{THH}^{(n)}(A; S^{n+m\varrho_C}) \wedge N^{\text{cy}}(\Pi) \rightarrow \text{THH}^{(n)}(A(\Pi); S^{n+m\varrho_C}).$$

This is the map $f_{n,m\varrho_C}$ of the statement. Here we have used that the geometric realization of a smash product of pointed simplicial spaces is canonically homeomorphic to the smash product of their geometric realizations.

We next give a similar description of the induced map of C -fixed points $(f_{n,m\varrho_C})^C$. The C -action on the domain and target of the map $f_{n,m\varrho_C}$, we recall, is the diagonal action induced from a natural $C \times C$ -action, where the action by one factor is induced by the C -action on ϱ_C , and where the other is the canonical action by the cyclic group $C \subset \mathbb{T}$ on the realization of a cyclic space. The latter C -action is not induced from simplicial C -action. However, this can be achieved by edge-wise subdivision, which now we recall.

Let X be a simplicial space, and let r be the order of C . Then by [7, lemma 1.1], there is a canonical (non-simplicial) homeomorphism

$$D_r: |[k] \mapsto (\mathrm{sd}_r X)_k| \xrightarrow{\sim} |[k] \mapsto X_k| ,$$

where $\mathrm{sd}_r X$ is the simplicial space with k -simplices given by

$$\mathrm{sd}_r(X)_k = X_{r(k+1)-1}$$

and simplicial structure maps, for $0 \leq i \leq k$, given by

$$d'_i: \mathrm{sd}_r(X_s)_k \rightarrow \mathrm{sd}_r(X_s)_{k-1} , \quad d'_i = d_i \circ d_{i+(k+1)} \circ \dots \circ d_{i+(r-1)(k+1)} ,$$

$$s'_i: \mathrm{sd}_r(X_s)_k \rightarrow \mathrm{sd}_r(X_s)_{k+1} , \quad s'_i = s_{i+(r-1)(k+2)} \circ \dots \circ s_{i+(k+2)} \circ s_i .$$

If X is a cyclic space, then the action by C on $\mathrm{sd}_r(X)_k$, where the generator $e^{2\pi i/r}$ acts as the operator $(t_{r(k+1)-1})^{k+1}$, is compatible with the simplicial structure maps, and hence induces a C -action on the geometric realization. Moreover, the homeomorphism D_r is C -equivariant, if we give the domain and target the C -action induced from the simplicial C -action and from the canonical $\mathbb{T}$ -action, respectively.

In the case at hand, we now consider

$$\mathrm{sd}_r \mathrm{THH} (A; S^{n+m\varrho_C})_k = \mathrm{THH}_{r(k+1)-1} (A; S^{n+m\varrho_C})$$

with the diagonal C -action induced by the $C \times C$ -action, where the generator $e^{2\pi i/r}$ of one C -factor acts as the operator $(t_{r(k+1)-1})^{k+1}$ on the right, and where the action by the other C -factor is induced from the C -action on ϱ_C . This action is not induced from an action on the individual terms of the homotopy colimit that defines the right-hand side. However, if we let $\Delta_{r,k}: I^{k+1} \rightarrow I^{(k+1)r}$ be the diagonal functor given by

$$\Delta_{r,k}(i_0, \dots, i_k) = (i_0, \dots, i_k, \dots, i_0, \dots, i_k) ,$$

then the canonical map of homotopy colimits

$$\mathrm{hocolim}_{I^{k+1}} G_{r(k+1)-1} (A; S^{n+m\varrho_C}) \circ \Delta_{r,k} \rightarrow \mathrm{hocolim}_{I^{r(k+1)}} G_{r(k+1)-1} (A; S^{n+m\varrho_C})$$

induces a homeomorphism of C -fixed sets. On the left, the group C acts trivially on the index category, and hence the action is induced from an action on the individual terms of the homotopy colimit. A typical term is canonically homeomorphic to

$$F((S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}, (\tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k})^{\wedge r} \wedge S^{n+m\varrho_C}) .$$

Moreover, the group C acts on the mapping space by the conjugation action induced from the action on the two r -fold smash products by cyclic permutation of the smash factors and from the action on $S^{n+m\varrho_C}$ induced from the one on ϱ_C . The canonical maps

$$\begin{aligned} & F((S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}, (\tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k})^{\wedge r} \wedge S^{n+m\varrho_C}) \wedge (\Pi^{\wedge(k+1)})^{\wedge r} \\ & \rightarrow F((S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}, (\tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge \Pi^{\wedge(k+1)})^{\wedge r} \wedge S^{n+m\varrho_C}) \\ & \rightarrow F((S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}, (\tilde{A}(\Pi)_{i_0} \wedge \dots \wedge \tilde{A}(\Pi)_{i_k})^{\wedge r} \wedge S^{n+m\varrho_C}) \end{aligned} \quad (3.10)$$

are C -equivariant and their composite defines a natural transformation of functors from I^{k+1} to pointed C -spaces

$$\begin{aligned} & \left(G_{(k+1)r-1}(A; S^{n+m\varrho_C}) \wedge N_{(k+1)r-1}^{\text{cy}}(\Pi) \right) \circ \Delta_{r,k} \\ & \rightarrow G_{(k+1)r-1}(A(\Pi); S^{n+m\varrho_C}) \circ \Delta_{r,k} . \end{aligned}$$

If we pre-compose both sides by the functor $(\sqcup_n)^{k+1}$, we get a similar natural transformation with $G_k^{(n)}$ in place of G_k . Taking C -fixed points and homotopy colimits over $(I^{(n)})^{k+1}$, we obtain the map

$$\begin{aligned} & \left(\text{sd}_r(\text{THH}^{(n)}(A; S^{n+m\varrho_C}) \wedge N^{(n)}(\Pi))_k \right)^C \\ & \rightarrow \left(\text{sd}_r \text{THH}^{(n)}(A(\Pi); S^{n+m\varrho_C})_k \right)^C . \end{aligned}$$

As k varies, these maps constitute a map of simplicial spaces, and hence we get an induced map of the associated geometric realizations. Finally, this map and the canonical homeomorphism D_r determine a map

$$(\text{THH}^{(n)}(A; S^{n+m\varrho_C}) \wedge N^{\text{cy}}(\Pi))^C \rightarrow \text{THH}^{(n)}(A(\Pi); S^{n+m\varrho_C})^C .$$

This is the map of C -fixed points induced by the map $f_{n,m\varrho_C}$ of the statement. Here we have used that geometric realization and finite limits (such as C -fixed points) commute [16, chap. 3, §3].

It remains to show that $(f_{n,m\varrho_C})^C$ is $(n+2m-1)$ -connected as stated. As we have just seen, this is the geometric realization of a map of simplicial spaces. It suffices to show that for the latter map, the induced map of k -simplices is $(n+2m-1)$ -connected, for all $k \geq 0$. Finally, by the approximation lemma (3.3), it suffices to show that for $i_0, \dots, i_k$ large, the maps of C -fixed points induced by the maps (3.10) are $(n+2m-1)$ -connected. Let $i = i_0 + \dots + i_k$.

We first consider the second map in (3.10). The canonical map

$$\tilde{A}_j \wedge \Pi \rightarrow \tilde{A}(\Pi)_j$$

is $(2j - 1)$ -connected. Indeed, this map is the inclusion of a wedge in the corresponding (weak) product. It follows that the canonical map

$$\begin{aligned} & (\tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k} \wedge \Pi^{\wedge(k+1)})^{\wedge r} \wedge S^{n+m\varrho_C} \\ & \rightarrow (\tilde{A}(\Pi)_{i_0} \wedge \dots \wedge \tilde{A}(\Pi)_{i_k})^{\wedge r} \wedge S^{n+m\varrho_C} \end{aligned}$$

is $(2ir + n + mr - 1)$ -connected. Let $C_s \subset C_r$ and $r = st$. Then the induced map of C_s -fixed points is $n(C_s)$ -connected with $n(C_s) = 2it + n + mt - 1$. The supremum of the dimension of the cells of $(S^{i_0} \wedge \dots \wedge S^{i_k})^{\wedge r}$ of orbit-type C_s is $d(C_s) = it$. Hence, by lemma 2, the map of C -fixed points induced from the second map in (3.10) is n -connected with

$$n = \inf \{n(C_s) - d(C_s) \mid C_s \subset C_r\} = i + n + m - 1.$$

Hence, this map is $(n + 2m - 1)$ -connected, if $i \geq m$.

Finally, we consider the first map in (3.10). We abbreviate

$$\begin{aligned} X &= S^{i_0} \wedge \dots \wedge S^{i_k}, & Y &= \tilde{A}_{i_0} \wedge \dots \wedge \tilde{A}_{i_k}, \\ Z &= S^{n+m\varrho_C}, & P &= \Pi^{\wedge(k+1)}, \end{aligned}$$

and consider the following diagram.

$$\begin{array}{ccc} F(X^{\wedge r}, Y^{\wedge r} \wedge Z) \wedge P^{\wedge r} & \longrightarrow & F(X^{\wedge r}, Y^{\wedge r} \wedge Z \wedge P^{\wedge r}) \\ \downarrow \tilde{\delta} & & \downarrow \tilde{\delta}_* \\ F(P^{\wedge r}, F(X^{\wedge r}, Y^{\wedge r} \wedge Z)) & \xrightarrow{\sim} & F(X^{\wedge r}, F(P^{\wedge r}, Y^{\wedge r} \wedge Z)) \end{array} \quad (3.11)$$

where the map $\tilde{\delta}$ is the adjoint of the (pointed) Kronecker delta function

$$\delta: P^{\wedge r} \wedge P^{\wedge r} \rightarrow S^0.$$

The top horizontal map in (3.11) is equal to the first map in (3.10). We wish to show that it induces an $(n + 2m - 1)$ -connected map of C -fixed points. We prove that the map of C -fixed points induced by the left-hand vertical map in (3.11) is $(n + 2m - 1)$ -connected and leave the analogous case of the right-hand vertical map to the reader. So consider the following diagram, where the map in question is the top horizontal map.

$$\begin{array}{ccc} F(X^{\wedge r}, Y^{\wedge r} \wedge Z)^C \wedge P & \xrightarrow{\tilde{\delta}^C} & F(P^{\wedge r}, F(X^{\wedge r}, Y^{\wedge r} \wedge Z))^C \\ \downarrow \tilde{\delta} & & \downarrow \Delta^* \\ F(P, F(X^{\wedge r}, Y^{\wedge r} \wedge Z)^C) & \xrightarrow{\sim} & F(P, F(X^{\wedge r}, Y^{\wedge r} \wedge Z))^C \end{array}$$

The left-hand vertical map is the inclusion of a wedge in the corresponding product. The summands $F(X^{\wedge r}, Y^{\wedge r} \wedge Z)^C$ are $(n + m - 1)$ -connected by lemma 2, and hence this map is $(2(n + m) - 1)$ -connected. The right-hand vertical map is induced from the inclusion $\Delta: P \rightarrow P^{\wedge r}$ of the diagonal. The map Δ^* has the homotopy lifting property, and the fiber over the base-point is canonically homeomorphic to the mapping space

$$F(P^{\wedge r}/\Delta(P), F(X^{\wedge r}, Y^{\wedge r} \wedge Z))^C.$$

The connectivity can be evaluated by using lemma 2. Since $P^{\wedge r}/\Delta(P)$ has no cells of orbit-type C/C , we find that this space is $(n + tm - 1)$ -connected, where t is the smallest non-trivial divisor in r . Since $t \geq 2$, we are done.

A similar argument shows the following result.

4

Proposition 4 Let A be a ring and Π a pointed monoid. Then the composite

$$H(A) \wedge N^{\text{cy}}(\Pi) \rightarrow H(A(\Pi)) \wedge N^{\text{cy}}(A(\Pi)) \xrightarrow{\mu} H(A(\Pi))$$

is an $\mathcal{F}$ -equivalence of $\mathbb{T}$ -spectra.

3.6

The Cyclic Bar-construction of Π_e

The $\mathbb{T}$ -equivariant homotopy type of the $\mathbb{T}$ -spaces $N^{\text{cy}}(\Pi_e)$ that occur in proposition 3 for truncated polynomial algebras was evaluated in [28]. The result, which we now recall, is quite simple, and it is this simplicity which, in turn, facilitates the understanding of the topological cyclic homology of truncated polynomial algebras.

There is a natural wedge decomposition

$$\bigvee_{i \in \mathbb{N}_0} N^{\text{cy}}(\Pi_e, i) \xrightarrow{\sim} N^{\text{cy}}(\Pi_e),$$

where $N^{\text{cy}}(\Pi_e, i)$ is the realization of the pointed cyclic subset $N^{\text{cy}}(\Pi_e, i)$ generated by the 0-simplex 1, if $i = 0$, and by the $(i - 1)$ -simplex $x \wedge \dots \wedge x$, if $i > 0$. The $\mathbb{T}$ -space $N^{\text{cy}}(\Pi, 0)$ is homeomorphic to the discrete space $\{0, 1\}$. For $i > 0$, we let $d = [(i - 1)/e]$ be the largest integer less than or equal to $(i - 1)/e$ and consider the complex $\mathbb{T}$ -representation

$$\lambda_d = \mathbb{C}(1) \oplus \mathbb{C}(2) \oplus \dots \oplus \mathbb{C}(d),$$

where $\mathbb{C}(t)$ denotes the representation of $\mathbb{T}$ on $\mathbb{C}$ through the t -fold power map. The following result is [28, theorem B].

Theorem 5 There is a canonical exact triangle of pointed $\mathbb{T}$ -spaces

5

$$S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+} \xrightarrow{\text{id} \wedge \text{pr}} S^{\lambda_d} \wedge \mathbb{T}/C_{i+} \rightarrow N^{\text{cy}}(\Pi_e, i) \xrightarrow{\partial} S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}[-1] ,$$

where $d = [(i-1)/e]$ and where the right and left-hand terms are understood to be a point, if e does not divide i .

We sketch the proof. By elementary cyclic theory, $N^{\text{cy}}(\Pi_e, i)$ is a quotient of the cyclic standard $(i-1)$ -simplex $\Delta^{i-1} = \Delta^{i-1} \times \mathbb{T}$. In fact, it is not difficult to see that there is a $\mathbb{T}$ -equivariant homeomorphism

$$(\Delta^{i-1}/C_i \cdot \Delta^{i-e}) \wedge_{C_i} \mathbb{T}_+ \xrightarrow{\sim} N^{\text{cy}}(\Pi_e, i) ,$$

where Δ^{i-1} is the standard $(i-1)$ -simplex with C_i acting by cyclically permuting the vertices and $\Delta^{i-e} \subset \Delta^{i-1}$ is the face spanned by the first $i-e+1$ vertices. It is also easy to understand the homology. Indeed, the reduced cellular complex of $N^{\text{cy}}(\Pi_e)$ is canonically isomorphic to the Hochschild complex of $\mathbb{Z}[x]/(x^e)$, and the homology of the latter was evaluated in [24]. If $e = 2$, $C_i \cdot \Delta^{i-2} = \partial \Delta^{i-1}$, and the result readily follows. If $e > 2$, however, the homological dimension $2d$ is smaller than the topological dimension $i-1$, and the main difficulty is to produce an equivariant map of degree one from the sphere S^{λ_d} to $N^{\text{cy}}(\Pi_e, i)$. It is the complete understanding of the combinatorial structure of the so-called cyclic polytopes [13, 17] that makes this possible.

Topological Cyclic Homology

3.7

We recall the definition of the cyclic homology of Connes [14] and the topological cyclic homology of Bökstedt–Hsiang–Madsen [7]. We first give the definition of the version of cyclic homology that was defined independently by Loday–Quillen [42] and Feigin–Tsygan [55] and that agrees with Connes’ original definition rationally.

In general, if G is a compact Lie group and $H \subset G$ a *finite* subgroup, then there is a canonical duality isomorphism in the G -stable category

$$[X \wedge G/H_+, Y \wedge S^{\mathfrak{g}}]_G \xrightarrow{\sim} [X, Y \wedge G/H_+]_G , \quad (3.12)$$

where $\mathfrak{g}$ denotes the Lie algebra of G with the adjoint action [29, section 8.1]. If $G = \mathbb{T}$, the adjoint representation $\mathfrak{t}$ is trivial, and we fix an isomorphism $\mathbb{R} \xrightarrow{\sim} \mathfrak{t}$.

Let E be a free contractible $\mathbb{T}$ -CW-complex; any two such $\mathbb{T}$ -CW-complexes are canonically $\mathbb{T}$ -homotopy equivalent. We use the unit sphere $E = S(\mathbb{C}^\infty)$ with the standard $\mathbb{T}$ -CW-structure with one cell in every even non-negative dimension. Then the cyclic homology of A is defined by

$$\text{HC}_q(A) = [S^{q+1}, H(A) \wedge E_+]_{\mathbb{T}} . \quad (3.13)$$

There is a canonical isomorphism

$$[S^{q+1}, H(A) \wedge E_+]_{\mathbb{T}} \xleftarrow{\sim} [S^{q+1}, (H(A) \wedge E_+)^{\mathbb{T}}] ,$$

and hence the group $\mathrm{HC}_q(A)$ is canonically isomorphic to the q th homotopy group of the $\mathbb{T}$ -group homology spectrum

$$\mathbb{H}.(\mathbb{T}, H(A)) = (H(A) \wedge E_+)^{\mathbb{T}} [+1]$$

(compare [40, theorem 7.1] for the dimension-shift). There is a canonical exact triangle of pointed $\mathbb{T}$ -CW-complexes

$$\mathbb{T}_+ = \mathrm{sk}_1(E_+) \rightarrow \mathrm{sk}_2(E_+) \rightarrow \mathbb{T}_+[-2] \xrightarrow{\partial} \mathbb{T}_+[-1] ,$$

which induces an exact triangle of $\mathbb{T}$ -spectra

$$H(A) \wedge \mathbb{T}_+ \rightarrow H(A) \wedge \mathrm{sk}_2(E_+) \rightarrow H(A) \wedge \mathbb{T}_+[-2] \xrightarrow{\partial} H(A) \wedge \mathbb{T}_+[-1] .$$

The boundary map induces Connes' (B -)operator

$$d: \mathrm{HH}_q(A) \rightarrow \mathrm{HH}_{q+1}(A) , \quad (3.14)$$

where we use the duality isomorphism (3.12) to identify

$$\mathrm{HH}_q(A) \xrightarrow{\sim} [S^{q+1} \wedge \mathbb{T}_+, H(A) \wedge S^t]_{\mathbb{T}} \xrightarrow{\sim} [S^{q+1}, H(A) \wedge \mathbb{T}_+]_{\mathbb{T}} .$$

The operator d is the d^2 -differential in the spectral sequence induced from the skeleton filtration of E . The spectral sequence is a first quadrant homology type spectral sequence with $E_{s,t}^2 = \mathrm{HH}_t(A)$, for s even, and zero, for s odd. Moreover, the groups $\mathrm{HH}_*(A)$ together with the operator d form a differential graded ring.

The definition (3.14) of the Connes' operator d makes sense for every $\mathbb{T}$ -spectrum. The operator d is a derivation, for every ring $\mathbb{T}$ -spectrum, but, in general, it is not a differential. Instead, one has

$$d \circ d = d \circ \iota = \iota \circ d ,$$

where ι is the map induced by the Hopf map $\eta: S^{q+1} \rightarrow S^q$.

We now explain the definition of topological cyclic homology, and refer to [20, 27, 29, 31] for details. Let p be a fixed prime and consider the groups

$$\mathrm{TR}_q^n(A; p) = [S^q \wedge \mathbb{T}/C_{p^{n-1}+}, T(A)]_{\mathbb{T}} . \quad (3.15)$$

There is a canonical isomorphism

$$[S^q \wedge \mathbb{T}/C_{p^{n-1}+}, T(A)]_{\mathbb{T}} \xleftarrow{\sim} [S^q, T(A)^{C_{p^{n-1}}}] ,$$

and hence the group $\mathrm{TR}_q^n(A; p)$ is canonically isomorphic to the q th homotopy group of the $C_{p^{n-1}}$ -fixed point spectrum

$$\mathrm{TR}^n(A; p) = T(A)^{C_{p^{n-1}}} .$$

By using the duality isomorphism (3.12), we can define a derivation

$$d: \mathrm{TR}_q^n(A; p) \rightarrow \mathrm{TR}_{q+1}^n(A; p)$$

in a manner similar to (3.14). The canonical projection from $\mathbb{T}/C_{p^{n-1}}$ to $\mathbb{T}/C_{p^{n-2}}$ induces a natural map

$$F: \mathrm{TR}_q^n(A; p) \rightarrow \mathrm{TR}_q^{n-1}(A; p) ,$$

called the Frobenius, and there is an associated transfer map

$$V: \mathrm{TR}_q^{n-1}(A; p) \rightarrow \mathrm{TR}_q^n(A; p) ,$$

called the Verschiebung. The two composites FV and VF are given by multiplication by the integer p and by the element $V(1)$, respectively. Moreover,

$$FdV = d + (p - 1)\iota .$$

There is an additional map

$$R: \mathrm{TR}_q^n(A; p) \rightarrow \mathrm{TR}_q^{n-1}(A; p) ,$$

called the restriction, whose definition we now explain.

In general, an isomorphism of compact Lie groups $f: G \xrightarrow{\sim} G'$ induces an equivalence of categories f^* from the G' -stable category to the G -stable category [40, II.1.7]. In particular, the isomorphism given by the r th root

$$\varrho_r: \mathbb{T} \xrightarrow{\sim} \mathbb{T}/C_r$$

induces an equivalence ϱ_r^* from the $\mathbb{T}/C_r$ -stable category to the $\mathbb{T}$ -stable category. We consider the following exact triangle of pointed $\mathbb{T}$ -CW-complexes.

$$E_+ \rightarrow S^0 \rightarrow \widetilde{E} \xrightarrow{\partial} E_+[-1] .$$

Here $\widetilde{E}$ is defined as the mapping cone of the left-hand map, which collapses E to the non-base point of S^0 . It induces an exact triangle of $\mathbb{T}$ -spectra

$$T(A) \wedge E_+ \rightarrow T(A) \rightarrow T(A) \wedge \widetilde{E} \xrightarrow{\partial} T(A) \wedge E_+[-1] .$$

The $\mathbb{T}$ -spectrum $T(A)$ has the additional property that there is a natural $\mathcal{F}$ -equivalence of $\mathbb{T}$ -spectra

$$r: \varrho_p^*(T(A) \wedge \widetilde{E})^{C_p} \xrightarrow{\sim} T(A) ,$$

and hence the exact triangle above induces an exact triangle of $\mathbb{T}$ -spectra

$$\varrho_p^*(T(A) \wedge E_+)^{C_p} \rightarrow \varrho_p^*T(A)^{C_p} \xrightarrow{R} T(A) \xrightarrow{\partial} \varrho_p^*(T(A) \wedge E_+)^{C_p}[-1] .$$

This induces an exact triangle of $C_{p^{n-2}}$ -fixed point spectra

$$\mathbb{H}_\bullet(C_{p^{n-1}}, T(A)) \rightarrow \mathrm{TR}^n(A; p) \xrightarrow{R} \mathrm{TR}^{n-1}(A; p) \xrightarrow{\partial} \mathbb{H}_\bullet(C_{p^{n-1}}, T(A))[-1],$$

which, in turn, gives rise to a long-exact sequence of homotopy groups

$$\dots \rightarrow \mathbb{H}_q(C_{p^{n-1}}, T(A)) \rightarrow \mathrm{TR}_q^n(A; p) \xrightarrow{R} \mathrm{TR}_q^{n-1}(A; p) \xrightarrow{\partial} \dots$$

Here the left-hand term is the $C_{p^{n-1}}$ -group homology spectrum

$$\mathbb{H}_\bullet(C_{p^{n-1}}, T(A)) = (T(A) \wedge E_+)^{C_{p^{n-1}}},$$

whose homotopy groups are the abutment of a first quadrant homology type spectral sequence

$$E_{s,t}^2 = H_s(C_{p^{n-1}}, \mathrm{THH}_t(A)) \Rightarrow \mathbb{H}_{s+t}(C_{p^{n-1}}, T(A)).$$

The spectral sequence is obtained from the skeleton filtration of E considered as a $C_{p^{n-1}}$ -CW-complex, and the identification the E^2 -term with the group homology of $C_{p^{n-1}}$ acting on $\mathrm{THH}_*(A)$ uses the duality isomorphism (3.12). We refer to [31, §4] for a detailed discussion.

One defines $\mathrm{TC}^n(A; p)$ as the homotopy equalizer of the maps

$$R, F: \mathrm{TR}^n(A; p) \rightarrow \mathrm{TR}^{n-1}(A; p)$$

and $\mathrm{TC}(A; p)$ as the homotopy limit of the spectra $\mathrm{TC}^n(A; p)$. Hence, there is a natural long-exact sequence of pro-abelian groups

$$\dots \rightarrow \mathrm{TC}_q^*(A; p) \rightarrow \mathrm{TR}_q^*(A; p) \xrightarrow{1-F} \mathrm{TR}_q^*(A; p) \xrightarrow{\partial} \mathrm{TC}_{q-1}^*(A; p) \rightarrow \dots$$

and a natural short-exact sequence

$$0 \rightarrow R^1 \lim \mathrm{TC}_{q+1}^*(A; p) \rightarrow \mathrm{TC}_q(A; p) \rightarrow \lim \mathrm{TC}_q^*(A; p) \rightarrow 0.$$

Here we use the restriction map as the structure map for the pro-systems, but we could just as well have used the Frobenius map. We shall use the notation $\mathrm{TF}_q^*(A; p)$ to denote the pro-abelian group consisting of the groups $\mathrm{TR}_q^n(A; p)$, for $n \geq 1$, with the Frobenius as the structure map.

3.8

The de Rham–Witt Complex

The Hochschild homology groups $\mathrm{HH}_*(A)$ form a differential graded ring, with the differential given by Connes' operator, and there is a canonical ring homomorphism $\lambda: A \rightarrow \mathrm{HH}_0(A)$. The de Rham complex Ω_A^* is the initial example of this algebraic structure, and hence there is a canonical map

$$\lambda: \Omega_A^q \rightarrow \mathrm{HH}_q(A).$$

Analogously, the groups $\mathrm{TR}_*^*(A; p)$ form a more complex algebraic structure called a Witt complex [27]. The de Rham–Witt complex $W.\Omega_A^*$ is the initial example of this algebraic structure, and hence there is a canonical map

$$\lambda: W_n.\Omega_A^q \rightarrow \mathrm{TR}_q^n(A; p) .$$

The construction of $W.\Omega_A^*$ was given first for $\mathbb{F}_p$ -algebras by Bloch–Deligne–Illusie [5, 35], who also showed that for k a perfect field of characteristic p and $X \rightarrow \mathrm{Spec} k$ smooth, there is a canonical isomorphism

$$H^q(X, W.\Omega_X^*) \xrightarrow{\sim} H_{\mathrm{crys}}^q(X/W(k))$$

of the (hyper-)cohomology of X with coefficients in $W.\Omega_X^*$ and the crystalline cohomology of X over $\mathrm{Spec} W(k)$ defined by Berthelot–Grothendieck [3]. Recent work by Madsen and the author [27, 31] has shown that the construction can be naturally extended to $\mathbb{Z}_{(p)}$ -algebras (where p is odd) and that the extended construction is strongly related to the p -adic K -theory of local number fields. See also [19, 26].

Suppose either that A is a $\mathbb{Z}_{(p)}$ -algebra with p odd or an $\mathbb{F}_2$ -algebra. Then we define a *Witt complex* over A to be the following (i)–(iii).

(i) A pro-differential graded ring E_*^* and a strict map of pro-rings

$$\lambda: W.(A) \rightarrow E_*^0$$

from the pro-ring of Witt vectors in A .

(ii) A strict map of pro-graded rings

$$F: E_*^* \rightarrow E_{*-1}^*$$

such that $\lambda F = F\lambda$ and such that for all $a \in A$,

$$Fd\lambda([a]_n) = \lambda([a]_{n-1})^{p-1} d\lambda([a]_{n-1}) ,$$

where $[a]_n = (a, 0, \dots, 0) \in W_n(A)$ is the multiplicative representative.

(iii) A strict map of graded E_*^* -modules

$$V: F_*E_{*-1}^* \rightarrow E_*^*$$

such that $\lambda V = V\lambda$ and such that $FdV = d$ and $FV = p$.

A map of Witt complexes over A is a strict map $f: E_*^* \rightarrow E_*^*$ of pro-differential graded rings such that $\lambda' = f\lambda$, $F'f = fF$ and $V'f = fV$. We call F the Frobenius, V the Verschiebung, and the structure map of the pro-differential graded ring the restriction.

It is proved in [27, theorem A] that there exists an initial Witt complex over A , $W.\Omega_A^*$, and that the canonical map $\Omega_{W_n(A)}^q \rightarrow W_n.\Omega_A^q$ is surjective. The following result, which we will need below, is [27, theorem B].

6

Theorem 6 Suppose either that A is a $\mathbb{Z}_{(p)}$ -algebra, with p an odd prime, or an $\mathbb{F}_2$ -algebra. Then every element $\omega^{(n)} \in W_n \Omega_{A[t]}^q$ can be written uniquely as a (direct) sum

$$\begin{aligned} \omega^{(n)} = & \sum_{j \in \mathbb{N}_0} a_{0,j}^{(n)} [t]_n^j + \sum_{j \in \mathbb{N}} b_{0,j}^{(n)} [t]_n^{j-1} d[t]_n \\ & + \sum_{s=1}^{n-1} \sum_{j \in I_p} \left(V^s \left(a_{s,j}^{(n-s)} [t]_{n-s}^j \right) + dV^s \left(b_{s,j}^{(n-s)} [t]_{n-s}^j \right) \right) \end{aligned}$$

with $a_{s,j}^{(n-s)} \in W_{n-s} \Omega_A^q$ and $b_{s,j}^{(n-s)} \in W_{n-s} \Omega_A^{q-1}$.

If A is a regular $\mathbb{F}_p$ -algebra, then the structure of the groups $W_n \Omega_A^q$ is well understood [35]. There is a multiplicative descending filtration by the differential graded ideals given by

$$\text{Fil}^s W_n \Omega_A^q = V^s W_{n-s} \Omega_A^q + dV^s W_{n-s} \Omega_A^{q-1}.$$

The filtration has length n and the filtration quotients can be expressed in terms of the de Rham complex of A and the Cartier operator [35, I.3.9].

3.9

Cyclic Homology of $A[x]/(x^e)$

As we recalled in the introduction, there is a natural isomorphism

$$K_q \left(A[x]/(x^e), (x) \right) \otimes \mathbb{Q} \approx \text{HC}_{q-1} \left(A[x]/(x^e), (x) \right) \otimes \mathbb{Q}.$$

We use the calculation of the $\mathbb{T}$ -equivariant homotopy type of $N^{\text{cy}}(\Pi_e)$ to derive a formula that expresses the left-hand groups in terms of the rational Hochschild homology groups of the ring A .

7

Proposition 7 There is a natural isomorphism, valid for all rings A ,

$$\bigoplus_{i \in \mathbb{N} \setminus e\mathbb{N}} \text{HH}_{q-2d}(A) \otimes \mathbb{Q} \xrightarrow{\sim} \text{HC}_q \left(A[x]/(x^e), (x) \right) \otimes \mathbb{Q},$$

where $d = [(i-1)/e]$.

Proof We recall that the composite

$$H(A) \wedge N^{\text{cy}}(\Pi_e) \rightarrow H \left(A[x]/(x^e) \right) \wedge N^{\text{cy}} \left(A[x]/(x^e) \right) \xrightarrow{\mu} H \left(A[x]/(x^e) \right)$$

is an equivalence of $\mathbb{T}$ -spectra. Hence, theorem 5 gives rise to an exact triangle of $\mathbb{T}$ -spectra

$$\begin{aligned} \bigvee_{i \in e\mathbb{N}} H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+} &\xrightarrow{\text{id} \wedge \text{pr}} \bigvee_{i \in \mathbb{N}} H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i+} \\ &\rightarrow H(A[x]/(x^e), (x)) \xrightarrow{\partial} \bigvee_{i \in e\mathbb{N}} H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}[-1], \end{aligned}$$

which induces an exact triangle of $\mathbb{T}$ -group homology spectra. The associated long-exact sequence of homotopy groups takes the form

$$\begin{aligned} \cdots \rightarrow \bigoplus_{i \in e\mathbb{N}} \mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}) &\rightarrow \bigoplus_{i \in \mathbb{N}} \mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i+}) \\ &\rightarrow \text{HC}_q(A[x]/(x^e), (x)) \xrightarrow{\partial} \bigoplus_{i \in e\mathbb{N}} \mathbb{H}_{q-1}(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}) \rightarrow \cdots \end{aligned}$$

There is a natural isomorphism

$$\mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i+}) \xleftarrow{\sim} \mathbb{H}_q(C_i, H(A) \wedge S^{\lambda_d}).$$

Moreover, the edge-homomorphism

$$H_0(C_i, \pi_q(H(A) \wedge S^{\lambda_d})) \rightarrow \mathbb{H}_q(C_i, T(A) \wedge S^{\lambda_d})$$

of the spectral sequence

$$E_{s,t}^2 = H_s(C_i, \pi_t(H(A) \wedge S^{\lambda_d}) \otimes \mathbb{Q}) \Rightarrow \mathbb{H}_{s+t}(C_i, H(A) \wedge S^{\lambda_d}) \otimes \mathbb{Q}$$

is an isomorphism. Indeed, for every C_i -module M , the composition

$$H_s(C_i, M) \xrightarrow{i^*} H_s(\{1\}, M) \xrightarrow{i_*} H_s(C_i, M)$$

is equal to multiplication by $i = [C_i : \{1\}]$. In the case at hand, this map is an isomorphism. But $H_s(\{1\}, M)$ is zero, for $s > 0$, and therefore also $H_s(C_i, M)$ is zero, for $s > 0$. Next, the action by C_i on $H(A) \wedge S^{\lambda_d}$ extends to an action by $\mathbb{T}$, and therefore, it induces the trivial action on homotopy groups. Hence, we further have an isomorphism

$$\text{HH}_{t-2d}(A) = \pi_t(H(A) \wedge S^{\lambda_d}) \xrightarrow{\sim} H_0(C_i, \pi_t(H(A) \wedge S^{\lambda_d})).$$

Finally, one sees in a similar manner that, after tensoring with $\mathbb{Q}$, the map

$$\text{pr}_* : \mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}) \rightarrow \mathbb{H}_q(\mathbb{T}, H(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i+})$$

becomes an isomorphism. Hence, after tensoring with $\mathbb{Q}$, the top left-hand term of the long-exact sequence above maps isomorphically onto the direct summands of the top right-hand term indexed by $i \in e\mathbb{N}$. The proposition follows.

Topological Cyclic Homology of $A[x]/(x^e)$

We recall that for every ring A and every prime p , the cyclotomic trace induces an isomorphism

$$K_q(A[x]/(x^e), (x), \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{TC}_q(A[x]/(x^e), (x), \mathbb{Z}_p) .$$

In this paragraph, we evaluate the topological cyclic homology groups on the right in terms of the groups

$$\mathrm{TR}_{q-\lambda}^n(A; p) = [S^q \wedge \mathbb{T}/C_{p^{n-1}+}, T(A) \wedge S^\lambda]_{\mathbb{T}} .$$

Indeed, we shall prove the following formula.

8

Proposition 8 Let $e = p^v e'$ with e' prime to p , and let A be an $\mathbb{F}_p$ -algebra. Then there is a natural long-exact sequence of abelian groups

$$\begin{aligned} \cdots \rightarrow \prod_{j \in e' I_p} \lim_R \mathrm{TR}_{q-1-\lambda_d}^{r-v}(A; p) &\xrightarrow{e' V^v} \prod_{j \in I_p} \lim_R \mathrm{TR}_{q-1-\lambda_d}^r(A; p) \\ \rightarrow \mathrm{TC}_q(A[x]/(x^e), (x); p) &\xrightarrow{\partial} \prod_{j \in e' I_p} \lim_R \mathrm{TR}_{q-2-\lambda_d}^{r-v}(A; p) \rightarrow \cdots \end{aligned}$$

where $d = [(p^{r-1}j - 1)/e]$. The analogous sequence for the homotopy groups with $\mathbb{Z}/p^v$ -coefficients is valid for every ring A .

Let λ be a finite dimensional orthogonal $\mathbb{T}$ -representation, and let λ' denote the $\mathbb{T}$ -representation $\mathfrak{o}_p^* \lambda^{C_p}$. Then the restriction map induces a map

$$R: \mathrm{TR}_{q-\lambda}^n(A; p) \rightarrow \mathrm{TR}_{q-\lambda'}^{n-1}(A; p) .$$

This is the structure map in the limits of proposition 8. We note that, if $j \in I_p$ and $r \in \mathbb{N}$, and if we let $d = [(p^{r-1}j - 1)/e]$ and $d' = [(p^{r-2}j - 1)/e]$, then $(\lambda_d)' = \lambda_{d'}$ as required. Moreover, by [29, theorem 2.2], there is a natural long-exact sequence

$$\cdots \rightarrow \mathbb{H}_q(C_{p^{n-1}}, T(A) \wedge S^\lambda) \rightarrow \mathrm{TR}_{q-\lambda}^n(A; p) \xrightarrow{R} \mathrm{TR}_{q-\lambda'}^{n-1}(A; p) \rightarrow \cdots$$

Since the left-hand term is zero, for $q < \dim_{\mathbb{R}}(\lambda)$, it follows that the map R in this sequence is an epimorphism, for $q \leq \dim_{\mathbb{R}}(\lambda)$, and an isomorphism, for $q < \dim_{\mathbb{R}}(\lambda)$. Hence, the limits of proposition 8 are attained. In addition, we see that the j th factor of the upper right-hand term of the long-exact sequence of proposition 8 is non-zero if and only if $q - 1 \geq 2[(j - 1)/e]$ and that the j th factor of the upper left-hand term is non-zero if and only if $q - 1 \geq 2[(p^v j - 1)/e]$. Hence, the products are finite in each degree q .

Proof (*Proof of proposition 8*) From proposition 3 and theorem 5, we get an exact triangle of $\mathbb{T}$ -spectra

$$\begin{aligned} \bigvee_{i \in e\mathbb{N}} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+} &\xrightarrow{\text{id} \wedge \text{pr}} \bigvee_{i \in \mathbb{N}} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i+} \\ &\rightarrow T(A[x]/(x^e), (x)) \xrightarrow{\partial} \bigvee_{i \in e\mathbb{N}} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{i|e+}[-1], \end{aligned} \quad (3.16)$$

where $d = [(i-1)/e]$. We wish to evaluate the map of homotopy groups of $C_{p^{n-1}}$ -fixed points induced by the map of $\mathbb{T}$ -spectra in the top line. We first consider the top right-hand term in (3.16). By re-indexing after the p -adic valuation of $i \in \mathbb{N}$, this term can be rewritten as

$$\bigvee_{j \in \mathbb{N}} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{p^{n-1}j+} \vee \bigvee_{r=1}^{n-1} \bigvee_{j \in I_p} T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{p^{r-1}j+},$$

and hence the $C_{p^{n-1}}$ -fixed point are expressed as a wedge sum

$$\begin{aligned} &\bigvee_{j \in \mathbb{N}} \varrho_{p^{n-1}}^* (T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{p^{n-1}j+})^{C_{p^{n-1}}} \\ &\vee \bigvee_{r=1}^{n-1} \bigvee_{j \in I_p} \varrho_{p^{n-r}}^* \left(\varrho_{p^{r-1}}^* (T(A) \wedge S^{\lambda_d} \wedge \mathbb{T}/C_{p^{r-1}j+})^{C_{p^{r-1}}} \right)^{C_{p^{n-r}}}. \end{aligned}$$

Moreover, for every $\mathbb{T}$ -spectrum T , there is a natural equivalence of $\mathbb{T}$ -spectra

$$\varrho_{p^m}^* T^{C_{p^m}} \wedge \varrho_{p^m}^* (\mathbb{T}/C_{p^m j+})^{C_{p^m}} \xrightarrow{\sim} \varrho_{p^m}^* (T \wedge \mathbb{T}/C_{p^m j+})^{C_{p^m}},$$

and the p^m th root defines a $\mathbb{T}$ -equivariant homeomorphism

$$\mathbb{T}/C_{j+} \xrightarrow{\sim} \varrho_{p^m}^* (\mathbb{T}/C_{p^m j+})^{C_{p^m}}.$$

Hence, the wedge sum above is canonically equivalent to the following wedge sum of $\mathbb{T}$ -spectra.

$$\begin{aligned} &\bigvee_{j \in \mathbb{N}} \varrho_{p^{n-1}}^* (T(A) \wedge S^{\lambda_d})^{C_{p^{n-1}}} \wedge \mathbb{T}/C_{j+} \\ &\vee \bigvee_{r=1}^{n-1} \bigvee_{j \in I_p} \varrho_{p^{n-r}}^* \left(\varrho_{p^{r-1}}^* (T(A) \wedge S^{\lambda_d})^{C_{p^{r-1}}} \wedge \mathbb{T}/C_{j+} \right)^{C_{p^{n-r}}}. \end{aligned} \quad (3.17)$$

Here, in the top line, $i = p^{n-1}j$, and in the bottom line, $i = p^{r-1}j$, and in both cases, $d = [(i-1)/e]$. The following easy result is [27, lemma 3.4.1].

9

Lemma 9 Let T be a $\mathbb{T}$ -spectrum, let $j \in I_p$, and let $\iota: C_j/C_j \rightarrow \mathbb{T}/C_j$ be the canonical inclusion. Then the map

$$V^m \iota_* + dV^m \iota_*: \pi_q(T) \oplus \pi_{q-1}(T) \rightarrow \pi_q\left(\mathfrak{C}_{p^m}^*(T \wedge \mathbb{T}/C_{j+})^{C_{p^m}}\right)$$

is an isomorphism.

The wedge decomposition (3.17) and lemma 9 gives rise to an isomorphism of the following direct sum onto the q th homotopy group of the top right-hand term of (3.16).

$$\begin{aligned} & \bigoplus_{j \in \mathbb{N}} \left(\mathrm{TR}_{q-\lambda_d}^n(A; p) \oplus \mathrm{TR}_{q-1-\lambda_d}^n(A; p) \right) \\ & \oplus \bigoplus_{j \in I_p} \bigoplus_{r=1}^{n-1} \left(\mathrm{TR}_{q-\lambda_d}^r(A; p) \oplus \mathrm{TR}_{q-1-\lambda_d}^r(A; p) \right). \end{aligned} \quad (3.18)$$

The same argument gives an isomorphism of the following direct sum onto the q th homotopy group of the top left-hand term of (3.16).

$$\begin{aligned} & \bigoplus_{j \in e' \mathbb{N}} \left(\mathrm{TR}_{q-\lambda_d}^m(A; p) \oplus \mathrm{TR}_{q-1-\lambda_d}^m(A; p) \right) \\ & \oplus \bigoplus_{j \in e' I_p} \bigoplus_{r=v+1}^{n-1} \left(\mathrm{TR}_{q-\lambda_d}^{r-v}(A; p) \oplus \mathrm{TR}_{q-1-\lambda_d}^{r-v}(A; p) \right). \end{aligned} \quad (3.19)$$

Here in the top line $m = \min\{n, n - v + v_p(j)\}$. Moreover, the map of q th homotopy groups induced by the map $\mathrm{id} \wedge \mathrm{pr}$ in (3.16) preserves the indices of the direct sum decompositions (3.18) and (3.19) of the q th homotopy groups of the target and domain. It is given on the summands in the bottom lines of (3.18) and (3.19) by the following maps [29, lemma 8.1].

$$V^v: \mathrm{TR}_{q-\lambda_d}^{r-v}(A; p) \rightarrow \mathrm{TR}_{q-\lambda_d}^r(A; p),$$

$$e' V^v: \mathrm{TR}_{q-1-\lambda_d}^{r-v}(A; p) \rightarrow \mathrm{TR}_{q-1-\lambda_d}^r(A; p).$$

We now assume that A is an $\mathbb{F}_p$ -algebra and consider the groups in (3.18) for varying $n \geq 1$ as a pro-abelian group with structure map given by the Frobenius map. The Frobenius map takes the summand with index $j \in \mathbb{N}$ in the top line of (3.18) for n to the summand with index $pj \in \mathbb{N}$ in the top line of (3.18) for $n - 1$. It takes the summand with index $j \in I_p$ and $1 \leq r < n - 1$ in the bottom line of (3.18) for n to the summand with the same indices in the bottom line of (3.18) for $n - 1$. Finally, it takes the summand with index $j \in I_p$ and $r = n - 1$ in the bottom line of (3.18) for n to the summand with index $j \in \mathbb{N}$ in the top line of (3.18) for $n - 1$. It follows that the projection onto the quotient-pro-abelian group given by the bottom line of (3.18) is an isomorphism of pro-abelian groups. Indeed, the sub-pro-abelian group given by the upper line of *loc.cit.* is Mittag-Leffler zero since

the sum is finite. The value of the Frobenius map on the summands in the bottom line of (3.18) follows immediately from lemma 9 and the relations $FV = p$ and $FdV = d$. We find that

$$F = p: \text{TR}_{q-\lambda_d}^r(A; p) \rightarrow \text{TR}_{q-\lambda_d}^r(A; p),$$

$$F = \text{id}: \text{TR}_{q-1-\lambda_d}^r(A; p) \rightarrow \text{TR}_{q-1-\lambda_d}^r(A; p).$$

It follows that the pro-abelian group with degree n term given by the direct sum (3.18) and with structure map given by the Frobenius is canonically isomorphic to the pro-abelian group with degree n term the direct sum

$$\bigoplus_{r=1}^{n-1} \bigoplus_{j \in I_p} \text{TR}_{q-1-\lambda_d}^r(A; p)$$

and with structure map the canonical projection. A similar argument shows that the pro-abelian group with degree n term given by the direct sum (3.19) and with structure map given by the Frobenius is canonically isomorphic to the pro-abelian group with degree n term the direct sum

$$\bigoplus_{r=v+1}^{n-1} \bigoplus_{j \in e'I_p} \text{TR}_{q-1-\lambda_d}^{r-v}(A; p)$$

and with structure map the canonical projection. Hence, we have a long-exact sequence of pro-abelian groups with degree n terms

$$\begin{aligned} \cdots \rightarrow \bigoplus_{r=v+1}^{n-1} \bigoplus_{j \in e'I_p} \text{TR}_{q-1-\lambda_d}^{r-v}(A; p) &\xrightarrow{e'V^v} \bigoplus_{r=1}^{n-1} \bigoplus_{j \in I_p} \text{TR}_{q-1-\lambda_d}^r(A; p) \\ &\rightarrow \text{TF}_q^n(A[x]/(x^e), (x); p) \xrightarrow{\partial} \bigoplus_{r=v+1}^{n-1} \bigoplus_{j \in e'I_p} \text{TR}_{q-2-\lambda_d}^{r-v}(A; p) \rightarrow \cdots \end{aligned} \quad (3.20)$$

and with the structure map given by the canonical projection in the two terms of the upper line and in the right-hand term of the lower line and by the Frobenius map in the left-hand term of the lower line. In particular, the pro-abelian group $\text{TF}_q^n(A[x]/(x^e), (x); p)$ satisfies the Mittag-Leffler condition, so the derived limit vanishes. Hence, we have an isomorphism

$$\text{TF}_q(A[x]/(x^e), (x); p) \xrightarrow{\sim} \lim_F \text{TF}_q^n(A[x]/(x^e), (x); p),$$

and the long-exact sequence (3.20) induces a long-exact sequence

$$\begin{aligned} \cdots \rightarrow \prod_{r \geq v+1} \prod_{j \in e'I_p} \text{TR}_{q-1-\lambda_d}^{r-v}(A; p) &\xrightarrow{e'V^v} \prod_{r \geq 1} \prod_{j \in I_p} \text{TR}_{q-1-\lambda_d}^r(A; p) \\ &\rightarrow \text{TF}_q(A[x]/(x^e), (x); p) \xrightarrow{\partial} \prod_{r \geq v+1} \prod_{j \in e'I_p} \text{TR}_{q-2-\lambda_d}^{r-v}(A; p) \rightarrow \cdots \end{aligned} \quad (3.21)$$

of the limits. Finally, the restriction map induces a self-map of the sequence (3.21) which on the two terms of the upper line and the right-hand term of the lower line is given by the map

$$R: \mathrm{TR}_{q-1-\lambda_d}^r(A; p) \rightarrow \mathrm{TR}_{q-1-\lambda'_d}^{r-1}(A; p) .$$

As we remarked after the statement of proposition 8, this map is an isomorphism for all but finitely many r . It follows that the map

$$\mathrm{TF}_q(A[x]/(x^e), (x); p) \xrightarrow{R-\mathrm{id}} \mathrm{TF}_q(A[x]/(x^e), (x); p)$$

is surjective and identifies $\mathrm{TC}_q(A[x]/(x^e), (x); p)$ with the kernel. Indeed, the self-map $R - \mathrm{id}$ of the sequence (3.21) is a split surjection with compatible sections on the remaining terms. Finally, the long-exact sequence of the statement is obtained as the long-exact sequence of kernels of the self-map $R - \mathrm{id}$ of the sequence (3.21).

3.11

The Characteristic Zero Case

We use the description of the cyclic homology and topological cyclic homology from the paragraphs above to prove the following result.

10

Theorem 10 Suppose that A is a regular noetherian ring and a $\mathbb{Q}$ -algebra. Then there is a natural isomorphism of abelian groups

$$K_{q-1}(A[x]/(x^e), (x)) \xleftarrow{\sim} \bigoplus_{m \geq 1} \left(\Omega_A^{q-2m} \right)^{e-1} ,$$

where the superscript $e - 1$ indicates product.

Proof We first show that the relative K -groups with $\mathbb{Z}_p$ -coefficients are zero. By McCarthy [47], the cyclotomic trace induces an isomorphism

$$K_q(A[x]/(x^e), (x), \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{TC}_q(A[x]/(x^e), (x); p, \mathbb{Z}_p)$$

and for every spectrum X , there is a natural short-exact sequence

$$0 \rightarrow R^1 \lim_v \pi_{q+1}(X, \mathbb{Z}/p^v) \rightarrow \pi_q(X, \mathbb{Z}_p) \rightarrow \lim_v \pi_q(X, \mathbb{Z}/p^v) \rightarrow 0 .$$

Hence, by proposition 8, it suffices to show that the groups $\mathrm{TR}_{q-\lambda}^n(A; p, \mathbb{Z}/p^v)$ are zero. Moreover, there is a natural long-exact sequence

$$\cdots \rightarrow \mathbb{H}_q(C_{p^{n-1}}, T(A) \wedge S^\lambda) \rightarrow \mathrm{TR}_{q-\lambda}^n(A; p) \xrightarrow{R} \mathrm{TR}_{q-\lambda'}^{n-1}(A; p) \rightarrow \cdots$$

and a natural spectral sequence

$$E_{s,t}^2 = H_s(C_{p^{n-1}}, \pi_t(T(A) \wedge S^\lambda)) \Rightarrow \mathbb{H}_{s+t}(C_{p^{n-1}}, T(A) \wedge S^\lambda) .$$

The same is true for the homotopy groups with $\mathbb{Z}/p^\nu$ -coefficients. Hence, it will be enough to show that the groups

$$\pi_t (T(A) \wedge S^\lambda, \mathbb{Z}/p^\nu) = \mathrm{THH}_{t-\dim_{\mathbb{R}}(A)}(A, \mathbb{Z}/p^\nu)$$

are zero. But these groups are at the same time A -modules and annihilated by p^ν . Therefore, they are zero, for every $\mathbb{Q}$ -algebra A . It follows that in the arithmetic square (which is homotopy-cartesian [8])

$$\begin{array}{ccc} K(A[x]/(x^e), (x)) & \longrightarrow & \prod_p K(A[x]/(x^e), (x))_p^\wedge \\ \downarrow & & \downarrow \\ K(A[x]/(x^e), (x))_{\mathbb{Q}} & \longrightarrow & \left(\prod_p K(A[x]/(x^e), (x))_p^\wedge \right)_{\mathbb{Q}}, \end{array} \quad (3.22)$$

the spectra on the right are trivial, and hence the left-hand vertical map is an equivalence. Hence, the canonical maps

$$K_q(A[x]/(x^e), (x)) \rightarrow \mathrm{HC}_q^-(A[x]/(x^e), (x)) \leftarrow \mathrm{HC}_{q-1}(A[x]/(x^e), (x))$$

are isomorphisms and the common group uniquely divisible. The statement now follows from proposition 7 and the fact that the canonical map

$$\Omega_A^* \rightarrow \mathrm{HH}_*(A)$$

is an isomorphism. The latter is true for A a smooth $\mathbb{Q}$ -algebra by Hochschild–Kostant–Rosenberg [32], and the general case follows from Popescu [50]. Indeed, the domain and target both commute with filtered colimits.

The Groups $\mathrm{TR}_{q-\lambda}^n(A; p)$

3.12

In this paragraph, we evaluate the groups

$$\mathrm{TR}_{q-\lambda}^n(A; p) = [S^q \wedge \mathbb{T}/C_{p^{n-1}+}, T(A) \wedge S^\lambda]_{\mathbb{T}}$$

for A a regular $\mathbb{F}_p$ -algebra. In the basic case of the field $\mathbb{F}_p$, or, more generally, a perfect field of characteristic p , the groups were evaluated in [29, proposition 9.1]. To state the result, we define

$$\ell_s = \dim_{\mathbb{C}}(\lambda^{C_{p^s}}),$$

for all $s \geq 0$, and $\ell_s = \infty$, for $s < 0$. Then there is a canonical isomorphism of graded abelian groups

$$\bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} W_r(k)[-2m] \xrightarrow{\sim} \mathrm{TR}_{*-\lambda}^n(k; p),$$

where the sum is over all integers m , and where r is the unique integer such that $\ell_{n-r} \leq m < \ell_{n-1-r}$. Equivalently, the group $\mathrm{TR}_{q-\lambda}^n(k; p)$ is isomorphic to $W_r(k)$, for $q = 2m$ and $\ell_{n-r} \leq m < \ell_{n-1-r}$, and is zero, for q is odd.

The groups $\mathrm{TR}_{*-\lambda}^n(A; p)$ naturally form a differential graded module over the differential graded ring $\mathrm{TR}_*^n(A; p)$. Hence, we have a natural pairing

$$W_n \Omega_A^* \otimes_{W_n(k)} \mathrm{TR}_{*-\lambda}^n(k; p) \rightarrow \mathrm{TR}_{*-\lambda}^n(A; p). \quad (3.23)$$

11

Theorem 11 Let A be a regular $\mathbb{F}_p$ -algebra, and let λ be a finite dimensional complex $\mathbb{T}$ -representation. Then the pairing (3.23) induces an isomorphism of graded abelian groups

$$\bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} W_r \Omega_A^*[-2m] \xrightarrow{\sim} \mathrm{TR}_{*-\lambda}^n(A; p).$$

Proof The domain and target of the map of the statement both commute with filtered colimits. Hence, by Popescu [50], we can assume that A is a smooth k -algebra. Moreover, if $f: A \rightarrow A'$ is an étale map of k -algebras, then the canonical map

$$W_n(A') \otimes_{W_n(A)} \mathrm{TR}_{q-\lambda}^n(A; p) \rightarrow \mathrm{TR}_{q-\lambda}^n(A'; p)$$

is an isomorphism. The analogous statement holds for the domain of the map of the statement. By a covering argument, we are reduced to consider the case where A is a polynomial algebra over k in a finite number of variables, see [30, lemma 2.2.8] for details. Hence, it suffices to show that the statement for A implies the statement for $A' = A[t]$.

The following result follows by an argument similar to [27, theorem C].

12

Proposition 12 Let λ be a finite dimensional orthogonal $\mathbb{T}$ -representation, and let A be a $\mathbb{Z}_{(p)}$ -algebra. Then every element $\omega^{(n)} \in \mathrm{TR}_{q-\lambda}^n(A[t]; p)$ can be written uniquely as a (direct) sum

$$\begin{aligned} \omega^{(n)} = & \sum_{j \in \mathbb{N}_0} a_{0,j}^{(n)} [t]_n^j + \sum_{j \in \mathbb{N}} b_{0,j}^{(n)} [t]_n^{j-1} d[t]_n \\ & + \sum_{s=1}^{n-1} \sum_{j \in I_p} \left(V^s \left(a_{s,j}^{(n-s)} [t]_{n-s}^j \right) + dV^s \left(b_{s,j}^{(n-s)} [t]_{n-s}^j \right) \right) \end{aligned}$$

with $a_{s,j}^{(n-s)} \in \mathrm{TR}_{q-\lambda}^{n-s}(A; p)$ and $b_{s,j}^{(n-s)} \in \mathrm{TR}_{q-1-\lambda}^{n-s}(A; p)$.

Proof We outline the proof. The group $\mathrm{TR}_{q-\lambda}^n(A[t]; p)$ is the q th homotopy group of the $\mathbb{T}$ -spectrum

$$\mathfrak{e}_{p^{n-1}}^* (T(A[t]) \wedge S^\lambda)^{C_{p^{n-1}}} \quad (3.24)$$

Let $\Pi = \{0, 1, t, t^2, \dots\}$ be the sub-pointed monoid of $A[t]$ generated by the variable. We recall from proposition 3 that the composite

$$T(A) \wedge N^{\mathrm{cy}}(\Pi) \xrightarrow{f \wedge 1} T(A[t]) \wedge N^{\mathrm{cy}}(A[t]) \xrightarrow{\mu} T(A[t])$$

is an $\mathcal{F}$ -equivalence of $\mathbb{T}$ -spectra. Moreover, the $\mathbb{T}$ -space $N^{\mathrm{cy}}(\Pi)$ decomposes as a wedge sum

$$\bigvee_{i \in \mathbb{N}_0} N^{\mathrm{cy}}(\Pi, i) \xrightarrow{\sim} N^{\mathrm{cy}}(\Pi),$$

where $N^{\mathrm{cy}}(\Pi, i)$ is the realization of the cyclic subset of $N^{\mathrm{cy}}(\Pi)$ generated by the 0-simplex 1, if $i = 0$, and by the $(i - 1)$ -simplex $t \wedge \dots \wedge t$, if $i > 0$. Hence, the $\mathbb{T}$ -spectrum (3.24) can be expressed, up to $\mathcal{F}$ -equivalence, as a wedge sum

$$\begin{aligned} & \bigvee_{j \in \mathbb{N}_0} \mathfrak{e}_{p^{n-1}}^* (T(A) \wedge S^\lambda \wedge N^{\mathrm{cy}}(\Pi, p^{n-1}j))^{C_{p^{n-1}}} \\ & \vee \bigvee_{s=1}^{n-1} \bigvee_{j \in I_p} \mathfrak{e}_{p^s}^* \left(\mathfrak{e}_{p^{n-1-s}}^* (T(A) \wedge S^\lambda \wedge N^{\mathrm{cy}}(\Pi, p^{n-1-s}j))^{C_{p^{n-1-s}}} \right)^{C_{p^s}}. \end{aligned}$$

In addition, there is a natural equivalence of $\mathbb{T}$ -spectra

$$\begin{aligned} & \mathfrak{e}_{p^m}^* (T(A) \wedge S^\lambda)^{C_{p^m}} \wedge \mathfrak{e}_{p^m}^* N^{\mathrm{cy}}(\Pi, p^m j)^{C_{p^m}} \\ & \xrightarrow{\sim} \mathfrak{e}_{p^m}^* (T(A) \wedge S^\lambda \wedge N^{\mathrm{cy}}(\Pi, p^m j))^{C_{p^m}}, \end{aligned}$$

and a $\mathbb{T}$ -equivariant homeomorphism

$$\Delta: N^{\mathrm{cy}}(\Pi, j) \xrightarrow{\sim} \mathfrak{e}_{p^m}^* N^{\mathrm{cy}}(\Pi, p^m j)^{C_{p^m}}.$$

Hence, we obtain the following wedge decomposition, up to $\mathcal{F}$ -equivalence, of the $\mathbb{T}$ -spectrum (3.24).

$$\begin{aligned} & \bigvee_{j \in \mathbb{N}_0} \mathfrak{e}_{p^{n-1}}^* (T(A) \wedge S^\lambda)^{C_{p^{n-1}}} \wedge N^{\mathrm{cy}}(\Pi, j) \\ & \vee \bigvee_{s=1}^{n-1} \bigvee_{j \in I_p} \mathfrak{e}_{p^s}^* \left(\mathfrak{e}_{p^{n-1-s}}^* (T(A) \wedge S^\lambda)^{C_{p^{n-1-s}}} \wedge N^{\mathrm{cy}}(\Pi, j) \right)^{C_{p^s}}. \end{aligned}$$

We claim that the induced direct sum decomposition of the q th homotopy group corresponds to the direct sum decomposition of the statement. To prove this, one first proves that the map

$$\mathrm{TR}_{*- \lambda}^n(A; p) \otimes \Omega_{\mathbb{Z}[t]}^* \rightarrow \mathrm{TR}_{*- \lambda}^n(A[t]; p)$$

that takes $a \otimes t^j$ to $f(a)[t]_n^j$ and $a \otimes t^{j-1} dt$ to $f(a)[t]_n^{j-1} d[t]_n$ is an isomorphism onto the direct summand $\pi_*(\mathcal{P}_{p^{n-1}}^*(T(A) \wedge S^\lambda)^{C_{p^{n-1}}} \wedge N^{\mathrm{cy}}(II))$. We refer to [27, lemma 3.3.1] for the proof. Secondly, one shows that if T is any $\mathbb{T}$ -spectrum, and if $j \in I_p$, then the map

$$V^s \iota + dV^s \iota: \pi_q(T) \oplus \pi_{q-1}(T) \rightarrow \pi_q\left(\mathcal{P}_{p^s}^*(T \wedge \mathbb{T}/C_{j+})^{C_{p^s}}\right)$$

is an isomorphism. Here $\iota: C_j/C_j \rightarrow \mathbb{T}/C_j$ is the canonical inclusion. This is lemma 9. This completes our outline of the proof of the proposition.

As we recalled in theorem 6 above, the de Rham–Witt groups of $A[t]$ can be similarly expressed in terms of those of A . We can now complete the proof of theorem 11. Let $E_n^q(A)$ denote the left-hand side of the statement. We claim that every element $\omega^{(n)} \in E_n^q(A[t])$ can be written uniquely as a direct sum

$$\begin{aligned} \omega^{(n)} = & \sum_{j \in \mathbb{N}_0} a_{0,j}^{(n)} [t]_n^j + \sum_{j \in \mathbb{N}} b_{0,j}^{(n)} [t]_n^{j-1} d[t]_n \\ & + \sum_{s=1}^{n-1} \sum_{j \in I_p} \left(V^s \left(a_{s,j}^{(n-s)} [t]_{n-s}^j \right) + dV^s \left(b_{s,j}^{(n-s)} [t]_{n-s}^j \right) \right) \end{aligned}$$

with $a_{s,j}^{(n-s)} \in E_{n-s}^q(A)$ and $b_{s,j}^{(n-s)} \in E_{n-s}^{q-1}(A)$, or equivalently, that the map

$$\bigoplus_{j \in \mathbb{N}_0} E_n^q(A) \oplus \bigoplus_{j \in \mathbb{N}} E_n^{q-1}(A) \oplus \bigoplus_{s=1}^{n-1} \bigoplus_{j \in I_p} (E_{n-s}^q(A) \oplus E_{n-s}^{q-1}(A)) \rightarrow E_n^q(A[t])$$

given by this formula is an isomorphism. On the one-hand, by theorem 6, the right-hand side is given by the direct sum

$$\begin{aligned} & \bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} \left(\bigoplus_{j \in \mathbb{N}_0} W_r \Omega_A^{q-2m} \oplus \bigoplus_{j \in \mathbb{N}} W_r \Omega_A^{q-1-2m} \right) \oplus \\ & \bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} \left(\bigoplus_{s=1}^{r-1} \bigoplus_{j \in I_p} \left(W_{r-s} \Omega_A^{q-2m} \oplus W_{r-s} \Omega_A^{q-1-2m} \right) \right), \end{aligned} \tag{3.25}$$

and on the other-hand, by the definition of $E_n^q(A)$, the left-hand side is given by the direct sum

$$\begin{aligned} & \bigoplus_{j \in \mathbb{N}_0} \bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} W_r \Omega_A^{q-2m} \oplus \bigoplus_{j \in \mathbb{N}} \bigoplus_{\ell_{n-r} \leq m < \ell_{n-1-r}} W_r \Omega_A^{q-1-2m} \oplus \\ & \bigoplus_{s=1}^{n-1} \bigoplus_{j \in I_p} \bigoplus_{\ell_{n-s-r} \leq m < \ell_{n-s-1-r}} (W_r \Omega_A^{q-2m} \oplus W_r \Omega_A^{q-1-2m}) . \end{aligned} \quad (3.26)$$

The top lines in (3.25) and (3.26) clearly are isomorphic, and the bottom lines both are seen to be isomorphic to the direct sum

$$\bigoplus_{r=1}^n \bigoplus_{\ell_{n-1-r} \leq m} \bigoplus_{j \in I_p} (W_r \Omega_A^{q-2m} \oplus W_r \Omega_A^{q-1-2m}) .$$

This proves the claim.

The Positive Characteristic Case

3.13

The following result was proved by Madsen and the author in [28, 30] but stated there in terms of big de Rham–Witt differential forms.

Theorem 13 Suppose that A is a regular noetherian ring and an $\mathbb{F}_p$ -algebra, and write $e = p^v e'$ with e' not divisible by p . Then there is a natural long-exact sequence of abelian groups

13

$$\begin{aligned} \cdots \rightarrow \bigoplus_{m \geq 1} \bigoplus_{j \in e' I_p} W_{s-v} \Omega_A^{q-2m} & \xrightarrow{e' V^v} \bigoplus_{m \geq 1} \bigoplus_{j \in I_p} W_s \Omega_A^{q-2m} \\ \rightarrow K_{q-1}(A[x]/(x^e), (x)) & \xrightarrow{\partial} \bigoplus_{m \geq 1} \bigoplus_{j \in e' I_p} W_{s-v} \Omega_A^{q-1-2m} \rightarrow \cdots , \end{aligned}$$

where $s = s(m, j)$ is the unique integer such that $p^{s-1}j \leq me < p^s j$.

Proof We recall that by Goodwillie [22], there is a canonical isomorphism

$$K_q(A[x]/(x^e), (x)) \otimes \mathbb{Q} \approx \mathrm{HC}_{q-1}(A[x]/(x^e), (x)) \otimes \mathbb{Q} .$$

But proposition 7 shows that the groups on the left are zero. Indeed, the groups $\mathrm{HH}_*(A)$ are A -modules, and therefore, annihilated by p . If ℓ is a prime, then by McCarthy [47], the cyclotomic trace induces an isomorphism

$$K_q(A[x]/(x^e), (x), \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{TC}_q(A[x]/(x^e), (x); \ell, \mathbb{Z}_\ell) ,$$

and for every spectrum X , there is a natural short-exact sequence

$$0 \rightarrow R^1 \lim_v \pi_{q+1}(X, \mathbb{Z}/\ell^v) \rightarrow \pi_q(X, \mathbb{Z}/\ell) \rightarrow \lim_v \pi_q(X, \mathbb{Z}/\ell^v) \rightarrow 0.$$

The groups $\mathrm{TC}_q(A[x]/(x^\ell), (x); \ell, \mathbb{Z}/\ell^v)$ are given by proposition 8 in terms of the groups $\mathrm{TR}_{q-\lambda}^n(A; \ell, \mathbb{Z}/\ell^v)$. We claim that the latter are zero, for $\ell \neq p$. In effect, we claim the slightly stronger statement that for every prime ℓ , the groups $\mathrm{TR}_{q-\lambda}^n(A; \ell)$ are p -groups of a bounded exponent (which depends on ℓ, q, n , and λ). The proof is by induction on $n \geq 1$ and uses the natural long-exact sequence

$$\cdots \rightarrow \mathbb{H}_q(C_{\ell^{n-1}}, T(A) \wedge S^\lambda) \rightarrow \mathrm{TR}_{q-\lambda}^n(A; \ell) \xrightarrow{R} \mathrm{TR}_{q-\lambda}^{n-1}(A; \ell) \rightarrow \cdots$$

and the natural spectral sequence

$$E_{s,t}^2 = H_s(C_{\ell^{n-1}}, \pi_t(T(A) \wedge S^\lambda)) \Rightarrow \mathbb{H}_{s+t}(C_{\ell^{n-1}}, T(A) \wedge S^\lambda).$$

Since the groups

$$\pi_t(T(A) \wedge S^\lambda) = \mathrm{THH}_{t-\dim_{\mathbb{R}}(\lambda)}(A)$$

are A -modules, and hence annihilated by p , the claim follows. We conclude from the arithmetic square (3.22) that the cyclotomic trace

$$K_q(A[x]/(x^\ell), (x)) \rightarrow \mathrm{TC}_q(A[x]/(x^\ell), (x); p)$$

is an isomorphism. The right-hand side can be read off from proposition 8 and theorem 11. For notational reasons we evaluate the group in degree $q-1$ rather than the one in degree q . We find that the j th factor of the upper right-hand term of the long-exact sequence of proposition 8 is given by

$$\lim_R \mathrm{TR}_{q-2-\lambda_d}^r(A; p) \xleftarrow{\sim} \bigoplus_{m \geq 0} W_s \Omega_A^{q-2(m+1)},$$

where s is the unique integer that satisfies

$$[(p^{s-1}j-1)/e] \leq m < [(p^s j-1)/e]$$

or equivalently,

$$p^{s-1}j \leq (m+1)e < p^s j.$$

Writing m instead of $m+1$, we obtain the upper right-hand term of the long-exact sequence of the statement. The upper left-hand term is evaluated similarly. Finally, because the isomorphism of theorem 11 is induced by the pairing (3.23), the map V^v in the long-exact sequence of proposition 8 induces the iterated Verschiebung V^v of the de Rham–Witt complex. The theorem follows.

We first proved theorem 13 in [28] in the case of a perfect field k of characteristic p , and the extension to all regular noetherian $\mathbb{F}_p$ -algebras was given in [30]. Our main motivation was that the more general case makes it possible to also evaluate the groups $NK_q(R) = K_q(R[t], (t))$, which occur in the fundamental theorem for non-regular rings, in the following situation.

Corollary 14 Let A be a regular noetherian ring that is also an $\mathbb{F}_p$ -algebra, and write $e = p^v e'$ with e' prime to p . Then there is a natural long-exact sequence of abelian groups

14

$$\begin{aligned} \cdots \rightarrow \bigoplus_{m \geq 1} \bigoplus_{j \in e' I_p} W_{s-v} \Omega_{(A[t], (t))}^{q-2m} &\xrightarrow{e' V^v} \bigoplus_{m \geq 1} \bigoplus_{j \in I_p} W_s \Omega_{(A[t], (t))}^{q-2m} \\ &\rightarrow NK_{q-1}(A[x]/(x^e)) \xrightarrow{\partial} \bigoplus_{m \geq 1} \bigoplus_{j \in e' I_p} W_{s-v} \Omega_{(A[t], (t))}^{q-1-2m} \rightarrow \cdots, \end{aligned}$$

where $s = s(m, j)$ is the unique integers such that $p^{s-1}j \leq me < p^s j$.

Proof We have a natural split-exact sequence

$$0 \rightarrow NK_q(A[x]/(x^e), (x)) \rightarrow NK_q(A[x]/(x^e)) \rightarrow NK_q(A) \rightarrow 0.$$

Since A is regular, the fundamental theorem shows that the right-hand term is zero. Hence, the left-hand map is an isomorphism, and the left-hand group is given by theorem 13.

We remark that Weibel [57] has shown that the groups $NK_*(A)$ are naturally modules over the big ring of Witt vectors $\mathbf{W}(A)$. We do not know if this module structure is compatible with the obvious $\mathbf{W}(A)$ -module structure on the remaining terms of the long-exact sequence of corollary 14.

Miscellaneous

3.14

As e varies, the spectra $K(A[x]/(x^e))$ are related by several maps. Let $e' = de$ and let $\pi_d: A[x]/(x^{e'}) \rightarrow A[x]/(x^e)$ and $\iota_d: A[x]/(x^e) \rightarrow A[x]/(x^{e'})$ be the maps of A -algebras that take x to x and x^d , respectively. Then there are maps of K -theory spectra

$$\begin{aligned} \pi_d^*, \iota_{d*}: K(A[x]/(x^{e'})) &\rightarrow K(A[x]/(x^e)), \\ \iota_d^*: K(A[x]/(x^e)) &\rightarrow K(A[x]/(x^{e'})), \end{aligned} \tag{3.27}$$

which are related in a manner similar to that of the restriction, Frobenius, and Verschiebung, respectively. It remains an unsolved problem to determine the value

of these maps under the isomorphisms of theorems 10 and 13 above. Another very interesting and open problem is to determine the multiplicative structure of the groups $K_*(A[x]/(x^e), \mathbb{Z}/p)$.

There are two limit cases, however, where the maps (3.27) are well understood. One case is a recent result by Betley–Schlichtkrull [4] which expresses topological cyclic homology in terms of K -theory of truncated polynomial algebras. To state it, let $\mathbb{I}$ be the category with objects the set of positive integers and with morphisms generated by morphisms r_d and f_d from $e' = de$ to e , for all positive integers e and d , subject to the following relations.

$$r_1 = f_1 = \text{id}, \quad r_d r_{d'} = r_{dd'}, \quad f_d f_{d'} = f_{dd'}, \quad r_d f_{d'} = f_{d'} r_d.$$

Then there is a functor from the category $\mathbb{I}$ to the category of symmetric spectra that to the object e assigns $K(A[x]/(x^e))$ and that to the morphisms r_d and f_d from $e' = de$ to e assign the maps π_d^* and ι_{d*} , respectively. The following result is [4, theorem 1.1].

15 Theorem 15 There is a natural weak equivalence of symmetric spectra

$$\text{TC}(A; p, \mathbb{Z}/p^v) \simeq \text{holim}_{\mathbb{I}} K(A[x]/(x^e), \mathbb{Z}/p^v) [1].$$

Another case concerns the limit over the maps π_d^* . This limit was considered first by Bloch [5] who used it to give a K -theoretical construction of the de Rham–Witt complex. This, in turn, led to the purely algebraic construction of the de Rham–Witt complex which we recalled in Sect. 3.8 above. The K -theoretical construction begins with the spectrum of curves on $K(A)$ defined to be the following homotopy limit with respect to the maps π_d^* .

$$C(A) = \text{holim}_e K(A[x]/(x^e), (x)) [1].$$

This is a ring spectrum in such a way that the ring of components is canonically isomorphic to the ring of big Witt vectors $\mathbf{W}(A)$. If A is a $\mathbb{Z}_{(p)}$ -algebra, the ring $\mathbf{W}(A)$ has a canonical idempotent decomposition as a product of copies of the ring of p -typical Witt vectors $W(A)$. These idempotents give rise to an analogous decomposition of the ring spectrum $C(A)$ as a product of copies of a ring spectrum $C(A; p)$ which is called the p -typical curves on $K(A)$. The following result was proved by the author in [25, theorem C]. The corresponding result for the symbolic part of K -theory was proved by Bloch [5] under the additional assumption that A be local of dimension less than p . The restriction on the dimension of A was removed by Kato [39].

16 Theorem 16 Let A be a smooth algebra over a perfect field of positive characteristic p . Then there are natural isomorphisms of abelian groups

$$C_*(A; p) \approx \text{TR}_*(A; p) \approx W\Omega_A^*.$$

Finally, we briefly discuss the K -theory of $\mathbb{Z}/p^e$ relative to the ideal generated by p . The cyclotomic trace also induces an isomorphism

$$K_q(\mathbb{Z}/p^e, (p)) \xrightarrow{\sim} \mathrm{TC}_q(\mathbb{Z}/p^e, (p); p) ,$$

and hence one may attempt to evaluate the relative K -groups by evaluating the relative topological cyclic homology groups on the right. The following result of Brun [11, 12] was proved by these methods. It generalizes earlier results by Evens and Friedlander [15] and by Aisbett, Lluís-Puebla and Snaith [2].

Theorem 17 Suppose that $0 \leq q \leq p - 3$. Then $K_q(\mathbb{Z}/p^e, (p))$ is a cyclic group of order $p^{j(e-1)}$, if $q = 2j - 1$ is odd, and is zero, if q is even. 17

We remark that the groups $K_q(\mathbb{F}_p[x]/(x^e), (x))$ and $K_q(\mathbb{Z}/p^e, (p))$ have the same order, for $0 \leq q \leq p - 3$. The order of the former group is given by the formula of theorem 17, for all non-negative integers q , but this cannot be true for the latter group. Indeed, by Suslin–Panin [48, 53], the canonical map

$$K_q(\mathbb{Z}_p, (p), \mathbb{Z}_p) \rightarrow \lim_e K_q(\mathbb{Z}/p^e, (p))$$

is an isomorphism, for all integers q . The left-hand group is non-zero, if q is even and divisible by $2(p - 1)$. Hence the groups $K_q(\mathbb{Z}/p^e, (p))$ cannot be zero for every even integer q and positive integer e . This comparison also shows that $K_q(\mathbb{Z}/p^e, (p))$ cannot be a cyclic group for every odd integer q and every positive integer e . But this also follows from the result of Geisser [18] that the group $K_3(\mathbb{Z}/9, (3))$ is isomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/3$.

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Bott Periodicity in Topological, Algebraic and Hermitian K -Theory

Max Karoubi

4.1	<i>Classical Bott Periodicity</i>	112
4.2	<i>Interpretation of Bott Periodicity via K-Theory</i>	114
	Banach Algebras.....	115
4.3	<i>The Role of Clifford Algebras</i>	118
	Elementary Proofs	119
4.4	<i>Negative K-Theory and Laurent Series</i>	120
	The K -Theory Spectrum	121
	Laurent Series Approximation	121
4.5	<i>Bott Periodicity, the Atiyah–Singer Index Theorem, and KR-Theory</i>	121
	$KR(X)$	122
4.6	<i>Bott Periodicity in Algebraic K-Theory</i>	123
	Periodicity with Finite Coefficients	123
	The Lichtenbaum–Quillen Conjecture	124
	Thomason’s Approach	125
	The Bloch–Kato Conjecture	126
4.7	<i>Bott Periodicity in Hermitian K-Theory</i>	129
	Orthogonal Groups	129
	Examples	132
4.8	<i>Conclusion</i>	133
	<i>References</i>	134

Summary. This paper is devoted to classical Bott periodicity, its history and more recent extensions in algebraic and Hermitian K -theory. However, it does not aim at completeness. For instance, the variants of Bott periodicity related to bivariant K -theory are described by Cuntz in this handbook. As another example, we don't emphasize here the relation between motivic homotopy theory and Bott periodicity since it is also described by other authors of this handbook (Grayson, Kahn, ...).

4.1

Classical Bott Periodicity

Bott periodicity [14] was discovered independently from K -theory, which started with the work of Grothendieck one year earlier [13]. In order to understand its great impact at the end of the 50's, one should notice that it was (and still is) quite hard to compute homotopy groups of spaces as simple as spheres. For example, it was proved by Serre that $\pi_i(S^n)$ is a finite group for $i \neq n$ and $i \neq 2n-1$ with n even, while $\pi_n(S^n) = \mathbb{Z}$ and $\pi_{2n-1}(S^n)$ is the direct sum of $\mathbb{Z}$ and a finite group for n even. All these finite groups are unknown in general (note however that $\pi_i(S^n) = 0$ for $i < n$). Since the classical groups $O(n)$ and $U(n)$ are built out of spheres through fibrations

$$\begin{aligned} O(n) &\rightarrow O(n+1) \rightarrow S^n \\ U(n) &\rightarrow U(n+1) \rightarrow S^{2n+1} \end{aligned}$$

it was thought that computing their homotopy groups would be harder. On the other hand, from these fibrations, it immediately follows that the homotopy groups of $O(n)$ and $U(n)$ stabilize. More precisely, $\pi_i(U(n)) \cong \pi_i(U(n+1))$ if $n > i/2$ and $\pi_i(O(n)) \cong \pi_i(O(n+1))$ if $n > i+1$. In this range of dimensions and degrees, we shall call $\pi_i(U)$ and $\pi_i(O)$ these stabilized homotopy groups: they are indeed homotopy groups of the “infinite” unitary and orthogonal groups

$$U = \operatorname{colim} U(n) \quad \text{and} \quad O = \operatorname{colim} O(n).$$

1

Theorem 1 [14] The homotopy groups $\pi_i(U)$ and $\pi_i(O)$ are periodic of period 2 and 8 respectively. More precisely, there exist homotopy equivalences¹

$$U \approx \Omega^2(U) \quad \text{and} \quad O \approx \Omega^8(O),$$

where Ω^t denotes the t -th iterated loop space.

2

Remark 2 Using polar decomposition of matrices, one may replace $O(n)$ and $U(n)$ by the general linear groups $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$, which have the same homotopy

¹ Throughout the paper, we use the symbol $\approx$ to denote a homotopy equivalence.

type respectively. Similarly, one may consider the infinite general linear group $GL(\mathbb{R}) = \text{colim } GL_n(\mathbb{R})$ and $GL(\mathbb{C}) = \text{colim } GL_n(\mathbb{C})$. Since the homotopy groups of $O \approx GL(\mathbb{R})$ and $U \approx GL(\mathbb{C})$ are periodic, it is enough to compute the first eight ones, which are given by the following table:

$$\begin{array}{cccccccc} i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \pi_i(U) & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} \\ \pi_i(O) & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} . \end{array}$$

In the same paper, Bott gave a more general theorem (in the real case), using the infinite homogeneous spaces related not only to the infinite orthogonal and unitary groups, but also to the infinite symplectic group Sp . More precisely, $Sp(n)$ is the compact Lie group associated to $GL_n(\mathbb{H})$ and $Sp = \text{colim } Sp(n)$, which has the same homotopy type as $GL(\mathbb{H}) = \text{colim } GL_n(\mathbb{H})$, where $\mathbb{H}$ is the skew field of quaternions [21].

Theorem 3 [14] We have the following homotopy equivalences (where BG denotes in general the classifying space of the topological group G):

3

$$\begin{aligned} \Omega(\mathbb{Z} \times BGL(\mathbb{R})) &\approx GL(\mathbb{R}) \\ \Omega(GL(\mathbb{R})) &\approx GL(\mathbb{R})/GL(\mathbb{C}) \\ \Omega(GL(\mathbb{R})/GL(\mathbb{C})) &\approx GL(\mathbb{C})/GL(\mathbb{H}) \\ \Omega(GL(\mathbb{C})/GL(\mathbb{H})) &\approx \mathbb{Z} \times BGL(\mathbb{H}) \\ \Omega(\mathbb{Z} \times BGL(\mathbb{H})) &\approx GL(\mathbb{H}) \\ \Omega(GL(\mathbb{H})) &\approx GL(\mathbb{H})/GL(\mathbb{C}) \\ \Omega(GL(\mathbb{H})/GL(\mathbb{C})) &\approx GL(\mathbb{C})/GL(\mathbb{R}) \\ \Omega(GL(\mathbb{C})/GL(\mathbb{R})) &\approx \mathbb{Z} \times BGL(\mathbb{R}) . \end{aligned}$$

In particular, we have the homotopy equivalences

$$\begin{aligned} \Omega^4(BGL(\mathbb{R})) &\approx \mathbb{Z} \times BGL(\mathbb{H}) \\ \Omega^4(BGL(\mathbb{H})) &\approx \mathbb{Z} \times BGL(\mathbb{R}) . \end{aligned}$$

Theorems 1 and 3 were not at all easy to prove (note however that $\Omega(BG) \approx G$ is a standard statement). One key ingredient was a heavy use of Morse theory. A detailed proof, starting from a short course in Riemannian geometry may be found in the beautiful book of Milnor [32]. The proof (in the complex case) is

based on two lemmas: one first shows that the space of minimal geodesics from I to $-I$ in the special unitary group $SU(2m)$ is homeomorphic to the Grassmannian $G_m(\mathbb{C}^{2m})$. In the other lemma one shows that every non-minimal geodesic from I to $-I$ has index $\geq 2m + 2$. These lemmas imply the following “unstable” theorem from which Bott periodicity follows easily:

4

Theorem 4 [32, p. 128] Let $G_m(\mathbb{C}^{2m})$ be the Grassmannian of m -dimensional subspaces of $\mathbb{C}^{2m}$. Then there is an inclusion map from $G_m(\mathbb{C}^{2m})$ into the space of paths² in $SU(2m)$ joining I and $-I$. For $i \leq 2m$, this map induces an isomorphism of homotopy groups

$$\pi_i(G_m(\mathbb{C}^{2m})) \cong \pi_{i+1}(SU(2m)) .$$

The algebraic topologists felt frustrated at that time by a proof using methods of Riemannian geometry in such an essential way. A special seminar [16] held in Paris by Cartan and Moore (1959/1960) was devoted not only to a detailed proof of Bott’s theorems, but also to another proof avoiding Morse theory and using more classical methods in algebraic topology. However, this second proof was still too complicated for an average mathematician to grasp (see however the sketch of some elementary proofs of the complex Bott periodicity in the Sect. 4.3.1 of this paper).

Interpretation of Bott Periodicity via K -Theory

4.2

Two years later, Atiyah and Hirzebruch [5] realized that Bott periodicity was related to the fundamental work of Grothendieck on *algebraic K-theory* [13]. By considering the category of topological vector bundles over a compact space X (instead of algebraic vector bundles), Atiyah and Hirzebruch defined a *topological K-theory* $K(X)$ following the same pattern as Grothendieck. As a new feature however, Atiyah and Hirzebruch managed to define “derived functors” $K^{-n}(X)$ by considering vector bundles over the n th suspension of X_+ (X with one point added outside). There are in fact two K -theories involved, whether one considers real or complex vector bundles. We shall denote them by $K_{\mathbb{R}}$ and $K_{\mathbb{C}}$ respectively if we want to be specific. Theorem 1 is then equivalent to the periodicity of the functors K^{-n} . More precisely,

$$K_{\mathbb{C}}^{-n}(X) \cong K_{\mathbb{C}}^{-n-2}(X) \quad \text{and} \quad K_{\mathbb{R}}^{-n-8}(X) \cong K_{\mathbb{R}}^{-n-8}(X) .$$

These isomorphisms enabled Atiyah and Hirzebruch to extend the definition of K^n for all $n \in \mathbb{Z}$ and define what we now call a “generalized cohomology theory” on the

² Note that this space of paths has the homotopy type of the loop space $\Omega(SU(2m))$.

category of compact spaces. Following the same spirit, Atiyah and Bott were able to give a quite elementary proof of the periodicity theorem in the complex case [6].

From the homotopy viewpoint, there exist two Ω -spectra defined by $\mathbb{Z} \times BGL(k)$, $k = \mathbb{R}$ or $\mathbb{C}$, and their iterated loop spaces. The periodicity theorems can be rephrased by saying that these Ω -spectra are periodic of period 2 or 8 in the stable homotopy category, according to the type of K -theory involved, depending on whether one considers real or complex vector bundles. For instance, we have the following formula (where $[\ , \]'$ means pointed homotopy classes of maps):

$$K_k^{-n}(X) \cong [X_+ \wedge S^n, \mathbb{Z} \times BGL(k)]' .$$

As it was noticed by many people in the 60's (Serre, Swan, Bass...), K -theory appears as a “homology theory” on the category of rings. More precisely, let $k = \mathbb{R}$ or $\mathbb{C}$ and let us consider a k -vector bundle E over a compact space X . Let A be the Banach algebra $C(X)$ of continuous functions $f : X \rightarrow k$ (with the Sup norm). If $M = \Gamma(X, E)$ denotes the vector space of continuous sections $s : X \rightarrow E$ of the vector bundle E , M is clearly a right A -module if we define $s.f$ to be the continuous section $x \mapsto s(x)f(x)$. Since X is compact, we may find another vector bundle E' such that the Whitney sum $E \oplus E'$ is trivial, say $X \times k^n$. Therefore, if we set $M' = \Gamma(X, E')$, we have $M \oplus M' \cong A^n$ as A -modules, which means that M is a finitely generated projective A -module. The theorem of Serre and Swan [28, 46] says precisely that the correspondence $E \mapsto M$ induces a functor from the category $\mathcal{E}(X)$ of vector bundles over X to the category $\mathcal{P}(A)$ of finitely generated projective (right) A -modules, which is an equivalence of categories. In particular, isomorphism classes of vector bundles correspond bijectively to isomorphism classes of finitely generated projective A -modules.

These considerations lead to the following definition of the K -theory of a ring with unit A : we just mimic the definition of $K(X)$ by replacing vector bundles by (finitely generated projective) A -modules. We call this group $K(A)$ by abuse of notation. It is clearly a *covariant* functor on the category of rings (through extension of scalars). We have of course $K(X) \cong K(A)$, when $A = C(X)$, thanks to the equivalence Γ above.

Banach Algebras

4.2.1

Similarly to what Atiyah and Hirzebruch did for $A = C(X)$, one would like to define new functors $K_n(A)$, $n \in \mathbb{Z}$, with nice formal properties starting from $K_0(A) = K(A)$. This task is in fact more difficult than it looks for general rings A and we shall concentrate at the beginning on the case when A is a (real or complex) Banach algebra.

Firstly, we extend the definition of $K(A)$ to non-unital algebras (over a commutative base ring k) by “adding a unit” to A . More precisely, we consider the k -module $\tilde{A} = k \oplus A$ provided with the following “twisted” multiplication

$$(\lambda, a).(\lambda', a') = (\lambda\lambda', \lambda.a' + a.\lambda' + a.a') .$$

The ring $\tilde{A}$ now has a unit that is $(1, 0)$. There is an obvious augmentation

$$\tilde{A} \rightarrow k$$

and $K(A)$ is just defined as the kernel of the induced homomorphism $K(\tilde{A}) \rightarrow K(k)$. It is easy to see that we recover the previous definition of $K(A)$ if A already has a unit and that the new K -functor is defined for maps between rings not necessarily having a unit. It is less easy to prove that this definition is in fact independent of k ; this follows from the excision property for the functor K_0 [8, 33].

Example 5. If $\mathcal{K}$ is the ideal of compact operators in a k -Hilbert space H (with $k = \mathbb{R}$ or $\mathbb{C}$), then the obvious inclusion from k to $\mathcal{K}$ induces an isomorphism $K(k) \cong K(\mathcal{K}) \cong \mathbb{Z}$. This is a classical result in operator theory (see for instance [39, section 2.2.10] and [28, exercise 6.15]).

Secondly, for $n \in \mathbb{N}$, we define $K_n(A)$ as $K(A_n)$, where $A_n = A(\mathbb{R}^n)$ is the Banach algebra of continuous functions $f = f(x)$ from $\mathbb{R}^n$ to A that vanish when x goes to ∞ . It is not too difficult to show that $K_{i+1}(A) \cong \text{colim } \pi_i(GL_r(A)) \cong \pi_i(GL(A))$, where $GL(A)$ is the direct limit of the $GL_r(A)$ with respect to the obvious inclusions $GL_r(A) \subset GL_{r+1}(A)$ (see for instance the argument in [28], p. 13).

As a fundamental example, let us come back to topology by taking $A = C(X)$, where X is compact. Let $Y = S^n(X_+)$ be the n -suspension of X_+ . Then $K(Y)$ is isomorphic to $K_n(A) \oplus \mathbb{Z}$. In order to show this, we notice that $C(Y)$ is isomorphic to $\tilde{C}(X \times \mathbb{R}^n)$. In particular $K(S^n)$ is isomorphic to $K_n(k) \oplus \mathbb{Z}$ ($k = \mathbb{R}$ or $\mathbb{C}$, according to the type of K -theory).

The following theorem, although not explicitly stated in this form in the literature (for the uniqueness part), is a direct consequence of the definitions.

6

Theorem 6 (compare with [28, exercise 6.14]). The functors $K_n(A)$, $n \in \mathbb{N}$, and A a Banach algebra, are characterized by the following properties

1. **Exactness:** for any exact sequence of Banach algebras (where A'' has the quotient norm and A' the induced norm)

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

we have an exact sequence of K -groups

$$K_{n+1}(A) \rightarrow K_{n+1}(A'') \rightarrow K_n(A') \rightarrow K_n(A) \rightarrow K_n(A'') .$$

2. **Homotopy invariance:** $K_n(A(I)) \cong K_n(A)$, where $A(I)$ is the ring of continuous functions on the unit interval I with values in A .
3. **Normalization:** $K_0(A) = K(A)$, the Grothendieck group defined above.

The functors $K_*(A)$ have other nice properties such as the following: a continuous bilinear pairing of Banach algebras

$$A \times C \rightarrow B$$

induces a “cup-product”

$$K_i(A) \otimes K_j(C) \rightarrow K_{i+j}(B) ,$$

which has associative and graded commutative properties [25, 28, 29]. In particular, if $C = k$, the field of real or complex numbers, and if $A = B$ is a k -Banach algebra, we have a pairing

$$K_i(A) \otimes K_j(k) \rightarrow K_{i+j}(A) .$$

We can now state the Bott periodicity theorem in the setting of Banach algebras.

Theorem 7 (Bott periodicity revisited, according to [25] and [53]).

1. Let A be a **complex** Banach algebra. Then the group $K_2(\mathbb{C})$ is isomorphic to $\mathbb{Z}$ and the cup-product with a generator u_2 induces an isomorphism $\beta_{\mathbb{C}} : K_n(A) \rightarrow K_{n+2}(A)$.
2. Let A be a **real** Banach algebra. Then the group $K_8(\mathbb{R})$ is isomorphic to $\mathbb{Z}$ and the cup-product with a generator u_8 induces an isomorphism $\beta_{\mathbb{R}} : K_n(A) \rightarrow K_{n+8}(A)$.

As we said in Sect. 4.2.1, this theorem implies the periodicity of the homotopy groups of the infinite general linear group in the more general setting of a Banach algebra, since $K_n(A) = K(A(\mathbb{R}^n))$ is isomorphic to the homotopy group $\pi_{n-1}(GL(A))$ (see the corollary below). If $A = \mathbb{C}$ for instance, the group $K(A(\mathbb{R}^n))$ is linked with the classification of stable complex vector bundles over the sphere S^n , which are determined by homotopy classes of “glueing functions”

$$f : S^{n-1} \rightarrow GL(A)$$

(see again the argument in [14, p. 13]). For a general Banach algebra A , one just has to consider vector bundles over the sphere whose fibers are the A -modules A^r instead of $\mathbb{C}^r$.

Corollary 8

(a) If A is a complex Banach algebra, we have

$$\pi_i(GL(A)) \cong \pi_{i+2}(GL(A)) \quad \text{and} \quad \pi_1(GL(A)) \cong K(A)$$

(b) If A is a real Banach algebra, we have

$$\pi_i(GL(A)) \cong \pi_{i+8}(GL(A)) \quad \text{and} \quad \pi_7(GL(A)) \cong K(A) .$$

Part (a) is essentially due to Atiyah and Bott [6], while part (b) is due to Wood [53] and the author [25]. In the complex case, we can easily see that the isomorphism $\pi_1(GL(A)) \cong K(A)$ implies the 2-periodicity of the homotopy groups of $GL(A)$. In order to prove this isomorphism, one essentially has to show that any loop in $GL(A)$ can be deformed into a loop of the type

$$\theta \mapsto pz + 1 - p$$

where p is an idempotent matrix of a certain size and $z = e^{i\theta}$. This is done via Fourier analysis and stabilization of matrices as explained with full details in [28], following the pattern initiated in [6]. Such an idempotent matrix p is of course associated to a finitely generated projective module. More conceptual proofs will be sketched later.

4.3

The Role of Clifford Algebras

One way to understand Bott periodicity in topology is to introduce Clifford algebras as it was pointed out by Atiyah, Bott and Shapiro [4]. Let us denote by C_n the Clifford algebra of $V = \mathbb{R}^n$ associated to the quadratic form $q(v) = (x_1)^2 + \cdots + (x_n)^2$, with $v = (x_1, \dots, x_n)$. We recall that C_n is the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by all relations of the form $v \otimes v - q(v) \cdot 1$. It is a finite dimensional semi-simple real algebra of dimension 2^n . There is a kind of “periodicity” of the C_n considered as $\mathbb{Z}/2$ -graded algebras: we have graded algebra isomorphisms³

$$C_{n+8} \cong M_{16}(C_n) .$$

On the other hand, the complexified Clifford algebras have a 2-periodicity

$$C_{n+2} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(C_n) \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(C_n \otimes_{\mathbb{R}} \mathbb{C}) .$$

These isomorphisms give rise to an “elementary” proof of the eight homotopy equivalences in Theorem 3 and the already stated results in Corollary 8 [25, 53]. Indeed, the afore-mentioned homotopy equivalences can be written in a uniform way (up to connected components) as follows:

$$GL(C_n)/GL(C_{n-1}) \approx \Omega[GL(C_{n+1})/GL(C_n)] .$$

In order to avoid the problem with connected components, let us introduce the “classifying space” $\mathcal{K}(A)$ of any (real or complex) Banach algebra A . As a first approximation⁴, it is the cartesian product

$$\mathcal{K}(A) = K(A) \times BGL(A) ,$$

³ Where $M_r(B)$ denotes in general the algebra of $r \times r$ matrices with coefficients in B .

⁴ This description is (non-canonically) homotopy equivalent to the “good” definition given later (see Sect. 4.4.1).

where $BGL(A)$ is the classifying space of the topological group $GL(A)$. This way, we have $\pi_i(\mathcal{K}(A)) = K_i(A)$ for $i \geq 0$. One could also consider the K -theory space $\mathcal{K}(A \otimes C_n)$, where tensor products are taken over $\mathbb{R}$, and the homotopy fiber $\mathcal{F}_n$ of the obvious inclusion map

$$\mathcal{K}(A \otimes C_{n-1}) \rightarrow \mathcal{K}(A \otimes C_n) .$$

We notice that the connected component of this homotopy fiber is precisely the connected component of the homogeneous space $GL(A \otimes C_n)/GL(A \otimes C_{n-1})$. The following theorem now includes all the versions of Bott periodicity quoted so far.

Theorem 9 We have natural homotopy equivalences

9

$$\mathcal{F}_n \approx \Omega(\mathcal{F}_{n+1}) .$$

Elementary Proofs

4.3.1

The proof of this theorem (see [25, 53], and [39, §3.5]) is still technical and requires easy but tedious lemmas. Therefore, it would be nice to have more conceptual proofs of Bott periodicity, at least in the complex case, as we already said at the end of Sect. 4.2. For the real case, we refer to 4.5.1 and 4.7.2.

A first approach (for $A = \mathbb{C}$) is an elegant proof of Suslin (unpublished, see also the independent proof of Harris [20]), using the machinery of Γ -spaces due to Segal [41]. Taking into account that the homotopy equivalence $U \cong \mathbb{Z} \times BU$ is a standard statement, the heart of the proof of Bott's periodicity is the homotopy equivalence

$$\mathbb{Z} \times BU \approx \Omega U .$$

Since $X_0 = \mathbb{Z} \times BU$ is a Γ -space, it can be infinitely delooped and there is an explicit recipe to build spaces X_n such that $\Omega^n(X_n) \cong X_0$ [41]. This explicit construction shows that X_1 is *homeomorphic* to the infinite unitary group and this ends the proof! However, it seems that this proof cannot be generalized to all Banach algebras since the deloopings via the machinery of Γ -spaces are connected: for instance X_2 is *not* homotopically equivalent to $\mathbb{Z} \times BU$.

Another conceptual approach (see [26, 29, 30]) leading in Sect. 4.4 to non connected deloopings of the space $\mathcal{K}(A)$ for any Banach algebra A , is to proceed as follows. Let us assume we have defined $K_n(A)$, not only for $n \in \mathbb{N}$, as we did before, but also for $n \in \mathbb{Z}$. We also assume that the pairing

$$K_i(A) \otimes K_j(C) \rightarrow K_{i+j}(B)$$

mentioned for i and $j \geq 0$, extends to all values of i and j in $\mathbb{Z}$ and has obvious associative and graded commutative properties. Finally, assume there exists a “negative Bott element” u_{-2} in $K_{-2}(\mathbb{C})$ whose cup-product with u_2 in $K_2(\mathbb{C})$ (as mentioned in 7) gives the unit element 1 in $K_0(\mathbb{C}) \cong \mathbb{Z}$.

Under these hypotheses, and assuming that A is a complex Banach algebra, we introduce an inverse homomorphism

$$\beta' : K_{n+2}(A) \rightarrow K_n(A)$$

of the Bott map

$$\beta : K_n(A) \rightarrow K_{n+2}(A) .$$

It is defined as the cup-product with u_{-2} . It is clear that the composites of β with β' both ways are the identity.

This formal approach may be extended to real Banach algebras as well, although the required elements in $K_8(\mathbb{R})$ and $K_{-8}(\mathbb{R})$ are less easy to construct. One should also compare this approach to the one described in Cuntz' paper in this handbook, using the machinery of KK -theory (for C^* -algebras at least).

4.4

Negative K -Theory and Laurent Series

The price to pay for this last conceptual proof is of course the construction of these “negative” K -groups $K_n(A)$, $n < 0$, which have some independent interest as we shall see later (A is now a *complex* or *real* Banach algebra). This may be done, using the notion of “suspension” of a ring, which is in some sense dual to the notion of suspension of a space. More precisely, we define the “cone” CA of a ring A to be the set of all infinite matrices $M = (a_{ij})$, $i, j \in \mathbb{N}$, such that each row and each column only contains a finite and bounded number of non zero elements in A (chosen among a finite number of elements in A). This clearly is a ring for the usual matrix operations. We make CA into a Banach algebra by completing it with respect to the following:

$$\|M\| = \sup_j \sum_i \|a_{ij}\|$$

(this is just an example; there are other ways to complete, leading to the same negative K -theory: see [26]). Finally we define $\overline{A}$, the “stabilization” of A , as the closure of the set of finite matrices⁵ in CA . It is a closed 2-sided ideal in CA and the suspension of A , denoted by SA , is the quotient ring $CA/\overline{A}$.

Let A be a Banach algebra. As in [26] we define the groups $K_{-n}(A)$ as $K(S^n A)$, where $S^n A$ is the n -th suspension of A for $n > 0$. Then $BGL(S^{n+1} A)$ is a delooping of $K(S^n A) \times BGL(S^n A)$, i.e. we have a homotopy equivalence (for any Banach algebra A)

$$\Omega(BGL(SA)) \approx K(A) \times BGL(A) .$$

Accordingly, to any exact sequence of Banach algebras as above

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

⁵ A matrix in CA is called finite if all but finitely many of its elements are 0.

we can associate an exact sequence of K -groups

$$K_{n+1}(A) \rightarrow K_{n+1}(A'') \rightarrow K_n(A') \rightarrow K_n(A) \rightarrow K_n(A'')$$

for $n \in \mathbb{Z}$.

The K -Theory Spectrum

4.4.1

As a matter of fact, this definition/theorem gives the *functorial* definition of the K -theory space $\mathcal{K}(A)$ mentioned above: it is nothing but the loop space $\Omega(BGL(SA)) \approx GL(SA)$. If $A = \mathbb{R}$ or $\mathbb{C}$, it is easy to see that this space has the homotopy type of the space of Fredholm operators in a Hilbert space modelled on A (see [2] or [10], appendix A).

Laurent Series Approximation

4.4.2

For a better understanding of SA , it may be interesting to notice that the ring of Laurent series $A\langle t, t^{-1} \rangle$ is a good approximation of the suspension. Any element of $A\langle t, t^{-1} \rangle$ is a series

$$S = \sum_{n \in \mathbb{Z}} a_n t^n$$

such that $\sum_{n \in \mathbb{Z}} \|a_n\| < +\infty$. There is a ring homomorphism $A\langle t, t^{-1} \rangle \rightarrow SA$ that associates to the series above the class of the following infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots \\ a_{-1} & a_0 & a_1 & \dots \\ a_{-2} & a_{-1} & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

For instance, in order to construct the element u_{-2} mentioned in Sect. 4.3.1, it is enough to describe a finitely generated projective module over the Banach algebra $C\langle t, u, t^{-1}, u^{-1} \rangle$, i.e. a non trivial complex vector bundle over the torus $S^1 \times S^1$, as we can see easily, using again the theory of Fourier series [29].

Bott Periodicity, the Atiyah–Singer Index Theorem, and KR -Theory

4.5

Many variants of topological K -theory have been considered since the 60's, also giving rise to more conceptual proofs of Bott periodicity and generalizing it. One of appealing interest is equivariant K -theory $K_G(X)$, introduced by Atiyah and Segal [40]. Here G is a compact Lie group acting on a compact space X and $K_G(X)$

is the Grothendieck group of the category of (real or complex) G -vector bundles on X . The analog of Bott periodicity in this context is the “Thom isomorphism”: we consider a complex G -vector bundle V on X and we would like to compute the complex equivariant K -theory of V , defined as the “reduced” K -theory $\tilde{K}_G(V_+)$, where V_+ is the one-point compactification of V . If we denote this group simply by $K_G(V)$, we have an isomorphism (due to Atiyah [3]) for complex equivariant K -theory

$$\beta_{\mathbb{C}} : K_G(X) \rightarrow K_G(V).$$

In a parallel way, if V is a *real* vector bundle of rank $8m$, provided with a spinorial structure⁶ Atiyah proved that we also have a “Thom isomorphism” for real equivariant K -theory

$$\beta_{\mathbb{R}} : K_G(X) \rightarrow K_G(V).$$

At this point, we should notice that if G is the trivial group and V a trivial vector bundle, the isomorphisms $\beta_{\mathbb{C}}$ and $\beta_{\mathbb{R}}$ are just restatements of the Bott periodicity theorems as in 1.

The isomorphisms $\beta_{\mathbb{C}}$ and $\beta_{\mathbb{R}}$ are not at all easy to prove. The algebraic ideas sketched in the previous sections are not sufficient (even if G a finite group!). One has to use the full strength of the Atiyah–Singer index theorem (KK -theory in modern terms) in order to construct a map going backwards

$$\beta' : K_G(V) \rightarrow K_G(X)$$

in the same spirit that led to the construction of the element u_{-2} in Sect. 4.4. The difficult part of the theorem is to show that $\beta' \cdot \beta = Id$. The fact that $\beta \cdot \beta' = Id$ follows from an ingenious trick due to Atiyah and described in [3].

4.5.1 $KR(X)$

The consideration of equivariant vector bundles led Atiyah to another elementary proof (i.e. more algebraic) of *real* Bott periodicity through the introduction of a new theory called $KR(X)$ and defined for any locally compact space X with an involution [1]. In our previous language, the group $KR(X)$ is just the K -theory of the Banach algebra A of continuous functions

$$f : X \rightarrow \mathbb{C}$$

such that $f(\sigma(x)) = \overline{f(x)}$ (where σ denotes the involution on X) and $f(x) \rightarrow 0$ when $x \rightarrow \infty$. The basic idea of this “Real” version of Bott periodicity is to prove (using Fourier analysis again) that $KR(X) \cong KR(X \times \mathbb{R}^{1,1})$ where $\mathbb{R}^{p,q}$ denotes in general

⁶ For a more general statement, using Clifford bundles, see the following paper of the author: *Equivariant K -theory of real vector spaces and real projective spaces*. Topology and its applications, 122, 531–546 (2002).

the Euclidean space $\mathbb{R}^{p+q} = \mathbb{R}^p \oplus \mathbb{R}^q$ with the involution $(x, y) \mapsto (-x, y)$. Using ingenious homeomorphisms between spaces with involution, Atiyah managed to show that this isomorphism implies that $K(A(\mathbb{R}^n))$ is periodic of period 8 with respect to n . As a consequence we recover the classical Bott periodicity (in the real case) if we restrict ourselves to spaces with trivial involution.

Bott Periodicity in Algebraic K -Theory

4.6

Let us now turn our interest to the *algebraic* K -theory of a *discrete* ring A . From now on, we change our notations and write $K_n(A)$ for the Quillen K -groups of a ring with unit A . We recall that for $n > 0$, $K_n(A)$ is the n th homotopy group of $BGL(A)^+$, where $BGL(A)^+$ is obtained from $BGL(A)$ by adding cells of dimensions 2 and 3 in such a way that the fundamental group becomes the quotient of $GL(A)$ by its commutator subgroup, without changing the homology.

If A is a Banach algebra, the K_n -groups considered before will be denoted by $K_n^{\text{top}}(A)$ in order to avoid confusion. We also have a definition of $K_n(A)$ for $n < 0$, due to Bass [8] and the author [26, 27]. More precisely, $K_{-r}(A) = K(S^r A)$, where SA denotes the suspension of the ring A (Beware: we do *not* take the Banach closure as in Sect. 4.4 since A is just a discrete ring). One interpretation of Bott periodicity, due to Bass [8] for $n \leq 0$ and Quillen [37] (for all n , assuming A to be regular Noetherian), is the following exact sequence:

$$0 \rightarrow K_{n+1}(A) \rightarrow K_{n+1}(A[t]) \oplus K_{n+1}(A[t^{-1}]) \rightarrow K_{n+1}(A[t, t^{-1}]) \rightarrow K_n(A) \rightarrow 0$$

[we replace the Laurent series considered in Sect. 4.4.2 by Laurent polynomials].

Periodicity with Finite Coefficients

4.6.1

However, the natural question to ask is whether we have some kind of “periodicity” for the groups $K_n(A)$, $n \in \mathbb{Z}$. Of course, this question is too naive: for instance, if A is the finite field F_q with q elements, we have $K_{2n-1}(F_q) \cong \mathbb{Z}/(q^n - 1)\mathbb{Z}$, as proved by Quillen [36]. The situation is much better if we consider algebraic K -theory with finite coefficients, introduced by Browder and the author in the 70’s [15, 42]. Since we know that algebraic K -theory is represented by a spectrum $\mathbb{K}(A)$ defined through the spaces $BGL(S^n A)^+$, as proved by Wagoner [51], there is a well known procedure in algebraic topology for constructing an associated mod n -spectrum for any positive integer $n > 1$. The homotopy groups of this spectrum are the K -theory groups of A with coefficients in $\mathbb{Z}/n$.

An alternative (dual) approach is to consider the Puppe sequence associated to a self-map of degree n of the sphere S^r

$$S^r \rightarrow S^r \rightarrow M(n, r) \rightarrow S^{r+1} \rightarrow S^{r+1}.$$

The space $M(n, r)$, known as a “Moore space”, has two cells of dimensions r and $r + 1$ respectively, the second cell being attached to the first by a map of degree n . If $n = 2$ and $r = 1$ for instance, $M(n, r)$ is the real projective space of dimension 2. We now define K -theory with coefficients in $\mathbb{Z}/n$, denoted by $K_{r+1}(A; \mathbb{Z}/n)$, for $r \geq 0$, as the group of pointed homotopy classes of maps from $M(n, r)$ to $\mathcal{K}(A)$. From the classical Puppe sequence in algebraic topology, we get an exact sequence

$$K_{r+1}(A) \rightarrow K_{r+1}(A) \rightarrow K_{r+1}(A; \mathbb{Z}/n) \rightarrow K_r(A) \rightarrow K_r(A) ,$$

where the arrows between the groups $K_i(A)$ are multiplication by n . Since $K_{r+t+1}(S^t A; \mathbb{Z}/n)$ is canonically isomorphic to $K_{r+1}(A; \mathbb{Z}/n)$, we may extend this definition of $K_{r+1}(A; \mathbb{Z}/n)$ to all values of $r \in \mathbb{Z}$, by putting $K_{r+1}(A; \mathbb{Z}/n) = K_{r+t+1}(S^t A; \mathbb{Z}/n)$ for t large enough so that $r + t \geq 0$. With certain restrictions on n or A (we omit the details), this K -theory mod n may be provided with a ring structure, having some nice properties [15].

Here is a fundamental theorem of Suslin [43] that is the true analog of complex Bott periodicity in algebraic K -theory.

10

Theorem 10 Let F be an algebraically closed field and let n be an integer prime to the characteristic of F . Then there is a canonical isomorphism between graded rings

$$K_{2r}(F; \mathbb{Z}/n) \cong (\mu_n)^{\otimes r} \quad \text{and} \quad K_{2r+1}(F; \mathbb{Z}/n) = 0 ,$$

where μ_n denotes the group of n th roots of unity in F (with $r \geq 0$).

4.6.2 The Lichtenbaum–Quillen Conjecture

Starting with this theorem one may naturally ask if there is a way to compute $K_*(F; \mathbb{Z}/n)$ for an arbitrary field F (not necessarily algebraically closed). If F is the field of real numbers, and if we work in topological K -theory instead, we know that it is not an easy task, since we get an 8-periodicity that looks mysterious compared to the 2-periodicity of the complex case. All these types of questions are in fact related to a “homotopy limit problem”⁷. More precisely, let us define in general $\mathbb{K}(A, n)$ as the spectrum of the K -theory of A mod n as in Sect. 4.6.1. It is easy to show that $\mathbb{K}(F, n)$ is the fixed point set of $\mathbb{K}(\overline{F}, n)$, where $\overline{F}$ denotes the separable closure of F , with respect to the action of the Galois group G (which is a profinite group). We have a fundamental map

$$\phi : \mathbb{K}(F, n) = \mathbb{K}(\overline{F}, n)^G \rightarrow \mathbb{K}(\overline{F}, n)^{hG} .$$

where $\mathbb{K}(\overline{F}, n)^{hG}$ is the “homotopy fixed point set” of $\mathbb{K}(\overline{F}, n)$, i.e. the set of equivariant maps $EG \rightarrow \mathbb{K}(\overline{F}, n)$ where EG is the “universal” principal G -bundle over BG .

⁷ See the paper of Thomason: *The homotopy limit problem*. Contemporary Mathematics Vol. 19, 407–419 (1983).

Let us denote by $K_r^{et}(F; \mathbb{Z}/n)$ the r -th homotopy group of this space of equivariant maps (which we call “étale K -theory” groups). According to Sect. 4.6.2 and the general theory of homotopy fixed point sets, there is a spectral sequence⁸

$$E_2^{p,q} = H^p(G; \mu_n^{\otimes(q/2)}) \Rightarrow K_{q-p}^{et}(F; \mathbb{Z}/n).$$

Let us assume now that the characteristic of the field does not divide n . A version of the “Lichtenbaum–Quillen conjecture”⁹, is that ϕ induces an isomorphism on homotopy groups π_r for $r > d_n$, where d_n is the n -cohomological dimension of G . In other words, the canonical map

$$K_r(F; \mathbb{Z}/n) \rightarrow K_r^{et}(F; \mathbb{Z}/n)$$

should be an isomorphism for $r > d_n$. The surjectivity of this map was investigated and proved in many cases by Soulé [42], Dwyer–Friedlander–Snaith–Thomason [18] in the 80’s.

Thomason’s Approach

4.6.3

In order to compare more systematically algebraic K -theory and étale K -theory (which is periodic, as we shall see), there is an elegant approach, initiated by Thomason [47]. If we stick to fields and to an odd prime p (for the sake of simplicity), there is a “Bott element” β belonging to the group $K_{2(p-1)}(F; \mathbb{Z}/p)$ as it was first shown by Browder and the author in the 70’s [15, 42]. One notice that the usual Bott element $u_2 \in K_2(\mathbb{C}; \mathbb{Z}/p)$ can be lifted to an element u of the group $K_2(A; \mathbb{Z}/p)$ via the homomorphism induced by the ring map $A \rightarrow \mathbb{C}$, where A is the ring of p -cyclotomic integers. By a transfer argument, one then shows that u^{p-1} is the image of a canonical element β of the group $K_{2(p-1)}(\mathbb{Z}; \mathbb{Z}/p)$ through the standard homomorphism $K_{2(p-1)}(\mathbb{Z}; \mathbb{Z}/p) \rightarrow K_{2(p-1)}(A; \mathbb{Z}/p)$. By abuse of notation, we shall also call β its image in $K_{2(p-1)}(F; \mathbb{Z}/p)$ and β_{et} its image in $K_{2(p-1)}^{et}(F; \mathbb{Z}/p)$. The important remark here is that β_{et} is invertible in the étale K -theory ring (which is a way to state that étale K -theory is periodic of period $2(p-1)$). Therefore, there is a factorisation

$$\begin{array}{ccc} K_*(F; \mathbb{Z}/p) & \xrightarrow{\quad\quad\quad} & K_*^{et}(F; \mathbb{Z}/p) \\ & \searrow \quad \quad \nearrow & \\ & K_*(F; \mathbb{Z}/p)[\beta^{-1}] & \end{array}$$

⁸ Note that $\pi_q(\mathbb{K}(\overline{F}), n) = \mu_n^{\otimes(q/2)}$ as stated in Sect. 4.6.2 where we put $\mu_n^{\otimes(q/2)} = 0$ for q odd.

⁹ Many mathematicians contributed to this formulation, which is quite different from the original one by Lichtenbaum and Quillen. We should mention the following names: Dwyer, Friedlander, Mitchell, Snaith, Thomason, ... For an overview, see for instance [47, p. 516–523].

11

Theorem 11 [47] Let us assume that F is of finite p -étale dimension and moreover satisfies the (mild) conditions of [47, Theorem 4.1]. Then the map

$$K_*(F; \mathbb{Z}/p)[\beta^{-1}] \rightarrow K_*^{\text{ét}}(F; \mathbb{Z}/p)$$

defined above is an isomorphism.

4.6.4 The Bloch–Kato Conjecture

In order to make more progress in the proof of the Lichtenbaum–Quillen conjecture, Bloch and Kato formulated in the 90's another conjecture that is now central to the current research in algebraic K -theory. This conjecture states that for any integer n , the Galois symbol from Milnor's K -theory mod n [33] to the corresponding Galois cohomology

$$K_r^M(F)/n \rightarrow H^r(F; \mu_n^{\otimes r}) = H^r(G; \mu_n^{\otimes r})$$

is an isomorphism. This conjecture was first proved by Voevodsky for $n = 2^k$: it is the classical Milnor's conjecture [48].

After this fundamental work of Voevodsky, putting as another ingredient a spectral sequence first established by Bloch and Lichtenbaum [11], many authors were able to solve the Lichtenbaum–Quillen conjecture for $n = 2^k$. We should mention Kahn [24], Rognes and Weibel [38], Ostvaer and Rosenschon (to appear).

At the present time (August 2003), there is some work in progress by Rost (unpublished) and Voevodsky [49] giving some evidence that the Bloch–Kato conjecture should be true for all values of n . Assuming this work accomplished, the Lichtenbaum–Quillen conjecture will then be proved in general!

There is another interesting consequence of the Bloch–Kato conjecture that is worth mentioning: we should have a “motivic” spectral sequence (first conjectured by Beilinson), different from the one written in Sect. 4.6.3:

$$E_2^{p,q} \Rightarrow K_{q-p}(F; \mathbb{Z}/n) .$$

Here, the term E_2 of the spectral sequence is the following:

$$E_2^{p,q} = H^p(G; \mu_n^{\otimes(q/2)}) \quad \text{for } p \leq q/2 \quad \text{and}$$

$$E_2^{p,q} = 0 \quad \text{for } p > q/2 .$$

This spectral sequence should degenerate in many cases, for instance if n is odd or if F is an exceptional field [24, p. 102].

Example 12. If F is a number field and if n is odd, we have $d_n = 1$ with the notations of Sect. 4.6.3. In this case, the degenerating spectral sequence above shows a direct link between algebraic K -theory and Galois cohomology, quite interesting in Number Theory.

The Bloch–Kato conjecture (if proved in general) also sheds a new light on Thomason’s localisation map considered in Sect. 4.6.3:

$$K_r(F; \mathbb{Z}/p) \rightarrow K_r(F; \mathbb{Z}/p)[\beta^{-1}] .$$

For instance, we can state Kahn’s Theorem 2 [24, p. 100], which gives quite general conditions of injectivity or surjectivity for this map in a certain range of degrees, not only for fields, but also for finite dimensional schemes.

We should notice in passing that a topological analog of the Lichtenbaum–Quillen conjecture is true: in this framework, one replace the classifying space of algebraic K -theory by the classifying space of (complex) topological K -theory $\mathcal{K}^{top}(A \otimes_{\mathbb{R}} \mathbb{C})$, where A is a real Banach algebra and $\mathbb{Z}/2$ acts by complex conjugation (compare with [44]). Then the fixed point set (i.e. the classifying space of the topological K -theory of A) has the homotopy type of the homotopy fixed point set [30].

Despite these recent breakthroughs, the groups $K_r(A)$ are still difficult to compute *explicitly*, even for rings as simple as the ring of integers in a number field (although we know these groups rationally from the work of Borel [12] on the rational cohomology of arithmetic groups). However, thanks to the work of Bökstedt, Rognes and Weibel [38], we can at least compute the 2-primary torsion of $K_r(\mathbb{Z})$ through the following homotopy cartesian square

$$\begin{array}{ccc} BGL(\mathbb{Z}[1/2])_{\#}^{+} & \longrightarrow & BGL(\mathbb{R})_{\#} \\ \downarrow & & \downarrow \\ BGL(\mathbb{F}_3)_{\#}^{+} & \longrightarrow & BGL(\mathbb{C})_{\#} . \end{array}$$

Here the symbol $\#$ means 2-adic completion, while $BGL(\mathbb{R})$ and $BGL(\mathbb{C})$ denote the classifying spaces of the *topological* groups $GL(\mathbb{R})$ and $GL(\mathbb{C})$ respectively. From this homotopy cartesian square, Rognes and Weibel [38], obtained the following results (modulo a finite odd torsion group and with $n > 0$ for the first 2 groups and $n \geq 0$ for the others):

$$K_{8n}(\mathbb{Z}) = 0$$

$$K_{8n+1}(\mathbb{Z}) = \mathbb{Z}/2$$

$$K_{8n+2}(\mathbb{Z}) = \mathbb{Z}/2$$

$$K_{8n+3}(\mathbb{Z}) = \mathbb{Z}/16$$

$$K_{8n+4}(\mathbb{Z}) = 0$$

$$K_{8n+5}(\mathbb{Z}) = \mathbb{Z}$$

$$K_{8n+6}(\mathbb{Z}) = 0$$

$$K_{8n+7}(\mathbb{Z}) = \mathbb{Z}/2^{r+2} ,$$

where 2^r is the 2 primary component of the number $4n + 4$. There is a (non published) conjecture of S.A. Mitchell about the groups $K_i(\mathbb{Z})$ in general (including the odd torsion), for $i \geq 2$. Let us write the k th Bernoulli number [28, pp. 297, 299] as an irreducible fraction c_k/d_k . Then we should have the following explicit computations of $K_r(\mathbb{Z})$ for $r \geq 2$, with $k = \left\lfloor \frac{r}{4} \right\rfloor + 1$:

$$\begin{aligned} K_{8n}(\mathbb{Z}) &= 0 \\ K_{8n+1}(\mathbb{Z}) &= \mathbb{Z} + \mathbb{Z}/2 \\ K_{8n+2}(\mathbb{Z}) &= \mathbb{Z}/2c_k (k = 2n + 1) \\ K_{8n+3}(\mathbb{Z}) &= \mathbb{Z}/8kd_k (k = 2n + 1) \\ K_{8n+4}(\mathbb{Z}) &= 0 \\ K_{8n+5}(\mathbb{Z}) &= \mathbb{Z} \\ K_{8n+6}(\mathbb{Z}) &= \mathbb{Z}/c_k (k = 2n + 2) \\ K_{8n+7}(\mathbb{Z}) &= \mathbb{Z}/4kd_k (k = 2n + 2) . \end{aligned}$$

Example 13. If we want to compute $K_{22}(\mathbb{Z})$ and $K_{23}(\mathbb{Z})$, we write

$$22 = 8 \cdot 2 + 6 \quad \text{and} \quad 23 = 8 \cdot 2 + 7 .$$

Hence $k = 6$ with an associated Bernoulli number $B_6 = 691/2730$. Therefore, according to the conjecture, we should have (compare with [42]):

$$K_{22}(\mathbb{Z}) \cong \mathbb{Z}/691 \quad K_{23}(\mathbb{Z}) \cong \mathbb{Z}/65\,520 .$$

Note that for the groups $K_r(\mathbb{Z})$ for $r \leq 6$, complete results (not just conjectures) are found among other theorems in the reference [19]. Here they are (the new ones are K_5 and K_6):

$$\begin{aligned} K_0(\mathbb{Z}) &= \mathbb{Z} \\ K_1(\mathbb{Z}) &= \mathbb{Z}/2 \\ K_2(\mathbb{Z}) &= \mathbb{Z}/2 \quad (\text{Milnor}) \\ K_3(\mathbb{Z}) &= \mathbb{Z}/48 \quad (\text{Lee and Szczarba}) \\ K_4(\mathbb{Z}) &= 0 \quad (\text{Rognes}) \\ K_5(\mathbb{Z}) &= \mathbb{Z} \\ K_6(\mathbb{Z}) &= 0 . \end{aligned}$$

Other rings with periodic algebraic K -theory, arising from a completely different point of view, are complex *stable* C^* -algebras A : by definition, they are isomorphic to their completed tensor product with the algebra $\mathcal{K}$ of compact operators in a complex Hilbert space. For example, the C^* -algebra associated to a foliation is stable [17]. Another example is the ring of continuous functions from a compact space X to the ring $\mathcal{K}$. These algebras are not unital but, as proved by Suslin and Wodzicki, they satisfy excision. A direct consequence of this excision property is the following theorem, conjectured by the author in the 70's (the tentative proof was based on the scheme described at the end of Sect. 4.3, assuming excision).

Theorem 14 [45] Let A be a complex stable C^* -algebra. Then the obvious map

14

$$K_n(A) \rightarrow K_n^{top}(A)$$

is an isomorphism. In particular, we have Bott periodicity

$$K_n(A) \approx K_{n+2}(A)$$

for the algebraic K -theory groups.

M. Wodzicki can provide many other examples of ideals $\mathcal{J}$ in the ring $\mathcal{B}(H)$ of bounded operators in a Hilbert space such that the tensor product (completed or not) $B \otimes \mathcal{J}$, for B any complex algebra, has a periodic ALGEBRAIC K -theory. These ideals are characterized by the fact that $\mathcal{J} = \mathcal{J}^2$ and that the commutator subgroup $[\mathcal{B}(H), \mathcal{J}]$ coincides with $\mathcal{J}$.

Bott Periodicity in Hermitian K -Theory

4.7

Orthogonal Groups

4.7.1

As we have seen, usual K -theory is deeply linked with the general linear group. One might also consider what happens for the other classical groups. Not only is it desirable, but this setting turns out to be quite suitable for a generalization of Bott periodicity and the computation of the homology of *discrete* orthogonal and symplectic groups in terms of classical Witt groups.

The starting point is a ring A with an antiinvolution $a \mapsto \bar{a}$, together with an element ε in the center of A such that $\varepsilon\bar{\varepsilon} = 1$. In most examples, $\varepsilon = \pm 1$. For reasons appearing later (see Theorem 19), we also assume the existence of an element λ in the center of A such that $\lambda + \bar{\lambda} = 1$ (if 2 is invertible in A , we may choose $\lambda = 1/2$). If M is a *right* finitely generated projective module over A , we define its dual M^* as the group of $\mathbb{Z}$ -linear maps $f : M \rightarrow A$ such that $f(m.a) = \bar{a}.f(m)$ for $m \in M$ and $a \in A$. It is again a right finitely generated projective A -module if we put $(f.b)(m) = f(m).b$ for $b \in A$. An ε -hermitian form on M is an A -linear

map: $\tilde{\phi} : M \rightarrow M^*$ satisfying some conditions of ε -symmetry ($\tilde{\phi} = \varepsilon \tilde{\phi}^*$ as written below). More precisely, it is given by a $\mathbb{Z}$ -bilinear map

$$\phi : M \times M \rightarrow A$$

such that

$$\begin{aligned}\phi(ma, m'b) &= \bar{a}\phi(m, m')b \\ \phi(m', m) &= \overline{\varepsilon\phi(m, m')}\end{aligned}$$

with obvious notations. Such a ϕ is called an ε -hermitian form and (M, ϕ) is an ε -hermitian module. The map

$$\tilde{\phi} : m' \mapsto [m \mapsto (m, m')]$$

does define a morphism from M to M^* and we say that ϕ is non-degenerate if $\tilde{\phi}$ is an isomorphism.

Fundamental example (the hyperbolic module). Let N be a finitely generated projective module and $M = N \oplus N^*$. A non-degenerate ε -hermitian form ϕ on M is given by the following formula

$$\phi((x, f), (x', f')) = \overline{f(x')} + \varepsilon f'(x).$$

We denote this module by $H(N)$. If $N = A^n$, we may identify N with its dual via the map $\gamma \mapsto f_\gamma$ with $f_\gamma(x) = \bar{x}\gamma$. The hermitian form on $A^n \oplus A^n$ may then be written as

$$\phi((x, \gamma), (x', \gamma')) = \bar{\gamma}x' + \varepsilon \bar{x}\gamma'.$$

There is an obvious definition of direct sum for non-degenerate ε -hermitian modules. We write ${}_\varepsilon L(A)$ or ${}_\varepsilon L_0(A)$ for the Grothendieck group constructed from such modules¹⁰.

Example 15. Let A be the ring of continuous functions on a compact space X with complex values. If A is provided with the trivial involution, ${}_1 L(A)$ is isomorphic to the real topological K -theory of X while ${}_{-1} L(A)$ is isomorphic to its quaternionic topological K -theory (see e.g. [28] p. 106, exercise 6.8).

In the Hermitian case, the analog of the general linear group is the ε -orthogonal group that is the group of automorphisms of $H(A^n)$, denoted by ${}_\varepsilon O_{n,n}(A)$: its elements may be described concretely in terms of $2n \times 2n$ matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

¹⁰ We use the letter L , which is quite convenient, but the reader should not mix up the present definition with the definition of the surgery groups, also denoted by the letter L (see the papers of Lück/Reich, Rosenberg and Williams in this handbook).

such that $M^*M = MM^* = I$, where

$$M^* = \begin{pmatrix} {}^t\overline{d} & \varepsilon {}^t\overline{b} \\ \overline{\varepsilon} {}^t\overline{c} & {}^t\overline{a} \end{pmatrix}.$$

Example 16. If A is the field of real numbers $\mathbb{R}$, ${}_1O_{n,n}(A)$ is the classical group $O(n, n)$, which has the homotopy type of $O(n) \times O(n)$. By contrast, ${}_{-1}O_{n,n}(A)$ is the classical group $Sp(2n, \mathbb{R})$, which has the homotopy type of the unitary group $U(n)$ [21].

The infinite orthogonal group

$${}_{\varepsilon}O(A) = \lim_{\varepsilon} O_{n,n}(A)$$

has a commutator subgroup that is perfect (similarly to GL). Therefore, we can perform the $+$ construction of Quillen as in [37].

Definition 17 The higher Hermitian K -theory of a ring A (for $n > 0$) is defined as

17

$${}_{\varepsilon}L_n(A) = \pi_n(B_{\varepsilon}O(A)^+).$$

Example 18. Let F be a field of characteristic different from 2 provided with the trivial involution. Then ${}_{\varepsilon}L_1(F) = 0$ if $\varepsilon = -1$ and ${}_{\varepsilon}L_1(F) = \mathbb{Z}/2 \times F^*/F^{*2}$ if $\varepsilon = +1$ (see e.g. [9]).

Notation We write

$$\mathcal{K}(A) = K(A) \times BGL(A)^+$$

for the classifying space of algebraic K -theory – as before – and

$${}_{\varepsilon}\mathcal{L}(A) = {}_{\varepsilon}L(A) \times B_{\varepsilon}O(A)^+$$

for the classifying space of Hermitian K -theory.

There are two interesting functors between Hermitian K -theory and algebraic K -theory. One is the forgetful functor from modules with hermitian forms to modules (with no forms) and the other one from modules to modules with forms, sending N to $H(N)$, the hyperbolic module associated to N . These functors induce two maps

$$F : {}_{\varepsilon}\mathcal{L}(A) \rightarrow \mathcal{K}(A) \quad \text{and} \quad H : \mathcal{K}(A) \rightarrow {}_{\varepsilon}\mathcal{L}(A).$$

We define ${}_{\varepsilon}\mathcal{V}(A)$ as the homotopy fiber of F and ${}_{\varepsilon}\mathcal{U}(A)$ as the homotopy fiber of H . We thus define two “relative” theories:

$${}_{\varepsilon}V_n(A) = \pi_n({}_{\varepsilon}\mathcal{V}(A)) \quad \text{and} \quad {}_{\varepsilon}U_n(A) = \pi_n({}_{\varepsilon}\mathcal{U}(A)).$$

19 **Theorem 19** (the fundamental theorem of Hermitian K -theory [29]). Let A be a **discrete** ring with the hypotheses in Sect. 4.7.1. Then there is a natural homotopy equivalence between ${}_{\varepsilon}\mathcal{V}(A)$ and the loop space of ${}_{-\varepsilon}\mathcal{U}(A)$. In particular,

$${}_{\varepsilon}V_n(A) \cong {}_{-\varepsilon}U_{n+1}(A) .$$

Moreover, if we work within the framework of Banach algebras with an antiinvolution, the same statement is valid for the topological analogs (i.e. replacing $BGL(A)^+$ by $BGL(A)^{top}$ and $B_{\varepsilon}O(A)^+$ by $B_{\varepsilon}O(A)^{top}$).

4.7.2 Examples

In order to get a feeling for this theorem, it is worthwhile to see what we get for the classical examples $A = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, with their usual topology and various antiinvolutions. Note in general that the connected component of ${}_{\varepsilon}\mathcal{V}(A)$ (resp. ${}_{-\varepsilon}\mathcal{U}(A)$) is the connected component of the homogeneous space $GL(A)/{}_{\varepsilon}O(A)$ (resp. ${}_{-\varepsilon}O(A)/GL(A)$). For instance, if $A = \mathbb{R}$, $\varepsilon = -1$, we get the homogeneous spaces

$$GL(\mathbb{R})/{}_{-1}O(\mathbb{R}) \quad \text{and} \quad {}_1O(\mathbb{R})/GL(\mathbb{R}) ,$$

which have the homotopy type of $GL(\mathbb{R})/GL(\mathbb{C})$, and $GL(\mathbb{R})$ respectively [21]. In this case, Theorem 19 implies that $GL(\mathbb{R})/GL(\mathbb{C})$ has the homotopy type of the loop space $\Omega GL(\mathbb{R})$, one of the eight homotopy equivalences of Bott (compare with Theorem 3). It is a pleasant exercise to recover the remaining seven homotopy equivalences by dealing with other classical groups and various inclusions between them. Since the list of classical groups is finite, it is “reasonable” to expect some periodicity...

An advantage of this viewpoint (compared to the Clifford algebra approach in Sect. 4.3 for instance) is the context of this theorem, valid in the discrete case, which implies Bott periodicity for “discrete” Hermitian K -theory. For instance, if we consider the higher Witt groups

$${}_{\varepsilon}W_n(A) = \text{Coker} \left(K_n(A) \rightarrow {}_{\varepsilon}L_n(A) \right) ,$$

the fundamental theorem implies a periodicity isomorphism (modulo 2 torsion)¹¹

$${}_{\varepsilon}W_n(A) \cong {}_{-\varepsilon}W_{n-2}(A) .$$

From this result we get some information about the homology of the *discrete* group ${}_{\varepsilon}O(A)$ (at least rationally) [29]. If we denote by ${}_{\varepsilon}W(A)$ the periodic graded vector space $\bigoplus_{n \in \mathbb{Z}} {}_{\varepsilon}W_n(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, we find that the homology with rational coefficients $H^*({}_{\varepsilon}O(A); \mathbb{Q})$ may be written as the tensor product of the symmetric algebra of

¹¹ As proved in [29], the hypotheses in Sect. 4.7.1 are no longer necessary for this statement and A might be an arbitrary ring.

${}_{\varepsilon}W(A)$ with a graded vector space¹² $T_*(A)$ such that $T_0(A) = \mathbb{Q}$. This result is of course related to the classical theorems of Borel [12], when A is the ring of S -integers in a number field.

In another direction, if A is a regular noetherian ring with 2 invertible in A , the isomorphism ${}_{\varepsilon}W_n(A) \approx {}_{-\varepsilon}W_{n-2}(A)$ is true for $n \leq 0$ with no restriction about the 2-torsion. This implies a 4-periodicity of these groups. As it was pointed in [23], these 4-periodic Witt groups are isomorphic to Balmer's triangular Witt groups in this case [7]. Note these Balmer's Witt groups are isomorphic to surgery groups as it was proved by Walter [52], even if the ring A is not regular. It would be interesting to relate – for non regular rings – the negative Witt groups of our paper to the classical surgery groups.

Finally, we should remark that the 2-primary torsion of the Hermitian K -theory of the ring $\mathbb{Z}[1/2]$ can be computed the same way as the 2-primary torsion of the algebraic K -theory of $\mathbb{Z}$ (compare with Sect. 4.6.4), thanks to the following homotopy cartesian square proved in [10]:

$$\begin{array}{ccc} B_{\varepsilon}O(\mathbb{Z}[1/2])_{\#}^{+} & \longrightarrow & B_{\varepsilon}O(\mathbb{R})_{\#} \\ \downarrow & & \downarrow \\ B_{\varepsilon}O(\mathbb{F}_3)_{\#}^{+} & \longrightarrow & B_{\varepsilon}O(\mathbb{C})_{\#} \end{array}$$

From this diagram, we deduce the following 2-adic computation of the groups ${}_{\varepsilon}L_i = {}_{\varepsilon}L_i(\mathbb{Z}[1/2])$ for $i \geq 2$, i an integer mod 8, and $\varepsilon = \pm 1$, in comparizon with the table of the $K_i = K_i(\mathbb{Z})$ in Sect. ?? (where 2^t is again the 2-primary component of $i + 1$):

i	0	1	2	3	4	5	6	7
K_i	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2^{t+1}$
${}_1L_i$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/8$	$\mathbb{Z}$	0	0	$\mathbb{Z}/2^{t+1}$
${}_{-1}L_i$	0	0	$\mathbb{Z}$	$\mathbb{Z}/16$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2^{t+1}$

Conclusion

Let us add a few words of conclusion about the relation between motivic ideas and Bott periodicity, although other papers in this handbook will develop this analogy with more details (see also 4.6.3).

Morel and Voevodsky [35] have proved that algebraic K -theory is representable by an infinite Grassmannian in the unstable motivic homotopy category. Moreover, Voevodsky [50] has shown that this, together with Quillen's computation of the K -theory of the projective line, implies that algebraic K -theory is representable in the

¹² $T^*(A)$ is the symmetric algebra of $K^+(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $K^+(A)$ is the part of K -theory that is invariant under the contragredient isomorphism (it is induced by the map sending a matrix M to ${}^t\overline{M}^{-1}$).

stable motivic homotopy category by a motivic $(2, 1)$ -periodic Ω -spectrum. This mysterious “periodicity” is linked with two algebraic analogs of the circle, already considered in [31] and [27]: one circle is the scheme of the subring of $A[t]$ consisting of polynomials $P(t)$ such that $P(0) = P(1)$. The second one is the multiplicative group $\mathbb{G}_m$, which is the scheme of the ring $A[t, t^{-1}]$, already considered in Sect. 4.6. The “smash product” (in the homotopy category of schemes) of these two models is the projective line $\mathbb{P}^1$. This $(2, 1)$ -periodicity referred to above is a consequence of Quillen’s computation of the K -theory of the projective line [34].

In the same spirit, Hornbostel [23] has shown that Hermitian K -theory is representable in the unstable motivic homotopy category. Combining this with Theorem 19 and the localisation theorem for Hermitian K -theory of Hornbostel–Schlichting [22] applied to $A[t, t^{-1}]$, Hornbostel deduces that Hermitian K -theory is representable by a motivic $(8, 4)$ -periodic Ω -spectrum in the stable homotopy category. This periodicity is linked with the computation of the Hermitian K -theory of the smash product of $\mathbb{P}^1$ four times with itself (which is S^8 from a homotopy viewpoint).

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Algebraic K -Theory of Rings of Integers in Local and Global Fields

Charles Weibel

5.1	<i>Introduction</i>	140
5.2	<i>Classical K-Theory of Number Fields</i>	143
5.3	<i>The e-Invariant</i>	147
5.4	<i>Étale Chern Classes</i>	153
5.5	<i>Global Fields of Finite Characteristic</i>	158
5.6	<i>Local Fields</i>	162
5.7	<i>Number Fields at Primes Where $cd = 2$</i>	167
5.8	<i>Real Number Fields at the Prime 2</i>	171
5.9	<i>The Odd Torsion in $K_*(\mathbb{Z})$</i>	180
	<i>References</i>	185

Summary. This survey describes the algebraic K -groups of local and global fields, and the K -groups of rings of integers in these fields. We have used the result of Rost and Voevodsky to determine the odd torsion in these groups.

5.1 Introduction

The problem of computing the higher K -theory of a number field F , and of its rings of integers $\mathcal{O}_F$, has a rich history. Since 1972, we have known that the groups $K_n(\mathcal{O}_F)$ are finitely generated [48], and known their ranks [7], but have only had conjectural knowledge about their torsion subgroups [5, 33, 34] until 1997 (starting with [76]). The resolutions of many of these conjectures by Suslin, Voevodsky, Rost and others have finally made it possible to describe the groups $K_*(\mathcal{O}_F)$. One of the goals of this survey is to give such a description; here is the odd half of the answer (the integers $w_i(F)$ are even, and are defined in Sect. 5.3):

1

Theorem 1 Let $\mathcal{O}_S$ be a ring of S -integers in a number field F . Then $K_n(\mathcal{O}_S) \cong K_n(F)$ for each odd $n \geq 3$, and these groups are determined only by the number r_1, r_2 of real and complex places of F and the integers $w_i(F)$:

- a) If F is totally imaginary, $K_n(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F)$;
- b) If F has $r_1 > 0$ real embeddings then, setting $i = (n + 1)/2$,

$$K_n(\mathcal{O}_S) \cong K_n(F) \cong \begin{cases} \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/w_i(F), & n \equiv 1 \pmod{8} \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/2w_i(F) \oplus (\mathbb{Z}/2)^{r_1-1}, & n \equiv 3 \pmod{8} \\ \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/\frac{1}{2}w_i(F), & n \equiv 5 \pmod{8} \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F), & n \equiv 7 \pmod{8} . \end{cases}$$

In particular, $K_n(\mathbb{Q}) \cong \mathbb{Z}$ for all $n \equiv 5 \pmod{8}$ (as $w_i = 2$; see Lemma 27). More generally, if F has a real embedding and $n \equiv 5 \pmod{8}$, then $K_n(F)$ has no 2-primary torsion (because $\frac{1}{2}w_i(F)$ is an odd integer; see Proposition 22).

The proof of Theorem 1 will be given in 70, 73, and Sect. 5.8 below.

We also know the order of the groups $K_n(\mathbb{Z})$ when $n \equiv 2 \pmod{4}$, and know that they are cyclic for $n < 20\,000$ (see Example 96 – conjecturally, they are cyclic for every $n \equiv 2$). If B_k denotes the k^{th} Bernoulli number (24), and c_k denotes the numerator of $B_k/4k$, then $|K_{4k-2}(\mathbb{Z})|$ is: c_k for k even, and $2c_k$ for k odd; see 95.

Although the groups $K_{4k}(\mathbb{Z})$ are conjectured to be zero, at present we only know that these groups have odd order, with no prime factors less than 10^7 . This conjecture follows from, and implies, Vandiver's conjecture in number theory (see 102 below). In Table 5.1, we have summarized what we know for $n < 20\,000$; conjecturally the same pattern holds for all n (see 105–107).

Table 5.1. The groups $K_n(\mathbb{Z})$, $n < 20\,000$. The notation '(0?)' refers to a finite group, conjecturally zero, whose order is a product of irregular primes $> 10^7$

$K_0(\mathbb{Z}) = \mathbb{Z}$	$K_8(\mathbb{Z}) = (0?)$	$K_{16}(\mathbb{Z}) = (0?)$
$K_1(\mathbb{Z}) = \mathbb{Z}/2$	$K_9(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_{17}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$
$K_2(\mathbb{Z}) = \mathbb{Z}/2$	$K_{10}(\mathbb{Z}) = \mathbb{Z}/2$	$K_{18}(\mathbb{Z}) = \mathbb{Z}/2$
$K_3(\mathbb{Z}) = \mathbb{Z}/48$	$K_{11}(\mathbb{Z}) = \mathbb{Z}/1008$	$K_{19}(\mathbb{Z}) = \mathbb{Z}/528$
$K_4(\mathbb{Z}) = 0$	$K_{12}(\mathbb{Z}) = (0?)$	$K_{20}(\mathbb{Z}) = (0?)$
$K_5(\mathbb{Z}) = \mathbb{Z}$	$K_{13}(\mathbb{Z}) = \mathbb{Z}$	$K_{21}(\mathbb{Z}) = \mathbb{Z}$
$K_6(\mathbb{Z}) = 0$	$K_{14}(\mathbb{Z}) = 0$	$K_{22}(\mathbb{Z}) = \mathbb{Z}/691$
$K_7(\mathbb{Z}) = \mathbb{Z}/240$	$K_{15}(\mathbb{Z}) = \mathbb{Z}/480$	$K_{23}(\mathbb{Z}) = \mathbb{Z}/65\,520$
$K_{8a}(\mathbb{Z}) = (0?)$	$K_{8a+4}(\mathbb{Z}) = (0?)$	
$K_{8a+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_{8a+5}(\mathbb{Z}) = \mathbb{Z}$	
$K_{8a+2}(\mathbb{Z}) = \mathbb{Z}/2c_{2a+1}$	$K_{8a+6}(\mathbb{Z}) = \mathbb{Z}/c_{2a+2}$	
$K_{8a+3}(\mathbb{Z}) = \mathbb{Z}/2w_{4a+2}$	$K_{8a+7}(\mathbb{Z}) = \mathbb{Z}/w_{4a+4}$	

For $n \leq 3$, the groups $K_n(\mathbb{Z})$ were known by the early 1970's; see Sect. 5.2. The right hand sides of Table 5.1 were also identified as subgroups of $K_n(\mathbb{Z})$ by the late 1970's; see Sects. 5.3 and 5.4. The 2-primary torsion was resolved in 1997 (Sect. 5.8), but the rest of Table 5.1 only follows from the recent Voevodsky-Rost theorem (Sects. 5.7 and 5.9).

The K -theory of local fields, and global fields of finite characteristic, is richly interconnected with this topic. The other main goal of this article is to survey the state of knowledge here too.

In Sect. 5.2, we describe the structure of $K_n(\mathcal{O}_F)$ for $n \leq 3$; this material is relatively classical, since these groups have presentations by generators and relations.

The cyclic summands in Theorem 1 are a special case of a more general construction, due to Harris and Segal. For all fields F , the odd-indexed groups $K_{2i-1}(F)$ have a finite cyclic summand, which, up to a factor of 2, is detected by a variation of Adams' e -invariant. These summands are discussed in Sect. 5.3.

There are also canonical free summands related to units, discovered by Borel, and (almost periodic) summands related to the Picard group of R , and the Brauer group of R . These summands were first discovered by Soulé, and are detected by étale Chern classes. They are discussed in Sect. 5.4.

The K -theory of a global field of finite characteristic is handled in Sect. 5.5. In this case, there is a smooth projective curve X whose higher K -groups are finite, and are related to the action of the Frobenius on the Jacobian variety of X . The orders of these groups are related to the values of the zeta function $\zeta_X(s)$ at negative integers.

The K -theory of a local field E containing $\mathbb{Q}_p$ is handled in Sect. 5.6. In this case, we understand the p -completion, but do not understand the actual groups $K_*(E)$.

In Sect. 5.7, we handle the odd torsion in the K -theory of a number field. This is a consequence of the Voevodsky-Rost theorem. These techniques also apply to the 2-primary torsion in totally imaginary number fields, and give 1(a).

The 2-primary torsion in real number fields (those with an embedding in $\mathbb{R}$) is handled in Sect. 5.8; this material is taken from [51], and uses Voevodsky's theorem in [69].

Finally, we consider the odd torsion in $K_{2i}(\mathbb{Z})$ in Sect. 5.9; the odd torsion in $K_{2i-1}(\mathbb{Z})$ is given by 1. The torsion occurring in the groups $K_{2i}(\mathbb{Z})$ only involves irregular primes, and is determined by Vandiver's conjecture (102). The lack of torsion for regular primes was first guessed by Soulé in [58].

The key technical tool that makes calculations possible for local and global fields is the motivic spectral sequence, from motivic cohomology H_M^* to algebraic K -theory. With coefficients $\mathbb{Z}/m$, the spectral sequence for X is:

$$E_2^{p,q} = H_M^{p-q}(X; \mathbb{Z}/m(-q)) \Rightarrow K_{-p-q}(X; \mathbb{Z}/m). \quad (5.1)$$

This formulation assumes that X is defined over a field [69]; a similar motivic spectral sequence was established by Levine in [32, (8.8)], over a Dedekind domain, in which the group $H_M^n(X, \mathbb{Z}(i))$ is defined to be the $(2i - n)$ -th hypercohomology on X of the complex of higher Chow group sheaves z^i .

When $1/m \in F$, Voevodsky and Rost proved in [69] ($m = 2^v$) and [68] (m odd) that $H_M^n(F, \mathbb{Z}/m(i))$ is isomorphic to $H_{\text{ét}}^n(F, \mu_m^{\otimes i})$ for $n \leq i$ and zero if $n > i$. That is, the E_2 -terms in this spectral sequence are just étale cohomology groups.

If $X = \text{Spec}(R)$, where R is a Dedekind domain with $F = \text{frac}(R)$ and $1/m \in R$, a comparison of the localization sequences for motivic and étale cohomology (see [32] and [58, p. 268]) shows that $H_M^n(X, \mathbb{Z}/m(i))$ is: $H_{\text{ét}}^n(X, \mu_m^{\otimes i})$ for $n \leq i$; the kernel of $H_{\text{ét}}^n(X, \mu_m^{\otimes i}) \rightarrow H_{\text{ét}}^n(F, \mu_m^{\otimes i})$ for $n = i + 1$; and zero if $n \geq i + 2$. That is, the E_2 -terms in the fourth quadrant are étale cohomology groups, but there are also modified terms in the column $p = +1$. For example, we have $E_2^{1,-1} = \text{Pic}(X)/m$. This is the only nonzero term in the column $p = +1$ when X has étale cohomological dimension at most two for ℓ -primary sheaves ($cd_\ell(X) \leq 2$), as will often occur in this article.

Writing $\mathbb{Z}/\ell^\infty(i)$ for the union of the étale sheaves $\mathbb{Z}/\ell^v(i)$, we also obtain a spectral sequence for every field F :

$$E_2^{p,q} = \begin{cases} H_{\text{ét}}^{p-q}(F; \mathbb{Z}/\ell^\infty(-q)) & \text{for } q \leq p \leq 0, \\ 0 & \text{otherwise} \end{cases} \Rightarrow K_{-p-q}(F; \mathbb{Z}/\ell^\infty), \quad (5.2)$$

and a similar spectral sequence for X , which can have nonzero entries in the column $p = +1$. If $cd_\ell(X) \leq 2$ it is:

$$E_2^{p,q} = \begin{cases} H_{\text{ét}}^{p-q}(X; \mathbb{Z}/\ell^\infty(-q)) & \text{for } q \leq p \leq 0, \\ \text{Pic}(X) \otimes \mathbb{Z}/\ell^\infty & \text{for } (p, q) = (+1, -1), \\ 0 & \text{otherwise} \end{cases} \Rightarrow K_{-p-q}(X; \mathbb{Z}/\ell^\infty). \quad (5.3)$$

Remark 2: (*Periodicity for $\ell = 2$.*) Pick a generator v_1^4 of $\pi^s(S^8; \mathbb{Z}/16) \cong \mathbb{Z}/16$; it defines a generator of $K_8(\mathbb{Z}[1/2]; \mathbb{Z}/16)$ and, by the edge map in (5.1), a canonical element of $H_{\text{ét}}^0(\mathbb{Z}[1/2]; \mu_{16}^{\otimes 4})$, which we shall also call v_1^4 . If X is any scheme, smooth over $\mathbb{Z}[1/2]$, the multiplicative pairing of v_1^4 (see [16, 32]) with the spectral sequence converging to $K_*(X; \mathbb{Z}/2)$ gives a morphism of spectral sequences $E_r^{p,q} \rightarrow E_r^{p-4, q-4}$ from (5.1) to itself. For $p \leq 0$ these maps are isomorphisms, induced by $E_2^{p,q} \cong H_{\text{ét}}^{p-q}(X, \mathbb{Z}/2)$; we shall refer to these isomorphisms as *periodicity isomorphisms*.

Since the Voevodsky-Rost result has not been published yet (see [68]), it is appropriate for us to indicate exactly where it has been invoked in this survey. In addition to Theorem 1, Table 5.1, (5.2) and (5.3), the Voevodsky-Rost theorem is used in Theorem 54, Sect. 5.7, 94–96, and in Sect. 5.9.

Acknowledgements Many people have graciously assisted me in assembling the information in this survey. The author specifically would like to thank Spencer Bloch, Bill Dwyer, Lars Hesselholt, Bruno Kahn, Steve Mitchell, Paul Østvær, John Rognes, Neil Sloane and Christophe Soulé.

The author would also like to thank the Newton Institute, for its hospitality during most of the writing process. The author was also partially supported by an NSA grant.

Classical K-Theory of Number Fields

5.2

Let F be a number field, *i.e.*, a finite extension of $\mathbb{Q}$, and let $\mathcal{O}_F$ denote the ring of integers in F , *i.e.*, the integral closure of $\mathbb{Z}$ in F . The first few K -groups of F and $\mathcal{O}_F$ have been known since the dawn of K -theory. We quickly review these calculations in this section.

When Grothendieck invented K_0 in the late 1950's, it was already known that over a Dedekind domain R (such as $\mathcal{O}_F$ or the ring $\mathcal{O}_S$ of S -integers in F) every projective module is the sum of ideals, each of which is projective and satisfies $I \oplus J \cong IJ \oplus R$. Therefore $K_0(R) = \mathbb{Z} \oplus \text{Pic}(R)$. Of course, $K_0(F) = \mathbb{Z}$.

In the case $R = \mathcal{O}_F$ the Picard group was already known as the *Class group* of F , and Dirichlet had proven that $\text{Pic}(\mathcal{O}_F)$ is finite. Although not completely understood to this day, computers can calculate the class group for millions of

number fields. For cyclotomic fields, we know that $\text{Pic}(\mathbb{Z}[\mu_p]) = 0$ only for $p \leq 19$, and that the size of $\text{Pic}(\mathbb{Z}[\mu_p])$ grows exponentially in p ; see [71].

Example 3. (*Regular primes.*) A prime p is called *regular* if $\text{Pic}(\mathbb{Z}[\mu_p])$ has no elements of exponent p , i.e., if p does not divide the order h_p of $\text{Pic}(\mathbb{Z}[\mu_p])$. Kummer proved that this is equivalent to the assertion that p does not divide the numerator of any Bernoulli number B_k , $k \leq (p-3)/2$ (see 24 and [71, 5.34]). Iwasawa proved that a prime p is regular if and only if $\text{Pic}(\mathbb{Z}[\mu_{p^v}])$ has no p -torsion for all v . The smallest irregular primes are $p = 37, 59, 67, 101, 103$ and 131. About 39% of the primes less than 4 million are irregular.

The historical interest in regular primes is that Kummer proved Fermat's Last Theorem for regular primes in 1847. For us, certain calculations of K -groups become easier at regular primes (see Sect. 5.9.)

We now turn to units. The valuations on F associated to the prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$ show that the group $F^\times$ is the product of the finite cyclic group $\mu(F)$ of roots of unity and a free abelian group of infinite rank. Dirichlet showed that the group of units of $\mathcal{O}_F$ is the product of $\mu(F)$ and a free abelian group of rank $r_1 + r_2 - 1$, where r_1 and r_2 are the number of embeddings of F into the real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$, respectively.

The relation of the units to the class group is given by the “divisor map” (of valuations) from $F^\times$ to the free abelian group on the set of prime ideals $\mathfrak{p}$ in $\mathcal{O}_F$. The divisor map fits into the “Units-Pic” sequence:

$$0 \rightarrow \mathcal{O}_F^\times \rightarrow F^\times \xrightarrow{\text{div}} \bigoplus_{\mathfrak{p}} \mathbb{Z} \rightarrow \text{Pic}(\mathcal{O}_F) \rightarrow 0.$$

If R is any commutative ring, the group $K_1(R)$ is the product of the group $R^\times$ of units and the group $SK_1(R) = SL(R)/[SL(R), SL(R)]$. Bass–Milnor–Serre proved in [3] that $SK_1(R) = 0$ for any ring of S -integers in any global field. Applying this to the number field F we obtain:

$$K_1(\mathcal{O}_F) = \mathcal{O}_F^\times \cong \mu(F) \times \mathbb{Z}^{r_1+r_2-1}. \quad (5.4)$$

For the ring $\mathcal{O}_S$ of S -integers in F , the sequence $1 \rightarrow \mathcal{O}_F^\times \rightarrow \mathcal{O}_S^\times \rightarrow \mathbb{Z}[S] \xrightarrow{\text{div}} \text{Pic}(\mathcal{O}_F) \rightarrow \text{Pic}(\mathcal{O}_S) \rightarrow 1$ yields:

$$K_1(\mathcal{O}_S) = \mathcal{O}_S^\times \cong \mu(F) \times \mathbb{Z}^{|S|+r_1+r_2-1}. \quad (5.5)$$

The 1967 Bass–Milnor–Serre paper [3] was instrumental in discovering the group K_2 and its role in number theory. Garland proved in [18] that $K_2(\mathcal{O}_F)$ is a finite group. By [49], we also know that it is related to $K_2(F)$ by the localization sequence:

$$0 \rightarrow K_2(\mathcal{O}_F) \rightarrow K_2(F) \xrightarrow{\partial} \bigoplus_{\mathfrak{p}} k(\mathfrak{p})^\times \rightarrow 0.$$

Since the map ∂ was called the *tame symbol*, the group $K_2(\mathcal{O}_F)$ was called the *tame kernel* in the early literature. Matsumoto's theorem allowed Tate to calculate

$K_2(\mathcal{O}_F)$ for the quadratic extensions $\mathbb{Q}(\sqrt{-d})$ of discriminant < 35 in [4]. In particular, we have $K_2(\mathbb{Z}) = K_2(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]) = \mathbb{Z}/2$ on $\{-1, -1\}$, and $K_2(\mathbb{Z}[i]) = 0$.

Tate's key breakthrough, published in [65], was the following result, which was generalized to all fields by Merkurjev and Suslin (in 1982).

Theorem 4: (*Tate [65]*) If F is a number field and R is a ring of S -integers in F such that $1/\ell \in R$ then $K_2(R)/m \cong H_{\text{ét}}^2(R, \mu_m^{\otimes 2})$ for every prime power $m = \ell^v$. The ℓ -primary subgroup of $K_2(R)$ is $H_{\text{ét}}^2(R, \mathbb{Z}_\ell(2))$, which equals $H_{\text{ét}}^2(R, \mu_m^{\otimes 2})$ for large v .

If F contains a primitive m^{th} root of unity ($m = \ell^v$), there is a split exact sequence:

$$0 \rightarrow \text{Pic}(R)/m \rightarrow K_2(R)/m \rightarrow {}_m\text{Br}(R) \rightarrow 0.$$

Here $\text{Br}(R)$ is the Brauer group and ${}_m\text{Br}(R)$ denotes $\{x \in \text{Br}(R) | mx = 0\}$. If we compose with the inclusion of $K_2(R)/m$ into $K_2(R; \mathbb{Z}/m)$, Tate's proof shows that the left map $\text{Pic}(R) \rightarrow K_2(R; \mathbb{Z}/m)$ is multiplication by the Bott element $\beta \in K_2(R; \mathbb{Z}/m)$ corresponding to a primitive m -th root of unity. The quotient ${}_m\text{Br}(R)$ of $K_2(R)$ is easily calculated from the sequence:

$$0 \rightarrow \text{Br}(R) \rightarrow (\mathbb{Z}/2)^{r_1} \oplus \prod_{\substack{v \in S \\ \text{finite}}} (\mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0. \quad (5.6)$$

Example 5. Let $F = \mathbb{Q}(\zeta_{\ell^v})$ and $R = \mathbb{Z}[\zeta_{\ell^v}, 1/\ell]$, where ℓ is an odd prime and ζ_{ℓ^v} is a primitive ℓ^v -th root of unity. Then R has one finite place, and $r_1 = 0$, so $\text{Br}(R) = 0$ via (5.6), and $K_2(R)/\ell \cong \text{Pic}(R)/\ell$. Hence the finite groups $K_2(\mathbb{Z}[\zeta_{\ell^v}, 1/\ell])$ and $K_2(\mathbb{Z}[\zeta_{\ell^v}])$ have ℓ -torsion if and only if ℓ is an irregular prime.

For the groups $K_n(\mathcal{O}_F)$, $n > 2$, different techniques come into play. Homological techniques were used by Quillen in [48] and Borel in [7] to prove the following result. Let r_1 (resp., r_2) denote the number of real (resp., complex) embeddings of F ; the resulting decomposition of $F \otimes_{\mathbb{Q}} \mathbb{R}$ shows that $[F : \mathbb{Q}] = r_1 + 2r_2$.

Theorem 6: (*Quillen–Borel*) Let F be a number field. Then the abelian groups $K_n(\mathcal{O}_F)$ are all finitely generated, and their ranks are given by the formula:

$$\text{rank } K_n(\mathcal{O}_F) = \begin{cases} r_1 + r_2, & \text{if } n \equiv 1 \pmod{4}; \\ r_2, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

In particular, if $n > 0$ is even then $K_n(\mathcal{O}_F)$ is a finite group. If $n = 2i - 1$, the rank of $K_n(\mathcal{O}_F)$ is the order of vanishing of the function ζ_F at $1 - i$.

There is a localization sequence relating the K -theory of $\mathcal{O}_F$, F and the finite fields $\mathcal{O}_F/\mathfrak{p}$; Soulé showed that the maps $K_n(\mathcal{O}_F) \rightarrow K_n(F)$ are injections. This proves the following result.

7

Theorem 7 Let F be a number field.

- a) If $n > 1$ is odd then $K_n(\mathcal{O}_F) \cong K_n(F)$.
- b) If $n > 1$ is even then $K_n(\mathcal{O}_F)$ is finite but $K_n(F)$ is an infinite torsion group fitting into the exact sequence

$$0 \rightarrow K_n(\mathcal{O}_F) \rightarrow K_n(F) \rightarrow \bigoplus_{\wp \subset \mathcal{O}_F} K_{n-1}(\mathcal{O}_F/\wp) \rightarrow 0.$$

For example, the groups $K_3(\mathcal{O}_F)$ and $K_3(F)$ are isomorphic, and hence the direct sum of $\mathbb{Z}^{r_2}$ and a finite group. The Milnor K -group $K_3^M(F)$ is isomorphic to $(\mathbb{Z}/2)^{r_1}$ by [4], and injects into $K_3(F)$ by [38].

The following theorem was proven by Merkurjev and Suslin in [38]. Recall that F is said to be *totally imaginary* if it cannot be embedded into $\mathbb{R}$, i.e., if $r_1 = 0$ and $r_2 = [F : \mathbb{Q}]/2$. The positive integer $w_2(F)$ is defined in Sect. 5.3 below, and is always divisible by 24.

8

Theorem 8: (*Structure of $K_3 F$.*) Let F be a number field, and set $w = w_2(F)$.

- a) If F is totally imaginary, then $K_3(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w$;
- b) If F has a real embedding then $K_3^M(F) \cong (\mathbb{Z}/2)^{r_1}$ is a subgroup of $K_3(F)$ and:

$$K_3(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/(2w) \oplus (\mathbb{Z}/2)^{r_1-1}.$$

Example 9.

- a) When $F = \mathbb{Q}$ we have $K_3(\mathbb{Z}) = K_3(\mathbb{Q}) \cong \mathbb{Z}/48$, because $w_2(F) = 24$. This group was first calculated by Lee and Szcza.
- b) When $F = \mathbb{Q}(i)$ we have $w_2(F) = 24$ and $K_3(\mathbb{Q}(i)) \cong \mathbb{Z} \oplus \mathbb{Z}/24$.
- c) When $F = \mathbb{Q}(\sqrt{\pm 2})$ we have $w_2(F) = 48$ because $F(i) = \mathbb{Q}(\zeta_8)$. For these two fields, $K_3(\mathbb{Q}(\sqrt{2})) \cong \mathbb{Z}/96 \oplus \mathbb{Z}/2$, while $K_3(\mathbb{Q}(\sqrt{-2})) \cong \mathbb{Z} \oplus \mathbb{Z}/48$.

Classical techniques have not been able to proceed much beyond this. Although Bass and Tate showed that the Milnor K -groups $K_n^M(F)$ are $(\mathbb{Z}/2)^{r_1}$ for all $n \geq 3$, and hence nonzero for every real number field (one embeddable in $\mathbb{R}$, so that $r_1 \neq 0$), we have the following discouraging result.

10

Lemma 10 Let F be a real number field. The map $K_4^M(F) \rightarrow K_4(F)$ is not injective, and the map $K_n^M(F) \rightarrow K_n(F)$ is zero for $n \geq 5$.

Proof The map $\pi_1^s \rightarrow K_1(\mathbb{Z})$ sends η to $[-1]$. Since $\pi_*^s \rightarrow K_*(\mathbb{Z})$ is a ring homomorphism and $\eta^4 = 0$, the Steinberg symbol $\{-1, -1, -1, -1\}$ must be zero in $K_4(\mathbb{Z})$. But the corresponding Milnor symbol is nonzero in $K_4^M(F)$, because it

is nonzero in $K_4^M(\mathbb{R})$. This proves the first assertion. Bass and Tate prove [4] that $K_n^M(F)$ is in the ideal generated by $\{-1, -1, -1, -1\}$ for all $n \geq 5$, which gives the second assertion.

Remark 11 Around the turn of the century, homological calculations by Rognes [53] and Elbaz-Vincent/Gangl/Soulé [15] proved that $K_4(\mathbb{Z}) = 0$, $K_5(\mathbb{Z}) = \mathbb{Z}$, and that $K_6(\mathbb{Z})$ has at most 3-torsion. These follow from a refinement of the calculations by Lee-Szczarba and Soulé in [57] that there is no p -torsion in $K_4(\mathbb{Z})$ or $K_5(\mathbb{Z})$ for $p > 3$, together with the calculation in [51] that there is no 2-torsion in $K_4(\mathbb{Z})$, $K_5(\mathbb{Z})$ or $K_6(\mathbb{Z})$.

The results of Rost and Voevodsky imply that $K_7(\mathbb{Z}) \cong \mathbb{Z}/240$ (see [76]). It is still an open question whether or not $K_8(\mathbb{Z}) = 0$.

11

The e -Invariant

5.3

The odd-indexed K -groups of a field F have a canonical torsion summand, discovered by Harris and Segal in [23]. It is detected by a map called the e -invariant, which we now define.

Let F be a field, with separable closure $\bar{F}$ and Galois group $G = \text{Gal}(\bar{F}/F)$. The abelian group μ of all roots of unity in $\bar{F}$ is a G -module. For all i , we shall write $\mu(i)$ for the abelian group μ , made into a G -module by letting $g \in G$ act as $\zeta \mapsto g^i(\zeta)$. (This modified G -module structure is called the i -th Tate twist of the usual structure.) Note that the abelian group underlying $\mu(i)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$ if $\text{char}(F) = 0$ and $\mathbb{Q}/\mathbb{Z}[1/p]$ if $\text{char}(F) = p \neq 0$. For each prime $\ell \neq \text{char}(F)$, we write $\mathbb{Z}/\ell^\infty(i)$ for the ℓ -primary G -submodule of $\mu(i)$, so that $\mu(i) = \bigoplus \mathbb{Z}/\ell^\infty(i)$.

For each odd $n = 2i - 1$, Suslin proved [60, 62] that the torsion subgroup of $K_{2i-1}(\bar{F})$ is naturally isomorphic to $\mu(i)$. It follows that there is a natural map

$$e: K_{2i-1}(F)_{\text{tors}} \rightarrow K_{2i-1}(\bar{F})_{\text{tors}}^G \cong \mu(i)^G. \quad (5.7)$$

If $\mu(i)^G$ is a finite group, write $w_i(F)$ for its order, so that $\mu(i)^G \cong \mathbb{Z}/w_i(F)$. This is the case for all local and global fields (by 14 below). We shall call e the e -invariant, since the composition $\pi_{2i-1}^s \rightarrow K_{2i-1}(\mathbb{Q}) \xrightarrow{e} \mathbb{Z}/w_i(\mathbb{Q})$ is Adams' complex e -invariant by [50].

The target group $\mu(i)^G$ is always the direct sum of its ℓ -primary Sylow subgroups $\mathbb{Z}/\ell^\infty(i)^G$. The orders of these subgroups are determined by the roots of unity in the cyclotomic extensions $F(\mu_{\ell^v})$. Here is the relevant definition.

Definition 12 Fix a prime ℓ . For any field F , define integers $w_i^{(\ell)}(F)$ by

12

$$w_i^{(\ell)}(F) = \max \{ \ell^v \mid \text{Gal}(F(\mu_{\ell^v})/F) \text{ has exponent dividing } i \}$$

for each integer i . If there is no maximum v we set $w_i^{(\ell)}(F) = \ell^\infty$.

13 Lemma 13 Let F be a field and set $G = \text{Gal}(\overline{F}/F)$. Then $\mathbb{Z}/\ell^\infty(i)^G$ is isomorphic to $\mathbb{Z}/w_i^{(\ell)}(F)$. Thus the target of the e -invariant is $\bigoplus_\ell \mathbb{Z}/w_i^{(\ell)}(F)$.

Suppose in addition that $w_i^{(\ell)}(F)$ is 1 for almost all ℓ , and is finite otherwise. Then the target of the e -invariant is $\mathbb{Z}/w_i(F)$, where $w_i(F) = \prod w_i^{(\ell)}(F)$.

Proof Let ζ be a primitive ℓ^v -th root of unity. Then $\zeta^{\otimes i}$ is invariant under $g \in G$ (the absolute Galois group) precisely when $g^i(\zeta) = \zeta$, and $\zeta^{\otimes i}$ is invariant under all of G precisely when the group $\text{Gal}(F(\mu_{\ell^v})/F)$ has exponent i .

14 Corollary 14 Suppose that $F(\mu_\ell)$ has only finitely many ℓ -primary roots of unity for all primes ℓ , and that $[F(\mu_\ell) : F]$ approaches ∞ as ℓ approaches ∞ . Then the $w_i(F)$ are finite for all $i \neq 0$.

This is the case for all local and global fields.

Proof For fixed $i \neq 0$, the formulas in 20 and 22 below show that each $w_i^{(\ell)}$ is finite, and equals one except when $[F(\mu_\ell) : F]$ divides i . By assumption, this exception happens for only finitely many ℓ . Hence $w_i(F)$ is finite.

Example 15. (*Finite fields.*) Consider a finite field $\mathbb{F}_q$. It is a pleasant exercise to show that $w_i(\mathbb{F}_q) = q^i - 1$ for all i . Quillen computed the K -theory of $\mathbb{F}_q$ in [47], showing that $K_{2i}(\mathbb{F}_q) = 0$ for $i > 0$ and that $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/w_i(\mathbb{F}_q)$. In this case, the e -invariant is an isomorphism.

The key part of the following theorem, *i.e.*, the existence of a $\mathbb{Z}/w_i$ summand, was discovered in the 1975 paper [23] by Harris and Segal; the splitting map was constructed in an ad hoc manner for number fields (see 18 below). The canonical nature of the splitting map was only established much later, in [11, 21, 28].

The summand does not always exist when $\ell = 2$; for example $K_5(\mathbb{Z}) = \mathbb{Z}$ but $w_3(\mathbb{Q}) = 2$. The Harris–Segal construction fails when the Galois groups of cyclotomic field extensions are not cyclic. With this in mind, we call a field F *non-exceptional* if the Galois groups $\text{Gal}(F(\mu_{2^v})/F)$ are cyclic for every v , and *exceptional* otherwise. There are no exceptional fields of finite characteristic. Both $\mathbb{R}$ and $\mathbb{Q}_2$ are exceptional, and so are each of their subfields. In particular, real number fields (like $\mathbb{Q}$) are exceptional, and so are some totally imaginary number fields, like $\mathbb{Q}(\sqrt{-7})$.

16 Theorem 16 Let R be an integrally closed domain containing $1/\ell$, and set $w_i = w_i^{(\ell)}(R)$. If $\ell = 2$, we suppose that R is non-exceptional. Then each $K_{2i-1}(R)$ has a canonical direct summand isomorphic to $\mathbb{Z}/w_i$, detected by the e -invariant.

The splitting $\mathbb{Z}/w_i \rightarrow K_{2i-1}(R)$ is called the *Harris–Segal map*, and its image is called the *Harris–Segal summand* of $K_{2i-1}(R)$.

Example 17. If R contains a primitive ℓ^v -th root of unity ζ , we can give a simple description of the subgroup $\mathbb{Z}/\ell^v$ of the Harris–Segal summand. In this case, $H_{\text{ét}}^0(R, \mu_{\ell^v}^{\otimes i}) \cong \mu_{\ell^v}^{\otimes i}$ is isomorphic to $\mathbb{Z}/\ell^v$, on generator $\zeta \otimes \cdots \otimes \zeta$. If $\beta \in K_2(R; \mathbb{Z}/\ell^v)$ is the *Bott element* corresponding to ζ , the *Bott map* $\mathbb{Z}/\ell^v \rightarrow K_{2i}(R; \mathbb{Z}/\ell^v)$ sends 1 to β^i . (This multiplication is defined unless $\ell^v = 2^1$.) The Harris–Segal map, restricted to $\mathbb{Z}/\ell^v \subseteq \mathbb{Z}/m$, is just the composition

$$\mu_{\ell^v}^{\otimes i} \cong \mathbb{Z}/\ell^v \xrightarrow{\text{Bott}} K_{2i}(R; \mathbb{Z}/\ell^v) \rightarrow K_{2i-1}(R).$$

Remark 18 Harris and Segal [23] originally constructed the Harris–Segal map by studying the homotopy groups of the space BN^+ , where N is the union of the wreath products $\mu \wr \Sigma_n$, $\mu = \mu_{\ell^v}$. Each wreath product embeds in $GL_n(R[\zeta_{\ell^v}])$ as the group of matrices whose entries are either zero or ℓ^v -th roots of unity, each row and column having at most one nonzero entry. Composing with the transfer, this gives a group map $N \rightarrow GL(R[\zeta_{\ell^v}]) \rightarrow GL(R)$ and hence a topological map $BN^+ \rightarrow GL(R)^+$.

From a topological point of view, BN^+ is the zeroth space of the spectrum $\Sigma^\infty(B\mu_+)$, and is also the K -theory space of the symmetric monoidal category of finite free μ -sets. The map of spectra underlying $BN^+ \rightarrow GL(R)^+$ is obtained by taking the K -theory of the free R -module functor from finite free μ -sets to free R -modules.

Harris and Segal split this map by choosing a prime p that is primitive mod ℓ , and is a topological generator of $\mathbb{Z}_\ell^\times$. Their argument may be interpreted as saying that if $\mathbb{F}_q = \mathbb{F}_p[\zeta_{\ell^v}]$ then the composite map $\Sigma^\infty(B\mu_+) \rightarrow K(R) \rightarrow K(\mathbb{F}_q)$ is an equivalence after KU -localization.

If F is an exceptional field, a transfer argument using $F(\sqrt{-1})$ shows that there is a cyclic summand in $K_{2i-1}(R)$ whose order is either $w_i(F)$, $2w_i(F)$ or $w_i(F)/2$. If F is a totally imaginary number field, we will see in 73 that the Harris–Segal summand is always $\mathbb{Z}/w_i(F)$. The following theorem, which follows from Theorem 82 below (see [51]), shows that all possibilities occur for real number fields, *i.e.*, number fields embeddable in $\mathbb{R}$.

Theorem 19 Let F be a real number field. Then the Harris–Segal summand in $K_{2i-1}(\mathcal{O}_F)$ is isomorphic to:

1. $\mathbb{Z}/w_i(F)$, if $i \equiv 0 \pmod{4}$ or $i \equiv 1 \pmod{4}$, *i.e.*, $2i - 1 \equiv \pm 1 \pmod{8}$;
2. $\mathbb{Z}/2w_i(F)$, if $i \equiv 2 \pmod{4}$, *i.e.*, $2i - 1 \equiv 3 \pmod{8}$;
3. $\mathbb{Z}/\frac{1}{2}w_i(F)$, if $i \equiv 3 \pmod{4}$, *i.e.*, $2i - 1 \equiv 5 \pmod{8}$.

Here are the formulas for the numbers $w_i^{(\ell)}(F)$, taken from [23, p. 28], and from [74, 6.3] when $\ell = 2$. Let $\log_\ell(n)$ be the maximal power of ℓ dividing n , i.e., the ℓ -adic valuation of n . By convention let $\log_\ell(0) = \infty$.

20

Proposition 20 Fix a prime $\ell \neq 2$, and let F be a field of characteristic $\neq \ell$. Let a be maximal such that $F(\mu_\ell)$ contains a primitive ℓ^a -th root of unity. Then if $r = [F(\mu_\ell) : F]$ and $b = \log_\ell(i)$ the numbers $w_i^{(\ell)} = w_i^{(\ell)}(F)$ are:

- (a) If $\mu_\ell \in F$ then $w_i^{(\ell)} = \ell^{a+b}$;
- (b) If $\mu_\ell \notin F$ and $i \equiv 0 \pmod{r}$ then $w_i^{(\ell)} = \ell^{a+b}$;
- (c) If $\mu_\ell \notin F$ and $i \not\equiv 0 \pmod{r}$ then $w_i^{(\ell)} = 1$.

Proof Since ℓ is odd, $G = \text{Gal}(F(\mu_{\ell^{a+v}})/F)$ is a cyclic group of order $r\ell^v$ for all $v \geq 0$. If a generator of G acts on $\mu_{\ell^{a+v}}$ by $\zeta \mapsto \zeta^g$ for some $g \in (\mathbb{Z}/\ell^{a+v})^\times$ then it acts on $\mu^{\otimes i}$ by $\zeta \mapsto \zeta^{g^i}$.

Example 21. If $F = \mathbb{Q}(\mu_{p^v})$ and $\ell \neq 2, p$ then $w_i^{(\ell)}(F) = w_i^{(\ell)}(\mathbb{Q})$ for all i . This number is 1 unless $(\ell - 1) \mid i$; if $(\ell - 1) \nmid i$ but $\ell \nmid i$ then $w_i^{(\ell)}(F) = \ell$. In particular, if $\ell = 3$ and $p \neq 3$ then $w_i^{(3)}(F) = 1$ for odd i , and $w_i^{(3)}(F) = 3$ exactly when $i \equiv 2, 4 \pmod{6}$. Of course, $p \mid w_i(F)$ for all i .

22

Proposition 22 ($\ell = 2$) Let F be a field of characteristic $\neq 2$. Let a be maximal such that $F(\sqrt{-1})$ contains a primitive 2^a -th root of unity. Let i be any integer, and let $b = \log_2(i)$. Then the 2-primary numbers $w_i^{(2)} = w_i^{(2)}(F)$ are:

- (a) If $\sqrt{-1} \in F$ then $w_i^{(2)} = 2^{a+b}$ for all i .
- (b) If $\sqrt{-1} \notin F$ and i is odd then $w_i^{(2)} = 2$.
- (c) If $\sqrt{-1} \notin F$, F is exceptional and i is even then $w_i^{(2)} = 2^{a+b}$.
- (d) If $\sqrt{-1} \notin F$, F is non-exceptional and i is even then $w_i^{(2)} = 2^{a+b-1}$.

Example 23. (*Local fields.*) If E is a local field, finite over $\mathbb{Q}_p$, then $w_i(E)$ is finite by 14. Suppose that the residue field is $\mathbb{F}_q$. Since (for $\ell \neq p$) the number of ℓ -primary roots of unity in $E(\mu_\ell)$ is the same as in $\mathbb{F}_q(\mu_\ell)$, we see from 20 and 22 that $w_i(E)$ is $w_i(\mathbb{F}_q) = q^i - 1$ times a power of p .

If $p > 2$ the p -adic rational numbers $\mathbb{Q}_p$ have $w_i(\mathbb{Q}_p) = q^i - 1$ unless $(p - 1) \mid i$; if $i = (p - 1)p^b m$ ($p \nmid m$) then $w_i(\mathbb{Q}_p) = (q^i - 1)p^{1+b}$.

For $p = 2$ we have $w_i(\mathbb{Q}_2) = 2(2^i - 1)$ for i odd, because $\mathbb{Q}_2$ is exceptional, and $w_i(\mathbb{Q}_2) = (2^i - 1)2^{2+b}$ for i even, $i = 2^b m$ with m odd.

Example 24. (*Bernoulli numbers.*) The numbers $w_i(\mathbb{Q})$ are related to the *Bernoulli numbers* B_k . These were defined by Jacob Bernoulli in 1713 as coefficients in the power series

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{t^{2k}}{(2k)!}.$$

(We use the topologists' B_k from [41], all of which are positive. Number theorists would write it as $(-1)^{k+1} B_{2k}$.) The first few Bernoulli numbers are:

$$\begin{aligned} B_1 &= \frac{1}{6}, & B_2 &= \frac{1}{30}, & B_3 &= \frac{1}{42}, & B_4 &= \frac{1}{30}, \\ B_5 &= \frac{5}{66}, & B_6 &= \frac{691}{2730}, & B_7 &= \frac{7}{6}, & B_8 &= \frac{3617}{510}. \end{aligned}$$

The denominator of B_k is always squarefree, divisible by 6, and equal to the product of all primes with $(p-1) \mid 2k$. Moreover, if $(p-1) \nmid 2k$ then p is not in the denominator of B_k/k even if $p \mid k$; see [41].

Although the numerator of B_k is difficult to describe, Kummer's congruences show that if p is regular it does not divide the numerator of *any* B_k/k (see [71, 5.14]). Thus only irregular primes can divide the numerator of B_k/k (see 3).

Remark 25 We have already remarked in 3 that if a prime p divides the numerator of some B_k/k then p divides the order of $\text{Pic}(\mathbb{Z}[\mu_p])$. Bernoulli numbers also arise as values of the Riemann zeta function. Euler proved (in 1735) that $\zeta_{\mathbb{Q}}(2k) = B_k(2\pi)^{2k}/2(2k)!$. By the functional equation, we have $\zeta_{\mathbb{Q}}(1-2k) = (-1)^k B_k/2k$. Thus the denominator of $\zeta(1-2k)$ is $\frac{1}{2}w_{2k}(\mathbb{Q})$.

25

Remark 26 The Bernoulli numbers are of interest to topologists because if $n = 4k-1$ the image of $J: \pi_n SO \rightarrow \pi_n^s$ is cyclic of order equal to the denominator of $B_k/4k$, and the numerator determines the number of exotic $(4k-1)$ -spheres that bound parallizable manifolds; (see [41], App. B).

26

From 24, 20 and 22 it is easy to verify the following important result.

Lemma 27 If i is odd then $w_i(\mathbb{Q}) = 2$ and $w_i(\mathbb{Q}(\sqrt{-1})) = 4$. If $i = 2k$ is even then $w_i(\mathbb{Q}) = w_i(\mathbb{Q}(\sqrt{-1}))$, and this integer is the denominator of $B_k/4k$. The prime ℓ divides $w_i(\mathbb{Q})$ exactly when $(\ell-1)$ divides i .

27

Example 28. For $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$, $w_2 = 24$, $w_4 = 240$, $w_6 = 504 = 2^3 \cdot 3^2 \cdot 7$, $w_8 = 480 = 2^5 \cdot 3 \cdot 5$, $w_{10} = 264 = 2^3 \cdot 3 \cdot 11$, and $w_{12} = 65\,520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$.

The w_i are the orders of the Harris–Segal summands of $K_3(\mathbb{Q}[\sqrt{-1}])$, $K_7(\mathbb{Q}[\sqrt{-1}])$, ..., $K_{23}(\mathbb{Q}[\sqrt{-1}])$ by 16. In fact, we will see in 73 that $K_{2i-1}(\mathbb{Q}[\sqrt{-1}]) \cong \mathbb{Z} \oplus \mathbb{Z}/w_i$ for all $i \geq 2$.

By 19, the orders of the Harris–Segal summands of $K_7(\mathbb{Q})$, $K_{15}(\mathbb{Q})$, $K_{23}(\mathbb{Q})$, ... are w_4 , w_8 , w_{12} , etc., and the orders of the Harris–Segal summands of $K_3(\mathbb{Q})$, $K_{11}(\mathbb{Q})$, $K_{19}(\mathbb{Q})$, ... are $2w_2 = 48$, $2w_6 = 1008$, $2w_{10} = 2640$, etc. In fact, these summands are exactly the torsion subgroups of the $K_{2i-1}(\mathbb{Q})$.

Example 29. The image of the natural maps $\pi_n^s \rightarrow K_n(\mathbb{Z})$ capture most of the Harris–Segal summands, and were analyzed by Quillen in [50]. When n is $8k+1$ or $8k+2$, there is a $\mathbb{Z}/2$ -summand in $K_n(\mathbb{Z})$, generated by the image of Adams' element μ_n . (It is the 2-torsion subgroup by [76].) Since $w_{4k+1}(\mathbb{Q}) = 2$, we may view it as the Harris–Segal summand when $n = 8k+1$. When $n = 8k+5$, the Harris–Segal summand is zero by 19. When $n = 8k+7$ the Harris–Segal summand of $K_n(\mathbb{Z})$ is isomorphic to the subgroup $J(\pi_n O) \cong \mathbb{Z}/w_{4k+4}(\mathbb{Q})$ of π_n^s .

When $n = 8k+3$, the subgroup $J(\pi_n O) \cong \mathbb{Z}/w_{4k+2}(\mathbb{Q})$ of π_n^s is contained in the Harris–Segal summand $\mathbb{Z}/(2w_i)$ of $K_n(\mathbb{Z})$; the injectivity was proven by Quillen in [50], and Browder showed that the order of the summand was $2w_i(\mathbb{Q})$.

Not all of the image of J injects into $K_*(\mathbb{Z})$. If $n = 0, 1 \pmod{8}$ then $J(\pi_n O) \cong \mathbb{Z}/2$, but Waldhausen showed (in 1982) that these elements map to zero in $K_n(\mathbb{Z})$.

Example 30. Let $F = \mathbb{Q}(\zeta + \zeta^{-1})$ be the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta)$, $\zeta^p = 1$ with p odd. Then $w_i(F) = 2$ for odd i , and $w_i(F) = w_i(\mathbb{Q}(\zeta))$ for even $i > 0$ by 20 and 22. Note that $p|w_i(F[\zeta])$ for all i , $p|w_i(F)$ if and only if i is even, and $p|w_i(\mathbb{Q})$ only when $(p-1)|i$. If $n \equiv 3 \pmod{4}$, the groups $K_n(\mathbb{Z}[\zeta + \zeta^{-1}]) = K_n(F)$ are finite by 6; the order of their Harris–Segal summands are given by Theorem 19, and have an extra p -primary factor not detected by the image of J when $n \not\equiv -1 \pmod{2p-2}$.

31

Conjecture 31: (*Birch–Tate Conjecture.*) If F is a number field, the zeta function $\zeta_F(s)$ has a pole of order r_2 at $s = -1$. Birch and Tate [64] conjectured that for totally real number fields ($r_2 = 0$) we have

$$\zeta_F(-1) = (-1)^{r_1} |K_2(\mathcal{O}_F)|/w_2(F) .$$

The odd part of this conjecture was proven by Wiles in [77], using Tate's Theorem 4. The two-primary part is still open, but it is known to be a consequence of the 2-adic Main Conjecture of Iwasawa Theory (see Kolster's appendix to [51]), which was proven by Wiles in *loc. cit.* for abelian extensions of $\mathbb{Q}$. Thus the full Birch–Tate Conjecture holds for all abelian extensions of $\mathbb{Q}$. For example, when $F = \mathbb{Q}$ we have $\zeta_{\mathbb{Q}}(-1) = -1/12$, $|K_2(\mathbb{Z})| = 2$ and $w_2(\mathbb{Q}) = 24$.

Étale Chern Classes

5.4

We have seen in (5.4) and 4 that $H_{\text{ét}}^1$ and $H_{\text{ét}}^2$ are related to K_1 and K_2 . In order to relate them to higher K -theory, it is useful to have well-behaved maps. In one direction, we use the étale Chern classes introduced in [58], but in the form found in Dwyer-Friedlander [14].

In this section, we construct the maps in the other direction. Our formulation is due to Kahn [26–28]; they were introduced in [27], where they were called “anti-Chern classes”. Kahn’s maps are an efficient reorganization of the constructions of Soulé [58] and Dwyer-Friedlander [14]. Of course, there are higher Kahn maps, but we do not need them for local or global fields so we omit them here.

If F is a field containing $1/\ell$, there is a canonical map from $K_{2i-1}(F; \mathbb{Z}/\ell^v)$ to $H_{\text{ét}}^1(F, \mu_{\ell^v}^{\otimes i})$, called the *first étale Chern class*. It is the composition of the map to the étale K -group $K_{2i-1}^{\text{ét}}(F; \mathbb{Z}/\ell^v)$ followed by the edge map in the Atiyah-Hirzebruch spectral sequence for étale K -theory [14]. For $i = 1$ it is the Kummer isomorphism from $K_1(F; \mathbb{Z}/\ell^v) = F^\times / F^{\times \ell^v}$ to $H_{\text{ét}}^1(F, \mu_{\ell^v})$.

For each i and v , we can construct a splitting of the first étale Chern class, at least if ℓ is odd (or $\ell = 2$ and F is non-exceptional). Let F_v denote the smallest field extension of F over which the Galois module $\mu_{\ell^v}^{\otimes i-1}$ is trivial, and let Γ_v denote the Galois group of F_v over F . Kahn proved in [26] that the transfer map induces an isomorphism $H_{\text{ét}}^1(F_v, \mu_{\ell^v}^{\otimes i})_{\Gamma_v} \xrightarrow{\cong} H_{\text{ét}}^1(F, \mu_{\ell^v}^{\otimes i})$. Note that because $H_{\text{ét}}^1(F_v, \mu_{\ell^v}) \cong F_v^\times / \ell^v$ we have an isomorphism of Γ_v -modules $H_{\text{ét}}^1(F_v, \mu_{\ell^v}^{\otimes i}) \cong (F_v^\times) \otimes \mu_{\ell^v}^{\otimes i-1}$.

Definition 32 The *Kahn map* $H_{\text{ét}}^1(F, \mu_{\ell^v}^{\otimes i}) \rightarrow K_{2i-1}(F; \mathbb{Z}/\ell^v)$ is the composition

32

$$\begin{aligned} H_{\text{ét}}^1(F, \mu_{\ell^v}^{\otimes i}) &\xleftarrow{\cong} H_{\text{ét}}^1(F_v, \mu_{\ell^v}^{\otimes i})_{\Gamma_v} = \left[F_v^\times \otimes \mu_{\ell^v}^{\otimes i-1} \right]_{\Gamma_v} \xrightarrow{\text{Harris-Segal}} \\ &\left[(F_v^\times) \otimes K_{2i-2}(F_v; \mathbb{Z}/\ell^v) \right]_{\Gamma_v} \xrightarrow{\cup} K_{2i-1}(F_v; \mathbb{Z}/\ell^v)_{\Gamma_v} \xrightarrow{\text{transfer}} K_{2i-1}(F; \mathbb{Z}/\ell^v). \end{aligned}$$

Compatibility 33 Let F be the quotient field of a discrete valuation ring whose residue field k contains $1/\ell$. Then the Kahn map is compatible with the Harris–Segal map in the sense that for $m = \ell^v$ the diagram commutes.

33

$$\begin{array}{ccc} H_{\text{ét}}^1(F, \mu_m^{\otimes i}) & \xrightarrow{\partial} & H_{\text{ét}}^0(k, \mu_m^{\otimes i-1}) \\ \text{Kahn} \downarrow & & \downarrow \text{Harris-Segal} \\ K_{2i-1}(F; \mathbb{Z}/m) & \xrightarrow{\partial} & K_{2i-2}(k; \mathbb{Z}/m). \end{array}$$

To see this, one immediately reduces to the case $F = F_v$. In this case, the Kahn map is the Harris–Segal map, tensored with the identification $H_{\text{ét}}^1(F, \mu_m) \cong F^\times/m$, and both maps ∂ amount to the reduction mod m of the valuation map $F^\times \rightarrow \mathbb{Z}$.

34 Theorem 34 Let F be a field containing $1/\ell$. If $\ell = 2$ we suppose that F is non-exceptional. Then for each i the Kahn map $H_{\text{ét}}^1(F, \mu_{\ell^v}^{\otimes i}) \rightarrow K_{2i-1}(F; \mathbb{Z}/\ell^v)$ is an injection, split by the first étale Chern class.

The Kahn map is compatible with change of coefficients. Hence it induces maps $H_{\text{ét}}^1(F, \mathbb{Z}_\ell(i)) \rightarrow K_{2i-1}(F; \mathbb{Z}_\ell)$ and $H_{\text{ét}}^1(F, \mathbb{Z}/\ell^\infty(i)) \rightarrow K_{2i-1}(F; \mathbb{Z}/\ell^\infty)$.

Proof When ℓ is odd (or $\ell = 2$ and $\sqrt{-1} \in F$), the proof that the Kahn map splits the étale Chern class is given in [27], and is essentially a reorganization of Soule’s proof in [58] that the first étale Chern class is a surjection up to factorials (cf. [14]). When $\ell = 2$ and F is non-exceptional, Kahn proves in [28] that this map is a split injection.

35 Corollary 35 Let $\mathcal{O}_S$ be a ring of S -integers in a number field F , with $1/\ell \in \mathcal{O}_S$. If $\ell = 2$, assume that F is non-exceptional. Then the Kahn maps for F induce injections $H_{\text{ét}}^1(\mathcal{O}_S, \mu_{\ell^v}^{\otimes i}) \rightarrow K_{2i-1}(\mathcal{O}_S; \mathbb{Z}/\ell^v)$, split by the first étale Chern class.

Proof Since $H_{\text{ét}}^1(\mathcal{O}_S, \mu_{\ell^v}^{\otimes i})$ is the kernel of $H_{\text{ét}}^1(F, \mu_{\ell^v}^{\otimes i}) \rightarrow \bigoplus_{\wp} H_{\text{ét}}^0(k(\wp), \mu_m^{\otimes i-1})$, and $K_{2i-1}(\mathcal{O}_S; \mathbb{Z}/\ell^v)$ is the kernel of $K_{2i-1}(F; \mathbb{Z}/\ell^v) \rightarrow \bigoplus_{\wp} K_{2i-2}(k(\wp); \mathbb{Z}/\ell^v)$ by 7, this follows formally from 33.

Example 36. If F is a number field, the first étale Chern class detects the torsion-free part of $K_{2i-1}(\mathcal{O}_F) = K_{2i-1}(F)$ described in 6. In fact, it induces isomorphisms $K_{2i-1}(\mathcal{O}_S) \otimes \mathbb{Q}_\ell \cong K_{2i-1}^{\text{ét}}(\mathcal{O}_S; \mathbb{Q}_\ell) \cong H_{\text{ét}}^1(\mathcal{O}_S, \mathbb{Q}_\ell(i))$.

To see this, choose S to contain all places over some odd prime ℓ . Then $1/\ell \in \mathcal{O}_S$, and $K_{2i-1}(\mathcal{O}_S) \cong K_{2i-1}(F)$. A theorem of Tate states that

$$\text{rank } H_{\text{ét}}^1(\mathcal{O}_S, \mathbb{Q}_\ell(i)) - \text{rank } H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Q}_\ell(i)) = \begin{cases} r_2, & i \text{ even} ; \\ r_1 + r_2, & i \text{ odd} . \end{cases}$$

We will see in 42 below that $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Q}_\ell(i)) = 0$. Comparing with 6, we see that the source and target of the first étale Chern class

$$K_{2i-1}(\mathcal{O}_S) \otimes \mathbb{Z}_\ell \rightarrow K_{2i-1}^{\text{ét}}(\mathcal{O}_S; \mathbb{Z}_\ell) \cong H_{\text{ét}}^1(\mathcal{O}_S, \mathbb{Z}_\ell(i))$$

have the same rank. By 34, this map is a split surjection (split by the Kahn map), whence the claim.

The second étale Chern class is constructed in a similar fashion. Assuming that ℓ is odd, or that $\ell = 2$ and F is non-exceptional, so that the e -invariant splits by 16, then for $i \geq 1$ there is also a canonical map

$$K_{2i}(F; \mathbb{Z}/\ell^\nu) \rightarrow H_{\text{ét}}^2(F, \mu_{\ell^\nu}^{\otimes i+1}),$$

called the *second étale Chern class*. It is the composition of the map to the étale K -group $K_{2i}^{\text{ét}}(F; \mathbb{Z}/\ell^\nu)$, or rather to the kernel of the edge map $K_{2i}^{\text{ét}}(F; \mathbb{Z}/\ell^\nu) \rightarrow H_{\text{ét}}^0(F, \mu_{\ell^\nu}^{\otimes i})$, followed by the secondary edge map in the Atiyah-Hirzebruch spectral sequence for étale K -theory [14].

Even if $\ell = 2$ and F is exceptional, this composition will define a family of second étale Chern classes $K_{2i}(F) \rightarrow H_{\text{ét}}^2(F, \mu_{\ell^\nu}^{\otimes i+1})$ and hence $K_{2i}(F) \rightarrow H_{\text{ét}}^2(F, \mathbb{Z}_\ell(i+1))$. This is because the e -invariant (5.7) factors through the map $K_{2i}(F; \mathbb{Z}/\ell^\nu) \rightarrow K_{2i-1}(F)$.

For $i = 1$, the second étale Chern class $K_2(F)/m \rightarrow H_{\text{ét}}^2(F, \mu_m^{\otimes 2})$ is just Tate's map, described in 4; it is an isomorphism for all F by the Merkurjev-Suslin theorem.

Using this case, Kahn proved in [26] that the transfer always induces an isomorphism $H_{\text{ét}}^2(F_v, \mu_{\ell^\nu}^{\otimes i})_{\Gamma_v} \xrightarrow{\cong} H_{\text{ét}}^2(F, \mu_{\ell^\nu}^{\otimes i})$. Here F_v and $\Gamma_v = \text{Gal}(F_v/F)$ are as in 32 above, and if $\ell = 2$ we assume that F is non-exceptional. As before, we have an isomorphism of Γ_v -modules $H_{\text{ét}}^2(F_v, \mu_{\ell^\nu}^{\otimes i+1}) \cong K_2(F_v) \otimes \mu_{\ell^\nu}^{\otimes i-1}$.

Definition 37 The Kahn map $H_{\text{ét}}^2(F, \mu_{\ell^\nu}^{\otimes i+1}) \rightarrow K_{2i}(F; \mathbb{Z}/\ell^\nu)$ is the composition

37

$$\begin{aligned} H_{\text{ét}}^2(F, \mu_{\ell^\nu}^{\otimes i+1}) &\xrightarrow{\cong} H_{\text{ét}}^2(F_v, \mu_{\ell^\nu}^{\otimes i+1})_{\Gamma_v} = \left[K_2(F_v) \otimes \mu_{\ell^\nu}^{\otimes i-1} \right]_{\Gamma_v} \xrightarrow{\text{Harris-Segal}} \\ &\left[K_2(F_v) \otimes K_{2i-2}(F_v; \mathbb{Z}/\ell^\nu) \right]_{\Gamma_v} \xrightarrow{\cup} K_{2i}(F_v; \mathbb{Z}/\ell^\nu)_{\Gamma_v} \xrightarrow{\text{transfer}} K_{2i}(F; \mathbb{Z}/\ell^\nu). \end{aligned}$$

Compatibility 38 Let F be the quotient field of a discrete valuation ring whose residue field k contains $1/\ell$. Then the first and second Kahn maps are compatible with the maps ∂ , from $H_{\text{ét}}^2(F)$ to $H_{\text{ét}}^1(k)$ and from $K_{2i}(F; \mathbb{Z}/m)$ to $K_{2i-1}(k; \mathbb{Z}/m)$. The argument here is the same as for 33.

38

As with 34, the following theorem was proven in [27, 28].

Theorem 39 Let F be a field containing $1/\ell$. If $\ell = 2$ we suppose that F is non-exceptional. Then for each $i \geq 1$ the Kahn map $H_{\text{ét}}^2(F, \mu_{\ell^\nu}^{\otimes i+1}) \rightarrow K_{2i}(F; \mathbb{Z}/\ell^\nu)$ is an injection, split by the second étale Chern class.

39

The Kahn map is compatible with change of coefficients. Hence it induces maps $H_{\text{ét}}^2(F, \mathbb{Z}_\ell(i+1)) \rightarrow K_{2i}(F; \mathbb{Z}_\ell)$ and $H_{\text{ét}}^2(F, \mathbb{Z}/\ell^\infty(i+1)) \rightarrow K_{2i}(F; \mathbb{Z}/\ell^\infty)$.

40 Corollary 40 Let $\mathcal{O}_S$ be a ring of S -integers in a number field F , with $1/\ell \in \mathcal{O}_S$. If $\ell = 2$, assume that F is non-exceptional. Then for each $i > 0$, the Kahn maps induce injections $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}_\ell(i+1)) \rightarrow K_{2i}(\mathcal{O}_S; \mathbb{Z}_\ell)$, split by the second étale Chern class.

Proof Since $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}_\ell(i+1))$ is the kernel of

$$H_{\text{ét}}^2(F, \mathbb{Z}_\ell(i+1)) \rightarrow \oplus_{\wp} H_{\text{ét}}^1(k(\wp), \mathbb{Z}_\ell(i)) ,$$

and $K_{2i}(\mathcal{O}_S; \mathbb{Z}_\ell)$ is the kernel of $K_{2i}(F; \mathbb{Z}_\ell) \rightarrow \oplus_{\wp} K_{2i-1}(k(\wp); \mathbb{Z}_\ell)$, this follows formally from 38.

41 Remark 41 For each v , $H_{\text{ét}}^2(\mathcal{O}_S, \mu_{\ell^v}^{\otimes i+1}) \rightarrow K_{2i}(\mathcal{O}_S; \mathbb{Z}/\ell^v)$ is also a split surjection, essentially because the map $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}_\ell(i+1)) \rightarrow H_{\text{ét}}^2(\mathcal{O}_S, \mu_{\ell^v}^{\otimes i+1})$ is onto; see [27, 5.2].

The summand $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}_\ell(i))$ is finite by the following calculation.

42 Proposition 42 Let $\mathcal{O}_S$ be a ring of S -integers in a number field F with $1/\ell \in \mathcal{O}_S$. Then for all $i \geq 2$, $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}_\ell(i))$ is a finite group, and $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Q}_\ell(i)) = 0$. Finally, $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}/\ell^\infty(i)) = 0$ if ℓ is odd, or if $\ell = 2$ and F is totally imaginary.

Proof If ℓ is odd or if $\ell = 2$ and F is totally imaginary, then $H_{\text{ét}}^3(\mathcal{O}_S, \mathbb{Z}_\ell(i)) = 0$ by Serre [55], so $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}/\ell^\infty(i))$ is a quotient of $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Q}_\ell(i))$. Since $H_{\text{ét}}^2(R, \mathbb{Q}_\ell(i)) = H_{\text{ét}}^2(R, \mathbb{Z}_\ell(i)) \otimes \mathbb{Q}$, it suffices to prove the first assertion for $i > 0$. But $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}_\ell(i))$ is a summand of $K_{2i-2}(\mathcal{O}_S) \otimes \mathbb{Z}_\ell$ for $i \geq 2$ by 40, which is a finite group by Theorem 6.

If $\ell = 2$ and F is exceptional, the usual transfer argument for $\mathcal{O}_S \subset \mathcal{O}_{S'} \subset F(\sqrt{-1})$ shows that the kernel A of $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}_2(i)) \rightarrow H_{\text{ét}}^2(\mathcal{O}_{S'}, \mathbb{Z}_2(i))$ has exponent 2. Since A must inject into the finite group $H_{\text{ét}}^2(\mathcal{O}_S, \mu_2)$, A must also be finite. Hence $H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}_2(i))$ is also finite, and $H_{\text{ét}}^2(R, \mathbb{Q}_\ell(i)) = H_{\text{ét}}^2(R, \mathbb{Z}_\ell(i)) \otimes \mathbb{Q} = 0$.

Taking the direct limit over all finite S yields:

43 Corollary 43 Let F be a number field. Then $H_{\text{ét}}^2(F, \mathbb{Z}/\ell^\infty(i)) = 0$ for all odd primes ℓ and all $i \geq 2$.

Example 44. The Main Conjecture of Iwasawa Theory, proved by Mazur and Wiles [36], implies that (for odd ℓ) the order of the finite group $H_{\text{ét}}^2(\mathbb{Z}[1/\ell], \mathbb{Z}_\ell(2k))$ is the ℓ -primary part of the numerator of $\zeta_{\mathbb{Q}}(1 - 2k)$. See [51, Appendix A] or [29, 4.2 and 6.3], for example. Note that by Euler's formula 25 this is also the ℓ -primary part of the numerator of $B_k/2k$, where B_k is the Bernoulli number discussed in 24.

Remark 45: (*Real number fields.*) If $\ell = 2$, the vanishing conclusion of corollary 43 still holds when F is totally imaginary. However, it fails when F has $r_1 > 0$ embeddings into $\mathbb{R}$:

45

$$H^2(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \cong H^2(F; \mathbb{Z}/2^\infty(i)) \cong \begin{cases} (\mathbb{Z}/2)^{r_1}, & i \geq 3 \text{ odd} \\ 0, & i \geq 2 \text{ even} . \end{cases}$$

One way to do this computation is to observe that, by 42, $H^2(\mathcal{O}_S; \mathbb{Z}/2^\infty(i))$ has exponent 2. Hence the Kummer sequence is:

$$0 \rightarrow H^2(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \rightarrow H^3(\mathcal{O}_S; \mathbb{Z}/2) \rightarrow H^3(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \rightarrow 0 .$$

Now plug in the values of the right two groups, which are known by Tate-Poitou duality: $H^3(\mathcal{O}_S; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{r_1}$, while $H^3(\mathcal{O}_S; \mathbb{Z}/2^\infty(i))$ is: $(\mathbb{Z}/2)^{r_1}$ for i even, and 0 for i odd.

Remark 46 Suppose that F is totally real ($r_2 = 0$), and set $w_i = w_i^{(\ell)}(F)$. If $i > 0$ is even then $H^1(\mathcal{O}_S, \mathbb{Z}_\ell(i)) \cong \mathbb{Z}/w_i$; this group is finite. If i is odd then $H^1(\mathcal{O}_S, \mathbb{Z}_\ell(i)) \cong \mathbb{Z}_\ell^{r_1} \oplus \mathbb{Z}/w_i$; this is infinite. These facts may be obtained by combining the rank calculations of 36 and 42 with (5.7) and universal coefficients.

46

Theorem 47 For every number field F , and all i , the Adams operation ψ^k acts on $K_{2i-1}(F) \otimes \mathbb{Q}$ as multiplication by k^i .

47

Proof The case $i = 1$ is well known, so we assume that $i \geq 2$. If S contains all places over some odd prime ℓ we saw in 36 that $K_{2i-1}(\mathcal{O}_S) \otimes \mathbb{Q}_\ell \cong K_{2i-1}^{\text{ét}}(\mathcal{O}_S; \mathbb{Q}_\ell) \cong H_{\text{ét}}^1(\mathcal{O}_S, \mathbb{Q}_\ell(i))$. Since this isomorphism commutes with the Adams operations, and Soulé has shown in [59] the $\psi^k = k^i$ on $H_{\text{ét}}^1(\mathcal{O}_S, \mathbb{Q}_\ell(i))$, the same must be true on $K_{2i-1}(\mathcal{O}_S) \otimes \mathbb{Q}_\ell = K_{2i-1}(F) \otimes \mathbb{Q}_\ell$.

Global Fields of Finite Characteristic

A global field of finite characteristic p is a finitely generated field F of transcendence degree one over $\mathbb{F}_p$; the algebraic closure of $\mathbb{F}_p$ in F is a finite field $\mathbb{F}_q$ of characteristic p . It is classical (see [24], I.6) that there is a unique smooth projective curve X over $\mathbb{F}_q$ whose function field is F . If S is a nonempty set of closed points of X , then $X - S$ is affine; we call the coordinate ring R of $X - S$ the ring of S -integers in F . In this section, we discuss the K -theory of F , of X and of the rings of S -integers of F .

The group $K_0(X) = \mathbb{Z} \oplus \text{Pic}(X)$ is finitely generated of rank two, by a theorem of Weil. In fact, there is a finite group $J(X)$ such that $\text{Pic}(X) \cong \mathbb{Z} \oplus J(X)$. For $K_1(X)$ and $K_2(X)$, the localization sequence of Quillen [49] implies that there is an exact sequence

$$0 \rightarrow K_2(X) \rightarrow K_2(F) \xrightarrow{\partial} \bigoplus_{x \in X} k(x)^\times \rightarrow K_1(X) \rightarrow \mathbb{F}_q^\times \rightarrow 0.$$

By classical Weil reciprocity, the cokernel of ∂ is $\mathbb{F}_q^\times$, so $K_1(X) \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. Bass and Tate proved in [4] that the kernel $K_2(X)$ of ∂ is finite of order prime to p . This establishes the low dimensional cases of the following theorem, first proven by Harder [22], using the method pioneered by Borel [7].

48

Theorem 48 Let X be a smooth projective curve over a finite field of characteristic p . For $n \geq 1$, the group $K_n(X)$ is finite of order prime to p .

Proof Tate proved that $K_n^M(F) = 0$ for all $n \geq 3$. By Geisser and Levine's theorem [19], the Quillen groups $K_n(F)$ are uniquely p -divisible for $n \geq 3$. For every closed point $x \in X$, the groups $K_n(x)$ are finite of order prime to p ($n > 0$) because $k(x)$ is a finite field extension of $\mathbb{F}_q$. From the localization sequence

$$\bigoplus_{x \in X} K_n(x) \rightarrow K_n(X) \rightarrow K_n(F) \rightarrow \bigoplus_{x \in X} K_{n-1}(x)$$

and a diagram chase, it follows that $K_n(X)$ is uniquely p -divisible. Now Quillen proved in [20] that the groups $K_n(X)$ are finitely generated abelian groups. A second diagram chase shows that the groups $K_n(X)$ must be finite.

49

Corollary 49 If R is the ring of S -integers in $F = \mathbb{F}_q(X)$ (and $S \neq \emptyset$) then:

- a) $K_1(R) \cong R^\times \cong \mathbb{F}_q^\times \times \mathbb{Z}^s$, $|S| = s + 1$;
- b) For $n \geq 2$, $K_n(R)$ is a finite group of order prime to p .

Proof Classically, $K_1(R) = R^\times \oplus SK_1(R)$ and the units of R are well known. The computation that $SK_1(R) = 0$ is proven in [3]. The rest follows from the localization sequence $K_n(X) \rightarrow K_n(X') \rightarrow \bigoplus_{x \in S} K_{n-1}(x)$.

Example 50. (*The e -invariant.*) The targets of the e -invariant of X and F are the same groups as for $\mathbb{F}_q$, because every root of unity is algebraic over $\mathbb{F}_q$. Hence the inclusions of $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)$ in $K_{2i-1}(X)$ and $K_{2i-1}(F)$ are split by the e -invariant, and this group is the Harris–Segal summand.

The inverse limit of the finite curves $X_v = X \times \text{Spec}(\mathbb{F}_{q^v})$ is the curve $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ over the algebraic closure $\bar{\mathbb{F}}_q$. To understand $K_n(\bar{X})$ for $n > 1$, it is useful to know not only what the groups $K_n(\bar{X})$ are, but how the (geometric) Frobenius $\varphi: x \mapsto x^q$ acts on them.

Classically, $K_0(\bar{X}) = \mathbb{Z} \oplus \mathbb{Z} \oplus J(\bar{X})$, where $J(\bar{X})$ is the group of points on the Jacobian variety over $\bar{\mathbb{F}}_q$; it is a divisible torsion group. If $\ell \neq p$, the ℓ -primary torsion subgroup $J(\bar{X})_\ell$ of $J(\bar{X})$ is isomorphic to the abelian group $(\mathbb{Z}/\ell^\infty)^{2g}$. The group $J(\bar{X})$ may or may not have p -torsion. For example, if X is an elliptic curve then the p -torsion in $J(\bar{X})$ is either 0 or $\mathbb{Z}/p^\infty$, depending on whether or not X is supersingular (see [24], Ex. IV.4.15). Note that the localization $J(\bar{X})[1/p]$ is the direct sum over all $\ell \neq p$ of the ℓ -primary groups $J(\bar{X})_\ell$.

Next, recall that the group of units $\bar{\mathbb{F}}_q^\times$ may be identified with the group μ of all roots of unity in $\bar{\mathbb{F}}_q$; its underlying abelian group is isomorphic to $\mathbb{Q}/\mathbb{Z}[1/p]$. Passing to the direct limit of the $K_1(X_v)$ yields $K_1(\bar{X}) \cong \mu \oplus \mu$.

For $n \geq 1$, the groups $K_n(\bar{X})$ are all torsion groups, of order prime to p , because this is true of each $K_n(X_v)$ by 48. The following theorem determines the abelian group structure of the $K_n(\bar{X})$ as well as the action of the Galois group on them. It depends upon Suslin's theorem (see [63]) that for $i \geq 1$ and $\ell \neq p$ the groups $H_M^n(\bar{X}, \mathbb{Z}/\ell^\infty(i))$ equal the groups $H_{\text{ét}}^n(\bar{X}, \mathbb{Z}/\ell^\infty(i))$.

Theorem 51 Let X be a smooth projective curve over $\mathbb{F}_q$. Then for all $n \geq 0$ we have isomorphisms of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ -modules:

51

$$K_n(\bar{X}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus J(\bar{X}), & n = 0 \\ \mu(i) \oplus \mu(i), & n = 2i - 1 > 0 \\ J(\bar{X})[1/p](i), & n = 2i > 0. \end{cases}$$

For $\ell \neq p$, the ℓ -primary subgroup of $K_{n-1}(\bar{X})$ is isomorphic to $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$, $n > 0$, whose Galois module structure is given by:

$$K_n(\bar{X}; \mathbb{Z}/\ell^\infty) \cong \begin{cases} \mathbb{Z}/\ell^\infty(i) \oplus \mathbb{Z}/\ell^\infty(i), & n = 2i \geq 0 \\ J(\bar{X})_\ell(i - 1), & n = 2i - 1 > 0. \end{cases}$$

Proof Since the groups $K_n(\bar{X})$ are torsion for all $n > 0$, the universal coefficient theorem shows that $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$ is isomorphic to the ℓ -primary subgroup of $K_{n-1}(\bar{X})$. Thus we only need to determine the Galois modules $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$. For

$n = 0, 1, 2$ they may be read off from the above discussion. For $n > 2$ we consider the motivic spectral sequence (5.3); by Suslin's theorem, the terms $E_2^{p,q}$ vanish for $q < 0$ unless $p = q, q+1, q+2$. There is no room for differentials, so the spectral sequence degenerates at E_2 to yield the groups $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$. There are no extension issues because the edge maps are the e -invariants $K_{2i}(X; \mathbb{Z}/\ell^\infty) \rightarrow H_{\text{ét}}^0(\bar{X}, \mathbb{Z}/\ell^\infty(i)) = \mathbb{Z}/\ell^\infty(i)$ of 50, and are therefore split surjections. Finally, we note that as Galois modules we have $H_{\text{ét}}^1(\bar{X}, \mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)$, and (by Poincaré Duality [39, V.2]) $H_{\text{ét}}^2(\bar{X}, \mathbb{Z}/\ell^\infty(i+1)) \cong \mathbb{Z}/\ell^\infty(i)$.

Passing to invariants under the group $G = \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, there is a natural map from $K_n(X)$ to $K_n(\bar{X})^G$. For odd n , we see from Theorem 51 and Example 15 that $K_{2i-1}(\bar{X})^G \cong \mathbb{Z}/(q^i-1) \oplus \mathbb{Z}/(q^i-1)$; for even n , we have the less concrete description $K_{2i}(\bar{X})^G \cong J(\bar{X})[1/p](i)^G$. One way of studying this group is to consider the action of the algebraic Frobenius φ^* (induced by φ^{-1}) on cohomology.

Example 52. φ^* acts trivially on $H_{\text{ét}}^0(\bar{X}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $H_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$. It acts as q^{-i} on the twisted groups $H_{\text{ét}}^0(\bar{X}, \mathbb{Q}_\ell(i))$ and $H_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(i+1))$.

Weil's proof in 1948 of the Riemann Hypothesis for Curves implies that the eigenvalues of φ^* acting on $H_{\text{ét}}^1(\bar{X}, \mathbb{Q}_\ell(i))$ have absolute value $q^{1/2-i}$.

Since $H_{\text{ét}}^n(X, \mathbb{Q}_\ell(i)) \cong H_{\text{ét}}^n(\bar{X}, \mathbb{Q}_\ell(i))^G$, a perusal of these cases shows that we have $H_{\text{ét}}^n(X, \mathbb{Q}_\ell(i)) = 0$ except when (n, i) is $(0, 0)$ or $(2, 1)$.

For any G -module M , we have an exact sequence [75, 6.1.4]

$$0 \rightarrow M^G \rightarrow M \xrightarrow{\varphi^{*-1}} M \rightarrow H^1(G, M) \rightarrow 0. \quad (5.8)$$

The case $i = 1$ of the following result reproduces Weil's theorem that the ℓ -primary torsion part of the Picard group of X is $J(\bar{X})_\ell^G$.

53

Lemma 53 For a smooth projective curve X over $\mathbb{F}_q$, $\ell \nmid q$ and $i \geq 2$, we have:

1. $H_{\text{ét}}^{n+1}(X, \mathbb{Z}_\ell(i)) \cong H_{\text{ét}}^n(X, \mathbb{Z}/\ell^\infty(i)) \cong H_{\text{ét}}^n(\bar{X}, \mathbb{Z}/\ell^\infty(i))^G$ for all n ;
2. $H_{\text{ét}}^0(X, \mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/w_i^{(\ell)}(F)$;
3. $H_{\text{ét}}^1(X, \mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)^G$;
4. $H_{\text{ét}}^2(X, \mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/w_{i-1}^{(\ell)}(F)$; and
5. $H_{\text{ét}}^n(X, \mathbb{Z}/\ell^\infty(i)) = 0$ for all $n \geq 3$.

Proof Since $i \geq 2$, we see from 52 that $H_{\text{ét}}^n(X, \mathbb{Q}_\ell(i)) = 0$. Since $\mathbb{Q}_\ell/\mathbb{Z}_\ell = \mathbb{Z}/\ell^\infty$, this yields $H_{\text{ét}}^n(X, \mathbb{Z}/\ell^\infty(i)) \cong H_{\text{ét}}^{n+1}(X, \mathbb{Z}_\ell(i))$ for all n .

Since each $H^n = H_{\text{ét}}^n(\bar{X}, \mathbb{Z}/\ell^\infty(i))$ is a quotient of $H_{\text{ét}}^n(\bar{X}, \mathbb{Q}_\ell(i))$, $\varphi^* - 1$ is a surjection, i.e., $H^1(G, H^n) = 0$. Since $H^n(G, -) = 0$ for $n > 1$, the Leray spectral sequence

for $\overline{X} \rightarrow X$ collapses for $i > 1$ to yield exact sequences

$$0 \rightarrow H_{\text{ét}}^n(X, \mathbb{Z}/\ell^\infty(i)) \rightarrow H_{\text{ét}}^n(\overline{X}, \mathbb{Z}/\ell^\infty(i)) \xrightarrow{\varphi^{*-1}} H_{\text{ét}}^n(\overline{X}, \mathbb{Z}/\ell^\infty(i)) \rightarrow 0. \quad (5.9)$$

In particular, $H_{\text{ét}}^n(X, \mathbb{Z}/\ell^\infty(i)) = 0$ for $n > 2$. Since $H_{\text{ét}}^2(\overline{X}, \mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/\ell^\infty(i-1)$ this yields $H_{\text{ét}}^2(X, \mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/\ell^\infty(i-1)^G = \mathbb{Z}/w_{i-1}$. We also see that $H_{\text{ét}}^1(X, \mathbb{Z}/\ell^\infty(i))$ is the group of invariants of the Frobenius, i.e., $J(X)_\ell(i-1)^{\varphi^*}$.

Given the calculation of $K_n(\overline{X})^G$ in 51 and that of $H_{\text{ét}}^n(X, \mathbb{Z}/\ell^\infty(i))$ in 53, we see that the natural map $K_n(X) \rightarrow K_n(\overline{X})^G$ is a surjection, split by the Kahn maps 34 and 39. Thus the real content of the following theorem is that $K_n(X) \rightarrow K_n(\overline{X})^G$ is an isomorphism.

Theorem 54 Let X be the smooth projective curve corresponding to a global field F over $\mathbb{F}_q$. Then $K_0(X) = \mathbb{Z} \oplus \text{Pic}(X)$, and the finite groups $K_n(X)$ for $n > 0$ are given by:

54

$$K_n(X) \cong K_n(\overline{X})^G \cong \begin{cases} K_n(\mathbb{F}_q) \oplus K_n(\mathbb{F}_q), & n \text{ odd}, \\ \bigoplus_{\ell \neq p} J(\overline{X})_\ell(i)^G, & n = 2i \text{ even}. \end{cases}$$

Proof We may assume that $n \neq 0$, so that the groups $K_n(X)$ are finite by 48. It suffices to calculate the ℓ -primary part $K_{n+1}(X; \mathbb{Z}/\ell^\infty)$ of $K_n(X)$. But this follows from the motivic spectral sequence (5.3), which degenerates by 53.

The Zeta Function of a Curve

We can relate the orders of the K -groups of the curve X to values of the zeta function $\zeta_X(s)$. By definition, $\zeta_X(s) = Z(X, q^{-s})$, where

$$Z(X, t) = \exp \left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{t^n}{n} \right).$$

Weil proved that $Z(X, t) = P(t)/(1-t)(1-qt)$ for every smooth projective curve X , where $P(t) \in \mathbb{Z}[t]$ is a polynomial of degree $2 \cdot \text{genus}(X)$ with all roots of absolute value $1/\sqrt{q}$. This formula is a restatement of Weil's proof of the Riemann Hypothesis for X (52 above), given Grothendieck's formula $P(t) = \det(1 - \varphi^*t)$, where φ^* is regarded as an endomorphism of $H_{\text{ét}}^1(\overline{X}; \mathbb{Q}_\ell)$. Note that by 52 the action of φ^* on $H_{\text{ét}}^0(\overline{X}; \mathbb{Q}_\ell)$ has $\det(1 - \varphi^*t) = (1-t)$, and the action on $H_{\text{ét}}^2(\overline{X}; \mathbb{Q}_\ell)$ has $\det(1 - \varphi^*t) = (1-qt)$.

Here is application of Theorem 54, which goes back to Thomason (see [67, (4.7)] and [35]). Let $\#A$ denote the order of a finite abelian group A .

55 **Corollary 55** If X is a smooth projective curve over $\mathbb{F}_q$ then for all $i \geq 2$,

$$\frac{\#K_{2i-2}(X) \cdot \#K_{2i-3}(\mathbb{F}_q)}{\#K_{2i-1}(\mathbb{F}_q) \cdot \#K_{2i-3}(X)} = \prod_{\ell} \frac{\#H_{\text{ét}}^2(X; \mathbb{Z}_{\ell}(i))}{\#H_{\text{ét}}^1(X; \mathbb{Z}_{\ell}(i)) \cdot \#H_{\text{ét}}^3(X; \mathbb{Z}_{\ell}(i))} = |\zeta_X(1-i)|.$$

Proof We have seen that all the groups appearing in this formula are finite. The first equality follows from 53 and 54. The second equality follows by the Weil-Grothendieck formula for $\zeta_X(1-i)$ mentioned a few lines above.

Iwasawa modules

The group $H_{\text{ét}}^1(X, \mathbb{Z}/\ell^{\infty}(i))$ is the (finite) group of invariants $M^{\#}(i)^{\varphi^*}$ of the i -th twist of the Pontrjagin dual $M^{\#}$ of the *Iwasawa module* $M = M_X$. By definition M_X is the Galois group of $\widehat{X}$ over $X_{\infty} = X \otimes_{\mathbb{F}_q} F_q(\infty)$, where the field $F_q(\infty)$ is obtained from $\mathbb{F}_q$ by adding all ℓ -primary roots of unity, and $\widehat{X}$ is the maximal unramified pro- ℓ abelian cover of X_{∞} . It is known that the Iwasawa module M_X is a finitely generated free $\mathbb{Z}_{\ell}$ -module, and that its dual $M^{\#}$ is a finite direct sum of copies of $\mathbb{Z}/\ell^{\infty}$ [12, 3.22]. This viewpoint was developed in [13], and the corresponding discussion of Iwasawa modules for number fields is in [42].

5.6 Local Fields

Let E be a local field of residue characteristic p , with (discrete) valuation ring V and residue field $\mathbb{F}_q$. It is well known that $K_0(V) = K_0(E) = \mathbb{Z}$ and $K_1(V) = V^{\times}$, $K_1(E) = E^{\times} \cong (V^{\times}) \times \mathbb{Z}$, where the factor $\mathbb{Z}$ is identified with the powers $\{\pi^m\}$ of a parameter π of V . It is well known that $V^{\times} \cong \mu(E) \times U_1$, where $\mu(E)$ is the group of roots of unity in E (or V), and where U_1 is a free $\mathbb{Z}_p$ -module.

In the equi-characteristic case, where $\text{char}(E) = p$, it is well known that $V \cong \mathbb{F}_q[[\pi]]$ and $E = \mathbb{F}_q((\pi))$ [55], so $\mu(E) = \mathbb{F}_q^{\times}$, and $U_1 = W(\mathbb{F}_q)$ has rank $[\mathbb{F}_q : \mathbb{F}_p]$ over $\mathbb{Z}_p = W(\mathbb{F}_p)$. The decomposition of $K_1(V) = V^{\times}$ is evident here. Here is a description of the abelian group structure on $K_n(V)$ for $n > 1$.

56 **Theorem 56** Let $V = \mathbb{F}_q[[\pi]]$ be the ring of integers in the local field $E = \mathbb{F}_q((\pi))$. For $n \geq 2$ there are uncountable, uniquely divisible abelian groups U_n so that

$$K_n(V) \cong K_n(\mathbb{F}_q) \oplus U_n, \quad K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q).$$

Proof The map $K_{n-1}(\mathbb{F}_q) \rightarrow K_n(E)$ sending x to $\{x, \pi\}$ splits the localization sequence, yielding the decomposition of $K_n(E)$. If U_n denotes the kernel of the canonical map $K_n(V) \rightarrow K_n(\mathbb{F}_q)$, then naturality yields $K_n(V) = U_n \oplus K_n(\mathbb{F}_q)$. By Gabber's rigidity theorem [17], U_n is uniquely ℓ -divisible for $\ell \neq p$ and $n > 0$. It suffices to show that U_n is uncountable and uniquely p -divisible when $n \geq 2$.

Tate showed that the Milnor groups $K_n^M(E)$ are uncountable, uniquely divisible for $n \geq 3$, and that the same is true for the kernel U_2 of the norm residue map $K_2(E) \rightarrow \mu(E)$; see [66]. If $n \geq 2$ then $K_n^M(E)$ is a summand of the Quillen K -group $K_n(E)$ by [61]. On the other hand, Geisser and Levine proved in [19] that the complementary summand is uniquely p -divisible.

In the mixed characteristic case, when $\text{char}(E) = 0$, even the structure of $V^\times$ is quite interesting. The torsion free part U_1 is a free $\mathbb{Z}_p$ -module of rank $[E : \mathbb{Q}_p]$; it is contained in $(1 + \pi V)^\times$ and injects into E by the convergent power series for $x \mapsto \ln(x)$.

The group $\mu(E)$ of roots of unity in E (or V) is identified with $(\mathbb{F}_q^\times) \times \mu_{p^\infty}(E)$, where the first factor arises from Teichmüller's theorem that $V^\times \rightarrow \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)$ has a unique splitting, and $\mu_{p^\infty}(E)$ denotes the finite group of p -primary roots of unity in E . There seems to be no simple formula for the order of the cyclic p -group $\mu_{p^\infty}(E)$.

For K_2 , there is a norm residue symbol $K_2(E) \rightarrow \mu(E)$ and we have the following result; see [75, III.6.6].

Theorem 57: (*Moore's Theorem.*) The group $K_2(E)$ is the product of a finite group, isomorphic to $\mu(E)$, and an uncountable, uniquely divisible abelian group U_2 . In addition,

$$K_2(V) \cong \mu_{p^\infty}(E) \times U_2.$$

Proof The fact that the kernel U_2 of the norm residue map is divisible is due to C. Moore, and is given in the Appendix to [40]. The fact that U_2 is torsion free (hence uniquely divisible) was proven by Tate [66] when $\text{char}(F) = p$, and by Merkurjev [37] when $\text{char}(F) = 0$.

Since the transcendence degree of E over $\mathbb{Q}$ is uncountable, it follows from Moore's theorem and the arguments in [40] that the Milnor K -groups $K_n^M(E)$ are uncountable, uniquely divisible abelian groups for $n \geq 3$. By [61], this is a summand of the Quillen K -group $K_n(E)$. As in the equicharacteristic case, $K_n(E)$ will contain an uncountable uniquely divisible summand about which we can say very little.

To understand the other factor, we typically proceed a prime at a time. This has the advantage of picking up the torsion subgroups of $K_n(E)$, and detecting the

groups $K_n(V)/\ell$. For p -adic fields, the following calculation reduces the problem to the prime p .

58

Proposition 58 If $i > 0$ there is a summand of $K_{2i-1}(V) \cong K_{2i-1}(E)$ isomorphic to $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)$, detected by the e -invariant. The complementary summand is uniquely ℓ -divisible for every prime $\ell \neq p$, i.e., a $\mathbb{Z}_{(p)}$ -module.

There is also a decomposition $K_{2i}(E) \cong K_{2i}(V) \oplus K_{2i-1}(\mathbb{F}_q)$, and the group $K_{2i}(V)$ is uniquely ℓ -divisible for every prime $\ell \neq p$, i.e., a $\mathbb{Z}_{(p)}$ -module.

Proof Pick a prime ℓ . We see from Gabber's rigidity theorem [17] that the groups $K_n(V; \mathbb{Z}/\ell^v)$ are isomorphic to $K_n(\mathbb{F}_q; \mathbb{Z}/\ell^v)$ for $n > 0$. Since the Bockstein spectral sequences are isomorphic, and detect all finite cyclic ℓ -primary summands of $K_n(V)$ and $K_n(\mathbb{F}_q)$ [72, 5.9.12], it follows that $K_{2i-1}(V)$ has a cyclic summand isomorphic to $\mathbb{Z}/w_i^{(\ell)}(E)$, and that the complement is uniquely ℓ -divisible. Since $K_n(V; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$, we also see that $K_{2i}(V)$ is uniquely ℓ -divisible. As ℓ varies, we get a cyclic summand of order $w_i(E)$ in $K_{2i-1}(V)$ whose complement is a $\mathbb{Z}_{(p)}$ -module.

If $x \in K_{2i-1}(V)$, the product $\{x, \pi\} \in K_{2i}(E)$ maps to the image of x in $K_{2i-1}(\mathbb{F}_q)$ under the boundary map ∂ in the localization sequence. Hence the summand of $K_{2i-1}(V)$ isomorphic to $K_{2i-1}(\mathbb{F}_q)$ lifts to a summand of $K_{2i}(E)$. This breaks the localization sequence up into the split short exact sequences $0 \rightarrow K_n(V) \rightarrow K_n(E) \rightarrow K_{n-1}(\mathbb{F}_q) \rightarrow 0$.

59

Completed K -theory 59 It will be convenient to fix a prime ℓ and pass to the ℓ -adic completion $\widehat{K}(R)$ of the K -theory space $K(R)$, where R is any ring. We also write $K_n(R; \mathbb{Z}_\ell)$ for $\pi_n \widehat{K}(R)$. Information about these groups tells us about the groups $K_n(R; \mathbb{Z}/\ell^v) = \pi_n(K(R); \mathbb{Z}/\ell^v)$, because these groups are isomorphic to $\pi_n(\widehat{K}(R); \mathbb{Z}/\ell^v)$ for all v .

If the groups $K_n(R; \mathbb{Z}/\ell^v)$ are finite, then $K_n(R; \mathbb{Z}_\ell)$ is an extension of the Tate module of $K_{n-1}(R)$ by the ℓ -adic completion of $K_n(R)$. (The *Tate module* of an abelian group A is the inverse limit of the groups $\text{Hom}(\mathbb{Z}/\ell^v, A)$.) For example, $K_n(\mathbb{C}; \mathbb{Z}_\ell)$ vanishes for odd n and for even n equals the Tate module $\mathbb{Z}_\ell$ of $K_{n-1}(\mathbb{C})$. If in addition the abelian groups $K_n(R)$ are finitely generated, there can be no Tate module and we have $K_n(R; \mathbb{Z}_\ell) \cong K_n(R) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \varprojlim K_n(R; \mathbb{Z}/\ell^v)$.

60

Warning 60 Even if we know $K_n(R; \mathbb{Z}_\ell)$ for all primes, we may not still be able to determine the underlying abelian group $K_n(R)$ exactly from this information. For example, consider the case $n = 1$, $R = \mathbb{Z}_p$. We know that $K_1(R; \mathbb{Z}_p) \cong (1 + pR)^\times \cong \mathbb{Z}_p$, $p \neq 2$, but this information does not even tell us that $K_1(R) \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_p$. To see why, note that the extension $0 \rightarrow \mathbb{Z}_{(p)} \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Z}_p/\mathbb{Z}_{(p)} \rightarrow 0$ doesn't split; there are

no p -divisible elements in $\mathbb{Z}_p$, yet $\mathbb{Z}_p/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Q}$ is a uniquely divisible abelian group.

We now consider the p -adic completion of $K(E)$. By 58, it suffices to consider the p -adic completion of $K(V)$.

Write w_i for the numbers $w_i = w_i^{(p)}(E)$, which were described in 23. For all i , and $\ell^v > w_i$, the étale cohomology group $H^1(E, \mu_{p^v}^{\otimes i})$ is isomorphic to $(\mathbb{Z}/p^v)^d \oplus \mathbb{Z}/w_i \oplus \mathbb{Z}/w_{i-1}$, $d = [E : \mathbb{Q}_p]$. By duality, the group $H^2(E, \mu_{p^v}^{\otimes i+1})$ is also isomorphic to $\mathbb{Z}/w_i$.

Theorem 61 Let E be a local field, of degree d over $\mathbb{Q}_p$, with ring of integers V . Then for $n \geq 2$ we have:

61

$$K_n(V; \mathbb{Z}_p) \cong K_n(E; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}/w_i^{(p)}(E), & n = 2i, \\ (\mathbb{Z}_p)^d \oplus \mathbb{Z}/w_i^{(p)}(E), & n = 2i - 1. \end{cases}$$

Moreover, the first étale Chern classes $K_{2i-1}(E; \mathbb{Z}/p^v) \cong H^1(E, \mu_{p^v}^{\otimes i})$ are natural isomorphisms for all i and v .

Finally, each $K_{2i}(V)$ is the direct sum of a uniquely divisible group, a divisible p -group and a subgroup isomorphic to $\mathbb{Z}/w_i^{(p)}(E)$.

Proof If $p > 2$ the first part is proven in [6] (see [25]). (It also follows from the spectral sequence (5.1) for E , using the Voevodsky-Rost theorem.) In this case, Theorem 34 and a count shows that the étale Chern classes $K_{2i-1}(E; \mathbb{Z}/p^v) \rightarrow H_{\text{ét}}^1(E; \mu_{p^v}^{\otimes i})$ are isomorphisms. If $p = 2$ this is proven in [51, (1.12)]; surprisingly, this implies that the Harris–Segal maps and Kahn maps are even defined when E is an exceptional 2-adic field.

Now fix i and set $w_i = w_i^{(p)}(E)$. Since the Tate module of any abelian group is torsion free, and $K_{2i}(E; \mathbb{Z}_p)$ is finite, we see that the Tate module of $K_{2i-1}(E)$ vanishes and the p -adic completion of $K_{2i}(E)$ is $\mathbb{Z}/w_i$. Since this is also the completion of the $\mathbb{Z}_{(p)}$ -module $K_{2i}(V)$ by 58, the decomposition follows from the structure of $\mathbb{Z}_{(p)}$ -modules. (This decomposition was first observed in [27, 6.2].)

Remark 62 The fact that these groups were finitely generated $\mathbb{Z}_p$ -modules of rank d was first obtained by Wagoner in [70], modulo the identification in [46] of Wagoner’s continuous K -groups with $K_*(E; \mathbb{Z}/p)$.

62

Unfortunately, I do not know how to reconstruct the “integral” homotopy groups $K_n(V)$ from the information in 61. Any of the $\mathbb{Z}_p$ ’s in $K_{2i-1}(V; \mathbb{Z}_p)$ could come from either a $\mathbb{Z}_{(p)}$ in $K_{2i-1}(V)$ or a $\mathbb{Z}/p^\infty$ in $K_{2i-2}(V)$. Here are some cases when I can

show that they come from torsion free elements; I do not know any example where a $\mathbb{Z}/p^\infty$ appears.

63

Corollary 63 $K_3(V)$ contains a torsion free subgroup isomorphic to $\mathbb{Z}_{(p)}^d$, whose p -adic completion is isomorphic to the torsion free part of $K_3(V; \mathbb{Z}_p) \cong (\mathbb{Z}_p)^d \oplus \mathbb{Z}/w_2^{(p)}$.

Proof Combine 61 with Moore's Theorem 57 and 58.

I doubt that the extension $0 \rightarrow \mathbb{Z}_{(p)}^d \rightarrow K_3(V) \rightarrow U_3 \rightarrow 0$ splits (see Warning 60).

Example 64. If $k > 0$, $K_{4k+1}(\mathbb{Z}_2)$ contains a subgroup T_k isomorphic to $\mathbb{Z}_{(2)} \times \mathbb{Z}/w_i$, and the quotient $K_{4k+1}(\mathbb{Z}_2)/T_k$ is uniquely divisible. (By 23, $w_i = 2(2^{2k+1} - 1)$.)

This follows from Rognes' theorem [52, 4.13] that the map from $K_{4k+1}(\mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus (\mathbb{Z}/2)$ to $K_{4k+1}(\mathbb{Z}_2; \mathbb{Z}_2)$ is an isomorphism for all $k > 1$. (The information about the torsion subgroups, missing in [52], follows from [51].) Since this map factors through $K_{4k+1}(\mathbb{Z}_2)$, the assertion follows.

Example 65. Let F be a totally imaginary number field of degree $d = 2r_2$ over $\mathbb{Q}$, and let $E_1, \dots, E_s$ be the completions of F at the prime ideals over p . There is a subgroup of $K_{2i-1}(F)$ isomorphic to $\mathbb{Z}^2$ by Theorem 6; its image in $\oplus K_{2i-1}(E_j)$ is a subgroup of rank at most r_2 , while $\oplus K_{2i-1}(E_j; \mathbb{Z}_p)$ has rank $d = \sum [E_j : \mathbb{Q}_p]$. So these subgroups of $K_{2i-1}(E_j)$ can account for at most half of the torsion free part of $\oplus K_{2i-1}(E_j; \mathbb{Z}_p)$.

Example 66. Suppose that F is a totally real number field, of degree $d = r_1$ over $\mathbb{Q}$, and let $E_1, \dots, E_s$ be the completions of F at the prime ideals over p . For $k > 0$, there is a subgroup of $K_{4k+1}(F)$ isomorphic to $\mathbb{Z}^d$ by Theorem 6; its image in $\oplus K_{4k+1}(E_j)$ is a subgroup of rank d , while $\oplus K_{4k+1}(E_j; \mathbb{Z}_p)$ has rank $d = \sum [E_j : \mathbb{Q}_p]$. However, this does not imply that the p -adic completion $\mathbb{Z}_p^d$ of the subgroup injects into $\oplus K_{4k+1}(E_j; \mathbb{Z}_p)$. Implications like this are related to Leopoldt's conjecture.

Leopoldt's conjecture states that the torsion free part $\mathbb{Z}_p^{d-1}$ of $(\mathcal{O}_F)^\times \otimes \mathbb{Z}_p$ injects into the torsion free part $\mathbb{Z}_p^d$ of $\prod_{j=1}^s \mathcal{O}_{E_j}^\times$; (see [71, 5.31]). This conjecture has been proven when F is an abelian extension of $\mathbb{Q}$; (see [71, 5.32]).

When F is a totally real abelian extension of $\mathbb{Q}$, and p is a regular prime, Soulé shows in [59, 3.1, 3.7] that the torsion free part $\mathbb{Z}_p^d$ of $K_{4k+1}(F) \otimes \mathbb{Z}_p$ injects into $\oplus K_{4k+1}(E_j; \mathbb{Z}_p) \cong (\mathbb{Z}_p)^d$, because the cokernel is determined by the Leopoldt p -adic L -function $L_p(F, \omega^{2k}, 2k+1)$, which is a p -adic unit in this favorable scenario. Therefore in this case we also have a subgroup $\mathbb{Z}_p^d$ in each of the groups $K_{4k+1}(E_j)$.

We conclude with a description of the topological type of $\widehat{K}(V)$ and $\widehat{K}(E)$, when p is odd. Recall that $F\Psi^k$ denotes the homotopy fiber of $\Psi^k - 1: \mathbb{Z} \times BU \rightarrow BU$. Since $\Psi^k = k^i$ on $\pi_{2i}(BU) = \mathbb{Z}$ for $i > 0$, and the other homotopy groups of BU vanish, we see that $\pi_{2i-1}F\Psi^k \cong \mathbb{Z}/(k^i - 1)$, and that all even homotopy groups of $F\Psi^k$ vanish, except for $\pi_0(F\Psi^k) = \mathbb{Z}$.

Theorem 67: (*Thm. D of [25].*) Let E be a local field, of degree d over $\mathbb{Q}_p$, with p odd. Then after p -completion, there is a number k (given below) so that

$$\widehat{K}(V) \simeq SU \times U^{d-1} \times F\Psi^k \times BF\Psi^k, \quad \widehat{K}(E) \simeq U^d \times F\Psi^k \times BF\Psi^k.$$

The number k is defined as follows. Set $r = [E(\mu_p) : E]$, and let p^a be the number of p -primary roots of unity in $E(\mu_p)$. If r is a topological generator of $\mathbb{Z}_p^\times$, then $k = r^n$, $n = p^{a-1}(p-1)/r$. It is an easy exercise, left to the reader, to check that $\pi_{2i-1}F\Psi^k \cong \mathbb{Z}_p/(k^i - 1)$ is $\mathbb{Z}/w_i$ for all i .

Number Fields at Primes Where $cd = 2$

In this section we quickly obtain a cohomological description of the odd torsion in the K -groups of a number field, and also the 2-primary torsion in the K -groups of a totally imaginary number field. These are the cases where $cd_\ell(\mathcal{O}_S) = 2$, which forces the motivic spectral sequence (5.3) to degenerate completely.

The following trick allows us to describe the torsion subgroup of the groups $K_n(R)$. Recall that the notation $A\{\ell\}$ denotes the ℓ -primary subgroup of an abelian group A .

Lemma 68 For a given prime ℓ , ring R and integer n , suppose that $K_n(R)$ is a finite group, and that $K_{n-1}(R)$ is a finitely generated group. Then $K_n(R)\{\ell\} \cong K_n(R; \mathbb{Z}_\ell)$ and $K_{n-1}(R)\{\ell\} \cong K_n(R; \mathbb{Z}/\ell^\infty)$.

Proof For large values of v , the finite group $K_n(R; \mathbb{Z}/\ell^v)$ is the sum of $K_n(R)\{\ell\}$ and $K_{n-1}(R)\{\ell\}$. The transition from coefficients $\mathbb{Z}/\ell^v$ to $\mathbb{Z}/\ell^{v-1}$ (resp., to $\mathbb{Z}/\ell^{v+1}$) is multiplication by 1 and ℓ (resp., by ℓ and 1) on the two summands. Taking the inverse limit, (resp., direct limit) yields the groups $K_n(R; \mathbb{Z}_\ell)$ and $K_n(R; \mathbb{Z}/\ell^\infty)$, respectively.

Example 69. By 7, the lemma applies to a ring $\mathcal{O}_S$ of integers in a number field F , with n even. For example, Theorem 4 says that $K_2(\mathcal{O}_S)\{\ell\} = K_2(\mathcal{O}_S; \mathbb{Z}_\ell) \cong H_{\text{ét}}^2(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(2))$, and of course $K_1(\mathcal{O}_S)\{\ell\} = K_2(\mathcal{O}_S; \mathbb{Z}/\ell^\infty)$ is the group $\mathbb{Z}/w_1^{(\ell)}(F)$ of ℓ -primary roots of unity in F .

We now turn to the odd torsion in the K -groups of a number field. The ℓ -primary torsion is described by the following result, which is based on [51] and uses the Voevodsky-Rost theorem. The notation $A_{(\ell)}$ will denote the localization of an abelian group A at the prime ℓ .

70 Theorem 70 Fix an odd prime ℓ . Let F be a number field, and let $\mathcal{O}_S$ be a ring of integers in F . If $R = \mathcal{O}_S[1/\ell]$, then for all $n \geq 2$:

$$K_n(\mathcal{O}_S)_{(\ell)} \cong \begin{cases} H_{\text{ét}}^2(R; \mathbb{Z}_{\ell}(i+1)) & \text{for } n = 2i > 0; \\ \mathbb{Z}_{(\ell)}^{r_2} \oplus \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i - 1, i \text{ even}; \\ \mathbb{Z}_{(\ell)}^{r_2+r_1} \oplus \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i - 1, i \text{ odd}. \end{cases}$$

Proof By 7 we may replace $\mathcal{O}_S$ by R without changing the ℓ -primary torsion. By 68 and 6, it suffices to show that $K_{2i}(R; \mathbb{Z}_{\ell}) \cong H_{\text{ét}}^2(R; \mathbb{Z}_{\ell}(i+1))$ and $K_{2i}(R; \mathbb{Z}/\ell^{\infty}) \cong \mathbb{Z}/w_i^{(\ell)}(F)$. Note that the formulas for $K_0(\mathcal{O}_S)$ and $K_1(\mathcal{O}_S)$ are different; see (5.5).

If F is a number field and $\ell \neq 2$, the étale ℓ -cohomological dimension of F (and of R) is 2. Since $H_{\text{ét}}^2(R; \mathbb{Z}/\ell^{\infty}(i)) = 0$ by 43, the Voevodsky-Rost theorem implies that the motivic spectral sequence (5.3) has only two nonzero diagonals, except in total degree zero, and collapses at E_2 . This gives

$$K_n(\mathcal{O}_S; \mathbb{Z}/\ell^{\infty}) \cong \begin{cases} H^0(R; \mathbb{Z}/\ell^{\infty}(i)) = \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i \geq 2, \\ H^1(R; \mathbb{Z}/\ell^{\infty}(i)) & \text{for } n = 2i - 1 \geq 1. \end{cases}$$

The description of $K_{2i-1}(\mathcal{O}_S)\{\ell\}$ follows from 68 and 6.

The same argument works for coefficients $\mathbb{Z}_{\ell}$. For $i > 0$ we see that $H_{\text{ét}}^n(R, \mathbb{Z}_{\ell}(i)) = 0$ for $n \neq 1, 2$, so the spectral sequence degenerates to yield $K_{2i}(R; \mathbb{Z}_{\ell}) \cong H_{\text{ét}}^2(R, \mathbb{Z}_{\ell}(i))$. (This is a finite group by 42.) The description of $K_{2i}(R)\{\ell\}$ follows from 68 and 6.

Because $H_{\text{ét}}^2(R, \mathbb{Z}_{\ell}(i+1))/\ell \cong H_{\text{ét}}^2(R, \mu_{\ell}^{\otimes i+1})$, we immediately deduce:

71 Corollary 71 For all odd ℓ and $i > 0$, $K_{2i}(\mathcal{O}_S)/\ell \cong H_{\text{ét}}^2(\mathcal{O}_S[1/\ell], \mu_{\ell}^{\otimes i+1})$.

72 Remark 72 Similarly, the mod- ℓ spectral sequence (5.1) collapses to yield the K -theory of $\mathcal{O}_S$ with coefficients $\mathbb{Z}/\ell$, ℓ odd. For example, if $\mathcal{O}_S$ contains a primitive ℓ -th root of unity and $1/\ell$ then $H^1(\mathcal{O}_S; \mu_{\ell}^{\otimes i}) \cong \mathcal{O}_S^{\times}/\mathcal{O}_S^{\times \ell} \oplus {}_{\ell}\text{Pic}(\mathcal{O}_S)$ and $H^2(\mathcal{O}_S; \mu_{\ell}^{\otimes i}) \cong \text{Pic}(\mathcal{O}_S)/\ell \oplus {}_{\ell}\text{Br}(\mathcal{O}_S)$ for all i , so

$$K_n(\mathcal{O}_S; \mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell \oplus \text{Pic}(\mathcal{O}_S)/\ell & \text{for } n = 0, \\ \mathcal{O}_S^\times / \mathcal{O}_S^{\times \ell} \oplus {}_\ell \text{Pic}(\mathcal{O}_S) & \text{for } n = 2i - 1 \geq 1, \\ \mathbb{Z}/\ell \oplus \text{Pic}(\mathcal{O}_S)/\ell \oplus {}_\ell \text{Br}(\mathcal{O}_S) & \text{for } n = 2i \geq 2. \end{cases}$$

The $\mathbb{Z}/\ell$ summands in degrees $2i$ are generated by the powers β^i of the Bott element $\beta \in K_2(\mathcal{O}_S; \mathbb{Z}/\ell)$ (see 17). In fact, $K_*(\mathcal{O}_S; \mathbb{Z}/\ell)$ is free as a graded $\mathbb{Z}[\beta]$ -module on $K_0(\mathcal{O}_S; \mathbb{Z}/\ell)$, $K_1(\mathcal{O}_S; \mathbb{Z}/\ell)$ and ${}_ell \text{Br}(\mathcal{O}_S) \in K_2(\mathcal{O}_S; \mathbb{Z}/\ell)$; this is immediate from the multiplicative properties of (5.1).

When F is totally imaginary, we have a complete description of $K_*(\mathcal{O}_S)$. The 2-primary torsion was first calculated in [51]; the odd torsion comes from Theorem 70. Write w_i for $w_i(F)$.

Theorem 73 Let F be a totally imaginary number field, and let $\mathcal{O}_S$ be the ring of S -integers in F for some set S of finite places. Then for all $n \geq 2$:

73

$$K_n(\mathcal{O}_S) \cong \begin{cases} \mathbb{Z} \oplus \text{Pic}(\mathcal{O}_S) & \text{for } n = 0; \\ \mathbb{Z}^{r_2 + |S| - 1} \oplus \mathbb{Z}/w_1 & \text{for } n = 1; \\ \oplus {}_\ell H_{\text{ét}}^2(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i + 1)) & \text{for } n = 2i \geq 2; \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i & \text{for } n = 2i - 1 \geq 3. \end{cases}$$

Proof The case $n = 1$ comes from (5.5), and the odd torsion comes from 70, so it suffices to check the 2-primary torsion. This does not change if we replace $\mathcal{O}_S$ by $R = \mathcal{O}_S[1/2]$, by 7. By 68 and 6, it suffices to show that $K_{2i}(R; \mathbb{Z}_2) \cong H_{\text{ét}}^2(R; \mathbb{Z}_2(i + 1))$ and $K_{2i}(R; \mathbb{Z}/2^\infty) \cong \mathbb{Z}/w_i^{(2)}(F)$.

Consider the mod 2^∞ motivic spectral sequence (5.3) for the ring R , converging to $K_*(R; \mathbb{Z}/2^\infty)$. It is known that $cd_2(R) = 2$, and $H_{\text{ét}}^2(R; \mathbb{Z}/2^\infty(i)) = 0$ by 43. Hence the spectral sequence collapses; except in total degree zero, the E_2 -terms are concentrated on the two diagonal lines where $p = q$, $p = q + 1$. This gives

$$K_n(R; \mathbb{Z}/2^\infty) \cong \begin{cases} H^0(R; \mathbb{Z}/2^\infty(i)) = \mathbb{Z}/w_i^{(2)}(F) & \text{for } n = 2i \geq 0, \\ H^1(R; \mathbb{Z}/2^\infty(i)) & \text{for } n = 2i - 1 \geq 1. \end{cases}$$

The description of $K_{2i-1}(R)\{2\}$ follows from 68 and 6.

The same argument works for coefficients $\mathbb{Z}_2$; for $i > 0$ and $n \neq 1, 2$ we have $H_{\text{ét}}^n(R, \mathbb{Z}_2(i)) = 0$, so (5.3) degenerates to yield $K_{2i}(R; \mathbb{Z}_2) \cong H_{\text{ét}}^2(R, \mathbb{Z}_2(i))$. (This is a finite group by 42). The description of $K_{2i}(R)\{2\}$ follows from 68 and 6.

Example 74. Let F be a number field containing a primitive ℓ -th root of unity, and let S be the set of primes over ℓ in $\mathcal{O}_F$. If t is the rank of $\text{Pic}(R)/\ell$, then $H_{\text{ét}}^2(R, \mathbb{Z}_\ell(i))/\ell \cong H_{\text{ét}}^2(R, \mu_\ell^{\otimes i}) \cong H_{\text{ét}}^2(R, \mu_\ell) \otimes \mu_\ell^{\otimes i-1}$ has rank $t + |S| - 1$ by (5.6). By 73, the ℓ -primary subgroup of $K_{2i}(\mathcal{O}_S)$ has $t + |S| - 1$ nonzero summands for each $i \geq 2$.

Example 75. If $\ell \neq 2$ is a regular prime, we claim that $K_{2i}(\mathbb{Z}[\zeta_\ell])$ has no ℓ -torsion. (The case K_0 is tautological by 3, and the classical case K_2 is 5.) Note that the group $K_{2i-1}(\mathbb{Z}[\zeta_\ell]) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F)$ always has ℓ -torsion, because $w_i^{(\ell)}(F) \geq \ell$ for all i by 20(a). Setting $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$, then by 73,

$$K_{2i}(\mathbb{Z}[\zeta_\ell]) \cong H_{\text{ét}}^2(R, \mathbb{Z}_\ell(i+1)) \oplus (\text{finite group without } \ell\text{-torsion}).$$

Since ℓ is regular, we have $\text{Pic}(R)/\ell = 0$, and we saw in 5 that $\text{Br}(R) = 0$ and $|S| = 1$. By 74, $H_{\text{ét}}^2(R, \mathbb{Z}_\ell(i+1)) = 0$ and the claim now follows.

We conclude with a comparison to the odd part of $\zeta_F(1 - 2k)$, generalizing the Birch–Tate Conjecture 31. If F is not totally real, $\zeta_F(s)$ has a pole of order r_2 at $s = 1 - 2k$. We need to invoke the following deep result of Wiles [77], which is often called the “Main Conjecture” of Iwasawa Theory.

76 Theorem 76 (Wiles) Let F be a totally real number field. If ℓ is odd and $\mathcal{O}_S = \mathcal{O}_F[1/\ell]$ then for all even $i = 2k > 0$:

$$\zeta_F(1 - i) = \frac{|H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{Z}_\ell(i))|}{|H_{\text{ét}}^1(\mathcal{O}_S, \mathbb{Z}_\ell(i))|} u_i,$$

where u_i is a rational number prime to ℓ .

The numerator and denominator on the right side are finite by 36. Lichtenbaum’s conjecture follows, up to a power of 2, by setting $i = 2k$:

77 Theorem 77 If F is totally real, then

$$\zeta_F(1 - 2k) = (-1)^{kr_1} \frac{|K_{4k-2}(\mathcal{O}_F)|}{|K_{4k-1}(\mathcal{O}_F)|} \quad \text{up to factors of 2}.$$

Proof By the functional equation, the sign of $\zeta_F(1 - 2k)$ is $(-1)^{kr_1}$. It suffices to show that the left and right sides of 77 have the same power of each odd prime ℓ . The group $H_{\text{ét}}^2(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(i))$ is the ℓ -primary part of $K_{2i-2}(\mathcal{O}_F)$ by 70. The group $H_{\text{ét}}^1(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(i))$ on the bottom of 76 is $\mathbb{Z}/w_i^{(\ell)}(F)$ by 46, and this is isomorphic to the ℓ -primary subgroup of $K_{2i-1}(\mathcal{O}_F)$ by Theorem 70.

Real Number Fields at the Prime 2

5.8

Let F be a real number field, i.e., F has $r_1 > 0$ embeddings into $\mathbb{R}$. The calculation of the algebraic K -theory of F at the prime 2 is somewhat different from the calculation at odd primes, for many reasons. One reason is that a real number field has infinite cohomological dimension, which complicates descent methods. A second reason is that the Galois group of a cyclotomic extension need not be cyclic, so that the e -invariant may not split (see 29). A final reason is that the groups $K_*(F; \mathbb{Z}/2)$ do not have a natural multiplication, because of the structure of the mod 2 Moore space $\mathbb{R}P^2$.

For the real numbers $\mathbb{R}$, the mod 2 motivic spectral sequence has $E_2^{p,q} = \mathbb{Z}/2$ for all p, q in the octant $q \leq p \leq 0$. In order to distinguish between the groups $E_2^{p,q}$, it is useful to label the nonzero elements of $H_{\text{ét}}^0(\mathbb{R}, \mathbb{Z}/2(i))$ as β_i , writing 1 for β_0 . Using the multiplicative pairing with (say) the spectral sequence $'E_2^{*,*}$ converging to $K_*(\mathbb{R}; \mathbb{Z}/16)$, multiplication by the element $\eta \in 'E_2^{0,-1}$ allows us to write the nonzero elements in the $-i$ -th column as $\eta^i \beta_i$ (see Table 5.2 below).

From Suslin's calculation of $K_n(\mathbb{R})$ in [62], we know that the groups $K_n(\mathbb{R}; \mathbb{Z}/2)$ are cyclic and 8-periodic (for $n \geq 0$) with orders 2, 2, 4, 2, 2, 0, 0, 0 (for $n = 0, 1, \dots, 7$).

Theorem 78 In the spectral sequence converging to $K_*(\mathbb{R}; \mathbb{Z}/2)$, all the d_2 differentials with nonzero source on the lines $p \equiv 1, 2 \pmod{4}$ are isomorphisms. Hence the spectral sequence degenerates at E_3 . The only extensions are the nontrivial extensions $\mathbb{Z}/4$ in $K_{8a+2}(\mathbb{R}; \mathbb{Z}/2)$.

78

Table 5.2. The mod 2 spectral sequence for $\mathbb{R}$

			1
		β_1	η
	β_2	$\eta\beta_1$	η^2
β_3	$\eta\beta_2$	$\eta^2\beta_1$	η^3
$\eta\beta_3$	$\eta^2\beta_2$	$\eta^3\beta_1$	η^4
The first 4 columns of E_2			
			1
		β_1	η
	0	$\eta\beta_1$	η^2
0	0	$\eta^2\beta_1$	0
0	0	0	0
The first 4 columns of E_3			

Proof Recall from Remark 2 that the mod 2 spectral sequence has periodicity isomorphisms $E_r^{p,q} \cong E_r^{p-4,q-4}$, $p \leq 0$. Therefore it suffices to work with the columns $-3 \leq p \leq 0$.

Because $K_3(\mathbb{R}; \mathbb{Z}/2) \cong \mathbb{Z}/2$, the differential closest to the origin, from β_2 to η^3 , must be nonzero. Since the pairing with $'E_2$ is multiplicative and $d_2(\eta) = 0$, we must have $d_2(\eta^j \beta_2) = \eta^{j+3}$ for all $j \geq 0$. Thus the column $p = -2$ of E_3 is zero, and every term in the column $p = 0$ of E_3 is zero except for $\{1, \eta, \eta^2\}$.

Similarly, we must have $d_2(\beta_3) = \eta^3 \beta_1$ because $K_5(\mathbb{R}; \mathbb{Z}/2) = 0$. By multiplicativity, this yields $d_2(\eta^j \beta_3) = \eta^{j+3} \beta_1$ for all $j \geq 0$. Thus the column $p = -3$ of E_3 is zero, and every term in the column $p = -1$ of E_3 is zero except for $\{\beta_1, \eta \beta_1, \eta^2 \beta_1\}$.

79 Variant 79 The analysis with coefficients $\mathbb{Z}/2^\infty$ is very similar, except that when $p > q$, $E_2^{p,q} = H_{\text{ét}}^{p-q}(\mathbb{R}; \mathbb{Z}/2^\infty(-q))$ is: 0 for p even; $\mathbb{Z}/2$ for p odd. If p is odd, the coefficient map $\mathbb{Z}/2 \rightarrow \mathbb{Z}/2^\infty$ induces isomorphisms on the $E_2^{p,q}$ terms, so by 78 all the d_2 differentials with nonzero source in the columns $p \equiv 1 \pmod{4}$ are isomorphisms. Again, the spectral sequence converging to $K_*(\mathbb{R}; \mathbb{Z}/2^\infty)$ degenerates at $E_3 = E_\infty$. The only extensions are the nontrivial extensions of $\mathbb{Z}/2^\infty$ by $\mathbb{Z}/2$ in $K_{8a+4}(\mathbb{R}; \mathbb{Z}/2^\infty) \cong \mathbb{Z}/2^\infty$.

80 Variant 80 The analysis with 2-adic coefficients is very similar, except that (a) $H^0(\mathbb{R}; \mathbb{Z}_2(i))$ is: $\mathbb{Z}_2$ for i even; 0 for i odd and (b) (for $p > q$) $E_2^{p,q} = H_{\text{ét}}^{p-q}(\mathbb{R}; \mathbb{Z}/2^\infty(-q))$ is: $\mathbb{Z}/2$ for p even; 0 for p odd. All differentials with nonzero source in the column $p \equiv 2 \pmod{4}$ are onto. Since there are no extensions to worry about, we omit the details.

In order to state the Theorem 82 below for a ring $\mathcal{O}_S$ of integers in a number field F , we consider the natural maps (for $n > 0$) induced by the r_1 real embeddings of F ,

$$\alpha_S^n(i): H^n(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \rightarrow \bigoplus_{r_1} H^n(\mathbb{R}; \mathbb{Z}/2^\infty(i)) \cong \begin{cases} (\mathbb{Z}/2)^{r_1}, & i - n \text{ odd} \\ 0, & i - n \text{ even} \end{cases} \quad (5.10)$$

This map is an isomorphism for all $n \geq 3$ by Tate-Poitou duality; by 45, it is also an isomorphism for $n = 2$ and $i \geq 2$. Write $\tilde{H}^1(\mathcal{O}_S; \mathbb{Z}/2^\infty(i))$ for the kernel of $\alpha_S^1(i)$.

81 Lemma 81 The map $H^1(F; \mathbb{Z}/2^\infty(i)) \xrightarrow{\alpha^1(i)} (\mathbb{Z}/2)^{r_1}$ is a split surjection for all even i . Hence $H^1(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \cong (\mathbb{Z}/2)^{r_1} \oplus \tilde{H}^1(\mathcal{O}_S; \mathbb{Z}/2^\infty(i))$ for sufficiently large S .

Proof By the strong approximation theorem for units of F , the left map vertical map is a split surjection in the diagram:

$$\begin{array}{ccccc}
 F^\times/F^{\times 2} & \xrightarrow{\cong} & H^1(F, \mathbb{Z}/2) & \longrightarrow & H^1(F, \mathbb{Z}/2^\infty(i)) \\
 \text{onto} \downarrow \oplus \sigma & & \downarrow & & \downarrow \alpha^1(i) \\
 (\mathbb{Z}/2)^{r_1} = \oplus \mathbb{R}^\times/\mathbb{R}^{\times 2} & \xrightarrow{\cong} & \oplus H^1(\mathbb{R}, \mathbb{Z}/2) & \xrightarrow{\cong} & \oplus H^1(\mathbb{R}, \mathbb{Z}/2^\infty(i)) .
 \end{array}$$

Since $F^\times/F^{\times 2}$ is the direct limit (over S) of the groups $\mathcal{O}_S^\times/\mathcal{O}_S^{\times 2}$, we may replace F by $\mathcal{O}_S$ for sufficiently large S .

We also write $A \rtimes B$ for an abelian group extension of B by A .

Theorem 82 ([51, 6.9]) Let F be a real number field, and let $R = \mathcal{O}_S$ be a ring of S -integers in F containing $\mathcal{O}_F[\frac{1}{2}]$. Then $\alpha_S^1(i)$ is onto when $i = 4k > 0$, and:

82

$$K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty) \cong \begin{cases} \mathbb{Z}/w_{4k}(F) & \text{for } n = 8a , \\ H^1(\mathcal{O}_S; \mathbb{Z}/2^\infty(4k+1)) & \text{for } n = 8a+1 , \\ \mathbb{Z}/2 & \text{for } n = 8a+2 , \\ H^1(\mathcal{O}_S; \mathbb{Z}/2^\infty(4k+2)) & \text{for } n = 8a+3 , \\ \mathbb{Z}/2w_{4k+2} \oplus (\mathbb{Z}/2)^{r_1-1} & \text{for } n = 8a+4 , \\ (\mathbb{Z}/2)^{r_1-1} \rtimes H^1(\mathcal{O}_S; \mathbb{Z}/2^\infty(4k+3)) & \text{for } n = 8a+5 , \\ 0 & \text{for } n = 8a+6 , \\ \tilde{H}^1(\mathcal{O}_S; \mathbb{Z}/2^\infty(4k+4)) & \text{for } n = 8a+7 . \end{cases}$$

Proof The morphism of spectral sequences (5.3), from that for $\mathcal{O}_S$ to the sum of r_1 copies of that for $\mathbb{R}$, is an isomorphism on $E_2^{p,q}$ except on the diagonal $p = q$ (where it is an injection) and $p = q+1$ (where we must show it is a surjection). When $p \equiv +1 \pmod{4}$, it follows from 79 that we may identify $d_2^{p,q}$ with α_S^{p-q} . Hence $d_2^{p,q}$ is an isomorphism if $p \geq 2+q$, and an injection if $p = q$. As in 79, the spectral sequence degenerates at E_3 , yielding $K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty)$ as proclaimed, except for two points: (a) the extension of $\mathbb{Z}/w_{4a+2}$ by $\mathbb{Z}/2^{r_1}$ when $n = 8a+4$ is seen to be nontrivial by comparison with the extension for $\mathbb{R}$, and (b) when $n = 8a+6$ it only shows that $K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty)$ is the cokernel of $\alpha_S^1(4a+4)$.

To resolve (b) we must show that $\alpha_S^1(4a+4)$ is onto when $a > 0$. Set $n = 8a+6$. Since $K_n(\mathcal{O}_S)$ is finite, $K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty)$ must equal the 2-primary subgroup of $K_{n-1}(\mathcal{O}_S)$, which is independent of S by Theorem 7. But for sufficiently large S , the map $\alpha^1(4a+4)$ is a surjection by 81, and hence $K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty) = 0$.

Proof of Theorem 1.

Let $n > 0$ be odd. By 6 and 7, it suffices to determine the torsion subgroup of $K_n(\mathcal{O}_S) = K_n(F)$. Since $K_{n+1}(\mathcal{O}_S)$ is finite, it follows that $K_{n+1}(\mathcal{O}_S; \mathbb{Z}/\ell^\infty)$ is the ℓ -primary subgroup of $K_n(\mathcal{O}_S)$. By 73, we may assume F has a real embedding. By 70, we need only worry about the 2-primary torsion, which we can read off from 82, recalling from 22(b) that $w_i^{(2)}(F) = 2$ for odd i .

To proceed further, we need to introduce the narrow Picard group and the signature defect of the ring $\mathcal{O}_S$.

83

Definition 83: (*Narrow Picard group.*) Each real embedding $\sigma_i: F \rightarrow \mathbb{R}$ determines a map $F^\times \rightarrow \mathbb{R}^\times \rightarrow \mathbb{Z}/2$, detecting the sign of units of F under that embedding. The sum of these maps is the *sign map* $\sigma: F^\times \rightarrow (\mathbb{Z}/2)^{r_1}$. The approximation theorem for F implies that σ is surjective. The group $F_+^\times$ of *totally positive units* in F is defined to be the kernel of σ .

Now let $R = \mathcal{O}_S$ be a ring of integers in F . The kernel of $\sigma|_R: R^\times \rightarrow F^\times \rightarrow (\mathbb{Z}/2)^{r_1}$ is the subgroup $R_+^\times$ of totally positive units in R . Since the sign map $\sigma|_R$ factors through $F^\times/2 = H^1(F, \mathbb{Z}/2)$, it also factors through $\alpha^1: H^1(R, \mathbb{Z}/2) \rightarrow (\mathbb{Z}/2)^{r_1}$. The *signature defect* $j(R)$ of R is defined to be the dimension of the cokernel of α^1 ; $0 \leq j(R) < r_1$ because $\sigma(-1) \neq 0$. Note that $j(F) = 0$, and that $j(R) \leq j(\mathcal{O}_F)$.

By definition, the *narrow Picard group* $\text{Pic}_+(R)$ is the cokernel of the restricted divisor map $F_+^\times \rightarrow \bigoplus_{\sigma \notin S} \mathbb{Z}$. (See [10, 5.2.7]. This definition is due to Weber; $\text{Pic}_+(\mathcal{O}_S)$ is also called the *ray class group* Cl_F^S ; see [45, VI.1].) The kernel of the restricted divisor map is clearly $R_+^\times$, and it is easy to see from this that there is an exact sequence

$$0 \rightarrow R_+^\times \rightarrow R^\times \xrightarrow{\sigma} (\mathbb{Z}/2)^{r_1} \rightarrow \text{Pic}_+(R) \rightarrow \text{Pic}(R) \rightarrow 0.$$

A diagram chase (performed in [51, 7.6]) shows that there is an exact sequence

$$0 \rightarrow \tilde{H}^1(R; \mathbb{Z}/2) \rightarrow H^1(R; \mathbb{Z}/2) \xrightarrow{\alpha^1} (\mathbb{Z}/2)^{r_1} \rightarrow \text{Pic}_+(R)/2 \rightarrow \text{Pic}(R)/2 \rightarrow 0. \quad (5.11)$$

($\tilde{H}^1(R; \mathbb{Z}/2)$ is defined as the kernel of α^1 .) Thus the signature defect $j(R)$ is also the dimension of the kernel of $\text{Pic}_+(R)/2 \rightarrow \text{Pic}(R)/2$. If we let t and u denote the dimensions of $\text{Pic}(R)/2$ and $\text{Pic}_+(R)/2$, respectively, then this means that $u = t + j(R)$.

If s denotes the number of finite places of $R = \mathcal{O}_S$, then $\dim H^1(R; \mathbb{Z}/2) = r_1 + r_2 + s + t$ and $\dim H^2(R; \mathbb{Z}/2) = r_1 + s + t - 1$. This follows from (5.5) and (5.6), using Kummer theory. As in (5.10) and (5.11), define $\tilde{H}^n(R; \mathbb{Z}/2)$ to be the kernel of $\alpha^n: H^n(R; \mathbb{Z}/2) \rightarrow H^n(\mathbb{R}; \mathbb{Z}/2)^{r_1} \cong (\mathbb{Z}/2)^{r_1}$.

Lemma 84 Suppose that $\frac{1}{2} \in R$. Then $\dim \tilde{H}^1(R; \mathbb{Z}/2) = r_2 + s + u$. Moreover, the map $\alpha^2: H^2(R; \mathbb{Z}/2) \rightarrow (\mathbb{Z}/2)^{r_1}$ is onto, and $\dim \tilde{H}^2(R; \mathbb{Z}/2) = t + s - 1$. 84

Proof The first assertion is immediate from (5.11). Since $H^2(R; \mathbb{Z}/2^\infty(3)) \cong (\mathbb{Z}/2)^{r_1}$ by 45, the coefficient sequence for $\mathbb{Z}/2 \subset \mathbb{Z}/2^\infty(3)$ shows that $H^2(R; \mathbb{Z}/2) \rightarrow H^2(R; \mathbb{Z}/2^\infty(3))$ is onto. The final two assertions follow.

Theorem 85 Let F be a real number field, and $\mathcal{O}_S$ a ring of integers containing $\frac{1}{2}$. If $j = j(\mathcal{O}_S)$ is the signature defect, then the mod 2 algebraic K-groups of $\mathcal{O}_S$ are given (up to extensions) for $n > 0$ as follows: 85

$$K_n(\mathcal{O}_S; \mathbb{Z}/2) \cong \begin{cases} \tilde{H}^2(\mathcal{O}_S; \mathbb{Z}/2) \oplus \mathbb{Z}/2 & \text{for } n = 8a, \\ H^1(\mathcal{O}_S; \mathbb{Z}/2) & \text{for } n = 8a + 1, \\ H^2(\mathcal{O}_S; \mathbb{Z}/2) \rtimes \mathbb{Z}/2 & \text{for } n = 8a + 2, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes H^1(\mathcal{O}_S; \mathbb{Z}/2) & \text{for } n = 8a + 3, \\ (\mathbb{Z}/2)^j \rtimes H^2(\mathcal{O}_S; \mathbb{Z}/2) & \text{for } n = 8a + 4, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes \tilde{H}^1(\mathcal{O}_S; \mathbb{Z}/2) & \text{for } n = 8a + 5, \\ (\mathbb{Z}/2)^j \oplus \tilde{H}^2(\mathcal{O}_S; \mathbb{Z}/2) & \text{for } n = 8a + 6, \\ \tilde{H}^1(\mathcal{O}_S; \mathbb{Z}/2) & \text{for } n = 8a + 7. \end{cases}$$

Table 5.3. The mod 2 spectral sequence for $\mathcal{O}_S$

			1
		β_1	H^1
	0	H^1	H^2
0	$\tilde{H}^1$	H^2	$(\mathbb{Z}/2)^{r_1-1}$
$\tilde{H}^1$	$\tilde{H}^2$	$(\mathbb{Z}/2)^{r_1-1}$	$(\mathbb{Z}/2)^j$
$\tilde{H}^2$	0	$(\mathbb{Z}/2)^j$	0
0	0	0	0

The first 4 columns ($-3 \leq p \leq 0$) of $E_3 = E_\infty$

Proof (cf. [51, 7.8]) As in the proof of Theorem 82, we compare the spectral sequence for $R = \mathcal{O}_S$ with the sum of r_1 copies of the spectral sequence for $\mathbb{R}$. For $n \geq 3$ we have $H^n(R; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{r_1}$. It is not hard to see that we may identify the differentials $d_2: H^n(R, \mathbb{Z}/2) \rightarrow H^{n+3}(R, \mathbb{Z}/2)$ with the maps α^n . Since these maps are described in 84, we see from Remark 2 that the columns $p \leq 0$ of E_3 are 4-periodic, and all nonzero entries are described by Table 5.3. (By the spectral sequence (5.3), there is only one nonzero entry for $p > 0$, $E_3^{+1, -1} = \text{Pic}(R)/2$, and it is only important for $n = 0$.) By inspection, $E_3 = E_\infty$, yielding the desired description of the groups $K_n(R, \mathbb{Z}/2)$ in terms of extensions. We omit the proof that the extensions split if $n \equiv 0, 6 \pmod{8}$.

The case $F = \mathbb{Q}$ has historical importance, because of its connection with the image of J (see Example 29 or [50]) and classical number theory. The following result was first established in [76]; the groups are not truly periodic only because the order of $K_{8a-1}(\mathbb{Z})$ depends upon a .

86

Corollary 86 For $n \geq 0$, the 2-primary subgroups of $K_n(\mathbb{Z})$ and $K_2(\mathbb{Z}[1/2])$ are essentially periodic, of period eight, and are given by the following table. (When $n \equiv 7 \pmod{8}$, we set $a = (n+1)/8$.)

$n \pmod{8}$	1	2	3	4	5	6	7	8
$K_n(\mathbb{Z})\{2\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	0	0	0	$\mathbb{Z}/16a$	0

In particular, $K_n(\mathbb{Z})$ and $K_n(\mathbb{Z}[1/2])$ have odd order for all $n \equiv 4, 6, 8 \pmod{8}$, and the finite group $K_{8a+2}(\mathbb{Z})$ is the sum of $\mathbb{Z}/2$ and a finite group of odd order. We will say more about the odd torsion in the next section.

Proof When n is odd, this is Theorem 1; $w_{4a}^{(2)}$ is the 2-primary part of $16a$ by 22(c). Since $s = 1$ and $t = u = 0$, we see from 84 that $\dim \tilde{H}^1(\mathbb{Z}[1/2]; \mathbb{Z}/2) = 1$ and that $\tilde{H}^2(\mathbb{Z}[1/2]; \mathbb{Z}/2) = 0$. By 85, the groups $K_n(\mathbb{Z}[1/2]; \mathbb{Z}/2)$ are periodic of orders 2, 4, 4, 4, 2, 2, 1, 2 for $n \equiv 0, 1, \dots, 7$ respectively. The groups $K_n(\mathbb{Z}[1/2])$ for n odd, given in Theorem 1, together with the $\mathbb{Z}/2$ summand in $K_{8a+2}(\mathbb{Z})$ provided by topology (see 29), account for all of $K_n(\mathbb{Z}[1/2]; \mathbb{Z}/2)$, and hence must contain all of the 2-primary torsion in $K_n(\mathbb{Z}[1/2])$.

Recall that the 2-rank of an abelian group A is the dimension of the vector space $\text{Hom}(\mathbb{Z}/2, A)$. We have already seen (in either Theorem 1 or 82) that for $n \equiv 1, 3, 5, 7 \pmod{8}$ the 2-ranks of $K_n(\mathcal{O}_S)$ are: 1, r_1 , 0 and 1, respectively.

87

Corollary 87 For $n \equiv 2, 4, 6, 8 \pmod{8}$, $n > 0$, the respective 2-ranks of the finite groups $K_n(\mathcal{O}_S)$ are: $r_1 + s + t - 1$, $j + s + t - 1$, $j + s + t - 1$ and $s + t - 1$.

Proof (cf. [51, 0.7]) Since $K_n(R; \mathbb{Z}/2)$ is an extension of $\text{Hom}(\mathbb{Z}/2, K_{n-1}R)$ by $K_n(R)/2$, and the dimensions of the odd groups are known, we can read this off from the list given in Theorem 85.

Example 88. Consider $F = \mathbb{Q}(\sqrt{p})$, where p is prime. When $p \equiv 1 \pmod{8}$, it is well known that $t = j = 0$ but $s = 2$. It follows that $K_{8a+2}(\mathcal{O}_F)$ has 2-rank 3, while the two-primary summand of $K_n(\mathcal{O}_F)$ is nonzero and cyclic when $n \equiv 4, 6, 8 \pmod{8}$.

When $p \equiv 7 \pmod{8}$, we have $j = 1$ for both $\mathcal{O}_F$ and $R = \mathcal{O}_F[1/2]$. Since $r_1 = 2$ and $s = 1$, the 2-ranks of the finite groups $K_n(R)$ are: $t + 2, t + 1, t + 1$ and t for $n \equiv 2, 4, 6, 8 \pmod{8}$ by 87. For example, if $t = 0$ ($\text{Pic}(R)/2 = 0$) then $K_n(R)$ has odd order for $n \equiv 8 \pmod{8}$, but the 2-primary summand of $K_n(R)$ is $(\mathbb{Z}/2)^2$ when $n \equiv 2$ and is cyclic when $n \equiv 4, 6$.

Example 89. (2-regular fields.) A number field F is said to be 2-regular if there is only one prime over 2 and the narrow Picard group $\text{Pic}_+(\mathcal{O}_F[\frac{1}{2}])$ is odd (i.e., $t = u = 0$ and $s = 1$). In this case, we see from 87 that $K_{8a+2}(\mathcal{O}_F)$ is the sum of $(\mathbb{Z}/2)^{r_1}$ and a finite odd group, while $K_n(\mathcal{O}_F)$ has odd order for all $n \equiv 4, 6, 8 \pmod{8}$ ($n > 0$). In particular, the map $K_4^M(F) \rightarrow K_4(F)$ must be zero, since it factors through the odd order group $K_4(\mathcal{O}_F)$, and $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$.

Browkin and Schinzel [8] and Rognes and Østvær [54] have studied this case. For example, when $F = \mathbb{Q}(\sqrt{m})$ and $m > 0$ ($r_1 = 2$), the field F is 2-regular exactly when $m = 2$, or $m = p$ or $m = 2p$ with $p \equiv 3, 5 \pmod{8}$ prime. (See [8].)

A useful example is $F = \mathbb{Q}(\sqrt{2})$. Note that the Steinberg symbols $\{-1, -1, -1, -1\}$ and $\{-1, -1, -1, 1 + \sqrt{2}\}$ generating $K_4^M(F) \cong (\mathbb{Z}/2)^2$ must both vanish in $K_4(\mathbb{Z}[\sqrt{2}])$, which we have seen has odd order. This is the case $j = \varrho = 0$ of the following result.

Corollary 90 Let F be a real number field. Then the rank ϱ of the image of $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$ in $K_4(F)$ satisfies $j(\mathcal{O}_F[1/2]) \leq \varrho \leq r_1 - 1$. The image $(\mathbb{Z}/2)^\varrho$ lies in the subgroup $K_4(\mathcal{O}_F)$ of $K_4(F)$, and its image in $K_4(\mathcal{O}_S)/2$ has rank $j(\mathcal{O}_S)$ for all $\mathcal{O}_S$ containing $1/2$. In particular, the image $(\mathbb{Z}/2)^\varrho$ lies in $2 \cdot K_4(F)$.

90

Proof By 10, we have $\varrho < r_1 = \text{rank } K_4^M(F)$. The assertion that $K_4^M(F) \rightarrow K_4(F)$ factors through $K_4(\mathcal{O}_F)$ follows from 8, by multiplying $K_3^M(F)$ and $K_3(\mathcal{O}_F) \cong K_3(F)$ by $[-1] \in K_1(\mathbb{Z})$. It is known [16, 15.5] that the edge map $H^n(F, \mathbb{Z}(n)) \rightarrow K_n(F)$ in the motivic spectral sequence agrees with the usual map $K_n^M(F) \rightarrow K_n(F)$. By Voevodsky's theorem, $K_n^M(F)/2^v \cong H^n(F, \mathbb{Z}(n))/2^v \cong H^n(F, \mathbb{Z}/2^v(n))$. For $n = 4$, the image of the edge map $H^4(\mathcal{O}_S, \mathbb{Z}/2^v(4)) \cong H^4(F, \mathbb{Z}/2^v(4)) \rightarrow K_4(\mathcal{O}_S; \mathbb{Z}/2)$ has rank j by table 5.3; this implies the assertion that the image in $K_4(\mathcal{O}_S)/2 \subset K_4(\mathcal{O}_S; \mathbb{Z}/2)$ has rank $j(\mathcal{O}_S)$. Finally, taking $\mathcal{O}_S = \mathcal{O}_F[1/2]$ yields the inequality $j(\mathcal{O}_S) \leq \varrho$.

Example 91. ($\varphi = 1$) Consider $F = \mathbb{Q}(\sqrt{7})$, $\mathcal{O}_F = \mathbb{Z}[\sqrt{7}]$ and $R = \mathcal{O}_F[1/2]$; here $s = 1$, $t = 0$ and $j(R) = \varphi = 1$ (the fundamental unit $u = 8 + 3\sqrt{7}$ is totally positive). Hence the image of $K_4^M(F) \cong (\mathbb{Z}/2)^2$ in $K_4(\mathbb{Z}[\sqrt{7}])$ is $\mathbb{Z}/2$ on the symbol $\sigma = \{-1, -1, -1, \sqrt{7}\}$, and this is all of the 2-primary torsion in $K_4(\mathbb{Z}[\sqrt{7}])$ by 87.

On the other hand, $\mathcal{O}_S = \mathbb{Z}[\sqrt{7}, 1/7]$ still has $\varphi = 1$, but now $j = 0$, and the 2-rank of $K_4(\mathcal{O}_S)$ is still one by 87. Hence the extension $0 \rightarrow K_4(\mathcal{O}_F) \rightarrow K_4(\mathcal{O}_S) \rightarrow \mathbb{Z}/48 \rightarrow 0$ of Theorem 7 cannot be split, implying that the 2-primary subgroup of $K_4(\mathcal{O}_S)$ must then be $\mathbb{Z}/32$.

In fact, the nonzero element σ is divisible in $K_4(F)$. This follows from the fact that if $p \equiv 3 \pmod{28}$ then there is an irreducible $q = a + b\sqrt{7}$ whose norm is $-p = q\bar{q}$. Hence $R' = \mathbb{Z}[\sqrt{7}, 1/2q]$ has $j(R') = 0$ but $\varphi = 1$, and the extension $0 \rightarrow K_4(\mathcal{O}_F) \rightarrow K_4(\mathcal{O}_S) \rightarrow \mathbb{Z}/(p^2 - 1) \rightarrow 0$ of Theorem 7 is not split. If in addition $p \equiv -1 \pmod{2^\nu}$ – there are infinitely many such p for each ν – then there is an element ν of $K_4(R')$ such that $2^{\nu+1}\nu = \sigma$. See [73] for details.

92

Question 92 Can φ be less than the minimum of $r_1 - 1$ and $j + s + t - 1$?

As in (5.10), when i is even we define $\tilde{H}^2(R; \mathbb{Z}_2(i))$ to be the kernel of $\alpha^2(i): H^2(R; \mathbb{Z}_2(i)) \rightarrow H^2(\mathbb{R}; \mathbb{Z}_2(i))^{r_1} \cong (\mathbb{Z}/2)^{r_1}$. By 84, $\tilde{H}^2(R; \mathbb{Z}_2(i))$ has 2-rank $s + t - 1$.

93

Theorem 93 ([51, 0.6]) Let F be a number field with at least one real embedding, and let $R = \mathcal{O}_S$ denote a ring of integers in F containing $1/2$. Let j be the signature defect of R , and write w_i for $w_i^{(2)}(F)$.

Then there is an integer φ , $j \leq \varphi < r_1$, such that, for all $n \geq 2$, the two-primary subgroup $K_n(\mathcal{O}_S)\{2\}$ of $K_n(\mathcal{O}_S)$ is isomorphic to:

$$K_n(\mathcal{O}_S)\{2\} \cong \begin{cases} H_{\text{ét}}^2(R; \mathbb{Z}_2(4a+1)) & \text{for } n = 8a, \\ \mathbb{Z}/2 & \text{for } n = 8a+1, \\ H_{\text{ét}}^2(R; \mathbb{Z}_2(4a+2)) & \text{for } n = 8a+2, \\ (\mathbb{Z}/2)^{r_1-1} \oplus \mathbb{Z}/2w_{4a+2} & \text{for } n = 8a+3, \\ (\mathbb{Z}/2)^\varphi \rtimes H_{\text{ét}}^2(R; \mathbb{Z}_2(4a+3)) & \text{for } n = 8a+4, \\ 0 & \text{for } n = 8a+5, \\ \tilde{H}_{\text{ét}}^2(R; \mathbb{Z}_2(4a+4)) & \text{for } n = 8a+6, \\ \mathbb{Z}/w_{4a+4} & \text{for } n = 8a+7. \end{cases}$$

Proof When $n = 2i - 1$ is odd, this is Theorem 1, since $w_i^{(2)}(F) = 2$ when $n \equiv 1 \pmod{4}$ by 22(b). When $n = 2$ it is Theorem 4. To determine the two-primary

subgroup $K_n(\mathcal{O}_S)\{2\}$ of the finite group $K_{2i+2}(\mathcal{O}_S)$ when $n = 2i + 2$, we use the universal coefficient sequence

$$0 \rightarrow (\mathbb{Z}/2^\infty)^r \rightarrow K_{2i+3}(\mathcal{O}_S; \mathbb{Z}/2^\infty) \rightarrow K_{2i+2}(\mathcal{O}_S)\{2\} \rightarrow 0,$$

where r is the rank of $K_{2i+3}(\mathcal{O}_S)$ and is given by Theorem 6 ($r = r_1 + r_2$ or r_2). To compare this with Theorem 82, we note that $H^1(\mathcal{O}_S, \mathbb{Z}/2^\infty(i))$ is the direct sum of $(\mathbb{Z}/2^\infty)^r$ and a finite group, which must be $H^2(\mathcal{O}_S, \mathbb{Z}_2(i))$ by universal coefficients; (see [51, 2.4(b)]). Since $\alpha_S^1(i) : H^1(R; \mathbb{Z}/2^\infty(i)) \rightarrow (\mathbb{Z}/2)^{r_1}$ must vanish on the divisible subgroup $(\mathbb{Z}/2^\infty)^r$, it induces the natural map $\alpha_S^2(i) : H_{\text{ét}}^2(\mathcal{O}_S; \mathbb{Z}_2(i)) \rightarrow (\mathbb{Z}/2)^{r_1}$ and

$$\tilde{H}^1(\mathcal{O}_S, \mathbb{Z}/2^\infty(i)) \cong (\mathbb{Z}/2^\infty)^r \oplus \tilde{H}^2(\mathcal{O}_S, \mathbb{Z}_2(i)).$$

This proves all of the theorem, except for the description of $K_n(\mathcal{O}_S)$, $n = 8a + 4$. By mod 2 periodicity (Remark 2) the integer φ of Corollary 90 equals the rank of the image of $H^4(\mathcal{O}_S, \mathbb{Z}/2(4)) \cong H^4(\mathcal{O}_S, \mathbb{Z}/2(4k + 4)) \cong (\mathbb{Z}/2)^{r_1}$ in $\text{Hom}(\mathbb{Z}/2, K_n(\mathcal{O}_S))$, considered as a quotient of $K_{n+1}(\mathcal{O}_S; \mathbb{Z}/2)$.

We can combine the 2-primary information in 93 with the odd torsion information in 70 and 77 to relate the orders of K -groups to the orders of étale cohomology groups. Up to a factor of 2^{r_1} , they were conjectured by Lichtenbaum in [34]. Let $|A|$ denote the order of a finite abelian group A .

Theorem 94 Let F be a totally real number field, with r_1 real embeddings, and let $\mathcal{O}_S$ be a ring of integers in F . Then for all even $i > 0$

94

$$2^{r_1} \cdot \frac{|K_{2i-2}(\mathcal{O}_S)|}{|K_{2i-1}(\mathcal{O}_S)|} = \frac{\prod_\ell |H_{\text{ét}}^2(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i))|}{\prod_\ell |H_{\text{ét}}^1(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i))|}.$$

Proof (cf. proof of 77) Since $2i - 1 \equiv 3 \pmod{4}$, all groups involved are finite (see 6, 42 and 46.) Write $h^{n,i}(\ell)$ for the order of $H_{\text{ét}}^n(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i))$. By 46, $h^{1,i}(\ell) = w_i^{(\ell)}(F)$. By 1, the ℓ -primary subgroup of $K_{2i-1}(\mathcal{O}_S)$ has order $h^{1,i}(\ell)$ for all odd ℓ and all even $i > 0$, and also for $\ell = 2$ with the exception that when $2i - 1 \equiv 3 \pmod{8}$ then the order is $2^{r_1} h^{1,i}(2)$.

By 70 and 93, the ℓ -primary subgroup of $K_{2i-2}(\mathcal{O}_S)$ has order $h^{2,i}(\ell)$ for all ℓ , except when $\ell = 2$ and $2i - 2 \equiv 6 \pmod{8}$ when it is $h^{1,i}(2)/2^{r_1}$. Combining these cases yields the formula asserted by the theorem.

Corollary 95 For $R = \mathbb{Z}$, the formula conjectured by Lichtenbaum in [34] holds up to exactly one factor of 2. That is, for $k \geq 1$,

95

$$\frac{|K_{4k-2}(\mathbb{Z})|}{|K_{4k-1}(\mathbb{Z})|} = \frac{B_k}{4k} = \frac{(-1)^k}{2} \zeta(1 - 2k).$$

Moreover, if c_k denotes the numerator of $B_k/4k$, then

$$|K_{4k-2}(\mathbb{Z})| = \begin{cases} c_k, & k \text{ even} \\ 2c_k, & k \text{ odd} . \end{cases}$$

Proof The equality $B_k/4k = (-1)^k \zeta(1-2k)/2$ comes from 25. By 77, the formula holds up to a factor of 2. By 27, the two-primary part of $B_k/4k$ is $1/w_{2k}^{(2)}$. By 22(c), this is also the two-primary part of $1/8k$. By 86, the two-primary part of the left-hand side of 95 is $2/16$ when k is odd, and the two-primary part of $1/8k$ when $k = 2a$ is even.

Example 96. ($K_{4k-2}(\mathbb{Z})$) The group $K_{4k-2}(\mathbb{Z})$ is cyclic of order c_k or $2c_k$ for all $k \leq 5000$. For small k we need only consult Example 24 to see that the groups $K_2(\mathbb{Z})$, $K_{10}(\mathbb{Z})$, $K_{18}(\mathbb{Z})$ and $K_{26}(\mathbb{Z})$ are isomorphic to $\mathbb{Z}/2$. We also have $K_6(\mathbb{Z}) = K_{14}(\mathbb{Z}) = 0$. (The calculation of $K_6(\mathbb{Z})$ up to 3-torsion was given in [15].) However, $c_6 = 691$, $c_8 = 3617$, $c_9 = 43\,867$ and $c_{13} = 657\,931$ are all prime, so we have $K_{22}(\mathbb{Z}) \cong \mathbb{Z}/691$, $K_{30}(\mathbb{Z}) \cong \mathbb{Z}/3617$, $K_{34}(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/43\,867$ and $K_{50} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/657\,931$.

The next hundred values of c_k are squarefree: $c_{10} = 283 \cdot 617$, $c_{11} = 131 \cdot 593$, $c_{12} = 103 \cdot 2\,294\,797$, $c_{14} = 9349 \cdot 362\,903$ and $c_{15} = 1721 \cdot 1\,001\,259\,881$ are all products of two primes, while $c_{16} = 37 \cdot 683 \cdot 305\,065\,927$ is a product of 3 primes. Hence $K_{38}(\mathbb{Z}) = \mathbb{Z}/c_{10}$, $K_{42}(\mathbb{Z}) = \mathbb{Z}/2c_{11}$, $K_{46} = \mathbb{Z}/c_{12}$, $K_{54}(\mathbb{Z}) = \mathbb{Z}/c_{14}$, $K_{58}(\mathbb{Z}) = \mathbb{Z}/2c_{15}$ and $K_{62}(\mathbb{Z}) = \mathbb{Z}/c_{16} = \mathbb{Z}/37 \oplus \mathbb{Z}/683 \oplus \mathbb{Z}/305\,065\,927$.

Thus the first occurrence of the smallest irregular prime (37) is in $K_{62}(\mathbb{Z})$; it also appears as a $\mathbb{Z}/37$ summand in $K_{134}(\mathbb{Z})$, $K_{206}(\mathbb{Z})$, ..., $K_{494}(\mathbb{Z})$. In fact, there is 37-torsion in every group $K_{72a+62}(\mathbb{Z})$ (see 105 below).

For $k < 5000$, only seven of the c_k are not square-free; see [56], A090943. The numerator c_k is divisible by ℓ^2 only for the following pairs (k, ℓ) : (114, 103), (142, 37), (457, 59), (717, 271), (1646, 67) and (2884, 101). However, $K_{4k-2}(\mathbb{Z})$ is still cyclic with one $\mathbb{Z}/\ell^2$ summand in these cases. To see this, we note that $\text{Pic}(R)/\ell \cong \mathbb{Z}/\ell$ for these ℓ , where $R = \mathbb{Z}[\zeta_\ell]$. Hence $K_{4k-2}(R)/\ell \cong H^2(R, \mathbb{Z}_\ell(2k))/\ell \cong H^2(R, \mathbb{Z}/\ell(2k)) \cong \text{Pic}(R) \cong \mathbb{Z}/\ell$. The usual transfer argument now shows that $K_{4k-2}(\mathbb{Z})/\ell$ is either zero or $\mathbb{Z}/\ell$ for all k .

5.9 The Odd Torsion in $K_*(\mathbb{Z})$

We now turn to the ℓ -primary torsion in the K -theory of $\mathbb{Z}$, where ℓ is an odd prime. By 27 and 70, the odd-indexed groups $K_{2i-1}(\mathbb{Z})$ have ℓ -torsion exactly when $i \equiv 0 \pmod{\ell-1}$. Thus we may restrict attention to the groups $K_{2i}(\mathbb{Z})$, whose ℓ -primary subgroups are $H_{\text{ét}}^2(\mathbb{Z}[1/\ell]; \mathbb{Z}_\ell(i+1))$ by 70.

Our method is to consider the cyclotomic extension $\mathbb{Z}[\zeta]$ of $\mathbb{Z}$, $\zeta = e^{2\pi i/\ell}$. Because the Galois group $G = \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is cyclic of order $\ell - 1$, prime to ℓ , the usual transfer argument shows that $K_*(\mathbb{Z}) \rightarrow K_*(\mathbb{Z}[\zeta])$ identifies $K_n(\mathbb{Z}) \otimes \mathbb{Z}_\ell$ with $K_n(\mathbb{Z}[\zeta])^G \otimes \mathbb{Z}_\ell$ for all n . Because $K_n(\mathbb{Z})$ and $K_n(\mathbb{Z}[1/\ell])$ have the same ℓ -torsion (by the localization sequence), it suffices to work with $\mathbb{Z}[1/\ell]$.

Proposition 97 When ℓ is an odd regular prime there is no ℓ -torsion in $K_{2i}(\mathbb{Z})$.

97

Proof Since ℓ is regular, we saw in Example 75 that the finite group $K_{2i}(\mathbb{Z}[\zeta])$ has no ℓ -torsion. Hence the same is true for its G -invariant subgroup, and also for $K_{2i}(\mathbb{Z})$.

It follows from this and 27 that $K_{2i}(\mathbb{Z}; \mathbb{Z}/\ell)$ contains only the Bockstein representatives of the Harris–Segal summands in $K_{2i-1}(\mathbb{Z})$, and this only when $2i \equiv 0 \pmod{2\ell - 2}$.

We can also describe the algebra structure of $K_*(\mathbb{Z}; \mathbb{Z}/\ell)$ using the action of the cyclic group $G = \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ on the ring $K_*(\mathbb{Z}[\zeta]; \mathbb{Z}/\ell)$. For simplicity, let us assume that ℓ is a regular prime. It is useful to set $R = \mathbb{Z}[\zeta, 1/\ell]$ and recall from 72 that $K_* = K_*(R; \mathbb{Z}/\ell)$ is a free graded $\mathbb{Z}/\ell[\beta]$ -module on the $(\ell + 1)/2$ generators of $R^\times/\ell \in K_1(R; \mathbb{Z}/\ell)$, together with $1 \in K_0(R; \mathbb{Z}/\ell)$.

By Maschke’s theorem, $\mathbb{Z}/\ell[G] \cong \prod_{i=0}^{\ell-2} \mathbb{Z}/\ell$ is a simple ring; every $\mathbb{Z}/\ell[G]$ -module has a unique decomposition as a sum of irreducible modules. Since μ_ℓ is an irreducible G -module, it is easy to see that the irreducible G -modules are $\mu_\ell^{\otimes i}$, $i = 0, 1, \dots, \ell - 2$. The “trivial” G -module is $\mu_\ell^{\otimes \ell-1} = \mu_\ell^{\otimes 0} = \mathbb{Z}/\ell$. By convention, $\mu_\ell^{\otimes -i} = \mu_\ell^{\otimes \ell-1-i}$.

For example, the G -module $\langle \beta^i \rangle$ of $K_{2i}(\mathbb{Z}[\zeta]; \mathbb{Z}/\ell)$ generated by β^i is isomorphic to $\mu_\ell^{\otimes i}$. It is a trivial G -module only when $(\ell - 1) | i$.

If A is any $\mathbb{Z}/\ell[G]$ -module, it is traditional to decompose $A = \bigoplus A^{[i]}$, where $A^{[i]}$ denotes the sum of all G -submodules isomorphic to $\mu_\ell^{\otimes i}$.

Example 98. Set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$. It is known that the torsion free part $R^\times/\mu_\ell \cong \mathbb{Z}^{(\ell-1)/2}$ of the units of R is isomorphic as a G -module to $\mathbb{Z}[G] \otimes_{\mathbb{Z}[c]} \mathbb{Z}$, where c is complex conjugation. (This is sometimes included as part of Dirichlet’s theorem on units.) It follows that as a G -module,

$$H_{\text{ét}}^1(R, \mu_\ell) = R^\times/R^{\times\ell} \cong \mu_\ell \oplus (\mathbb{Z}/\ell) \oplus \mu_\ell^{\otimes 2} \oplus \cdots \oplus \mu_\ell^{\otimes \ell-3}.$$

The root of unity ζ generates the G -submodule μ_ℓ , and the class of the unit ℓ of R generates the trivial submodule of $R^\times/R^{\times\ell}$.

Tensoring with $\mu_\ell^{\otimes i-1}$ yields the G -module decomposition of $R^\times \otimes \mu_\ell^{\otimes i-1}$. If ℓ is regular this is $K_{2i-1}(R; \mathbb{Z}/\ell) \cong H_{\text{ét}}^1(R, \mu_\ell^{\otimes i})$ by 72. If i is odd, exactly one term is $\mathbb{Z}/\ell$; if i is even, $\mathbb{Z}/\ell$ occurs only when $i \equiv 0 \pmod{\ell - 1}$.

Notation Set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$. For $i = 0, \dots, (\ell - 3)/2$, pick a generator x_i of the G -submodule of $R^\times / R^{\times \ell}$ isomorphic to $\mu_\ell^{\otimes -2i}$. The indexing is set up so that $y_i = \beta^{2i} x_i$ is a G -invariant element of $K_{4i+1}(R; \mathbb{Z}/\ell) \cong H_{\text{ét}}^1(R, \mu_\ell^{\otimes 2i+1})$. We may arrange that $x_0 = y_0$ is the unit $[\ell]$ in $K_1(R; \mathbb{Z}/\ell)$.

The elements $\beta^{\ell-1}$ of $H_{\text{ét}}^0(R, \mu_\ell^{\otimes \ell-1})$ and $v = \beta^{\ell-2}[\zeta]$ of $H_{\text{ét}}^1(R, \mu_\ell^{\otimes \ell-1})$ are also G -invariant. By abuse of notation, we shall also write $\beta^{\ell-1}$ and v , respectively, for the corresponding elements of $K_{2\ell-2}(\mathbb{Z}[1/\ell]; \mathbb{Z}/\ell)$ and $K_{2\ell-3}(\mathbb{Z}[1/\ell]; \mathbb{Z}/\ell)$.

99

Theorem 99 If ℓ is an odd regular prime then $K_* = K_*(\mathbb{Z}[1/\ell]; \mathbb{Z}/\ell)$ is a free graded module over the polynomial ring $\mathbb{Z}/\ell[\beta^{\ell-1}]$. It has $(\ell + 3)/2$ generators: $1 \in K_0$, $v \in K_{2\ell-3}$, and $y_i \in K_{4i+1}$ ($i = 0, \dots, (\ell - 3)/2$).

Similarly, $K_*(\mathbb{Z}; \mathbb{Z}/\ell)$ is a free graded module over $\mathbb{Z}/\ell[\beta^{\ell-1}]$; a generating set is obtained from the generators of K_* by replacing y_0 by $y_0 \beta^{\ell-1}$.

The submodule generated by v and $\beta^{\ell-1}$ comes from the Harris–Segal summands of $K_{2\ell-3}(\mathbb{Z})$. The submodule generated by the y 's comes from the $\mathbb{Z}$ summands in $K_{4i+1}(\mathbb{Z})$.

Proof $K_*(\mathbb{Z}[1/\ell]; \mathbb{Z}/\ell)$ is the G -invariant subalgebra of $K_*(R; \mathbb{Z}/\ell)$. Given 98, it is not very hard to check that this is just the subalgebra described in the theorem.

Example 100. When $\ell = 3$, the groups $K_* = K_*(\mathbb{Z}[1/3]; \mathbb{Z}/3)$ are 4-periodic of ranks 1, 1, 0, 1, generated by an appropriate power of β^2 times one of $\{1, [3], v\}$.

When $\ell = 5$, the groups $K_* = K_*(\mathbb{Z}[1/5]; \mathbb{Z}/5)$ are 8-periodic, with respective ranks 1, 1, 0, 0, 0, 1, 0, 1 ($*$ = 0, ..., 7), generated by an appropriate power of β^4 times one of $\{1, [5], y_1, v\}$.

Now suppose that ℓ is an irregular prime, so that $\text{Pic}(R)$ has ℓ -torsion for $R = \mathbb{Z}[\zeta, 1/\ell]$. Then $H_{\text{ét}}^1(R, \mu_\ell)$ is $R^\times / \ell \oplus {}_\ell \text{Pic}(R)$ and $H_{\text{ét}}^2(R, \mu_\ell) \cong \text{Pic}(R)/\ell$ by Kummer theory. This yields $K_*(R; \mathbb{Z}/\ell)$ by 72.

Example 101. Set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$ and $P = \text{Pic}(R)/\ell$. If ℓ is regular then $P = 0$ by definition; see 3. When ℓ is irregular, the G -module structure of P is not fully understood; see Vandiver's conjecture 102 below. However, the following arguments show that $P^{[i]} = 0$, i.e., P contains no summands isomorphic to $\mu_\ell^{\otimes i}$, for $i = 0, -1, -2, -3$.

The usual transfer argument shows that $P^G \cong \text{Pic}(\mathbb{Z}[1/\ell])/\ell = 0$. Hence P contains no summands isomorphic to $\mathbb{Z}/\ell$. By 5, we have a G -module isomorphism $(P \otimes \mu_\ell) \cong K_2(R)/\ell$. Since $K_2(R)/\ell^G \cong K_2(\mathbb{Z}[1/\ell])/\ell = 0$, $(P \otimes \mu_\ell)$ has no $\mathbb{Z}/\ell$ summands – and hence P contains no summands isomorphic to $\mu_\ell^{\otimes -1}$.

Finally, we have $(P \otimes \mu_\ell^{\otimes 2}) \cong K_4(R)/\ell$ and $(P \otimes \mu_\ell^{\otimes 3}) \cong K_6(R)/\ell$ by 73. Again, the transfer argument shows that $K_n(R)/\ell^G \cong K_n(\mathbb{Z}[1/\ell])/\ell$ for $n = 4, 6$. These groups are known to be zero by [53] and [15]; see 11. It follows that P contains no summands isomorphic to $\mu_\ell^{\otimes -2}$ or $\mu_\ell^{\otimes -3}$.

Conjecture 102: (*Vandiver's Conjecture.*) If ℓ is an irregular prime number, then the group $\text{Pic}(\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}])$ has no ℓ -torsion. Equivalently, the natural representation of $G = \text{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$ on $\text{Pic}(\mathbb{Z}[\zeta_\ell])/\ell$ is a sum of G -modules $\mu_\ell^{\otimes i}$ with i odd.

102

This means that complex conjugation c acts as multiplication by -1 on the ℓ -primary subgroup of $\text{Pic}(\mathbb{Z}[\zeta_\ell])$, because c is the unique element of G of order 2.

As partial evidence for this conjecture, we mention that Vandiver's conjecture has been verified for all primes up to 12 million; see [9]. We also know from 101 that $\mu_\ell^{\otimes i}$ does not occur as a summand of $\text{Pic}(R)/\ell$ for $i = 0, -2$.

Remark 103 The Herbrand-Ribet theorem [71, 6.17–18] states that $\ell | B_k$ if and only if $\text{Pic}(R)/\ell^{[\ell-2k]} \neq 0$. Among irregular primes < 4000 , this happens for at most 3 values of k . For example, $37 | c_{16}$ (see 96), so $\text{Pic}(R)/\ell^{[5]} = \mathbb{Z}/37$ and $\text{Pic}(R)/\ell^{[k]} = 0$ for $k \neq 5$.

103

Historical Remark 104 What we now call ‘Vandiver's conjecture’ was actually discussed by Kummer and Kronecker in 1849–1853; Harry Vandiver was not born until 1882 and made his conjecture no earlier than circa 1920. In 1849, Kronecker asked if Kummer conjectured that a certain lemma [71, 5.36] held for all p , and that therefore p never divided h^+ (i.e., Vandiver's conjecture holds). Kummer's reply [30, pp. 114–115] pointed out that the Lemma could not hold for irregular p , and then called the assertion [Vandiver's conjecture] “a theorem still to be proven.” Kummer also pointed out some of its consequences. In an 1853 letter (see [30], p. 23), Kummer wrote to Kronecker that in spite of months of effort, the assertion [Vandiver's conjecture] was still unproven.

104

For the rest of this paper, we set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$, where $\zeta_\ell^\ell = 1$.

Theorem 105 (Kurihara [31]) Let ℓ be an irregular prime number. Then the following are equivalent for every k between 1 and $(\ell - 1)/2$:

105

1. $\text{Pic}(\mathbb{Z}[\zeta])/\ell^{[-2k]} = 0$.
2. $K_{4k}(\mathbb{Z})$ has no ℓ -torsion;
3. $K_{2a(\ell-1)+4k}(\mathbb{Z})$ has no ℓ -torsion for all $a \geq 0$;
4. $H^2(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1}) = 0$.

In particular, Vandiver's conjecture for ℓ is equivalent to the assertion that $K_{4k}(\mathbb{Z})$ has no ℓ -torsion for all $k < (\ell - 1)/2$, and implies that $K_{4k}(\mathbb{Z})$ has no ℓ -torsion for all k .

Proof Set $P = \text{Pic}(R)/\ell$. By Kummer theory (see 5), $P \cong H^2(R, \mu_\ell)$ and hence $P \otimes \mu_\ell^{\otimes 2k} \cong H^2(R, \mu_\ell^{\otimes 2k+1})$ as G -modules. Taking G -invariant subgroups shows that $H^2(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1}) \cong (P \otimes \mu_\ell^{\otimes 2k})^G \cong P^{[-2k]}$. Hence (1) and (4) are equivalent.

By 71, $K_{4k}(\mathbb{Z})/\ell \cong H^2(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1})$ for all $k > 0$. Since $\mu_\ell^{\otimes b} = \mu_\ell^{\otimes a(\ell-1)+b}$ for all a and b , this shows that (2) and (3) are separately equivalent to (4).

106

Theorem 106 If Vandiver's conjecture holds for ℓ then the ℓ -primary torsion subgroup of $K_{4k-2}(\mathbb{Z})$ is cyclic for all k .

If Vandiver's conjecture holds for all ℓ , the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all k .

(We know that the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all $k < 500$, by 96.)

Proof Set $P = \text{Pic}(R)/\ell$. Vandiver's conjecture also implies that each of the "odd" summands $P^{[1-2k]} = P^{[\ell-2k]}$ of P is cyclic, and isomorphic to $\mathbb{Z}_\ell/c_k$; (see [71, 10.15]) and 44 above. Since $\text{Pic}(R) \otimes \mu_\ell^{\otimes 2k-1} \cong H^2(R, \mu_\ell^{\otimes 2k})$, taking G -invariant subgroups shows that $P^{[1-2k]} \cong H^2(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k})$. By Theorem 70, this group is the ℓ -primary torsion in $K_{4k-2}(\mathbb{Z}[1/\ell])$.

Using 24 and 27 we may write the Bernoulli number $B_k/4k$ as c_k/w_{2k} in reduced terms, with c_k odd. The following result, which follows from Theorems 1, 105 and 106, was observed independently by Kurihara [31] and Mitchell [44].

107

Corollary 107 If Vandiver's conjecture holds, then $K_n(\mathbb{Z})$ is given by Table 5.4, for all $n \geq 2$. Here k is the integer part of $1 + \frac{n}{4}$.

Table 5.4. The K -theory of $\mathbb{Z}$, assuming Vandiver's Conjecture

$n \pmod{8}$	1	2	3	4	5	6	7	8
$K_n(\mathbb{Z})$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2c_k$	$\mathbb{Z}/2w_{2k}$	0	$\mathbb{Z}$	$\mathbb{Z}/c_k$	$\mathbb{Z}/w_{2k}$	0

108

Remark 108 The elements of $K_{2i}(\mathbb{Z})$ of odd order become divisible in the larger group $K_{2i}(\mathbb{Q})$. (The assertion that an element a is divisible in A means that for every m there is an element b so that $a = mb$.) This was proven by Banaszak and Kolster for i odd (see [1], thm. 2), and for i even by Banaszak and Gajda [2, proof of prop. 8].

For example, recall from 96 that $K_{22}(\mathbb{Z}) = \mathbb{Z}/691$ and $K_{30}(\mathbb{Z}) \cong \mathbb{Z}/3617$. Banaszak observed [1] that these groups are divisible in $K_{22}(\mathbb{Q})$ and $K_{30}(\mathbb{Q})$, i.e., that the inclusions $K_{22}(\mathbb{Z}) \subset K_{22}(\mathbb{Q})$ and $K_{30}(\mathbb{Z}) \subset K_{30}(\mathbb{Q})$ do not split.

There are no divisible elements of even order in $K_{2i}(\mathbb{Q})$, because by 29 and 86 the only elements of exponent 2 in $K_{2i}(\mathbb{Z})$ are the Adams elements when $2i \equiv 2 \pmod{8}$. Divisible elements in $K_{2i}(F)$ do exist for other number fields, as we saw in 91, and are described in [73].

Let t_j and s_j be respective generators of the summand of $\text{Pic}(R)/\ell$ and $K_1(R; \mathbb{Z}/\ell)$ isomorphic to $\mu_\ell^{\otimes -j}$. The following result follows easily from 72 and 98, using the proof of 99, 105 and 106. It was originally proven in [44]; another proof is given in the article [42] in this Handbook. (The generators $s_j \beta^j$ were left out in [43, 6.13].)

Theorem 109 If ℓ is an irregular prime for which Vandiver's conjecture holds, then $K_* = K_*(\mathbb{Z}; \mathbb{Z}/\ell)$ is a free module over $\mathbb{Z}/\ell[\beta^{\ell-1}]$ on the $(\ell-3)/2$ generators y_i described in 99, together with the generators $t_j \beta^j \in K_{2j}$ and $s_j \beta^j \in K_{2j+1}$.

109

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Part II

***K*-Theory and Algebraic Geometry**

Motivic Cohomology, K -Theory
and Topological Cyclic
Homology*

II.1

Thomas Geisser

1.1	<i>Introduction</i>	195
1.2	<i>Motivic Cohomology</i>	195
	Definition	196
	Hyper-cohomology	196
	Functoriality	197
	Localization	198
	Gersten Resolution	199
	Products	199
	Affine and Projective Bundles, Blow-ups	200
	Milnor K -Theory	201
	Beilinson–Lichtenbaum Conjecture	202
	Cycle Map	202
	Rational Coefficients	203
1.3	<i>Sheaf Theoretic Properties</i>	204
	Invertible Coefficients	204
	Characteristic Coefficients	204
	Projective Bundle and Blow-up	206
	Mixed Characteristic	207
1.4	<i>K-Theory</i>	208
	Basic Properties	209
	Localization	210

* Supported in part by the NSF, and the Alfred P. Sloan Foundation

	Gersten Resolution	211
	Motivic Cohomology and K -Theory	211
	Sheaf Theoretic Properties	212
1.5	<i>Etale K-Theory and Topological Cyclic Homology</i>	213
	Continuous Hyper-cohomology.....	214
	Hyper-cohomology of K -Theory	214
	The Lichtenbaum–Quillen Conjecture	215
	Topological Cyclic Homology	216
	Comparison	218
A	<i>Appendix: Basic Intersection Theory</i>	219
	Proper Intersection	219
	Intersection with Divisors	220
	Pull-back Along a Regular Embedding.....	222
	Flat Pull-back.....	225
	Proper Push-forward.....	226
	<i>References</i>	228

Introduction

1.1

We give a survey on motivic cohomology, higher algebraic K -theory, and topological cyclic homology. We concentrate on results which are relevant for applications in arithmetic algebraic geometry (in particular, we do not discuss non-commutative rings), and our main focus lies on sheaf theoretic results for smooth schemes, which then lead to global results using local-to-global methods.

In the first half of the paper we discuss properties of motivic cohomology for smooth varieties over a field or Dedekind ring. The construction of motivic cohomology by Suslin and Voevodsky has several technical advantages over Bloch's construction, in particular it gives the correct theory for singular schemes. But because it is only well understood for varieties over fields, and does not give well-behaved étale motivic cohomology groups, we discuss Bloch's higher Chow groups. We give a list of basic properties together with the identification of the motivic cohomology sheaves with finite coefficients.

In the second half of the paper, we discuss algebraic K -theory, étale K -theory and topological cyclic homology. We sketch the definition, and give a list of basic properties of algebraic K -theory, sketch Thomason's hyper-cohomology construction of étale K -theory, and the construction of topological cyclic homology. We then give a short overview of the sheaf theoretic properties, and relationships between the three theories (in many situations, étale K -theory with p -adic coefficients and topological cyclic homology agree).

In an appendix we collect some facts on intersection theory which are needed to work with higher Chow groups. The results can be found in the literature, but we thought it would be useful to find them concentrated in one article.

Acknowledgements: We received valuable comments from L. Hesselholt, B. Kahn, M. Levine, C. Weibel, and the referee. Part of this paper was written while the author was visiting the University of Tokyo, which we thank for their hospitality.

Motivic Cohomology

1.2

The existence of a complex of sheaves whose cohomology groups are related to special values of L -functions was first conjectured by Beilinson [1], [2, §5] (for the Zariski topology) and Lichtenbaum [62, 63] (for the étale topology). Consequently, the conjectural relationship between these complexes of sheaves is called the Beilinson–Lichtenbaum conjecture. The most commonly used constructions of motivic cohomology are the ones of Bloch [5] and Suslin–Voevodsky [89, 91, 99, 100]. Bloch's higher Chow groups are defined for any scheme, but they have properties analogous to a Borel–Moore homology theory in topology. In particular, they behave like a cohomology theory only for smooth schemes over a field. Voevodsky's motivic cohomology groups has good properties for non-smooth schemes, but their basic properties are only established for schemes of finite type

over a field, and they do not give a good étale theory (étale hypercohomology vanishes with mod p -coefficients over a field of characteristic p). By a theorem of Voevodsky [102], his motivic cohomology groups agree with Bloch's higher Chow groups for smooth varieties over a field. Since we want to include varieties over Dedekind rings into our discussion, we discuss Bloch's higher Chow groups.

1.2.1 Definition

Let X be separated scheme, which is essentially of finite type over a quotient of a regular ring of finite Krull dimension (we need this condition in order to have a well-behaved concept of dimension, see the appendix). We also assume for simplicity that X is equi-dimensional, i.e. every irreducible component has the same dimension (otherwise one has to replace codimension by dimension in the following discussion). Then *Bloch's higher Chow groups* are defined as the cohomology of the following complex of abelian groups. Let Δ^r be the algebraic r -simplex $\text{Spec } \mathbb{Z}[t_0, \dots, t_r]/(1 - \sum_j t_j)$. It is non-canonically isomorphic to the affine space $\mathbb{A}_{\mathbb{Z}}^r$, but has distinguished subvarieties of codimension s , given by $t_{i_1} = t_{i_2} = \dots = t_{i_s} = 0$, $0 \leq i_1 < \dots < i_s \leq r$. These subvarieties are called faces of Δ^r , and they are isomorphic to Δ^{r-s} . The group $z^n(X, i)$ is the free abelian group generated by closed integral subschemes $Z \subset \Delta^i \times X$ of codimension n , such that for every face F of codimension s of Δ^i , every irreducible component of the intersection $Z \cap (F \times X)$ has codimension s in $F \times X$. This ensures that the intersection with a face of codimension 1 gives an element of $z^n(X, i-1)$, and we show in the appendix that taking the alternating sum of these intersections makes $z^n(X, *)$ a chain complex. Replacing a cycle in $z^n(X, *)$ by another cycle which differs by a boundary is called *moving the cycle*. We let $H^i(X, \mathbb{Z}(n))$ be the cohomology of the cochain complex with $z^n(X, 2n-i)$ in degree i . For dimension reasons it is clear from the definition that $H^i(X, \mathbb{Z}(n)) = 0$ for $i > \min\{2n, n + \dim X\}$. In particular, if X is the spectrum of a field F , then $H^i(F, \mathbb{Z}(n)) = 0$ for $i > n$. It is a conjecture of Beilinson and Soulé that $H^i(X, \mathbb{Z}(n)) = 0$ for $i < 0$.

The motivic cohomology with coefficients in an abelian group A is defined as the cohomology of the complex $A(n) := \mathbb{Z}(n) \otimes A$. In particular, motivic cohomology groups with finite coefficients fit into a long exact sequences

$$\dots \rightarrow H^i(X, \mathbb{Z}(n)) \xrightarrow{\times m} H^i(X, \mathbb{Z}(n)) \rightarrow H^i(X, \mathbb{Z}/m(n)) \rightarrow \dots$$

1.2.2 Hypercohomology

By varying X , one can view $\mathbb{Z}(n) := z^n(-, 2n - *)$ as a complex of presheaves on X . It turns out that this is in fact a complex of sheaves for the Zariski, Nisnevich and étale topology on X [5] [28, Lemma 3.1]. It is clear from the definition that there is a canonical quasi-isomorphism $\mathbb{Z}(0) \simeq \mathbb{Z}$ of complexes of Zariski-sheaves if X is integral. Since étale covers of a normal scheme are normal, the same quasi-isomorphism holds for the Nisnevich and étale topology if X is normal. For X

smooth of finite type over a field or Dedekind ring, there is a quasi-isomorphism $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$ for all three topologies. This has been shown by Bloch [5] for quasi-projective X over a field, and follows for general X by the localization below. We define $H^i(X_{\text{Zar}}, \mathbb{Z}(n))$, $H^i(X_{\text{Nis}}, \mathbb{Z}(n))$ and $H^i(X_{\text{ét}}, \mathbb{Z}(n))$ as the hyper-cohomology with coefficients in $\mathbb{Z}(n)$ for the Zariski, Nisnevich and étale topology of X , respectively. One defines motivic cohomology with coefficients in the abelian group A as the hyper-cohomology groups of $\mathbb{Z}(n) \otimes A$.

If X is of finite type over a finite field $\mathbb{F}_q$, then it is worthwhile to consider motivic cohomology groups $H^i(X_W, \mathbb{Z}(n))$ for the Weil-étale topology [29]. This is a topology introduced by Lichtenbaum [64], which is finer than the étale topology. Weil-étale motivic cohomology groups are related to étale motivic cohomology groups via the long exact sequence

$$\rightarrow H^i(X_{\text{ét}}, \mathbb{Z}(n)) \rightarrow H^i(X_W, \mathbb{Z}(n)) \rightarrow H^{i-1}(X_{\text{ét}}, \mathbb{Q}(n)) \xrightarrow{\delta} H^{i+1}(X_{\text{ét}}, \mathbb{Z}(n)) \rightarrow .$$

The map δ is the composition

$$H^{i-1}(X_{\text{ét}}, \mathbb{Q}(n)) \rightarrow H^{i-1}(X_{\text{ét}}, \mathbb{Q}[\mathbb{Z}(n)]) \xrightarrow{\cup e} H^i(X_{\text{ét}}, \mathbb{Q}[\mathbb{Z}(n)]) \xrightarrow{\beta} H^{i+1}(X_{\text{ét}}, \mathbb{Z}(n)) ,$$

where $e \in H^1(\mathbb{F}_q, \mathbb{Z}) \cong \text{Ext}_{\mathbb{Z}[\mathbb{Z}]}^1(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ is a generator, and β the Bockstein homomorphism. Consequently, the sequence breaks up into short exact sequences upon tensoring with $\mathbb{Q}$. Weil-étale motivic cohomology groups are expected to be an integral model for l -adic cohomology, and are expected to be finitely generated for smooth and projective varieties over finite fields [29].

Most of the properties of motivic cohomology which follow are due to Bloch [5] for varieties over fields, and to Levine [59] for varieties over discrete valuation rings.

Functoriality

1.2.3

We show in the appendix that a flat, equidimensional map $f : X \rightarrow Y$ induces a map of complexes $z^n(Y, *) \rightarrow z^n(X, *)$, hence a map $f^* : H^i(Y, \mathbb{Z}(n)) \rightarrow H^i(X, \mathbb{Z}(n))$. The resulting map of complexes of sheaves $f^*\mathbb{Z}(n)_Y \rightarrow \mathbb{Z}(n)_X$ induces a map on hyper-cohomology groups.

We also show in the appendix that there is a map of cycle complexes $z^n(X, *) \rightarrow z^{n-c}(Y, *)$ for a proper map $f : X \rightarrow Y$ of relative dimension c of schemes of finite type over an excellent ring (for example, over a Dedekind ring of characteristic 0, or a field). This induces a map $f_* : H^i(X, \mathbb{Z}(n)) \rightarrow H^{i-2c}(Y, \mathbb{Z}(n-c))$ and a map of sheaves $f_*\mathbb{Z}(n)_X \rightarrow \mathbb{Z}(n-c)_Y[-2c]$, which induces a map of hyper-cohomology groups if $f_* = Rf_*$.

Motivic cohomology groups are contravariantly functorial for arbitrary maps between smooth schemes over a field [5, §4], [57, II §3.5] or discrete valuation ring [59]. This requires a moving lemma, because the pull-back of cycles may not meet faces properly. Hence one needs to show that every cycle c is equivalent to a cycle c' whose pull-back does meet faces properly, and that the pull-back does not depend on the choice of c' . This moving lemma is known for affine schemes, and one can employ the Mayer-Vietoris property below to reduce to this situation by a covering of X .

1.2.4 Localization

Let X be a scheme of finite type over a discrete valuation ring, and let $i : Z \rightarrow X$ be a closed subscheme of pure codimension c with open complement $j : U \rightarrow X$. Then the exact sequence of complexes

$$0 \rightarrow z^{n-c}(Z, *) \xrightarrow{i_*} z^n(X, *) \xrightarrow{j^*} z^n(U, *)$$

gives rise to a distinguished triangle in the derived category of abelian groups [8, 58], i.e. the cokernel of j^* is acyclic. In particular, there are long exact *localization sequences* of cohomology groups

$$\cdots \rightarrow H^{i-2c}(Z, \mathbb{Z}(n-c)) \rightarrow H^i(X, \mathbb{Z}(n)) \rightarrow H^i(U, \mathbb{Z}(n)) \rightarrow \cdots$$

and the complexes $\mathbb{Z}(n)$ satisfy the *Mayer–Vietoris property*, i.e. if $X = U \cup V$ is a covering of X by two Zariski open subsets, then there is a long exact sequence

$$\cdots \rightarrow H^i(X, \mathbb{Z}(n)) \rightarrow H^i(U, \mathbb{Z}(n)) \oplus H^i(V, \mathbb{Z}(n)) \rightarrow H^i(U \cap V, \mathbb{Z}(n)) \rightarrow \cdots$$

If X is a separated noetherian scheme of finite Krull dimension, then the argument of Brown–Gersten [15, 17, 95] shows that whenever the cohomology groups $H^i(C(-))$ of a complex of presheaves C satisfies the Mayer–Vietoris property, then the cohomology groups $H^i(C(X))$ and hyper-cohomology groups $H^i(X_{\text{Zar}}, \tilde{C})$ of the associated complex of sheaves agree. Note that hyper-cohomology, i.e. an injective resolution of C , always satisfies the Mayer–Vietoris property, but the Mayer–Vietoris property is not preserved by quasi-isomorphisms. For example, $\mathbb{Z}(1)$ satisfies the Mayer–Vietoris property, but the quasi-isomorphic sheaf $\mathbb{G}_m$ does not. As a consequence of the theorem of Brown–Gersten, cohomology and hyper-cohomology of $\mathbb{Z}(n)$ agree, $H^i(X, \mathbb{Z}(n)) \cong H^i(X_{\text{Zar}}, \mathbb{Z}(n))$, and the spectral sequence for the hyper-cohomology of a complex gives

$$E_2^{s,t} = H^s(X_{\text{Zar}}, \mathcal{H}^t(\mathbb{Z}(n))) \Rightarrow H^{s+t}(X, \mathbb{Z}(n)) . \quad (1.1)$$

The argument of Brown–Gersten has been generalized by Nisnevich [75], see also [17, Thm. 7.5.2], replacing the Mayer–Vietoris property by the étale excision property. This property is satisfied by Bloch’s higher Chow groups in view of localization, hence motivic cohomology agrees with its Nisnevich hyper-cohomology $H^i(X, \mathbb{Z}(n)) \cong H^i(X_{\text{Nis}}, \mathbb{Z}(n))$, and we get a spectral sequence

$$E_2^{s,t} = H^s(X_{\text{Nis}}, \mathcal{H}^t(\mathbb{Z}(n))) \Rightarrow H^{s+t}(X, \mathbb{Z}(n)) . \quad (1.2)$$

For smooth schemes, the spectral sequences (1.1) and (1.2) are isomorphic [17]. On the other hand, if X is the node $\text{Spec } k[x, y]/(y^2 - x(x+1))$, then $H^1(X_{\text{Nis}}, \mathbb{Z}) \cong \mathbb{Z}$, but localization shows that $H^1(X, \mathbb{Z}(0)) = 0$. Hence $\mathbb{Z}(0)$ is not quasi-isomorphic to $\mathbb{Z}$ even for integral schemes.

Gersten Resolution

1.2.5

In order to study the sheaf $\mathcal{H}^t(\mathbb{Z}(n))$, one considers the spectral sequence coming from the filtration of $z^n(X, *)$ by *coniveau* [5, §10]. The complex $F^s z^n(X, *)$ is the subcomplex generated by closed integral subschemes such that the projection to X has codimension at least s . The localization property implies that the pull-back map $gr^s z^n(X, *) \rightarrow \bigoplus_{x \in X^{(s)}} z^n(k(x), *)$ is a quasi-isomorphism, where $X^{(s)}$ denotes the set of points $x \in X$ such that the closure of x has codimension s , and $k(x)$ is the residue field of x . The spectral sequence for a filtration of a complex then takes the form:

$$E_1^{s,t} = \bigoplus_{x \in X^{(s)}} H^{t-s}(k(x), \mathbb{Z}(n-s)) \Rightarrow H^{s+t}(X, \mathbb{Z}(n)) . \quad (1.3)$$

The spectral sequence degenerates at E_2 for an essentially smooth semi-local ring over a field. Hence, for X a smooth scheme over a field, the E_1 -terms and differentials gives rise to an exact sequence of Zariski sheaves, the *Gersten resolution* [5, Thm. 10.1]

$$\begin{aligned} 0 \rightarrow \mathcal{H}^t(\mathbb{Z}(n)) &\rightarrow \bigoplus_{x \in X^{(0)}} (i_x)_* H^t(k(x), \mathbb{Z}(n)) \\ &\rightarrow \bigoplus_{x \in X^{(1)}} (i_x)_* H^{t-1}(k(x), \mathbb{Z}(n-1)) \rightarrow \cdots . \end{aligned} \quad (1.4)$$

Here $(i_x)_* G$ is the skyscraper sheaf with group G at the point x . The same argument works for motivic cohomology with coefficients. Since skyscraper sheaves are flabby, one can calculate the cohomology of $\mathcal{H}^t(\mathbb{Z}(n))$ with the complex (1.4), and gets $E_2^{s,t} = H^s(X_{\text{Zar}}, \mathcal{H}^t(\mathbb{Z}(n)))$.

If X is smooth over a discrete valuation ring V , there is a conditional result. Assume that for any discrete valuation ring R , essentially of finite type over V , with quotient field K of R , the map $H^t(R, \mathbb{Z}(n)) \rightarrow H^t(K, \mathbb{Z}(n))$ is injective (this is a special case of (1.4)). Then the sequence (1.4) is exact on X [28]. The analogous statement holds with arbitrary coefficients. Since the hypothesis is satisfied with mod p -coefficients if p is the residue characteristic of V (see below), we get a Gersten resolution for $\mathcal{H}^t(\mathbb{Z}/p^r(n))$. As a corollary of the proof of (1.4), one can show [28] that the complex $\mathbb{Z}(n)$ is acyclic in degrees above n .

Products

1.2.6

For X and Y varieties over a field, there is an external product structure, see [34, Appendix] [5, Section 5] [107].

$$z^n(X, *) \otimes z^m(Y, *) \rightarrow z^{n+m}(X \times Y, *) ,$$

which induces an associative and (graded) commutative product on cohomology. If $Z_1 \subseteq X \times \Delta^i$ and $Z_2 \subseteq Y \times \Delta^j$ are generators of $z^n(X, i)$ and $z^m(Y, j)$, respectively,

then the map sends $Z_1 \otimes Z_2$ to $Z_1 \times Z_2 \subseteq X \times Y \times \Delta^i \times \Delta^j$. One then *triangulates* $\Delta^p \times \Delta^q$, i.e. covers it with a union of copies of Δ^{p+q} . The complication is to move cycles such that the pull-back along the maps $\Delta^{p+q} \rightarrow \Delta^p \times \Delta^q$ intersects faces properly. Sheafifying the above construction on $X \times Y$, we get a pairing of complexes of sheaves

$$p_X^* \mathbb{Z}(n)^X \otimes p_Y^* \mathbb{Z}(m)^Y \rightarrow \mathbb{Z}(n+m)^{X \times Y}.$$

which in turn induces a pairing of hyper-cohomology groups. For smooth X , the external product induces, via pull-back along the diagonal, an internal product

$$H^i(X, \mathbb{Z}(n)) \otimes H^j(X, \mathbb{Z}(m)) \rightarrow H^{i+j}(X, \mathbb{Z}(n+m)).$$

Over a discrete valuation ring, we do not know how to construct a product structure in general. The problem is that the product of cycles lying in the closed fiber will not have the correct codimension, and we don't know how to move a cycle in the closed fiber to a cycle which is flat over the base. Sometimes one can get by with the following construction of Levine [59]. Let B be the spectrum of a discrete valuation ring, let Y be flat over B , and consider the subcomplex $z^m(Y/B, *) \subseteq z^m(Y, *)$ generated by cycles whose intersections with all faces are equidimensional over B . A similar construction as over fields gives a product structure

$$z^n(X, *) \otimes z^m(Y/B, *) \rightarrow z^{n+m}(X \times_B Y, *).$$

This is helpful because often the cohomology classes one wants to multiply with can be represented by cycles in $z^m(Y/B, *)$.

Levine conjectures that the inclusion $z^m(Y/B, *) \subseteq z^m(Y, *)$ induces a quasi-isomorphism of Zariski sheaves. If B is the spectrum of a Dedekind ring, it would be interesting to study the cohomology groups of the complex $z^m(Y/B, *)$.

1.2.7 Affine and Projective Bundles, Blow-ups

Let X be of finite type over a field or discrete valuation ring, and let $p : E \rightarrow X$ be a flat map such that for each point $x \in X$ the fiber is isomorphic to $\mathbb{A}_{k(x)}^n$. Then the pull back map induced by the projection $E \rightarrow X$ induces a quasi-isomorphism $z^n(X, *) \rightarrow z^n(E, *)$. This was first proved by Bloch [5, Thm. 2.1] over a field, and can be generalized using localization. Note that the analogous statement for étale hyper-cohomology of the motivic complex is wrong. For example, one can see with Artin-Schreier theory that $H^2(\mathbb{A}_{\mathbb{F}_p, \text{ét}}^1, \mathbb{Z}(0))$ has a very large p -part. The localization sequence

$$\cdots \rightarrow H^{i-2}(\mathbb{P}_X^{n-1}, \mathbb{Z}(n-1)) \rightarrow H^i(\mathbb{P}_X^n, \mathbb{Z}(n)) \rightarrow H^i(\mathbb{A}_X^n, \mathbb{Z}(n)) \rightarrow \cdots$$

is split by the following composition of pull-back maps

$$H^i(\mathbb{A}_X^n, \mathbb{Z}(n)) \xleftarrow{\sim} H^i(X, \mathbb{Z}(n)) \rightarrow H^i(\mathbb{P}_X^n, \mathbb{Z}(n)).$$

This, together with induction, gives a canonical isomorphism

$$H^i(\mathbb{P}_X^m, \mathbb{Z}(n)) \cong \bigoplus_{j=0}^m H^{i-2j}(X, \mathbb{Z}(n-j)) . \quad (1.5)$$

If X' is the blow-up of the smooth scheme X along the smooth subscheme Z of codimension c , then we have the blow-up formula

$$H^i(X', \mathbb{Z}(n)) \cong H^i(X, \mathbb{Z}(n)) \oplus \bigoplus_{j=1}^{c-1} H^{i-2j}(Z, \mathbb{Z}(i-j)) . \quad (1.6)$$

The case X over a field is treated in [57, Lemma IV 3.1.1] and carries over to X over a Dedekind ring using the localization sequence.

Milnor K -Theory

1.2.8

The *Milnor K -groups* [73] of a field F are defined as the quotient of the tensor algebra of the multiplicative group of units $F^\times$ by the ideal generated by the Steinberg relation $a \otimes (1-a) = 0$,

$$K_*^M(F) = T_{\mathbb{Z}}^*(F^\times) / (a \otimes (1-a) | a \in F - \{0, 1\}) .$$

For R a regular semi-local ring over a field k , we define the Milnor K -theory of R as the kernel

$$K_n^M(R) = \ker \left(\bigoplus_{x \in R^{(0)}} K_n^M(k(x)) \xrightarrow{\delta} \bigoplus_{y \in R^{(1)}} K_{n-1}^M(k(y)) \right) . \quad (1.7)$$

Here δ_y is defined as follows [73, Lemma 2.1]. The localization V_y of R at the prime corresponding to y is a discrete valuation ring with quotient field $k(x)$ for some $x \in R^{(0)}$ and residue field $k(y)$. Choose a uniformizer $\pi \in V_y$, and if $u_j \in V_y^\times$ has reduction $\bar{u}_j \in k(y)$, set $\delta_y(\{u_1, \dots, u_n\}) = 0$ and $\delta_y(\{u_1, \dots, u_{n-1}, \pi\}) = \{\bar{u}_1, \dots, \bar{u}_{n-1}\}$. By multilinearity of symbols this determines δ_y . Since there exists a universally exact Gersten resolution for Milnor K -theory [17, Example 7.3(5)], the equation (1.7) still holds after tensoring all terms with an abelian group.

For any ring, one can define a graded ring $\overline{K}_*^M(R)$ by generators and relations as above (including the extra relation $a \otimes (-a) = 0$, which follows from the Steinberg relation if R is a field). If R is a regular semi-local ring over a field, there is a canonical map $\overline{K}_i^M(R) \rightarrow K_i^M(R)$, and Gabber proved that this map is surjective, provided the base field is infinite, see also [19].

For a field F , $H^i(F, \mathbb{Z}(n)) = 0$ for $i > n$, and in the highest degree we have the isomorphism of Nesterenko–Suslin [74, Thm. 4.9] and Totaro [98]:

$$K_n^M(F) \xrightarrow{\sim} H^n(F, \mathbb{Z}(n)) . \quad (1.8)$$

The map is given by

$$\{u_1, \dots, u_n\} \mapsto \left(\frac{-u_1}{1 - \sum u_i}, \dots, \frac{-u_n}{1 - \sum u_i}, \frac{1}{1 - \sum u_i} \right) \in (\Delta_F^n)^{(n)}. \quad (1.9)$$

If m is relatively prime to the characteristic of F , it follows from Kummer theory that $K_1^M(F)/m \cong F^\times / (F^\times)^m \cong H^1(F_{\text{ét}}, \mu_m)$. Since the cup-product on Galois cohomology satisfies the Steinberg relation [93, Thm. 3.1], we get the symbol map from Milnor K -theory to Galois cohomology

$$K_n^M(F)/m \rightarrow H^n(F_{\text{ét}}, \mu_m^{\otimes n}). \quad (1.10)$$

The *Bloch–Kato conjecture* [9] states that the symbol map is an isomorphism. Voevodsky [101] proved the conjecture for m a power of 2, and has announced a proof for general m in [103].

1.2.9 Beilinson–Lichtenbaum Conjecture

Motivic cohomology groups for the étale and Zariski topology are different. For example, for a field F , we have $H^3(F, \mathbb{Z}(1)) = 0$, but $H^3(F_{\text{ét}}, \mathbb{Z}(1)) \cong H^2(F_{\text{ét}}, \mathbb{G}_m) \cong \text{Br } F$. The *Beilinson–Lichtenbaum conjecture* states that this phenomenon only occurs in higher degrees. More precisely, let X be a smooth scheme over a field, and let $\varepsilon : X_{\text{ét}} \rightarrow X_{\text{Zar}}$ be the canonical map of sites. Then the Beilinson–Lichtenbaum conjecture states that the canonical map

$$\mathbb{Z}(n) \xrightarrow{\sim} \tau_{\leq n+1} R\varepsilon_* \mathbb{Z}(n). \quad (1.11)$$

is a quasi-isomorphism, or more concretely, that for every smooth scheme X over a field,

$$H^i(X, \mathbb{Z}(n)) \cong H^i(X_{\text{ét}}, \mathbb{Z}(n)) \quad \text{for } i \leq n+1.$$

In [91], Suslin and Voevodsky show that, assuming resolution of singularities, the Bloch–Kato conjecture (1.10) implies the Beilinson–Lichtenbaum conjecture (1.11) with mod m -coefficients; in [34] the hypothesis on resolution of singularities is removed.

1.2.10 Cycle Map

Let $H^i(X, n)$ be a bigraded cohomology theory which is the hyper-cohomology of a complex of sheaves $C(n)$; important examples are étale cohomology $H^i(X, \mu_m^{\otimes n})$ and Deligne cohomology $H_{\mathcal{D}}^i(X, \mathbb{Z}(n))$. Assume that $C(n)$ is contravariantly functorial, i.e. for $f : X \rightarrow Y$ there exists a map $f^* C(n)_X \rightarrow C(n)_Y$ in the derived category, compatible with composition. Assume furthermore that $C(n)$ admits a cycle class map $\text{CH}^n(X) \rightarrow H^{2n}(X, n)$, is homotopy invariant, and satisfies

a weak form of purity. Then Bloch constructs in [7] (see also [34]) a natural map

$$H^i(X, \mathbb{Z}(n)) \rightarrow H^i(X, n). \quad (1.12)$$

Unfortunately, this construction does not work for cohomology theories which satisfy the projective bundle formula, but are not homotopy invariant, like crystalline cohomology, de Rham cohomology or syntomic cohomology.

Rational Coefficients

1.2.11

If X is a separated noetherian scheme of finite Krull dimension, then

$$H^i(X, \mathbb{Q}(n)) \cong H^i(X_{\text{ét}}, \mathbb{Q}(n)). \quad (1.13)$$

Indeed, in view of $H^i(X, \mathbb{Z}(n)) \cong H^i(X_{\text{Nis}}, \mathbb{Z}(n))$, it suffices to observe that for any sheaf $\mathcal{F}_{\mathbb{Q}}$ of $\mathbb{Q}$ -vector spaces, $H^i(X_{\text{Nis}}, \mathcal{F}_{\mathbb{Q}}) \cong H^i(X_{\text{ét}}, \mathcal{F}_{\mathbb{Q}})$. But for any henselian local ring R with residue field k , and $i > 0$, $H^i(R_{\text{ét}}, \mathcal{F}_{\mathbb{Q}}) \cong H^i(k_{\text{ét}}, \mathcal{F}_{\mathbb{Q}}) = 0$ because higher Galois cohomology is torsion. Hence $R^i\alpha_* = 0$ for $\alpha : X_{\text{ét}} \rightarrow X_N$ is the canonical morphism of sites.

Parshin conjectured that for X smooth and projective over a finite field,

$$H^i(X, \mathbb{Q}(n)) = 0 \quad \text{for } i \neq 2n.$$

Using (1.3) and induction, this implies that for any field F of characteristic p , $H^i(F, \mathbb{Q}(n)) = 0$ for $i \neq n$. From the sequence (1.4), it follows then that for any smooth X over a field k of characteristic p , $H^i(X, \mathbb{Q}(n)) = 0$ unless $n \leq i \leq \min\{2n, n + d\}$. To give some credibility to Parshin's conjecture, one can show [26] that it is a consequence of the conjunction of the strong form of Tate's conjecture, and a conjecture of Beilinson stating that over finite fields numerical and rational equivalence agree up to torsion. The argument is inspired by Soulé [85] and goes as follows. Since the category of motives for numerical equivalence is semi-simple by Jannsen [52], we can break up X into simple motives M . By Beilinson's conjecture, Grothendieck motives for rational and numerical equivalence agree, and we can break up $H^i(X, \mathbb{Q}(n))$ correspondingly into a direct sum of $H^i(M, \mathbb{Q}(n))$. By results of Milne [72], Tate's conjecture implies that a simple motive M is characterized by the eigenvalue e_M of the Frobenius endomorphism φ_M of M . By Soule [85], φ_M acts on $H^i(M, \mathbb{Q}(n))$ as p^n , so this group can only be non-zero if $e_M = p^n$, which implies that $M \subseteq \mathbb{P}^n$. But the projective space satisfies Parshin's conjecture by the projective bundle formula (1.5).

In contrast, if K is a number field, then

$$H^1(K, \mathbb{Q}(n)) = \begin{cases} \mathbb{Q}^{r_1+r_2} & 1 < n \equiv 1 \pmod{4} \\ \mathbb{Q}^{r_2} & n \equiv 3 \pmod{4}, \end{cases}$$

where r_1 and r_2 are the number of real and complex embeddings of K , respectively.

Sheaf Theoretic Properties

In this section we discuss sheaf theoretic properties of the motivic complex.

Invertible Coefficients

Let X be a smooth scheme over k , and m prime to the characteristic k . Then there is a quasi-isomorphism

$$\mathbb{Z}/m(n)_{\text{ét}} \simeq \mu_m^{\otimes n} \quad (1.14)$$

of the étale motivic complex with mod m -coefficients. The map is the map of (1.12), and it is an isomorphism, because for a henselian local ring R of X with residue field κ , there is a commutative diagram

$$\begin{array}{ccc} H^i(R_{\text{ét}}, \mathbb{Z}/m(n)) & \rightarrow & H^i(R_{\text{ét}}, \mu_m^{\otimes n}) \\ \sim \downarrow & & \sim \downarrow \\ H^i(\kappa_{\text{ét}}, \mathbb{Z}/m(n)) & \rightarrow & H^i(\kappa_{\text{ét}}, \mu_m^{\otimes n}). \end{array}$$

The left vertical map is an isomorphism by Bloch [5, Lemma 11.1] (he assumes, but does not use, that R is strictly henselian), the right horizontal map by rigidity for étale cohomology (Gabber [22]). Finally, the lower horizontal map is an isomorphism for separably closed κ by Suslin [87].

In view of (1.14), the Beilinson–Lichtenbaum conjecture with mod m -coefficients takes the more familiar form

$$\mathbb{Z}/m(n) \xrightarrow{\sim} \tau_{\leq n} R\mathcal{E}_* \mu_m^{\otimes n},$$

or more concretely,

$$H^i(X, \mathbb{Z}/m(n)) \xrightarrow{\sim} H^i(X_{\text{ét}}, \mu_m^{\otimes n}) \quad \text{for } i \leq n.$$

One could say that motivic cohomology is completely determined by étale cohomology for $i \leq n$, whereas for $i > n$ the difference encodes deep arithmetic properties of X . For example, the above map for $i = 2n$ is the cycle map.

Characteristic Coefficients

We now consider the motivic complex mod p^r , where $p = \text{char } k$. For simplicity we assume that k is perfect. This is no serious restriction because the functors we study commute with filtered colimits of rings. Let $W(k)$ be the ring of Witt vectors of k , and K the field of quotients of $W(k)$. For a smooth variety X over k , Illusie [50], based on ideas of Bloch [4] and Deligne, defines the *de Rham–Witt pro-complex* $W.\Omega_X^*$. It generalizes Witt vectors $W.\mathcal{O}_X$, and the de Rham complex Ω_X^* , and comes equipped with operators $F : W_r\Omega_X^* \rightarrow W_{r-1}\Omega_X^*$ and $V : W_r\Omega_X^* \rightarrow W_{r+1}\Omega_X^*$, which

generalize the Frobenius and Verschiebung maps on Witt vectors. The hypercohomology of the de Rham–Witt complex calculates the crystalline cohomology $H_{crys}^*(X/W(k))$ of X , hence it can be used to analyze crystalline cohomology using the *slope spectral sequence*

$$E_1^{s,t} = H^t(X, W.\Omega^s) \Rightarrow H_{crys}^{s+t}(X/W(k)) .$$

This spectral sequence degenerates at E_1 up to torsion [50, II Thm. 3.2], and if we denote by $H_{crys}^i(X/W(k))_K^{[s,s+1]}$ the part of the F -crystal $H_{crys}^i(X/W(k)) \otimes_{W(k)} K$ with slopes in the interval $[s, s+1[$, then [50, II (3.5.4)]

$$H_{crys}^t(X/W(k))_K^{[s,s+1]} = H^{t-s}(X, W.\Omega_X^s) \otimes_{W(k)} K .$$

The (étale) *logarithmic de Rham–Witt sheaf* $v_r^n = W_r.\Omega_{X,\log}^n$ is defined as the subsheaf of $W_r.\Omega_X^n$ generated locally for the étale topology by $d \log \bar{x}_1 \wedge \cdots \wedge d \log \bar{x}_n$, where $\bar{x} \in W_r.\mathcal{O}_X$ are Teichmüller lifts of units. See [40, 69] for basic properties. For example, $v_r^0 \cong \mathbb{Z}/p^r$, and there is a short exact sequence of étale sheaves

$$0 \rightarrow \mathbb{G}_m \xrightarrow{p^r} \mathbb{G}_m \rightarrow v_r^1 \rightarrow 0 .$$

There are short exact sequences of pro-sheaves on the small étale site of X , [50, Théorème 5.7.2]

$$0 \rightarrow v_r^n \rightarrow W.\Omega_X^n \xrightarrow{F-1} W.\Omega_X^n \rightarrow 0 . \quad (1.15)$$

For a quasi-coherent sheaf (or pro-sheaf) of $\mathcal{O}_X$ -modules such as $W.\Omega_X^n$, the higher direct images $R^i \varepsilon_* W.\Omega_X^i$ of its associated étale sheaf are zero for $i > 0$. Thus we get an exact sequence of pro-Zariski-sheaves

$$0 \rightarrow v_r^n \rightarrow W.\Omega_X^n \xrightarrow{F-1} W.\Omega_X^n \rightarrow R^1 \varepsilon_* v_r^n \rightarrow 0 .$$

and $R^i \varepsilon_* v_r^n = 0$ for $i \geq 2$. By Gros and Suwa [41], the sheaves v_r^n have a Gersten resolution on smooth schemes X over k ,

$$0 \rightarrow v_r^n \rightarrow \bigoplus_{x \in X^{(0)}} (i_x)_* v_r^n(k(x)) \rightarrow \bigoplus_{x \in X^{(1)}} (i_x)_* v_r^{n-1}(k(x)) \rightarrow \cdots .$$

In particular, $H^i(X_{Zar}, v_r^n) = 0$ for $i > n$. Milnor K -theory and logarithmic de Rham–Witt sheaves of a field F of characteristic p are isomorphic by the theorem of Bloch–Gabber–Kato [9, 55]

$$d \log : K_n^M(F)/p^r \xrightarrow{\sim} v_r^n(F) . \quad (1.16)$$

Since motivic cohomology, Milnor K -theory and logarithmic de Rham–Witt cohomology all admit Gersten resolutions, we get as a corollary of (1.16) and (1.8) that for a semi-localization R of a regular k -algebra of finite type, there are isomorphisms

$$H^n(R, \mathbb{Z}/p^r(n)) \xleftarrow{\sim} K_n^M(R)/p^r \xrightarrow{\sim} v_r^n(R) .$$

Using this as a base step for induction, one can show [33] that for any field K of characteristic p ,

$$H^i(K, \mathbb{Z}/p^r(n)) = 0 \quad \text{for } i \neq n. \quad (1.17)$$

Consequently, for a smooth variety X over k , there is a quasi-isomorphism of complexes of sheaves for the Zariski (hence also the étale) topology,

$$\mathbb{Z}/p^r(n) \cong v_r^n[-n]. \quad (1.18)$$

so that

$$\begin{aligned} H^{s+n}(X, \mathbb{Z}/p^r(n)) &\cong H^s(X_{\text{Zar}}, v_r^n), \\ H^{s+n}(X_{\text{ét}}, \mathbb{Z}/p^r(n)) &\cong H^s(X_{\text{ét}}, v_r^n). \end{aligned}$$

Since $dF = pFd$ on the de Rham–Witt complex, we can define a map of truncated complexes $F : W.\Omega_X^{*\geq n} \rightarrow W.\Omega_X^{*\geq n}$ by letting $F = p^{j-n}F$ on $W.\Omega_X^j$. Since $p^jF - \text{id}$ is an automorphism on $W_r.\Omega_X^j$ for every $j \geq 1$ [50, Lemma I 3.30], the sequence (1.15) gives rise to an exact sequence of pro-complexes of étale sheaves

$$0 \rightarrow v_r^n[-n] \rightarrow W.\Omega_X^{*\geq n} \xrightarrow{F-\text{id}} W.\Omega_X^{*\geq n} \rightarrow 0.$$

The Frobenius endomorphism φ of X induces the map p^iF on $W_r.\Omega_X^i$ [50, I 2.19], hence composing with the inclusion $W.\Omega_X^{*\geq n} \rightarrow W.\Omega_X^*$ and using (1.18), we get a map to crystalline cohomology [69]

$$H^s(X, \mathbb{Z}(n)) \rightarrow H^{s-n}(X_{\text{Zar}}, v_r^n) \rightarrow H_{\text{crys}}^s(X/W_r(k))^{\varphi-p^n}.$$

This generalizes the crystalline cycle map of Gros [40].

1.3.3 Projective Bundle and Blow-up

If X is smooth over a field of characteristic p , let $\mathbb{Q}/\mathbb{Z}(n)' = \text{colim}_{p|m} \mu_m^{\otimes n}$, and for $n < 0$ define negative étale motivic cohomology to be $H^i(X_{\text{ét}}, \mathbb{Z}(n)) = H^{i-1}(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z}(n)')$. Then the formula (1.5) has the analog

$$H^i((\mathbb{P}_X^m)_{\text{ét}}, \mathbb{Z}(n)) \cong \bigoplus_{j=0}^m H^{i-2j}(X_{\text{ét}}, \mathbb{Z}(n-j)). \quad (1.19)$$

Indeed, it suffices to show this after tensoring with $\mathbb{Q}$, and with finite coefficients $\mathbb{Z}/l^r$ for all primes l . Rationally, the formula holds by (1.5) and (1.13). With $\mathbb{Z}/l^r$ -coefficients, it follows by (1.14) and (1.18) from the projective bundle formula for étale cohomology [71, Prop. VI 10.1] and logarithmic de Rham–Witt cohomology [40].

Similarly, the formula (1.6) for the blow-up X' of a smooth variety X in a smooth subscheme Z of codimension c has the analog

$$H^i(X'_{\text{ét}}, \mathbb{Z}(n)) \cong H^i(X_{\text{ét}}, \mathbb{Z}(n)) \oplus \bigoplus_{j=1}^{d-1} H^{i-2j}(Z_{\text{ét}}, \mathbb{Z}(i-j)). \quad (1.20)$$

This follows rationally from (1.6), and with finite coefficients by the proper base-change for $\mu_m^{\otimes n}$ and [40, Cor. IV 1.3.6].

Mixed Characteristic

1.3.4

If X is an essentially smooth scheme over the spectrum B of a Dedekind ring, one can show that the Bloch–Kato conjecture (1.10) implies the following sheaf theoretic properties of the motivic complex [28].

Purity: Let $i : Y \rightarrow X$ be the inclusion of one of the closed fibers. Then the map induced by adjointness from the natural inclusion map is a quasi-isomorphism

$$\mathbb{Z}(n-1)^{\text{ét}}[-2] \rightarrow \tau_{\leq n+1} R i^! \mathbb{Z}(n)^{\text{ét}}. \quad (1.21)$$

Beilinson–Lichtenbaum: The canonical map is a quasi-isomorphism

$$\mathbb{Z}(n)^{\text{Zar}} \xrightarrow{\sim} \tau_{\leq n+1} R \varepsilon_* \mathbb{Z}(n)^{\text{ét}}.$$

Rigidity: For an essentially smooth henselian local ring R over B with residue field k and $m \in k^\times$, the canonical map is a quasi-isomorphism

$$H^i(R, \mathbb{Z}/m(n)) \xrightarrow{\sim} H^i(k, \mathbb{Z}/m(n)).$$

Étale sheaf: There is a quasi-isomorphism of complexes of étale sheaves on $X \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{m}]$

$$\mathbb{Z}/m(n)^{\text{ét}} \cong \mu_m^{\otimes n}.$$

Gersten resolution: For any m , there is an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{H}^s(\mathbb{Z}/m(n)^{\text{Zar}}) &\rightarrow \bigoplus_{x \in X^{(0)}} (i_x)_* H^s(k(x), \mathbb{Z}/m(n)) \\ &\rightarrow \bigoplus_{x \in X^{(1)}} (i_x)_* H^{s-1}(k(x), \mathbb{Z}/m(n-1)) \rightarrow \cdots. \end{aligned}$$

Combining the above, one gets a Gersten resolution for the sheaf $R^s \varepsilon_* \mu_m^{\otimes n}$ for $s \leq n$, m invertible on X , and $\varepsilon : X_{\text{ét}} \rightarrow X_{\text{Zar}}$ the canonical map. This extends the result of Bloch and Ogus [11], who consider smooth schemes over a field.

If X is a smooth scheme over a discrete valuation ring V of mixed characteristic $(0, p)$ with closed fiber $i : Z \rightarrow X$ and generic fiber $j : U \rightarrow X$, and if (1.21) is a quasi-isomorphism, then the (truncated) decomposition triangle $R i^! \rightarrow i^* \rightarrow i^* R j_* j^*$ gives a distinguished triangle

$$\cdots \rightarrow i^* \mathbb{Z}/p^r(n)^{\text{ét}} \rightarrow \tau_{\leq n} i^* R j_* \mu_{p^r}^{\otimes n} \rightarrow v_r^{n-1}[-n] \rightarrow \cdots. \quad (1.22)$$

By a result of Kato and Kurihara [56], for $n < p-1$ the *syntomic complex* $S_r(n)$ of Fontaine–Messing [20] fits into a similar triangle

$$\cdots \rightarrow S_r(n) \rightarrow \tau_{\leq n} i^* R j_* \mu_{p^r}^{\otimes n} \xrightarrow{\kappa} v_r^{n-1}[-n] \rightarrow \cdots. \quad (1.23)$$

Here κ is the composition of the projection $\tau_{\leq n} i^* Rj_* \mu_{p^r}^{\otimes n} \rightarrow i^* R^n j_* \mu_{p^r}^{\otimes n}[-n]$ with the symbol map of [9, §6.6]. More precisely, $i^* R^n j_* \mu_{p^r}^{\otimes n}[-n]$ is locally generated by symbols $\{f_1, \dots, f_n\}$, for $f_i \in i^* j_* \mathcal{O}_U^\times$ by [9, Cor. 6.1.1]. By multilinearity, each such symbol can be written as a sum of symbols of the form $\{f_1, \dots, f_n\}$ and $\{f_1, \dots, f_{n-1}, \pi\}$, for $f_i \in i^* \mathcal{O}_U^\times$ and π a uniformizer of V . Then κ sends the former to zero, and the latter to $d \log \bar{f}_1 \wedge \dots \wedge d \log \bar{f}_{n-1}$, where $\bar{f}_i$ is the reduction of f_i to $\mathcal{O}_Y^\times$.

For $n \geq p-1$, we extend the definition of the syntomic complex $S_r(n)$ by defining it as the cone of the map κ . This cone has been studied by Sato [81]. Comparing the triangles (1.22) and (1.23), one can show [28] that there is a unique map

$$i^* \mathbb{Z}/p^r(n)^{\text{ét}} \rightarrow S_r(n)$$

in the derived category of sheaves on $Y_{\text{ét}}$, which is compatible with the maps of both complexes to $\tau_{\leq n} i^* Rj_* \mu_{p^r}^{\otimes n}$. The map is a quasi-isomorphism provided that the Bloch–Kato conjecture with mod p -coefficients holds. Thus motivic cohomology can be thought of as a generalization of syntomic cohomology, as anticipated by Milne [70, Remark 2.7] and Schneider [82]. As a special case, we get for smooth and projective X over V the *syntomic cycle map*

$$H^i(X, \mathbb{Z}(n)) \rightarrow H^i(X_{\text{Zar}}, S_r(n)) .$$

1.4

K-Theory

The first satisfactory construction of algebraic K -groups of schemes was the Q -construction of Quillen [79]. Given a scheme X , one starts with the category $\mathcal{P}$ of locally free $\mathcal{O}_X$ -modules of finite rank on X , and defines an intermediate category $Q\mathcal{P}$ with the same objects, and where a morphism $P \rightarrow P'$ is defined to be an isomorphism of P with a sub-quotient of P' . Any (small) category $\mathcal{C}$ gives rise to a simplicial set, the *nerve* $N\mathcal{C}$. An n -simplex of the nerve is a sequence of maps $C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n$ in $\mathcal{C}$, and the degeneracy and face maps are defined by including an identity $C_i \xrightarrow{\text{id}} C_i$, and contracting $C_i \xrightarrow{f_i} C_{i+1} \xrightarrow{f_{i+1}} C_{i+2}$ to $C_i \xrightarrow{f_{i+1} \circ f_i} C_{i+2}$, respectively. The K -groups $K_i(X)$ of X are the homotopy groups $\pi_{i+1} BQ\mathcal{P}$ of the geometric realization $BQ\mathcal{P}$ of the nerve $NQ\mathcal{P}$ of $Q\mathcal{P}$. Algebraic K' -groups are defined similarly using the category of coherent $\mathcal{O}_X$ -modules on X . If X is regular, then $K(X)$ and $K'(X)$ are homotopy equivalent, because every coherent $\mathcal{O}_X$ -module has a finite resolution by finitely generated locally free $\mathcal{O}_X$ -modules, hence one can apply the resolution theorem of Quillen [79, §4, Cor. 2]. The functor K' has properties analogous to the properties of Bloch's higher Chow groups.

For a ring R , a different construction of $K_i(R)$ is the $+$ -construction [78]. It is defined by modifying the classifying space $BGL(R)$ of the infinite general linear group $GL(R) = \text{colim}_i GL_i(R)$ to a space $BGL^+(R)$, which has the same homology groups as $BGL(R)$, but abelian fundamental group. The K -groups of R

are $K_i(R) = \pi_i BGL(R)^+$, and by [38] they agree with the groups defined using the Q -construction for $X = \text{Spec } R$. The Q -construction has better functorial properties, whereas the $+$ -construction is more accessible to calculations. For example, Quillen calculates the K -theory of finite fields in [78] using the $+$ -construction. This was the only type of ring for which the K -theory was completely known, until 25 years later the K -theory of truncated polynomial algebras over finite fields was calculated [47]. It takes deep results on topological cyclic homology and 25 pages of calculations to determine $K_3(\mathbb{Z}/9\mathbb{Z})$ using the $+$ -construction [27].

Waldhausen [105] gave an improved version of the Q -construction, called the S -construction, which gives a symmetric spectrum in the sense of [49], see [30, Appendix]. It also allows categories with more general weak equivalences than isomorphisms as input, for example categories of complexes and quasi-isomorphisms. Using this and ideas from [3], Thomason [97, §3] gave the following, better behaved definition of K -theory. For simplicity we assume that the scheme X is noetherian. The K' -theory of X is the Waldhausen K -theory of the category of complexes, which are quasi-isomorphic to a bounded complex of coherent $\mathcal{O}_X$ -modules. This definition gives the same homotopy groups as Quillen's construction. The K -theory of X is the Waldhausen K -theory of the category of perfect complexes, i.e. complexes quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$ -modules of finite rank. If X has an ample line bundle, which holds for example if X is quasi-projective over an affine scheme, or separated, regular and noetherian [97, §2], then this agrees with the definition of Quillen.

The K -groups with coefficients are the homotopy groups of the smash product $K/m(X) := K(X) \wedge M_m$ of the K -theory spectrum and the Moore spectrum. There is a long exact sequence

$$\cdots \rightarrow K_i(X) \xrightarrow{\times m} K_i(X) \rightarrow K_i(X, \mathbb{Z}/m) \rightarrow K_{i-1}(X) \rightarrow \cdots$$

and similarly for K' -theory. We let $K_i(X, \mathbb{Z}_p)$ be the homotopy groups of the homotopy limit $\text{holim}_n K/p^n(X)$. Then the homotopy groups are related by the Milnor exact sequence [14]

$$0 \rightarrow \lim_n^1 K_{i+1}(X, \mathbb{Z}/p^n) \rightarrow K_i(X, \mathbb{Z}_p) \rightarrow \lim_n K_i(X, \mathbb{Z}/p^n) \rightarrow 0. \quad (1.24)$$

The K -groups with coefficients satisfy all of the properties given below for K -groups, except the product structure in case that m is divisible by 2 but not by 4, or by 3 but not by 9.

Basic Properties

1.4.1

By [79, §7.2] and [97, §3], the functor K is contravariantly functorial, and the functor K' is contravariantly functorial for maps $f : X \rightarrow Y$ of finite Tor-dimension, i.e. $\mathcal{O}_X$ is of finite Tor-dimension as a module over $f^{-1}\mathcal{O}_Y$. The functor K' is covariant functorial for proper maps, and K is covariant for proper maps of finite Tor-dimension.

Waldhausen S-construction gives maps of symmetric spectra [30, 104, §9]

$$K(X) \wedge K(Y) \rightarrow K(X \times Y)$$

$$K(X) \wedge K'(Y) \rightarrow K'(X \times Y),$$

which induces a product structure on algebraic K -theory, and an action of K -theory on K' -theory, respectively. If we want to define a product using the Q -construction, then because $K_i(X) = \pi_{i+1} BQ\mathcal{P}_X$, one needs a map from $BQ\mathcal{P}_X \times BQ\mathcal{P}_Y$ to a space $\mathcal{C}$ such that $K_i(X \times Y) = \pi_{i+2}\mathcal{C}$, i.e. a delooping $\mathcal{C}$ of $BQ\mathcal{P}_{X \times Y}$. Thus products can be defined more easily with the Thomason–Waldhausen construction. There is a product formula [97, §3]: If $f : X \rightarrow Y$ is proper, $y \in K_i(Y)$ and $x \in K'_j(X)$, then $f_*(f^*y \cdot x) = y \cdot f_*x$. The analogous result holds for $x \in K_j(X)$, if f is proper and of finite Tor-dimension.

The functor K' is *homotopy invariant* [79, §7, Pro. 4.1], i.e. for a flat map $f : E \rightarrow X$ whose fibers are affine spaces, the pull-back map induces an isomorphism $f^* : K'_i(X) \xrightarrow{\sim} K'_i(E)$. The projective bundle formula holds for K' and K : If $\mathcal{E}$ is a vector bundle of rank n on a noetherian separated scheme X , and $\mathbb{P}\mathcal{E} \xrightarrow{p} X$ the corresponding projective space, then there is an isomorphism [79, §7, Prop. 4.3]

$$K'_i(X)^n \xrightarrow{\sim} K'_i(\mathbb{P}\mathcal{E})$$

$$(x_i) \mapsto \sum_{i=0}^{n-1} p^*(x_i)[\mathcal{O}(-i)].$$

If X is a quasi-compact scheme then the analog formula holds for K -theory [79, §8, Thm. 2.1].

If $i : Z \rightarrow X$ is a regular embedding of codimension c (see the appendix for a definition), and X' is the blow-up of X along Z and $Z' = Z \times_X X'$, then we have the blow-up formula [96]

$$K_n(X') \cong K_n(X) \oplus K_n(Z)^{\oplus c-1}.$$

1.4.2 Localization

For Y a closed subscheme of X with open complement U , there is a localization sequence for K' -theory [79, §7, Prop. 3.2]

$$\cdots \rightarrow K'_{i+1}(U) \rightarrow K'_i(Y) \rightarrow K'_i(X) \rightarrow K'_i(U) \rightarrow \cdots$$

In particular, K' -theory satisfies the Mayer–Vietoris property. If X is a noetherian and finite dimensional scheme, the construction of Brown and Gersten [15] then gives a spectral sequence

$$E_2^{s,t} = H^s(X_{\text{Zar}}, \mathcal{K}'_{-t}) \Rightarrow K'_{-s-t}(X). \quad (1.25)$$

Here $\mathcal{K}'_i$ is the sheaf associated to the presheaf $U \mapsto K'_i(U)$. A consequence of the main result of Thomason [97, Thm. 8.1] is that the modified K -groups K^B also satisfy the Mayer–Vietoris property, hence there is a spectral sequence analogous to (1.25). Here K^B is Bass- K -theory, which can have negative homotopy groups, but satisfies $K_i(X) \cong K_i^B(X)$ for $i \geq 0$. See Carlson’s article [16] in this handbook for more on negative K -groups.

Gersten Resolution

1.4.3

As in (1.3), filtration by coniveau gives a spectral sequence [79, §7, Thm. 5.4]

$$E_1^{s,t} = \bigoplus_{x \in X^{(s)}} K_{-s-t}(k(x)) \Rightarrow K'_{-s-t}(X). \quad (1.26)$$

If X is smooth over a field, then as in (1.4), the spectral sequence (1.26) degenerates at E_2 for every semi-local ring of X , and we get the Gersten resolution [79, §7, Prop. 5.8, Thm. 5.11]

$$0 \rightarrow \mathcal{K}_i \rightarrow \bigoplus_{x \in X^{(0)}} i_* K_i(k(x)) \rightarrow \bigoplus_{x \in X^{(1)}} i_* K_{i-1}(k(x)) \rightarrow \cdots. \quad (1.27)$$

Because skyscraper sheaves are flabby, one can calculate cohomology of $\mathcal{K}_i$ with (1.27) and gets $E_2^{s,t} = H^s(X_{\text{Zar}}, \mathcal{K}_{-t})$. By [37, Thm. 2, 4 (iv)] there is always a map from the spectral sequence (1.25) to the spectral sequence (1.26), and for smooth X the two spectral sequences agree from E_2 on. If X is essentially smooth over a discrete valuation ring V of mixed characteristic $(0, p)$, then the Gersten resolution exists with finite coefficients. The case $p \nmid m$ was treated by Gillet and Levine [35, 36], and the case $m = p^r$ in [33]. The corresponding result is unknown for rational, hence integral coefficients.

The product structure of K -theory induces a canonical map from Milnor K -theory of fields to Quillen K -theory. Via the Gersten resolutions, this gives rise to a map $K_n^M(R) \rightarrow K_n(R)$ for any regular semi-local ring essentially of finite type over a field.

Motivic Cohomology and K-Theory

1.4.4

If X is of finite type over a discrete valuation ring, then Bloch’s higher Chow groups and algebraic K' -theory are related by a spectral sequence

$$E_2^{s,t} = H^{s-t}(X, \mathbb{Z}(-t)) \Rightarrow K'_{-s-t}(X). \quad (1.28)$$

This is an analog of the Atiyah–Hirzebruch spectral sequence from singular cohomology to topological K -theory. Consequences of the existence of the spectral sequence for K -theory had been observed previous to the definition of motivic cohomology. By [37, Thm. 7], there are *Adams* operators acting on the E_r -terms of the spectral sequence compatible with the action on the abutment. In particular,

the spectral sequence degenerates after tensoring with $\mathbb{Q}$ by the argument of [86], and the induced filtration agrees with the γ -filtration. The resulting graded pieces have been used as a substitute for motivic cohomology before its definition.

The existence of the spectral sequence has been conjectured by Beilinson [2], and first been proved by Bloch and Lichtenbaum for fields [10]. Friedlander–Suslin [21] and Levine [59] used their result to generalize this to varieties over fields and discrete valuation rings, respectively. There are different methods of constructing the spectral by Grayson–Suslin [39, 90] and Levine [60], which do not use the theorem of Bloch and Lichtenbaum.

Using the spectral sequence (1.28), we can translate results on motivic cohomology into results on K -theory. For example, Parshin’s conjecture states that for a smooth projective variety over a finite field, $K_i(X)$ is torsion for $i > 0$, and this implies that for a field F of characteristic p , $K_i^M(F) \otimes \mathbb{Q} \cong K_i(F) \otimes \mathbb{Q}$.

1.4.5 Sheaf Theoretic Properties

It is a result of Gabber [24] and Suslin [88] that for every henselian pair (A, I) with m invertible in A , and for all $i \geq 0$,

$$K_i(A, \mathbb{Z}/m) \xrightarrow{\sim} K_i(A/I, \mathbb{Z}/m). \quad (1.29)$$

Together with Suslin’s calculation of the K -theory of an algebraically closed field [88], this implies that if m is invertible on X , then the étale K -theory sheaf with coefficients in the sheaf $(\mathcal{K}/m)_n$, associated to the presheaf $U \mapsto K_m(U, \mathbb{Z}/m)$ can be described as follows:

$$(\mathcal{K}/m)_n = \begin{cases} \mu_m^{\otimes \frac{n}{2}} & n \geq 0 \text{ even,} \\ 0 & n \text{ odd.} \end{cases} \quad (1.30)$$

One can use the spectral sequence (1.28) to deduce this from (1.14), but historically the results for K -theory were proved first, and then the analogous results for motivic cohomology followed. However, for mod p -coefficients, Theorem (1.17) was proved first, and using the spectral sequence (1.28), it has the following consequences for K -theory: For any field F of characteristic p , the groups $K_n^M(F)$ and $K_n(F)$ are p -torsion free. The natural map $K_n^M(F)/p^r \rightarrow K_n(F, \mathbb{Z}/p^r)$ is an isomorphism, and the natural map $K_n^M(F) \rightarrow K_n(F)$ is an isomorphism up to uniquely p -divisible groups. Finally, for a smooth variety X over a perfect field of characteristic p , the K -theory sheaf for the Zariski or étale topology is given by

$$(\mathcal{K}/p^r)_n \cong v_r^n. \quad (1.31)$$

In particular, the spectral sequence (1.25) takes the form

$$E_2^{s,t} = H^s(X_{\text{Zar}}, v_r^{-t}) \Rightarrow K_{-s-t}(X, \mathbb{Z}/p^r).$$

and $K_n(X, \mathbb{Z}/p^r) = 0$ for $n > \dim X$.

In a generalization of (1.29) to the case where m is not invertible, Suslin and Panin [76, 88] show that for a henselian valuation ring V of mixed characteristic $(0, p)$ with maximal ideal I , one has an isomorphism of pro-abelian groups

$$K_i(V, \mathbb{Z}/p^r) \cong K_i\left(\lim_s V/I^s, \mathbb{Z}/p^r\right) \cong \{K_i(V/I^s, \mathbb{Z}/p^r)\}_s.$$

In [31], the method of Suslin has been used in the following situation. Let R be a local ring, such that (R, pR) is a henselian pair, and such that p is not a zero divisor. Then the reduction map

$$K_i(R, \mathbb{Z}/p^r) \rightarrow \{K_i(R/p^s, \mathbb{Z}/p^r)\}_s \quad (1.32)$$

is an isomorphism of pro-abelian groups.

Étale K-Theory and Topological Cyclic Homology

1.5

We give a short survey of Thomason's construction of *hyper-cohomology spectra* [95, §1], which is a generalization of Godement's construction of hyper-cohomology of a complex of sheaves. Let X_r be a site with Grothendieck topology τ on the scheme X , $X_r^\sim$ the category of sheaves of sets on X_r , and $\mathcal{S}ets$ the category of sets. A *point* of X consists of a pair of adjoint functors $p^* : X_r^\sim \rightarrow \mathcal{S}ets$ and $p_* : \mathcal{S}ets \rightarrow X_r^\sim$, such that the left adjoint p^* commutes with finite limits. We say $X_{\text{étale}}$ has enough points if we can find a set of points $\mathcal{P}$ such that a morphism α of sheaves is an isomorphism provided that $p^*\alpha$ is an isomorphism for all points $p \in \mathcal{P}$. For example, the Zariski site X_{Zar} and étale site $X_{\text{ét}}$ on a scheme X have enough points (points of X_{Zar} are points of X and points of $X_{\text{ét}}$ are geometric points of X , p^* is the pull-back and p_* the push-forward along the inclusion map $p : \text{Spec } k \rightarrow X$ of the residue field).

Let $\mathcal{F}$ be a presheaf of spectra on X_r . Given $p \in \mathcal{P}$, we can consider the presheaf of spectra $p_*p^*\mathcal{F}$, and the endo-functor on the category of presheaves of spectra $T\mathcal{F} = \prod_{p \in \mathcal{P}} p_*p^*\mathcal{F}$. The adjunction morphisms $\eta : \text{id} \rightarrow p_*p^*$ and $\varepsilon : p^*p_* \rightarrow \text{id}$ induce natural transformation $\eta : \text{id} \rightarrow T$ and $\mu = p_*\varepsilon p^* : TT \rightarrow T$. Thomason defines $T\mathcal{F}$ as the cosimplicial presheaf of spectra $n \mapsto T^{n+1}\mathcal{F}$, where the coface maps are $d_n^i = T^i\eta T^{n+1-i}$ and the codegeneracy maps are $s_n^i = T^i\mu T^{n-i}$. The map η induces an augmentation $\eta : \mathcal{F} \rightarrow T\mathcal{F}$, and since $T\mathcal{F}$ only depends on the stalks of $\mathcal{F}$, $T\mathcal{F}$ for a presheaf $\mathcal{F}$ only depends on the sheafification of $\mathcal{F}$.

The hyper-cohomology spectrum of $\mathcal{F}$ is defined to be the homotopy limit of the simplicial spectrum $T\mathcal{F}(X)$,

$$\mathbb{H}'(X_r, \mathcal{F}) := \text{holim } T\mathcal{F}(X).$$

It comes equipped with a natural augmentation $\eta : \mathcal{F}(X) \rightarrow \mathbb{H}^*(X_r, \mathcal{F})$, and if $\mathcal{F}$ is contravariant in X_r , then so is $\mathbb{H}^*(X_r, \mathcal{F})$. One important feature of $\mathbb{H}^*(X_r, \mathcal{F})$ is that it admits a spectral sequence [95, Prop. 1.36]

$$E_2^{s,t} = H^s(X_r, \widetilde{\pi}_{-t}\mathcal{F}) \Rightarrow \pi_{-s-t}\mathbb{H}^*(X_r, \mathcal{F}), \quad (1.33)$$

where $\widetilde{\pi}_i\mathcal{F}$ is the sheaf associated to the presheaf of homotopy groups $U \mapsto \pi_i\mathcal{F}(U)$. The spectral sequence converges strongly, if, for example, X_r has finite cohomological dimension.

1.5.1 Continuous Hyper-cohomology

Let $\mathcal{A}$ be the category of (complexes of) sheaves of abelian groups on X_r and consider the category $\mathcal{A}^{\mathbb{N}}$ of pro-sheaves. A pro-sheaf on the site X_r is the same as a sheaf on the site $X_r \times \mathbb{N}$, where $\mathbb{N}$ is the category with objects $[n]$, a unique map $[n] \rightarrow [m]$ if $n \leq m$, and identity maps as coverings. If $\mathcal{A}$ has enough injectives, then so does $\mathcal{A}^{\mathbb{N}}$, and Jannsen [51] defines the *continuous cohomology group* $H_{cont}^j(X_r, A')$ to be the j -th derived functor of $A' \mapsto \lim_r \Gamma(X, A')$. There are exact sequences

$$0 \rightarrow \lim_r^1 H^{j-1}(X_r, A') \rightarrow H_{cont}^j(X_r, A') \rightarrow \lim_r H^j(X_r, A') \rightarrow 0. \quad (1.34)$$

If $\mathbb{Z}/l'(n)$ is the (étale) motivic complex mod l' , then we abbreviate

$$H^i(X_{\text{ét}}, \mathbb{Z}/l'(n)) := H_{cont}^i(X_{\text{ét}}, \mathbb{Z}/l'(n)). \quad (1.35)$$

In view of (1.14) and (1.18), this is consistent with the usual definition of the left hand side.

Given a pro-presheaf $\mathcal{F}^*$ of spectra on X_r , one gets the hyper-cohomology spectrum $\mathbb{H}^*(X_r, \mathcal{F}^*) := \text{holim}_r \mathbb{H}^*(X_r, \mathcal{F}^r)$. The corresponding spectral sequence takes the form [30]

$$E_2^{s,t} = H_{cont}^s(X_r, \widetilde{\pi}_{-t}\mathcal{F}^*) \Rightarrow \pi_{-s-t}\mathbb{H}^*(X_r, \mathcal{F}^*). \quad (1.36)$$

1.5.2 Hyper-cohomology of K -Theory

If X_{Zar} is the Zariski site of a noetherian scheme of finite dimension, then by Thomason [95, 2.4] [97, Thm. 10.3], the augmentation maps

$$\begin{aligned} \eta : K'(X) &\rightarrow \mathbb{H}^*(X_{\text{Zar}}, K') \\ \eta : K^B(X) &\rightarrow \mathbb{H}^*(X_{\text{Zar}}, K^B) \end{aligned}$$

are homotopy equivalences, and the spectral sequence (1.25) and (1.33) agree. By Nisnevich [75], the analogous result holds for the Nisnevich topology.

If $X_{\text{ét}}$ is the small étale site of the scheme X , then we write $K^{\text{ét}}(X)$ for $\mathbb{H}^*(X_{\text{ét}}, K)$ and $K^{\text{ét}}(X, \mathbb{Z}_p)$ for $\text{holim}_r \mathbb{H}^*(X_{\text{ét}}, K/p^r)$. A different construction of étale K -theory was given by Dwyer and Friedlander [18], but by Thomason [95, Thm. 4.11] the

two theories agree for a separated, noetherian, and regular scheme X of finite Krull dimension with l invertible on X . If m and l are invertible on X , then in view of (1.30), the spectral sequences (1.33) and (1.36) take the form

$$E_2^{s,t} = H^s(X_{\text{ét}}, \mu_m^{\otimes -(t/2)}) \Rightarrow K_{-s-t}^{\text{ét}}(X, \mathbb{Z}/m) \quad (1.37)$$

$$E_2^{s,t} = H^s(X_{\text{ét}}, \mathbb{Z}_l(-\frac{t}{2})) \Rightarrow K_{-s-t}^{\text{ét}}(X, \mathbb{Z}_l). \quad (1.38)$$

Similarly, if X is smooth over a field of characteristic p , then by (1.31) and (1.35) there are spectral sequences

$$E_2^{s,t} = H^s(X_{\text{ét}}, v_r^{-t}) \Rightarrow K_{-s-t}^{\text{ét}}(X, \mathbb{Z}/p^r), \quad (1.39)$$

$$E_2^{s,t} = H^{s-t}(X_{\text{ét}}, \mathbb{Z}_p(-t)) \Rightarrow K_{-s-t}^{\text{ét}}(X, \mathbb{Z}_p). \quad (1.40)$$

The Lichtenbaum–Quillen Conjecture

1.5.3

The *Lichtenbaum–Quillen conjecture* (although never published by either of them in this generality) is the K -theory version of the Beilinson–Lichtenbaum conjecture, and predates it by more than 20 years. It states that on a regular scheme X , the canonical map from K -theory to étale K -theory

$$K_i(X) \rightarrow K_i^{\text{ét}}(X)$$

is an isomorphism for sufficiently large i (the cohomological dimension of X is expected to suffice). Since rationally, K -theory and étale K -theory agree [95, Thm. 2.15], one can restrict oneself to finite coefficients. If X is smooth over a field k of characteristic p , then with mod p^r -coefficients, the conjecture is true by (1.31) for $i > \text{cd}_p k + \dim X$, because both sides vanish. Here $\text{cd}_p k$ is the p -cohomological dimension of k , which is the cardinality of a p -base of k (plus one in certain cases). For example, $\text{cd}_p k = 0$ if k is perfect.

Levine has announced a proof of an étale analog of the spectral sequence from Bloch's higher Chow groups to algebraic K' -theory for a smooth scheme X over a discrete valuation ring (and similarly with coefficients)

$$E_2^{s,t} = H^{s-t}(X_{\text{ét}}, \mathbb{Z}(-t)) \Rightarrow K_{-s-t}^{\text{ét}}(X). \quad (1.41)$$

Comparing the mod m -version of the spectral sequences (1.28) and (1.41), one can deduce that the Beilinson–Lichtenbaum conjecture implies the Lichtenbaum–Quillen conjecture with mod m -coefficients for $i \geq \text{cd}_m X_{\text{ét}}$.

If one observes that in the spectral sequence (1.37) $E_2^{s,t} = E_3^{s,t}$ and reindexes $s = p - q$, $t = 2q$, then we get a spectral sequence with the same E_2 -term as (1.41) with mod m -coefficients, but we don't know if the spectral sequences agree. Similarly, the E_2 -terms of the spectral sequences (1.39) and (1.41) with mod p^r -coefficients agree, but we don't know if the spectral sequences agree.

Topological Cyclic Homology

Bökstedt, Hsiang and Madsen [12] define *topological cyclic homology* for a ring A . Bökstedt first defines topological Hochschild homology $\mathrm{TH}(A)$, which can be thought of as a topological analog of Hochschild homology, see [30] [45, §1]. It is the realization of a *cyclic spectrum* (i.e. a simplicial spectrum together with maps $\tau : [n] \rightarrow [n]$ satisfying certain compatibility conditions with respect to the face and degeneracy maps), hence $\mathrm{TH}(A)$ comes equipped with an action of the circle group S^1 [65]. Using Thomason's hyper-cohomology construction, one can extend this definition to schemes [30]: On a site X_r , one considers the presheaf of spectra

$$\mathrm{TH} : U \mapsto \mathrm{TH}(\Gamma(U, \mathcal{O}_U)) .$$

and defines

$$\mathrm{TH}(X_r) = \mathbb{H}'(X_r, \mathrm{TH}) . \quad (1.42)$$

If the Grothendieck topology τ on the scheme X is coarser than or equal to the étale topology, then $\mathrm{TH}(X_r)$ is independent of the topology [30, Cor. 3.3.3], and accordingly we drop τ from the notation. If X is the spectrum of a ring A , then $\mathrm{TH}(A) \xrightarrow{\sim} \mathrm{TH}(X)$, [30, Cor. 3.2.2].

To define topological cyclic homology [45, §6], we let $\mathrm{TR}^m(X; p)$ be the fixed point spectrum under the cyclic subgroup of roots of unity $\mu_{p^{m-1}} \subseteq S^1$ acting on $\mathrm{TH}(X)$. If X is the spectrum of a ring A , the group of components $\pi_0 \mathrm{TR}^m(A; p)$ is isomorphic to the Witt vectors $W_m(A)$ of length m of A [46, Thm. F]. The maps F, V, R on $W_m(A) = \pi_0 \mathrm{TR}^m(A; p)$ are induced by maps of spectra: The inclusion of fixed points induces the map

$$F : \mathrm{TR}^m(X; p) \rightarrow \mathrm{TR}^{m-1}(X; p)$$

called *Frobenius*, and one can construct the *restriction map*

$$R : \mathrm{TR}^m(X; p) \rightarrow \mathrm{TR}^{m-1}(X; p)$$

and the *Verschiebung map*

$$V : \mathrm{TR}^m(X; p) \rightarrow \mathrm{TR}^{m+1}(X; p) .$$

Note that F, V exist for all cyclic spectra, whereas the existence of R is particular to the topological Hochschild spectrum. The two composites FV and VF induce multiplication by p and $V(1) \in \pi_0 \mathrm{TR}^m(X; p)$, respectively, on homotopy groups. *Topological cyclic homology* $\mathrm{TC}^m(X; p)$ is the homotopy equalizer of the maps

$$F, R : \mathrm{TR}^m(X; p) \rightarrow \mathrm{TR}^{m-1}(X; p) .$$

We will be mainly interested in the version with coefficients

$$\mathrm{TR}^m(X; p, \mathbb{Z}/p^r) = \mathrm{TR}^m(X; p) \wedge M_{p^r},$$

$$\mathrm{TC}^m(X; p, \mathbb{Z}/p^r) = \mathrm{TC}^m(X; p) \wedge M_{p^r},$$

and its homotopy groups

$$\mathrm{TR}_i^m(X; p, \mathbb{Z}/p^r) = \pi_i \mathrm{TR}^m(X; p, \mathbb{Z}/p^r),$$

$$\mathrm{TC}_i^m(X; p, \mathbb{Z}/p^r) = \pi_i \mathrm{TC}^m(X; p, \mathbb{Z}/p^r).$$

We view $\mathrm{TR}^*(X; p, \mathbb{Z}/p^r)$ and $\mathrm{TC}^*(X; p, \mathbb{Z}/p^r)$ as pro-spectra with R as the structure map. Then F and V induce endomorphisms of the pro-spectrum $\mathrm{TR}^*(X; p, \mathbb{Z}/p^r)$. The homotopy groups fit into a long exact sequence of pro-abelian groups

$$\cdots \rightarrow \mathrm{TC}_i^*(X; p, \mathbb{Z}/p^r) \rightarrow \mathrm{TR}_i^*(X; p, \mathbb{Z}/p^r) \xrightarrow{F-1} \mathrm{TR}_i^*(X; p, \mathbb{Z}/p^r) \rightarrow \cdots.$$

Finally, one can take the homotopy limit and define

$$\mathrm{TR}(X; p, \mathbb{Z}_p) = \mathrm{holim}_{m,r} \mathrm{TR}^m(X; p, \mathbb{Z}/p^r),$$

$$\mathrm{TC}(X; p, \mathbb{Z}_p) = \mathrm{holim}_{m,r} \mathrm{TC}^m(X; p, \mathbb{Z}/p^r).$$

The corresponding homotopy groups again fit into a long exact sequence. If we let $(\mathrm{TC}^m/p^r)_i$ be the sheaf associated to the presheaf $U \mapsto \mathrm{TC}_i^m(U; p, \mathbb{Z}/p^r)$, then (1.33) and (1.36) take the form

$$E_2^{s,t} = H^s(X_r, (\mathrm{TC}^m/p^r)_{-t}) \Rightarrow \mathrm{TC}_{-s-t}^m(X; p, \mathbb{Z}/p^r), \quad (1.43)$$

$$E_2^{s,t} = H_{\mathrm{cont}}^s(X_r, (\mathrm{TC}^m/p^r)_{-t}) \Rightarrow \mathrm{TC}_{-s-t}^m(X; p, \mathbb{Z}_p), \quad (1.44)$$

and similarly for TR . The spectral sequences differ for different Grothendieck topologies τ on X , even though the abutment does not [30].

Topological cyclic homology comes equipped with the *cyclotomic trace map*

$$\mathrm{tr} : K(X, \mathbb{Z}_p) \rightarrow \mathrm{TC}(X; p, \mathbb{Z}_p),$$

which factors through étale K -theory because topological cyclic homology for the Zariski and the étale topology agree. The map induces an isomorphism of homotopy groups in many cases, a fact which is useful to calculate K -groups. To show that the trace map is an isomorphism, the following result of McCarthy is the starting point [67]. If R is a ring and I a nilpotent ideal, then the following diagram is homotopy cartesian

$$\begin{array}{ccc} K(R, \mathbb{Z}_p) & \xrightarrow{\mathrm{tr}} & \mathrm{TC}(R; p, \mathbb{Z}_p) \\ \downarrow & & \downarrow \\ K(R/I, \mathbb{Z}_p) & \xrightarrow{\mathrm{tr}} & \mathrm{TC}(R/I; p, \mathbb{Z}_p). \end{array} \quad (1.45)$$

In particular, if the lower map is a homotopy equivalence, then so is the upper map. In [46], Hesselholt and Madsen use this to show that the trace map is an isomorphism in non-negative degrees for a finite algebra over the Witt ring of a perfect field. They apply this in [47] to calculate the K -theory with mod p coefficients of truncated polynomial algebras $k[t]/(t^i)$ over perfect fields k of characteristic p . In [48], they calculate the topological cyclic homology of a local number ring ($p \neq 2$), and verify the Lichtenbaum–Quillen conjecture for its quotient field with p -adic coefficients (prime to p -coefficients were treated in [9]). This has been generalized to certain discrete valuation rings with non-perfect residue fields in [32]. See [44] for a survey of these results.

1.5.5 Comparison

Hesselholt has shown in [43] that for a regular $\mathbb{F}_p$ -algebra A , there is an isomorphism of pro-abelian groups

$$W.\Omega_A^i \xrightarrow{\sim} \mathrm{TR}_i^*(A; p),$$

which is compatible with the Frobenius endomorphism on both sides. In [43] the result is stated for a smooth $\mathbb{F}_p$ -algebra, but any regular $\mathbb{F}_p$ -algebra is a filtered colimit of smooth ones [77, 92], and the functors on both sides are compatible with filtered colimits (at this point it is essential to work with pro-sheaves and not take the inverse limit). In particular, for a smooth scheme X over a field of characteristic p , we get from (1.15) the following diagram of pro-sheaves for the étale topology,

$$\begin{array}{ccccccc} 0 & \rightarrow & v_*^i & \rightarrow & W.\Omega_X^i & \xrightarrow{F-1} & W.\Omega_X^i \rightarrow 0 \\ & & \downarrow & & \cong \downarrow & & \cong \downarrow \\ \cdots & \xrightarrow{\delta} & (\mathrm{TC}/p)_i & \rightarrow & (\mathrm{TR}^*/p)_i & \xrightarrow{F-1} & (\mathrm{TR}^*/p)_i \xrightarrow{\delta} \cdots \end{array}$$

This shows that $\delta = 0$, and that there is an isomorphism of pro-étale sheaves

$$\{v_r^i\} \cong \{(\mathrm{TC}^m/p^r)_i\}_{m,r}. \quad (1.46)$$

The spectral sequence (1.43) becomes

$$E_2^{s,t} = H_{\mathrm{cont}}^{s,t}(X, v_*^{-t}) \Rightarrow \mathrm{TC}_{-s-t}(X; p, \mathbb{Z}_p). \quad (1.47)$$

and the cyclotomic trace map from étale K -theory to topological cyclic homology

$$K_i^{\mathrm{ét}}(X, \mathbb{Z}_p) \xrightarrow{\mathrm{tr}} \mathrm{TC}_i(X; p, \mathbb{Z}_p) \quad (1.48)$$

is an isomorphism, because the induced map on the strongly converging spectral sequences (1.40) and (1.47) is an isomorphism on E_2 -terms.

If X is not smooth, then the above method does not work because we cannot identify the K -theory and TC -sheaves. However, we do not know of any example of

a finitely generated algebra over a perfect field of characteristic p where the trace map (1.48) is not an isomorphism.

One can extend the isomorphism (1.48) to smooth, proper schemes X over a henselian discrete valuation ring V of mixed characteristic $(0, p)$, [31]. Comparing the hyper-cohomology spectral sequences (1.36), it suffices to show that the cyclotomic trace map induces an isomorphism on E_2 -terms. If $i : Y \rightarrow X$ is the embedding of the closed fiber, then because V is henselian, the proper base change theorem together with (1.34) implies that for every pro-sheaf $\mathcal{F}^\bullet$ on X , $H_{\text{cont}}^i(X_{\text{ét}}, \mathcal{F}^\bullet) \cong H_{\text{cont}}^i(Y_{\text{ét}}, i^* \mathcal{F}^\bullet)$. Hence it suffices to show that the cyclotomic trace map induces a pro-isomorphism of homotopy groups for an essentially smooth, strictly henselian local ring R over V . If π is a uniformizer of V , then from the characteristic p case we know that the trace map is an isomorphism for R/π . Using (a pro-version of) the method of McCarthy (1.45), this implies that the trace map is an isomorphism for R/π^s , $s \geq 1$. By (1.32), the p -adic K -theory of R is determined by the K -theory of the system $\{R/\pi^s\}_s$. The analogous statement for topological cyclic homology holds more generally: If R is a ring such that p is not a zero-divisor, then the reduction map

$$\mathrm{TC}_i^m(R; p, \mathbb{Z}/p^r) \rightarrow \left\{ \mathrm{TC}_i^m(R/p^s; p, \mathbb{Z}/p^r) \right\}_s$$

is an isomorphism of pro-abelian groups.

If the trace map (1.48) is an isomorphism for every finitely generated algebra over a perfect field of characteristic p (or any normal crossing scheme), then the same argument shows that (1.48) is an isomorphism for every proper V -scheme of finite type (or semi-stable scheme).

Appendix: Basic Intersection Theory

A

In this appendix we collect some facts of intersection theory needed to work with higher Chow groups. When considering higher Chow groups, one always assumes that all intersections are proper intersections; furthermore one does not have to deal with rational equivalence. Thus the intersection theory used for higher Chow groups becomes simpler, and can be concentrated on a few pages. We hope that our treatment will allow a beginner to start working with higher Chow groups more quickly. For a comprehensive treatment, the reader should refer to the books of Fulton [23], Roberts [80], and Serre [83].

Proper Intersection

A.1

To define intersections on a noetherian, separated scheme, we can always reduce ourselves to an open, affine neighborhood of the generic points of irreducible components of the intersection, hence assume that we deal with the spectrum of a noetherian ring A .

If A is a finitely generated algebra over the quotient of a regular ring R of finite Krull dimension (this is certainly true if we consider rings of finite type over a field or Dedekind ring), then A is catenary [66, Thm. 17.9]. This implies that the length of a maximal chain of prime ideals between two primes $\mathfrak{p} \subseteq \mathcal{P}$ does not depend on the chain. In particular, if A is local and equidimensional, i.e. $\dim A/\mathfrak{q}$ is equal for all minimal prime ideals $\mathfrak{q}$, then for every prime $\mathfrak{p}$ of A , the dimension equality holds

$$\text{ht } \mathfrak{p} + \dim A/\mathfrak{p} = \dim A. \quad (1.49)$$

In order for (1.49) to extend to rings A that are *localizations* of finitely generated algebras over the quotient of a regular ring R of finite Krull dimension d , it is necessary to modify the definition of dimension as follows, see [80, §4.3]:

$$\dim A/\mathfrak{p} := \text{trdeg}(k(\mathfrak{p})/k(\mathfrak{q})) - \text{ht}_R \mathfrak{q} + d. \quad (1.50)$$

Here $\mathfrak{q}$ is the inverse image in R of the prime ideal $\mathfrak{p}$ of A . With this modified definition, (1.49) still holds. If A is a quotient of a regular ring, or is of finite type over a field, then (1.50) agrees with the Krull dimension of $A/\mathfrak{p}$.

1

Definition 1 We say that two closed subschemes V and W of X intersect properly, if for every irreducible component C of $V \times_X W$,

$$\dim C \leq \dim V + \dim W - \dim X.$$

If $\text{Spec } A \subseteq X$ is a neighborhood of the generic point of C , and if $V = \text{Spec } A/\mathfrak{a}$ and $W = \text{Spec } A/\mathfrak{b}$, then V and W meet properly, if for every minimal prime ideal $\mathcal{P} \supseteq \mathfrak{a} + \mathfrak{b}$

$$\dim A/\mathcal{P} \leq \dim A/\mathfrak{a} + \dim A/\mathfrak{b} - \dim A. \quad (1.51)$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are primes in an equidimensional local ring, then by (1.49) this means $\text{ht } \mathcal{P} \geq \text{ht } \mathfrak{a} + \text{ht } \mathfrak{b}$, i.e. the codimension of the intersection is at least the sum of the codimensions of the two irreducible subvarieties.

If A is regular, then it is a theorem of Serre [83, Thm. V.B.3] that the left hand side of (1.51) is always greater than or equal to the right hand side. We will see below (1.53) that the same is true if $\mathfrak{a}$ or $\mathfrak{b}$ can be generated by a regular sequence. On the other hand, in the 3-dimensional ring $k[X, Y, Z, W]/(XY - ZW)$, the subschemes defined by (X, Z) and (Y, W) both have dimension 2, but their intersection has dimension 0.

A.2 Intersection with Divisors

If $a \in A$ is neither a zero divisor nor a unit, then the divisor $D = \text{Spec } A/aA$ has codimension 1 by Krull's principal ideal theorem. If $V = \text{Spec } A/\mathfrak{p}$ is an irreducible subscheme of dimension r , then V and D meet properly if and only if $a \notin \mathfrak{p}$, if and

only if V is not contained in D , if and only if $V \times_X D$ is empty or has dimension $r - 1$.

Definition 2 If V and D intersect properly, then we define the intersection to be the cycle

2

$$[A/\mathfrak{p}] \cdot (a) = \sum_{\mathfrak{q}} \text{lgth}_{A_{\mathfrak{q}}} (A_{\mathfrak{q}}/(\mathfrak{p}_{\mathfrak{q}} + aA_{\mathfrak{q}})) [A/\mathfrak{q}] .$$

Here $\mathfrak{q}$ runs through the prime ideals such that $\dim A/\mathfrak{q} = r - 1$.

Note that only prime ideals $\mathfrak{q}$ containing $\mathfrak{p} + aA$ can have non-zero coefficient, and the definition makes sense because of [80, Cor. 2.3.3]:

Lemma 3 If $(A, \mathfrak{m})$ is a local ring, then $A/\mathfrak{a}$ has finite length if and only if $\mathfrak{m}$ is the only prime ideal of A containing $\mathfrak{a}$.

3

A sequence of elements $a_1, \dots, a_n$ in a ring A is called a *regular sequence*, if the ideal $(a_1, \dots, a_n)$ is a proper ideal of A , and if a_i is not a zero-divisor in $A/(a_1, \dots, a_{i-1})$. In particular, a_i is not contained in any minimal prime ideal of $A/(a_1, \dots, a_{i-1})$, and $\dim A/(a_1, \dots, a_i) = \dim A - i$. If A is local, then the regularity of the sequence is independent of the order [66, Cor. 16.3]. Our principal example is the regular sequence $(t_1, \dots, t_i)$ in the ring $A[t_0, \dots, t_i]/(1 - \sum t_i)$.

Given a regular sequence $a_1, \dots, a_n$, assume that the closed subscheme $V = \text{Spec } A/\mathfrak{p}$ of dimension r meets all subschemes $\text{Spec } A/(a_{i_1}, \dots, a_{i_j})$ properly. This amounts to saying that the dimension of every irreducible component of $A/(\mathfrak{p}, a_{i_1}, \dots, a_{i_j})$ has dimension $r - j$ if $r \geq j$, and is empty otherwise. In this case, we can inductively define the intersection $[V] \cdot (a_1) \cdots (a_n)$, a cycle of dimension $r - n$, using Definition 2.

Proposition 4 Let $a_1, \dots, a_n$ be a regular sequence in A . Then the cycle $[V] \cdot (a_1) \cdots (a_n)$ is independent of the order of the a_i .

4

Proof See also [23, Thm. 2.4]. Clearly it suffices to show $[V] \cdot (a) \cdot (b) = [V] \cdot (b) \cdot (a)$. Let $\mathfrak{m}$ be a minimal ideal containing $\mathfrak{p} + aA + bA$. To calculate the multiplicity of $\mathfrak{m}$ on both sides, we can localize at $\mathfrak{m}$ and divide out $\mathfrak{p}$, which amounts to replacing X by V . Thus we can assume that A is a two-dimensional local integral domain, and (a, b) is not contained in any prime ideal except $\mathfrak{m}$. If $\mathcal{P}$ runs through the minimal prime ideals of A containing a , then the cycle $[V] \cap (a)$ is $\sum_{\mathcal{P}} \text{lgth}_{A_{\mathcal{P}}} (A_{\mathcal{P}}/aA_{\mathcal{P}}) [A/\mathcal{P}]$, and if we intersect this with (b) we get the multiplicity

$$\sum_{\mathcal{P}} \text{lgth}_{A_{\mathcal{P}}} (A_{\mathcal{P}}/aA_{\mathcal{P}}) \cdot \text{lgth}_{A/\mathcal{P}} (A/(\mathcal{P} + bA)) .$$

The following lemma applied to $A/(a)$ shows that this is $\operatorname{lgth}_A(A/(a, b)) - \operatorname{lgth}_A({}_b(A/aA))$. This is symmetric in a and b , because since A has no zero-divisors, we have the bijection ${}_b(A/aA) \rightarrow {}_a(A/bA)$, sending a b -torsion element x of A/aA to the unique $y \in A$ with $bx = ay$. The map is well-defined because $x + za$ maps to $y + zb$.

5

Lemma 5 Let $(B, \mathfrak{m})$ be a one-dimensional local ring with minimal primes $\mathcal{P}_1, \dots, \mathcal{P}_r$, and let b be an element of B not contained in any of the $\mathcal{P}_i$. Then for every finitely generated B -module M , we have

$$\operatorname{lgth}_B(M/bM) - \operatorname{lgth}_B({}_bM) = \sum_i \operatorname{lgth}_{B_{\mathcal{P}_i}}(M_{\mathcal{P}_i}) \cdot \operatorname{lgth}_B(B/(\mathcal{P}_i + bB)) .$$

Proof See also [23, Lemma A.2.7]. Every finitely generated module M over a noetherian ring admits a finite filtration by submodules M_i such that the quotients M_i/M_{i+1} are isomorphic to $B/\mathfrak{p}$ for prime ideals $\mathfrak{p}$ of B . Since both sides are additive on short exact sequence of B -modules, we can consider the case $M = B/\mathcal{P}_i$ or $M = B/\mathfrak{m}$. If $M = B/\mathfrak{m}$, then $\operatorname{lgth}_B(M/bM) = \operatorname{lgth}_B({}_bM) = 0$ or 1 for $b \notin \mathfrak{m}$ and $b \in \mathfrak{m}$, respectively, and $M_{\mathcal{P}_i} = 0$ for all i .

For $M = B/\mathcal{P}_i$, we have $\operatorname{lgth}_{B_{\mathcal{P}_j}}(M_{\mathcal{P}_j}) = 0$ or 1 for $j \neq i$ and $j = i$, respectively, and $b \notin \mathcal{P}_i$ implies ${}_b(B/\mathcal{P}) = 0$, so that $\operatorname{lgth}_B(M/bM) - \operatorname{lgth}_B({}_bM) = \operatorname{lgth}_B(B/(\mathcal{P}_i + bB))$.

6

Corollary 6 The cycle complex $z^n(X, *)$ is a complex.

Proof Recall that the differentials are alternating sums of intersection with face maps. Since intersecting with two faces does not depend on the order of intersection, it follows from the simplicial identities that the composition of two differentials is the zero map.

A.3

Pull-back Along a Regular Embedding

A closed embedding $i : Z \rightarrow X$ of schemes is called a *regular embedding of codimension c* , if every point of Z has an affine neighborhood $\operatorname{Spec} A$ in X such that the ideal $\mathfrak{a}$ of A defining Z is generated by a regular sequence of length c . If Z is smooth over a base S , then i is regular if and only if X is smooth over S in some neighborhood of Z by EGA IV.17.12.1 [42]. In particular, if Y is smooth over S of relative dimension n , then for any morphism $X \rightarrow Y$, the graph map $X \rightarrow X \times_S Y$ is a regular embedding of codimension n . Indeed, the graph map is a closed embedding of smooth schemes over X . If Y is smooth and X is flat over S , this allows us to factor $X \rightarrow Y$ into a regular embedding followed by a flat map.

Let $i : Z \rightarrow X$ be a closed embedding and V a closed irreducible subscheme of X which intersects Z properly. (In practice, the difficult part is to find cycles which intersects Z properly.) In order to define the pull-back i^*V of V along i , we need to determine the multiplicity m_W of each irreducible component W of $Z \times_X V$. If we localize at the point corresponding to W , we can assume that X is the spectrum of a local ring $(A, \mathfrak{m})$, Z is defined by an ideal $\mathfrak{a}$ generated by a regular sequence $a_1, \dots, a_c$, and V is defined by a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{m}$ is the only prime containing $\mathfrak{a} + \mathfrak{p}$. The intersection multiplicity m_W at W is defined as the multiplicity of the iterated intersection with divisors

$$m_W \cdot [A/\mathfrak{m}] = [A/\mathfrak{p}] \cdot (a_1) \cdots (a_c) .$$

Corollary 7 If V and Z intersect properly, then the pull-back of V along the regular embedding $i : Z \rightarrow X$ is compatible with the boundary maps in the cycle complex. 7

Proof It is easy to see that if $(a_1, \dots, a_c)$ is a regular sequence in A , then $(a_1, \dots, a_c, t_{i_1}, \dots, t_{i_j})$ is a regular sequence in $A[t_0, \dots, t_s]$ for any $j \leq s$ and pairwise different indices i_l . Hence the corollary follows from Proposition 4.

The Koszul complex $K(x_1, \dots, x_n)$ for the elements $x_1, \dots, x_n$ in a ring A is defined to be the total complex of the tensor product of chain complexes

$$K(x_1, \dots, x_n) := \bigotimes_{i=1}^n (A \xrightarrow{\times x_i} A) .$$

Here the source and target are in degrees 1 and 0, respectively.

Proposition 8 Let $(A, \mathfrak{m})$ be a local ring, $\mathfrak{a} \subset A$ be an ideal generated by the regular sequence $a_1, \dots, a_c$. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{m}$ is minimal over $\mathfrak{a} + \mathfrak{p}$, and such that $A/\mathfrak{a}$ and $A/\mathfrak{p}$ meet properly. Then 8

$$[A/\mathfrak{p}] \cdot (a_1) \cdots (a_c) = \sum_{i=0}^c (-1)^i \operatorname{lgth}_A (H_i(K(a_1, \dots, a_c) \otimes A/\mathfrak{p})) \cdot [A/\mathfrak{m}] . \quad (1.52)$$

Proof By Krull's principal ideal theorem, any minimal prime divisor $\mathfrak{q}$ of $(a_1, \dots, a_i) + \mathfrak{p}$ in $A/\mathfrak{p}$ has height at most i , so that using (1.49) we get $\dim A/(\mathfrak{p}, a_1, \dots, a_i) = \dim A/\mathfrak{p} - \operatorname{ht} \mathfrak{q} \geq \dim A/\mathfrak{p} - i$. On the other hand, since $A/\mathfrak{a}$ and $A/\mathfrak{p}$ meet properly, $\dim A/(\mathfrak{p} + \mathfrak{a}) \leq \dim A/\mathfrak{p} + \dim A/\mathfrak{a} - \dim A = \dim A/\mathfrak{p} - c$. This can only happen if we have equality everywhere, i.e.

$$\dim A/(\mathfrak{p}, a_1, \dots, a_i) = \dim A/\mathfrak{p} - \operatorname{ht} \mathfrak{q} = \dim A/\mathfrak{p} - i . \quad (1.53)$$

We now proceed by induction on c . For $c = 1$, by regularity, $a_1(A/\mathfrak{p}) = 0$, hence both sides equal $\operatorname{lgth}_A(A/(\mathfrak{p} + a_1A)) \cdot [A/\mathfrak{m}]$. If we denote $H_i(K(a_1, \dots, a_l) \otimes A/\mathfrak{p})$ by $H_i(l)$, then by definition of the Koszul complex, there is a short exact sequence of A -modules

$$0 \rightarrow H_i(c-1)/a_c \rightarrow H_i(c) \rightarrow {}_{a_c}H_{i-1}(c-1) \rightarrow 0.$$

Hence we get for the multiplicity of the right hand side of (1.52)

$$\begin{aligned} \sum_{i=0}^c (-1)^i \operatorname{lgth}_A(H_i(c)) &= \sum_{i=0}^c (-1)^i (\operatorname{lgth}_A(H_i(c-1)/a_c) + \operatorname{lgth}_A({}_{a_c}H_{i-1}(c-1))) \\ &= \sum_{i=0}^{c-1} (-1)^i (\operatorname{lgth}_A(H_i(c-1)/a_c) - \operatorname{lgth}_A({}_{a_c}H_i(c-1))). \end{aligned}$$

If $\mathfrak{q}$ runs through the minimal ideals of $A/(\mathfrak{p}, a_1, \dots, a_{c-1})$, then by Lemma 5, this can be rewritten as

$$\sum_{i=0}^{c-1} (-1)^i \sum_{\mathfrak{q}} \operatorname{lgth}_{A_{\mathfrak{q}}}(H_i(c-1)_{\mathfrak{q}}) \cdot \operatorname{lgth}_A(A/(\mathfrak{q} + a_cA)).$$

By induction, we can assume that the multiplicity $[A/\mathfrak{p}] \cdot (a_1) \cdots (a_{c-1})$ at $\mathfrak{q}$ is $\sum_{i=0}^{c-1} (-1)^i \operatorname{lgth}_{A_{\mathfrak{q}}}(H_i(c-1)_{\mathfrak{q}})$, hence we get for the left hand side of (1.52)

$$\begin{aligned} [A/\mathfrak{p}] \cdot (a_1) \cdots (a_{c-1}) \cdot (a_c) &= \left(\sum_{\mathfrak{q}} \sum_{i=0}^{c-1} (-1)^i \operatorname{lgth}_{A_{\mathfrak{q}}}(H_i(c-1)_{\mathfrak{q}}) \cdot [A/\mathfrak{q}] \right) \cdot (a_c) \\ &= \sum_{\mathfrak{q}} \sum_{i=0}^{c-1} (-1)^i \operatorname{lgth}_{A_{\mathfrak{q}}}(H_i(c-1)_{\mathfrak{q}}) \cdot \operatorname{lgth}_A(A/(\mathfrak{q} + a_cA)) \cdot [A/\mathfrak{m}]. \end{aligned}$$

9

Corollary 9 The pull-back of V along the regular embedding $i : Z \rightarrow X$ agrees with Serre's intersection multiplicity

$$[A/\mathfrak{p}] \cdot (a_1) \cdots (a_c) = \sum_{i=0}^c (-1)^i \operatorname{lgth}_A(\operatorname{Tor}_i^A(A/\mathfrak{a}, A/\mathfrak{p})) \cdot [A/\mathfrak{m}].$$

In particular, it is independent of the choice of the regular sequence.

Proof If $(x_1, \dots, x_n)$ are elements in a local ring A , then by [80, Thm. 3.3.4] the Koszul-complex $K(x_1, \dots, x_n)$ is acyclic above degree 0, if and only if $(x_1, \dots, x_n)$ is

a regular sequence. Hence $K(a_1, \dots, a_c)$ is a free resolution of $A/\mathfrak{a}$. Tensoring the Koszul complex with $A/\mathfrak{p}$ and taking cohomology gives

$$H_i(K(a_1, \dots, a_c) \otimes A/\mathfrak{p}) \cong \mathrm{Tor}_i^A(A/\mathfrak{a}, A/\mathfrak{p}).$$

Flat Pull-back

A.4

Let $f : X \rightarrow Y$ be a flat morphism. We assume that f is of relative dimension n , i.e. for each subvariety V of Y and every irreducible component W of $X \times_Y V$, $\dim W = \dim V + n$. This is satisfied if for example Y is irreducible and every irreducible component of X has dimension equal to $\dim Y + n$, see EGA IV.14.2. In particular, the hypothesis implies that the pull-back of subschemes which intersect properly also intersects properly. For every closed integral subscheme V of Y , we define the pull-back

$$f^*[V] = \sum_W \mathrm{lgth}_{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W} \otimes_{\mathcal{O}_{Y,V}} k(V)) \cdot [W],$$

where W runs through the irreducible components of $V \times_Y X$.

Given $f : X \rightarrow Y$ and a subscheme D of Y locally defined by $a \in A$ on $\mathrm{Spec} A \subseteq Y$, then we can define the subscheme $f^{-1}D$ of X on any $\mathrm{Spec} B \subseteq X$ mapping to $\mathrm{Spec} A$ by $f^*a \in B$. If f is flat and D is a divisor, then so is $f^{-1}D$, because f^* sends non-zero divisors to non-zero divisors.

Proposition 10 Let $f : X \rightarrow Y$ be a flat map. Then intersection with an effective principal divisor D of Y is compatible with flat pull-back, i.e. if V is a closed subscheme of Y not contained in D , then $f^*[V] \cdot f^{-1}D = f^*[V \cdot D]$ as cycles on X .

10

Proof Intersection with a divisor was defined in Definition 2. It suffices to compare the multiplicities of $f^*[V] \cdot f^{-1}D$ and $f^*[V \cdot D]$ at each irreducible component $\mathcal{Q}$ of $V \times_Y D \times_Y X$. Let $\mathrm{Spec} A \subseteq Y$ be an affine neighborhood of the generic point $\mathfrak{p}$ of V and $\mathrm{Spec} B \subseteq X$ an affine neighborhood of $\mathcal{Q}$ mapping to $\mathrm{Spec} A$. We denote the induced flat map $A \rightarrow B$ by g . We can replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}$; this corresponds to replacing Y by V and X by $f^{-1}(V)$. We can also localize B at $\mathcal{Q}$ and A at $\mathfrak{q} = g^{-1}\mathcal{Q}$ ($\mathfrak{q}$ is the generic point of the irreducible component of $V \times_X D$ to which $\mathcal{Q}$ maps). If $a \in A$ defines D , then a is non-zero because V was not contained in D . Let $\mathcal{P}_1, \dots, \mathcal{P}_r$ be the finitely many minimal primes of B corresponding to the irreducible components of $f^{-1}(V)$ passing through $\mathcal{Q}$; for all i , $\mathcal{Q}$ is minimal among the primes of B containing $\mathcal{P}_i + aB$. We are thus in the following situation:

$(A, \mathfrak{q})$ is a one-dimensional local integral domain, $(B, \mathcal{Q})$ a one-dimensional local ring with minimal prime ideals $\mathcal{P}_1, \dots, \mathcal{P}_r$, a is a non-zero divisor in A , and because B is flat over A , a is also a non-zero divisor in B .

The pull-back $f^*[V]$ of V is given by the cycle $\sum_i \text{lgth}_{B_{\mathcal{P}_i}}(B_{\mathcal{P}_i}) \cdot [B/\mathcal{P}_i]$. The multiplicity of the intersection of $B/\mathcal{P}_i$ with $f^{-1}D$ (at its only point $\mathcal{Q}$) is $\text{lgth}_B(B/(\mathcal{P}_i + aB))$, hence the multiplicity of $f^*[V] \cdot f^{-1}D$ at $\mathcal{Q}$ is:

$$\sum_i \text{lgth}_{B_{\mathcal{P}_i}}(B_{\mathcal{P}_i}) \cdot \text{lgth}_B(B/(\mathcal{P}_i + aB)) . \quad (1.54)$$

The multiplicity of the intersection of V with D (at its only point q) is $\text{lgth}_A(A/aA)$, and the pull-back of the point q of A has multiplicity $\text{lgth}_B(B/q)$, hence the multiplicity of $f^*[V \cdot D]$ is $\text{lgth}_A(A/aA) \cdot \text{lgth}_B(B/q)$, which by the following lemma, applied to A/a and B/aB , agrees with $\text{lgth}_B(B/aB)$. Noting that a is not a zero-divisor in B , the latter agrees with (1.54) by Lemma 5.

11 Lemma 11 Let $A \rightarrow B$ be a flat homomorphism of zero-dimensional artinian local rings, then $\text{lgth}_B(B) = \text{lgth}_A(A) \cdot \text{lgth}_B(B/\mathfrak{m}_A B)$.

Proof See also [23, Lemma A.4.1]. There is a finite sequence of ideals I_i of A , say of length r , such that the quotients I_i/I_{i+1} are isomorphic to $A/\mathfrak{m}_A$. Then $r = \text{lgth}_A A$, and tensoring with B , we get a chain of ideals $B \otimes_A I_i$ of B with quotients $B \otimes_A A/\mathfrak{m}_A \cong B/\mathfrak{m}_A B$. Thus $\text{lgth}_B(B) = r \cdot \text{lgth}_B(B/\mathfrak{m}_A B)$.

A.5 Proper Push-forward

In this section we suppose that our schemes are of finite type over an excellent base. This holds for example if the base is the spectrum of a Dedekind ring of characteristic 0, or a field. Given a proper map $f : X \rightarrow Y$ and a cycle V in X , we define the proper push-forward to be

$$f_*[V] = \begin{cases} [k(V) : k(f(V))] \cdot [f(V)] & \text{if } \dim V = \dim f(V) \\ 0 & \text{if } \dim V > \dim f(V) . \end{cases}$$

12 Proposition 12 Let $f : X \rightarrow Y$ be a proper map. Then intersection with an effective principal divisor D of Y is compatible with push-forward, i.e. if V is a closed subscheme of X not contained in $f^{-1}D$, then $f_*[V \cdot f^{-1}D] = f_*[V] \cdot D$ as cycles on Y .

Proof Let $\text{Spec } A \subseteq Y$ be an affine neighborhood of the generic point of $f(V)$, $\text{Spec } B \subseteq X$ an affine neighborhood of the generic point of V mapping to $\text{Spec } A$, and let $a \in A$ be an equation for D . Let $\mathcal{P} \subseteq B$ and $\mathfrak{p} \subseteq A$ be the prime ideals corresponding to V and $f(V)$, respectively. If we denote the map $A \rightarrow B$ by g , then $\mathfrak{p} = g^{-1}\mathcal{P}$.

First assume that $f_*[V] = 0$, i.e. $\dim B/\mathcal{P} > \dim A/\mathfrak{p}$. Since V is not contained in $f^{-1}D$, we have $a \notin \mathcal{P}$, hence by Krull's principal ideal theorem, we get for any minimal ideal $\mathcal{Q}$ of B containing $\mathcal{P} + aB$ (corresponding to a component of $V \cap f^{-1}D$) with inverse image $\mathfrak{q} = g^{-1}\mathcal{Q}$ in A ,

$$\dim B/\mathcal{Q} \geq \dim B/\mathcal{P} - 1 > \dim A/\mathfrak{p} - 1 \geq \dim A/\mathfrak{q}.$$

This implies $f_*[V \cdot f^{-1}D] = 0$.

In general, we can divide out $\mathfrak{p}$ and $\mathcal{P}$ from A and B , respectively (which amounts to replacing X by V , and Y by the closed subscheme $f(V)$), and assume that A and B are integral domains. It suffices to consider the multiplicities at each irreducible component of $f(V) \cap D$, i.e. we can localize A and B at a minimal ideal $\mathfrak{q}$ of A containing a . Then A is a one-dimensional local integral domain and B a one-dimensional semi-local integral domain with maximal ideals $\mathcal{Q}_1, \dots, \mathcal{Q}_r$ (corresponding to the irreducible components of $V \cap f^{-1}D$ which map to the component of $f(V) \cap D$ corresponding to $\mathfrak{q}$). Let K and L be the fields of quotients of A and B , respectively. We need to show that

$$\mathrm{lgth}_A(A/aA) \cdot [L : K] = \sum_{\mathcal{Q}} \mathrm{lgth}_{B_{\mathcal{Q}}}(B_{\mathcal{Q}}/aB_{\mathcal{Q}}) \cdot [B_{\mathcal{Q}}/\mathcal{Q} : A/\mathfrak{q}]. \quad (1.55)$$

Let $\tilde{A}$ and $\tilde{B}$ be the integral closures of A and B in K and L , respectively. Note that $\tilde{A}$ and $\tilde{B}$ are finitely generated over A because the base is supposed to be excellent, hence a Nagata ring. Let $\mathfrak{m}$ and $\mathfrak{n}$ run through the maximal ideals of $\tilde{A}$ and $\tilde{B}$, respectively. The localizations $\tilde{A}_{\mathfrak{m}}$ and $\tilde{B}_{\mathfrak{n}}$ are discrete valuation rings. The following lemma, applied to A and K gives

$$\mathrm{lgth}_A(A/aA) = \sum_{\mathfrak{m}} \mathrm{lgth}_{\tilde{A}_{\mathfrak{m}}}(\tilde{A}_{\mathfrak{m}}/a\tilde{A}_{\mathfrak{m}}) \cdot [\tilde{A}_{\mathfrak{m}}/\mathfrak{m} : A/\mathfrak{q}].$$

Applying the lemma to $\tilde{A}_{\mathfrak{m}}$ and L , we get

$$[L : K] \cdot \mathrm{lgth}_{\tilde{A}_{\mathfrak{m}}}(\tilde{A}_{\mathfrak{m}}/a\tilde{A}_{\mathfrak{m}}) = \sum_{\mathfrak{n}|\mathfrak{m}} \mathrm{lgth}_{\tilde{B}_{\mathfrak{n}}}(\tilde{B}_{\mathfrak{n}}/a\tilde{B}_{\mathfrak{n}}) \cdot [\tilde{B}_{\mathfrak{n}}/\mathfrak{n} : \tilde{A}_{\mathfrak{m}}/\mathfrak{m}].$$

By multiplicativity of the degree of field extensions, the left hand side of (1.55) is

$$\sum_{\mathfrak{m}} \sum_{\mathfrak{n}|\mathfrak{m}} \mathrm{lgth}_{\tilde{B}_{\mathfrak{n}}}(\tilde{B}_{\mathfrak{n}}/a\tilde{B}_{\mathfrak{n}}) \cdot [\tilde{B}_{\mathfrak{n}}/\mathfrak{n} : A/\mathfrak{q}].$$

On the other hand, if we apply the lemma to $B_{\mathcal{Q}}$ and L , we get

$$\mathrm{lgth}_{B_{\mathcal{Q}}}(B_{\mathcal{Q}}/aB_{\mathcal{Q}}) = \sum_{\mathfrak{n}|\mathcal{Q}} \mathrm{lgth}_{\tilde{B}_{\mathfrak{n}}}(\tilde{B}_{\mathfrak{n}}/a\tilde{B}_{\mathfrak{n}}) \cdot [\tilde{B}_{\mathfrak{n}}/\mathfrak{n} : B/\mathcal{Q}].$$

Hence the right hand side of (1.55) becomes

$$\sum_{\mathcal{Q}} \sum_{\mathfrak{n}|\mathcal{Q}} \mathrm{lgth}_{\tilde{B}_{\mathfrak{n}}}(\tilde{B}_{\mathfrak{n}}/a\tilde{B}_{\mathfrak{n}}) \cdot [\tilde{B}_{\mathfrak{n}}/\mathfrak{n} : A/\mathfrak{q}].$$

We only need to show that for every maximal ideal $\mathfrak{n}$ of $\tilde{B}$, there is a maximal ideal $\mathcal{Q}$ of B which it divides. This follows from the valuation property of properness. Indeed, consider the commutative square

$$\begin{array}{ccc} \mathrm{Spec} L & \rightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec} \tilde{B}_{\mathfrak{n}} & \rightarrow & Y \end{array}$$

The image in X of the closed point $\mathfrak{n}$ of $\mathrm{Spec} \tilde{B}_{\mathfrak{n}}$ under the unique lift $\mathrm{Spec} \tilde{B}_{\mathfrak{n}} \rightarrow X$ provides an ideal $\mathcal{Q}$ in B with $\mathfrak{n}|\mathcal{Q}$.

13

Lemma 13 Let A be a one-dimensional local integral domain with maximal ideal $\mathfrak{p}$ and quotient field K . Let L be an extension of K of degree n and $\tilde{A}$ be the integral closure of A in L . Assume that $\tilde{A}$ is finitely generated over A . Then for any $a \in A - \{0\}$,

$$n \cdot \mathrm{lgth}_A(A/aA) = \mathrm{lgth}_A(\tilde{A}/a\tilde{A}) = \sum_{\mathfrak{m}|\mathfrak{p}} \mathrm{lgth}_{\tilde{A}_{\mathfrak{m}}}(\tilde{A}_{\mathfrak{m}}/a\tilde{A}_{\mathfrak{m}}) \cdot [\tilde{A}_{\mathfrak{m}}/\mathfrak{m} : A/\mathfrak{p}] .$$

Proof Choose a K -basis of L consisting of elements of $\tilde{A}$. This basis generates a free A -submodule F of rank n of $\tilde{A}$, with finitely generated torsion quotient $\tilde{A}/F$. Any finitely generated torsion A -module M has a composition series with graded pieces $A/\mathfrak{p}$, hence is of finite length. Then the sequence

$$0 \rightarrow {}_aM \rightarrow M \rightarrow M \rightarrow M/a \rightarrow 0$$

shows that $\mathrm{lgth}_A({}_aM) = \mathrm{lgth}_A(M/a)$. Mapping the short exact sequence $0 \rightarrow F \rightarrow \tilde{A} \rightarrow \tilde{A}/F \rightarrow 0$ to itself by multiplication by a , we have $\mathrm{lgth}_A(\tilde{A}/a\tilde{A}) = \mathrm{lgth}_A(F/aF) = n \cdot \mathrm{lgth}_A(A/aA)$ by the snake lemma.

We now show more generally that for any $\tilde{A}$ -module M of finite length,

$$\mathrm{lgth}_A(M) = \sum_{\mathfrak{m}|\mathfrak{p}} \mathrm{lgth}_{\tilde{A}_{\mathfrak{m}}}(M_{\mathfrak{m}}) \cdot [\tilde{A}_{\mathfrak{m}}/\mathfrak{m} : A/\mathfrak{p}] .$$

The statement is additive on exact sequences, so we can reduce to the case $M = \tilde{A}/\mathfrak{m} \cong \tilde{A}_{\mathfrak{m}}/\mathfrak{m}$ for one of the maximal ideals of $\tilde{A}$. But in this case,

$$\mathrm{lgth}_A(\tilde{A}/\mathfrak{m}) = \mathrm{lgth}_{A/\mathfrak{p}}(\tilde{A}_{\mathfrak{m}}/\mathfrak{m}) = [\tilde{A}_{\mathfrak{m}}/\mathfrak{m} : A/\mathfrak{p}] .$$

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K-Theory and Intersection Theory

Henri Gillet

2.1	<i>Introduction</i>	237
	Conventions.....	239
2.2	<i>Chow groups</i>	240
	Dimension and Codimension	240
	Cycles	240
	Dimension Relative to a Base	242
	Cartier Divisors	243
	Cap Products with Cartier Divisors and the Divisor Homomorphism	245
	Rational Equivalence.....	246
	Basic Properties of Chow Groups	249
2.3	<i>K-Theory and Intersection Multiplicities</i>	253
	Serre's tor Formula	253
	K_0 with Supports	254
	The Coniveau Filtration and the γ -Filtration	258
2.4	<i>Complexes Computing Chow Groups</i>	259
	Higher Rational Equivalence and Milnor K -Theory	259
	Rost's Axiomatics	261
2.5	<i>Higher Algebraic K-Theory and Chow Groups</i>	266
	Stable Homotopy Theory	266
	Filtrations on the Cohomology of Simplicial Sheaves	268
	Review of Basic Notions of K -Theory	270
	Quillen's Spectral Sequence	272
	K -Theory as Sheaf Hypercohomology	274

Gersten's Conjecture, Bloch's Formula and the Comparison of Spectral Sequences	275
The Coniveau Spectral Sequence for Other Cohomology Theories.....	277
Compatibility with Products and Localized Intersections	278
Other Cases of Gersten's Conjecture	279
Operations on the Quillen Spectral Sequence	280
The Multiplicativity of the Coniveau Filtration: a Proof Using Deformation to the Normal Cone.....	281
2.6 <i>Bloch's Formula and Singular Varieties</i>	284
Cohomology Versus Homology	284
Local Complete Intersection Subschemes and Other Cocyles	286
Chow Groups of Singular Surfaces	287
Intersection Theory on Stacks	287
2.7 <i>Deformation to the Normal Cone</i>	288
2.8 <i>Envelopes and Hyperenvelopes</i>	288
<i>References</i>	289

Introduction

The problem of defining intersection products on the Chow groups of schemes has a long history. Perhaps the first example of a theorem in intersection theory is Bézout's theorem, which tells us that two projective plane curves C and D , of degrees c and d and which have no components in common, meet in at most cd points. Furthermore if one counts the points of $C \cap D$ with multiplicity, there are *exactly* cd points. Bezout's theorem can be extended to closed subvarieties Y and Z of projective space over a field k , $\mathbb{P}_k^n$, with $\dim(Y) + \dim(Z) = n$ and for which $Y \cap Z$ consists of a finite number of points.

When the ground field $k = \mathbb{C}$, Bezout's theorem can be proved using integral cohomology. However, prior to the development of étale cohomology for curves over fields of characteristic p , one had to use algebraic methods to prove Bezout's theorem, and there is still no cohomology theory which makes proving similar theorems over an arbitrary base, including $\text{Spec}(\mathbb{Z})$, possible.

In this chapter we shall outline the two approaches to intersection theory that are currently available. One method is to reduce the problem of defining the intersection product of arbitrary cycles to intersections with divisors. The other method is to use the product in K -theory to define the product of cycles. The difference between the two perspectives on intersection theory is already apparent in the possible definitions of intersection multiplicities. The first definition, due to Weil and Samuel, first defines the multiplicities of the components of the intersection of two subvarieties Y and Z which intersect properly in a variety X , when Y is a local complete intersection in X , by reduction to the case of intersection with divisors. For general Y and Z intersecting properly in a smooth variety X , the multiplicities are defined to be the multiplicities of the components of the intersection of $Y \times Z$ with the diagonal Δ_X in $X \times X$. The key point here is that Δ_X is a local complete intersection in $X \times X$. The second construction is Serre's "tor formula", which is equivalent to taking the product, in K -theory with supports, of the classes $[\mathcal{O}_Y] \in K_0^Y(X)$ and $[\mathcal{O}_Z] \in K_0^Z(X)$. This definition works for an arbitrary regular scheme X , since it does not involve reduction to the diagonal.

The Chow ring of a smooth projective variety X over a field was first constructed using the moving lemma, which tells us that, given two arbitrary closed subvarieties Y and Z of X , Z is rationally equivalent to a cycle $\sum_i n_i [W_i]$ in which all the W_i meet Y properly. One drawback of using the moving lemma is that one expects that $[Y].[Z]$ should be able to be constructed as a cycle on $Y \cap Z$, since (for example) using cohomology with supports gives a cohomology class supported on the intersection. A perhaps less important drawback is that it does not apply to non-quasi-projective varieties.

This problem was solved, by Fulton and others, by replacing the moving lemma by reduction to the diagonal and deformation to the tangent bundle. One can then prove that intersection theory for varieties over fields is determined by intersections with Cartier divisors, see Fulton's book *Intersection Theory* [17] for details.

For a general regular scheme X , $X \times X$ will not be regular, and the diagonal map $\Delta_X \rightarrow X \times X$ will not be a regular immersion. Hence we cannot use deformation

to the normal cone to construct a product on the Chow groups. In SGA6, [2], Grothendieck and his collaborators showed that, when X is regular $\mathrm{CH}^*(X)_{\mathbb{Q}} \simeq \mathrm{Gr}_Y^*(K_0(X))_{\mathbb{Q}}$, which has a natural ring structure, and hence one can use the product on K -theory, which is induced by the tensor product of locally free sheaves, to define the product on $\mathrm{CH}^*(X)_{\mathbb{Q}}$. Here $\mathrm{Gr}_Y^*(K_0(X))$ is the graded ring associated to the Y -filtration $F_Y(K_0(X))$. By construction, this filtration is automatically compatible with the product structure on $K_0(X)$, and was introduced because the filtration that is more naturally related to the Chow groups, the coniveau or codimension filtration $F_{\mathrm{cod}}(K_0(X))$, is not tautologically compatible with products. However, in SGA6 Grothendieck proved, using the moving lemma, that if X is a smooth quasi-projective variety over a field, then $F_{\mathrm{cod}}(K_0(X))$ is compatible with products. In this chapter, using deformation to the normal cone, we give a new proof of the more general result that the coniveau filtration $F_{\mathrm{cod}}(K_*(X))$ on the entire K -theory ring is compatible with products.

Instead of looking at the group $K_0(X)$, we can instead filter the category of coherent sheaves on X by codimension of support. We then get a filtration on the K -theory spectrum of X and an associated spectral sequence called the Quillen, or coniveau, spectral sequence. The Chow groups appear as part of the E_2 -term of this spectral sequence, while $\mathrm{Gr}_{\mathrm{cod}}^* K_*(X)$ is the E_{∞} term. The natural map $\mathrm{CH}^*(X) \rightarrow \mathrm{Gr}_{\mathrm{cod}}^*(K_0(X))$ then becomes an edge homomorphism in this spectral sequence. The E_1 -terms of this spectral sequence form a family of complexes $R_q^*(X)$ for $q \geq 0$, with $H^q(R_q^*(X)) \simeq \mathrm{CH}^q(X)$.

Let us write $\mathcal{R}_{X,q}^*$ for the complex of sheaves with $\mathcal{R}_{X,q}^*(U) \simeq R_q^*(U)$ for $U \subset X$ an open subset; we shall refer to these as the Gersten complexes. Gersten's conjecture (Sect. 2.5.6) implies that the natural augmentation $K_q(\mathcal{O}_X) \rightarrow \mathcal{R}_{X,q}^*$ is a quasi-isomorphism, which in turn implies Bloch's formula:

$$H^q(X, K_q(\mathcal{O}_X)) \simeq \mathrm{CH}^q(X).$$

If X is a regular variety over a field, Quillen proved Gersten's conjecture, so that Bloch's formula is true in that case. (Bloch proved the $q = 2$ case by different methods.) For regular varieties over a field, this then gives another construction of a product on $\mathrm{CH}^*(X)$, which one may prove is compatible with the product defined geometrically.

If X is a regular scheme of dimension greater than 0, the E_1 -term of the associated Quillen spectral sequence does not have an obvious multiplicative structure. (Having such a product would imply that one can choose intersection *cycles* in a fashion compatible with rational equivalence.) However, there is another spectral sequence (the Brown spectral sequence) associated to the Postnikov tower of the presheaf of K -theory spectra on X which is naturally multiplicative. In general there is map from the Brown spectral sequence to the Quillen spectral sequence, which maps E_r to E_{r+1} . If this map is an isomorphism, then the Quillen spectral sequence is compatible with the product on K -theory from E_2 on, and it follows that the coniveau filtration on $K_*(X)$ is also compatible with the ring structure on K -theory. This map of spectral sequences is a quasi-isomorphism if Gersten's con-

jecture is true. Thus we have another proof of the multiplicativity of the coniveau filtration $F_{\text{cod}}(K_0(X))$, which depends on Gersten's conjecture, rather than using deformation to the normal cone.

The groups $H_Y^p(X, K_q(\mathcal{O}_X))$, for all p and q , and for all pairs $Y \subset X$ with Y a closed subset of a regular variety X , form a bigraded cohomology theory with nice properties, including homotopy invariance and long exact sequences, for pairs $Y \subset X$ with Y closed in X :

$$\begin{aligned} \dots \rightarrow H^{p-1}(X - Y, K_q(\mathcal{O}_X)) &\xrightarrow{\partial} H_Y^p(X, K_q(\mathcal{O}_X)) \\ &\rightarrow H^p(X, K_q(\mathcal{O}_X)) \rightarrow H^p(X - Y, K_q(\mathcal{O}_X)) \rightarrow \dots \end{aligned}$$

However, from the perspective of intersection theory, these groups contain a lot of extraneous information; if one looks at the weights of the action of the Adams operations on the Quillen spectral sequence, then it was shown in [64] that after tensoring with $\mathbb{Q}$, the spectral sequence breaks up into a sum of spectral sequences, all but one of which gives no information about the Chow groups, and that the E_1 term of the summand which computes the Chow groups can be described using Milnor K -theory tensored with the rational numbers.

It is natural to ask whether one can build a “smallest” family of complexes with the same formal properties that the Gersten complexes have, and which still computes the Chow groups. As explained in Sect. 2.4.1, the “obvious” relations that must hold in a theory of “higher rational equivalence” are also the relations that define the Milnor K -theory ring as quotient of the exterior algebra of the units in a field. The remarkable fact, proved by Rost, is that for smooth varieties over a field, the obvious relations are enough, *i.e.*, the cycle complexes constructed using Milnor K -theory have all the properties that one wants. Deformation to the normal cone plays a key role in constructing the product on Rost's cycle complexes. This result is strong confirmation that to build intersection theory for smooth varieties over a field, one needs only the theory of intersections of divisors, together with deformation to the tangent bundle. Rost also proves, though this is *not* needed for his result, that the analog of Gersten's conjecture holds for the complexes built out of Milnor K -theory.

Thus for smooth varieties over a field, we have a good theory of Chow groups and higher rational equivalence, whether we use Gersten's conjecture or deformation to the normal cone. For general regular schemes, there is not an obvious analog of reduction to the diagonal and deformation to the tangent bundle. Gersten's conjecture still makes sense; however a new idea is needed in order to prove it. It is perhaps worth noting that a weaker conjecture, that for a regular local ring R , $\text{CH}^p(\text{Spec}(R)) = 0$ for all $p > 0$ (see [12]), is still open.

Conventions

2.1.1

- Schemes will be assumed to be separated, noetherian, finite dimensional, and excellent. See EGA IV.7.8, [36], for a discussion of excellent schemes. We shall refer to these conditions as the “standard assumptions”.

Any separated scheme which is of finite type over a field or $\mathrm{Spec}(\mathbb{Z})$ automatically satisfies these hypotheses.

- By a variety we will simply mean a scheme which is of finite type over a field.
- A scheme is said to be integral if it is reduced and irreducible.
- The natural numbers are $1, 2, \dots$.

2.2 Chow groups

In this section we shall give the basic properties of divisors and Chow groups on general schemes, and sketch the two geometric constructions of the intersection product for varieties over fields, via the moving lemma and via deformation to the normal cone.

2.2.1 Dimension and Codimension

Normally one is used to seeing the group of cycles on a scheme equipped with a grading – however it is important to remember that dimension is not always a well behaved concept. In particular, for general noetherian schemes, while one may think of cycles as *homological* objects, it is the grading by *codimension* that is well defined.

The best reference for the dimension theory of general schemes is EGA IV, §5, [36]. We shall summarize here some of the main points.

Recall that any noetherian local ring has finite Krull dimension. Therefore, if X is a noetherian scheme, any integral subscheme $Z \subset X$ with generic point $\zeta \in X$, has finite codimension, equal to the Krull dimension of the noetherian local ring $\mathcal{O}_{X,\zeta}$. We will also refer to this as the codimension of the point ζ .

1

Definition 1 A scheme (or more generally a topological space) X is said to be: *Catenary*, if given irreducible closed subsets $Y \subset Z \subset X$, all maximal chains of closed subsets between Y and Z have the same length.

Finite Dimensional, if there is a (finite) upper bound on the length of chains of irreducible closed subsets.

A scheme S , is said to be *universally catenary* if every scheme of finite type over S is catenary. Any excellent scheme is universally catenary.

2.2.2 Cycles

2

Definition 2 Let X be a scheme (not necessarily satisfying the standard assumptions). A cycle on X is an element of the free abelian group on the set of closed integral subschemes of X . We denote the group of cycles by $Z(X)$.

Since the closed integral subschemes of X are in one to one correspondence with the points of X , with an integral subscheme $Z \subset X$ corresponding to its generic point ζ , we have:

$$Z(X) := \bigoplus_{\zeta \in X} \mathbb{Z}.$$

For a noetherian scheme, this group may be graded by codimension, and we write $Z^p(X)$ for the subgroup consisting of the free abelian group on the set of closed integral subschemes of codimension p in X . If every integral subscheme of X is finite dimensional, we can also grade the group $Z(X)$ by dimension, writing $Z_p(X)$ for the free abelian group on the set of closed integral subschemes of dimension p in X . If X is a noetherian, catenary and finite dimensional scheme, which is also equidimensional (*i.e.*, all the components of X have the same dimension), of dimension d then the two gradings are just renumberings of each other: $Z_{d-p}(X) = Z^p(X)$. However if X is not equidimensional, then codimension and dimension do *not* give equivalent gradings:

Example 3. Suppose that k is a field, and that $X = T \cup S$, with $T := \mathbb{A}_k^1$ and $S := \mathbb{A}_k^2$, is the union of the affine line and the affine plane, with $T \cap S = \{P\}$ a single (closed) point P . Then X is two dimensional. However any closed point in T , other than P has dimension 0, and codimension 1, while P has dimension 0 and codimension 2.

Definition 4 If X is a general noetherian scheme, we write $X^{(p)}$ for the set of points $x \in X$, which are of “codimension p ”, *i.e.*, such that the integral closed subscheme $\overline{\{x\}} \subset X$ has codimension p , or equivalently, the local ring $\mathcal{O}_{X,x}$ has Krull dimension p .

We also write $X_{(p)}$ for the set of points $x \in X$ such that the closed subset $\overline{\{x\}} \subset X$ is finite dimensional of dimension p .

Observe that

$$Z^p(X) \simeq \bigoplus_{x \in X^{(p)}} \mathbb{Z},$$

while, if X is finite dimensional,

$$Z_q(X) \simeq \bigoplus_{x \in X_{(q)}} \mathbb{Z}.$$

Cycles of codimension 1, *i.e.* elements of $Z^1(X)$, are also referred to as Weil divisors.

If $Z \subset X$ is an closed integral subscheme, we will write $[Z]$ for the associated cycle, and will refer to it as a “prime cycle”. An element $\zeta \in Z(X)$ will then be written $\zeta = \sum_i n_i [Z_i]$.

If $\mathcal{M}$ is a coherent sheaf of $\mathcal{O}_X$ -modules, let us write $\text{supp}(\mathcal{M}) \subset X$ for its support. For each irreducible component Z of the closed subset $\text{supp}(\mathcal{M}) \subset X$, the stalk $\mathcal{M}_\zeta$ of $\mathcal{M}$ at the generic point ζ of Z is an $\mathcal{O}_{X,\zeta}$ -module of finite length.

Definition 5 The cycle associated to $\mathcal{M}$ is:

$$[\mathcal{M}] := \sum_{\zeta} \ell(\mathcal{M}_\zeta) [Z],$$

where the sum runs over the generic points ζ of the irreducible components Z of $\text{supp}(\mathcal{M})$, and $\ell(\mathcal{M}_\zeta)$ denotes the length of the Artinian $\mathcal{O}_\zeta$ module $\mathcal{M}_\zeta$.

If $Y \subset X$ is a closed subscheme, we set $[Y] := [\mathcal{O}_Y]$; notice that if Y is an integral closed subscheme then this is just the prime cycle $[Y]$.

It will also be convenient to have:

Definition 6 Let $W \subset X$ be a closed subset. Then $Z_W^p(X) \subset Z^p(X)$ is the subgroup generated by those cycles supported in W , i.e., of the form $\sum_i n_i [Z_i]$ with $Z_i \subset W$.

If $U \subset V \subset X$ are Zariski open subsets, and $\zeta = \sum_i n_i [Z_i] \in Z^p(V)$ is a codimension p cycle, then $\zeta|_U := \sum_i n_i [Z_i \cap U]$ is a codimension p cycle on U . The maps $Z^p(V) \rightarrow Z^p(U)$, for all pairs $U \subset V$ define a sheaf on the Zariski topology of X , $\mathcal{Z}_X^p$ which is clearly flasque. Note that $Z_W^p(X) = H_W^0(X, \mathcal{Z}_X^p)$, and also that if U is empty then $Z^p(U)$ is the free abelian group on the empty set, i.e., $Z^p(U) \simeq 0$.

2.2.3 Dimension Relative to a Base

Dimension can behave in ways that seem counter-intuitive. For example, if $U \subset X$ is a dense open subset of a scheme, U may have strictly smaller dimension than X . The simplest example of this phenomenon is if X is the spectrum of a discrete valuation ring, so $\dim(X) = 1$, and U is the Zariski open set consisting of the generic point, so $\dim(U) = 0$. One consequence of this phenomenon is that the long exact sequence of Chow groups associated to the inclusion of an open subset into a scheme will not preserve the grading by dimension.

However, dimension is well behaved with respect to proper morphisms:

Theorem 7 Let $f : W \rightarrow X$ be a proper surjective morphism between integral schemes which satisfy our standing hypotheses. Then $\dim(W) = \dim(X) + \text{tr. deg.}_{k(X)} k(W)$. In particular, if f is birational and proper, then $\dim(W) = \dim(X)$.

Proof [36], proposition 5.6.5.

This leads to the notion of *relative dimension*. By a theorem of Nagata, any morphism $f : X \rightarrow S$ of finite type is compactifiable. *I.e.*, it may be factored as $\bar{f} \cdot i$:

$$\begin{array}{ccc} X & \xrightarrow{i} & \bar{X} \\ & \searrow \bar{f} & \downarrow \\ & f & S \end{array}$$

with i an open immersion with dense image, and $\bar{f}$ proper.

Lemma 8 If S and X are as above, the dimension of $\bar{X}$ minus the dimension of S is independent of the choice of compactification $\bar{X}$.

8

Proof This is a straightforward consequence of Theorem 7.

Therefore we may make the following definition:

Definition 9 Let S be a fixed base, satisfying our standing hypotheses. If X is a scheme of finite type over S , we set $\dim_S(X) := \dim(\bar{X}) - \dim(S)$, where $\bar{X}$ is any compactification of X over S .

9

The key feature of relative dimension is that if X is a scheme of finite type over S and $U \subset X$ is a dense open, then $\dim_S(X) = \dim_S(U)$. It follows that the grading of the Chow groups of schemes of finite type over S by dimension relative to S is compatible with proper push-forward. In this respect, Chow homology behaves like homology with locally compact supports; see [15], where this is referred to as LC-homology.

Note that if the base scheme S is the spectrum of a field, or of the ring of integers in a number field, relative dimension and dimension give equivalent gradings, differing by the dimension of the base, on the cycle groups.

An equivalent approach to the definition of $\dim_S(X)$ may be found in Fulton's book on intersection theory ([17]) using transcendence degree.

Cartier Divisors

2.2.4

The starting point for intersection theory from the geometric point of view is the definition of intersection with a Cartier divisor.

Let X be a scheme. If $U \subset X$ is an open set, a section $f \in \mathcal{O}_X(U)$ is said to be *regular* if, for every $x \in U$, its image in the stalk $\mathcal{O}_{X,x}$ is a non-zero divisor. The

regular sections clearly form a subsheaf $\mathcal{O}_{X,\text{reg}}$ of the sheaf of monoids (with respect to multiplication) $\mathcal{O}_X$. The sheaf of total quotient rings $\mathcal{K}_X$ is the localization of $\mathcal{O}_X$ with respect to $\mathcal{O}_{X,\text{reg}}$. Note that the sheaf of units $\mathcal{K}_X^*$ is the sheaf of groups associated to the sheaf of monoids $\mathcal{O}_{X,\text{reg}}$, and that the natural map $\mathcal{O}_X \rightarrow \mathcal{K}_X$ is injective.

10

Definition 10 Write $\mathcal{D}\text{iv}_X$ for the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. The group $\text{Div}(X)$ of Cartier divisors on X is defined to be $H^0(X, \mathcal{D}\text{iv}_X)$. Note that we will view this as an *additive* group.

If $D \in \text{Div}(X)$, we write $|D|$ for the *support* of D , which is of codimension 1 in X if D is non-zero. A Cartier divisor D is said to be *effective* if it lies in the image of

$$H^0(X, \mathcal{O}_{X,\text{reg}}) \rightarrow H^0(X, \mathcal{D}\text{iv}_X \simeq \mathcal{K}_X^*/\mathcal{O}_X^*).$$

For details, see [37] IV part 4, §21. See also the article of Kleiman [43] for pathologies related to the sheaf of total quotient rings on a non-reduced scheme.

There is a long exact sequence:

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}_X^*) \rightarrow H^0(X, \mathcal{K}_X^*) \rightarrow H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \rightarrow \\ H^1(X, \mathcal{O}_X^*) \rightarrow H^1(X, \mathcal{K}_X^*) \rightarrow \dots \end{aligned}$$

Recall that a Cartier divisor is said to be *principal* if it is in the image of

$$H^0(X, \mathcal{K}_X^*) \rightarrow H^0(X, \mathcal{D}\text{iv}_X = \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Two Cartier divisors are said to be linearly equivalent if their difference is principal. From the long exact sequence above we see that there is always an *injection* of the group of linear equivalence classes of Cartier divisors into the Picard group $H^1(X, \mathcal{O}_X^*)$ of isomorphism classes of rank one locally free sheaves. If $H^1(X, \mathcal{K}_X^*) \simeq 0$ (for example if X is reduced), this injection becomes an isomorphism. Note that there are examples of schemes for which the map from the group of Cartier divisors to $\text{Pic}(X)$ is *not* surjective. See the paper [44] of Kleiman for an example.

More generally, if $W \subset X$ is a closed subset, we can consider $H_W^1(X, \mathcal{O}_X^*)$, i.e., the group of isomorphism classes of pairs $(\mathcal{L}, s)$ consisting of an invertible sheaf $\mathcal{L}$ and a non-vanishing section $s \in H^0(X - W, \mathcal{L})$. Then one has:

11

Lemma 11 If X is reduced and irreducible, so that $\mathcal{K}_X^*$ is the constant sheaf, and W if has codimension at least one, then

$$H_W^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \simeq H_W^1(X, \mathcal{O}_X^*).$$

If D is a Cartier divisor on X , the subsheaf of $\mathcal{K}_X^*$ which is the inverse image of D is an $\mathcal{O}_X^*$ torsor – the sheaf of equations of D . The $\mathcal{O}_X$ submodule of $\mathcal{K}_X$ generated by this subsheaf is invertible; i.e., it is a fractional ideal. The *inverse* of this sheaf is denoted $\mathcal{O}_X(D)$ and its class is the image of D in $H^1(X, \mathcal{O}_X^*)$ under the boundary map, see [37].

Cap Products with Cartier Divisors and the Divisor Homomorphism

2.2.5

There is a natural map from the group of Cartier divisors to the group of Weil divisors:

Lemma 12 Let X be a scheme. Then there is a unique homomorphism of sheaves:

12

$$\mathrm{div} : \mathcal{D}\mathrm{iv}_X \rightarrow \mathcal{Z}_X^1,$$

such that, if $U \subset X$ is an open set and $f \in \mathcal{O}_{X,\mathrm{reg}}(U)$ is a regular element, then $\mathrm{div}(f) = [\mathcal{O}_U/(f)]$ – the cycle associated to the codimension one subscheme with equation f .

Proof See [37] §21.6.

If X is regular, or more generally locally factorial, one can show that this map is an isomorphism.

If X is an integral scheme, then since $\mathcal{K}_X$ is the constant sheaf associated to the function field $\mathbf{k}(X)$ of X , we get a homomorphism, also denoted div :

$$\mathrm{div} : \mathbf{k}(X)^* \rightarrow \mathcal{Z}^1(X)$$

Remark 13 Observe that if X is a scheme and $D \subset X$ is a codimension 1 subscheme, the ideal sheaf $\mathcal{I}_D$ of which is locally principal, then the Cartier divisor given by the local generators of $\mathcal{I}_D$ has divisor equal to the cycle $[D] = [\mathcal{O}_D]$.

13

Definition 14 Suppose that $D \in \mathrm{Div}(X)$ is a Cartier divisor, and that $Z \subset X$ is an irreducible subvariety, such that $|D| \cap Z$ is a proper subset of Z . The Cartier divisor D determines an invertible sheaf $\mathcal{O}_X(D)$, equipped with a trivialization outside of $|D|$. Restricting $\mathcal{O}_X(D)$ to Z , we get an invertible sheaf $\mathcal{L}$ equipped with a trivialization s on $Z - (Z \cap |D|)$. Since Z is irreducible, and $Z \cap |D|$ has codimension at least 1 in Z , by Lemma 11 $H_{Z \cap |D|}^1(Z, \mathcal{O}_Z^*) \simeq H_{Z \cap |D|}^0(Z, \mathcal{D}\mathrm{iv}_Z)$, and hence the pair $(\mathcal{L}, s)$ determines a Cartier divisor on Z , which we write $D|_Z$, and which we call the *restriction* of D to Z .

14

Definition 15 We define the *cap product* $D \cap [Z]$ to be $\mathrm{div}(D|_Z) \in \mathcal{Z}^1(Z)$.

15

Rational Equivalence

16 **Definition 16** If X is a scheme, the direct sum

$$R(X) := \bigoplus_{\zeta \in X} \mathbf{k}(\zeta)^*$$

where $\mathbf{k}(\zeta)^*$ is the group of units in the residue field of the point ζ , will be called the group of “ K_1 -chains” on X . For a noetherian scheme, this group has a natural grading, in which

$$R^q(X) := \bigoplus_{x \in X^{(q)}} \mathbf{k}(x)^* .$$

We call this the group of codimension q K_1 -chains.

If X is finite dimensional, then we can also grade $R(X)$ by dimension:

$$R_p(X) := \bigoplus_{x \in X_{(p)}} \mathbf{k}(x)^* .$$

If X is catenary and equidimensional, these gradings are equivalent.

The sum of the homomorphisms $\text{div} : \mathbf{k}(Z)^* \rightarrow Z^1(Z)$, as Z runs through all integral subschemes $Z \subset X$, induces a homomorphism, for which we use the same notation:

$$\text{div} : R(X) \rightarrow Z(X) .$$

We say that a cycle in the image of div is *rationally equivalent to zero*.

17 **Definition 17** If X is a general scheme, then we set the (ungraded) Chow group of X equal to:

$$\text{CH}(X) := \text{coker}(\text{div}) .$$

Now suppose that X satisfies our standing assumptions. The homomorphism div is of pure degree -1 with respect to the grading by dimension (or by relative dimension for schemes over a fixed base), and we set $\text{CH}_q(X)$ equal to the cokernel of

$$\text{div} : R_{q+1}(X) \rightarrow Z_q(X) .$$

The homomorphism div is not in general of pure degree $+1$ with respect to the grading by codimension, unless X is equidimensional, but it does increase

codimension by at least one, and so we define $\mathrm{CH}^p(X)$ to be the cokernel of the induced map:

$$\begin{aligned} \mathrm{div} : \bigoplus_{x \in X^{(p-1)}} \mathbf{k}(x)^* &\rightarrow \bigoplus_{x \in X^{(p)}} \mathbb{Z} \\ f = \sum_x \{f_x\} &\mapsto \sum_x \mathrm{div}(\{f_x\}) . \end{aligned}$$

We refer to these as the Chow groups of *dimension* q and *codimension* p cycles on X , respectively.

If X is equidimensional of dimension d , then dimension and codimension are compatible, i.e., $\mathrm{CH}^p(X) \simeq \mathrm{CH}_{d-p}(X)$.

The classical definition of rational equivalence of cycles on a variety over a field was that two cycles α and β were rationally equivalent if there was a family of cycles ζ_t , parameterized by $t \in \mathbb{P}^1$, with $\zeta_0 = \alpha$ and $\zeta_\infty = \beta$. More precisely, suppose that W is an irreducible closed subvariety of $X \times \mathbb{P}^1$, which is flat over $\mathbb{P}^1$ (i.e., not contained in a fiber of $X \times \{t\}$, for $t \in \mathbb{P}^1$). For each $t \in \mathbb{P}^1$, $(X \times \{t\})$ is a Cartier Divisor, which is the pull back, via the projection to $\mathbb{P}^1$, of the Cartier divisor $[t]$ corresponding to the point $t \in \mathbb{P}^1$. Then one sets, for $t \in \mathbb{P}^1$, $W_t := [W] \cdot (X \times \{t\})$. (Note that $[W]$ and the Cartier divisor $(X \times \{t\})$ intersect properly). The cycle $W_\infty - W_0$ is then said to be rationally equivalent to zero. (Note that $[0]$ and $[\infty]$ are linearly equivalent Cartier divisors.) More generally a cycle is rationally equivalent to zero if it is the sum of such cycles.

It can be shown that this definition agrees with the one given previously, though we shall not use this fact here.

Just as we sheafified the cycle functors to get flasque sheaves $\mathcal{Z}_X^p$, we have flasque sheaves $\mathcal{R}_X^q$, with $\mathcal{R}_X^q(U) = R^q(U)$. The divisor homomorphism then gives a homomorphism:

$$\mathrm{div} : \mathcal{R}_X^{q-1} \rightarrow \mathcal{Z}_X^q$$

and an isomorphism:

$$\mathrm{CH}^q(X) \simeq \mathbb{H}^1(X, \mathcal{R}_X^{q-1} \rightarrow \mathcal{Z}_X^q) .$$

Lemma 18 The map $\mathrm{Div}(X) \rightarrow \mathrm{CH}^1(X)$ induced by div factors through $\mathrm{Pic}(X)$, and vanishes on principal divisors.

18

Proof If $x \in X^{(0)}$ is a generic point of X , then the local ring $\mathcal{O}_{X,x}$ is Artinian, and $\mathcal{O}_{X,\mathrm{reg},x} = \mathcal{O}_{X,x}^*$. Hence $\mathcal{K}_{X,x} = \mathcal{O}_{X,x}$, and so there is a natural homomorphism

$\mathcal{K}_{X,x}^* \rightarrow \mathbf{k}(x)^*$. Therefore there is a commutative diagram of maps of sheaves of abelian groups:

$$\begin{array}{ccccc}
 \mathcal{O}_X^* & \longrightarrow & \mathcal{K}_X^* & \longrightarrow & \mathcal{R}_X^0 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{D}\mathrm{iv}_X & \xrightarrow{\mathrm{div}} & \mathcal{Z}_X^1
 \end{array}$$

Hence we get maps:

$$\begin{aligned}
 \mathrm{Div}(X) &\rightarrow H^1(X, \mathcal{O}_X^*) \simeq \mathbb{H}^1(X, \mathcal{K}_X^* \rightarrow \mathcal{D}\mathrm{iv}_X) \\
 &\rightarrow \mathbb{H}^1(X, \mathcal{R}_X^0 \rightarrow \mathcal{Z}_X^1) \simeq \mathrm{CH}^1(X) .
 \end{aligned}$$

It follows by a diagram chase that the map from Cartier divisors on X to Weil divisors X induces a map from linear equivalence classes of Cartier divisors to rational equivalence classes of Weil divisors which factors through $\mathrm{Pic}(X)$.

From this lemma and the intrinsic contravariance of $H_W^1(X, \mathcal{O}_X^*)$, we get the following proposition.

19

Proposition 19 Let $f : Y \rightarrow X$ be a morphism of varieties. Let $W \subset X$ be a closed subset, suppose that $\phi \in H_W^0(X, \mathcal{D}\mathrm{iv}_X)$ is a Cartier divisor with supports in W , and that $\zeta \in Z_T^p(Y)$ is a cycle supported in a closed subset $T \subset Y$. Then there is a natural “cap” product cycle class $\phi \cap \zeta \in \mathrm{CH}_{(T \cap f^{-1}(W))}^{p+1}(Y)$.

Note that if ϕ above is a principal effective divisor, given by a regular element $g \in \Gamma(X, \mathcal{O}_X)$, which is invertible on $X - W$, and if $\zeta = [Z]$ is the cycle associated to a reduced irreducible subvariety $Z \subset X$, with $f(Z) \not\subset W$, then $f^*(g)|_Z$ is again a regular element, and $\phi \cap \zeta$ is the divisor associated to $f^*(g)|_Z$ discussed in Definition 15.

Finally, the other situation in which we can pull back Cartier divisors is if $f : X \rightarrow Y$ is a flat morphism of schemes. Since f is flat, if $x \in X$, and $t \in \mathcal{O}_{Y,f(x)}$ is a regular element, then $f^*(t)$ is a regular element in $\mathcal{O}_{X,x}$. It follows that there is a pull-back

$$f^* : f^{-1}\mathcal{K}_Y \rightarrow \mathcal{K}_X ,$$

and hence an induced map on Cartier divisors.

Basic Properties of Chow Groups

2.2.7

Functoriality

Let $f : X \rightarrow Y$ be a morphism of schemes.

If f is *flat*, then there is a pull-back map $f^* : Z^p(Y) \rightarrow Z^p(X)$, preserving codimension. If $Z \subset X$ is an closed integral subscheme, then:

$$f^* : [Z] \mapsto [\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Z] ,$$

which is then extended to the full group of cycles by linearity.

If f is *proper*, then there is a push-forward map on cycles; if Z is a k -dimensional closed integral subscheme, then:

$$f_*([Z]) = \begin{cases} [k(Z) : k(f(Z))] [f(Z)] & \text{if } \dim(f(Z)) = \dim(Z) \\ 0 & \text{if } \dim(f(Z)) < \dim(Z) . \end{cases}$$

Push-forward preserves dimension, or dimension relative to a fixed base S , by Theorem 7.

Proposition 20 Both flat pull-back and proper push-forward are compatible with rational equivalence, and therefore induce maps on Chow groups. *I.e.*, if $f : X \rightarrow Y$ is a morphism of schemes, we have maps:

20

$$f^* : \mathrm{CH}^p(Y) \rightarrow \mathrm{CH}^p(X)$$

for f flat, and

$$f_* : \mathrm{CH}_q(X) \rightarrow \mathrm{CH}_q(Y)$$

for f proper.

See [17], for proofs, at least for maps between varieties over fields. For general schemes, there is a proof in [24] using algebraic K -theory.

Intersection Multiplicities and the Moving Lemma

The product structure on the Chow groups of a smooth quasi-projective variety over a field was first constructed in the 1950's. See Séminaire Chevalley ([1]), exposés 2 and 3. Two key steps in the construction are:

- Defining intersection multiplicities.
- The Moving Lemma.

Suppose that X is a noetherian scheme. Two closed subsets Y and Z of X , of codimensions p and q , respectively, are said to *intersect properly* if every irreducible component W of $Y \cap Z$ has codimension $p + q$ – note that this is vacuously true if Y and Z do not intersect. Two cycles $\eta = \sum_i m_i [Y_i]$ and $\zeta = \sum_j n_j [Z_j]$ are said

to *meet properly* if their supports intersect properly. If two prime cycles $[Y]$ and $[Z]$ meet properly, then the problem of *intersection multiplicities* is to assign an integer $\mu_W(Y, Z)$ to each irreducible component W of $Y \cap Z$. These multiplicities should have the property that if one defines the product of two prime cycles $[Y]$ and $[Z]$ to be $[Y] \cdot [Z] = \sum_W \mu_W(Y, Z)[W]$, then one gets a ring structure on the Chow groups.

To get a well defined product, one would certainly require that if η and ζ are two cycles which meet properly, and η (resp. ζ) is rationally equivalent to a cycle η' (resp. ζ'), such that η' and ζ' meet properly, then $\eta \cdot \zeta$ and $\eta' \cdot \zeta'$ are rationally equivalent, and that every pair of cycles η and ζ is rationally equivalent to a pair η' and ζ' which meet properly. Finally, one requires, that the product with divisors be consistent with intersection with divisors.

Since $\eta \cdot \zeta$ is defined when η is a divisor, then it will also be defined when $\eta = \alpha_1 \cdot \dots \cdot \alpha_p$ is the successive intersection product of divisors. (Of course one should worry whether this product is independent of the choice of the α_i .) Thus one will be able to define $[Y] \cdot \zeta$ when $Y \subset X$ is a closed subvariety which is globally a complete intersection in X . If Y is only a local complete intersection, and if W is an irreducible component of $Y \cap Z$ with generic point w , then one defines the intersection multiplicity $\mu_W(Y, Z)$ by using the fact that Y is a complete intersection in a Zariski open neighborhood of w .

The original definition of the intersection multiplicities of the components of the intersection of two closed integral subschemes $Y \subset X$ and $Z \subset X$ which meet properly on a smooth variety X over a field, was given by Samuel [59], when one of them, Y say, is a local complete intersection subscheme of X . One then defines the multiplicities for general integral subschemes Y and Z of a smooth variety X , by observing that $Y \cap Z = \Delta_X \cap (Y \times_k Z)$, where $\Delta_X \subset X \times_k X$ is the diagonal, and then setting the $\mu_W(Y, Z) = \mu_{\Delta_W}(\Delta_X, Y \times_k Z)$. Note that Δ_X is an l.c.i. subvariety if and only if X is smooth.

Once given a definition of multiplicity, one has an intersection product for cycles which meet properly. The next step is:

21

Theorem 21: *Chow's Moving Lemma* Suppose that X is a smooth quasi-projective variety over a field k , and that Y and Z are closed integral subschemes of X . Then the cycle $[Y]$ is rationally equivalent to a cycle η which meets Z properly.

Proof See [1] and [55].

22

Theorem 22 Let X be a smooth quasi-projective variety over a field k . Then one has:

- Given elements α and β in the Chow ring, let η and ζ be cycles representing them which meet properly (these exist by the moving lemma). Then the class in $\text{CH}^*(X)$ of $\eta \cdot \zeta$ is independent of the choice of representatives η and ζ , and depends only on α and β .

- The product on $\mathrm{CH}^*(X)$ that this defines is commutative and associative.
- Given an arbitrary (*i.e.*, not necessarily flat) morphism $f : X \rightarrow Y$ between smooth projective varieties, there is a pull back map $\mathrm{CH}^*(Y) \rightarrow \mathrm{CH}^*(X)$, making $X \mapsto \mathrm{CH}^*(X)$ is a contravariant functor from the category of quasi-projective smooth varieties to the category of commutative rings.

Proof See Séminaire Chevalley, exposés 2 and 3 in [1].

There are several drawbacks to this method of constructing the product on the Chow ring:

- It only works for X smooth and quasi-projective over a field.
- It does not respect supports. It is reasonable to expect that the intersection product of two cycles should be a cycle supported on the set-theoretic intersection of the support of the two cycles, but the Moving Lemma does not “respect supports”.
- It requires a substantial amount of work to check that this product is well defined and has all the properties that one requires.

In the next section, we shall see an alternative geometric approach to this problem.

Intersection via Deformation to the Normal Cone

Let us recall the goal: one wishes to put a ring structure on $\mathrm{CH}^*(X)$, for X a smooth quasi-projective variety, and one wants this ring structure to have various properties, including compatibility with intersections with Cartier divisors. The approach of Fulton, [17], as it applies to smooth varieties over a field, can be summarized in the following theorem:

Theorem 23 On the category of smooth, not necessarily quasi-projective, varieties over a field, there is a *unique* contravariant graded ring structure on CH^* such that:

23

1. It agrees with flat pull-back of cycles when $f : X \rightarrow Y$ is flat.
2. It agrees with the product $\mathrm{CH}^1(X) \times \mathrm{CH}^p(X) \rightarrow \mathrm{CH}^{p+1}(X)$ induced by intersection with Cartier divisors, for all X and p .
3. If V and W are arbitrary integral closed subschemes a smooth variety X , then we have an equality of cycles on $X \times_k X$:

$$[V \times_k X] \cdot [X \times_k W] = [V \times_k W]$$

4. If $f : X \rightarrow Y$ is a proper map between nonsingular varieties, and $\alpha \in \mathrm{CH}^*(X), \beta \in \mathrm{CH}^*(Y)$, then

$$f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta.$$

(The projection formula)

5. If $p : V \rightarrow X$ is a vector bundle over a variety X , then the flat pull-back map $p^* : \mathrm{CH}^*(X) \rightarrow \mathrm{CH}^*(V)$ is an isomorphism. (Homotopy Invariance).

Sketch of Proof

Any map $f : Y \rightarrow X$ of smooth varieties can be factored into $Y \xrightarrow{\Gamma_f} Y \times X \xrightarrow{\pi_X} X$, with Γ_f the graph of f , and π_X the projection map. Since π_X is flat, to define $f^* : \mathrm{CH}^*(X) \rightarrow \mathrm{CH}^*(Y)$ we need only define Γ_f^* .

Therefore we need only construct the pull-back map for a general regular immersion $f : Y \rightarrow X$. First we need (see [17], sects. 2.3 and 2.4, especially corollary 2.4.1):

24

Lemma 24: *Specialization* Let $D \subset S$ be a principal divisor in the scheme S . Since $\mathcal{O}_S(D)|_D$ is trivial, intersection with D , $\cap[D] : \mathrm{CH}^*(D) \rightarrow \mathrm{CH}^{*-1}(D)$, is zero. It follows that $\cap[D] : \mathrm{CH}^*(S) \rightarrow \mathrm{CH}^*(D)$ factors through $\mathrm{CH}^*(S - D)$.

Let $W_{Y/X}$ be the associated deformation to the normal bundle space (see [3] and appendix 2.7). Since the special fiber $W_0 \subset W_{Y/X}$ is a principal divisor, there is an associated specialization map

$$\sigma : \mathrm{CH}^*(W_{Y/X} - N_{Y/X}) \simeq X \times \mathbb{G}_m \rightarrow \mathrm{CH}^*(W_0) .$$

Composing with the flat pull-back:

$$\mathrm{CH}^*(X) \rightarrow \mathrm{CH}^*(X \times \mathbb{G}_m) ,$$

we get a map

$$\mathrm{CH}^*(X) \rightarrow \mathrm{CH}^*(N_{Y/X}) \simeq \mathrm{CH}^*(Y) ,$$

where $\mathrm{CH}^*(N_{Y/X}) \simeq \mathrm{CH}^*(Y)$ by homotopy invariance.

It is not difficult to show, using homotopy invariance, that this must agree with the pull-back map.

Finally, to get the product, one simply composes the pull-back along the inclusion of the diagonal with the external product

$$\mathrm{CH}^*(X) \times \mathrm{CH}^*(X) \rightarrow \mathrm{CH}^*(X \times X) .$$

This geometric construction avoids any need to give a definition of intersection multiplicity, and also shows that for any two cycles Y and Z , $Y \cdot Z$ is naturally a cycle on the intersection $\mathrm{supp}(Y) \cap \mathrm{supp}(Z)$.

Corresponding to this cohomology theory on the category of non-singular varieties over a field k , we also have *Chow Homology* groups, defined for *all* varieties over k :

Definition 25 If X is a (possibly singular) variety over a field, let $Z_p(X)$ be the group of *dimension* p cycles on X , and $\text{CH}_p(X)$ the corresponding quotient by rational equivalence. These groups are *covariant* functors with respect to *proper* morphisms between varieties, and *contravariant* with respect to flat maps (but with a degree shift by the relative dimension).

25

K-Theory and Intersection Multiplicities

2.3

Serre's tor Formula

2.3.1

While deformation to the normal cone tells us that intersection theory is unique, given a collection of reasonable axioms, one can ask if there is an intrinsic, purely algebraic, description of intersection multiplicities, and in particular a definition that is valid on any regular scheme. A solution to this problem was given by Serre in his book [61].

If R is a noetherian local ring, an R -module has finite length if and only if it is supported at the closed point of $\text{Spec}(R)$, and K_0 of the category of modules of finite length is isomorphic, by dévissage, to K_0 of the category of vector spaces over the residue field k of R , *i.e.*, to $\mathbb{Z}$. Given an R -module M of finite length, we write $\ell(M)$, for its length.

Definition 26 Suppose that R is a regular local ring, and that M and N are finitely generated R -modules, the supports of which intersect only at the closed point of $\text{Spec}(R)$, then Serre defines their intersection multiplicity:

26

$$\chi(M, N) := \sum_{i \geq 0} (-1)^i \ell(\text{Tor}_i^R(M, N)) .$$

In his book, Serre proved:

Theorem 27 The multiplicity defined above agrees with Samuel's multiplicity, when that is defined.

27

Theorem 28 If R is essentially of finite type over a field, and if the codimensions of the supports of M and N sum to more than the dimension of R , then the intersection multiplicity vanishes, while if the sum is *equal* to the dimension of R , the intersection multiplicity is (strictly) positive.

28

Idea of Proof

There are two key points:

- **Reduction to the diagonal.** If R is a k -algebra, and M and N are R -modules, then $M \otimes_R N \simeq R \otimes_{R \otimes_k R} (M \otimes_k N)$. Thus if M and N are flat k -modules, as is the case when k is a field, to understand $\mathrm{Tor}_*^R(M, N)$, it is enough to understand $\mathrm{Tor}_*^{R \otimes_k R}(R, \_)$.
- **Koszul Complexes.** If R is regular local ring which is a localization of an algebra which is smooth over a field k , then a choice of system of parameters for R determines a finite free resolution of R as an $R \otimes_k R$ -algebra by a Koszul complex. One proves positivity, first for intersections with principal effective Cartier divisors, and then using induction on the number of parameters.

Serre conjectured:

29 Conjecture 29 The conclusion of the theorem holds for any regular local ring.

The vanishing conjecture was proved in 1985 by Roberts [56] and independently by Gillet and Soulé [21]. Non-negativity (but not strict positivity) was proved by Gabber in 1996. Gabber's proof uses, in an essential fashion, de Jong's theorem [14] on the existence of non-singular alterations of varieties over discrete valuation rings. Gabber did not publish his proof, but there are various expositions of it, for example by Berthelot in his Bourbaki exposé on the work of de Jong [4] and by Roberts [57].

2.3.2 K_0 with Supports

Serre's definition of intersection multiplicity can be rephrased using K_0 with supports. (We shall discuss higher K -theory with supports later).

Let X be a noetherian scheme. Then if $Y \subset X$ is a closed subset, we define $K_0^Y(X)$ to be the quotient of the Grothendieck group of bounded complexes of locally free coherent sheaves of $\mathcal{O}_X$ -modules, having cohomology with supports in Y , by the subgroup of classes of acyclic complexes.

There is a natural map

$$K_0^Y(X) \rightarrow G_0(Y)$$

$$[\mathcal{E}^*] \mapsto \sum_i (-1)^i [\mathcal{H}^i(\mathcal{E}^*)] .$$

If X is a regular noetherian scheme this map is an isomorphism, because every coherent sheaf has a resolution by locally free sheaves.

If Y and Z are closed subsets of X , then there is a natural product

$$K_0^Y(X) \otimes K_0^Z(X) \rightarrow K_0^{Y \cap Z}(X)$$

given by

$$[\mathcal{E}^*] \otimes [\mathcal{F}^*] \mapsto [\mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{F}^*] .$$

(The definition of the tensor product of two complexes may be found, for example, in [70]).

If X is regular then this induces a pairing:

$$G_0(Y) \otimes G_0(Z) \rightarrow G_0(Y \cap Z)$$

$$[\mathcal{E}] \otimes [\mathcal{F}] \mapsto \sum_i (-1)^i [\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})] .$$

Therefore we see that Serre's intersection multiplicity is a special case of the product in K -theory with supports. *I.e.*, if $X = \text{Spec}(R)$, with R a regular local ring, and if M and N are finitely generated R -modules with supports $Y \subset X$ and $Z \subset X$ respectively, such that $Y \cap Z = \{x\}$, with $x \in X$ the closed point, then $\chi(M, N) = [M] \cdot [N] \in K_0^{(x)}(X) \simeq \mathbb{Z}$, where $[M] \in K_0^Y(X)$ is the class of any projective resolution of M , and similarly for $[N]$.

The Filtration by Codimension of Supports

Definition 30 A family of supports on a topological space X is a collection Φ of closed subsets of X which is closed under finite unions, and such that any closed subset of a member of Φ is also in Φ . Given two families of supports Φ and Ψ we set $\Phi \wedge \Psi$ equal to the family generated by the intersections $Y \cap Z$ with $Y \in \Phi$ and $Z \in \Psi$.

30

Definition 31 Let X be a scheme, and let Φ be a family of supports on X . Then

31

$$K_0^\Phi(X) := \lim_{\rightarrow Y \in \Phi} K_0^Y(X)$$

Clearly there is a product

$$K_0^\Phi(X) \otimes K_0^\Psi(X) \rightarrow K_0^{\Phi \wedge \Psi}(X) .$$

For intersection theory, the most important families of supports are $X^{\geq i}$, the closed subsets of codimension at least i , and $X_{\leq j}$, the subsets of dimension at most j .

32 **Definition 32** The *filtration by codimension of supports*, or *coniveau filtration*, is the decreasing filtration, for $i \geq 0$:

$$F_{\text{cod}}^i(K_0(X)) := \text{Image} \left(K_0^{X \geq i}(X) \rightarrow K_0(X) \right) .$$

Similarly, we can consider the coniveau filtration on $G_0(X)$ where $F^i(G_0(X))$ is the subgroup of $G_0(X)$ generated by the classes of those $\mathcal{O}_X$ -modules $[\mathcal{M}]$ for which $\text{codim}(\text{Supp}(\mathcal{M})) \geq i$.

We shall write $\text{Gr}_{\text{cod}}^*(K_0(X))$ and $\text{Gr}_{\text{cod}}^*(G_0(X))$ for the associated graded groups.

The Coniveau Filtration and Chow Groups

If $Y \subset X$ is a codimension p subscheme of a Noetherian scheme, then $[\mathcal{O}_Y] \in F^p(G_0(X))$. Thus we have a map:

$$Z^p(X) \rightarrow F_{\text{cod}}^p(G_0(X)) .$$

By dévissage, *i.e.*, the fact that every coherent sheaf has a filtration with quotients which are coherent sheaves on, and have supports equal to, closed integral subschemes, ([2] appendix to exp. 0, prop. 2.6.), we have:

33 **Lemma 33** If X is a noetherian scheme, the induced map

$$Z^p(X) \rightarrow \text{Gr}_{\text{cod}}^p(G_0(X))$$

is *surjective*.

34 **Theorem 34** For an arbitrary noetherian scheme, this map factors through $\text{CH}^p(X)$.

Proof The original proof due to Grothendieck, is in *op. cit.*, appendix to exp. 0, Theorem 2.12. One can also observe that the homomorphism of Lemma 33 is simply an edge homomorphism from $Z^p(X) = E_1^{p,-p}$ to $\text{Gr}_{\text{cod}}^p(G_0(X)) = E_{\infty}^{p,-p}$ in the Quillen spectral sequence (Sect. 2.5.4 below) which factors through $E_2^{p,-p} \simeq \text{CH}^p(X)$.

Because there is such a close relationship between the Chow groups and $\text{Gr}_{\text{cod}}^* K_0$, including the fact that Serre's definition of intersection multiplicities is via the product in K -theory, it is reasonable to ask whether the product structure on K -theory is compatible with the coniveau filtration, *i.e.*, whether $F_{\text{cod}}^p(K_0(X)) \cdot F_{\text{cod}}^q(K_0(X)) \subset F_{\text{cod}}^{p+q}(K_0(X))$.

Observe that this is not true at the level of modules. *I.e.*, if $\mathcal{E}^*$ and $\mathcal{F}^*$ are complexes of locally free sheaves of $\mathcal{O}_X$ -modules which have their cohomology

sheaves supported in codimensions p and q respectively, then it is *not* in general true that $\mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{F}^*$ has its cohomology sheaves in codimension at least $p + q$.

The following theorem was proved by Grothendieck, using Chow's moving lemma; see [2], appendix to exp. 0, §4, corollary 1 to theorem 2.12.

Theorem 35 If X is a smooth quasi-projective variety over a field, then the product structure on $K_0(X)$ is compatible with the coniveau filtration. 35

Using the Riemann–Roch theorem for a closed immersion between smooth (not necessarily quasi-projective) varieties, one can extend Grothendieck's result to all smooth varieties. See [32] for details. Later, in Sect. 2.5.11, we shall prove the analogous result for $K_p(X)$ for all $p \geq 0$, again for general smooth varieties over a field, using deformation to the normal cone rather than the moving lemma. There is also another proof of this more general result, which uses Quillen's theorem that Gersten's conjecture is true for non-singular varieties, together with the homotopy theory of sheaves of spectra, in Sect. 2.5.6 below.

In general, one conjectures:

Conjecture 36: *Multiplicativity of the coniveau filtration* If X is a regular noetherian scheme, the product on K -theory respects the filtration by codimension of supports, and $*$ is the product on K -theory: 36

$$F_{\text{cod}}^i(K_0^Y(X)) * F_{\text{cod}}^j(K_0^Z(X)) \subset F_{\text{cod}}^{i+j}(K_0^{Y \cap Z}(X))$$

Note that:

Proposition 37 Conjecture 36 implies Serre's vanishing conjecture. 37

Proof Suppose that R is a regular local ring of dimension n , and that M and N are finitely generated R -modules, supported on closed subsets Y (of codimension p) and Z (of codimension q) of $X = \text{Spec}(R)$. Suppose also that $Y \cap Z = \{x\}$, where $x \in X$ is the closed point. Then $[M] \in F_{\text{cod}}^p(K_0^Y(X))$, $[N] \in F_{\text{cod}}^q(K_0^Z(X))$, and

$$\chi(M, N) = [M] \cup [N] \in F_{\text{cod}}^{p+q}(K_0^{\{x\}}(X)) .$$

Now $K_0^{\{x\}}(X) \simeq \mathbb{Z}[\mathbf{k}(x)]$, with $[\mathbf{k}(x)] \in F_{\text{cod}}^n(K_0^{\{x\}}(X)) \setminus F_{\text{cod}}^{n+1}(K_0^{\{x\}}(X))$. Therefore if $p + q > n$, we have

$$\chi(M, N) = [M] \cup [N] \in F_{\text{cod}}^{n+1}(K_0^{\{x\}}(X)) \simeq 0 .$$

In the next section, we shall sketch how Conjecture 36 can be proved, *after tensoring with $\mathbb{Q}$* , following the method of [21].

2.3.3

The Coniveau Filtration and the γ -Filtration

In [2] Grothendieck constructed a product on $\mathrm{CH}^*(X)_{\mathbb{Q}}$, for X a regular scheme, by constructing a *multiplicative* filtration $F_{\gamma}^*(K_0(X))$ on K_0 , such that there are Chern classes with values in the graded ring $\mathrm{Gr}_{\gamma}^*(K_0(X))_{\mathbb{Q}}$. He then used the Chern classes to define an isomorphism:

$$\mathrm{CH}^*(X)_{\mathbb{Q}} \simeq \mathrm{Gr}_{\gamma}^* K_0(X)_{\mathbb{Q}} ,$$

and hence a product on $\mathrm{CH}^*(X)_{\mathbb{Q}}$.

For any scheme, there are operations $\lambda^i : K_0(X) \rightarrow K_0(X)$, defined by taking exterior powers: $\lambda^i([\mathcal{E}]) = [\bigwedge^i(\mathcal{E})]$. Note that these are not group homomorphisms, but rather $\lambda^n(x + y) = \sum_{i=0}^n \lambda^i(x) \lambda^{(n-i)}(y)$.

38 Definition 38 The γ -operations are defined by:

$$\gamma^n : K_0(X) \rightarrow K_0(X)$$

$$\gamma^n : x \mapsto \lambda^n(x + (n-1)[\mathcal{O}_X]) .$$

The γ -filtration $F_{\gamma}^*(K_0(X))$ is defined by setting $F_{\gamma}^1(K_0(X))$ equal to the subgroup generated by classes that are (locally) of rank zero, and then requiring that if $x \in F_{\gamma}^1(K_0(X))$ then $\gamma^i(x) \in F_{\gamma}^i(K_0(X))$, and that the filtration be multiplicative, i.e., $F_{\gamma}^i(K_0(X)) \cdot F_{\gamma}^j(K_0(X)) \subset F_{\gamma}^{i+j}(K_0(X))$, so that the associated graded object $\mathrm{Gr}_{\gamma}^* K_0(X)$ is a commutative ring, which is contravariant with respect to X .

Recall, following [38], that a theory of Chern classes with values in a cohomology theory A^* associates to every locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$ -modules on a scheme X , classes $C_k(\mathcal{E}) \in A^k(X)$, for $k \geq 0$ such that

1. $C_0(\mathcal{E}) = 1$.
2. The map $\mathcal{L} \rightarrow C_1(\mathcal{L}) \in A^1(X)$ defines a natural transformation $\mathrm{Pic} \rightarrow A^1$.
3. If

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

is an exact sequence of locally free sheaves, then, for all $n \geq 0$, we have the *Whitney sum formula*:

$$C_n(\mathcal{F}) = \sum_{i=0}^n C_i(\mathcal{E}) C_{n-i}(\mathcal{G}) .$$

39 Proposition 39: *Grothendieck, [2]* The natural transformations which assign to a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$ -modules, the elements, for $k \geq 1$,

$$C_k(\mathcal{E}) := \gamma^k([\mathcal{E}] - \mathrm{rk}(\mathcal{E})) \in \mathrm{Gr}_{\gamma}^k K_0(X) ,$$

satisfy the axioms for Chern classes.

Corresponding to these classes there is a Chern character which is a natural transformation:

$$ch : K_0(X) \rightarrow \mathrm{Gr}_Y^*(K_0(X))_{\mathbb{Q}}$$

The following theorem is proved in [2], theorem VII 4.11, where there is the extra assumption that X possesses an ample sheaf. However as explained in [32], the same proof, with only minor modifications, works for general regular schemes. This theorem also follows from the results of Soulé ([64]), where the result is also proved for K_1 and K_2 , by studying the action of the Adams operations on the Quillen spectral sequence; see Theorem 78 below.

Theorem 40 If X is a regular scheme, the Chern character induces an isomorphism:

40

$$\mathrm{Gr}_{cod}^*(K_0(X))_{\mathbb{Q}} \rightarrow \mathrm{Gr}_Y^*(K_0(X))_{\mathbb{Q}}$$

Furthermore, there is an isomorphism, for each $k \geq 0$

$$\mathrm{CH}^k(X)_{\mathbb{Q}} \rightarrow \mathrm{Gr}_Y^k(K_0(X))_{\mathbb{Q}}$$

$$[Y] \mapsto ch_k([\mathcal{O}_Y])$$

Corollary 41 If X is a regular Noetherian scheme, the coniveau filtration on $K_0(X)_{\mathbb{Q}}$ is multiplicative.

41

In [21], it is shown that one can also construct lambda operations on K -theory with supports, and that, after tensoring with $\mathbb{Q}$, the coniveau filtration and γ -filtrations on the K_0 , with supports in a closed subset, of a finite dimensional regular noetherian scheme are isomorphic, and hence the coniveau filtration (tensor $\mathbb{Q}$) is multiplicative. An immediate consequence of this result is Serre's vanishing conjecture for general regular local rings. Robert's proof in [56] used Fulton's operational Chow groups, which give an alternative method of constructing the product on $\mathrm{CH}^*(X)_{\mathbb{Q}}$ for a general regular Noetherian scheme.

Complexes Computing Chow Groups

2.4

Higher Rational Equivalence and Milnor K -Theory

2.4.1

Suppose that X is a noetherian scheme, and that $Y \subset X$ is a closed subset with complement $U = X - Y$. Then we have short exact sequences (note that $Z(Y)$ and $R(Y)$ are independent of the particular subscheme structure we put on Y):

$$0 \rightarrow R(Y) \rightarrow R(X) \rightarrow R(U) \rightarrow 0$$

and

$$0 \rightarrow Z(Y) \rightarrow Z(X) \rightarrow Z(U) \rightarrow 0$$

and hence an exact sequence:

$$\text{Ker}(\text{div} : R(U) \rightarrow Z(U)) \rightarrow \text{CH}(Y) \rightarrow \text{CH}(X) \rightarrow \text{CH}(U) \rightarrow 0.$$

It is natural to ask if this sequence can be extended to the left, and whether there is a natural notion of rational equivalence between K_1 -chains. In particular are there elements in the kernel $\text{Ker}(\text{div} : R(U) \rightarrow Z(U))$ that obviously have trivial divisor, and so will map to zero in $\text{CH}(Y)$?

Let us start by asking, given a scheme X , whether there are elements in $R(X)$ which obviously have divisor 0. First of all, any $f \in k(x)^*$ which has valuation zero for all discrete valuations of the field $k(x)$ is in the kernel of the divisor map. However since we are dealing with general schemes, the only elements of $k(x)$ which we can be sure are of this form are ± 1 .

Suppose now that X is an integral scheme, and that f and g are two rational functions on X , i.e., elements of $k(X)^*$, such that the Weil divisors $\text{div}(f)$ and $\text{div}(g)$ have no components in common. Writing $\{f\}$ and $\{g\}$ for the two (principal) Cartier divisors defined by these rational functions, we can consider the two cap products:

$$\{f\} \cap \text{div}(g) = \text{div}\{f|_{\text{div}(g)}\}$$

and

$$\{g\} \cap \text{div}(f) = \text{div}\{g|_{\text{div}(f)}\}.$$

Here $\{f\}|_{\sum_i [Y_i]} := \sum_i \{f\}|_{[Y_i]}$, where $\{f\}|_{[Y_i]}$ denotes, equivalently, the restriction of f either as a Cartier divisor (Definition 14), or simply as a rational function which is regular at the generic point of Y_i . If one supposes that the cap product between Chow cohomology and Chow homology is to be associative, and that the product in Chow cohomology is to be commutative, then we should have:

$$\begin{aligned} \{f\} \cap (\{g\} \cap [X]) &= (\{f\} * \{g\}) \cap [X] \\ &= (\{g\} * \{f\}) \cap [X] \\ &= \{g\} \cap (\{f\} \cap [X]). \end{aligned}$$

i.e. the K_1 -chain

$$g|_{\text{div}(f)} - f|_{\text{div}(g)}$$

should have divisor zero.

That this is indeed the case follows from the following general result, in which $\{f\}$ and $\{g\}$ are replaced by general Cartier divisors.

Proposition 42: *Commutativity of Intersections of Cartier Divisors* Let X be an integral scheme, and suppose that ϕ and ψ are two Cartier divisors, with $\text{div}(\phi) = \sum_i n_i [Y_i]$ and $\text{div}(\psi) = \sum_j m_j [Z_j]$ their associated Weil Divisors. Then

$$\sum_i n_i \text{div}(\psi|_{Y_i}) = \sum_j m_j \text{div}(\phi|_{Z_j}) .$$

The original proof of this result, in [24], used higher algebraic K -theory, and depended on the properties of the coniveau spectral sequence for K -theory ([53]). However if one wants to avoid proofs using K -theory, then for varieties over fields this is proved in Fulton's book ([17], theorem 2.4), and there is a purely algebraic proof of the general case by Kresch in [46].

Thus every pair of rational functions (f, g) , as above, gives rise to a K_1 -chain with trivial divisor. This suggests that one could view such K_1 -chains as being rationally equivalent to zero, *i.e.* that one should extend the complex $R(X) \rightarrow Z(X)$ to the left by $\bigoplus_x k(x)^* \otimes k(x)^*$, with “div” $(f \otimes g) = f|_{\text{div}(g)} - g|_{\text{div}(f)}$. Since

$$\text{“div” } (f \otimes g) = -\text{“div” } (g \otimes f) ,$$

it seems reasonable to impose the relation $f \otimes g + g \otimes f = 0$. Again it is natural to ask what elements are obviously in the kernel of this map. An element $f \otimes g \in k(x)^* \otimes k(x)^*$ such that $f \equiv 1 \pmod{g}$, and $g \equiv 1 \pmod{f}$, will be in the kernel of the map “div”, and the elements of $k(x)^* \otimes k(x)^*$ that we can be sure are of this type are those of the form $f \otimes g$ with $f + g = 1$.

This leads naturally to the quotient of the exterior algebra $\bigwedge^*(F) = \bigoplus_n \bigwedge_{\mathbb{Z}}^n F^*$ (of F^* viewed as a $\mathbb{Z}$ -module) by the two-sided ideal I generated by elements of the form $f \otimes g$ with $f + g = 1$:

Definition 43 If F is a field, its Milnor K -theory is defined to be:

$$K_*^M(F) := \bigwedge^*(F)/I .$$

Note that the relation $\{f, g\} = -\{g, f\} \in K_2^M(F)$ can be deduced from the relation $\{f, 1 - f\} = 0$, and hence one can equally write:

$$K_*^M(F) := T^*(F)/I ,$$

where $T^*(F)$ is the tensor algebra of F .

Rost's Axiomatics

The fact that Milnor K -theory appears so naturally when trying to construct a complex that computes the Chow groups is fully explored in the paper [58] of Rost, where he proves that one need impose no more relations, or add any more generators, to get a theory which has very nice properties.

Rost considers a more general structure, which includes Milnor K -theory as a special case.

44

Definition 44 A (graded) cycle module is a covariant functor M from the category of fields (over some fixed base scheme S) to the category of $\mathbb{Z}$ -graded (or $\mathbb{Z}/2$ -graded) Abelian groups, together with:

1. Transfers $\mathrm{tr}_{E/F} : M(E) \rightarrow M(F)$, of degree zero, for every finite extension $F \subset E$.
2. For every discrete valuation v of a field F a residue or boundary map $\partial_v : M(F) \rightarrow M(k(v))$ of degree -1 .
3. A pairing, for every F , $F^* \times M(F) \rightarrow M(F)$ of degree 1, which extends to a pairing $K_*^M(F) \times M(F) \rightarrow M(F)$ which makes $M(F)$ a graded module over the Milnor- K -theory ring.

These data are required to satisfy axioms which may be found in *op. cit.*, Definitions 1.1 and 2.1.

A cycle module M is said to be a cycle module with *ring structure* if there is a pairing $M \times M \rightarrow M$, respecting the grading, which is compatible with the cycle module structure; see *op. cit.* Definition 1.2.

Milnor K -theory itself is a cycle module with ring structure. This follows from results of Bass and Tate, of Kato, and of Milnor; see [58], theorem 1.4. We shall see later (Theorem 65) that the same holds for the Quillen K -theory of fields.

45

Definition 45 Let X be a variety over a field. Then $C^*(X, M, q)$ is the complex:

$$C^p(X, M, q) := \bigoplus_{x \in X^{(p)}} M_{q-p}(k(x))$$

with the differential $C^p(X, M, q) \rightarrow C^{p+1}(X, M, q)$ induced by the maps $\partial_v : M_{q-p}(k(x)) \rightarrow M_{q-p-1}(k(v))$ for each discrete valuation v of $k(x)$ which is trivial on the ground field. (Here, if k is a field, $M_n(k)$ is the degree n component of $M(k)$.)

Similarly, one defines the homological complex:

$$C_p(X, M, q) := \bigoplus_{x \in X^{(p)}} M_{q-p}(k(x)) .$$

These complexes are the “cycle complexes” associated to the cycle module M .

One then defines:

46

Definition 46

$$A_p(X, M, q) := H_p(C_*(X, M, q))$$

and

$$A^p(X, M, q) := H^p(C^*(X, M, q)) .$$

It is easy to prove that the cohomological complex for Milnor K -theory is contravariant with respect to flat maps. To prove that the corresponding homological complex is covariant with respect to proper maps, one uses Weil reciprocity for curves, see [58], proposition 4.6.; a similar argument for Quillen K -theory is also in [24]. Therefore the groups $A^*(X, M, q)$ are contravariant with respect to flat morphisms, while the groups $A_*(X, M, q)$ are covariant with respect to proper morphisms.

Remark 47 One can consider bases more general than a field. In [58] Rost fixes a base B which is a scheme over a field, and then considers schemes X of finite type over B . More generally, it is easy to see that the homological theory can be defined for schemes of finite type over a fixed excellent base B , so long as one grades the complexes by $\dim_B$ (see Definition 9).

47

Since, if F is a field, $K_1(F) = K_1^M(F) = F^*$, and $K_0(F) = K_0^M(F) = \mathbb{Z}$, we see that the last two terms in $C_*(X, M, p)$ are the groups $R_{p-1}(X)$ and $Z_p(X)$ of dimension $p-1$ K_1 -chains and dimension p cycles on X . Therefore we have ([58], remark 5.1):

Proposition 48 If M_* is Milnor K -theory (or Quillen K -theory):

48

$$\begin{aligned} A_p(X, K^M, -p) &\simeq \mathrm{CH}_p(X) \\ A^p(X, K^M, p) &\simeq \mathrm{CH}^p(X). \end{aligned}$$

Rost shows:

Theorem 49

49

1. For any M_* , the cohomology groups $A^*(X, M, *)$ are homotopy invariant, *i.e.*, for any flat morphism $\pi : E \rightarrow X$ with fibres affine spaces, $\pi^* : A^*(X, M, *) \rightarrow A^*(E, M, *)$ is an isomorphism.
2. For any M_* , if $f : X \rightarrow S$ is a flat morphism with S the spectrum of a Dedekind domain Λ , and t is a regular element of Λ , there is a specialization map $\sigma_t : A^*(X_t, M, *) \rightarrow A^*(X_0, M, *)$, which preserves the bigrading. Here $X_t = X \times_S \mathrm{Spec}(\Lambda[1/t])$ and $X_0 = X \times_S \mathrm{Spec}(\Lambda/(t))$.
3. If M_* is a cycle module with ring structure, and $f : Y \rightarrow X$ is a regular immersion, then there is a Gysin homomorphism $f^* : A^*(X, M, *) \rightarrow A^*(Y, M, *)$. This Gysin homomorphism is compatible with flat pull-back:
 - a) If $p : Z \rightarrow X$ is flat, and $i : Y \rightarrow X$ is a regular immersion, then $p_X^* \cdot i^* = i_Y^* \cdot p^*$, where $p_X : X \times_Y Z \rightarrow X$ and $i_Z : X \times_Y Z \rightarrow Z$ are the projections in the fiber product over Y .
 - b) If $p : X \rightarrow Y$ is flat, and $i : Y \rightarrow X$ is a section of p which is a regular immersion, then $i^* \cdot p^* = 1_Y^*$.
4. If M_* is a cycle module with ring structure, and if X is a smooth variety over a field, then there is a product structure on $A^*(X, M, *)$.

The map σ_t is the composition of the cup-product by $\{t\} \in H^1(X_t, \mathcal{O}_X^*)$ (which is defined since M_* is a module over Milnor K -theory) with the boundary map in the localization sequence for the open subset $X_t \subset X$ with complement X_0 . See [58] sect. 11, as well as [24], where a similar construction is used for the case when M_* is Quillen K -theory. The construction of the Gysin map uses deformation to the normal cone, specialization, and homotopy invariance. The product is constructed as the composition of the external product $A^*(X, M, *) \times A^*(X, M, *) \rightarrow A^*(X \times X, M, *)$ with the Gysin morphism associated to the diagonal map $\Delta : X \rightarrow X \times X$.

50 **Corollary 50** For all $p \geq 0$ and $q \geq 0$, $X \mapsto A^p(X, M, q)$ is a contravariant abelian group valued functor on the category of smooth varieties over k .

Sketch of Proof

If $f : X \rightarrow Y$ is a map of smooth varieties over k , we can factor $f = p \cdot \gamma_f$ with $p : X \times_k Y \rightarrow Y$ the projection, and $\gamma_f : X \rightarrow X \times_k Y$ the graph of f . We then define $f^* = (\gamma_f)^* \cdot p^*$, where p^* is defined since p is flat, and $(\gamma_f)^*$ is defined since γ_f is a regular immersion. To prove that this is compatible with composition, one uses parts a) and b) part 3 of the theorem.

More generally f^* can be defined for any local complete intersection morphism between (not necessarily regular) varieties over k , using the methods of [20].

Let us write $\mathcal{M}_q$ for the sheaf $X \mapsto A^0(X, M, q)$ on the big Zariski site of regular varieties over k . Rost also shows

51 **Theorem 51** If X is the spectrum of regular semi-local ring, which is a localization of an algebra of finite type over the ground field, then for all p , the complex $C^*(X, M, p)$ only has cohomology in degrees $i = 0$.

The proof is variation on the proofs of Gersten's conjecture by Quillen [53] and Gabber [19].

52 **Corollary 52** If X is a regular variety over k , then $H^p(C^*(X, M, q)) \simeq H^p(X, \mathcal{M}_q)$.

We then get immediately, the following variation on Bloch's formula:

53 **Corollary 53** For a variety X as above: $\mathrm{CH}^p(X) \simeq H^p(X, \mathcal{M}_p)$.

Thus Rost's paper shows us that one can construct intersection theory, together with higher "rational equivalence", *i.e.* the higher homology of the cycle complexes,

building just on properties of divisors, and that Milnor K -theory arises naturally in this process.

To construct Chern classes, we can follow the method of [24]. Start by observing that since M_* is a K_*^M -module, there are products

$$H^1(X, \mathcal{O}_X^*) \otimes A^p(X, M, q) \rightarrow A^{p+1}(X, M, q+1) .$$

Proposition 54 Let M_* be a cycle module. Then if X is a variety over k , and $\pi : E \rightarrow X$ is a vector bundle of constant rank n , there is an isomorphism $A^p(\mathbb{P}(E), M, q) \simeq \bigoplus_{i=0}^{n-1} A^{p-i}(\mathbb{P}(E), M, q-i) \xi^i$, where $\xi \in H^1(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}^*)$ is the class of $\mathcal{O}_{\mathbb{P}(E)}(1)$.

54

Proof By a standard spectral sequence argument, this may be reduced to the case when X is a point, and the bundle is trivial, so that $\mathbb{P}(E) \simeq \mathbb{P}^n$. Let $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ be the hyperplane at infinity. It is easy to see that there is a short exact sequence:

$$0 \rightarrow C^*(\mathbb{P}^{n-1}, M, q-1)[1] \rightarrow C^*(\mathbb{P}^n, M, q) \rightarrow C^*(\mathbb{A}^n, M, q) \rightarrow 0 ,$$

which gives rise to a long exact sequence:

$$\dots \rightarrow A^{p-1}(\mathbb{P}^{n-1}, M, q-1) \xrightarrow{j_*} A^p(\mathbb{P}^n, M, q) \xrightarrow{i^*} A^p(\mathbb{A}^n, M, q) \rightarrow \dots ,$$

in which j_* is the Gysin homomorphism, and i^* is the pull back map associated to the inclusion of the open subset $i : \mathbb{A}^n \rightarrow \mathbb{P}^n$. Let $\pi : \mathbb{P}^n \rightarrow \text{Spec}(k)$ be the projection. By homotopy invariance, the map

$$(\pi \cdot i)^* : A^p(\text{Spec}(k), M, q) \rightarrow A^p(\mathbb{A}^n, M, q)$$

is an isomorphism, and so i^* is a split monomorphism, while j^* is a split epimorphism. Next, one may show that $i_* \cdot i^* : A^p(\mathbb{P}^n, M, q) \rightarrow A^{p+1}(\mathbb{P}^n, M, q+1)$ is the same as cap product by ξ . Since i^* is defined using deformation to the normal cone this is not completely tautologous. The proof then finishes by induction on n .

One may now apply the axiomatic framework of [24], to obtain:

Theorem 55 There is a theory of Chern classes for vector bundles, and also for higher algebraic K -theory, on the category of regular varieties over k , with values in Zariski cohomology with coefficients in the Milnor K -theory sheaf:

55

$$C_n : K_p(X) \rightarrow H^{n-p}(X, \mathcal{K}_n^M)$$

which satisfies the properties of *op. cit.*

These classes seem not to be in the literature, though they are known to the experts, and they induce homomorphisms $K_n(F) \rightarrow K_n^M(F)$ which are presumably the same as the homomorphisms defined by Suslin in [65].

56 Remark 56 One can also construct the universal Chern classes $C_p \in H^p(B, \mathbb{G}L_n, \mathcal{K}_p^M)$ by explicitly computing $H^p(B, \mathbb{G}L_n, \mathcal{K}_q^M)$ for all p and all q . To do this one first computes $H^p(\mathbb{G}L_n, \mathcal{K}_q^M)$, using the cellular decomposition of the general linear groups, and then applies a standard spectral sequence argument, to get:

$$H^*(B, \mathbb{G}L_n, \mathcal{K}_*^M) \simeq K_*^M(k)[C_1, C_2, \dots] .$$

Higher Algebraic K -Theory and Chow Groups

2.5

The connection between higher K -theory and Chow groups has at its root the relationship between two different filtrations on the K -theory spectrum of a regular scheme. One of these, the Brown filtration, is intrinsically functorial and compatible with the product structure on K -theory. The other is the coniveau filtration, or filtration by codimension, which is directly related to the Chow groups.

Gersten's conjecture implies that there should be an isomorphism of the corresponding spectral sequences, and hence that these two different filtrations of the K -theory spectrum should induce the same filtration on the K -theory groups. At the E_2 level, this isomorphism of spectral sequences includes Bloch's formula:

$$CH^p(X) \simeq H^p(X, K_p(\mathcal{O}_X)) .$$

The equality of these two filtrations on the K -theory groups tells us that, if Gersten's conjecture holds, then the product on the K -theory of a regular scheme is compatible with the coniveau filtration. Recall that this compatibility implies Serre's conjecture on the vanishing of intersection multiplicities.

At the moment Gersten's conjecture is only known for regular varieties over a field. As we saw in the last section, one can also develop intersection theory for smooth varieties over a field, using deformation to the normal cone. At the end of this section, we use deformation to the normal cone to give a new proof, which does not depend on Gersten's conjecture, that the product on the K -theory of a smooth variety is compatible with the coniveau filtration.

2.5.1 Stable Homotopy Theory

Before discussing higher algebraic K -theory, we should fix some basic ideas of stable homotopy theory and of the homotopy theory of presheaves of spectra.

There are various versions of the stable homotopy category available, such as the category of symmetric spectra of [40] and the category of S -modules of [16]. It is shown in [60] that these are essentially equivalent.

For us, spectra have two advantages. The first is that cofibration sequences and fibration sequences are equivalent (see Theorem 3.1.14 of [40]), and the second is that the product in K -theory can be described via smash products of spectra. In particular we will need the following lemma which gives information about the stable homotopy groups of smash products:

Lemma 57 Suppose that E and F are spectra with $\pi_i(E) = 0$ if $i < p$ and $\pi_i(F) = 0$ if $i < q$. Then $\pi_i(E \wedge F) = 0$ if $i < p + q$.

57

Proof This follows from the spectral sequence

$$\mathrm{Tor}^{\pi_*(S)}(\pi_*(E), \pi_*(F)) \Rightarrow \pi_*(E \wedge F),$$

see [16], Chapt. II, Theorem 4.5.

Following the paper [41] of Jardine, the category of presheaves of spectra on (the Zariski topology of) a scheme X may be given a closed model structure in the sense of [54], in which the weak equivalences are the maps of presheaves which induce weak equivalences stalkwise. See also the papers [10] of Brown and [11] of Brown and Gersten, as well as [23].

If E is a presheaf of spectra on X , we define $R\Gamma(X, E)$ to be $\Gamma(X, \tilde{E})$, where $i : E \rightarrow \tilde{E}$ is a *fibrant resolution* of E , i.e., i is a trivial cofibration and $\tilde{E}$ is fibrant. If $Y \subset X$ is a closed subset, or if Φ is a family of supports, we define $R\Gamma_Y(X, E)$ and $R\Gamma_\Phi(X, E)$ similarly. By a standard argument, one can show that $R\Gamma(X, E)$ is a fibrant spectrum which is, up to weak equivalence, independent of the choice of fibrant resolution.

One also defines

$$H^n(X, F) := \pi_{-n}(R\Gamma(X, F)),$$

and

$$H^n_\Phi(X, F) := \pi_{-n}(R\Gamma_\Phi(X, F)).$$

Note that these are abelian groups.

The reason for this notation is that if $\mathcal{A}$ is a sheaf of abelian groups on X and $\Pi(\mathcal{A}, n)$ is the corresponding sheaf of Eilenberg-MacLane spectra, which has $\pi_i(\Pi(\mathcal{A}, n)) = \mathcal{A}$ if $i = n$, and equal to 0 otherwise, we have:

$$H^p(X, \Pi(\mathcal{A}, n)) \simeq H^{n-p}(X, \mathcal{A})$$

for $n \geq p$.

Filtrations on the Cohomology of Simplicial Sheaves

If $\mathbf{E}$ is a spectrum, and $F^\bullet \mathbf{E}$ is a decreasing filtration of $\mathbf{E}$ by subspectra, there is an associated spectral sequence with

$$E_1^{p,q} = \pi_{-p-q}(F^p \mathbf{E} / F^{p+1} \mathbf{E}) .$$

See [23] for a detailed construction of this spectral sequence, and the associated exact couple.

Given a scheme X satisfying our standard assumptions, and a presheaf of spectra $\mathbf{E}$ on X , one can consider two different filtrations on $R\Gamma(X, \mathbf{E})$.

The first is the “Brown” or hypercohomology filtration:

58

Definition 58 Let $\mathbf{E}$ be a fibrant simplicial presheaf on the scheme X . Let $\mathbf{E}(\infty, k) \subset \mathbf{E}$ be the sub-presheaf with sections over an open $U \subset X$ consisting of those simplices which have all of their faces of dimension less than k trivial. Since the stalks of $\mathbf{E}$ are fibrant (*i.e.* are Kan simplicial sets), the stalk of $\mathbf{E}(\infty, k)$ at $x \in X$ is the fibre of the map from $\mathbf{E}_x$ to the k -th stage of its Postnikov tower.

If $\mathbf{E} = (E_i)_{i \in \mathbb{N}}$ is a fibrant presheaf of spectra, then we can define similarly its Postnikov tower:

$$\mathbf{E}(\infty, k)_i := E_i(\infty, k + i) .$$

In either case, we set

$$F_B^k R\Gamma(\mathbf{E}) := R\Gamma(X, \mathbf{E}(\infty, k)) ,$$

and, if Φ is a family of supports on X ,

$$F_B^k R\Gamma_\Phi(\mathbf{E}) := R\Gamma_\Phi(X, \mathbf{E}(\infty, k)) .$$

Associated to this filtration we have a spectral sequence:

59

Proposition 59 If X is a scheme (which as usual, we assume to be finite dimensional), and if $\mathbf{E}$ is a presheaf of connective spectra on X , there is a hypercohomology spectral sequence:

$$E_1^{p,q} = \pi_{-p-q}(R\Gamma(X, \mathbf{E}(p))) \simeq H^q(X, \pi_{-p}(\mathbf{E})) \Rightarrow H^{p+q}(X, \mathbf{E}) .$$

Here $\mathbf{E}(p)$ denotes the cofiber of $\mathbf{E}(\infty, p+1) \rightarrow \mathbf{E}(\infty, p)$, which is weakly equivalent to the presheaf of Eilenberg-MacLane spectra with homotopy groups $\pi_k(\mathbf{E}(p)) = \pi_p(\mathbf{E})$ if $k = p$ and 0 otherwise. This spectral sequence is concentrated in degrees $q \leq 0$ and $0 \leq p \leq \dim(X)$.

More generally, if Φ is a family of supports on X , then we have

$$E_1^{p,q} = \pi_{-p-q}(R\Gamma_\Phi(X, \mathbf{E}(p))) \simeq H_\Phi^q(X, \pi_{-p}(\mathbf{E})) \Rightarrow H_\Phi^{p+q}(X, \mathbf{E}) .$$

Proof See [11] and [23].

Definition 60 We shall refer to the spectral sequence of the previous proposition as the *Brown spectral sequence*, and the corresponding filtration on the groups $H^*(X, \mathbf{E})$ and $H_\Phi^*(X, \mathbf{E})$ as the Brown filtration.

60

The second filtration on $R\Gamma(X, \mathbf{E})$ is the coniveau filtration:

Definition 61 Recall that $\Gamma_{X \geq k}$ denotes sections with support of codimension at least k ; then we set, for $\mathbf{E}$ a presheaf either of connective spectra,

61

$$F^k R\Gamma(\mathbf{E}) := R\Gamma_{X \geq k}(X, \mathbf{E}) .$$

The resulting spectral sequence

$$E_1^{p,q} = E_{1, \text{cod}}^{p,q}(X, \mathbf{E}) \simeq \pi_{-p-q}(R\Gamma_{X^{(p)}}(X, \mathbf{E})) \Rightarrow H^{p+q}(X, \mathbf{E})$$

converges to the coniveau filtration on $H^*(X, \mathbf{E})$.

We can also require everything to have supports in a family of supports Φ :

$$F_{\text{cod}}^k R\Gamma_\Phi(\mathbf{E}) := R\Gamma_{X \geq k \cap \Phi}(X, \mathbf{E}) ,$$

to obtain a spectral sequence:

$$E_1^{p,q} = E_{1, \text{cod}, \Phi}^{p,q}(X, \mathbf{E}) \simeq \pi_{-p-q}(R\Gamma_{X^{(p)} \cap \Phi}(X, \mathbf{E})) \Rightarrow H_\Phi^{p+q}(X, \mathbf{E}) .$$

It was shown in [23] that these spectral sequences are related. First note that we can renumber the Brown spectral sequence so that it starts at E_2 :

$$\widehat{E}_2^{p,q}(X, \mathbf{E}) := \pi_{-p-q}(R\Gamma(X, \mathbf{E}(-q))) .$$

Theorem 62 With X and $\mathbf{E}$ as above, There is a map of spectral sequences, for $r \geq 2$,

62

$$\widehat{E}_r^{p,q}(X, \mathbf{E}) \rightarrow E_{r, \text{cod}}^{p,q}(X, \mathbf{E}) ,$$

and more generally, given a family of supports Φ ,

$$\widehat{E}_{r, \Phi}^{p,q}(X, \mathbf{E}) \rightarrow E_{r, \text{cod}, \Phi}^{p,q}(X, \mathbf{E}) ,$$

Proof See [23], Theorem 2, Sect. 2.2.4. The proof there uses a generalization to sheaves of simplicial groups of the techniques that Deligne used in a unpublished proof of the analogous result for complexes of sheaves of abelian groups (see [9]). There is a discussion of Deligne's result in [51]. A key point in the proof is that

$X - X^{\geq p}$ has dimension $p - 1$, so that $H_{X - X^{\geq p}}^i(X, \mathcal{A}) = 0$ for all $i \geq q$ and any sheaf of abelian groups $\mathcal{A}$.

This theorem may be viewed as an analog, in the homotopy theory of simplicial presheaves, of a result of Maunder [50], in which he compared the two different ways of defining the Atiyah-Hirzebruch spectral sequence for the generalized cohomology of a CW complex.

Looking at the map on E_∞ terms, we get:

63 Corollary 63 With X and E as above,

$$F^k H^*(X, E) \subset F_{\text{cod}}^k H^*(X, E) ,$$

for all $k \geq 0$. If Φ is a family of supports on X , then:

$$F^k H_\Phi^*(X, E) \subset F_{\text{cod}}^k H_\Phi^*(X, E) .$$

We shall see that for the algebraic K -theory of regular schemes over a field, as well as other cohomology theories for which one can prove Gersten's conjecture, that these two spectral sequences are isomorphic, and hence the filtrations that they converge to are equal.

2.5.3 Review of Basic Notions of K -Theory

Recall that if $\mathcal{E}$ is an *exact category* the K -theory groups $K_p(\mathcal{E})$, for $p \geq 0$ of $\mathcal{E}$ were originally defined by Quillen in [53], to be the homotopy groups $\pi_{p+1}(BQ\mathcal{E})$ of the classifying space of the category $Q\mathcal{E}$ defined in *op.cit.* (Here one takes the zero object of the category as a base point.)

An alternative construction, which gives a space which can be shown to be a deformation retract of $BQ\mathcal{E}$, is Waldhausen's S .-construction. This associates to the exact category $\mathcal{E}$ a simplicial set $S.\mathcal{E}$. The iterates of S .-construction then give a sequence of deloopings $S_*^k \mathcal{E}$ of $S.\mathcal{E}$. See Sect. 1.4 of the article of Carlsson in this volume for details. It is straightforward to check that these deloopings may be used to define a symmetric spectrum which we will denote $\mathbf{K}(\mathcal{E})$, with $K_*(\mathcal{E}) \simeq \pi_*(\mathbf{K}(\mathcal{E}))$. We may then think of K -theory as a functor from the category of exact categories and exact functors to the category of spectra.

Any bi-exact functor $\Phi : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ induces pairings

$$S_*^p(\mathcal{A}) \wedge S_*^q(\mathcal{B}) \rightarrow S_*^{p+q}(\mathcal{C}) ,$$

which are compatible with the actions of the relevant symmetric groups, and hence induce a pairing:

$$\mathbf{K}(\Phi) : \mathbf{K}(\mathcal{A}) \wedge \mathbf{K}(\mathcal{B}) \rightarrow \mathbf{K}(\mathcal{C}) .$$

If X is a scheme we can consider the abelian category $\mathcal{M}(X)$ of all coherent sheaves on X , and the exact subcategory $\mathcal{P}(X) \subset \mathcal{M}(X)$ of locally free coherent

sheaves on X . We shall denote the K -theory spectra of these categories by $\mathbf{G}(X)$ and $\mathbf{K}(X)$ respectively, and the K -theory groups by $G_*(X)$ and $K_*(X)$ respectively. (Note that in [53], $G_*(X)$ is written $K'_*(X)$.) These groups have the following basic properties:

- $X \mapsto K_*(X)$ is a contravariant functor from schemes to graded (anti-)commutative rings. See [53]. The product is induced by the bi-exact functor

$$\begin{aligned}\mathcal{P}(X) \times \mathcal{P}(X) &\rightarrow \mathcal{P}(X) \\ (\mathcal{F}, \mathcal{G}) &\mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G},\end{aligned}$$

which induces a pairing of spectra:

$$\mathbf{K}(X) \wedge \mathbf{K}(X) \rightarrow \mathbf{K}(X).$$

- $X \mapsto G_*(X)$ is a covariant functor from the category of proper morphisms between schemes to the category of graded abelian groups. The covariance of $G_*(X)$ is proved in [53] for projective morphisms, while for general proper morphisms it is proved in [27] and [66].
- G_* is also contravariant for flat maps: if $f : X \rightarrow Y$ is a flat morphism, then the pull-back functor $f^* : \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$ is exact. More generally, $G_*(X)$ is contravariant with respect any morphism of schemes $f : X \rightarrow Y$ which is of finite tor-dimension. This is proven in [53] when Y has an ample line bundle. The general case may deduced from this case by using the fact that the pull back exists locally on Y , since all affine schemes have an ample line bundle, together with the weak equivalence $\mathbf{G}(Y) \simeq Rf(Y, \mathbf{G}_Y)$ discussed in Theorem 66 below. Alternatively, one may show that the pull-back for general f (of finite tor-dimension) exists by the methods of Thomason [66].
- There is a “cap product” $K_*(X) \otimes G_*(X) \rightarrow G_*(X)$, which makes $G_*(X)$ a graded $K_*(X)$ -module. If $f : X \rightarrow Y$ is a proper morphism of schemes, then $f_* : G_*(X) \rightarrow G_*(Y)$ is a homomorphism of $K_*(Y)$ -modules, where $G_*(X)$ is a $K_*(Y)$ -module via the ring homomorphism $f^* : K_*(Y) \rightarrow K_*(X)$. This fact is known as the projection formula. The cap product is induced by the bi-exact functor:

$$\begin{aligned}\mathcal{P}(X) \times \mathcal{M}(X) &\rightarrow \mathcal{M}(X) \\ (\mathcal{F}, \mathcal{G}) &\mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G},\end{aligned}$$

which induces a pairing of spectra:

$$\mathbf{K}(X) \wedge \mathbf{G}(X) \rightarrow \mathbf{G}(X).$$

- If X is regular, then the inclusion $\mathcal{P}(X) \subset \mathcal{M}(X)$ induces an isomorphism on K -theory, $K_*(X) \simeq G_*(X)$.

While $X \mapsto K_*(X)$ is a functor, the operation $X \mapsto \mathcal{P}(X)$ is *not* a functor, since given maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, the functors $(g \cdot f)^*$ and $f^* \cdot g^*$ are only isomorphic, rather than equal. There are standard ways of replacing such

a ‘pseudo’-functor or ‘lax’-functor by an equivalent strict functor; in this case we can replace the category $\mathcal{P}(X)$ of locally free sheaves on X by the equivalent category $\mathcal{P}_{\text{Big}}(X)$ of locally free sheaves on the *Big Zariski Site* over X . An object in this category consists of giving, for every morphism $f : U \rightarrow X$, a locally free sheaf $\mathcal{F}_f$ of $\mathcal{O}_U$ -modules, and for every triple $(f : U \rightarrow X, g : V \rightarrow X, h : U \rightarrow V)$ such that $g \cdot h = f$, an isomorphism $g^* : g^*(\mathcal{F}_h) \rightarrow \mathcal{F}_f$. These data are required to satisfy the obvious compatibilities. Then if $\mathcal{F} \in \mathcal{P}_{\text{Big}}(X)$, and $f : Y \rightarrow X$ is a morphism, $(f^*\mathcal{F}_g)$, for $g : U \rightarrow Y$ is set equal to $\mathcal{F}_{f \cdot g}$. It is a straightforward exercise to show that $X \mapsto \mathcal{P}_{\text{Big}}(X)$ is a strict functor.

We may then view $X \mapsto \mathbf{K}(X)$ as a contravariant functor from schemes to spectra. When we restrict this functor to a single scheme X , we get a presheaf of spectra which we denote $\mathbf{K}_X$. Similarly we have the presheaf $\mathbf{G}_X$ associated to G theory, together with pairings of presheaves:

$$\mathbf{K}_X \wedge \mathbf{K}_X \rightarrow \mathbf{K}_X$$

$$\mathbf{K}_X \wedge \mathbf{G}_X \rightarrow \mathbf{G}_X .$$

64

Definition 64 Let X be a scheme. Given an open subset $U \subset X$, we define $\mathbf{K}(X, U)$ to be the homotopy fibre of the restriction $\mathbf{K}(X) \rightarrow \mathbf{K}(U)$. If $Y \subset X$ is a closed subset, then we also write $\mathbf{K}^Y(X) = \mathbf{K}(X, X - Y)$, and $K_*(X, U)$, $K_*^Y(X)$, for the corresponding groups.

We can perform similar constructions for G -theory. However by Quillen’s localization and dévissage theorems ([53]) if $i : Y \rightarrow X$ is the inclusion of a closed subset of a scheme X , with its structure as a closed reduced subscheme, the exact functor $i_* : \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$ induces a map $i_* : \mathbf{G}(Y) \rightarrow \mathbf{G}_Y(X)$ which is a homotopy equivalence. More generally, if Φ is any family of supports on X , then $\mathbf{G}_\Phi(X) \simeq \mathbf{K}(\mathcal{M}_\Phi(X))$, the K -theory of the category of coherent sheaves of $\mathcal{O}_X$ -modules with support belonging to Φ .

2.5.4 Quillen’s Spectral Sequence

For a general noetherian scheme X , the exact category $\mathcal{M}(X)$ of coherent sheaves of $\mathcal{O}_X$ -modules has a decreasing filtration

$$\mathcal{M}(X) = \mathcal{M}^{\geq 0}(X) \supset \dots \supset \mathcal{M}^{\geq i}(X) \supset \mathcal{M}^{\geq i+1}(X) \supset \dots ,$$

in which $\mathcal{M}^{\geq i}(X)$ is the Serre subcategory consisting of those sheaves which have supports of codimension at least i . Applying the K -theory functor, we get a filtration of the G -theory spectrum by

$$\dots \subset \mathbf{G}^{X^{\geq i+1}}(X) \subset \mathbf{G}^{X^{\geq i}}(X) \subset \dots \subset \mathbf{G}(X) .$$

We shall refer to the corresponding spectral sequence

$$\begin{aligned} E_1^{p,q}(X) &= \pi_{-p-q} \left(\mathbf{G}^{X \geq p}(X) / \mathbf{G}^{X \geq p+1}(X) \right) \\ &\simeq \bigoplus_{x \in X^{(p)}} K_{-p-q}(\mathbf{k}(x)) \Rightarrow F_{\text{cod}}(K_{-p-q}(X)) . \end{aligned}$$

as the *Quillen spectral sequence*. The identification of the $E_1^{p,q}$ -term follows from a combination of localization and dévissage; see [53] for details.

Observe that $E_1^{p,-p}(X) \simeq Z^p(X)$ and $E_1^{p-1,-p}(X) \simeq R^p(X)$. One may also prove that the differential $E_1^{p-1,-p}(X) \rightarrow E_1^{p,-p}(X)$ is simply the divisor map. Hence $E_2^{p,-p}(X) \simeq \text{CH}^p(X)$.

Thus for each $p \geq 0$, we get a complex $R_q^*(X)$, which we shall call the Gersten complex:

$$R_q^p(X) := E_1^{p,-q}(X) = \bigoplus_{x \in X^{(p)}} K_{q-p}(\mathbf{k}(x)) .$$

We may also filter the spectrum $\mathbf{G}(X)$ by *dimension* of supports:

$$\dots \mathbf{G}^{X \leq p-1}(X) \subset \mathbf{G}^{X \leq p}(X) \subset \dots \subset \mathbf{G}(X) .$$

The corresponding spectral sequence is:

$$\begin{aligned} E_{p+q}^1(X) &= \pi_{p+q} \left(\mathbf{G}^{X \leq p}(X) / \mathbf{G}^{X \leq p-1}(X) \right) \\ &\simeq \bigoplus_{x \in X_{(p)}} K_{p+q}(\mathbf{k}(x)) \Rightarrow F_{\text{dim}} K_{p+q}(X) . \end{aligned}$$

We also have the corresponding homological Gersten complex:

$$R_{p,q}(X) := E_{p,q}^1(X) .$$

Again, we have that $E_{p,p}^2 \simeq \text{CH}_p(X)$.

If $f : X \rightarrow Y$ is a flat morphism, and $Z \subset Y$ has codimension p , then $f^{-1}(Z)$ has codimension p in X and hence $f^{-1}(Y^{\geq p}) \subset X^{\geq p}$. If f is proper, and $W \subset X$ has dimension q , then $f(W)$ has dimension at most q and hence $f(X_{\leq q}) \subset Y_{\leq q}$. It follows that if f is flat, flat pull-back induces a map of coniveau spectral sequences, and hence of Gersten complexes. If f is proper, then push-forward induces a map of spectral sequences, and hence of Gersten complexes:

$$f_* : R_{*,q}(X) \rightarrow R_{*,q}(Y) .$$

Notice that this automatically gives the covariance of the Chow groups with respect to proper maps.

One can extend these results to prove:

65 **Theorem 65** Quillen K -theory of fields is a cycle module, and the Gersten complexes are the associated cycle complexes.

Proof See [53], [62] and [24].

2.5.5 K -Theory as Sheaf Hypercohomology

Recall that if X is a scheme, $\mathbf{G}_X$ denotes the presheaf of G -theory spectra on X .

66 **Theorem 66** If $Y \subset X$ is a closed subset, the natural map

$$\mathbf{G}(Y) \simeq \mathbf{G}^Y(X) \rightarrow R\Gamma_Y(X, \mathbf{G}_X) ,$$

is a weak homotopy equivalence.

Proof It is enough to prove that this is true for $Y = X$. The general result then follows by comparing the fibration sequences:

$$\mathbf{G}_Y(X) \rightarrow \mathbf{G}(X) \rightarrow \mathbf{G}(X - Y)$$

and

$$R\Gamma_Y(X, \mathbf{G}_X) \rightarrow R\Gamma(X, \mathbf{G}_X) \rightarrow R\Gamma(X - Y, \mathbf{G}_X) \simeq R\Gamma(X - Y, \mathbf{G}_{X-Y}) .$$

The result for X is a consequence of the Mayer-Vietoris property of G -theory. See [11].

67 **Corollary 67** The Quillen spectral sequence for $G_*(X)$ is the same as the coniveau spectral sequence for the sheaf of spectra $\mathbf{G}_X$, and both converge to the coniveau filtration on G -theory.

Let X be a scheme, and suppose that $Y \subset X$ and $Z \subset X$ are closed subsets. Then, using the fact that smash products preserve cofibration sequences, one may easily check that the K -theory product respects supports:

$$\mathbf{K}^Y(X) \wedge \mathbf{K}^Z(X) \rightarrow \mathbf{K}^{Y \cap Z}(X) .$$

When X is regular, then this may be identified with the pairing on generalized sheaf cohomology:

$$R\Gamma_Y(X, \mathbf{K}_X) \wedge R\Gamma_Z(X, \mathbf{K}_X) \rightarrow R\Gamma_{Y \cap Z}(X, \mathbf{K}_X) .$$

Corollary 68 Let X be a scheme. Then the Brown spectral sequence

68

$$E_2^{p,-q}(X, \mathbf{G}) = H^p(X, G_q(\mathcal{O}_X)) \Rightarrow G_{-p-q}(X)$$

determines a filtration $F^*(G_*(X))$. By Corollary 63, we have an inclusion of filtrations $F^*(G_i(X)) \subset F_{\text{cod}}^*(G_i(X))$.

When X is regular, we then get a filtration $F^k K_*(X)$, which we will still call the Brown filtration, and which has nice properties:

Theorem 69 Let X be a regular scheme. Then the Brown filtration on $K(X)$ is compatible with the product on K -theory, and is (contravariant) functorial in X . *I.e.*, if $*$ denotes the K -theory product,

69

$$F^i K_p(X) * F^j K_q(X) \subset F^{i+j} K_{p+q}(X)$$

and if Y and Z are closed subsets of X , then

$$F^i K_{Y,p}(X) * F^j K_{Z,q}(X) \subset F^{i+j} K_{Y \cap Z, p+q}(X).$$

If $f : X \rightarrow Y$ is a map of regular schemes, and $W \subset Y$ is a closed subset then

$$f^*(F^i K_{W,p}(Y)) \subset F^i K_{f^{-1}(W),p}(X).$$

Proof We have $F^i K_p(X) = \text{Image}(H^{-p}(X, K_X(\infty, i)) \rightarrow H^{-p}(X, K_X))$. Hence it suffices to know that the map

$$K_X(\infty, i) \wedge K_X(\infty, j) \rightarrow K_X$$

induced by the K -theory product factors, up to homotopy, through $K_X(\infty, i+j)$, and this is a straightforward consequence of the universal coefficient theorem 57.

The compatibility of the filtration with pull-backs is a consequence of the functoriality of the Postnikov tower.

As we will see below Gersten's conjecture implies that for a regular scheme X , the Brown and coniveau filtrations coincide, and hence Gersten's conjecture implies that the coniveau filtration is multiplicative.

Gersten's Conjecture, Bloch's Formula and the Comparison of Spectral Sequences

2.5.6

We have seen on that on a nonsingular variety, a divisor corresponds to an element of $H^1(X, K_1(\mathcal{O}_X))$, which is determined by the local equations of the divisor. In the seminal paper [8], Bloch showed that on a smooth algebraic surface, the fact

that a point is given locally by a *pair* of equations could be used to provide an isomorphism $\mathrm{CH}^2(X) \simeq H^2(X, K_2(\mathcal{O}_X))$.

Quillen's generalization of Bloch's formula to all codimensions, starts from:

70 **Conjecture 70:** *Gersten's conjecture* Suppose that R is a regular local ring. Then for all $i > 0$, the map

$$\mathcal{M}^{(i)}(\mathrm{Spec}(R)) \subset \mathcal{M}^{(i-1)}(\mathrm{Spec}(R))$$

induces zero on K -theory.

71 **Proposition 71** If Gersten's conjecture holds for a given regular local ring R , then, for all p , the following complex is exact:

$$0 \rightarrow K_p(R) \rightarrow K_p(F) \rightarrow \bigoplus_{x \in X^{(1)}} K_{p-1}(\mathbf{k}(x)) \rightarrow \dots \rightarrow K_{p-\dim(R)}(k) \rightarrow 0$$

Here $X := \mathrm{Spec}(R)$, while F and k are the fraction and residue fields of R respectively.

Note that this implies that $\mathrm{CH}^p(\mathrm{Spec}(R)) \simeq 0$, if $p > 0$, for R a regular local ring; this is a conjecture of Fossum, [12].

72 **Corollary 72** If X is regular scheme, and Gersten's conjecture holds for all the local rings on X , then the augmentation:

$$K_p(\mathcal{O}_X) \rightarrow \mathcal{R}_{p,X}$$

is a quasi-isomorphism, where $\mathcal{R}_{p,X}$ is the sheaf of Gersten complexes $U \mapsto R_p^*(U)$. Hence, since the $\mathcal{R}_{p,X}$ are flasque, we have *Bloch's formula*:

$$H^p(X, K_p(\mathcal{O}_X)) \simeq H^p(R_p^*(X)) \simeq \mathrm{CH}^p(X) .$$

If $Y \subset X$ is a subset of pure codimension r , then:

$$H_Y^p(X, K_p(\mathcal{O}_X)) \simeq H_Y^p(X, \mathcal{R}_{p,X}) \simeq H^{p-r}(R_{p-r}^*(Y)) \simeq \mathrm{CH}^{p-r}(Y) .$$

Finally,

$$H_Y^p(X, K_q(\mathcal{O}_X)) \simeq 0$$

if $p > q$.

73 **Theorem 73:** *Quillen, [53]* If X is a regular variety over a field, then Gersten's conjecture is true for all the local rings on X .

Gersten's conjecture may be viewed as a "local" version of the moving lemma, and versions of it play a key role elsewhere, such as in proving the local acyclicity of the motivic cohomology complexes.

The key point in Quillen's proof of Gersten's conjecture is that if X is a regular affine variety over a field k , then given a divisor $D \subset X$, and a point $x \in D$, the map $i : D \hookrightarrow X$ is "homotopic" to zero in a neighborhood of x . Quillen uses a variant of Noether normalization to show that there is a map $U \rightarrow \mathbb{A}_k^{d-1}$, with domain an affine open $U \subset X$ neighborhood of x , which is smooth and which has finite restriction to $D \cap U$. A variation on Quillen's proof may be found in the paper [19] of Gabber, where he proves Gersten's conjecture for Milnor K -theory. See also [13].

Corollary 74 If X is a regular variety of finite type over a field, and Φ is a family of supports on X , then the map of Theorem 62 from the Brown spectral sequence to the coniveau spectral sequence is an isomorphism from E_2 onward, and hence the Brown and the coniveau filtrations on the groups $K_*^\Phi(X)$ agree.

74

Proof See [27] and [23]. The key point is that the map on E_2 -terms: $\widehat{E}_{2,\Phi}^{p,q}(X) = H_\Phi^p(X, K_{-q}(\mathcal{O}_X)) \rightarrow E_{2,\text{cod},\Phi}^{p,q}(X) \simeq H_\Phi^p(X, \mathcal{R}_{-q,X})$ is the same as the map induced by the augmentation $K_{-q}(\mathcal{O}_X) \rightarrow \mathcal{R}_{-q,X}$.

Since, by Theorem 69, the Brown filtration is multiplicative, we get:

Corollary 75 If X is a regular variety over a field, the coniveau filtration on K -theory with supports is multiplicative.

75

We shall give another proof of this result in Sect. 2.5.11, using deformation to the normal cone.

The Coniveau Spectral Sequence for Other Cohomology Theories

2.5.7

We can replace the presheaf of spectra F in the previous section by a complex of sheaves of abelian groups $\mathcal{F}_X^*$. Then the Brown spectral sequence is the standard hypercohomology spectral sequence, and one also has the coniveau spectral sequence. In [9], Bloch and Ogus considered (graded) cohomology theories $X \mapsto \mathcal{F}_X^*(*)$, satisfying suitable axioms, and showed that the analog of Gersten's conjecture holds in these cases. Examples of theories satisfying the Bloch-Ogus axioms are étale cohomology (in which case the Brown spectral sequence is the Leray spectral sequence for the map from the étale site to the Zariski site) and Deligne-Beilinson cohomology ([5], see also [26]).

If $X \mapsto H^*(X, *) = \mathbb{H}^*(X, \mathcal{F}_X^*(*))$ is a theory satisfying the axioms of Bloch and Ogus, then the analog of Gersten's conjecture implies that the E_2 term of the coniveau spectral sequence is isomorphic to $H^p(X, \mathcal{H}^q(*))$, where $\mathcal{H}^q(*)$ is the Zariski sheaf associated to $X \mapsto H^q(X, *)$. Deligne then showed, in an unpublished note:

76 **Theorem 76** The coniveau and hypercohomology spectral sequences agree from E^2 on.

Proof A version of Deligne's proof may be found in the paper [51]. In addition, the proof in the paper [23] of the analogous result for K -theory, is based on the methods of Deligne.

2.5.8 Compatibility with Products and Localized Intersections

One of the great virtues of Bloch's formula is that the K -cohomology groups have a product structure, induced by the K -theory product.

Let us write $\eta : \mathrm{CH}^p(X) \rightarrow H^p(X, K_p(\mathcal{O}_X))$ for the isomorphism induced by the Gersten resolution of the K -theory sheaf. Grayson proved in [34],

77 **Theorem 77** Let X be a smooth variety over a field k . If $\alpha \in Z^p(X)$ and $\beta \in Z^q(X)$ are two cycles which intersect properly, then

$$\eta(\alpha)\eta(\beta) = (-1)^{\frac{p(p-1)}{2} \frac{q(q-1)}{2}} \eta(\alpha.\beta)$$

where $\alpha.\beta$ is the product defined by using the intersection multiplicities of Serre, (see Definition 26), and hence with the intersection product defined by Samuel.

Proof By additivity, one can reduce to the case in which $\alpha = [Y]$ and $\beta = [Z]$, where Y and Z are two integral subschemes of X which meet properly. From Quillen's proof of Gersten's conjecture, we have:

$$H_Y^p(X, K_p(\mathcal{O}_X)) \simeq \mathrm{CH}^0(Y) = \mathbb{Z}[Y] ,$$

$$H_Z^q(X, K_q(\mathcal{O}_X)) \simeq \mathrm{CH}^0(Z) = \mathbb{Z}[Z] ,$$

and

$$H_{Y \cap Z}^{p+q}(X, K_{p+q}(\mathcal{O}_X)) \simeq \mathrm{CH}^0(Y \cap Z) = \bigoplus_S \mathbb{Z} ,$$

where the direct sum is over the irreducible components S of $Y \cap Z$.

Using the equality of the Brown and coniveau spectral sequences, Theorem 74, we can identify these isomorphisms with the edge homomorphisms

$$H_Y^p(X, K_p(\mathcal{O}_X)) \rightarrow Gr^p K_0^Y(X) = \mathbb{Z}[\mathcal{O}_Y] ,$$

$$H_Z^q(X, K_q(\mathcal{O}_X)) \rightarrow Gr^q K_0^Z(X) = \mathbb{Z}[\mathcal{O}_Z] ,$$

and

$$H_{Y \cap Z}^{p+q}(X, K_{p+q}(\mathcal{O}_X)) \rightarrow Gr^{p+q} K_0^{Y \cap Z}(X) = \bigoplus_S \mathbb{Z}[\mathcal{O}_S] ,$$

in the Brown spectral sequences for K -theory with supports in Y , Z , and $Y \cap Z$ respectively. By the multiplicativity of the Brown spectral sequence, these edge homomorphisms are compatible with products, and so the product of the cycles associated to the classes $[\mathcal{O}_Y]$ and $[\mathcal{O}_Z]$ maps to the cycle associated to the K -theory product $[\mathcal{O}_Y] \cdot [\mathcal{O}_Z]$, which is non other than the cycle defined using Serre's definition of intersection multiplicities.

The sign comes from the fact that the isomorphism

$$\widehat{E}_{2,Y}^{p,p}(X) \simeq H^{2p}(X, K_p(\mathcal{O}_X)[p])$$

preserves products, while the isomorphism

$$H^{2p}(X, K_p(\mathcal{O}_X)[p]) \simeq H^p(X, K_p(\mathcal{O}_X))$$

only preserves products up to the factor $(-1)^{\frac{p(p-1)}{2} \frac{q(q-1)}{2}}$.

By Quillen's proof of Bloch's formula, if $Y \subset X$ is a closed set, and X is equidimensional of dimension n , then

$$\mathrm{CH}_{n-p}(Y) \simeq H_Y^p(X, K_p(\mathcal{O}_X)) .$$

It follows that purely by the formalism of cohomology with supports, that we get a product, for $Y \subset X$ and $Z \subset X$ closed subsets,

$$\mathrm{CH}_k(Y) \times \mathrm{CH}_l(Z) \rightarrow \mathrm{CH}_{k+l-n}(Y \cap Z) ,$$

which may be shown to agree with the product with supports on Chow homology constructed by Fulton and MacPherson ([17]). See [24] and [30].

Other Cases of Gersten's Conjecture

2.5.9

For non-geometric regular local rings, the only case for which Gersten's conjecture is known is that of henselian discrete valuation rings Λ with finite residue field k , a result due to Sherman ([63]). The idea of Sherman's proof is that since the general linear group of a finite field is finite, one can use Brauer lifting to show that the restriction map

$$K_*(\Lambda) \rightarrow K_*(k)$$

is surjective. A variation of this result is that if Λ is a discrete valuation ring, the conjecture is true for K -theory with coefficients $\mathbb{Z}/n$ of order prime to the characteristic of k ([29]). The proof depends on the result of Gabber ([18]), and of Gillet and Thomason ([33]), that if R is a Henselian discrete valuation ring then the restriction map

$$K_*(\Lambda, \mathbb{Z}/n) \rightarrow K_*(k, \mathbb{Z}/n)$$

is an isomorphism.

If R is a regular local ring which is smooth over a discrete valuation ring Λ with maximal ideal $\pi\Lambda \subset \Lambda$, then one can consider relative versions of Gersten's conjecture, in which one considers not all R -modules, but only those which are flat over Λ . See [7], and [31], where it is shown that this "relative" version of Gersten's conjecture implies that Gersten's conjecture is true for R if it is true for the discrete valuation ring associated to the ideal πR .

2.5.10 Operations on the Quillen Spectral Sequence

One can show that the λ -operations on K_0 of Sect. 2.3.3 can be extended to the higher K -theory of rings and of regular schemes (see the papers of Kratzer ([45]) and Soulé ([64])). Of particular use are the Adams operations ψ^p for $p \in \mathbb{N}$. These are defined as follows. For $x \in K_*(X)$, consider the formal power series $\lambda_t(x) := \sum_i t^i \lambda^i(x) \in K_*(X)[[t]]$. Then the ψ^i are defined by

$$\frac{d}{dt} (\lambda_t(x)) / \lambda_t(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \psi^k(x) t^{k-1}.$$

One can show (*op. cit.*) that, if X is a regular scheme satisfying our standing assumptions, then the action of the Adams operations on $K_*(X)_{\mathbb{Q}}$ can be diagonalized, so that ψ^p acts with eigenvalue k^p on a subspace which is isomorphic to $\mathrm{Gr}_Y^k(K_*(X)_{\mathbb{Q}})$.

Note that if X is a variety over the finite field $\mathbb{F}_p$, then the action of ψ^p on K -theory is the same as the action induced by the Frobenius endomorphism of X . The Adams operation act compatibly with supports and hence act on the Quillen spectral sequence. Using a variant of the Riemann–Roch theorem for higher K -theory of [24], Soulé (*op. cit.*) identified the action on the E^2 -term, and could thereby deduce by weight considerations, that for a regular scheme, the differentials into the $E_{p,-p}^2$ and $E_{p,1-p}^2$ terms are torsion, and therefore:

78 Theorem 78 If X is a regular scheme, there are isomorphisms, for all $p \geq 0$:

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \simeq \mathrm{Gr}_{\mathrm{cod}}^p(K_0(X))_{\mathbb{Q}} \simeq \mathrm{Gr}_Y^p(K_0(X))_{\mathbb{Q}}.$$

This extends a result that was proved in SGA6 [2]; the first isomorphism was proved in in *op. cit.* for smooth varieties over a field, see *op. cit.* Sect. 4.2 of expose é XIV,

while the second was proved with the (unnecessary) assumption that there is an ample invertible sheaf on X ; see Theorem 40.

It is natural to ask what the relationship between the γ -filtration and the other filtrations on higher K -theory is. In [23], we prove:

Theorem 79 Let X be a variety over a field k . Then

79

$$F_Y^p K_m(X) \subset F_{\text{cod}}^{p-m} K_m(X) .$$

Note that in [2], exposé X, Jussila proved:

Theorem 80 Let X be a noetherian scheme. Then

80

$$F_Y^p K_0(X) \subset F_{\text{cod}}^p K_0(X) .$$

It is therefore natural to ask:

Question 81 Can one extend Theorem 79 to the case of general noetherian schemes?

81

While this would follow from Gersten's conjecture, there may be other ways to approach this problem, such as the construction of a filtration on the K -theory of a regular ring constructed by Grayson ([35]) using commuting automorphisms, which is conjecturally related to the γ -filtration.

The Multiplicativity of the Coniveau Filtration: a Proof Using Deformation to the Normal Cone.

2.5.11

In this section we shall give a proof of the multiplicativity of the coniveau filtration on higher K -theory for arbitrary smooth varieties over a field which uses deformation to the normal cone, rather than hypercohomology of sheaves.

Let X be a smooth variety over a field k . We can decompose the product on $K_*(X)$ into the composition of the external product:

$$\square : K_*(X) \otimes K_*(X) \rightarrow K_*(X \times X)$$

and pull-back via the diagonal map $\Delta : X \rightarrow X \times X$:

$$\Delta^* : K_*(X \times X) \rightarrow K_*(X) .$$

Lemma 82 The coniveau filtration is multiplicative with respect to the external product.

82

Proof The external product is induced by the bi-exact functor:

$$\begin{aligned}\mathcal{M}(X) \times \mathcal{M}(X) &\rightarrow \mathcal{M}(X \times X) \\ (\mathcal{F}, \mathcal{G}) &\mapsto \mathcal{F} \otimes_k \mathcal{G}\end{aligned}$$

Let $\pi_i : X \times X \rightarrow X$, for $i = 1, 2$ be the two projections. Suppose that $Y \subset X$ and $Z \subset X$ are closed subsets of X of codimensions p and q respectively. If $\mathcal{F}$ is a coherent sheaf supported on Y , and $\mathcal{G}$ is a coherent sheaf supported on Z , then $\mathcal{F} \otimes_k \mathcal{G}$ is supported on $\pi_1^{-1}(Y) \cap \pi_2^{-1}(Z)$, which has codimension $p + q$ in $X \times X$. Hence the product $\square : (G_i(Y) \otimes G_j(Z)) \rightarrow K_*(X \times X)$ factors through $G_{i+j}(Y \times Z)$, and hence its image lies in $F^{p+q}K_{i+j}(X \times X)$.

Therefore we need only show that pull back by the diagonal preserves the coniveau filtration. More generally, we have:

83 Theorem 83 If $f : Y \hookrightarrow X$ is a regular immersion of schemes satisfying our standing hypotheses, then $f^*(F_{\text{cod}}^i(G_p(X))) \subset F_{\text{cod}}^i(G_p(Y))$.

Proof The pull back map $f^* : G_*(X) \rightarrow G_*(Y)$ is defined because f is a morphism of finite Tor-dimension. However, it can also be constructed using deformation to the normal bundle.

First we need two lemmas:

84 Lemma 84 Let $f : X \rightarrow Y$ be a flat morphism. Then

$$f^*(F^p(G_q(Y))) \subset F_{\text{cod}}^p(G_q(X)) .$$

Proof The pull-back map f^* is induced by the exact functor $f^* : \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$, and if a coherent sheaf $\mathcal{F}$ on Y is supported on closed subset $Z \subset Y$ of codimension p , then $f^*(\mathcal{F})$ is supported on $f^{-1}(Z)$ which has codimension p in X .

85 Lemma 85 Let $p : N \rightarrow Y$ be a vector bundle. Then the map $p^* : F_{\text{cod}}^p(G_q(Y)) \rightarrow F_{\text{cod}}^p(G_q(N))$ is an isomorphism for all p and for all q .

Proof Since p is flat, the functor $p^* : \mathcal{M}(X) \rightarrow \mathcal{M}(N)$ preserves codimension of supports, and so we get a map of filtered spectra $F^*\mathbf{G}(X) \rightarrow F^*\mathbf{G}(N)$ which induces a map

$$p^* : E_r^{p,q}(X) \rightarrow E_r^{p,q}(N)$$

of coniveau spectral sequences. However it is shown in [62] (and also [24]) that the

cohomology of the E^2 term of the Quillen spectral sequence is homotopy invariant, and hence the coniveau filtration is homotopy invariant.

Consider the deformation to the normal cone space $W := W_{Y/X}$. Then we have maps:

Pull Back $\pi^* : K_*(X) \rightarrow K_*(W \setminus W_0 \simeq X \times \mathbb{G}_m)$.

Specialization If t is the parameter on $\mathbb{G}_m$, i.e., the equation of the principal divisor W_0 , then we have maps, for all $p \geq 0$:

$$\sigma_t : K_p(W \setminus W_0) \rightarrow K_p(W_0 \simeq N_{Y/X})$$

$$\alpha \mapsto \partial(\alpha * \{t\})$$

Here ∂ is the boundary map in the localization sequence

$$\dots \rightarrow K_{p+1}(W) \rightarrow K_{p+1}(W \setminus W_0) \xrightarrow{\partial} K_p(W_0) \rightarrow \dots$$

Homotopy Invariance If $p : N_{Y/X} \rightarrow Y$ is the projection, the pull back map $p^* : K_*(Y) \rightarrow K_*(N_{Y/X})$ is an isomorphism.

Proposition 86 With the notation above, we have

86

$$f^* = (p^*)^{-1} \cdot \sigma_t \cdot \pi^*.$$

Proof See [24].

By Lemmas 84 and 85, π^* and $(p^*)^{-1}$ preserve the coniveau filtration. It remains to show that the coniveau filtration is preserved by specialization.

Suppose that $Z \subset X$ is a codimension p closed, reduced, subscheme. If we are given an element α in $G_p(X)$ which is supported on Z – i.e., its restriction to $X - Z$ vanishes, we know from the localization sequence that it is the image of an element $\gamma \in G_p(Z)$. It will be enough to show that $\sigma_t(\alpha)$ is supported on a closed subset of codimension p in $N_{Y/X}$.

By Lemma 90 below, we know that the Zariski closure in W of $Z \times \mathbb{G}_m \subset X \times \mathbb{G}_m \subset W$ is isomorphic to the deformation to the normal cone space $W_{Z/(Y \cap Z)}$ associated to the subscheme $Y \cap Z \subset Z$. Furthermore, the special fibre of $W_{Z/(Y \cap Z)}$ is the normal cone $C_{Z/(Y \cap Z)} \subset N_{Y/X}$, which has codimension p in the normal bundle.

Since $C_{Z/Y \cap Z} = W_{Z/Y \cap Z} \cap N_{Y/X}$, if we write $j : W_{Z/Y \cap Z} \rightarrow W_{Y/X}$ for the inclusion, j_* induces a map of localization sequences, and in particular a commutative square:

$$\begin{array}{ccc} G_{p+1}(Z \times \mathbb{G}_m) & \xrightarrow{\partial} & G_p(C_{Z/Y \cap Z}) \\ j_* \downarrow & & j_* \downarrow \\ G_{p+1}(X \times \mathbb{G}_m) & \xrightarrow{\partial} & G_p(N_{Y/X}). \end{array}$$

By the projection formula, we also have a commutative square:

$$\begin{array}{ccc}
 G_p(Z \times \mathbb{G}_m) & \xrightarrow{* \{t\}} & G_{p+1}(Z \times \mathbb{G}_m) \\
 j_* \downarrow & & j_* \downarrow \\
 G_p(X \times \mathbb{G}_m) & \xrightarrow{* \{t\}} & G_{p+1}(X \times \mathbb{G}_m) .
 \end{array}$$

Putting these together, we get a commutative diagram:

$$\begin{array}{ccc}
 G_p(Z \times \mathbb{G}_m) & \xrightarrow{\sigma_t} & G_p(C_{Z/Y \cap Z}) \\
 j_* \downarrow & & j_* \downarrow \\
 G_p(X \times \mathbb{G}_m) & \xrightarrow{\sigma_t} & G_p(N_{Y/X}) ,
 \end{array}$$

as desired. Hence $\sigma_t(\alpha) \in j_*(G_p(C_{Z/Y \cap Z})) \subset F_{\text{cod}}^p(G_p(N_{Y/X}))$.

This completes the proof of the proposition, and hence of Theorem 35.

Since every regular variety over a field is a localization of smooth variety over the prime field, it also follows that the theorem is true for regular varieties.

A similar reduction to the diagonal argument may also be used to prove the theorem for schemes which are smooth over the spectrum of a discrete valuation ring.

2.6

Bloch's Formula and Singular Varieties

2.6.1

Cohomology Versus Homology

If X is a general CW complex which is not a manifold, it will no longer be the case that there is a Poincaré duality isomorphism $H^*(X, \mathbb{Z}) \simeq H_*(X, \mathbb{Z})$. If X is a singular variety, the Chow groups of cycles modulo rational equivalence are analogous (even when graded by codimension) to the singular homology of a CW complex. It is natural to ask if there is appropriate theory of *Chow cohomology*.

One answer to this question is given by Fulton in his book [17], in which he defines the cohomology groups to be the *operational Chow groups* $\text{CH}_{\text{op}}^*(X)$. An element $\alpha \in \text{CH}_{\text{op}}^p(X)$ consists, essentially, of giving homomorphisms, for every map $f : Y \rightarrow X$ of varieties and every $q \geq 0$, $\cap \alpha : \text{CH}_q(Y) \rightarrow \text{CH}_{q-p}(Y)$, which satisfy various compatibilities. Fulton's operational groups are the target of a theory of Chern classes for vector bundles, and any regularly immersed codimension p

closed subscheme $Y \subset X$ has a cycle class $[Y] \in \mathrm{CH}_{\mathrm{op}}^p(X)$. (The existence of this class uses deformation to the normal bundle.) Any Chow cohomology theory that has reasonable properties, in particular that contains Chern classes for vector bundles, and which has cap products with Chow homology for which the projection formula holds, will map to the operational groups. However the operational groups, while they have many virtues, also miss some information. For example, if X is a nodal cubic curve over a field k , one can prove that $\mathrm{CH}_{\mathrm{op}}^1(X) \simeq \mathbb{Z}$. However the group of Cartier divisors is isomorphic to $\mathbb{Z} \oplus k^*$, which carries more information about the motive of X . Even if $\mathrm{Pic}(X)$ gave a “good” definition of codimension 1 Chow cohomology, it is not clear what should happen in higher codimensions – *i.e.* is there a theory of codimension p “Cartier Cycles”?

It is natural to consider the groups $H^p(X, K_p(\mathcal{O}_X))$, because of Bloch’s formula and by analogy with $\mathrm{Pic}(X) \simeq H^1(X, K_1(\mathcal{O}_X))$.

Arguments in favor of this choice are:

1. The functor $X \mapsto \bigoplus_p H^p(X, K_p(\mathcal{O}_X))$ is a contravariant functor from varieties over a given field k to commutative graded rings with unit.
2. There are cap products $H^p(X, K_p(\mathcal{O}_X)) \otimes \mathrm{CH}_q(X) \rightarrow \mathrm{CH}_{q-p}(X)$, which satisfy the projection formula.
3. There are Chern classes for vector bundles: $C_p(E) \in H^p(X, K_p(\mathcal{O}_X))$, for E is a vector bundle on X , which are functorial and satisfy the Whitney sum formula for exact sequences of bundles.
4. Any codimension p subscheme $Y \subset X$ of which is a local complete intersection has a fundamental class $[Y] \in H^p(X, K_p(\mathcal{O}_X))$.

This approach to Chow cohomology is developed in [24] and [25].

A strong argument against this choice is that these groups (including $\mathrm{Pic}(X)$, see [69]) are not homotopy invariant.

One way to get a homotopy invariant theory, if the ground field k has characteristic zero, is given a singular variety X , to take a nonsingular simplicial hyperenvelope $\tilde{X} \rightarrow X$ (see Appendix 2.8) and take $H^p(\tilde{X}, K_p(\mathcal{O}_{\tilde{X}}))$ (or $H^p(\tilde{X}, \mathcal{K}_p^M)$) as the definition of Chow cohomology. One can show (at least for the usual K -cohomology, see *op. cit.*) that these groups are independent of the choice of hyperenvelope, are homotopy invariant, will have cap products with the homology of the Gersten complexes, and will be the target of a theory of Chern classes.

Given a variety X and a nonsingular simplicial hyperenvelope $\tilde{X} \rightarrow X$ there will be, for each $q \geq 0$, a spectral sequence:

$$E_1^{i,j} = H^i(\tilde{X}_j, K_q(\mathcal{O}_{\tilde{X}_j})) \Rightarrow H^{i+j}(\tilde{X}, K_q(\mathcal{O}_{\tilde{X}})) .$$

One can show, by the method of [27], that the E_2 term of this spectral sequence is independent of the choice of hyperenvelope, and hence gives a filtration on the groups $H^p(\tilde{X}, K_q(\mathcal{O}_{\tilde{X}}))$. This filtration is a K -cohomology version of the weight filtration of mixed Hodge theory.

Note that if X is a (projective) nodal cubic, and $\tilde{X} \rightarrow X$ is a hyperenvelope, then

$$H^1(\tilde{X}, K_1(\mathcal{O}_{\tilde{X}})) \simeq \mathrm{Pic}(X) ,$$

while if X is a cuspidal cubic, then

$$H^1(\tilde{X}, K_1(\mathcal{O}_{\tilde{X}})) \simeq \mathbb{Z} \neq \text{Pic}(X).$$

It follows from the following theorem that the weight zero part of these groups are Fulton's operational Chow groups:

87 **Theorem 87:** *Kimura, [42]* Let X be variety over a field. Given nonsingular envelopes $p_0 : \tilde{X}_0 \rightarrow X$ and $p_1 : \tilde{X}_1 \rightarrow \tilde{X}_0 \times_X \tilde{X}_0$ we have an exact sequence:

$$0 \rightarrow \text{CH}_{\text{op}}^p(X) \xrightarrow{p_0^*} \text{CH}^p(\tilde{X}_0) \xrightarrow{\delta} \text{CH}^p(\tilde{X}_1),$$

where $\delta = p_1^*(\pi_1^* - \pi_0^*)$, with $\pi_i : \tilde{X}_0 \times_X \tilde{X}_0 \rightarrow \tilde{X}_0$ the projections.

Local Complete Intersection Subschemes and Other Cocycles

If X is a scheme, we know that any subscheme the ideal of which is generated locally by nonzero divisors defines a Cartier divisor and is a “codimension 1” cocycle. What about higher codimension?

Let $Y \subset X$ be a codimension p regularly immersed subscheme. Recall that there is an operational class $[Y]_{\text{op}} \in \text{CH}_{\text{op}}^p$ corresponding to the “pull-back” operation constructed using deformation to the normal cone.

88 **Theorem 88:** [25] Let X be a variety over a field k , and suppose that $Y \subset X$ is closed subscheme which is a codimension p local complete intersection. Then there is a natural class $[Y] \in H^p(X, K_p(\mathcal{O}_X))$, such that cap product with $[Y]$ induces the operational product by $[Y]_{\text{op}}$.

Idea of Proof

While the theorem is stated using K -cohomology, it really holds for almost any cohomology theory constructed using sheaf cohomology, that has a theory of cycle classes with supports. The key point is that for a given $p > 0$, there is a pair of simplicial schemes $V. \hookrightarrow U.$ smooth over the base, which is a “universal” codimension p local complete intersection. That is, given $Y \subset X$, a codimension p local complete intersection, there is a Zariski open cover $\mathcal{W}$ of X , and a map of simplicial schemes $\eta : N.(\mathcal{W}) \rightarrow U.$ such that $\eta^{-1}(V.) = Y \cap N.(\mathcal{W})$. Then one may construct a universal class in $\alpha \in H_V^p(U., K_p(\mathcal{O}_{V.}))$, and define $[Y] := \eta^*(\alpha)$.

The same principle also is true for codimension 2 subschemes $Y \subset X$ for which the sheaf of ideals $\mathcal{I}_{Y/X}$ is locally of projective dimension 2. Such ideals are deter-

minantal, *i.e.*, locally they are generated by the maximal minors of an $n \times (n - 1)$ matrix.

Chow Groups of Singular Surfaces

2.6.3

Bloch's formula for codimension two cycles on a singular surface over a field, at least if it has isolated singularities, is fairly well understood, thanks to the work of Collino, Levine, Pedrini, Srinivas, Weibel, and others.

In particular, if X is a reduced quasi-projective surface X over an algebraically closed field k , Biswas and Srinivas, [6] have constructed a Chow ring

$$CH^*(X) = CH^0(X) \oplus CH^1(X) \oplus CH^2(X)$$

satisfying the usual properties of intersection theory for smooth varieties. In particular, there are Chern class maps $C_i: K_0(X) \rightarrow CH^i(X)$ satisfying the Riemann–Roch formula such that, if $F_0K_0(X)$ denotes the subgroup generated by the classes of the structure sheaves of smooth points of X , then $C_2: F_0K_0(X) \rightarrow CH^2(X)$ is an isomorphism, inverse (up to sign) to the cycle map $CH^2(X) \rightarrow K_0(X)$.

The definition of the group of 0-cycles modulo rational equivalence for a singular variety X follows the one given by Levine and Weibel ([49]), *i.e.* as the Chow group $CH_0(X, Y)$ of X relative to its singular locus Y . This is the group generated by closed points on $X - Y$, with rational equivalence defined using rational functions on Cartier curves, *i.e.* every point of $Z \cap Y$ lies in an open neighborhood U where $Z \cap U$ is defined by a regular sequence. See also [52], [48].

Intersection Theory on Stacks

2.6.4

If X is a smooth Deligne Mumford stack over a field, then one can define Chow groups $CH^p(X)$, where $Z^p(X)$ is the free abelian group on the reduced irreducible substacks, and rational equivalence is defined using rational functions on substacks. Bloch's formula remains true, though one is forced to take rational coefficients, and to replace the Zariski topology with the étale topology:

$$CH^p(X)_{\mathbb{Q}} \simeq H_{\text{ét}}^p(X, K_p(\mathcal{O}_X))_{\mathbb{Q}}.$$

This leads to a K -theoretic construction of an intersection product on X . See [28]. One should note that there are other approaches intersection theory on stacks, using operational Chow groups, by Vistoli ([68]), Kresch ([47]) and others.

If $\overline{X}$ is the coarse moduli space, or quotient, of the stack, then one can show that the quotient map $\pi: X \rightarrow \overline{X}$ induces an isomorphism:

$$\pi_*: CH(X)_{\mathbb{Q}} \rightarrow CH(\overline{X})_{\mathbb{Q}},$$

and hence a product structure on the Chow groups, with rational coefficients of the singular variety $\overline{X}$. This is analogous to the construction of the rational cohomology ring of an orbifold.

Deformation to the Normal Cone

This section is based on the expositions in [17] and [67].

89

Definition 89 Let X be a scheme, satisfying our standing assumptions, and suppose that $Y \hookrightarrow X$, is a closed subscheme, defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. Let $W_{Y/X}$ be the scheme obtained by blowing up $\mathbb{A}_X^1 = \text{Spec}(\mathcal{O}_X[t])$ with respect to the sheaf of ideals $(\mathcal{I}, t) \subset \mathcal{O}_{\mathbb{A}_X^1}$ (i.e., along the subscheme $Y \times \{0\}$), and then deleting the divisor (isomorphic to the blow up of X along Y) which is the strict transform of $X \times \{0\}$.

Observe that $t \in \Gamma(W_{Y/X}, \mathcal{O}_{W_{Y/X}})$ is a regular element and so defines a (principal effective Cartier) divisor $W_{Y/X,0} \subset W_{Y/X}$, which is isomorphic to the normal cone $C_{Y/X} = \text{Spec}(\bigoplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1})$. Also note that $W_{Y/X} \setminus W_{Y/X,0} \simeq \mathbb{G}_{m,X} = \text{Spec}(\mathcal{O}_X[t, t^{-1}])$; we shall write $\pi : \mathbb{G}_{m,X} \rightarrow X$ for the natural projection.

We write $p : W_{Y/X,0} = C_{Y/X} \rightarrow Y$ for the natural projection.

If $Y \hookrightarrow X$ is a regular immersion, in the sense of EGA IV ([37]) 16.9.2, then $W_{Y/X,0} = C_{Y/X} \simeq N_{Y/X}$ is a vector bundle over Y .

There is a natural inclusion $\mathbb{A}_Y^1 \hookrightarrow W$, because $Y \times \{0\}$ is principal divisor in $\mathbb{A}_Y^1$.

$$\begin{array}{ccccc}
 W_0 \simeq C_{Y/X} & \hookrightarrow & W_{Y/X} & \longleftarrow & \mathbb{G}_{m,X} \\
 \uparrow & & \uparrow & & \uparrow \\
 Y & \hookrightarrow & \mathbb{A}_Y^1 & \longleftarrow & \mathbb{G}_{m,Y}
 \end{array}$$

We will need the following lemma, which is a straightforward consequence of the basic properties of blow-ups (see [39], II.7):

90

Lemma 90 Suppose that $Z \subset X$ is a closed subscheme. Then $W_{Y \cap Z/Z}$ is a closed subscheme of $W_{Y/X}$, indeed it is the strict transform of $\mathbb{A}_Z^1 \subset \mathbb{A}_X^1$ with respect to the blow up, and $W_{Y \cap Z/Z} \cap W_{Y/X,0} = W_{Y \cap Z/Z,0}$

Envelopes and Hyperenvelopes

91

Definition 91 A map $f : X \rightarrow Y$ is said to be an *envelope* if it is proper and if for every field F , $X(F) \rightarrow Y(F)$ is surjective – or equivalently, for every integral subscheme $Z \subset Y$, there is an integral subscheme $\tilde{Z} \subset X$ such that $f(\tilde{Z}) = Z$, and $f|_{\tilde{Z}} \rightarrow Z$ is birational.

Theorem 92 Suppose that resolution of singularities holds for the category of varieties over the field k . Then, for every variety X over k , there is an envelope $p : \tilde{X} \rightarrow X$ with $\tilde{X}$ nonsingular.

Proof The proof is by induction on the dimension of X . If $\dim(X) = 0$, then X is already nonsingular. Suppose the theorem is true for all varieties of dimension less than $d > 0$. If $\dim(X) = d$, let $p_1 : \tilde{X}_1 \rightarrow X$ be a resolution of singularities of X such that there is a subvariety $Y \subset X$ such that p_1 is an isomorphism over $X - Y$, with $\dim(X) < d$. By the induction hypothesis there is an envelope $q : \tilde{Y} \rightarrow Y$. Now set $\tilde{X} := \tilde{X}_1 \sqcup \tilde{Y}$, and $p := p_1 \sqcup q$.

Definition 93 We say that a map of simplicial schemes $f : X \rightarrow Y$ is *hyperenvelope* if for all fields F , $f_*(F) : X_*(F) \rightarrow Y_*(F)$ is a trivial Kan fibration between simplicial sets. Alternatively, f is a hypercovering in the topology for which envelopes are the coverings. See [22] for more details.

It follows from Theorem 92 that if X is a variety over a field of characteristic zero, then there is a non-singular hyperenvelope $\tilde{X} \rightarrow X$.

Notice that this argument also works for schemes of dimension d over a base S , if resolution of singularities holds for such schemes.

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Regulators*

II.3

Alexander B. Goncharov

To Steve Lichtenbaum for his 65th birthday

3.1	<i>Introduction</i>	297
	Special Values of the Riemann ζ -Function and Their Motivic Nature	297
	The Class Number Formula	298
	Special Values of the Dedekind ζ -Functions, Borel Regulators and Polylogarithms	300
	Beilinson's Conjectures and Arakelov Motivic Complexes	304
3.2	<i>Arakelov Motivic Complexes</i>	307
	Bloch's Cycle Complex	307
	The Deligne Cohomology and Deligne's Complex	307
	The Regulator Map	309
	The Higher Arakelov Chow Groups	312
	Remarks on the Special Values of the Dedekind ζ -Functions	313
3.3	<i>Grassmannian Polylogarithms and Borel's Regulator</i>	315
	The Grassmannian Polylogarithm [30]	315
	The Function ψ_n	318
	The Grassmannian Polylogarithm $\mathcal{L}_n^G$ is the Boundary Value of the Function ψ_n [33]	319
	Construction of the Borel Regulator Map	320
	$\mathbb{P}^1 - \{0, \infty\}$ as a Special Stratum in the Configuration Space of $2n$ Points in $\mathbb{P}^{n-1}$ [30]	323
	Computations of the Grassmannian n -Logarithms	324

* This work was partially supported by IHES (Bures sur Yvette), MPI (Bonn), and the NSF grant DMS-0099390. I am grateful to the referee for useful comments.

3.4	<i>Polylogarithmic Motivic Complexes</i>	325
	The Groups $\mathcal{B}_n(F)$ and Polylogarithmic Motivic Complexes ([27, 28])	325
	The Residue Homomorphism for Complexes $I(F, n)$ [27, 1.14].....	327
	A Variation of Mixed Hodge Structures on $P^1(\mathbb{C}) - \{0, 1, \infty\}$ Corresponding to the Classical Polylogarithm $\text{Li}_n(z)$ ([19])	327
	A Motivic Proof of the Weak Version of Zagier's Conjecture	328
	A Construction of the Motivic ζ -Elements (3.1)	332
	A Geometric Construction of the Motivic Classical Polylogarithm $\text{Li}_n^{\mathcal{M}}(z)$...	333
3.5	<i>Regulator Maps on the Polylogarithmic Motivic Complexes</i>	334
	The Regulator Map on the Polylogarithmic Motivic Complexes at the Generic Point of a Complex Algebraic Variety	334
	The General Case.....	337
	Weight One.....	338
	Weight Two.....	338
	Weight Three	339
	Classical Polylogarithms and Special Values of ζ -Functions of Algebraic Varieties	340
	Coda: Polylogarithms on Curves, Feynman Integrals and Special Values of L -Functions	341
3.6	<i>Motivic Lie Algebras and Grassmannian Polylogarithms</i>	342
	Motivic Lie Coalgebras and Motivic Complexes.....	342
	Grassmannian Approach of the Motivic Lie Coalgebra	343
	The Motivic Grassmannian Tetralogarithm.....	345
	<i>References</i>	346

Introduction

3.1

The ζ -function is one of the most deep and mysterious objects in mathematics. During the last two centuries it has served as a key source of new ideas and concepts in arithmetic algebraic geometry. The ζ -function seems to be created to guide mathematicians into the right directions. To illustrate this, let me recall three themes in the 20th century mathematics which emerged from the study of the most basic properties of ζ -functions: their zeros, analytic properties and special values.

- Weil's conjectures on ζ -functions of varieties over finite fields inspired Grothendieck's revolution in algebraic geometry and led Grothendieck to the concept of motives, and Deligne to the yoga of weight filtrations. In fact (pure) motives over $\mathbb{Q}$ can be viewed as the simplest pieces of algebraic varieties for which the L -function can be defined. Conjecturally the L -function characterizes a motive.
- Langlands' conjectures predict that n -dimensional representations of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ correspond to automorphic representations of $GL(n)_{/\mathbb{Q}}$. The relationship between these seemingly unrelated objects was manifested by L -functions: the Artin L -function of the Galois representation coincides with the automorphic L -function of the corresponding representation of $GL(n)$.
- Investigation of the behavior of L -functions of arithmetic schemes at integer points, culminated in Beilinson's conjectures, led to the discovery of the key principles of the theory of mixed motives.

In this survey we elaborate on a single aspect of the third theme: regulators. We focus on the analytic and geometric aspects of the story, and explore several different approaches to motivic complexes and regulator maps. We neither touch the seminal Birch–Swinnerton–Dyer conjecture and the progress made in its direction nor do we consider the vast generalization of this conjecture, due to Bloch and Kato [13].

Special Values of the Riemann ζ -Function and Their Motivic Nature

3.1.1

Euler proved the famous formula for the special values of the Riemann ζ -function at positive even integers:

$$\zeta(2n) = (-1)^k \pi^{2k} \frac{2^{2k-1}}{(2k-1)!} \left(\frac{-B_{2k}}{2k} \right).$$

All attempts to find a similar formula expressing $\zeta(3), \zeta(5), \dots$ via some known quantities failed. The reason became clear only in the recent time: First, the special values $\zeta(n)$ are periods of certain elements

$$\zeta^{\mathcal{M}}(n) \in \text{Ext}_{\mathcal{M}_T(\mathbb{Z})}^1(\mathbb{Z}(0), \mathbb{Z}(n)), \quad n = 2, 3, 4, \dots \quad (3.1)$$

where on the right stays the extension group in the abelian category $\mathcal{M}_T(\mathbb{Z})$ of mixed Tate motives over $\text{Spec}(\mathbb{Z})$ (as defined in [20]). Second, the fact that $\zeta^{\mathcal{M}}(2n-1)$ are non torsion elements should imply, according to a version of Grothendieck's conjecture on periods, that π and $\zeta(3), \zeta(5), \dots$ are algebraically independent over $\mathbb{Q}$. The analytic manifestation of the motivic nature of the special values is the formula

$$\zeta(n) = \int_{0 < t_1 < \dots < t_n < 1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2} \wedge \dots \wedge \frac{dt_n}{t_n} \quad (3.2)$$

discovered by Leibniz. This formula presents $\zeta(n)$ as a length n iterated integral. The existence of such a formula seems to be a specific property of the L -values at integer points. A geometric construction of the motivic ζ -element (3.1) using the moduli space $\mathcal{M}_{0,n+3}$ is given in Chapt. 4.4.

Beilinson conjectured ([1]) a similar picture for special values of L -functions of motives at integer points. In particular, his conjectures imply that these special values should be periods (in fact a very special kind of periods). For the Riemann ζ -function this is given by the formula (3.2). In general a period is a number given by an integral

$$\int_{\Delta_B} \Omega_A,$$

where Ω_A is a differential form on a variety X with singularities at a divisor A , Δ_B is a chain with boundary at a divisor B , and X, A, B are defined over $\mathbb{Q}$. So far we can write L -values at integer points as periods only in a few cases. Nevertheless, in all cases when Beilinson's conjecture was confirmed, we have such a presentation. More specifically, such a presentation for the special values of the Dedekind ζ -function this comes from the Tamagawa Number formula and Borel's work [16], and in the other cases it is given by Rankin–Selberg type formulas. In general the mechanism staying behind this phenomenon remains a mystery.

Let us now turn to another classical example: the residue of the Dedekind ζ -function at $s = 1$.

3.1.2 The Class Number Formula

Let F be a number field with r_1 real and r_2 complex places, so that $[F : \mathbb{Q}] = 2r_1 + r_2$. Let $\zeta_F(s)$ be the Dedekind ζ -function of F . Then according to Dirichlet and Dedekind one has

$$\text{Res}_{s=1} \zeta_F(s) = \frac{2^{r_1+r_2} \pi^{r_2} R_F h_F}{w_F |D_F|^{1/2}}. \quad (3.3)$$

Here w_F is the number of roots of unity in F , D_F is the discriminant, h_F is the class number, and R_F is the regulator of F , whose definition we recall below. Using the functional equation for $\zeta_F(s)$, S. Lichtenbaum [52] wrote (3.3) as

$$\lim_{s \rightarrow 0} s^{-(r_1+r_2-1)} \zeta_F(s) = -\frac{h_F R_F}{w_F}. \quad (3.4)$$

Let us interpret the right hand side of this formula via the weight one Arakelov motivic complex for $\text{Spec}(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers in F . Let us define the following diagram, where $\mathcal{P}$ runs through all prime ideals of $\mathcal{O}_F$:

$$\begin{array}{ccc} \mathbb{R}^{r_1+r_2} & \xrightarrow{\Sigma} & \mathbb{R} \\ R_1 \uparrow & & \uparrow l \\ F^* & \xrightarrow{\text{div}} & \bigoplus_{\mathcal{P}} \mathbb{Z}. \end{array} \quad (3.5)$$

In this diagram, the maps are given as follows: If $\text{val}_{\mathcal{P}}$ is the canonical valuation defined on F by $\mathcal{P}$ and $|\mathcal{P}|$ is the norm of $\mathcal{P}$, then

$$\text{div}(x) = \sum \text{val}_{\mathcal{P}}(x)[\mathcal{P}], \quad l : [\mathcal{P}] \mapsto -\log |\mathcal{P}|, \quad \Sigma : (x_1, \dots, x_n) \mapsto \sum x_i.$$

The regulator map R_1 is defined by $x \in F^* \mapsto (\log |x|_{\sigma_1}, \dots, \log |x|_{\sigma_{r_1+r_2}})$, where $\{\sigma_1, \dots, \sigma_{r_1+r_2}\}$ is the set of all archimedean places of F and $|\cdot|_{\sigma}$ is the valuation defined σ , $(|x|_{\sigma} := |\sigma(x)|^2 \text{ for a complex place } \sigma)$.

The product formula tells us $\Sigma \circ R_1 + l \circ \text{div} = 0$. Therefore summing up the groups over the diagonals in (3.5) we get a complex. The first two groups of this complex, placed in degrees $[1, 2]$, form the *weight one Arakelov motivic complex* $\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1)$ of $\mathcal{O}_F$. There is a map

$$H^2 \Gamma_{\mathcal{A}}(\mathcal{O}_F, 1) \rightarrow \mathbb{R}.$$

Let $\tilde{H}^2 \Gamma_{\mathcal{A}}(\mathcal{O}_F, 1)$ be its kernel. Then there is an exact sequence

$$0 \rightarrow \frac{\text{Ker}(\Sigma)}{R_1(\mathcal{O}_F^*)} \rightarrow \tilde{H}^2 \Gamma_{\mathcal{A}}(\mathcal{O}_F, 1) \rightarrow \text{Cl}_F \rightarrow 0.$$

Further, $R_1(\mathcal{O}_F^*)$ is a lattice in $\text{Ker}(\Sigma) = \mathbb{R}^{r_1+r_2-1}$, its volume with respect to the measure $\delta(\sum x_i) dx_1 \wedge \dots \wedge dx_{r_1+r_2}$ is R_F , and $h_F = |\text{Cl}_F|$. Therefore

$$\text{vol} \tilde{H}^2 \Gamma_{\mathcal{A}}(\mathcal{O}_F, 1) = R_F h_F; \quad H^1 \tilde{\Gamma}_{\mathcal{A}}(\mathcal{O}_F, 1) = \mu_F$$

Now the class number formula (3.4) reads

$$\lim_{s \rightarrow 0} s^{-(r_1+r_2-1)} \zeta_F(s) = -\frac{\text{vol} \tilde{H}^2 \Gamma_{\mathcal{A}}(\mathcal{O}_F, 1)}{H^1 \tilde{\Gamma}_{\mathcal{A}}(\mathcal{O}_F, 1)}. \quad (3.6)$$

The right hand side is a volume of the determinant of a complex, see Chapt. 2.5. I do not know the cohomological origin of the sign in (3.6).

Special Values of the Dedekind ζ -Functions, Borel Regulators and Polylogarithms

3.1.3

B. Birch and J. Tate [62] proposed a generalization of the class number formula for totally real fields using Milnor's K_2 -group of $\mathcal{O}_F$:

$$\zeta_F(-1) = \pm \frac{|K_2(\mathcal{O}_F)|}{w_2(F)}$$

Here $w_2(F)$ is the largest integer m such that $\text{Gal}(\bar{F}/F)$ acts trivially on $\mu_\infty^{\otimes 2}$. Up to a power of 2, the above formula follows from the Iwasawa main conjecture for totally real fields, proved by B. Mazur and A. Wiles for $\mathbb{Q}$ [54], and by A. Wiles [65] in general.

S. Lichtenbaum [52] suggested that for $\zeta_F(n)$ there should be a formula similar to (3.4) with a higher regulator defined using Quillen's K -groups $K_*(F)$ of F . Such a formula for $\zeta_F(n)$, considered up to a non zero rational factor, has been established soon after in the fundamental work of A. Borel [15, 16]. Let us discuss it in more detail. The rational K -groups of a field can be defined as the primitive part in the homology of GL . Even better, one can show that

$$K_{2n-1}(F) \otimes \mathbb{Q} \cong \text{Prim} H_{2n-1}(GL_{2n-1}(F), \mathbb{Q}) .$$

Let $\mathbb{R}(n) := (2\pi i)^n \mathbb{R}$. There is a distinguished class, called the Borel class,

$$B_n \in H_c^{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{R}(n-1))$$

in the continuous cohomology of the Lie group $GL_{2n-1}(\mathbb{C})$. Pairing with this class provides the Borel regulator map

$$r_n^{\text{Bo}} : K_{2n-1}(\mathbb{C}) \rightarrow \mathbb{R}(n-1) .$$

Let $X_F := \mathbb{Z}^{\text{Hom}(F, \mathbb{C})}$. The Borel regulator map on $K_{2n-1}(F)$ is the composition

$$K_{2n-1}(F) \rightarrow \oplus_{\text{Hom}(F, \mathbb{C})} K_{2n-1}(\mathbb{C}) \rightarrow X_F \otimes \mathbb{R}(n-1)$$

The image of this map is invariant under complex conjugation acting both on $\text{Hom}(F, \mathbb{C})$ and $\mathbb{R}(n-1)$. So we get the map

$$R_n^{\text{Bo}} : K_{2n-1}(F) \rightarrow (X_F \otimes \mathbb{R}(n-1))^+ . \quad (3.7)$$

Here $+$ means the invariants under the complex conjugation. Borel proved that for $n > 1$ the image of this map is a lattice, and the volume $R_n(F)$ of this lattice is related to the Dedekind ζ -function as follows:

$$R_n(F) \sim_{\mathbb{Q}^*} \lim_{s \rightarrow 1-n} (s-1+n)^{-d_n} \zeta_F(s) .$$

Here $a \sim_{\mathbb{Q}^*} b$ means $a = \lambda b$ for some $\lambda \in \mathbb{Q}^*$, and

$$d_n = \dim (X_F \otimes \mathbb{R}(n-1))^+ = \begin{cases} r_1 + r_2 : & n > 1 \text{ odd} \\ r_2 & : n \geq 2 \text{ even} . \end{cases}$$

Using the functional equation for $\zeta_F(s)$ it tells us about $\sim_{\mathbb{Q}^*} \zeta_F(n)$. However Licht-enbaum's original conjecture was stronger since it was about $\zeta_F(n)$ itself.

In 1977 S. Bloch discovered [11, 12] that the regulator map on $K_3(\mathbb{C})$ can be explicitly defined using the dilogarithm. Here is how the story looks today. The dilogarithm is a multivalued analytic function on $\mathbb{CP}^1 - \{0, 1, \infty\}$:

$$\text{Li}_2(z) := - \int_0^z \log(1-z) \frac{dz}{z}; \quad \text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \quad \text{for } |z| \leq 1 .$$

The dilogarithm has a single-valued version, called the Bloch–Wigner function:

$$\mathcal{L}_2(z) := \text{Im} \left(\text{Li}_2(z) + \log(1-z) \log |z| \right) .$$

It vanishes on the real line. Denote by $r(z_1, \dots, z_4)$ the cross-ratio of the four points $z_1, \dots, z_4$ on the projective line. The Bloch–Wigner function satisfies Abel's five term relation: for any five points $z_1, \dots, z_5$ on $\mathbb{CP}^1$ one has

$$\sum_{i=1}^5 (-1)^i \mathcal{L}_2 \left(r(z_1, \dots, \widehat{z}_i, \dots, z_5) \right) = 0 . \quad (3.8)$$

Let $\mathcal{H}_3$ be the hyperbolic three space. Its absolute is identified with the Riemann sphere $\mathbb{CP}^1$. Let $I(z_1, \dots, z_4)$ be the ideal geodesic simplex with the vertices at the points $z_1, \dots, z_4$ at the absolute. Lobachevsky proved that

$$\text{vol} I(z_1, \dots, z_4) = \mathcal{L}_2 \left(r(z_1, \dots, z_4) \right) .$$

(He got this in a different but equivalent form). Lobachevsky's formula makes Abel's equation obvious: the alternating sum of the geodesic simplices with the vertices at $z_1, \dots, \widehat{z}_i, \dots, z_5$ is empty.

Abel's equation can be interpreted as follows: for any $z \in \mathbb{CP}^1$ the function

$$\mathcal{L}_2 \left(r(g_1 z, \dots, g_4 z) \right), \quad g_i \in GL_2(\mathbb{C}) \quad (3.9)$$

is a measurable 3-cocycle of the Lie group $GL_2(\mathbb{C})$. The cohomology class of this cocycle is non trivial. The simplest way to see it is this. The function

$$\text{vol} I(g_1 x, \dots, g_4 x), \quad g_i \in GL_2(\mathbb{C}), x \in \mathcal{H}^3$$

provides a smooth 3-cocycle of $GL_2(\mathbb{C})$. Its cohomology class is nontrivial: indeed, its infinitesimal version is provided by the volume form in $\mathcal{H}_3$. Since the cohomology class does not depend on x , we can take x at the absolute, proving the claim.

Let F be an arbitrary field. Denote by $\mathbb{Z}[F^*]$ the free abelian group generated by the set F^* . Let $R_2(F)$ be the subgroup of $\mathbb{Z}[F^*]$ generated by the elements

$$\sum_{i=1}^5 (-1)^i \{r(z_1, \dots, \widehat{z_i}, \dots, z_5)\}.$$

Let $B_2(F)$ be the quotient of $\mathbb{Z}[F^*]$ by the subgroup $R_2(F)$. Then one shows that the map

$$\mathbb{Z}[F^*] \rightarrow \Lambda^2 F^*; \quad \{z\} \mapsto (1-z) \wedge z$$

kills the subgroup $R_2(F)$, providing a complex (called the Bloch–Suslin complex)

$$\delta_2 : B_2(F) \rightarrow \Lambda^2 F^*. \quad (3.10)$$

By Matsumoto's theorem $\text{Coker} \delta_2 = K_2(F)$. Let us define the Milnor ring $K_*^M(F)$ of F as the quotient of the tensor algebra of the abelian group F^* by the two sided ideal generated by the Steinberg elements $(1-x) \otimes x$ where $x \in F^* - 1$. The product map in the K -theory provides a map

$$\otimes^n K_1(F) = \otimes^n F^* \rightarrow K_n(F).$$

It kills the Steinberg elements, and thus provides a map $K_n^M(F) \rightarrow K_n(F)$. Set

$$K_3^{\text{ind}}(F) := \text{Coker} (K_3^M(F) \rightarrow K_3(F)).$$

A.A. Suslin [61] proved that there is an exact sequence

$$0 \rightarrow \text{Tor}(F^*, F^*)^\sim \rightarrow K_3^{\text{ind}}(F) \rightarrow \text{Ker} \delta_2 \rightarrow 0, \quad (3.11)$$

where $\text{Tor}(F^*, F^*)^\sim$ is a nontrivial extension of $\mathbb{Z}/2\mathbb{Z}$ by $\text{Tor}(F^*, F^*)$.

Abel's relation provides a well defined homomorphism

$$\mathcal{L}_2 : B_2(\mathbb{C}) \rightarrow \mathbb{R}; \quad \{z\}_2 \mapsto \mathcal{L}_2(z).$$

Restricting it to the subgroup $\text{Ker} \delta_2 \subset B_2(\mathbb{C})$, and using (3.11), we get a map $K_3^{\text{ind}}(\mathbb{C}) \rightarrow \mathbb{R}$. Using the interpretation of the cohomology class of the cocycle (3.9) as a volume of geodesic simplex, one can show that it is essentially the Borel regulator. Combining this with Borel's theorem we get an explicit formula for $\zeta_F(2)$ for an arbitrary number field F .

How to generalize this beautiful story? Recall the classical polylogarithms

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad |z| \leq 1; \quad \text{Li}_n(z) = \int_0^z \text{Li}_{n-1}(z) \frac{dz}{z}.$$

D. Zagier [67] formulated a precise conjecture expressing $\zeta_F(n)$ via classical polylogarithms, see the survey [24]. It was proved for $n = 3$ in [27, 28], but it is not

known for $n > 3$, although its easier part has been proved in [5, 46] and, in a different way, in Chapt. 4.4 below.

Most of the $\zeta_F(2)$ picture has been generalized to the case of $\zeta_F(3)$ in [27, 28] and [29]. Namely, there is a single valued version of the trilogarithm:

$$\mathcal{L}_3(z) := \operatorname{Re} \left(\operatorname{Li}_3(z) - \operatorname{Li}_2(z) \log |z| + \frac{1}{6} \operatorname{Li}_1(z) \log^2 |z| \right).$$

It satisfies the following functional equation which generalizes (3.8). Let us define the generalized cross-ratio of 6 points $x_0, \dots, x_5$ in P^2 as follows. We present P^2 as a projectivization of the three dimensional vector space V_3 and choose the vectors $l_i \in V_3$ projecting to the points x_i . Let us choose a volume form $\omega \in \det V_3^*$ and set $\Delta(a, b, c) := \langle a \wedge b \wedge c, \omega \rangle$. Set

$$r_3(x_0, \dots, x_5) := \operatorname{Alt}_6 \left\{ \frac{\Delta(l_0, l_1, l_3) \Delta(l_1, l_2, l_4) \Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4) \Delta(l_1, l_2, l_5) \Delta(l_2, l_0, l_3)} \right\} \in \mathbb{Z}[F^*].$$

Here Alt_6 denotes the alternation of $l_0, \dots, l_5$. The function $\mathcal{L}_3$ extends by linearity to a homomorphism $\mathcal{L}_3 : \mathbb{Z}[\mathbb{C}^*] \rightarrow \mathbb{R}$, and there is a generalization of Abel's equation to the case of the trilogarithm (see [28] and the appendix to [29]):

$$\sum_{i=1}^7 (-1)^i \mathcal{L}_3(r_3(x_1, \dots, \widehat{x_i}, \dots, x_7)) = 0. \quad (3.12)$$

We define the group $B_3(F)$ as the quotient of $\mathbb{Z}[F^*]$ by the subgroup generated by the functional equations (3.12) for the trilogarithm. Then there is complex

$$B_3(F) \xrightarrow{\delta_3} B_2(F) \otimes F^* \xrightarrow{\delta_2 \wedge \operatorname{Id}} \Lambda^3 F^*; \quad \delta_3 : \{x\}_3 \mapsto \{x\}_2 \otimes x. \quad (3.13)$$

There is a map $K_5(F) \rightarrow \operatorname{Ker} \delta_3$ such that in the case $F = \mathbb{C}$ the composition

$$K_5(\mathbb{C}) \rightarrow \operatorname{Ker} \delta_3 \hookrightarrow B_3(\mathbb{C}) \xrightarrow{\mathcal{L}_3} \mathbb{R}$$

coincides with the Borel regulator (see [28] and appendix in [29]). This plus Borel's theorem leads to an explicit formula expressing $\zeta_F(3)$ via the trilogarithm conjectured by Zagier [67].

In Chapt. 3 we define, following [30], the Grassmannian n -logarithm function $\mathcal{L}_n^G$. It is a function on the configurations of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. One of its functional equations generalizes Abel's equation:

$$\sum_{i=1}^{2n+1} (-1)^i \mathcal{L}_n^G(h_1, \dots, \widehat{h_i}, \dots, h_{2n+1}) = 0.$$

It means that for a given hyperplane h the function $\mathcal{L}_n^G(g_1 h, \dots, g_{2n} h)$, $g_i \in GL_n(\mathbb{C})$, is a measurable cocycle of $GL_n(\mathbb{C})$. Its cohomology class essentially coincides with the restriction of the Borel class B_n to $GL_n(\mathbb{C})$. To prove this we show that $\mathcal{L}_n^G$ is

the boundary value of a certain function defined on configurations of $2n$ points in the symmetric space $SL_n(\mathbb{C})/SU(n)$. Using this we express the Borel regulator via the Grassmannian polylogarithm. We show that for $n = 2$ we recover the dilogarithm story: $\mathcal{L}_2^G$ coincides with the Bloch–Wigner function, $SL_2(\mathbb{C})/SU(2)$ is the hyperbolic space, and the extension of $\mathcal{L}_2^G$ is given by the volume of geodesic simplices. The proofs can be found in [33].

Beilinson's Conjectures and Arakelov Motivic Complexes

3.1.4

A conjectural generalization of the class number formula (3.3) to the case of elliptic curves was suggested in the seminal work of Birch and Swinnerton–Dyer. Several years later J. Tate formulated conjectures relating algebraic cycles to the poles of zeta functions of algebraic varieties.

Let X be a regular algebraic variety over a number field F . Generalizing the previous works of Bloch [11] and P. Deligne [18], A.A. Beilinson [1] suggested a fantastic picture unifying all the above conjectures. Beilinson defined the rational motivic cohomology of X via the algebraic K -theory of X by

$$H_{Mot}^i(X, \mathbb{Q}(n)) := gr_n^\gamma K_{2n-i}(X) \otimes \mathbb{Q}.$$

Here γ is the Adams γ -filtration.

Let us assume that X is projective. For schemes which admit regular models over $\mathbb{Z}$, Beilinson [1] defined a $\mathbb{Q}$ -vector subspace

$$H_{Mot/\mathbb{Z}}^i(X, \mathbb{Q}(n)) \subset H_{Mot}^i(X, \mathbb{Q}(n)), \quad (3.14)$$

called *integral part in the motivic cohomology*. Using alterations, A. Scholl [60] extended this definition to arbitrary regular projective schemes over a number field F . For every regular, projective and flat over $\mathbb{Z}$ model X' of X the subspace (3.14) coincides with the image of the map $H_{Mot}^i(X', \mathbb{Q}(n)) \rightarrow H_{Mot}^i(X, \mathbb{Q}(n))$.

For a regular complex projective variety X Beilinson constructed in [1] the regulator map to the Deligne cohomology of X :

$$H_{Mot}^i(X, \mathbb{Q}(n)) \rightarrow H_{\mathcal{D}}^i(X(\mathbb{C}), \mathbb{Z}(n)). \quad (3.15)$$

There is a natural projection $H_{\mathcal{D}}^i(X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow H_{\mathcal{D}}^i(X(\mathbb{C}), \mathbb{R}(n))$. Having in mind applications to the special values of L-functions, we compose the map (3.15) with this projection, getting a regulator map

$$H_{Mot}^i(X, \mathbb{Q}(n)) \rightarrow H_{\mathcal{D}}^i(X(\mathbb{C}), \mathbb{R}(n)).$$

Now let X again be a regular projective scheme over a number field F . We will view it as a scheme over $\mathbb{Q}$ via the projection $X \rightarrow \text{Spec}(F) \rightarrow \text{Spec}(\mathbb{Q})$. We define the *real Deligne cohomology of X* as the following $\mathbb{R}$ -vector space:

$$H_{\mathcal{D}}^i((X \otimes_{\mathbb{Q}} \mathbb{R})(\mathbb{C}), \mathbb{R}(n))^{\bar{F}_{\infty}},$$

where $\bar{F}_\infty$ is an involution given by the composition of complex conjugation acting on $X(\mathbb{C})$ and on the coefficients. Then, restricting the map (3.15) to the integral part in the motivic cohomology (3.14) and projecting the image onto the real Deligne cohomology of X , we get a regulator map

$$r_{Be} : H_{Mot/\mathbb{Z}}^i(X, \mathbb{Q}(n)) \rightarrow H_{\mathcal{D}}^i((X \otimes_{\mathbb{Q}} \mathbb{R})(\mathbb{C}), \mathbb{R}(n))^{\bar{F}_\infty}.$$

Beilinson formulated a conjecture relating the special values $L(h^{i-1}(X), n)$ of the L -function of Grothendieck's motive $h^{i-1}(X)$ to values of this regulator map, up to a nonzero rational factor; see [1], the survey [58] and the book [57] for the original version of the conjecture, and the survey by J. Nekovář [56] for a motivic reformulation. For $X = \text{Spec}(F)$, where F is a number field, it boils down to Borel's theorem. A precise Tamagawa Number conjecture about the special values $L(h^i(X), n)$ was suggested by Bloch and Kato [13].

Beilinson [2] and Lichtenbaum [53] conjectured that the weight n integral motivic cohomology of a scheme X should appear as cohomology of some complexes of abelian groups $\mathbb{Z}_X^\bullet(n)$, called the weight n motivic complexes of X . One must have

$$H_{Mot}^i(X, \mathbb{Q}(n)) = H^i \mathbb{Z}_X^\bullet(n) \otimes \mathbb{Q}.$$

Motivic complexes are objects of the derived category. Beilinson conjectured [2] that there exists an abelian category $\mathcal{MS}_X$ of mixed motivic sheaves on X , and that one should have an isomorphism in the derived category

$$\mathbb{Z}_X^\bullet(n) = R\text{Hom}_{\mathcal{MS}_X}(\mathbb{Q}(0)_X, \mathbb{Q}(n)_X). \quad (3.16)$$

Here $\mathbb{Q}(n)_X := p^* \mathbb{Q}(n)$, where $p : X \rightarrow \text{Spec}(F)$ is the structure morphism, is a motivic sheaf on X obtained by pull back of the Tate motive $\mathbb{Q}(n)$ over the point $\text{Spec}(F)$. This formula implies the Beilinson–Soulé vanishing conjecture:

$$H_{Mot}^i(X, \mathbb{Q}(n)) = 0 \quad \text{for } i < 0 \text{ and } i = 0, n > 0.$$

Indeed, the negative Ext's between objects of an abelian category are zero, and we assume that the objects $\mathbb{Q}(n)_X$ are mutually non-isomorphic. Therefore it is quite natural to look for representatives of motivic complexes which are zero in the negative degrees, as well as in the degree zero for $n = 0$.

Motivic complexes are more fundamental, and in fact simpler objects than rational K -groups. Several constructions of motivic complexes are known.

- i) Bloch [8,9] suggested a construction of the motivic complexes, called the Higher Chow complexes, using algebraic cycles. The weight n cycle complex appears in a very natural way as a “resolution” for the codimension n Chow groups on X modulo rational equivalence, see Sect. 3.2.1 below. These complexes as well as their versions defined by Suslin and Voevodsky played an essential role in the construction of triangulated categories of mixed motives [47, 51, 63]. However they are unbounded from the left.

- ii) Here is a totally different construction of the first few of motivic complexes. One has $\mathbb{Z}_X^\bullet(0) := \mathbb{Z}$. Let $X^{(k)}$ be the set of all irreducible codimension k subschemes of a scheme X . Then $\mathbb{Z}_X^\bullet(1)$ is the complex

$$\mathcal{O}_X^* \xrightarrow{\partial} \bigoplus_{Y \in X(1)} \mathbb{Z}; \quad \partial(f) := \operatorname{div}(f).$$

The complex $\mathbb{Q}_X^\bullet(2)$ is defined as follows. First, using the Bloch–Suslin complex (3.10), we define the following complex

$$B_2(\mathbb{Q}(X)) \xrightarrow{\delta_2} \Lambda^2 \mathbb{Q}(X)^* \xrightarrow{\partial_1} \bigoplus_{Y \in X(1)} \mathbb{Q}(Y)^* \xrightarrow{\partial_2} \bigoplus_{Y \in X(2)} \mathbb{Z}.$$

Then tensoring it by $\mathbb{Q}$ we get $\mathbb{Q}_X^\bullet(2)$. Here ∂_2 is the tame symbol, and ∂_1 is given by the divisor of a function on Y . Similarly one can define a complex $\mathbb{Q}_X^\bullet(3)$ using the complex (3.13). Unlike the cycle complexes, these complexes are concentrated exactly in the degrees where they might have nontrivial cohomology. It is amazing that motivic complexes have two so different and beautiful incarnations.

Generalizing this, we introduce in Chapt. 4 the *polylogarithmic motivic complexes*, which are conjectured to be the motivic complexes for an arbitrary field ([27, 28]). Then we give a motivic proof of the weak version of Zagier’s conjecture for a number field F . In Chapt. 5 we discuss how to define motivic complexes for an arbitrary regular variety X using the polylogarithmic motivic complexes of its points.

In Chaps. 2 and 5 we discuss constructions of the regulator map on the level of complexes. Precisely, we want to define for a regular complex projective variety X a homomorphism of complexes of abelian groups

$$\{\text{weight } n \text{ motivic complex of } X\} \rightarrow \{\text{weight } n \text{ Deligne complex of } X(\mathbb{C})\}. \quad (3.17)$$

In Chapt. 2 we present a construction of a regulator map on the Higher Chow complexes given in [30, 33]. In Chapt. 5 we define, following [32], a regulator map on polylogarithmic complexes. It is given explicitly in terms of the classical polylogarithms. Combining this with Beilinson’s conjectures we arrive at explicit conjectures expressing the special values of L -functions via classical polylogarithms. If $X = \operatorname{Spec}(F)$, where F is a number field, this boils down to Zagier’s conjecture.

The cone of the map (3.17), shifted by -1 , defines the *weight n Arakelov motivic complex*, and so its cohomology are the *weight n Arakelov motivic cohomology* of X .

The weight n Arakelov motivic complex should be considered as an ingredient of a definition of the *weight n arithmetic motivic complex*. Namely, one should exist a complex computing the weight n integral motivic cohomology of X , and a natural map from this complex to the weight n Deligne complex. The cone of this map, shifted by -1 , would give the weight n arithmetic motivic complex. The weight one arithmetic motivic complex is the complex $\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1)$.

In Chapt. 6 we discuss a yet another approach to motivic complexes of fields: as standard cochain complexes of the motivic Lie algebras. We also discuss a relationship between the motivic Lie algebra of a field and the (motivic) Grassmannian polylogarithms.

Arakelov Motivic Complexes

3.2

In this section we define a regulator map from the weight n motivic complex, understood as Bloch's Higher Chow groups complex [8], to the weight n Deligne complex. This map was defined in [30], and elaborated in detail in [33]. The construction can be immediately adopted to the Suslin–Voevodsky motivic complexes.

Bloch's Cycle Complex

3.2.1

A non degenerate simplex in $\mathbb{P}^m$ is an ordered collection of hyperplanes $L_0, \dots, L_m$ with empty intersection. Let us choose in $\mathbb{P}^m$ a simplex L and a generic hyperplane H . Then L provides a simplex in the affine space $\mathbb{A}^m := \mathbb{P}^m - H$.

Let X be a regular scheme over a field. Let $I = (i_1, \dots, i_k)$ and $L_I := L_{i_1} \cap \dots \cap L_{i_k}$. Let $\mathcal{Z}_m(X; n)$ be the free abelian group generated by irreducible codimension n algebraic subvarieties in $X \times \mathbb{A}^m$ which intersect properly (i.e. the intersection has the right dimension) all faces $X \times L_I$. Intersection with the codimension 1 face $X \times L_i$ provides a group homomorphism $\partial_i : \mathcal{Z}_m(X; n) \rightarrow \mathcal{Z}_{m-1}(X; n)$. Set $\partial := \sum_{i=0}^m (-1)^i \partial_i$. Then $\partial^2 = 0$, so $(\mathcal{Z}_\bullet(X; n), \partial)$ is a homological complex. Consider the cohomological complex $\mathcal{Z}^\bullet(X; n) := \mathcal{Z}_{2r-\bullet}(X; n)$. Its cohomology give the motivic cohomology of X :

$$H_{\mathcal{M}}^i(X, \mathbb{Z}(n)) := H^i(\mathcal{Z}^\bullet(X; n)) .$$

According to the fundamental theorem of Bloch ([8, 9])

$$H^i(\mathcal{Z}^\bullet(X; n) \otimes \mathbb{Q}) = gr_n^Y K_{2n-i}(X) \otimes \mathbb{Q} .$$

The Deligne Cohomology and Deligne's Complex

3.2.2

Let X be a regular projective variety over $\mathbb{C}$. The Beilinson–Deligne complex $\mathbb{R}^\bullet(X; n)_{\mathcal{D}}$ is the following complex of sheaves in the classical topology on $X(\mathbb{C})$:

$$\mathbb{R}(n) \rightarrow \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow \Omega_X^2 \rightarrow \dots \rightarrow \Omega_X^{n-1} .$$

Here the constant sheaf $\mathbb{R}(n) := (2\pi i)^n \mathbb{R}$ is in the degree zero. The hypercohomology of this complex of sheaves is called the *weight n Deligne cohomology* of $X(\mathbb{C})$. They are finite dimensional real vector spaces. Beilinson proved [3] that the truncated weight n Deligne cohomology, which are obtained by putting the weight n Deligne cohomology equal to zero in the degrees $> 2n$, can be interpreted as the absolute Hodge cohomology of $X(\mathbb{C})$.

One can replace the above complex of sheaves by a quasiisomorphic one, defined as the total complex associated with the following bicomplex:

$$\begin{array}{ccccccc}
 (\mathcal{D}_X^0 & \xrightarrow{d} & \mathcal{D}_X^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathcal{D}_X^n & \xrightarrow{d} & \mathcal{D}_X^{n+1} & \xrightarrow{d} & \cdots) \otimes \mathbb{R}(n-1) \\
 & & & & & & \uparrow \pi_n & & \uparrow \pi_n & & \\
 & & & & & & \Omega_X^n & \xrightarrow{\partial} & \Omega_X^{n+1} & \xrightarrow{\partial} &
 \end{array}$$

Here $\mathcal{D}_X^k$ is the sheaf of real k -distributions on $X(\mathbb{C})$, that is k -forms with the generalized function coefficients. Further,

$$\pi_n : \mathcal{D}_X^p \otimes \mathbb{C} \rightarrow \mathcal{D}_X^p \otimes \mathbb{R}(n-1)$$

is the projection induced by the one $\mathbb{C} = \mathbb{R}(n-1) \oplus \mathbb{R}(n) \rightarrow \mathbb{R}(n-1)$, the sheaf $\mathcal{D}_X^0$ is placed in degree 1, and $(\Omega_X^\bullet, \partial)$ is the De Rham complex of sheaves of holomorphic forms.

To calculate the hypercohomology with coefficients in this complex we replace the holomorphic de Rham complex by its Doulbeut resolution, take the global sections of the obtained complex, and calculate its cohomology. Taking the canonical truncation of this complex in the degrees $[0, 2n]$ we get a complex calculating the absolute Hodge cohomology of $X(\mathbb{C})$. Let us define, following Deligne, yet another complex of abelian groups quasiisomorphic to the latter complex.

Let $\mathcal{D}_X^{p,q} = \mathcal{D}^{p,q}$ be the abelian group of complex valued distributions of type (p, q) on $X(\mathbb{C})$. Consider the following cohomological bicomplex, where $\mathcal{D}_d^{n,n}$ is the subspace of closed distributions, and $\mathcal{D}^{0,0}$ is in degree 1:

$$\begin{array}{ccccccc}
 & & & & & & \mathcal{D}_d^{n,n} \\
 & & & & & \nearrow 2\bar{\partial}\partial & \\
 \mathcal{D}^{0,n-1} & \xrightarrow{\partial} & \mathcal{D}^{1,n-1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathcal{D}^{n-1,n-1} \\
 \bar{\partial} \uparrow & & \bar{\partial} \uparrow & & & & \bar{\partial} \uparrow \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 \bar{\partial} \uparrow & & \bar{\partial} \uparrow & & & & \bar{\partial} \uparrow \\
 \mathcal{D}^{0,1} & \xrightarrow{\partial} & \mathcal{D}^{1,1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathcal{D}^{n-1,1} \\
 \bar{\partial} \uparrow & & \bar{\partial} \uparrow & & & & \bar{\partial} \uparrow \\
 \mathcal{D}^{0,0} & \xrightarrow{\partial} & \mathcal{D}^{1,0} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathcal{D}^{n-1,0}
 \end{array}$$

The complex $C_{\mathcal{D}}^{\bullet}(X; n)$ is a subcomplex of the total complex of this bicomplex provided by the $\mathbb{R}(n-1)$ -valued distributions in the $n \times n$ square of the diagram and the subspace $\mathcal{D}_{\mathbb{R}, cl}^{n, n}(n) \subset \mathcal{D}_{cl}^{n, n}$ of the $\mathbb{R}(n)$ -valued distributions of type (n, n) . Notice that $\bar{\partial}\partial$ sends $\mathbb{R}(n-1)$ -valued distributions to $\mathbb{R}(n)$ -valued distributions. The cohomology of this complex of abelian groups is the absolute Hodge cohomology of $X(\mathbb{C})$, see Proposition 2.1 of [33]. Now if X is a variety over $\mathbb{R}$, then

$$C_{\mathcal{D}}^{\bullet}(X_{/\mathbb{R}}; n) := C_{\mathcal{D}}^{\bullet}(X; n)^{\bar{F}_{\infty}}; \quad H_{\mathcal{D}}^i(X_{/\mathbb{R}}, \mathbb{R}(n)) = H^i C_{\mathcal{D}}^{\bullet}(X_{/\mathbb{R}}; n)$$

where $\bar{F}_{\infty}$ is the composition of the involution F_{∞} on $X(\mathbb{C})$ induced by the complex conjugation with the complex conjugation of coefficients.

The Regulator Map

3.2.3

Theorem-Construction 1 Let X be a regular complex projective variety. Then there exists *canonical* homomorphism of complexes

1

$$\mathcal{P}^{\bullet}(n) : \mathcal{Z}^{\bullet}(X; n) \rightarrow C_{\mathcal{D}}^{\bullet}(X; n).$$

If X is defined over $\mathbb{R}$ then its image lies in the subcomplex $C_{\mathcal{D}}^{\bullet}(X_{/\mathbb{R}}; n)$.

To define this homomorphism we need the following construction. Let X be a variety over $\mathbb{C}$ and $f_1, \dots, f_m$ be m rational functions on X . The form

$$\pi_m(d \log f_1 \wedge \dots \wedge d \log f_m),$$

where $\pi_n(a + ib) = a$ if n odd, and $\pi_n(a + ib) = ib$ if n even, has zero periods. It has a canonical primitive defined as follows. Consider the following $(m-1)$ -form on $X(\mathbb{C})$:

$$\begin{aligned} r_{m-1}(f_1 \wedge \dots \wedge f_m) &:= \\ \text{Alt}_m \sum_{j \geq 0} c_{j,m} \log |f_1| d \log |f_2| \wedge \dots \wedge d \log |f_{2j+1}| \wedge di \arg f_{2j+2} \wedge \dots \wedge di \arg f_m. \end{aligned} \quad (3.18)$$

Here $c_{j,m} := ((2j+1)!(m-2j-1)!)^{-1}$ and Alt_m is the operation of alternation:

$$\text{Alt}_m F(x_1, \dots, x_m) := \sum_{\sigma \in S_m} (-1)^{|\sigma|} F(x_{\sigma(1)}, \dots, x_{\sigma(m)}).$$

So $r_{m-1}(f_1 \wedge \dots \wedge f_m)$ is an $\mathbb{R}(m-1)$ -valued $(m-1)$ -form. One has

$$dr_{m-1}(f_1 \wedge \dots \wedge f_m) = \pi_m(d \log f_1 \wedge \dots \wedge d \log f_m).$$

It is sometimes convenient to write the form (3.18) as a multiple of

$$\text{Alt}_m \sum_{i=1}^m (-1)^i \log |f_1| d \log f_2 \wedge \dots \wedge d \log f_i \wedge d \log \bar{f}_{i+1} \wedge \dots \wedge d \log \bar{f}_m.$$

Precisely, let $\mathcal{A}^i(M)$ be the space of smooth i -forms on a real smooth manifold M . Consider the following map

$$\omega_{m-1} : \Lambda^m \mathcal{A}^0(M) \rightarrow \mathcal{A}^{m-1}(M) \quad (3.19)$$

$$\omega_{m-1}(\varphi_1 \wedge \dots \wedge \varphi_m) :=$$

$$\frac{1}{m!} \text{Alt}_m \left(\sum_{k=1}^m (-1)^{k-1} \varphi_1 \partial \varphi_2 \wedge \dots \wedge \partial \varphi_k \wedge \bar{\partial} \varphi_{k+1} \wedge \dots \wedge \bar{\partial} \varphi_m \right).$$

For example

$$\omega_0(\varphi_1) = \varphi_1; \quad \omega_1(\varphi_1 \wedge \varphi_2) = \frac{1}{2}(\varphi_1 \partial \varphi_2 - \varphi_2 \partial \varphi_1 - \varphi_1 \bar{\partial} \varphi_2 + \varphi_2 \bar{\partial} \varphi_1)$$

Then one easily checks that

$$\begin{aligned} d\omega_{m-1}(\varphi_1 \wedge \dots \wedge \varphi_m) &= \partial \varphi_1 \wedge \dots \wedge \partial \varphi_m + (-1)^m \bar{\partial} \varphi_1 \wedge \dots \wedge \bar{\partial} \varphi_m \\ &\quad + \sum_{i=1}^m (-1)^i \bar{\partial} \partial \varphi_i \wedge \omega_{m-2}(\varphi_1 \wedge \dots \wedge \widehat{\varphi}_i \wedge \dots \wedge \varphi_m). \end{aligned} \quad (3.20)$$

Now let f_i be rational functions on a complex algebraic variety X . Set $M := X^0(\mathbb{C})$, where X^0 is the open part of X where the functions f_i are regular. Then $\varphi_i := \log |f_i|$ are smooth functions on M , and we have

$$\omega_{m-1}(\log |f_1| \wedge \dots \wedge \log |f_m|) = r_{m-1}(f_1 \wedge \dots \wedge f_m).$$

Denote by $\mathcal{D}_{X, \mathbb{R}}^*(k) = \mathcal{D}_{\mathbb{R}}^*(k)$ the subspace of $\mathbb{R}(k)$ -valued distributions in $\mathcal{D}_{X(\mathbb{C})}^*$.

Let Y^0 be the nonsingular part of Y , and $i_Y^0 : Y^0(\mathbb{C}) \hookrightarrow Y(\mathbb{C})$ the canonical embedding.

2

Proposition 2 Let Y be an arbitrary irreducible subvariety of a smooth complex variety X and $f_1, \dots, f_m \in \mathcal{O}^*(Y)$. Then for any smooth differential form ω with compact support on $X(\mathbb{C})$ the following integral is convergent:

$$\int_{Y^0(\mathbb{C})} r_{m-1}(f_1 \wedge \dots \wedge f_m) \wedge i_Y^0 \omega.$$

Thus there is a distribution $r_{m-1}(f_1 \wedge \dots \wedge f_m) \delta_Y$ on $X(\mathbb{C})$:

$$\langle r_{m-1}(f_1 \wedge \dots \wedge f_m) \delta_Y, \omega \rangle := \int_{Y^0(\mathbb{C})} r_{m-1}(f_1 \wedge \dots \wedge f_m) \wedge i_Y^0 \omega.$$

It provides a group homomorphism

$$r_{m-1} : \Lambda^m \mathbb{C}(Y)^* \rightarrow \mathcal{D}_{X, \mathbb{R}}^{m-1}(m-1). \quad (3.21)$$

Construction 3 We have to construct a morphism of complexes

3

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \mathcal{Z}^1(X; n) & \longrightarrow & \cdots & \longrightarrow & \mathcal{Z}^{2n-1}(X; n) \longrightarrow \mathcal{Z}^{2n}(X; n) \\
 & & \downarrow \mathcal{P}^1(n) & & \vdots & & \downarrow \mathcal{P}^{2n-1}(n) \quad \downarrow \mathcal{P}^{2n}(n) \\
 0 & \longrightarrow & \mathcal{D}_{\mathbb{R}}^{0,0}(n-1) & \longrightarrow & \cdots & \longrightarrow & \mathcal{D}_{\mathbb{R}}^{n-1,n-1}(n-1) \xrightarrow{2\bar{\partial}} \mathcal{D}_{\mathbb{R}}^{n,n}(n)
 \end{array}$$

Here at the bottom stays the complex $\mathcal{C}_{\mathcal{D}}^{\bullet}(X; n)$.

Let $Y \in \mathcal{Z}^{2n}(X; n)$ be a codimension n cycle in X . By definition

$$\mathcal{P}^{2n}(n)(Y) := (2\pi i)^n \delta_Y.$$

Let us construct homomorphisms

$$\mathcal{P}^{2n-k}(n) : \mathcal{Z}^{2n-k}(X; n) \rightarrow \mathcal{D}_X^{2n-k-1}, \quad k > 0.$$

Denote by $\pi_{\mathbb{A}^k}$ (resp. π_X) the projection of $X \times \mathbb{A}^k$ to $\mathbb{A}^k$ (resp. X), and by $\bar{\pi}_{\mathbb{A}^k}$ (resp. $\bar{\pi}_X$) the projection of $X(\mathbb{C}) \times \mathbb{CP}^k$ to $\mathbb{CP}^k$ (resp. $X(\mathbb{C})$).

The pair (L, H) in $\mathbb{P}^k$ defines uniquely homogeneous coordinates $(z_0 : \dots : z_k)$ in $\mathbb{P}^k$ such that the hyperplane L_i is given by equation $\{z_i = 0\}$ and the hyperplane H is $\{\sum_{i=1}^k z_i = z_0\}$. Then there is an element

$$\frac{z_1}{z_0} \wedge \dots \wedge \frac{z_k}{z_0} \in \Lambda^{k-1} \mathbb{C}(\mathbb{A}^k)^*. \quad (3.22)$$

Let $Y \in \mathcal{Z}^{2n-k}(X; n)$. Restricting to Y the inverse image of the element (3.22) by $\pi_{\mathbb{A}^k}^*$ we get an element

$$g_1 \wedge \dots \wedge g_k \in \Lambda^k \mathbb{C}(Y)^*. \quad (3.23)$$

Observe that this works if and only if the cycle Y intersects properly all codimension one faces of $X \times L$. Indeed, if Y does not intersect properly one of the faces, then the equation of this face restricts to zero to Y , and so (3.23) does not make sense.

The element (3.23) provides, by Proposition 2, a distribution on $X(\mathbb{C}) \times \mathbb{CP}^k$. Pushing it down by $(2\pi i)^{n-k} \cdot \bar{\pi}_X$ we get the distribution $\mathcal{P}^{2n-k}(n)(Y)$:

Definition 4 $\mathcal{P}^{2n-k}(n)(Y) := (2\pi i)^{n-k} \cdot \bar{\pi}_{X*} r_{k-1}(g_1 \wedge \dots \wedge g_k).$

4

In other words, the following distribution makes sense:

$$\mathcal{P}^{2n-k}(n)(Y) = (2\pi i)^{n-k} \bar{\pi}_{X*} \left(\delta_Y \wedge \bar{\pi}_{\mathbb{A}^k}^* r_{k-1} \left(\frac{z_1}{z_0} \wedge \dots \wedge \frac{z_k}{z_0} \right) \right).$$

One can rewrite Definition 4 more explicitly as an integral over $Y(\mathbb{C})$. Namely, let ω be a smooth form on $X(\mathbb{C})$ and $i_Y : Y \hookrightarrow X \times \mathbb{P}^k$. Then

$$\langle \mathcal{P}^{2n-k}(n)(Y), \omega \rangle = (2\pi i)^{n-k} \int_{Y^0(\mathbb{C})} r_{k-1}(g_1 \wedge \dots \wedge g_k) \wedge i_{Y^0(\mathbb{C})}^* \overline{\pi}_X^* \omega.$$

It is easy to check that $\mathcal{P}^{2n-k}(n)(Y)$ lies in $\mathcal{C}_{\mathcal{D}}^{2n-k}(X; n)$. Therefore we defined the maps $\mathcal{P}^k(n)$. It was proved in Theorem 2.12 in [33] that $\mathcal{P}^\bullet(n)$ is a homomorphism of complexes.

3.2.4 The Higher Arakelov Chow Groups

Let X be a regular complex variety. Denote by $\widetilde{\mathcal{C}}_{\mathcal{D}}^\bullet(X; n)$ the quotient of the complex $\mathcal{C}_{\mathcal{D}}^\bullet(X; n)$ along the subgroup $\mathcal{A}_{cl, \mathbb{R}}^{n,n}(n) \subset \mathcal{D}_{cl, \mathbb{R}}^{n,n}(n)$ of closed smooth forms. The cone of the homomorphism $\mathcal{P}^\bullet(n)$ shifted by -1 is the Arakelov motivic complex:

$$\widehat{\mathcal{Z}}^\bullet(X; n) := \text{Cone}(\mathcal{Z}^\bullet(X; n) \rightarrow \widetilde{\mathcal{C}}_{\mathcal{D}}^\bullet(X; n))[-1].$$

The Higher Arakelov Chow groups are its cohomology:

$$\widehat{CH}^n(X; i) := H^{2n-i}(\widehat{\mathcal{Z}}^\bullet(X; n)).$$

Recall the arithmetic Chow groups defined by Gillet–Soulé [26] as follows:

$$\widehat{CH}^n(X) := \frac{\left\{ (Z, g); \frac{\partial \bar{\partial}}{\pi i} g + \delta_Z \in \mathcal{A}^{n,n} \right\}}{\left\{ (0, \partial u + \bar{\partial} v); (\text{div} f, -\log |f|), f \in \mathbb{C}(Y), \text{codim}(Y) = n-1 \right\}}. \quad (3.24)$$

Here Z is a divisor in X , f is a rational function on a divisor Y in X ,

$$g \in \mathcal{D}_{\mathbb{R}}^{n-1, n-1}(n-1), \quad (u, v) \in \mathcal{C}_{\mathcal{D}}^{2n-2}(X; n) = (\mathcal{D}^{n-2, n-1} \oplus \mathcal{D}^{n-1, n-2})_{\mathbb{R}}(n-1).$$

5

Proposition 5 $\widehat{CH}^n(X; 0) = \widehat{CH}^n(X)$.

Proof Let us look at the very right part of the complex $\widehat{\mathcal{Z}}^\bullet(X; n)$:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathcal{Z}^{2n-1}(X; n) & \longrightarrow & \mathcal{Z}^{2n}(X; n) & & \\ & & \downarrow \mathcal{P}^{2n-1}(n) & & \downarrow \mathcal{P}^{2n}(n) & & \\ & & (\partial, \bar{\partial}) & & 2\bar{\partial}\partial & & \\ (\mathcal{D}^{n-2, n-1} \oplus \mathcal{D}^{n-1, n-2})_{\mathbb{R}}(n-1) & \longrightarrow & \mathcal{D}_{\mathbb{R}}^{n-1, n-1}(n-1) & \longrightarrow & \mathcal{D}_{\mathbb{R}}^{n,n}(n) / \mathcal{A}_{\mathbb{R}}^{n,n}(n) & & \end{array}$$

Consider the stupid truncation of the Gersten complex on X :

$$\prod_{Y \in X_{n-1}} \mathbb{C}(Y)^* \rightarrow \mathcal{Z}_0(X; n). \quad (3.25)$$

It maps to the stupid truncation $\sigma_{\geq 2n-1} \widehat{\mathcal{Z}}^\bullet(X; n)$ of the cycle complex as follows. The isomorphism $\mathcal{Z}_0(X; n) = \mathcal{Z}^{2n}(X; n)$ provides the right component of the map. A pair (Y, f) where Y is an irreducible codimension $n-1$ subvariety of X maps to the cycle $(y, f(y)) \subset X \times (\mathbb{P}^1 - \{1\})$. It is well known that such cycles $(y, f(y))$ plus $\partial \widehat{\mathcal{Z}}^{2n-2}(X; n)$ generate $\widehat{\mathcal{Z}}^{2n-1}(X; n)$. Computing the composition of this map with the homomorphism $\mathcal{P}^\bullet(n)$ we end up precisely with formula (3.24). The proposition is proved.

Remarks on the Special Values of the Dedekind ζ -Functions

3.2.5

Let $\widetilde{\Gamma}_{\mathcal{A}}(\mathcal{O}_F, 1)$ be the three term complex (3.5). It consists of locally compact abelian groups. Each of them is equipped with a natural Haar measure. Indeed, the measure of a discrete group is normalized so that the measure of the identity element is 1; the group $\mathbb{R}$ has the canonical measure dx ; and we use the product measure for the products. We need the following general observation.

Lemma-Definition 6 Let

6

$$A^\bullet = \dots \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots$$

be a complex of locally compact abelian groups such that

- i) Each of the groups A_i is equipped with an invariant Haar measure μ_i .
- ii) The cohomology groups are compact.
- iii) Only finite number of the cohomology groups are nontrivial, and almost all groups A_i are discrete groups with canonical measures.

Then there is a naturally defined number $R_\mu A^\bullet$, and $R_\mu A^\bullet[1] = (R_\mu A^\bullet)^{-1}$.

Construction 7 Let

7

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \quad (3.26)$$

be an exact sequence of locally compact abelian groups. Then a choice of Haar measure for any two of the groups A, B, C determines naturally the third one. For example Haar measures μ_A, μ_C on A and C determine the following Haar measure $\mu_{A,C}$ on B . Take a compact subset of C and its section $K_C \subset B$, and take a compact subset K_A . Then $\mu_{A,C}(K_A \cdot K_C) := \mu_A(K_A) \mu_C(K_C)$.

For the complex (3.26) placed in degrees $[0, 2]$ we put $R_\mu(3.26) = \mu_{A,C}/\mu_B$.

Let us treat first the case when $A^\bullet$ is a finite complex. Then we define the invariant $R_\mu A^\bullet$ by induction. If $A^\bullet = A[-i]$ is concentrated in just one degree, i , that A is compact and equipped with Haar measure μ . We put $R_\mu A^\bullet := \mu(A)^{(-1)^i}$. Assume $A^\bullet$ starts from A_0 . One has

$$\mathrm{Ker}f_0 \rightarrow A_0 \rightarrow \mathrm{Im}f_0 \rightarrow 0; \quad \mathrm{Im}f_0 \rightarrow A_1 \rightarrow A_1/\mathrm{Im}f_0 \rightarrow 0.$$

Since $\mathrm{Ker}f_0$ is compact, we can choose the volume one Haar measure on it. This measure and the measure μ_0 on A_0 provides, via the first short exact sequence, a measure on $\mathrm{Im}f_0$. Similarly using this and the second exact sequence we get a measure on $A_1/\mathrm{Im}f_0$. Therefore we have the measures on the truncated complex $\tau_{\geq 1}A^\bullet$. Now we define

$$R_\mu A^\bullet := \mu_0(A_0) \cdot R_\mu \tau_{\geq 1}A^\bullet.$$

If $A^\bullet$ is an infinite complex we set $R_\mu A^\bullet := R_\mu(\tau_{-N,N}A^\bullet)$ for sufficiently big N . Here $\tau_{-N,N}$ is the canonical truncation functor. Thanks to iii) this does not depend on the choice of N . The lemma is proved.

Now the class number formula (3.4) reads

$$\lim_{s \rightarrow 0} s^{-(r_1+r_2-1)} \zeta_F(s) = -R_\mu \tilde{\Gamma}_{\mathcal{A}}(\mathcal{O}_F, 1). \quad (3.27)$$

Lichtenbaum's conjectures [52] on the special values of the Dedekind ζ -functions can be reformulated in a similar way:

$$\zeta_F(1-n) \stackrel{?}{=} \pm R_\mu \Gamma_{\mathcal{A}}(\mathcal{O}_F, n); \quad n > 1.$$

It is not quite clear what is the most natural normalization of the regulator map. In the classical $n = 1$ case this formula needs modification, as was explained in Sect. 3.1.1, to take into account the pole of the ζ -function.

Example 8. The group $H^2 \Gamma_{\mathcal{A}}(\mathcal{O}_F, 2)$ sits in the exact sequence

$$0 \rightarrow R_2(F) \rightarrow H^2 \Gamma(\mathcal{O}_F, 2) \rightarrow K_2(\mathcal{O}_F) \rightarrow 0.$$

To calculate it let us define the Bloch–Suslin complex for $\mathrm{Spec}(\mathcal{O}_F)$:

$$B(\mathcal{O}_F, 2) : \quad B_2(F) \rightarrow \Lambda^2 F^* \rightarrow \prod_{\mathcal{P}} k_{\mathcal{P}}^* \quad (3.28)$$

Its Arakelov version is the total complex of the following bicomplex, where the vertical map is given by the dilogarithm: $\{x\}_2 \mapsto (\mathcal{L}_2(\sigma_1(x)), \dots, \mathcal{L}_2(\sigma_{r_2}(x)))$.

$$\begin{array}{ccc}
 & \mathbb{R}^{r_2} & \\
 & \uparrow & \\
 B_2(F) & \longrightarrow & \Lambda^2 F^* \longrightarrow \prod_{\mathcal{P}} k_{\mathcal{P}}^* .
 \end{array} \tag{3.29}$$

Then $H^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2) = H^2B_{\mathcal{A}}(\mathcal{O}_F, 2)$. The second map in (3.28) is surjective ([55], Cor. 16.2). Using this one can check that

$$H^i\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2) = 0 \quad \text{for } i \geq 3$$

(Notice that $H^1B_{\mathcal{A}}(\mathcal{O}_F, 2) \neq H^1\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2)$). Summarizing, we should have

$$\zeta_F(-1) \stackrel{?}{=} \pm R_{\mu}\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2) = \pm \frac{\text{vol}H^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2)}{|H^1\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2)|}.$$

For totally real fields it is a version of the Birch–Tate conjecture. It would be very interesting to compare this with the Bloch–Kato conjecture.

Grassmannian Polylogarithms and Borel’s Regulator

3.3

The Grassmannian Polylogarithm [30]

3.3.1

Let $h_1, \dots, h_{2n}$ be arbitrary $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. Choose an additional hyperplane h_0 . Let f_i be a rational function on $\mathbb{CP}^{n-1}$ with divisor $h_i - h_0$. It is defined up to a scalar factor. Set

$$\mathcal{L}_n^G(h_1, \dots, h_{2n}) := (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-2} \left(\sum_{j=1}^{2n} (-1)^j f_1 \wedge \dots \wedge \widehat{f_j} \wedge \dots \wedge f_{2n} \right).$$

It is skew-symmetric by definition. It is easy to see that it does not depend on the choice of scalar in the definition of f_i . To check that it does not depend on the choice of h_0 observe that

$$\sum_{j=1}^{2n} (-1)^j f_1 \wedge \dots \wedge \widehat{f_j} \wedge \dots \wedge f_{2n} = \frac{f_1}{f_{2n}} \wedge \frac{f_2}{f_{2n}} \wedge \dots \wedge \frac{f_{2n-1}}{f_{2n}}.$$

So if we choose rational functions $g_1, \dots, g_{2n-1}$ such that $\text{div} g_i = h_i - h_{2n}$ then

$$\mathcal{L}_n^G(h_1, \dots, h_{2n}) = (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-2}(g_1 \wedge \dots \wedge g_{2n-1}).$$

9

Remark 9 The function $\mathcal{L}_n^G$ is defined on the set of all configurations of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. However it is not even continuous on this set. It is real analytic on the submanifold of generic configurations.

10

Theorem 10 The function $\mathcal{L}_n^G$ satisfies the following functional equations:

a) For any $2n + 1$ hyperplanes $h_1, \dots, h_{2n+1}$ in $\mathbb{CP}^n$ one has

$$\sum_{j=1}^{2n+1} (-1)^j \mathcal{L}_n^G(h_j \cap h_1, \dots, h_j \cap h_{2n+1}) = 0. \quad (3.30)$$

b) For any $2n + 1$ hyperplanes $h_1, \dots, h_{2n+1}$ in $\mathbb{CP}^{n-1}$ one has

$$\sum_{j=1}^{2n+1} (-1)^j \mathcal{L}_n^G(h_1, \dots, \widehat{h_j}, \dots, h_{2n+1}) = 0. \quad (3.31)$$

Proof

a) Let $f_1, \dots, f_{2n+1}$ be rational functions on $\mathbb{CP}^n$ as above. Then

$$\begin{aligned} dr_{2n-1} \left(\sum_{j=1}^{2n+1} (-1)^j f_1 \wedge \dots \wedge \widehat{f_j} \wedge \dots \wedge f_{2n+1} \right) = \\ \sum_{j \neq i} (-1)^{j+i-1} 2\pi i \delta(f_j) r_{2n-2} (f_1 \wedge \dots \wedge \widehat{f_i} \wedge \dots \wedge \widehat{f_j} \wedge \dots \wedge f_{2n+1}) \end{aligned} \quad (3.32)$$

(Notice that $d \log f_1 \wedge \dots \wedge \widehat{d \log f_j} \wedge \dots \wedge d \log f_{2n+1} = 0$ on $\mathbb{CP}^n$). Integrating (3.32) over $\mathbb{CP}^n$ we get a).

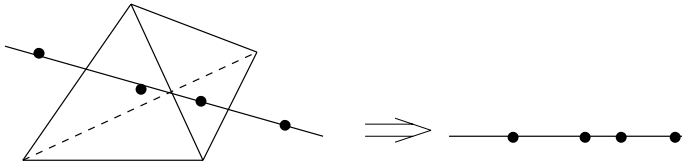
b) is obvious: we apply r_{2n-1} to zero element. The theorem is proved.

Let us choose a hyperplane H in $\mathbb{P}^{2n-1}$. Then the complement to H is an affine space $\mathbb{A}^{2n-1}$. Let L be a non-degenerate simplex which does not lie in a hyperplane, generic with respect to H . Observe that all such pairs (H, L) are projectively equivalent, and the complement $\mathbb{P}^{2n-1} - L$ is identified with $(\mathbb{G}_m^*)^{2n-1}$ canonically as soon as the numbering of the hyperplanes is chosen.

Let PG_n^{n-1} denote the quotient of the set of $(n-1)$ -planes in $\mathbb{P}^{2n-1}$ in generic position to L , modulo the action of the group $(\mathbb{G}_m^*)^{2n-1}$. There is a natural bijection

$$PG_n^{n-1} \cong \{\text{Configurations of } 2n \text{ generic hyperplanes in } \mathbb{P}^{n-1}\}$$

given by intersecting an $(n-1)$ -plane h with the codimension one faces of L .



Thus $\mathcal{L}_n^G$ is a function on the torus quotient PG_n^{n-1} of the generic part of Grassmannian.

Gelfand and MacPherson [25] suggested a beautiful construction of the real valued version of $2n$ -logarithms on $PG_{2n}^{2n-1}(\mathbb{R})$. The construction uses the Pontryagin form. These functions generalize the Rogers dilogarithm.

The defined above functions $\mathcal{L}_n^G$ on complex Grassmannians generalize the Bloch–Wigner dilogarithm. They are related to the Chern classes. It would be very interesting to find a link between the construction in [25] with our construction.

The existence of the multivalued analytic Grassmannian n -logarithms on complex Grassmannians was conjectured in [6]. They were constructed in [48, 49] and, as a particular case of the analytic Chow polylogarithms, in [30].

Recall the following general construction. Let X be a G -set and F a G -invariant function on X^n satisfying

$$\sum_{i=1}^n (-1)^i F(x_1, \dots, \widehat{x_i}, \dots, x_n) = 0.$$

Choose a point $x \in X$. Then there is an $(n-1)$ -cocycle of the group G :

$$f_x(g_1, \dots, g_n) := F(g_1 x, \dots, g_n x).$$

Lemma 11 The cohomology class of the cocycle f_x does not depend on x .

11

Thus thanks to (3.31) the function $\mathcal{L}_n^G$ provides a measurable cocycle of $GL_n(\mathbb{C})$. We want to determine its cohomology class, but a priori it is not even clear that it is non zero. To handle this problem we will show below that the function $\mathcal{L}_n^G$ is a boundary value of a certain function ψ_n defined on the configurations of $2n$ points inside of the symmetric space $SL_n(\mathbb{C})/SU(n)$. The cohomology class of $SL_n(\mathbb{C})$ provided by this function is obviously related to the so-called Borel class. Using this we will show that the Grassmannian n -logarithm function $\mathcal{L}_n^G$ provides the Borel class, and moreover can be used to define the Borel regulator.

Finally, the restriction of the function $\mathcal{L}_n^G$ to certain special stratum in the configuration space of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$ provides a single valued version of the classical n -logarithm function, see Sects. 3.4–3.5 below.

3.3.2 The Function ψ_n

Let V_n be an n -dimensional complex vector space. Let

$$\begin{aligned}\mathbb{H}_n &:= \{\text{positive definite Hermitian forms in } V_n\} / \mathbb{R}_+^* = SL_n(\mathbb{C}) / SU(n) \\ &= \{\text{positive definite Hermitian forms in } V_n \text{ with determinant} = 1\}\end{aligned}$$

It is a symmetric space of rank $n - 1$. For example $\mathbb{H}_2 = \mathcal{H}_3$ is the hyperbolic 3-space. Replacing positive definite by non negative definite Hermitian forms we get a compactification $\overline{\mathbb{H}}_n$ of the symmetric space $\mathbb{H}_n$.

Let G_x be the subgroup of $SL_n(\mathbb{C})$ stabilizing the point $x \in \mathbb{H}_n$. A point x defines a one dimensional vector space M_x :

$$x \in \mathbb{H}_n \mapsto M_x := \{\text{measures on } \mathbb{CP}^{n-1} \text{ invariant under } G_x\}.$$

Namely, a point x corresponds to a hermitian metric in V_n . This metric provides the Fubini-Study Kahler form on $\mathbb{CP}^{n-1} = P(V_n)$. Its imaginary part is a symplectic form. Raising it to $(n - 1)$ -th power we get the Fubini-Study volume form. The elements of M_x are its multiples.

Let $x_0, \dots, x_{2n-1}$ be points of the symmetric space $SL_n(\mathbb{C})/SU(n)$. Consider the following function

$$\psi_n(x_0, \dots, x_{2n-1}) := \int_{\mathbb{CP}^{n-1}} \log \left| \frac{\mu_{x_1}}{\mu_{x_0}} \right| d \log \left| \frac{\mu_{x_2}}{\mu_{x_0}} \right| \wedge \dots \wedge d \log \left| \frac{\mu_{x_{2n-1}}}{\mu_{x_0}} \right|.$$

More generally, let X be an m -dimensional manifold. For any $m + 2$ measures $\mu_0, \dots, \mu_{m+1}$ on X such that $\frac{\mu_i}{\mu_j}$ are smooth functions consider the following differential m -form on X :

$$\bar{r}_m(\mu_0 : \dots : \mu_{m+1}) := \log \left| \frac{\mu_1}{\mu_0} \right| d \log \left| \frac{\mu_2}{\mu_0} \right| \wedge \dots \wedge d \log \left| \frac{\mu_{m+1}}{\mu_0} \right|.$$

12 Proposition 12 The integral $\int_X \bar{r}_m(\mu_0 : \dots : \mu_{m+1})$ satisfies the following properties:

- 1) Skew symmetry with respect to the permutations of μ_i .
- 2) Homogeneity: if $\lambda_i \in \mathbb{R}^*$ then

$$\int_X \bar{r}_m(\lambda_0 \mu_0 : \dots : \lambda_{m+1} \mu_{m+1}) = \int_X \bar{r}_m(\mu_0 : \dots : \mu_{m+1})$$

- 3) Additivity: for any $m + 3$ measures μ_i on X one has

$$\sum_{i=0}^{m+2} (-1)^i \int_X \bar{r}_m(\mu_0 : \dots : \widehat{\mu}_i : \dots : \mu_{m+2}) = 0$$

- 4) Let g be a diffeomorphism of X . Then

$$\int_X \bar{r}_m(g^* \mu_0 : \dots : g^* \mu_{m+1}) = \int_X \bar{r}_m(\mu_0 : \dots : \mu_{m+1})$$

The Grassmannian Polylogarithm $\mathcal{L}_n^G$ is the Boundary Value of the Function ψ_n [33]

3.3.3

Let $(z_0 : \dots : z_{n-1})$ be homogeneous coordinates in $\mathbb{P}^{n-1}$. Let

$$\sigma_n(z, dz) := \sum_{i=0}^{n-1} (-1)^i z_i dz_0 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_{n-1}$$

$$\omega_{FS}(H) := \frac{1}{(2\pi i)^{n-1}} \frac{\sigma_n(z, dz) \wedge \sigma_n(\bar{z}, d\bar{z})}{H(z, \bar{z})^n} \quad (3.33)$$

This form is clearly invariant under the group preserving the Hermitian form H . In fact it is the Fubini-Study volume form.

Take any $2n$ non zero nonnegative definite Hermitian forms $H_0, \dots, H_{2n-1}$, possibly degenerate. For each of the forms H_i choose a multiple μ_{H_i} of the Fubini-Study form given by formula (3.33). It is a volume form with singularities along the projectivization of kernel of H_i .

Lemma 13 The following integral is convergent

13

$$\psi_n(H_0, \dots, H_{2n-1}) := \int_{\mathbb{CP}^{n-1}} \log \left| \frac{\mu_{H_1}}{\mu_{H_0}} \right| d \log \left| \frac{\mu_{H_2}}{\mu_{H_0}} \right| \wedge \dots \wedge d \log \left| \frac{\mu_{H_{2n-1}}}{\mu_{H_0}} \right|.$$

This integral does not change if we multiply one of the Hermitian forms by a positive scalar. Therefore we can extend ψ_n to a function on the configuration space of $2n$ points in the compactification $\overline{\mathbb{H}}_{n-1}$. This function is discontinuous.

One can realize $\mathbb{CP}^{n-1}$ as the smallest stratum of the boundary of $\mathbb{H}_n$. Indeed, let $\mathbb{CP}^{n-1} = P(V_n)$. For a hyperplane $h \in V_n$ let

$$F_h := \{ \text{nonnegative definite hermitian forms in } V_n \text{ with kernel } h \} / \mathbb{R}_+^*$$

The set of hermitian forms in V_n with the kernel h is isomorphic to $\mathbb{R}_+^*$, so F_h defines a point on the boundary of $\overline{\mathbb{H}}_n$. Therefore Lemma 13 provides a function

$$\psi_n(h_0, \dots, h_{2n-1}) := \psi_n(F_{h_0}, \dots, F_{h_{2n-1}}) \quad (3.34)$$

Applying Lemma 11 to the case when X is $\overline{\mathbb{H}}_n$ and using only the fact that the function $\psi_n(x_0, \dots, x_{2n-1})$ is well defined for *any* $2n$ points in $\overline{\mathbb{H}}_n$ and satisfies the cocycle condition for any $2n + 1$ of them we get

Proposition 14 Let $x \in \mathbb{H}_n$ and h is a hyperplane in $\mathbb{CP}^{n-1}$. Then the cohomology classes of the following cocycles coincide:

14

$$\psi_n(g_0 x, \dots, g_{2n-1} x) \quad \text{and} \quad \psi_n(g_0 h, \dots, g_{2n-1} h)$$

The Fubini-Study volume form corresponding to a hermitian form from the set F_h is a Lebesgue measure on the affine space $\mathbb{CP}^{n-1} - h$. Indeed, if $h_0 = \{z_0 = 0\}$ then (3.33) specializes to

$$\frac{1}{(2\pi i)^{n-1}} d\frac{z_1}{z_0} \wedge \dots \wedge d\frac{z_{n-1}}{z_0} \wedge d\frac{\bar{z}_1}{\bar{z}_0} \wedge \dots \wedge d\frac{\bar{z}_{n-1}}{\bar{z}_0}.$$

Using this it is easy to prove the following proposition.

15 Proposition 15 For any $2n$ hyperplanes $h_0, \dots, h_{2n-1}$ in $\mathbb{CP}^{n-1}$ one has

$$\psi_n(h_0, \dots, h_{2n-1}) = (-4)^{-n} \cdot (2\pi i)^{n-1} (2n)^{2n-1} \binom{2n-2}{n-1} \cdot \mathcal{L}_n^G(h_0, \dots, h_{2n-1}).$$

3.3.4 Construction of the Borel Regulator Map

Let us first normalize the Borel class ψ_n .

Denote by $H_c^*(G, \mathbb{R})$ the continuous cohomology of a Lie group G . Let us define an isomorphism

$$\gamma_{\text{DR}} : H_{\text{DR}}^k(SL_n(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} H_c^k(SL_n(\mathbb{C}), \mathbb{C}).$$

We do it in two steps. First, let us define an isomorphism

$$\alpha : H_{\text{DR}}^k(SL_n(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} \mathcal{A}^k(SL_n(\mathbb{C})/SU(n))^{SL_n(\mathbb{C})} \otimes \mathbb{C}.$$

It is well known that any cohomology class on the left is represented by a biinvariant, and hence closed, differential k -form Ω on $SL_n(\mathbb{C})$. Let us restrict it first to the Lie algebra, and then to the orthogonal complement $su(n)^\perp$ to the Lie subalgebra $su(n) \subset sl_n(\mathbb{C})$. Let e be the point of $\mathbb{H}_n$ corresponding to the subgroup $SU(n)$. We identify the $\mathbb{R}$ -vector spaces $T_e \mathbb{H}_n$ and $su(n)^\perp$. The obtained exterior form on $T_e \mathbb{H}_n$ is restriction of an invariant closed differential form ω on the symmetric space $\mathbb{H}_n$.

Now let us construct, following J. Dupont [22], an isomorphism

$$\beta : \mathcal{A}^k(SL_n(\mathbb{C})/SU(n))^{SL_n(\mathbb{C})} \xrightarrow{\sim} H_c^k(SL_n(\mathbb{C}), \mathbb{R}).$$

For any ordered $m+1$ points $x_1, \dots, x_{m+1}$ in $\mathbb{H}_n$ there is a geodesic simplex $I(x_1, \dots, x_{m+1})$ in $\mathbb{H}_n$. It is constructed inductively as follows. Let $I(x_1, x_2)$ be the geodesic from x_1 to x_2 . The geodesics from x_3 to the points of $I(x_1, x_2)$ form a geodesic triangle $I(x_1, x_2, x_3)$, and so on. If $n > 2$ the geodesic simplex $I(x_1, \dots, x_k)$ depends on the order of vertices.

Let ω be an invariant differential m -form on $SL_n(\mathbb{C})/SU(n)$. Then it is closed, and provides a volume of the geodesic simplex:

$$\text{vol}_\omega I(x_1, \dots, x_{m+1}) := \int_{I(x_1, \dots, x_{m+1})} \omega$$

The boundary of the simplex $I(x_1, \dots, x_{m+2})$ is the alternated sum of simplices $I(x_1, \dots, \widehat{x}_i, \dots, x_{m+2})$. Since the form ω is closed, the Stokes theorem yields

$$\sum_{i=1}^{m+2} (-1)^i \int_{I(x_1, \dots, \widehat{x}_i, \dots, x_{m+2})} \omega = \int_{I(x_1, \dots, x_{m+2})} d\omega = 0.$$

This just means that for a given point x the function $\text{vol}_\omega I(g_1 x, \dots, g_{m+1} x)$ is a smooth m -cocycle of the Lie group $SL_n(\mathbb{C})$. By Lemma 11 cocycles corresponding to different points x are canonically cohomologous. The obtained cohomology class is the class $\beta(\omega)$. Set $\gamma_{\text{DR}} := \beta \circ \alpha$.

It is known that

$$H_{\text{DR}}^*(SL_n(\mathbb{C}), \mathbb{Q}) = \Lambda_{\mathbb{Q}}^*(C_3, \dots, C_{2n-1})$$

where

$$C_{2n-1} := \text{tr}(g^{-1} dg)^{2n-1} \in \Omega^{2n-1}(SL).$$

The Hodge considerations shows that $[C_{2n-1}] \in H_{\text{Betti}}^{2n-1}(SL_n(\mathbb{C}), \mathbb{Q}(n))$.

Lemma 16 $\alpha(C_{2n-1})$ is an $\mathbb{R}(n-1)$ -valued differential form. So it provides a cohomology class

16

$$b_n := \gamma_{\text{DR}}(C_{2n-1}) \in H_c^{2n-1}(SL_n(\mathbb{C}), \mathbb{R}(n-1)).$$

We call the cohomology class provided by this lemma the Borel class, and use it below to construct the Borel regulator.

It is not hard to show that the cohomology class of the cocycle

$$\psi_n(g_0 x, \dots, g_{2n-1} x)$$

is a non zero multiple of the Borel class. So thanks to Propositions 14 and 15 the same is true for the cohomology class provided by the Grassmannian n -logarithm. The final result will be stated in Theorem 17 below.

Now let us proceed to the construction of the Borel regulator map.

Let G be a group. The diagonal map $\Delta : G \rightarrow G \times G$ provides a homomorphism $\Delta_* : H_n(G) \rightarrow H_n(G \times G)$. Recall that

$$\text{Prim} H_n G := \{x \in H_n(G) \mid \Delta_*(x) = x \otimes 1 + 1 \otimes x\}.$$

Set $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. One has

$$K_n(F)_{\mathbb{Q}} = \text{Prim} H_n GL(F)_{\mathbb{Q}} = \text{Prim} H_n GL_n(F)_{\mathbb{Q}},$$

where the second isomorphism is provided by Suslin's stabilization theorem. Let

$$B_n \in H_c^{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{R}(n-1))$$

be a cohomology which goes to b_n under the restriction map to GL_n . We define

the Borel regulator map by restricting the class B_n to the subspace $K_{2n-1}(\mathbb{C})_{\mathbb{Q}}$ of $H_{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{Q})$:

$$r_n^{\text{Bo}}(b_n) := \langle B_n, * \rangle : K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \rightarrow \mathbb{R}(n-1)$$

It does not depend on the choice of B_n .

Recall the Grassmannian complex $C_*(n)$

$$\dots \xrightarrow{d} C_{2n-1}(n) \xrightarrow{d} C_{2n-2}(n) \xrightarrow{d} \dots \xrightarrow{d} C_0(n),$$

where $C_k(n)$ is the free abelian group generated by configurations, i.e. $GL(V)$ -coinvariants, of $k+1$ vectors $(l_0, \dots, l_k)$ in generic position in an n -dimensional vector space V over a field F , and d is given by the standard formula

$$(l_0, \dots, l_k) \mapsto \sum_{i=0}^k (-1)^i (l_0, \dots, \widehat{l_i}, \dots, l_k). \quad (3.35)$$

The group $C_k(n)$ is in degree k . Since it is a homological resolution of the trivial $GL_n(F)$ -module $\mathbb{Z}$ (see Lemma 3.1 in [28]), there is canonical homomorphism

$$\varphi_{2n-1}^n : H_{2n-1}(GL_n(F)) \rightarrow H_{2n-1}(C_*(n)).$$

Thanks to Lemma 16 the Grassmannian n -logarithm function provides a homomorphism

$$\mathcal{L}_n^G : C_{2n-1}(n) \rightarrow \mathbb{R}(n-1); \quad (l_0, \dots, l_{2n-1}) \mapsto \mathcal{L}_n^G(l_0, \dots, l_{2n-1}). \quad (3.36)$$

Thanks to the functional equation (3.30) for $\mathcal{L}_n^G$ it is zero on the subgroup $dC_{2n}(n)$. So it induces a homomorphism

$$\mathcal{L}_n^G : H_{2n-1}(C_*(n)) \rightarrow \mathbb{R}(n-1);$$

Let us extend the map $\varphi_{2n-1}^n \circ \mathcal{L}_n^G$ to obtain a homomorphism from $H_{2n-1}(GL_{2n-1}(\mathbb{C}))$ to $\mathbb{R}(n-1)$, by following [28, 3.10]. Consider the following bicomplex:

$$\begin{array}{ccccccc}
 & & & & & \xrightarrow{d} & C_{2n-1}(2n-1) \\
 & & & & & & \downarrow \\
 & & & & \dots & & \dots \\
 & & & \downarrow & & & \downarrow \\
 \dots & \xrightarrow{d} & \dots & \xrightarrow{d} & C_{2n-1}(n+1) & \xrightarrow{d} & \dots & \xrightarrow{d} & C_{n+1}(n) \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 \dots & \xrightarrow{d} & C_{2n-1}(n) & \xrightarrow{d} & C_{2n-2}(n) & \xrightarrow{d} & \dots & \xrightarrow{d} & C_n(n)
 \end{array}$$

The horizontal differentials are given by formula (3.35), and the vertical by

$$(l_0, \dots, l_k) \mapsto \sum_{i=0}^k (-1)^i (l_i | l_1, \dots, \widehat{l_i}, \dots, l_k) .$$

Here $(l_i | l_1, \dots, \widehat{l_i}, \dots, l_k)$ means projection of the configuration $(l_1, \dots, \widehat{l_i}, \dots, l_k)$ to the quotient $V/\langle l_i \rangle$. The total complex of this bicomplex is called the weight n bi-Grassmannian complex $BC_*(n)$.

Let us extend homomorphism (3.36) to a homomorphism

$$\mathcal{L}_n^G : BC_{2n-1}(n) \rightarrow \mathbb{R}(n-1)$$

by setting it zero on the groups $C_{2n-1}(n+i)$ for $i > 0$. The functional equation (3.31) for the Grassmannian n -logarithm just means that the composition

$$C_{2n}(n+1) \rightarrow C_{2n-1}(n) \xrightarrow{\mathcal{L}_n^G} \mathbb{R}(n-1) ,$$

where the first map is a vertical arrow in $BC_*(n)$, is zero. Therefore we get a homomorphism

$$\mathcal{L}_n^G : H_{2n-1}(BC_*(n)) \rightarrow \mathbb{R}(n-1)$$

The bottom row of the Grassmannian bicomplex is the stupid truncation of the Grassmannian complex at the group $C_n(n)$. So there is a homomorphism

$$H_{2n-1}(C_*(n)) \rightarrow H_{2n-1}(BC_*(n)) \quad (3.37)$$

In [27, 28] we proved that there are homomorphisms

$$\varphi_{2n-1}^m : H_{2n-1}(GL_m(F)) \rightarrow H_{2n-1}(BC_*(n)), \quad m \geq n$$

whose restriction to the subgroup $GL_n(F)$ coincides with the composition

$$H_{2n-1}(GL_n(F)) \xrightarrow{\varphi_{2n-1}^n} H_{2n-1}(C_*(n)) \xrightarrow{(3.37)} H_{2n-1}(BC_*(n)) .$$

Theorem 17 The composition

17

$K_{2n-1}(\mathbb{C}) \xrightarrow{\sim} \text{Prim} H_{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{Q}) \xrightarrow{\varphi_{2n-1}^{2n-1}} H_{2n-1}(BC_*(n)_{\mathbb{Q}}) \xrightarrow{\mathcal{L}_n^G} \mathbb{R}(n-1)$
equals

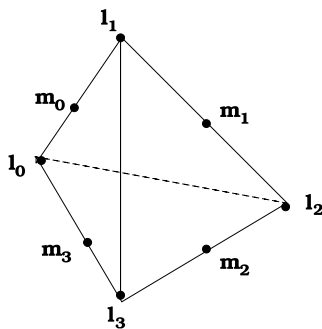
$$-(-1)^{n(n+1)/2} \cdot \frac{(n-1)!^2}{n(2n-2)!(2n-1)!} r_n^{\text{Bo}}(b_n) .$$

$\mathbb{P}^1 - \{0, \infty\}$ as a Special Stratum in the Configuration Space of $2n$ Points in $\mathbb{P}^{n-1}$ [30]

3.3.5

A *special configuration* is a configuration of $2n$ points

$$(l_0, \dots, l_{n-1}, m_0, \dots, m_{n-1}) \quad (3.38)$$

Figure 3.1. A special configuration of 8 points in $\mathbb{P}^3$

in $\mathbb{P}^{n-1}$ such that $l_0, \dots, l_{n-1}$ are vertices of a simplex in $\mathbb{P}^{n-1}$ and m_i is a point on the edge $l_i l_{i+1}$ of the simplex different from l_i and l_{i+1} , as on Fig. 3.1.

18 Proposition 18 The set of special configurations of $2n$ points in $\mathbb{P}^{n-1}$ is canonically identified with $\mathbb{P}^1 \setminus \{0, \infty\}$.

19 Construction 19 Let $\widehat{m}_i$ be the point of intersection of the line $l_i l_{i+1}$ with the hyperplane passing through all the points m_j except m_i . Let $r(x_1, \dots, x_4)$ be the cross-ratio of the four points on $\mathbb{P}^1$. Let us define the generalized cross-ratio by

$$r(l_0, \dots, l_{n-1}, m_0, \dots, m_{n-1}) := r(l_i, l_{i+1}, m_i, \widehat{m}_{i+1}) \in F^*.$$

It does not depend on i , and provides the desired isomorphism. Here is a different definition, which makes obvious the fact that the generalized cross-ratio is cyclically invariant. Consider the one dimensional subspaces L_i, M_j in the $(n+1)$ -dimensional vector space projecting to l_i, m_j . Then L_i, M_i, L_{i+1} belong to a two dimensional subspace. The subspace M_i provides a linear map $L_i \rightarrow L_{i+1}$. The composition of these maps is a linear map $L_0 \rightarrow L_0$. The element of F^* describing this map is the generalized cross-ratio.

3.3.6 Computations of the Grassmannian n -Logarithms

Restriction of the Grassmannian n -Logarithm to the Special Stratum

The n -logarithm function $Li_n(z)$ has a single-valued version ([67])

$$\mathcal{L}_n(z) := \begin{matrix} \text{Re } (n : \text{odd}) \\ \text{Im } (n : \text{even}) \end{matrix} \left(\sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot Li_{n-k}(z) \right), \quad n \geq 2$$

It is continuous on $\mathbb{CP}^1$. Here $\frac{2x}{e^{2x}-1} = \sum_{k=0}^{\infty} \beta_k x^k$, so $\beta_k = \frac{2^k B_k}{k!}$ where B_k are Bernoulli numbers. For example $\mathcal{L}_2(z)$ is the Bloch–Wigner function.

Let us consider the following modification of the function $\mathcal{L}_n(z)$ proposed by A.M. Levin in [50]:

$$\tilde{\mathcal{L}}_n(x) := \frac{(2n-3)}{(2n-2)} \sum_{\substack{k \text{ even}; 0 \leq k \leq n-2}} \frac{2^k (n-2)! (2n-k-3)!}{(2n-3)! (k+1)! (n-k-2)!} \mathcal{L}_{n-k}(x) \log^k |x| \quad (3.39)$$

For example $\tilde{\mathcal{L}}_n(x) = \mathcal{L}_n(x)$ for $n \leq 3$, but already $\tilde{\mathcal{L}}_4(x)$ is different from $\mathcal{L}_4(x)$. A direct integration shows that

$$-(2\pi i)^{n-1} (-1)^{(n-1)(n-2)/2} \tilde{\mathcal{L}}_n(x) = \int_{\mathbb{CP}^{n-1}} \log |1 - z_1| \prod_{i=1}^{n-1} d \log |z_i| \wedge \prod_{i=1}^{n-2} d \log |z_i - z_{i+1}| \wedge d \log |z_{n-1} - a|$$

For $n = 2$ this has been done in [30, 33]. For general n it was done in Proposition 4.4.1 of [50].

This combined with Proposition 10 below implies

Theorem 20 The value of function $\mathcal{L}_n^G$ on special configuration (3.38) equals

20

$$-(-1)^{n(n-1)/2} 4^{n-1} \binom{2n-2}{n-1}^{-1} \tilde{\mathcal{L}}_n(a); \quad a = r(l_0, \dots, l_{n-1}, m_0, \dots, m_{n-1})$$

It follows from Theorem 20 that $\mathcal{L}_2^G(l_1, \dots, l_4) = -2\mathcal{L}_2(r(l_1, \dots, l_4))$.

It was proved in Theorem 1.3 of [41] that

$$\begin{aligned} \mathcal{L}_3^G(l_0, \dots, l_5) &= \frac{1}{90} \text{Alt}_6 \mathcal{L}_3(r_3(l_0, \dots, l_5)) \\ &+ \frac{1}{9} \text{Alt}_6 (\log |\Delta(l_0, l_1, l_2)| \log |\Delta(l_1, l_2, l_3)| \Delta(l_2, l_3, l_4)|) \end{aligned} \quad (3.40)$$

We will continue this discussion in Sect. 3.5.

The functions $\mathcal{L}_n^G$ for $n > 3$ can not be expressed via classical polylogarithms.

Polylogarithmic Motivic Complexes

3.4

The Groups $\mathcal{B}_n(F)$ and Polylogarithmic Motivic Complexes ([27, 28])

3.4.1

For a set X denote by $\mathbb{Z}[X]$ the free abelian group generated by symbols $\{x\}$ where x run through all elements of the set X . Let F be an arbitrary field. We define

inductively subgroups $\mathcal{R}_n(F)$ of $\mathbb{Z}[P_F^1]$, $n \geq 1$ and set

$$\mathcal{B}_n(F) := \mathbb{Z}[P_F^1] / \mathcal{R}_n(F) .$$

One has

$$\mathcal{R}_1(F) := (\{x\} + \{y\} - \{xy\}, (x, y \in F^*); \{0\}; \{\infty\}) ; \quad \mathcal{B}_1(F) = F^* .$$

Let $\{x\}_n$ be the image of $\{x\}$ in $\mathcal{B}_n(F)$. Consider homomorphisms

$$\mathbb{Z}[P_F^1] \xrightarrow{\delta_n} \begin{cases} \mathcal{B}_{n-1}(F) \otimes F^* : & n \geq 3 \\ \Lambda^2 F^* & : & n = 2 \end{cases} \quad (3.41)$$

$$\delta_n : \{x\} \mapsto \begin{cases} \{x\}_{n-1} \otimes x : & n \geq 3 \\ (1-x) \wedge x : & n = 2 \end{cases} \quad \delta_n : \{\infty\}, \{0\}, \{1\} \mapsto 0 \quad (3.42)$$

Set $\mathcal{A}_n(F) := \text{Ker} \delta_n$. Any element $\alpha(t) = \sum n_i \{f_i(t)\} \in \mathbb{Z}[P_{F(t)}^1]$ has a specialization $\alpha(t_0) := \sum n_i \{f_i(t_0)\} \in \mathbb{Z}[P_F^1]$ at each point $t_0 \in P_F^1$.

21

Definition 21 $\mathcal{R}_n(F)$ is generated by elements $\{\infty\}, \{0\}$ and $\alpha(0) - \alpha(1)$ where $\alpha(t)$ runs through all elements of $\mathcal{A}_n(F(t))$.

Then $\delta_n(\mathcal{R}_n(F)) = 0$ ([27], 1.16). So we get homomorphisms

$$\delta_n : \mathcal{B}_n(F) \rightarrow \mathcal{B}_{n-1}(F) \otimes F^*, \quad n \geq 3; \quad \delta_2 : \mathcal{B}_2(F) \rightarrow \Lambda^2 F^*$$

and finally the *polylogarithmic motivic complex* $\Gamma(F, n)$:

$$\mathcal{B}_n \xrightarrow{\delta} \mathcal{B}_{n-1} \otimes F^* \xrightarrow{\delta} \mathcal{B}_{n-2} \otimes \Lambda^2 F^* \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{B}_2 \otimes \Lambda^{n-2} F^* \xrightarrow{\delta} \Lambda^n F^*$$

where $\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} y_i \rightarrow \delta_p(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_i$ and $\mathcal{B}_n$ is in degree 1.

22

Conjecture 22 $H^i \Gamma(F, n) \otimes \mathbb{Q} = \text{gr}_n^Y K_{2n-i}(F) \otimes \mathbb{Q}$.

Denote by $\widehat{\mathcal{L}}_n$ the function $\mathcal{L}_n$, multiplied by i for even n and unchanged for odd n . There is a well defined homomorphism ([28], Theorem 1.13):

$$\widehat{\mathcal{L}}_n : \mathcal{B}_n(\mathbb{C}) \rightarrow \mathbb{R}(n-1); \quad \widehat{\mathcal{L}}_n \left(\sum m_i \{z_i\}_n \right) := \sum m_i \widehat{\mathcal{L}}_n(z_i)$$

There are canonical homomorphisms

$$\mathcal{B}_n(F) \rightarrow \mathcal{B}_n(F); \quad \{x\}_n \mapsto \{x\}_n, \quad n = 1, 2, 3 . \quad (3.43)$$

They are isomorphisms for $n = 1, 2$ and expect to be an isomorphism for $n = 3$, at least modulo torsion.

The Residue Homomorphism for Complexes $\Gamma(F, n)$ [27, 1.14]

3.4.2

Let $F = K$ be a field with a discrete valuation v , the residue field k_v and the group of units U . Let $u \rightarrow \bar{u}$ be the projection $U \rightarrow k_v^*$. Choose a uniformizer π . There is a homomorphism $\theta : \Lambda^n K^* \rightarrow \Lambda^{n-1} k_v^*$ uniquely defined by the following properties ($u_i \in U$):

$$\theta(\pi \wedge u_1 \wedge \cdots \wedge u_{n-1}) = \bar{u}_1 \wedge \cdots \wedge \bar{u}_{n-1}; \quad \theta(u_1 \wedge \cdots \wedge u_n) = 0.$$

It is clearly independent of π . Define a homomorphism $s_v : \mathbb{Z}[P_K^1] \rightarrow \mathbb{Z}[P_{k_v}^1]$ by setting $s_v\{x\} = \{\bar{x}\}$ if x is a unit and 0 otherwise. It induces a homomorphism $s_v : \mathcal{B}_m(K) \rightarrow \mathcal{B}_m(k_v)$. Put

$$\partial_v := s_v \otimes \theta : \mathcal{B}_m(K) \otimes \Lambda^{n-m} K^* \rightarrow \mathcal{B}_m(k_v) \otimes \Lambda^{n-m-1} k_v^*.$$

It defines a morphism of complexes $\partial_v : \Gamma(K, n) \rightarrow \Gamma(k_v, n-1)[-1]$.

A Variation of Mixed Hodge Structures on $P^1(\mathbb{C}) - \{0, 1, \infty\}$ Corresponding to the Classical Polylogarithm $\text{Li}_n(z)$ ([19])

3.4.3

Its fiber $H(z)$ over a point z is described via the period matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \text{Li}_1(z) & 2\pi i & 0 & \cdots & 0 \\ \text{Li}_2(z) & 2\pi i \log z & (2\pi i)^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \text{Li}_n(z) & 2\pi i \frac{\log^{n-1} z}{(n-1)!} & (2\pi i)^2 \frac{\log^{n-2} z}{(n-2)!} & \cdots & (2\pi i)^n \end{pmatrix}. \quad (3.44)$$

Its entries are defined using analytic continuation to the point z along a path γ from a given point in $\mathbb{C}$ where all the entries are defined by power series expansions, say the point $1/2$.

Here is a more natural way to define the entries. Consider the following regularized iterated integrals along a certain fixed path γ between 0 to z :

$$\text{Li}_n(z) = \int_0^z \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n-1 \text{ times}}; \quad \frac{\log^n z}{n!} = \int_0^z \underbrace{\frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n \text{ times}}.$$

To regularize the divergent integrals we take the lower limit of integration to be ε . Then it is easy to show that the integral has an asymptotic expansion of type $I_0(\varepsilon) + I_1(\varepsilon) \log \varepsilon + \cdots + I_k(\varepsilon) \log^k \varepsilon$, where all the functions $I_i(\varepsilon)$ are smooth at $\varepsilon = 0$. Then we take $I_0(0)$ to be the regularized value.

Now let us define a mixed Hodge structure $H(z)$. Let $\mathbb{C}^{n+1}$ be the standard vector space with basis $(e_0, \dots, e_n)$, and V_{n+1} the $\mathbb{Q}$ -vector subspace spanned by the

columns of the matrix (3.44). Let $W_{-2(n+1-k)}V_{n+1}$ be the subspace spanned by the first k columns, counted from the right to the left. One shows that these subspaces do not depend on the choice of the path γ , i.e. they are well-defined inspite of the multivalued nature of the entries of the matrix. Then $W_{\bullet}V_{n+1}$ is the weight filtration. We define the Hodge filtration $F^{\bullet}\mathbb{C}^{n+1}$ by setting $F^{-k}\mathbb{C}^{n+1} := \langle e_0, \dots, e_k \rangle_{\mathbb{C}}$. It is opposite to the weight filtration. We get a Hodge–Tate structure, i.e. $h^{pq} = 0$ unless $p \neq q$. One checks that the family of Hodge–Tate structures $H(z)$ forms a unipotent variation of Hodge–Tate structures on $P^1(\mathbb{C}) - \{0, 1, \infty\}$.

Let $n \geq 0$. An n -framed Hodge–Tate structure H is a triple (H, ν_0, f_n) , where $\nu_0 : \mathbb{Q}(0) \rightarrow gr_0^W H$ and $f_n : gr_{-2n}^W H \rightarrow \mathbb{Q}(n)$ are nonzero morphisms. A framing plus a choice of a splitting of the weight filtration determines a period of a Hodge–Tate structure. Consider the coarsest equivalence relation on the set of all n -framed Hodge–Tate structures for which $M_1 \sim M_2$ if there is a map $M_1 \rightarrow M_2$ respecting the frames. Then the set $\mathcal{H}_n$ of the equivalence classes has a natural abelian group structure. Moreover

$$\mathcal{H}_{\bullet} := \bigoplus_{n \geq 0} \mathcal{H}_n \quad (3.45)$$

has a natural Hopf algebra structure with a coproduct Δ , see the Appendix of [38].

Observe that $Gr_{-2k}^W H_n(z) = \mathbb{Q}(k)$ for $-n \leq k \leq 0$. Therefore $H_n(z)$ has a natural framing such that the corresponding period is given by the function $Li_n(z)$. The obtained framed object is denoted by $Li_n^{\mathcal{H}}(z)$. To define it for $z = 0, 1, \infty$ we use the specialization functor to the punctured tangent space at z , and then take the fiber over the tangent vector corresponding to the parameter z on P^1 . It is straitforward to see that the coproduct $\Delta Li_n^{\mathcal{H}}(z)$ is computed by the formula

$$\Delta Li_n^{\mathcal{H}}(z) = \sum_{k=0}^n Li_{n-k}^{\mathcal{H}}(z) \otimes \frac{\log^{\mathcal{H}}(z)^k}{k!}, \quad (3.46)$$

where $\log^{\mathcal{H}}(z)$ is the 1-framed Hodge–Tate structure corresponding to $\log(z)$.

A Motivic Proof of the Weak Version of Zagier's Conjecture

3.4.4

Our goal is the following result, which was proved in [46] and in the unfinished manuscript [5]. The proof below uses a different set of ideas. It follows the framework described in Chapt. 13 of [34], and quite close to the approach outlined in [4], although it is formulated a bit differently, using the polylogarithmic motivic complexes.

Theorem 23 Let F be a number field. Then there exists a homomorphism

$$l_n : H^1(\Gamma(F, n) \otimes \mathbb{Q}) \rightarrow K_{2n-1}(F) \otimes \mathbb{Q}$$

such that for any embedding $\sigma : F \hookrightarrow \mathbb{C}$ one has the following commutative diagram

$$\begin{array}{ccccc}
H^1(\Gamma(F, n) \otimes \mathbb{Q}) & \xrightarrow{\sigma_*} & H^1(\Gamma(\mathbb{C}, n) \otimes \mathbb{Q}) & \xrightarrow{\mathcal{L}_n} & \mathbb{R}(n-1) \\
\downarrow l_n & & & & \downarrow = \\
K_{2n-1}(F) \otimes \mathbb{Q} & \xrightarrow{\sigma_*} & K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q} & \xrightarrow{r_{Bo}} & \mathbb{R}(n-1)
\end{array} \quad (3.47)$$

Proof We will use the following background material and facts:

- i) The existence of the abelian tensor category $\mathcal{M}_T(F)$ of mixed Tate motives over a number field F , satisfying all the desired properties, including Beilinson's formula expressing the Ext groups via the rational K -theory of F and the Hodge realization functor. See [20] and the references there.
- ii) The formalism of mixed Tate categories, including the description of the fundamental Hopf algebra $\mathcal{A}_\bullet(\mathcal{M})$ of a mixed Tate category via framed objects in $\mathcal{M}$. See [38], Sect. 8. The fundamental Hopf algebra of the category $\mathcal{M}_T(F)$ is denoted $\mathcal{A}_\bullet(F)$. For example for the category of mixed Hodge–Tate structures fundamental Hopf algebra is the one $\mathcal{H}_\bullet$ from (3.45). Let

$$\Delta : \mathcal{A}_\bullet(F) \rightarrow \mathcal{A}_\bullet(F)^{\otimes 2}$$

be the coproduct in the Hopf algebra $\mathcal{A}_\bullet(F)$, and $\Delta' := \Delta - \text{Id} \otimes 1 + 1 \otimes \text{Id}$ is the restricted coproduct. The key fact is a canonical isomorphism

$$\text{Ker} \Delta' \cap \mathcal{A}_n(F) \stackrel{\sim}{=} K_{2n-1}(F) \otimes \mathbb{Q}. \quad (3.48)$$

Since $\mathcal{A}_\bullet(F)$ is graded by ≥ 0 integers, and $\mathcal{A}_0(F) = \mathbb{Q}$, formula (3.48) for $n = 1$ reduces to an isomorphism

$$\mathcal{A}_1(F) \stackrel{\sim}{=} F^* \otimes \mathbb{Q}. \quad (3.49)$$

- iii) The existence of the motivic classical polylogarithms

$$\text{Li}_n^{\mathcal{M}}(z) \in \mathcal{A}_n(F), \quad z \in P^1(F) \quad (3.50)$$

They were defined in Sect. 3.6 of [38] using either a geometric construction of [35], or a construction of the motivic fundamental torsor of path between the tangential base points given in [20]. In particular one has $\text{Li}_n^{\mathcal{M}}(0) = \text{Li}_n^{\mathcal{M}}(\infty) = 0$. A natural construction of the elements (3.50) using the moduli space $\overline{\mathcal{M}}_{0,n+3}$ is given in Chapt. 4.6 below.

- iv) The Hodge realization of the element (3.50) is equivalent to the framed Hodge–Tate structure $\text{Li}_n^{\mathcal{H}}(z)$ from Chapt. 4.3. This fact is more or less straightforward if one uses the fundamental torsor of path on the punctured projective line to define the element $\text{Li}_n^{\mathcal{M}}(z)$, and follows from the general specialization theorem proved in [35] if one uses the approach of loc. cit. This implies that the Lie-

period of $\mathrm{Li}_n^{\mathcal{H}}(z)$ equals to $\mathcal{L}_n(z)$. Indeed, for the Hodge–Tate structure $H(z)$ assigned to $\mathrm{Li}_n(z)$ this was shown in [4].

v) The crucial formula (Sect. 6.3 of [38]):

$$\Delta \mathrm{Li}_n^{\mathcal{M}}(z) = \sum_{k=0}^n \mathrm{Li}_{n-k}^{\mathcal{M}}(z) \otimes \frac{\log^{\mathcal{M}}(z)^k}{k!} \quad (3.51)$$

where $\log^{\mathcal{M}}(z) \in \mathcal{A}_1(F)$ is the element corresponding to z under the isomorphism (3.48), and $\log^{\mathcal{M}}(z)^k \in \mathcal{A}_k(F)$ is its k -th power. It follows from formula (3.46) using the standard trick based on Borel’s theorem to reduce a motivic claim to the corresponding Hodge one.

vi) The Borel regulator map on $K_{2n-1}(F) \otimes \mathbb{Q}$, which sits via (3.48) inside of $\mathcal{A}_n(F)$, is induced by the Hodge realization functor on the category $\mathcal{M}_T(F)$.

Having this background, we proceed as follows. Namely, let

$$\mathcal{L}_{\bullet}(F) := \frac{\mathcal{A}_{>0}(F)}{\mathcal{A}_{>0}(F)^2}$$

be the fundamental Lie coalgebra of $\mathcal{M}_T(F)$. Its cobracket δ is induced by Δ . Projecting the element (3.50) into $\mathcal{L}_n(F)$ we get an element $l_n^{\mathcal{M}}(z) \in \mathcal{L}_n(F)$ such that

$$\delta l_n^{\mathcal{M}}(z) = l_{n-1}^{\mathcal{M}}(z) \wedge z. \quad (3.52)$$

Consider the following map

$$\tilde{l}_n : \mathbb{Q}[P^1(F)] \rightarrow \mathcal{L}_n(F), \quad \{z\} \mapsto l_n^{\mathcal{M}}(z)$$

24

Proposition 24 The map $\tilde{l}_n$ kills the subspace $\mathcal{R}_n(F)$, providing a well defined homomorphism

$$l_n : \mathcal{B}_n(F) \rightarrow \mathcal{L}_n(F); \quad \{z\}_n \rightarrow \tilde{l}_n(z).$$

Proof We proceed by the induction on n . The case $n = 1$ is self-obvious. Suppose we are done for $n - 1$. Then there is the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}[F] & \xrightarrow{\delta_n} & \mathcal{B}_{n-1}(F) \otimes F^* \\ \downarrow \tilde{l}_n & & \downarrow l_{n-1} \wedge \mathrm{Id} \\ \mathcal{L}_n(F) & \xrightarrow{\delta} & \oplus_{k \leq n/2} \mathcal{L}_{n-k}(F) \wedge \mathcal{L}_k(F) \end{array}$$

Indeed, its commutativity is equivalent to the basic formula (3.52).

Let $x \in P^1(F)$. Recall the specialization at x homomorphism

$$s_x : \mathcal{B}_n(F(T)) \rightarrow \mathcal{B}_n(F), \quad \{f(T)\}_n \mapsto \{f(x)\}_n$$

It gives rise to the specialization homomorphism

$$s'_x : \mathcal{B}_{n-1}(F(T)) \otimes F(T)^* \rightarrow \mathcal{B}_{n-1}(F) \otimes F^*, \quad (3.53)$$

$$\{f(T)\}_{n-1} \otimes g(T) \mapsto \{f(x)\}_{n-1} \otimes \frac{g(T)}{(T-x)^{v_x(g)}}(x) \quad (3.54)$$

(Use the local parameter T^{-1} when $x = \infty$).

Now let

$$\alpha(T) \in \text{Ker}(\delta_n : \mathbb{Q}[F(T)] \rightarrow \mathcal{B}_{n-1}(F(T)) \otimes F(T)^*)$$

Using $\text{Li}_n^{\mathcal{M}}(0) = \text{Li}_n^{\mathcal{M}}(\infty) = 0$, for any $x \in P^1(F)$ one has $\delta(\tilde{l}_n(\alpha(x))) = 0$. Thus

$$\tilde{l}_n(\alpha(x)) \in K_{2n-1}(F) \otimes \mathbb{Q} \subset \mathcal{L}_n(F)$$

Let us show that this element is zero. Given an embedding $\sigma : F \hookrightarrow \mathbb{C}$, write $\sigma(\alpha(T)) = \sum_i n_i \{f_i^\sigma(T)\}$. Applying the Lie-period map to this element and using v) we get $\sum_i n_i \mathcal{L}_n f_i^\sigma(z)$. By Theorem 1.13 in [28] the condition on $\alpha(T)$ implies that this function is constant on $\mathbb{CP}^1$. Thus the difference of its values at $\sigma(x_1)$ and $\sigma(x_2)$, where $x_0, x_1 \in P^1(F)$, is zero. On the other hand thanks to v) and vi) it coincides with the Borel regulator map applied to the corresponding element of $K_{2n-1}(F) \otimes \mathbb{Q}$. Thus the injectivity of the Borel regulator map proves the claim. So $\tilde{l}_n(\alpha(x_0) - \alpha(x_1)) = 0$. The proposition is proved.

Proposition 24 implies that we get a homomorphism of complexes

$$\begin{array}{ccc} \mathcal{B}_n(F) & \xrightarrow{\delta} & \mathcal{B}_{n-1}(F) \otimes F^* \\ \downarrow l_n & & \downarrow l_{n-1} \wedge \text{Id} \\ \mathcal{L}_n(F) & \xrightarrow{\delta} & \oplus_{k \leq n/2} \mathcal{L}_{n-k}(F) \wedge \mathcal{L}_k(F) . \end{array}$$

The theorem follows immediately from this. Indeed, it remains to check commutativity of the diagram (3.47), and it follows from v) and vi).

If we assume the existence of the hypothetical abelian category of mixed Tate motives over an arbitrary field F , the same argumentation as above (see Chapt. 6.1) implies the following result: one should have canonical homomorphisms

$$H^i(\Gamma(F, n)) \otimes \mathbb{Q} \rightarrow \text{gr}_n^Y K_{2n-i}(F) \otimes \mathbb{Q} .$$

The most difficult part of Conjecture 22 says that this maps are supposed to be isomorphisms.

In the next section we define a regulator map on the polylogarithmic motivic complexes. Combined with these maps, it should give an explicit construction of the regulator map.

3.4.5

A Construction of the Motivic ζ -Elements (3.1)

The formula (3.2) leads to the motivic extension $\zeta^{\mathcal{M}}(n)$ as follows ([40]). Recall the moduli space $\overline{\mathcal{M}}_{n+3}$ parametrising stable curves of genus zero with $n+3$ marked points. It contains as an open subset the space $\mathcal{M}_{n+3}$ parametrising the $(n+3)$ -tuples of distinct points on $\mathbb{P}^1$ modulo $\text{Aut}(\mathbb{P}^1)$. Then the complement $\partial\mathcal{M}_{n+3} := \overline{\mathcal{M}}_{n+3} - \mathcal{M}_{n+3}$ is a normal crossing divisor, and the pair $(\mathcal{M}_{n+3}, \partial\mathcal{M}_{n+3})$ is defined over $\mathbb{Z}$. Let us identify sequences $(t_1, \dots, t_n)$ of distinct complex numbers different from 0 and 1 with the points $(0, t_1, \dots, t_n, 1, \infty)$ of $\mathcal{M}_{n+3}(\mathbb{C})$. Let us consider the integrand in (3.2) as a holomorphic form on $\mathcal{M}_{n+3}(\mathbb{C})$. Meromorphically extending it to $\overline{\mathcal{M}}_{n+3}$ we get a differential form with logarithmic singularities Ω_n . Let A_n be its divisor. Similarly, embed the integration simplex $0 < t_1 < \dots < t_n < 1$ into the set of real points of $\overline{\mathcal{M}}_{n+3}(\mathbb{R})$, take its closure Δ_n there, and consider the Zariski closure B_n of its boundary $\partial\Delta_n$. Then one can show that the mixed motive

$$H^n(\overline{\mathcal{M}}_{n+3} - A_n, B_n - (A_n \cap B_n)) \quad (3.55)$$

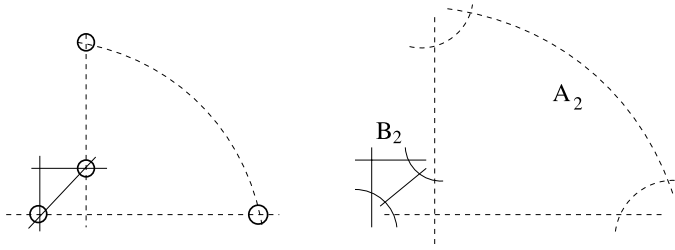
is a mixed Tate motive over $\text{Spec}(\mathbb{Z})$. Indeed, it is easy to prove that its l -adic realization is unramified outside l , and is glued from the Tate modules of different weights, and then refer to [20]. The mixed motive (3.55) comes equipped with an additional data, framing, given by non-zero morphisms

$$[\Omega_n] : \mathbb{Z}(-n) \rightarrow \text{gr}_{2n}^W H^n(\overline{\mathcal{M}}_{n+3} - A_n, B_n - (A_n \cap B_n)) , \quad (3.56)$$

$$[\Delta_n] : \text{gr}_0^W H^n(\overline{\mathcal{M}}_{n+3} - A_n, B_n - (A_n \cap B_n)) \rightarrow \mathbb{Z}(0) . \quad (3.57)$$

There exists the minimal subquotient of the mixed motive (3.55) which inherits non-zero framing. It delivers the extension class $\zeta^{\mathcal{M}}(n)$. Leibniz formula (3.2) just means that $\zeta(n)$ is its period.

Example 25. To construct $\zeta^{\mathcal{M}}(2)$, take the pair of triangles in P^2 shown on the left of Fig. 3.2. The triangle shown by the punctured lines is the divisor of poles of the differential $d \log(1 - t_1) \wedge d \log t_2$ in (3.2), and the second triangle is the algebraic closure of the boundary of the integration cycle $0 \leq t_1 \leq t_2 \leq 1$. The corresponding configuration of six lines is defined uniquely up to a projective equivalence. Blowing up the four points shown by little circles on Fig. 3.2 (they are the triple intersection points of the lines), we get the moduli

Figure 3.2. Constructing $\zeta^{\mathcal{M}}(2)$

space $\overline{\mathcal{M}}_{0,5}$. Its boundary is the union of the two pentagons, A_2 and B_2 , projecting to the two triangles on P^2 . Then $\zeta^{\mathcal{M}}(2)$ is a subquotient of the mixed Tate motive $H^2(\overline{\mathcal{M}}_{0,5} - A_2, B_2 - (A_2 \cap B_2))$ over $\text{Spec}(\mathbb{Z})$. Observe that an attempt to use a similar construction for the pair of triangles in P^2 fails since there is no non-zero morphism $[\Delta_2]$ in this case. Indeed, there are two vertices of the B -triangle shown on the left of Fig. 3.2 lying at the sides of the A -triangle, and therefore the chain $0 \leq t_1 \leq t_2 \leq 1$ does not give rise to a relative class in $H_2(P^2(\mathbb{C}) - A, B - (A \cap B))$.

A Geometric Construction of the Motivic Classical Polylogarithm $\text{Li}_n^{\mathcal{M}}(z)$

3.4.6

The above construction is easily generalized to the case of the classical polylogarithm. Let $A_n(z)$ be the divisor of the meromorphic differential form

$$\Omega_n(z) := \frac{dt_1}{z^{-1} - t_1} \wedge \frac{dt_2}{t_2} \wedge \dots \wedge \frac{dt_n}{t_n}$$

extended as a rational form to $\overline{\mathcal{M}}_{0,n+3}$. Suppose that F is a number field and $z \in F$. Then there is the following mixed Tate motive over F :

$$H^n(\overline{\mathcal{M}}_{n+3} - A_n(z), B_n - (A_n(z) \cap B_n)) \quad (3.58)$$

One checks that the vertices (that is the zero-dimensional strata) of the divisor B_n are disjoint with the divisor $A_n(z)$. Using this we define a framing $([\Omega_n(z)], [\Delta_n])$ on the mixed motive (3.58) similar to the one (3.56)–(3.57). The geometric condition on the divisor $A_n(z)$ is used to show that the framing morphism $[\Delta_n]$ is non-zero. We define $\text{Li}_n^{\mathcal{M}}(z)$ as the framed mixed Tate motive (3.58) with the framing $([\Omega_n(z)], [\Delta_n])$. A similar construction for the multiple polylogarithms was worked out in the Ph. D. thesis of Q. Wang [64].

It follows from a general result in [35] that the Hodge realization of $\text{Li}_n^{\mathcal{M}}(z)$ is equivalent to the framed mixed Hodge structure $\text{Li}_n^{\mathcal{H}}(z)$. If $n = 2$ the two corresponding mixed Hodge structures are isomorphic ([64]), while in general they are not isomorphic but equivalent as framed mixed Hodge structures.

Regulator Maps on the Polylogarithmic Motivic Complexes

In this section we define explicitly these regulator maps via classical polylogarithm functions following [32] and [28]. This implies how the special values of ζ -functions of algebraic varieties outside of the critical strip should be expressed using the classical polylogarithms.

The Regulator Map on the Polylogarithmic Motivic Complexes at the Generic Point of a Complex Algebraic Variety

The numbers $\beta_{n,k}$.

Define for any integers $p \geq 1$ and $k \geq 0$ the numbers

$$\beta_{k,p} := (-1)^p (p-1)! \sum_{0 \leq i \leq \lfloor \frac{p-1}{2} \rfloor} \frac{1}{(2i+1)!} \beta_{k+p-2i}.$$

For instance $\beta_{k,1} = -\beta_{k+1}$; $\beta_{k,2} = \beta_{k+2}$; $\beta_{k,3} = -2\beta_{k+3} - \frac{1}{3}\beta_{k+1}$.

One has recursions

$$2p \cdot \beta_{k+1,2p} = -\beta_{k,2p+1} - \frac{1}{2p+1} \beta_{k+1}; \quad (2p-1) \cdot \beta_{k+1,2p-1} = -\beta_{k,2p} \quad (3.59)$$

These recursions together with $\beta_{k,1} = -\beta_{k+1}$ determine the numbers $\beta_{k,p}$.

Let $m \geq 1$. Then one can show that

$$\beta_{0,2m} = \beta_{0,2m+1} = \frac{1}{2m+1}, \quad \beta_{1,2m-1} = -\frac{1}{(2m-1)(2m+1)}, \quad \beta_{1,2m} = 0.$$

Let us now proceed to the definition of the regulator map.

Let us define differential 1-forms $\widehat{\mathcal{L}}_{p,q}$ on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ for $q \geq 1$ as follows:

$$\widehat{\mathcal{L}}_{p,q}(z) := \widehat{\mathcal{L}}_p(z) \log^{q-1} |z| \cdot d \log |z|, \quad p \geq 2 \quad (3.60)$$

$$\widehat{\mathcal{L}}_{1,q}(z) := \alpha(1-z, z) \log^{q-1} |z|$$

For any rational function f on a complex variety X the 1-form $\widehat{\mathcal{L}}_{p,q}(f)$ provides a distribution on $X(\mathbb{C})$. Set

$$\mathcal{A}_m \left\{ \bigwedge_{i=1}^{2p} d \log |g_i| \wedge \bigwedge_{i=2p+1}^m di \arg g_j \right\} :=$$

$$\text{Alt}_m \left\{ \frac{1}{(2p)!(m-2p)!} \bigwedge_{i=1}^{2p} d \log |g_i| \wedge \bigwedge_{i=2p+1}^m di \arg g_j \right\}$$

and

$$\mathcal{A}_m \left\{ \log |g_1| \cdot \bigwedge_{i=2}^p d \log |g_i| \wedge \bigwedge_{i=p+1}^m di \arg g_i \right\} :=$$

$$\text{Alt}_m \left\{ \frac{1}{(p-1)!(m-p)!} \log |g_1| \cdot \bigwedge_{i=2}^p d \log |g_i| \wedge \bigwedge_{i=p+1}^m di \arg g_i \right\}.$$

So $\mathcal{A}_m(F(g_1, \dots, g_m))$ is a weighted alternation (we divide by the order of the stabilizer of the term we alternate).

Let $f, g_1, \dots, g_m$ be rational functions on a complex variety X . Set

$$r_{n+m}(m+1) : \{f\}_n \otimes g_1 \wedge \dots \wedge g_m \mapsto$$

$$\widehat{\mathcal{L}}_n(f) \cdot \mathcal{A}_m \left\{ \sum_{p \geq 0} \frac{1}{2^p + 1} \bigwedge_{i=1}^{2p} d \log |g_i| \wedge \bigwedge_{j=2p+1}^m di \arg g_j \right\} \quad (3.61)$$

$$+ \sum_{k \geq 1} \sum_{1 \leq p \leq m} \beta_{k,p} \widehat{\mathcal{L}}_{n-k,k}(f) \wedge \mathcal{A}_m \left\{ \log |g_1| \bigwedge_{i=2}^p d \log |g_i| \wedge \bigwedge_{j=p+1}^m di \arg g_j \right\}. \quad (3.62)$$

Proposition 26 The differential form $r_{n+m}(m+1)(\{f\}_n \otimes g_1 \wedge \dots \wedge g_m)$ defines a distribution on $X(\mathbb{C})$.

26

Example 27. $r_n(1)(\{f\}_n) = \widehat{\mathcal{L}}_n(f)$.

Example 28. $r_n(n)(g_1 \wedge \dots \wedge g_n) = r_{n-1}(g_1 \wedge \dots \wedge g_n)$.

Example 29. $m = 1, n$ is arbitrary. Then

$$r_{n+1}(2) : \{f\}_n \otimes g \mapsto \widehat{\mathcal{L}}_n(f) di \arg g - \sum_{k=1}^{n-1} \beta_{k+1} \widehat{\mathcal{L}}_{n-k,k}(f) \cdot \log |g|$$

Example 30. $m = 2, n$ is arbitrary.

$$r_{n+2}(3) : \{f\}_n \otimes g_1 \wedge g_2 \mapsto$$

$$\widehat{\mathcal{L}}_n(f) \left\{ di \arg g_1 \wedge di \arg g_2 + \frac{1}{3} d \log |g_1| \wedge d \log |g_2| \right\}$$

$$- \sum_{k=1}^{n-1} \beta_{k+1} \widehat{\mathcal{L}}_{n-k,k}(f) \wedge (\log |g_1| di \arg g_2 - \log |g_2| di \arg g_1)$$

$$+ \sum_{k \geq 1} \beta_{k+2} \widehat{\mathcal{L}}_{n-k,k}(f) \wedge (\log |g_1| d \log |g_2| - \log |g_2| d \log |g_1|)$$

Let $\mathcal{A}^i(\eta_X)$ be the space of real smooth i -forms at the generic point $\eta_X := \text{Spec} \mathbb{C}(X)$ of a complex variety X . Let $\mathcal{D}$ be the de Rham differential on distributions on $X(\mathbb{C})$, and d the de Rham differential on $\mathcal{A}^i(\eta_X)$. For example:

$$d(di \arg z) = 0; \quad \mathcal{D}(di \arg z) = 2\pi i \delta(z)$$

Recall the residue homomorphisms defined in Chapt. 4.2.

31

Theorem 31

a) The maps $r_n(\cdot)$ provide a homomorphism of complexes

$$\begin{array}{ccccccc} \mathcal{B}_n(\mathbb{C}(X)) & \xrightarrow{\delta} & \mathcal{B}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & \bigwedge^n \mathbb{C}(X)^* \\ \downarrow r_n(1) & & \downarrow r_n(2) & & & & \downarrow r_n(n) \\ \mathcal{A}^0(\eta_X)(n-1) & \xrightarrow{d} & \mathcal{A}^1(\eta_X)(n-1) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathcal{A}^{n-1}(\eta_X)(n-1) \end{array}$$

b) The maps $r_n(m)$ are compatible with the residues:

$$\mathcal{D} \circ r_n(m) - r_n(m+1) \circ \delta = 2\pi i \cdot \sum_{Y \in X^{(1)}} r_{n-1}(m-1) \circ \partial_{v_Y}, \quad m < n$$

$$\mathcal{D} \circ r_n(n) - \pi_n(d \log f_1 \wedge \cdots \wedge d \log f_n) = 2\pi i \cdot \sum_{Y \in X^{(1)}} r_{n-1}(n-1) \circ \partial_{v_Y}$$

where v_Y is the valuation on the field $\mathbb{C}(X)$ defined by a divisor Y .

Let X be a regular variety over $\mathbb{C}$. Recall the n -th Beilinson–Deligne complex $\underline{\mathbb{R}}_{\mathcal{D}}(n)_X$ defined as a total complex associated with the following bicomplex of sheaves in classical topology on $X(\mathbb{C})$:

$$\begin{array}{ccccccc} (\underline{\mathcal{D}}_X^0 & \xrightarrow{d} & \underline{\mathcal{D}}_X^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \underline{\mathcal{D}}_X^n & \xrightarrow{d} & \underline{\mathcal{D}}_X^{n+1} & \xrightarrow{d} & \cdots) \otimes \mathbb{R}(n-1) \\ & & & & & & \uparrow \pi_n & & \uparrow \pi_n & & \\ & & & & & & \Omega_{X,\log}^n & \xrightarrow{\partial} & \Omega_{X,\log}^{n+1} & \xrightarrow{\partial} & \end{array}$$

Here $\underline{\mathcal{D}}_X^0$ is in degree 1 and $(\Omega_{X,\log}^\bullet, \partial)$ is the de Rham complex of holomorphic forms with logarithmic singularities at infinity. We will denote by $\underline{\mathbb{R}}_{\mathcal{D}}(n)(U)$ the complex of the global sections.

Theorem 31 can be reformulated as follows. Set $\tilde{r}_n(i) := r_n(i)$ for $i < n$ and

$$\begin{aligned} \tilde{r}_n(n) : \Lambda^n \mathbb{C}(X)^* &\rightarrow \mathcal{A}^{n-1}(\eta_X)(n-1) \oplus \Omega_{\log}^n(\eta_X) \\ f_1 \wedge \dots \wedge f_n &\mapsto r_n(n)(f_1 \wedge \dots \wedge f_n) + d \log f_1 \wedge \dots \wedge d \log f_n \end{aligned} \quad (3.63)$$

Theorem 32 Let X be a complex algebraic variety. Then there is a homomorphism of complexes 32

$$\tilde{r}_n(\cdot) : \Gamma(\mathbb{C}(X); n) \rightarrow \underline{\mathbb{R}}_{\mathcal{D}}(n)(\mathrm{Spec} \mathbb{C}(X)) \quad (3.64)$$

compatible with the residues as explained in the part b) of Theorem 31.

The General Case

3.5.2

Let X be a regular projective variety over a field F . Let $d := \dim X$. Then the complex $\Gamma(X; n)$ should be defined as the total complex of the following bicomplex:

$$\begin{aligned} \Gamma(F(X); n) &\rightarrow \bigoplus_{Y_1 \in X^{(1)}} \Gamma(F(Y_1); n-1)[-1] \rightarrow \\ &\bigoplus_{Y_2 \in X^{(2)}} \Gamma(F(Y_2); n-2)[-1] \rightarrow \dots \rightarrow \bigoplus_{Y_d \in X^{(d)}} \Gamma(F(Y_d); n-d)[-d], \end{aligned}$$

where the arrows are provided by the residue maps, see [27], pp. 239–240. The complex $\Gamma(X; n) \otimes \mathbb{Q}$ should be quasiisomorphic to the weight n motivic complex.

However there is a serious difficulty in the definition of the complex $\Gamma(X; n)$ for a general variety X and $n > 3$, ([27], p. 240). It would be resolved if homotopy invariance of the polylogarithmic complexes were known (Conjecture 1.39 in [27]). As a result we have an unconditional definition of the polylogarithmic complexes $\Gamma(X; n)$ only in the following cases:

- a) $X = \mathrm{Spec}(F)$, F is an arbitrary field.
- b) X is an regular curve over any field, and n is arbitrary.
- c) X is an arbitrary regular scheme, but $n \leq 3$.

Now let F be a subfield of $\mathbb{C}$. Having in mind applications to arithmetic, we will restrict ourself by the case when $F = \mathbb{Q}$. Assuming we are working with one of the above cases, or assuming the above difficulty has been resolved, let us define the regulator map

$$\Gamma(X; n) \rightarrow \mathcal{C}_{\mathcal{D}}(X(\mathbb{C}); \mathbb{R}(n)) .$$

We specify it for each of $\Gamma(\mathbb{Q}(Y_k); n-k)[-k]$ where $k = 0, \dots, d$. Namely, we take the homomorphism $r_{n-k}(\cdot)$ for $\mathrm{Spec}(\mathbb{Q}(Y_k))$ and multiply it by $(2\pi i)^{n-k} \delta_{Y_{n-k}}$. Notice that the distribution δ_Y depends only on the generic point of a subvariety Y . Then Theorem 31, and in particular its part b), providing *compatibility with the residues* property, guarantee that we get a homomorphism of complexes. Here are some examples.

3.5.3 Weight One

The regulator map on the weight one motivic complex looks as follows:

$$\begin{array}{ccc}
 \mathbb{Q}(X)^* & \longrightarrow & \oplus_{Y \in X(1)} \mathbb{Z} \\
 \downarrow r_1(1) & & \downarrow r_1(2) \\
 \mathcal{D}^{0,0} & \xrightarrow{2\bar{\partial}\partial} & \mathcal{D}_{cl, \mathbb{R}}^{1,1}(1)
 \end{array}$$

$$r_1(2) : Y_1 \mapsto 2\pi i \cdot \delta_{Y_1}, \quad r_1(1) : f \mapsto \log |f|$$

Here the top line is the weight 1 motivic complex, sitting in degrees $[1, 2]$.

3.5.4 Weight Two

The regulator map on the weight two motivic complexes looks as follows.

$$\begin{array}{ccccccc}
 \mathcal{B}_2(\mathbb{Q}(X)) & \xrightarrow{\delta} & \Lambda^2 \mathbb{Q}(X)^* & \xrightarrow{\partial} & \oplus_{Y \in X(1)} \mathbb{Q}(Y)^* & \xrightarrow{\partial} & \oplus_{Y \in X(2)} \mathbb{Z} \\
 \downarrow r_2(1) & & \downarrow r_2(2) & & \downarrow r_2(3) & & \downarrow r_2(4) \\
 \mathcal{D}_{\mathbb{R}}^{0,0}(1) & \xrightarrow{D} & (\mathcal{D}^{0,1} \oplus \mathcal{D}^{1,0})_{\mathbb{R}}(1) & \xrightarrow{D} & \mathcal{D}_{\mathbb{R}}^{1,1}(1) & \xrightarrow{2\bar{\partial}\partial} & \mathcal{D}_{cl, \mathbb{R}}^{2,2}(2)
 \end{array}$$

where D is the differential in $\mathcal{C}_{\mathcal{D}}(X, \mathbb{R}(2))$

$$r_2(4) : Y_2 \mapsto (2\pi i)^2 \cdot \delta_{Y_2}; \quad r_2(3) : (Y_1, f) \mapsto 2\pi i \cdot \log |f| \delta_{Y_1}$$

$$r_2(2) : f \wedge g \mapsto -\log |f| di \arg g + \log |g| di \arg f; \quad r_2(1) : \{f\}_2 \mapsto \widehat{\mathcal{L}}_2(f)$$

To prove that we get a morphism of complexes we use Theorem 31. The following argument is needed to check the commutativity of the second square. The de Rham differential of the distribution $r_2(2)(f \wedge g)$ is

$$\begin{aligned}
 & \mathcal{D}(-\log |f| di \arg g + \log |g| di \arg f) = \\
 & \pi_2(d \log f \wedge \log g) + 2\pi i \cdot (\log |g| \delta(f) - \log |f| \delta(g)).
 \end{aligned}$$

This *does not* coincide with $r_2(3) \circ \partial(f \wedge g)$, but the difference is

$$(\mathcal{D} \circ r_2(2) - r_2(3) \circ \partial)(f \wedge g) = \pi_2(d \log f \wedge \log g) \in (\mathcal{D}^{0,2} \oplus \mathcal{D}^{2,0})_{\mathbb{R}}(1).$$

Defining the differential D on the second group of the complex $\mathcal{C}_{\mathcal{D}}(X, \mathbb{R}(2))$ we take the de Rham differential and throw away from it precisely these components. Therefore the middle square is commutative.

Weight Three

3.5.5

The weight three motivic complex $\Gamma(X; 3)$ is the total complex of the following bicomplex: (the first group is in degree 1)

$$\begin{array}{ccccc}
 \mathcal{B}_3(\mathbb{Q}(X)) & \longrightarrow & \mathcal{B}_2(\mathbb{Q}(X)) \otimes \mathbb{Q}(X)^* & \longrightarrow & \Lambda^3 \mathbb{Q}(X)^* \\
 & & \downarrow & & \downarrow \\
 & & \oplus_{Y_1 \in X^{(1)}} \mathcal{B}_2(\mathbb{Q}(Y_1)) & \longrightarrow & \oplus_{Y_1 \in X^{(1)}} \Lambda^2 \mathbb{Q}(Y_1)^* \\
 & & & & \downarrow \\
 & & & & \oplus_{Y_2 \in X^{(2)}} \mathbb{Q}(Y_2)^* \\
 & & & & \downarrow \\
 & & & & \oplus_{Y_3 \in X^{(3)}} \mathbb{Q}(Y_3)^*
 \end{array}$$

The Deligne complex $\mathcal{C}_{\mathcal{D}}(X, \mathbb{R}(3))$ looks as follows:

$$\begin{array}{ccccc}
 & & & & \mathcal{D}_{cl, \mathbb{R}}^{3,3}(3) \\
 & & & \nearrow 2\bar{\partial}\partial & \\
 \mathcal{D}^{0,2} & \xrightarrow{\partial} & \mathcal{D}^{1,2} & \xrightarrow{\partial} & \mathcal{D}^{2,2} \\
 \uparrow \bar{\partial} & & \uparrow \bar{\partial} & & \uparrow \bar{\partial} \\
 \mathcal{D}^{0,1} & \xrightarrow{\partial} & \mathcal{D}^{1,1} & \xrightarrow{\partial} & \mathcal{D}^{1,2} \\
 \uparrow \bar{\partial} & & \uparrow \bar{\partial} & & \uparrow \bar{\partial} \\
 \mathcal{D}^{0,0} & \xrightarrow{\partial} & \mathcal{D}^{1,0} & \xrightarrow{\partial} & \mathcal{D}^{2,0}
 \end{array}$$

We construct the regulator map $\Gamma(X; 3) \rightarrow \mathcal{C}_{\mathcal{D}}(X, \mathbb{R}(3))$ by setting

$$\begin{aligned}
 r_3(6) : Y_3 &\mapsto (2\pi i)^3 \cdot \delta_{Y_3} \\
 r_3(5) : (Y_2, f) &\mapsto (2\pi i)^2 \cdot \log |f| \delta_{Y_2} \\
 r_3(4) : (Y_1, f \wedge g) &\mapsto 2\pi i \cdot (-\log |f| di \arg g + \log |g| di \arg f) \delta_{Y_1} \\
 r_3(3) : (Y_1, \{f\}_2) &\mapsto 2\pi i \cdot \widehat{\mathcal{L}}_2(f) \delta_{Y_1} \\
 r_3(3) : f_1 \wedge f_2 \wedge f_3 &\mapsto \text{Alt}_3 \left(\frac{1}{6} \log |f_1| d \log |f_2| \wedge d \log |f_3| \right. \\
 &\quad \left. + \frac{1}{2} \log |f_1| di \arg f_2 \wedge di \arg f_3 \right) \\
 r_3(2) : \{f\}_2 \otimes g &\mapsto \widehat{\mathcal{L}}_2(f) di \arg g \\
 &\quad - \frac{1}{3} \log |g| \cdot \left(-\log |1-f| d \log |f| + \log |f| d \log |1-f| \right) \\
 r_3(1) : \{f\}_3 &\mapsto \widehat{\mathcal{L}}_3(f)
 \end{aligned}$$

Classical Polylogarithms and Special Values of ζ -Functions of Algebraic Varieties

3.5.6

We conjecture that the polylogarithmic motivic complex $\Gamma(X; n) \otimes \mathbb{Q}$ should be quasiisomorphic to the weight n motivic complex, and Beilinson's regulator map under this quasiisomorphism should be equal to the defined above regulator map on $\Gamma(X; n)$. This implies that the special values of ζ -functions of algebraic varieties outside of the critical strip should be expressed via classical polylogarithms.

The very special case of this conjecture when $X = \text{Spec}(F)$ where F is a number field is equivalent to Zagier's conjecture [67].

The next interesting case is when X is a curve over a number field. The conjecture in this case was elaborated in [29], see also [34] for a survey. For example if X is an elliptic curve this conjecture suggests that the special values $L(X, n)$ for $n \geq 2$ are expressed via certain generalized Eisenstein–Kronecker series. For $n = 2$ this was discovered of S. Bloch [11], and for $n = 3$ it implies Deninger's conjecture [21].

A homomorphism from K -theory to $H^i(\Gamma(X; n)) \otimes \mathbb{Q}$ has been constructed in the following cases:

- (1) F is an arbitrary field, $n \leq 3$ ([27, 28, 31]) and $n = 4$, $i > 1$ (to appear).
- (2) X is a curve over a number field, $n \leq 3$ ([31]) and $n = 4$, $i > 1$ (to appear).

In all these cases we proved that this homomorphism followed by the regulator map on polylogarithmic complexes (when $F = \mathbb{C}(X)$ in (1)) coincides with Beilinson's regulator. This proves the difficult “surjectivity” property: the image of the

regulator map on these polylogarithmic complexes *contains* the image of Beilinson's regulator map in the Deligne cohomology. The main ingredient of the proof is an explicit construction of the motivic cohomology class

$$H^{2n}(BGL_{\bullet}, I(*; n)) .$$

The results (2) combined with the results of R. de Jeu [44, 45] prove that the image of Beilinson's regulator map in the Deligne cohomology *coincides* with the image of the regulator map on polylogarithmic complexes in the case (2).

Coda: Polylogarithms on Curves, Feynman Integrals and Special Values of L -Functions

3.5.7

The classical polylogarithm functions admit the following generalization. Let X be a regular complex algebraic curve. Let us assume first that X is projective. Let us choose a metric on $X(\mathbb{C})$. Denote by $G(x, y)$ the Green function provided by this metric. Then one can define real-valued functions $G_n(x, y)$ depending on a pair of points x, y of $X(\mathbb{C})$ with values in the complex vector space $S^{n-1}H_1(X(\mathbb{C}), \mathbb{C})(1)$, see [36], Chapt. 9.1. By definition $G_1(x, y)$ is the Green function $G(x, y)$. It has a singularity at the diagonal, but provides a generalized function. For $n > 1$ the function $G_n(x, y)$ is well-defined on $X(\mathbb{C}) \times X(\mathbb{C})$. Let $X = E$ be an elliptic curve. Then the functions $G_n(x, y)$ are translation invariant, and thus reduced to a single variable functions: $G_n(x, y) = G_n(x - y)$. The function $G_n(x)$ is given by the classical Kronecker–Eisenstein series. Finally, adjusting this construction to the case when $X = \mathbb{C}^*$, we get a function $G_n(x, y)$ invariant under the shifts on the group $\mathbb{C}^*$, i.e. one has $G_n(x, y) = G_n(x/y)$. The function $G_n(z)$ is given by the single-valued version (3.39) of the classical n -logarithm function.

Let us extend the function $G_n(x, y)$ by linearity to a function $G_n(D_1, D_2)$ depending on a pair of divisors on $X(\mathbb{C})$. Although the function $G_n(x, y)$ depends on the choice of metric on $X(\mathbb{C})$, the restriction of the function $G_n(D_1, D_2)$ to the subgroups of the degree zero divisors is independent of the choice of the metric.

Let X be a curve over a number field F . In [36, section 9.1] we proposed a conjecture which allows us to express the special values $L(\mathrm{Sym}^{n-1}H^1(X), n)$ via determinants whose entries are given by the values of the function $G_n(D_1, D_2)$, where D_1, D_2 are degree zero divisors on $(X(\bar{F}))$ invariant under the action of the Galois group $\mathrm{Gal}(\bar{F}/F)$. In the case when X is an elliptic curve it boils down to the so-called elliptic analog of Zagier's conjecture, see [37, 66]. It has been proved for $n = 2$ at [39], and a part of this proof (minus surjectivity) can be transformed to the case of arbitrary X .

It was conjectured in the Sect. 9.3 of [36] that the special values $L(\mathrm{Sym}^{n-1}H^1(X), n + m - 1)$, where $m \geq 1$, should be expressed similarly via the special values of the depth m multiple polylogarithms on the curve X , defined in the Sect. 9.2 of loc. cit. One can show that in the case when $X = E$ is an elliptic curve, and $n = m = 2$, this conjecture reduces to Deninger's conjecture [21] on $L(E, 3)$, which has been proved

in [29] An interesting aspect of this story is that the multiple polylogarithms on curves are introduced via mathematically well defined Feynman integrals. This seems to be the first application of Feynman integrals in number theory.

Motivic Lie Algebras and Grassmannian Polylogarithms

3.6

3.6.1

Motivic Lie Coalgebras and Motivic Complexes

Beilinson conjectured that for an arbitrary field F there exists an abelian $\mathbb{Q}$ -category $\mathcal{M}_T(F)$ of mixed Tate motives over F . This category is supposed to be a mixed Tate category, see Sect. 8 of [38] for the background. Then the Tannakian formalism implies that there exists a positively graded Lie coalgebra $\mathcal{L}_\bullet(F)$ such that the category of finite dimensional graded comodules over $\mathcal{L}_\bullet(F)$ is naturally equivalent to the category $\mathcal{M}_T(F)$.

Moreover $\mathcal{L}_\bullet(F)$ depends functorially on F . This combined with Beilinson's conjectural formula for the Ext groups in the category $\mathcal{M}_T(F)$ imply that the cohomology of this Lie coalgebra are computed by the formula

$$H_{(n)}^i(\mathcal{L}_\bullet(F)) = \mathrm{gr}_n^Y K_{2n-i}(F) \otimes \mathbb{Q} \quad (3.65)$$

where $H_{(n)}^i$ means the degree n part of H^i . This conjecture provides a new point of view on algebraic K -theory of fields, suggesting and explaining several conjectures and results, see ch. 1 [28].

The degree n part of the standard cochain complex

$$\mathcal{L}_\bullet(F) \xrightarrow{\delta} \Lambda^2 \mathcal{L}_\bullet(F) \xrightarrow{\delta \wedge \mathrm{Id} - \mathrm{Id} \wedge \delta} \Lambda^3 \mathcal{L}_\bullet(F) \rightarrow \dots$$

of the Lie coalgebra $\mathcal{L}_\bullet(F)$ is supposed to be quasiisomorphic to the weight n motivic complex of F , providing yet another point of view on motivic complexes. For example the first four of the motivic complexes should look as follows:

$$\mathcal{L}_1(F); \quad \mathcal{L}_2(F) \rightarrow \Lambda^2 \mathcal{L}_1(F); \quad \mathcal{L}_3(F) \rightarrow \mathcal{L}_2(F) \otimes \mathcal{L}_1(F) \rightarrow \Lambda^3 \mathcal{L}_1(F)$$

$$\mathcal{L}_4(F) \rightarrow \mathcal{L}_3(F) \otimes \mathcal{L}_1(F) \oplus \Lambda^2 \mathcal{L}_2(F) \rightarrow \mathcal{L}_2(F) \otimes \Lambda^2 \mathcal{L}_1(F) \rightarrow \Lambda^4 \mathcal{L}_1(F).$$

Comparing this with the formula (3.65), the shape of complexes (3.10) and (3.13), and the known information relating their cohomology with algebraic K -theory we conclude the following. One must have

$$\mathcal{L}_1(F) = F^* \otimes \mathbb{Q}; \quad \mathcal{L}_2(F) = B_2(F) \otimes \mathbb{Q}$$

and we expect to have an isomorphism

$$B_3(F) \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{L}_3(F).$$

Moreover the motivic complexes (3.10) and (3.13) are simply the degree n parts of the standard cochain complex of the Lie coalgebra $\mathcal{L}_\bullet(F)$. More generally, the very existence for an arbitrary field F of the motivic classical polylogarithms (3.50) implies that one should have canonical homomorphisms

$$l_n : \mathcal{B}_n(F) \rightarrow \mathcal{L}_n(F) .$$

These homomorphisms are expected to satisfy the basic relation 3.52. Therefore the maps l_n give rise to a canonical homomorphism from the weight n polylogarithmic complex of F to the degree n part of the standard cochain complex of $\mathcal{L}_\bullet(F)$.

$$\begin{array}{ccccccc} \mathcal{B}_n(F) & \longrightarrow & \mathcal{B}_{n-1}(F) \otimes F^* & \longrightarrow & \cdots & \longrightarrow & \Lambda^n F^* \\ \downarrow l_n & & \downarrow l_{n-1} \wedge l_1 & & & & \downarrow = \\ \mathcal{L}_n(F) & \longrightarrow & \Lambda_{(n)}^2 \mathcal{L}_\bullet(F) & \longrightarrow & \cdots & \longrightarrow & \Lambda^n \mathcal{L}_1(F) \end{array} \quad (3.66)$$

where $\Lambda_{(n)}^2$ denotes the degree n part of $\Lambda_{(n)}^2$. For $n = 1, 2, 3$, these maps, combined with the ones (3.43), lead to the maps above. For $n > 3$ the map of complexes (3.66) will not be an isomorphism. The conjecture that it is a quasisomorphism is equivalent to the Freeness conjecture for the Lie coalgebra $\mathcal{L}_\bullet(F)$, see [27, 28].

Therefore we have two different points of view on the groups $\mathcal{B}_n(F)$ for $n = 1, 2, 3$: according to one of them they are the particular cases of the groups $\mathcal{B}_n(F)$, and according to the other they provide an explicit computation of the first three of the groups $\mathcal{L}_n(F)$. It would be very interesting to find an explicit construction of the groups $\mathcal{L}_n(F)$ for $n > 3$ generalizing the definition of the groups $\mathcal{B}_n(F)$. More specifically, we would like to have a “finite dimensional” construction of all vector spaces $\mathcal{L}_n(F)$, i.e. for every n there should exist a finite number of finite dimensional algebraic varieties X_n^i and R_n^j such that

$$\mathcal{L}_n(F) = \text{Coker} \left(\bigoplus_j \mathbb{Q}[R_n^j(F)] \rightarrow \bigoplus_i \mathbb{Q}[X_n^i(F)] \right) .$$

Such a construction would be provided by the scissor congruence motivic Hopf algebra of F ([6, 7]). However so far the problem in the definition of the coproduct in loc. cit. for non generic generators has not been resolved. A beautiful construction of the motivic Lie coalgebra of a field F was suggested by Bloch and Kriz in [14]. However it is not finite dimensional in the above sense.

Grassmannian Approach of the Motivic Lie Coalgebra

3.6.2

We suggested in [31] that there should exist a construction of the Lie coalgebra $\mathcal{L}_\bullet(F)$ such that the variety X_n is the variety of configurations of $2n$ points in P^{n-1} and the relations varieties R_n^j are provided by the functional equations for the *motivic Grassmannian n -logarithm*. Let us explain this in more detail.

Let $\tilde{\mathcal{L}}_n(F)$ be the free abelian group generated by $2n$ -tuples of points $(l_1, \dots, l_{2n})$ in $P^{n-1}(F)$ subject to the following relations:

- 1) $(l_1, \dots, l_{2n}) = (gl_1, \dots, gl_{2n})$ for any $g \in PGL_n(F)$.
- 2) $(l_1, \dots, l_{2n}) = (-1)^{|\sigma|} (l_{\sigma(1)}, \dots, l_{\sigma(2n)})$ for any permutation $\sigma \in S_{2n}$.
- 3) for any $2n+1$ points $(l_0, \dots, l_{2n})$ in $P^{n-1}(F)$ one has

$$\sum_{i=0}^{2n} (-1)^i (l_0, \dots, \widehat{l_i}, \dots, l_{2n}) = 0.$$

- 4) for any $2n+1$ points $(l_0, \dots, l_{2n})$ in $P^n(F)$ one has

$$\sum_{i=0}^{2n} (-1)^i (l_i | l_0, \dots, \widehat{l_i}, \dots, l_{2n}) = 0.$$

We conjecture that $\mathcal{L}_n(F)$ is a quotient of $\tilde{\mathcal{L}}_n(F)$. It is a nontrivial quotient already for $n = 3$. Then to define the Lie coalgebra $\mathcal{L}_\bullet(F)$ one needs to produce a cobracket

$$\delta : \mathcal{L}_n(F) \rightarrow \oplus_i \mathcal{L}_i(F) \wedge \mathcal{L}_{n-i}(F).$$

Here is how to do this in the first nontrivial case, $n = 4$.

Let us define a homomorphism

$$\tilde{\mathcal{L}}_4(F) \xrightarrow{\delta} B_3(F) \otimes F^* \oplus B_2(F) \wedge B_2(F)$$

by setting $\delta = (\delta_{3,1}, \delta_{2,2})$ where

$$\begin{aligned} \delta_{3,1}(l_1, \dots, l_8) := & -\frac{1}{9} \text{Alt}_8 \left((r_3(l_1 | l_2, l_3, l_4; l_5, l_6, l_7) + \{r(l_1 l_2 | l_3, l_6, l_4, l_5)\}_3 - \right. \\ & \left. - \{r(l_1 l_2 | l_3, l_5, l_4, l_6)\}_3) \wedge \Delta(l_5, l_6, l_7, l_8) \right) \in B_3(F) \wedge F^* \end{aligned}$$

$$\begin{aligned} \delta_{2,2}(l_1, \dots, l_8) := & \frac{1}{7} \text{Alt}_8 \left(\{r_2(l_1, l_2 | l_3, l_4, l_5, l_6)\}_2 \wedge \{r_2(l_3, l_4 | l_1, l_2, l_5, l_7)\}_2 \right) \\ & \in \Lambda^2 B_2(F). \end{aligned}$$

These formulae are obtained by combining the definitions at pp. 136–137 and p. 156 of [31]. The following key result is Theorem 5.1 in loc. cit.

Theorem 33

- a) The homomorphism δ kills the relations 1)–4).
- b) The following composition equals to zero:

$$\tilde{\mathcal{L}}_4(F) \xrightarrow{\delta} B_3(F) \otimes F^* \oplus B_2(F) \wedge B_2(F) \xrightarrow{\delta} B_2(F) \otimes \Lambda^2 F^*$$

Here the second differential is defined by the Leibniz rule, using the differentials in complexes (3.10) and (3.13).

Taking the “connected component” of zero in $\text{Ker } \delta$ (as we did in Sect. 3.4.1 above, or in [27]) we should get the set of defining relations for the group $\mathcal{L}_4(F)$. An explicit construction of them is not known.

The Motivic Grassmannian Tetralogarithm

3.6.3

Let F be a function field on a complex variety X . There is a morphism of complexes

$$\begin{array}{ccccc}
 B_3(F) \otimes F^* \oplus B_2(F) \wedge B_2(F) & \xrightarrow{\delta} & B_2(F) \otimes \Lambda^2 F^* & \xrightarrow{\delta} & \Lambda^4 F^* \\
 \downarrow R_4(2) & & \downarrow R_4(3) & & \downarrow R_4(4) \\
 \mathcal{A}^1(\text{Spec } F) & \xrightarrow{d} & \mathcal{A}^2(\text{Spec } F) & \xrightarrow{d} & \mathcal{A}^3(\text{Spec } F)
 \end{array}$$

extending the homomorphism $r_4(*)$ from Chapt. 5. Namely, $R_4(*) = r_4(*)$ for $* = 3, 4$ and $R_4(2) = (r_4(2), r'_4(2))$ where

$$\begin{aligned}
 r'_4(2) : \Lambda^2 B_2(F) &\rightarrow \mathcal{A}^1(\text{Spec } F) \\
 \{f\}_2 \wedge \{g\}_2 &\mapsto \frac{1}{3} \cdot (\widehat{\mathcal{L}}_2(g) \cdot \alpha(1-f, f) - \widehat{\mathcal{L}}_2(f) \cdot \alpha(1-g, g))
 \end{aligned}$$

It follows from Theorem 33 that the composition $R_4(2) \circ \delta(l_1, \dots, l_8)$ is a closed 1-form on the configuration space of 8 points in $\mathbb{CP}^3$. One can show (Proposition 5.3 in [31]) that integrating this 1-form we get a single valued function on the configuration space, denoted $\mathcal{L}_4^M$ and called the *motivic Grassmannian tetralogarithm*. It would be very interesting to compute the difference $\mathcal{L}_4^M - \mathcal{L}_4^{\mathcal{L}}$, similarly to the formula (3.40) in the case $n = 3$. We expect that is expressed as a sum of products of functions $\mathcal{L}_3, \mathcal{L}_2$ and $\log |*|$.

Theorem 34

34

a) There exists a canonical map

$$K_7^{[3]}(F) \otimes \mathbb{Q} \rightarrow \text{Ker} \left(\widetilde{\mathcal{L}}_4(F) \xrightarrow{\delta} B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F) \right)_{\mathbb{Q}}$$

b) In the case $F = \mathbb{C}$ the composition $K_7(\mathbb{C}) \rightarrow \widetilde{\mathcal{L}}_4(\mathbb{C}) \xrightarrow{\mathcal{L}_4^M} \mathbb{R}$ coincides with a nonzero rational multiple of the Borel regulator map.

The generalization of the above picture to the case $n > 4$ is unknown. It would be very interesting at least to define the motivic Grassmannian polylogarithms via the Grassmannian polylogarithms.

For $n = 3$ the motivic Grassmannian trilogarithm $\mathcal{L}_3^M$ is given by the first term of the formula (3.40). It is known ([41]) that $\mathcal{L}_3^L$ does not provide a homomorphism $\mathcal{L}_3(\mathbb{C}) \rightarrow \mathbb{R}$ since the second term in (3.40) does not have this property. Indeed, the second term in (3.40) vanishes on the special configuration of 6 points in P^2 , but does not vanish at the generic configuration. On the other hand the defining relations in $\mathcal{L}_3(F)$ allow to express any configuration as a linear combination of the special ones.

We expect the same situation in general: for $n > 2$ the Grassmannian n -logarithms should not satisfy all the functional equations for the motivic Grassmannian n -logarithms. It would be very interesting to find a conceptual explanation of this surprising phenomena.

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Algebraic K-Theory, Algebraic Cycles and Arithmetic Geometry

Bruno Kahn

4.1	<i>Introduction</i>	353
4.2	<i>The Picture in Algebraic Topology</i>	354
4.3	<i>The Picture in Algebraic Geometry</i>	356
	Algebraic K-Theory	356
	Bloch's Cycle Complexes.....	359
	Suslin–Voevodsky's Motivic Cohomology	362
	The Beilinson–Soulé Conjecture	363
4.4	<i>Review of Motives</i>	366
	Pure Motives	366
	The Conjectural Abelian Category of Mixed Motives.....	370
	The Nonconjectural Triangulated Categories of Mixed Motives	370
	Nori's Category	372
	$\mathbf{A}^1$ -Homotopy and Stable Homotopy Categories	374
4.5	<i>Comparisons</i>	376
	Étale topology.....	376
	Zariski Topology	376
	Back to the Beilinson–Soulé Conjecture.....	379
	Borel–Moore Étale Motivic Homology	379
4.6	<i>Applications: Local Structure of Algebraic K-Groups and Finiteness Theorems</i>	382

4.7	<i>The Picture in Arithmetic Geometry</i>	386
	Finite Generation Theorems	386
	Ranks, Torsion and Zeta Functions	389
	Soulé's Conjecture	390
	Lichtenbaum's First Conjecture: Rings of Algebraic Integers	393
	The Tate, Beilinson and Parshin Conjectures.....	395
	Lichtenbaum's Second Conjecture: Varieties over Finite Fields.....	397
	Motivic Reformulation of the Tate and Beilinson Conjectures	398
	Lichtenbaum's Weil-étale Topology; Reformulation of his Second Conjecture.....	404
4.8	<i>Unconditional Results:</i> <i>Varieties of Abelian Type over Finite Fields</i>	410
	Main Result	410
	The Soulé–Geisser Argument	411
	The Kimura–O'Sullivan Theory	412
	The Proof	413
	Justification of Example 75 1)	413
4.9	<i>Questions and Speculations</i>	415
	The Finite Generation Issue	415
	Characteristic 0.....	417
	<i>References</i>	418

Introduction

Warning: This chapter is full of conjectures. If you are allergic to them it may be harmful to your health. Parts of them are proven, though.

In algebraic geometry, one encounters two important kinds of objects: vector bundles and algebraic cycles. The first lead to algebraic K -theory while the second lead to motivic cohomology. They are related via the Chern character and Atiyah–Hirzebruch-like spectral sequences.

Vector bundles and algebraic cycles offer very different features. On the one hand, it is often more powerful and easier to work with vector bundles: for example Quillen’s localisation theorem in algebraic K -theory is considerably easier to prove than the corresponding localisation theorem of Bloch for higher Chow groups. In short, no moving lemmas are needed to handle vector bundles. In appropriate cases they can be classified by moduli spaces, which underlies the proof of finiteness theorems like Tate’s theorem [190] and Faltings’ proof of the Mordell conjecture, or Quillen’s finite generation theorem for K -groups of curves over a finite field [66]. They also have a better functoriality than algebraic cycles, and this has been used for example by Takeshi Saito in [164, Proof of Lemma 2.4.2] to establish functoriality properties of the weight spectral sequences for smooth projective varieties over $\mathbf{Q}_p$.

On the other hand, it is fundamental to work with motivic cohomology: as E_2 -terms of a spectral sequence converging to K -theory, the groups involved contain finer torsion information than algebraic K -groups (see Remark 3), they appear naturally as Hom groups in triangulated categories of motives and they appear naturally, rather than K -groups, in the arithmetic conjectures of Lichtenbaum on special values of zeta functions.

In this survey we shall try and clarify for the reader the interaction between these two mathematical objects and give a state of the art of the (many) conjectures involving them, and especially the various implications between these conjectures. We shall also explain some unconditional results.

Sections 4.2 to 4.5 are included for the reader’s convenience and are much more developed in other chapters of this Handbook: the reader is invited to refer to those for more details. These sections are also used for reference purposes. The heart of the chapter is in Sects. 4.7 and 4.8: in the first we try and explain in much detail the conjectures of Soulé and Lichtenbaum on the order of zeroes and special values of zeta functions of schemes of finite type over $\text{Spec } \mathbf{Z}$, and an approach to prove them, in characteristic p (we don’t touch the much more delicate Beilinson conjectures on special values of L -functions of $\mathbf{Q}$ -varieties and their refinements by Bloch–Kato and Fontaine–Perrin–Riou; these have been excellently exposed in many places of the literature anyway). In the second, we indicate some cases where they can be proven, following [95].

There are two sources for the formulation of (ii) in Theorem 54. One is Soulé’s article [176] and the article of Geisser building on it [56], which led to the equivalence of (ii) with (i). The other source is the formulation of the Beilinson–Lichtenbaum conjecture and its treatment by Suslin–Voevodsky in (the first version of) [187],

which led to the equivalence of (ii) with (iii). As the Bloch–Kato conjecture and the Beilinson–Lichtenbaum conjecture both take their roots in number theory and arithmetic geometry, it is a bit of a paradox that they seem irrelevant to the conjectures on special values of zeta functions after all: see the discussion in Sect. 4.9.1.

I have tried to keep the exposition as light as possible without sacrificing rigour. This means that many proofs are omitted and replaced by references to the literature. Many others are just sketches, although hopefully these should be enough for the interested reader to reconstitute them. Some proofs are complete, though. I hope this will not create any frustration or confusion.

I wish to thank Yves André, Antoine Chambert-Loir, Pierre Colmez, Thomas Geisser, Dan Grayson, Manfred Kolster, Max Karoubi, Fabien Morel, Thong Nguyen Quang Do, Joël Riou, Jean-Pierre Serre, V. Srinivas, Chuck Weibel and the referee for helpful comments on the first versions of this text. This work was partly revised during a stay at TIFR, Mumbai, under CEFIPRA Project 2501-1 *Algebraic groups in arithmetic and geometry*, and I gratefully acknowledge its support.

4.2

The Picture in Algebraic Topology

(For complements on this section, see Geisser’s chapter, §4, Karoubi’s chapter in this Handbook and also [99, ch. V].)

The picture in algebraic topology is quite simple: singular cohomology and complex K -theory are related both via the *Atiyah–Hirzebruch spectral sequence* and via the *Chern character*. The latter lets the Atiyah–Hirzebruch spectral sequence degenerate rationally, something the Adams operations also perform.

More precisely, let X be a reasonable topological space: say, X has the homotopy type of a CW -complex. The *singular cohomology* of X , $H^*(X, \mathbf{Z})$, is the cohomology of the cochain complex $\text{Hom}(C_*(X), \mathbf{Z})$, where

$$C_i(X) = \mathbf{Z} [\mathcal{C}^0(\Delta^i, X)] ,$$

where $\mathcal{C}^0$ denotes continuous functions and Δ^i is the standard i -simplex. The differential $C_i(X) \rightarrow C_{i-1}(X)$ is defined as the alternating sum of the restrictions of a given map to $(i-1)$ -dimensional faces. The functors $H^i(-, \mathbf{Z})$ are representable in the sense that

$$H^i(X, \mathbf{Z}) = [X, K(\mathbf{Z}, i)]$$

where $K(\mathbf{Z}, i)$ is the i -th *Eilenberg–Mac Lane space* of $\mathbf{Z}$.

On the other hand, complex K -theory of X may be defined as

$$K^i(X) = \begin{cases} [X, \mathbf{Z} \times BU] & \text{if } i \text{ is even} \\ [X, U] & \text{if } i \text{ is odd} , \end{cases}$$

where U is the infinite unitary group. Bott periodicity gives canonical isomorphisms $K^i(X) \simeq K^{i+2}(X)$. If X is compact, $K^0(X)$ (stably) classifies complex vector bundles over X .

The Atiyah–Hirzebruch spectral sequence has the following form:

$$E_2^{p,q} = \begin{cases} H^p(X, \mathbf{Z}) & \text{if } q \text{ is even} \\ 0 & \text{if } q \text{ is odd} \end{cases} \Rightarrow K^{p+q}(X) \quad (4.1)$$

while the Chern character has the form

$$\begin{aligned} ch : K^0(X) \otimes \mathbf{Q} &\rightarrow \prod_{i \geq 0} H^{2i}(X, \mathbf{Q}) \\ ch : K^1(X) \otimes \mathbf{Q} &\rightarrow \prod_{i \geq 0} H^{2i+1}(X, \mathbf{Q}) . \end{aligned}$$

These are isomorphisms for X finite dimensional, and they can be used to prove that (4.1) $\otimes \mathbf{Q}$ degenerates in this case. An alternate proof is to use the *Adams operations*

$$\Psi^k : K^i(X) \rightarrow K^i(X) .$$

One shows that Ψ^k acts on (4.1) and induces multiplication by q^k on $E_2^{p,2q}$. Hence all differentials in (4.1) are torsion, with explicit upper bounds; this yields in particular the formula:

$$K^i(X)^{(j)} \simeq H^{2j+i}(X, \mathbf{Z}) \quad \text{up to groups of finite exponent} \quad (4.2)$$

where $K^i(X)^{(j)}$ stands for the common eigenspace of weight j on $K^i(X)$ for all Adams operations.

If X is a finite CW-complex, its singular cohomology $H^*(X, \mathbf{Z})$ is finitely generated. This can be proven easily by induction on the number of cells, using the well-known cohomology of spheres:

$$H^i(S^n, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = 0, n \text{ (except } i = n = 0!) \\ 0 & \text{otherwise} . \end{cases}$$

By (4.1), this implies that the groups $K^i(X)$ are also finitely generated in this case. Conversely, suppose that for a given space X we know that the $K^i(X)$ are finitely generated. Then using the partial degeneration of (4.1), we can deduce that $H^*(X, \mathbf{Z})$ is finitely generated up to some small torsion. This approach will fail if we want to get finite generation of $H^*(X, \mathbf{Z})$ on the nose, unless X has small cohomological dimension.

The Picture in Algebraic Geometry

In algebraic geometry, the picture is a bit more complicated.

Algebraic K -Theory

Historically, the theory that was defined first was *algebraic K -theory*, denoted with lower indices. The definition of $K_0(X)$ (X a scheme) goes back to Grothendieck (late fifties) and actually predates topological K -theory: $K_0(X)$ classifies algebraic vector bundles over X , the relations being given by short exact sequences. Among the many proposed definitions for the higher K -groups, the one that was the most useful was that of Quillen around 1971/72 [154]: to any noetherian scheme he associates the category $\mathcal{M}(X)$ of coherent sheaves on X and the full subcategory $\mathcal{P}(X)$ of locally free sheaves: $\mathcal{M}(X)$ is abelian and $\mathcal{P}(X)$ is an *exact* subcategory. Then

$$K_i(X) = \pi_i(\Omega BQ\mathcal{P}(X))$$

$$K'_i(X) = \pi_i(\Omega BQ\mathcal{M}(X))$$

where $Q\mathcal{E}$ is Quillen's Q -construction on an exact category $\mathcal{E}$ and B denotes the *classifying space* (or nerve) construction. For $i = 0$, $K'_0(X)$ classifies coherent sheaves on X with respect to short exact sequences, a definition which also goes back to Grothendieck. There is always a map

$$K_*(X) \rightarrow K'_*(X) ,$$

which is an isomorphism when X is regular (“Poincaré duality”).

Two fundamental additions to the foundations of algebraic K -theory were the works of Waldhausen in the early eighties [208] and Thomason–Trobaugh in the late eighties [196]. In particular, Thomason–Trobaugh slightly modifies the definition of Quillen's algebraic K -theory so as to obtain functoriality missing in Quillen's definition. His K -groups will be denoted here by K^{TT} to distinguish them from those of Quillen: there is always a map

$$K_*(X) \rightarrow K_*^{TT}(X)$$

and this map is an isomorphism as soon as X has an ample family of vector bundles, for example if X is quasi-projective over an affine base, or is regular.

On the other hand, motivated by Matsumoto's theorem giving a presentation of K_2 of a field k [130], Milnor introduced in [141] a graded commutative ring

$$K_*^M(k) = T(k^*)/I ,$$

where $T(k^*)$ is the tensor algebra of the multiplicative group of k and I is the two-sided ideal generated by the $x \otimes (1 - x)$ for $x \neq 0, 1$; its graded pieces are called

the *Milnor K-groups* of k . Since algebraic K -theory has a product structure, there are natural homomorphisms

$$K_i^M(k) \rightarrow K_i(k),$$

which are isomorphisms for $i \leq 2$ (the case $i = 2$ being Matsumoto's theorem) but not for $i \geq 3$. While Milnor stresses that his definition is purely ad hoc, it came as a surprise to many mathematicians that Milnor's K -groups are in fact not ad hoc at all and are fundamental objects in our story. See Theorem 2 below as a sample.

We might say that algebraic K -theory is an algebro-geometric analogue of complex K -theory. It took about 10 more years to get a correct definition of the corresponding analogue of singular cohomology for smooth schemes over a field, and a further 6 or 7 years for arbitrary schemes over a field of characteristic 0. See the beautiful introduction of [14].

However, early on, Quillen already looked for a strengthening of this analogue and conjectured the following version of an Atiyah–Hirzebruch spectral sequence:

Conjecture 1: (*Quillen Conjecture [156]*) Let A be a finitely generated regular $\mathbb{Z}$ -algebra of Krull dimension d and l a prime number. Then there exists a spectral sequence with E_2 -terms

$$E_2^{p,q} = \begin{cases} 0 & \text{if } q \text{ is odd} \\ H_{\text{ét}}^p(A[l^{-1}], \mathbb{Z}_l(i)) & \text{if } q = -2i \end{cases}$$

and whose abutment is isomorphic to $K_{-p-q}(A) \otimes \mathbb{Z}_l$ at least for $-p - q \geq 1 + d$.

In this conjecture, the étale cohomology groups may be defined as inverse limits of étale cohomology groups with finite coefficients; they coincide with the continuous étale cohomology groups of Dwyer–Friedlander [44] and Jannsen [76] by Deligne's finiteness theorems [SGA 4 1/2, Th. finitude]. Note that if A is a ring of integers of a number field, then such a spectral sequence must degenerate for cohomological dimension reasons when l is odd or A is totally imaginary, as pointed out by Quillen.

The Quillen conjecture and a complementary conjecture of Lichtenbaum relating algebraic K -theory and the Dedekind zeta function when A is the ring of integers of a number field (see Conjecture 46 below) have inspired much of the development of the arithmetico-geometric side of algebraic K -theory ever since. For the benefit of the reader, let us shortly describe the progress towards this still unproven conjecture:

1. In [155] and [66], Quillen proved that the K -groups of rings of algebraic integers or of smooth curves over a finite field are finitely generated.
2. In [173], following a suggestion of Quillen, Soulé constructed higher Chern classes from algebraic K -theory with finite coefficients to étale cohomology. He proved that the corresponding l -adic homomorphisms are surjective up

to finite torsion in the case of rings of integers in a global field. This gave the first application of algebraic K -theory to an algebraic problem outside the field (finiteness of certain l -adic cohomology groups).

3. In [48] and [49], Friedlander introduced étale K -theory in the geometric case, inspired by ideas from Artin–Mazur’s étale homotopy (compare [8, p. 0.5]). He formulated a conjecture in [49] for complex varieties, related to Quillen’s Conjecture 1 and that he christened the Quillen–Lichtenbaum conjecture.
4. In [44], Dwyer and Friedlander defined continuous étale cohomology and [continuous] étale K -theory in full generality. They then did two quite different things:
 - a) They constructed a spectral sequence with E_2 -terms the former, converging to the latter, for any $\mathbf{Z}[t^{-1}]$ -scheme of finite étale cohomological l -dimension.
 - b) They defined a natural transformation

$$K_i(X) \otimes \mathbf{Z}_l \rightarrow K_i^{\text{ét}}(\widehat{X}) \quad (4.3)$$

(where the right hand side is l -adic étale K -theory) and proved that this map is surjective when X is the spectrum of a ring of integers of a global field.

This last result refined the result of Soulé, because in this case the spectral sequence of a) degenerates for dimension reasons. They reinterpreted Quillen’s conjecture by conjecturing that the version of (4.3) with finite coefficients is an isomorphism for i large enough, and christened this the “Lichtenbaum–Quillen conjecture”.

5. In [43], Dwyer, Friedlander, Snaith and Thomason introduced algebraic K -theory with the Bott element inverted, proved that it maps to a version of étale K -theory and that, under some technical assumptions, this map is surjective. So “algebraic K -theory eventually surjects onto étale K -theory”. (To the best of my knowledge, one can trace back the idea of using roots of unity to define an algebraic version of the Bott element, and to invert it, to Snaith [172].)
6. In [193], Thomason went a step further by showing that, at least for nice enough schemes, étale K -theory is obtained from algebraic K -theory by “inverting the Bott element”. Hence, in this case, the spectral sequence of 4 a) converges to something close to algebraic K -theory. This refined the Dwyer–Friedlander result a). (Actually, Thomason constructs *a priori* a spectral sequence like that one converging to K -theory with the Bott element inverted, and uses it to show that this coincides with Dwyer–Friedlander’s étale K -theory.)
7. Meanwhile, the Milnor conjecture and the more general Bloch–Kato conjecture (see Conjecture 16 below) showed up. The latter was proven in degree 2 by Merkurjev–Suslin [132]: this was the second application of algebraic K -theory to an algebraic problem outside algebraic K -theory (structure of central simple algebras). Then it was proven in degree 3 for $l = 2$ by Merkurjev–Suslin [134] and independently Rost [161].

8. At the same time, Merkurjev–Suslin [133] and Levine [113] independently studied the indecomposable part of K_3 of a field F (i.e. $K_3(F)/K_3^M(F)$). This was the first instance of work in the direction “the Bloch–Kato conjecture implies the Beilinson–Lichtenbaum conjecture”.
9. In [114], Levine went a step further and proved that a form of the Bloch–Kato conjecture for (possibly singular) semi-local rings implies a form of the Quillen–Lichtenbaum conjecture, expressed in terms of Soulé’s higher Chern classes.
10. In [83] and [85], Kahn introduced anti-Chern classes going from étale cohomology to algebraic K -theory and étale K -theory, defined when the Bloch–Kato conjecture is true; he recovered the results of Dwyer–Friedlander in this way.
11. In [74] (unfortunately unpublished), Hoobler proved that the Bloch–Kato conjecture for *regular* semi-local rings implies the same conjecture for arbitrary semi-local rings. A previous argument of Lichtenbaum [125], relying on Gersten’s conjecture, showed that the Bloch–Kato conjecture for regular semi-local rings of geometric origin follows from the Bloch–Kato conjecture for fields.
12. Meanwhile, motivic cohomology started being introduced, first by Bloch and then by Friedlander, Suslin and Voevodsky. Spectral sequences from motivic cohomology to algebraic K -theory were constructed by Bloch–Lichtenbaum [22], Friedlander–Suslin [51] and Levine [119], and with different ideas by Grayson [67] and Hopkins–Morel [75].
13. In [187], Suslin and Voevodsky formulated a Beilinson–Lichtenbaum conjecture for motivic cohomology (see Conjecture 17 below) and proved that, under resolution of singularities, it follows from the Bloch–Kato conjecture. In [61] and [62], Geisser and Levine removed the resolution of singularities assumption and also covered the case where the coefficients are a power of the characteristic.
14. Voevodsky proved the Bloch–Kato conjecture at the prime 2 [204] and conditionally at any prime [206].
15. Following this, the Quillen–Lichtenbaum conjecture at the prime 2 was proven by various authors [88, 91, 121, 152, 160]. Conditionally, the same proofs work at an odd prime (and are in fact simpler). If one had finite generation results for motivic cohomology, Conjecture 1 would follow from all this work.

Ironically, Thomason strongly disbelieved the Bloch–Kato Conjecture [195, p. 409], while it was the key result that led to proofs of the Quillen–Lichtenbaum conjecture!

This concludes our short and necessarily incomplete survey. More details on parts of it will be found in the next sections.

Bloch’s Cycle Complexes

4.3.2

See §2 and Appendix in Geisser’s chapter, §§7, 8 in Grayson’s chapter and §2.3 in Levine’s chapter for more details in this subsection.

For X a quasi-projective scheme over a field k , Bloch introduced the *cycle complexes* in 1984 [20]. Denote by $\Delta^\bullet$ the standard cosimplicial scheme over k with

$$\Delta^p = \operatorname{Spec} k[t_0, \dots, t_p] / \left(\sum t_i - 1 \right) .$$

We define the *homological cycle complex of dimension n* of X as the chain complex $z_n(X, *)$ associated to the simplicial abelian group $z_n(X, \bullet)$, where, for all p , $z_n(X, p)$ is the group of cycles of dimension $n + p$ on $X \times \Delta^p$ meeting all faces properly; the faces and degeneracies are induced by those of $\Delta^\bullet$. The homology groups of $z_n(X, *)$ are called the *higher Chow groups* of X :

$$CH_n(X, p) = H_p(z_n(X, *)) .$$

These groups are 0 for $p < 0$ or $n + p < 0$ by dimension reasons. Beyond these trivial examples, let us give two other characteristic ones. First, for $p = 0$, one recovers the classical Chow group $CH_n(X)$, as is easily seen: this justifies the terminology. Less easy is the following theorem, due independently to Nesterenko–Suslin and Totaro, when $X = \operatorname{Spec} k$ for k a field.

2

Theorem 2: ([149, 197]) $CH_{-n}(k, n) \simeq K_n^M(k)$.

Higher Chow groups form, not a cohomology theory, but a *Borel–Moore homology theory* on k -schemes of finite type. For example, if Z is a closed subset of X with open complement U , then one has a long *localisation exact sequence* [20, 21]

$$\dots \rightarrow CH_n(Z, p) \rightarrow CH_n(X, p) \rightarrow CH_n(U, p) \rightarrow CH_n(Z, p-1) \rightarrow \dots \quad (4.4)$$

This is a hard theorem. Using it, one derives Mayer–Vietoris long exact sequences for open covers, which allows one to extend the definition of higher Chow groups to arbitrary k -schemes X essentially of finite type by the formula

$$CH_n(X, p) = \mathbb{H}_{\operatorname{Zar}}^{-p}(X, \underline{z}_n(-, *)) ,$$

where $\underline{z}_n(-, *)$ is the sheaf of complexes associated to the presheaf $U \mapsto z_n(U, *)$.

Even harder is the “Atiyah–Hirzebruch” spectral sequence (Bloch–Lichtenbaum [22], Friedlander–Suslin [51], Levine [119, 122])

$$E_{p,q}^2 = H_p^{BM}(X, \mathbf{Z}(-q/2)) \Rightarrow K'_{p+q}(X) , \quad (4.5)$$

where we have renumbered the higher Chow groups by the rule

$$H_p^{BM}(X, \mathbf{Z}(n)) := CH_n(X, p - 2n)$$

and $H_p^{BM}(X, \mathbf{Z}(-q/2))$ is defined to be 0 if q is odd.

If X is smooth of pure dimension d , we again change the notation by setting

$$\begin{aligned} CH^n(X, p) &:= CH_{d-n}(X, p) \\ H^p(X, \mathbf{Z}(n)) &:= CH^n(X, 2n - p) = H_{2d-p}^{BM}(X, \mathbf{Z}(d - n)) . \end{aligned}$$

We then extend this definition by additivity to the case where X is not equidimensional. Given the isomorphism $K_*(X) \xrightarrow{\sim} K'_*(X)$, the spectral sequence (4.5) may now be rewritten

$$E_2^{p,q} = H^p(X, \mathbf{Z}(-q/2)) \Rightarrow K_{-p-q}(X) \quad (4.6)$$

resembling (4.1) closely. For future reference, let us record the mod n version of this spectral sequence:

$$E_2^{p,q} = H^p(X, \mathbf{Z}/n(-q/2)) \Rightarrow K_{-p-q}(X, \mathbf{Z}/n) . \quad (4.7)$$

Rather easier to prove are the Chern character isomorphisms (Bloch [20], Levine [116, 117])

$$K'_i(X) \otimes \mathbf{Q} \xrightarrow{\sim} \bigoplus_{n \in \mathbf{Z}} H_{2n-i}^{BM}(X, \mathbf{Q}(n)) \quad (4.8)$$

$$K_i(X) \otimes \mathbf{Q} \xrightarrow{\sim} \bigoplus_{n \in \mathbf{Z}} H^{2n-i}(X, \mathbf{Q}(n)) \quad (X \text{ smooth}) . \quad (4.9)$$

They might be used to prove the degeneration of (4.5) and (4.6) up to small torsion, although I don't think this has been done yet. However, one may use Adams operations for this, just as in topology (Soulé [178], Gillet–Soulé [65, §7]), which yields the formula analogous to (4.2):

$$K_i(X)^{(j)} \simeq H^{2j-i}(X, \mathbf{Z}(i)) \quad (X \text{ smooth}) \quad (4.10)$$

$$K'_i(X)_{(j)} \simeq H_{2j+i}^{BM}(X, \mathbf{Z}(i)) \quad (4.11)$$

up to groups of finite exponent, where $K'_i(X)_{(j)}$ are the eigenspaces of the homological Adams operations [177, Th. 7 p. 533].

The above picture may be extended to schemes of finite type (*resp.* regular of finite type) over a Dedekind scheme (Levine [119, 121], see also Geisser [59] and §3.3 of Geisser's chapter in this Handbook).

Remark 3 Let $f : X \rightarrow Y$ be a morphism of schemes, taken smooth over a field to fix ideas. Suppose that f induces an isomorphism on motivic cohomology groups. Then the spectral sequence (4.6) shows that f also induces an isomorphism on K -groups. By (4.10), the converse is true up to small torsion, but I doubt that it is true on the nose, except in small-dimensional cases. The situation is quite similar to that one in Thomason's proof of absolute cohomological purity for étale

cohomology with torsion coefficients [194]: Thomason's proof gave the result only up to small torsion, and it required delicate further work by Gabber to get the theorem in its full precision (see Fujiwara's exposition of Gabber's proof in [53]).

4.3.3 Suslin–Voevodsky's Motivic Cohomology

See §2.4 in Levine's chapter and also Friedlander's Bourbaki talk [50] for more details on this subsection.

Of course, defining a cohomology theory on smooth schemes as a renumbering of a Borel–Moore homology theory is a kind of a cheat, and it does not define motivic cohomology for non-smooth varieties. Friedlander, Suslin and Voevodsky (1987–1997, [52, 186, 187, 199]) have associated to any scheme of finite type X over a field k a *motivic complex* of sheaves $C_*(X) = C_*(L(X))$ on the category Sm/k of smooth k -varieties provided with the Nisnevich topology, where, for $U \in Sm/k$, $L(X)(U)$ is the free abelian group with basis the closed integral subschemes of $X \times U$ which are finite and surjective over a component of U , and for a presheaf $\mathcal{F}$, $C_*(\mathcal{F})$ is the complex of presheaves with $C_n(\mathcal{F})$ defined by

$$C_n(\mathcal{F})(U) = \mathcal{F}(U \times \Delta^n).$$

Then they define for each n a sheaf $L(\mathbb{G}_m^{\wedge n})$ as the cokernel of the map

$$\bigoplus_{i=1}^{n-1} L(\mathbb{G}_m^{n-1}) \rightarrow L(\mathbb{G}_m^n)$$

induced by the embeddings of the form

$$(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, 1, \dots, x_{n-1})$$

(it is in fact a direct summand). Finally, they set $\mathbf{Z}_{SV}(n) := C_*(L(\mathbb{G}_m^{\wedge n}))[-n]$ (the index SV is ours, in order to avoid notational confusion).

If X is smooth, then there are canonical isomorphisms [201]

$$H^i(X, \mathbf{Z}(n)) \simeq \mathbb{H}_{\text{Nis}}^i(X, \mathbf{Z}_{SV}(n)).$$

However, in general one does not expect the right-hand-side group to have good functorial properties. For this, one has to replace the Nisnevich topology by the stronger cdh topology. If $\text{char } k = 0$, resolution of singularities implies that, for X smooth, the natural maps

$$H_{\text{Nis}}^i(X, \mathbf{Z}_{SV}(n)) \rightarrow \mathbb{H}_{\text{cdh}}^i(X, \mathbf{Z}_{SV}(n)) \quad (4.12)$$

are isomorphisms [199]. However this is not known in characteristic p . One may therefore say that the Suslin–Voevodsky approach yields the “correct” motivic cohomology for all schemes in characteristic 0, but does so only conjecturally

in characteristic p , because (unlike Bloch's approach) it relies fundamentally on resolution of singularities. For this reason and others, we shall mainly work with Bloch's cycle complexes in the sequel.

Using these ideas, Suslin has recently proven that a spectral sequence constructed by Grayson [67] based on ideas of Goodwillie and Lichtenbaum and converging to algebraic K -theory for X smooth semi-local essentially of finite type over a field has its E_2 -terms isomorphic to motivic cohomology [185]. Thus we get a spectral sequence like (4.6), independent of the Bloch–Lichtenbaum construction. It is not clear, however, that the two spectral sequences coincide.

One cannot expect a spectral sequence like (4.6) for arbitrary schemes of finite type, even over a field of characteristic 0, nor Chern character isomorphisms as (4.9). Indeed, motivic cohomology is homotopy invariant while algebraic K -theory is not. One can however expect that (4.6) and (4.9) generalise to arbitrary schemes of finite type X by replacing algebraic K -theory by Weibel's homotopy invariant algebraic K -theory $KH(X)$ [210]. This has been done recently in characteristic 0 by Christian Haesemeyer, who produced a spectral sequence (reproduced here up to the indexing)

$$H^p(X, \mathbf{Z}(-q/2)) \Rightarrow KH_{-p-q}(X) \quad (4.13)$$

for X of finite type over a field of characteristic 0 [70, Th. 7.11]. This goes some way towards the following general conjecture:

Conjecture 4: (*cf. Beilinson [12, 5.10 B and C (vi)]*) Let $m \geq 1$ and let X be a Noetherian separated $\mathbf{Z}[1/m]$ -scheme of finite Krull dimension. Then there is a spectral sequence

$$E_2^{p,q} = H_{\text{Zar}}^p(X, B/m(-q/2)) \Rightarrow K_{-p-q}^{TT}(X, \mathbf{Z}/m)$$

degenerating up to small torsion.

Here $B/m(n) = \tau_{\leq n} R\alpha_* \mu_m^{\otimes n}$ where α is the projection of the big étale site of $\text{Spec } \mathbf{Z}[1/m]$ onto its big Zariski site, and K^{TT} is Thomason–Trobeaugh K -theory, *cf.* 4.3.1. (Note that $K_*^{TT}(X, \mathbf{Z}/m)$ is homotopy invariant: Weibel [209, Cons. 1.1], Thomason [196, Th. 9.5 a].) See Corollary 22 below for an explanation of this conjecture, and Theorem 32 for evidence to it.

The Beilinson–Soulé Conjecture

4.3.4

As a basic difference between algebraic topology and algebraic geometry, the analogue of the following conjecture is trivially true for singular cohomology:

Conjecture 5: (*Beilinson–Soulé Conjecture [10], [177, Conj. p. 501]*) For X regular, $H^i(X, \mathbf{Z}(n)) = 0$ for $n \geq 0$ and $i < 0$ (even for $i = 0$ when $n > 0$).

5

This conjecture is central in the developments of the theory of motives, and we shall come back to it in this survey every now and then.

Let us toy with the Beilinson–Soulé conjecture a little. Let Z be a regular closed subset of X of codimension c and U be the open complement. The Gysin exact sequence for motivic cohomology (an equivalent form of (4.4)) reads

$$\begin{aligned} \cdots \rightarrow H^{i-2c}(Z, \mathbf{Z}(n-c)) \rightarrow H^i(X, \mathbf{Z}(n)) \rightarrow H^i(U, \mathbf{Z}(n)) \\ \rightarrow H^{i-2c+1}(Z, \mathbf{Z}(n-c)) \rightarrow \cdots \end{aligned} \quad (4.14)$$

Suppose we have found an inductive proof of the conjecture: induction could be either on n or on $\dim X$, or on both. In each case we find inductively that the map $H^i(X, \mathbf{Z}(n)) \rightarrow H^i(U, \mathbf{Z}(n))$ is an isomorphism. On the other hand, motivic cohomology transforms filtering inverse limits of regular schemes with affine transition morphisms into direct limits. From this, one deduces easily:

6

Lemma 6 The following conditions are equivalent:

- (i) The Beilinson–Soulé conjecture is true.
- (ii) The Beilinson–Soulé conjecture is true for all fields.
- (iii) The Beilinson–Soulé conjecture is true for all finitely generated fields.
- (iv) The Beilinson–Soulé conjecture is true for all regular schemes of finite type over $\mathrm{Spec} \mathbf{Z}$. □

If one inputs Hironaka’s resolution of singularities or de Jong’s alteration theorems [82, Th. 4.1 and 8.2], one gets stronger results:

7

Lemma 7 a) If we restrict to regular schemes over $\mathbf{Q}$, the following condition is equivalent to those of the previous lemma:

- (v) The Beilinson–Soulé conjecture is true for all smooth projective varieties over $\mathrm{Spec} \mathbf{Q}$.
- b) If we restrict to regular schemes over $\mathbf{F}_p$ and tensor groups with $\mathbf{Q}$, the following condition is equivalent to those of the previous lemma:
- (vi) The Beilinson–Soulé conjecture is true for all smooth projective varieties over $\mathrm{Spec} \mathbf{F}_p$.
- c) If we tensor groups with $\mathbf{Q}$, the following condition is equivalent to those of the previous lemma:
- (vii) The Beilinson–Soulé conjecture is true for all regular projective schemes over $\mathrm{Spec} \mathbf{Z}$, generically smooth over a suitable ring of integers of a number field and with strict semi-stable reduction outside the smooth locus.

The dévissage arguments for this are standard and I shall only recall them sketchily: for more details see Geisser’s survey article on applications of de Jong’s theorem [57], or also [94]. There are two main steps:

1. Given X regular and $Z \subset X$ closed regular with open complement U , the exact sequence (4.14) shows inductively that the conjecture is true for X if and only if it is true for U . If U is now any open subset of X , a suitable stratification of $X - U$ reduces us to the first case, in characteristic p because finite fields are perfect and over $\mathbf{Z}$ by [EGA IV, cor. 6.12.6].
2. By Hironaka in characteristic 0, any smooth variety X contains an open subset U which is an open subset of a smooth projective variety. By de Jong in characteristic p (*resp.* over $\mathbf{Z}$), any regular X contains an U such that there exists a finite étale map $f : \tilde{U} \rightarrow U$ such that $\tilde{U}$ is contained in a smooth projective variety (*resp.* in a scheme as in (vii)). A transfer argument finishes the proof (and uses coefficients $\mathbf{Q}$). $\square$

We shall see in Sect. 4.5.3 that tensoring groups with $\mathbf{Q}$ is not a very serious restriction by now.

The Beilinson–Soulé conjecture is true for $n = 0$ because $\mathbf{Z}(0) = \mathbf{Z}$ and for $n = 1$ because $\mathbf{Z}(1) \simeq \mathbb{G}_m[-1]$. It is also true for X finitely generated over $\mathbf{Z}$ of Krull dimension ≤ 1 (see Lemma 41), and for some smooth projective varieties over $\mathbf{F}_p$ (see Sect. 4.8). Although some mathematicians have doubted its validity in recent years, it is my belief that it is true in general and will be proven by arithmetic means, analytic means or a combination of both. Evidence for this will appear in Sects. 4.7 and 4.8.

Finally, there is a companion conjecture to the Beilinson–Soulé conjecture. For fields, it was formulated in weight 2 by Merkurjev and Suslin [134, Conj. 11.7], and we take the liberty to attribute it to them in general.¹

Conjecture 8: (*Merkurjev–Suslin Conjecture*) For any regular scheme X , let X_0 be the “scheme of constants” of X , that is, the normalisation of $\mathrm{Spec} \mathbf{Z}$ into X (for example if $X = \mathrm{Spec} F$ with F a field, then $X_0 = \mathrm{Spec} F_0$ where F_0 is the algebraic closure in F of its prime subfield). Then for all $n \geq 2$, the map $H^1(X_0, \mathbf{Z}(n)) \rightarrow H^1(X, \mathbf{Z}(n))$ is an isomorphism.

8

The same reductions as for the Beilinson–Soulé conjecture apply to this conjecture.

¹ In this we follow a well-established tradition in algebraic K -theory which consists in attributing conjectures to people who did not really formulate them in those terms. For example, Beilinson and Soulé did not actually formulate Conjecture 5 as it stands, because at the time motivic cohomology had not been defined. However they formulated it in terms of the gamma filtration on algebraic K -theory and Beilinson envisioned a motivic cohomology which would explain this conjecture. Similarly, the Beilinson–Lichtenbaum conjecture 17 was formulated by Beilinson and Lichtenbaum as a conjecture about a yet conjectural motivic cohomology. One last example is the name “Quillen–Lichtenbaum” given to the conjecture asserting that algebraic and étale K -theories with finite coefficients should agree in large enough degrees, while étale K -theory had been neither invented at the time when they made the corresponding conjectures relating algebraic K -theory and étale cohomology, nor even envisioned by them!

4.4 Review of Motives

The aim of this section is to present a state-of-the-art description of the current understanding of the “theory of motives”, whose idea is fully due to Grothendieck in the mid sixties (see [69]). This description will be very sketchy: a thorough one would go far beyond the scope of this survey.

For an overlapping but different and much more detailed survey, we invite the reader to consult Marc Levine’s chapter in this Handbook.

We work over a base field k . We won’t enter the description of categories of motives over other bases.

4.4.1 Pure Motives

For more background on this subsection, we refer the reader to [129], [163, Ch. VI §4], [41], [108], [167] and [3].

To define a category of pure motives, one needs

1. a commutative ring A of coefficients;
2. an “adequate” equivalence relation $\sim$ on algebraic cycles (on smooth projective varieties) with coefficients in A : roughly, modulo $\sim$, direct and inverse images and intersection products can be defined.

We shall refer to a pair $(A, \sim)$ as above as to an *adequate pair*. For X smooth projective, the groups of cycles on X modulo $\sim$ will be denoted by $\mathcal{Z}_{\sim}^*(X, A)$.

The finest adequate equivalence relation is rational (= linear) equivalence rat , and when A contains $\mathbf{Q}$, the coarsest one is numerical equivalence num . Between the two, one finds other interesting adequate equivalence relations (in increasing order of coarseness):

- algebraic equivalence;
- Voevodsky’s smash-nilpotence equivalence [198] (the observation that it defines an adequate equivalence relation is due to Y. André);
- homological equivalence relative to a given Weil cohomology theory.

By definition, one has $\mathcal{Z}_{\text{rat}}^*(X, \mathbf{Z}) = CH^*(X)$.

Now, given an adequate pair $(A, \sim)$, one first constructs the *category of $(A, \sim)$ -correspondences* $\text{Cor}_{\sim}(k, A)$. Objects are smooth projective k -varieties; for X smooth projective we denote by $[X]$ its image in $\text{Cor}_{\sim}(k, A)$. For X, Y two smooth projective varieties one defines

$$\text{Hom}([X], [Y]) = \mathcal{Z}_{\sim}^{\dim Y}(X \times Y, A) .$$

Composition of correspondences $\alpha : X \rightarrow Y$ and $\beta : Y \rightarrow Z$ is given by the semipiternal formula

$$\beta \circ \alpha = (p_{XZ})_*(p_{XY}^* \alpha \cdot p_{YZ}^* \beta) .$$

For a morphism $f : X \rightarrow Y$, denote by Γ_f its graph, viewed as an algebraic cycle on $X \times Y$. Then $f \mapsto [\Gamma_f]$ defines a functor $\text{SmProj}(k) \rightarrow \text{Cor}_{\sim}(k, A)$. One should be careful that this functor is *covariant* here, which is the convention in Fulton [54] and Voevodsky [199] but is opposite to that of Grothendieck.

We may put on $\text{Cor}_{\sim}(k, A)$ a symmetric monoidal structure by the rule $[X] \otimes [Y] = [X \times Y]$, the tensor product of correspondences being given by their external product. Then $\mathbf{1} = h(\text{Spec } k)$ is the unit object of this structure.

Once the category of correspondences is defined, we get to the category of pure motives by two formal categorical operations:

- **Effective pure motives:** take the pseudo-abelian (karoubian) envelope of $\text{Cor}_{\sim}(k, A)$. This amounts to formally adjoin kernels to idempotent endomorphisms. The resulting category $\text{Mot}_{\sim}^{\text{eff}}(k, A)$ is still monoidal symmetric. We write $h(X)$ for the image of $[X]$ in $\text{Mot}_{\sim}^{\text{eff}}(k, A)$.
- **Pure motives:** in $\text{Mot}_{\sim}^{\text{eff}}(k, A)$, we have a decomposition $h(\mathbf{P}^1) = \mathbf{1} \oplus L$ given by the choice of any rational point: L is called the *Lefschetz motive*. Tensor product by L is fully faithful. We may then formally invert L for the monoidal structure: the resulting category $\text{Mot}_{\sim}(k, A)$ inherits a symmetric monoidal structure because the permutation (123) acts on $L^{\otimes 3}$ by the identity (this necessary and sufficient condition which does not appear for instance in Saavedra [163] was first noticed by Voevodsky)². Here too we shall depart from a more traditional notation by writing $M(n)$ for $M \otimes L^{\otimes n}$: this object is usually written $M(-n)$ in the Grothendieck tradition.

The category $\text{Mot}_{\sim}(k, A)$ is *rigid*: any object M has a dual $M^{\vee}$ and any object is canonically isomorphic to its double dual. If X has pure dimension d , then $h(X)^{\vee} = h(X)(-d)$ and the unit and counit of the duality are both given by the class of the diagonal in $\text{Hom}(\mathbf{1}, h(X) \otimes h(X)^{\vee}) \simeq \text{Hom}(h(X)^{\vee} \otimes h(X), \mathbf{1}) = \mathbb{Z}_{\sim}^d(X \times X, A)$. All this theory is purely formal beyond the projective line formula:

$$\mathbb{Z}_{\sim}^n(X \times \mathbf{P}^1, A) = \mathbb{Z}_{\sim}^n(X, A) \oplus \mathbb{Z}_{\sim}^{n-1}(X, A) .$$

Then one is interested in finer properties of $\text{Mot}_{\sim}(k, A)$. The most important result is Jannsen's theorem, which was for a long time a standard conjecture:

Theorem 9: (Jannsen [79]) The category $\text{Mot}_{\sim}(k, \mathbb{Q})$ is semi-simple if and only if $\sim = \text{num}$.

9

What we don't Know about Pure Motives

What is missing from a rigid tensor category to be tannakian (hence to be classified by a gerbe) is a *fibre functor* (detailing these notions would go beyond the scope of this survey and we can only refer the reader to the excellent refer-

² In fact we even have that the permutation (12) acts on $L^{\otimes 2}$ by the identity.

ences [163], [39], [28]). Morally, such a fibre functor on $\text{Mot}_{\text{num}}(k, \mathbf{Q})$ should be given by any Weil cohomology theory H . However there are two unsolved problems here:

- We don't know whether homological equivalence (relative to H) equals numerical equivalence.
- Even so, there is a more subtle problem that the category of pure motives as defined above is “false”: it cannot be tannakian! Namely, let $\dim M$ be the “rigid dimension” of an object M , defined as the trace of 1_M . In any tannakian category, the dimension is a nonnegative integer because it can be computed as the dimension of a vector space via a fibre functor. Now if $M = h(X)$, computing $\dim X$ via H gives $\dim X = \chi(X)$, the Euler–Poincaré characteristic of X , which can be any kind of integer (it is $2 - 2g$ if X is a curve of genus g).

The second problem is a matter of the commutativity constraint in $\text{Mot}_H(k, \mathbf{Q})$. To solve it, one looks at the *Künneth projectors*, i.e. the projectors π_i from $H^*(X)$ onto its direct summands $H^i(X)$. In case the π_i are given by algebraic correspondences, one can use them to modify the commutativity constraint (by changing signs) and transform the Euler–Poincaré characteristic into the sum of dimensions of the $H^i(X)$, which is certainly a nonnegative integer. To do this, it suffices that the sum of the π_{2i} be algebraic.

The two conjectures:

(HN) Homological and numerical equivalences coincide

(C) For any X and H , the Künneth projectors are algebraic

are part of Grothendieck's *standard conjectures*³ [68]. This is not the place to elaborate on them: see also [107], [109], [163, p. 380] and [2, §1 and Appendix] for more details. Let us just mention that there is another conjecture B (the “Lefschetz-type conjecture”) and that

$$\text{Conjecture HN} \Rightarrow \text{Conjecture B} \Rightarrow \text{Conjecture C},$$

where the first implication is an equivalence in characteristic 0.

Under these conjectures, one can modify $\text{Mot}_{\text{num}}(k, \mathbf{Q})$ and make it a tannakian category, semi-simple by Theorem 9. To the fibre functor H will then correspond its “motivic Galois group” (*ibid.*), a proreductive group defined over the field of coefficients of H , that one can then use to do “motivic Galois theory”. Moreover the Künneth components yield a *weight grading* on $\text{Mot}_{\text{num}}(k, \mathbf{Q})$.

Lieberman [128] proved Conjecture B for abelian varieties (over $\mathbf{C}$; see Kleiman [107] for a write-up over an arbitrary field), and Katz–Messing [104] proved Conjecture C when k is finite for l -adic and crystalline cohomology, as a consequence of Deligne's proof of the Weil conjecture (Riemann hypothesis) [34]. Besides these special cases and a few more easy cases, these conjectures are still open now.

³ The terminology “standard conjectures” is not limited to Grothendieck: Serre used it in [170] with a quite different, although closely related, meaning.

Suppose that k is finite, and take $H = H_l$, l -adic cohomology (where l is a prime number different from $\text{char } k$). Still by Katz–Messing, for any X the ideal

$$\text{Ker} \left(\text{End}_{\text{Mot}_H}(h(X)) \rightarrow \text{End}_{\text{Mot}_{\text{num}}}(h(X)) \right)$$

is nilpotent: this implies that the functor $\text{Mot}_H(k, \mathbb{Q}) \rightarrow \text{Mot}_{\text{num}}(k, \mathbb{Q})$ is *essentially surjective*. It follows that one can “push down” from $\text{Mot}_H(k, \mathbb{Q})$ to $\text{Mot}_{\text{num}}(k, \mathbb{Q})$ the change of commutativity constraint. Then the new $\otimes$ -category $\text{Mot}_{\text{num}}(k, \mathbb{Q})$ is semi-simple rigid and any object has a nonnegative dimension: a theorem of Deligne [SGA 4 1/2] then implies that *it is (abstractly) tannakian*. For details, see Jannsen [79].

Finally we should mention Voevodsky’s conjecture:

Conjecture 10: ([198]) Smash-nilpotence and numerical equivalence coincide.

10

It is stronger than the first standard conjecture above, and has the advantage not to single out any Weil cohomology.

Getting around the Standard Conjectures

There are two ways to make the approach above both more unconditional and more explicit. The first one was initiated by Deligne in [35] using (in characteristic 0) the notion of an *absolute Hodge cycle* on a smooth projective variety X , which is a system of cohomology classes in all “classical” cohomology theories applied to X , corresponding to each other via the comparison isomorphisms: the classes of a given algebraic cycle define an absolute Hodge cycle and the Hodge conjecture asserts that there are no others. This approach was refined and made almost algebraic by Yves André in [2]. Like Deligne, André’s idea is to adjoin cycles that one hopes eventually to be algebraic: but he just takes the inverses of the Lefschetz operators in the graded cohomology ring $H^*(X)$ (for some classical Weil cohomology H) and shows that, if $\text{char } k = 0$, he gets a semi-simple tannakian category (with fibre functor given by H), a priori with larger Hom groups than $\text{Mot}_H(k, \mathbb{Q})$.

The second and rather opposite approach, due to André and the author [5], consists of restricting *a priori* to the full $\otimes$ -subcategory $\text{Mot}_H^+(k, \mathbb{Q})$ formed of those homological motives whose even Künneth projectors are algebraic. After showing that its image $\text{Mot}_{\text{num}}^+(k, \mathbb{Q})$ in $\text{Mot}_{\text{num}}(k, \mathbb{Q})$ does not depend on the choice of a “classical” Weil cohomology H ,⁴ we show that the projection functor $\text{Mot}_H^+(k, \mathbb{Q}) \rightarrow \text{Mot}_{\text{num}}^+(k, \mathbb{Q})$ has monoidal sections, unique up to monoidal conjugation: this depends on the results of [6]. Then H defines an essentially unique fibre functor on $\text{Mot}_{\text{num}}^+(k, \mathbb{Q})$ after the suitable modification of the commutativity constraints.

⁴ If $\text{char } k = 0$ this is obvious via the comparison theorems; in positive characteristic it depends on the Weil conjectures (Riemann hypothesis).

The Conjectural Abelian Category of Mixed Motives

See §3 in Levine's chapter for more details on this subsection.

What about varieties X that are not smooth projective? Elementary cases show that their cohomology, viewed in enriched categories, is not in general semi-simple. For example, the l -adic cohomology of X is not in general semi-simple as a G_k representation. Or, if $k = \mathbb{C}$, the Betti cohomology of X is not in general a semi-simple mixed Hodge structure.

Therefore one cannot expect that general varieties are classified by a semi-simple tannakian category. One still hopes, however, that they are classified by a (not semi-simple) tannakian category $\mathrm{MMot}(k, \mathbb{Q})$: see especially [12, 5.10]. Here are some conjectural properties of this conjectural category (the list is not exhaustive of course):

- $\mathrm{MMot}(k, \mathbb{Q})$ is tannakian and every object has finite length.
- The socle of $\mathrm{MMot}(k, \mathbb{Q})$ (i.e. the full subcategory of semisimple objects) is $\mathrm{Mot}_{\mathrm{num}}(k, \mathbb{Q})$.
- There is a weight filtration on $\mathrm{MMot}(k, \mathbb{Q})$ which extends the weight grading of $\mathrm{Mot}_{\mathrm{num}}(k, \mathbb{Q})$; its associated graded (for any object) belongs to $\mathrm{Mot}_{\mathrm{num}}(k, \mathbb{Q})$.
- Any variety X has cohomology objects and cohomology objects with compact supports $h^i(X), h_c^i(X) \in \mathrm{MMot}(k, \mathbb{Q})$, with Künneth formulas; h^* is contravariant for all morphisms while h_c^* is contravariant for proper morphisms. There are canonical morphisms $h_c^i(X) \rightarrow h^i(X)$ which are isomorphisms for X proper.
- There are blow-up exact sequences for h^* , localisation exact sequences for h_c^* and Mayer–Vietoris exact sequences for both.
- For any X , the natural maps $h^*(X) \rightarrow h^*(X \times \mathbb{A}^1)$ are isomorphisms.
- If X is smooth of pure dimension d , one has canonical isomorphisms $h_c^i(X)^\vee \simeq h^{2d-i}(X)(-d)$.
- For all X, n with X smooth there is a spectral sequence $\mathrm{Ext}_{\mathrm{MMot}}^p(\mathbf{1}, h^q(X)(n)) \Rightarrow H^{p+q}(X, \mathbb{Q}(n))$; if X is smooth projective, it degenerates up to torsion and yields, for $p + q = 2n$, the famous Bloch–Beilinson–Murre filtration on Chow groups, cf. [12, 5.10 C]. (For more details on this filtration, see Jannsen [81]).

Note that the last property contains the Beilinson–Soulé Conjecture 5, since $\mathrm{Ext}^p = 0$ for $p < 0$. See Levine's chapter, Conjecture 3.4, for a slightly different set of properties.

The Nonconjectural Triangulated Categories of Mixed Motives

One expects that $\mathrm{MMot}(k, \mathbb{Q})$ will arise as the heart of a t -structure on a *tensor triangulated category*. There are currently 3 constructions of such a category:

- Hanamura’s construction [71], [72];
- Levine’s construction [118];
- Voevodsky’s construction [199].

For a discussion of the constructions of these categories and their comparisons, we refer the reader to §4 in Marc Levine’s chapter. Here are briefly some of their common features (for simplicity, let us write $D(k)$ for any of these categories):

1. $D(k)$ is a $\mathbf{Z}$ -linear tensor triangulated category; it is rigid if $\text{char } k = 0$, or after tensoring with $\mathbf{Q}$ in characteristic p (Hanamura’s category has rational coefficients anyway).
2. There is a canonical fully faithful tensor functor

$$\delta : \text{Mot}_{\text{rat}}(k, \mathbf{Z}) \rightarrow D(k) \quad (4.15)$$

(for Hanamura, tensor with $\mathbf{Q}$). The image of L under this functor is denoted by $\mathbf{Z}(1)[2]$ by Voevodsky. (Note that Voevodsky calls $\mathbf{Z}(1)$ the “Tate object”, so that δ sends the Lefschetz motive to a shift of the Tate object!)

3. Any smooth variety X (smooth projective for Hanamura) has a “motive” $M(X) \in D(k)$, which is contravariant in X in Levine and Hanamura, covariant in Voevodsky; there are Mayer–Vietoris exact triangles (for open covers) and $M(X)$ is homotopy invariant. If $\text{char } k = 0$, any variety has a motive $M(X)$ and a Borel–Moore motive $M^c(X)$; on top of the above properties, there are blow-up and localisation exact triangles. There is a canonical morphism $M(X) \rightarrow M^c(X)$ which is an isomorphism when X is proper.
4. If X is smooth of pure dimension d , there is a “Poincaré duality” isomorphism $M^c(X) \simeq M(X)^\vee(d)[2d]$.
5. For any X (smooth in characteristic p) one has canonical isomorphisms $\text{Hom}(\mathbf{Z}(p)[q], M^c(X)) = H_q^{BM}(X, \mathbf{Z}(p))$. For X smooth, one has canonical isomorphisms $\text{Hom}(M(X), \mathbf{Z}(p)[q]) = H^q(X, \mathbf{Z}(p))$ (here we take the variance of Voevodsky).

As pointed out at the end of the last subsection, the existence of a “motivic” t -structure on $D(k) \otimes \mathbf{Q}$ depends at least on the Beilinson–Soulé Conjecture 5. In [71, Part III], Hanamura gives a very nice proof of the existence of this t -structure for his category, assuming an extension of this conjecture plus Grothendieck’s standard conjectures and the Bloch–Beilinson–Murre filtration.

Naturally, in the notation of the previous subsection, one should have $h^i(X) = H^i(M(X)^\vee)$ and $h_c^i(X) = H^i(M^c(X)^\vee)$ for the motivic t -structure. The spectral sequence mentioned would just be the corresponding hypercohomology spectral sequence, and its degeneracy for smooth projective X would follow from Grothendieck’s standard conjecture B (“Lefschetz type”, see [109]) and Deligne’s degeneracy criterion [38].

What about a motivic t -structure on $D(k)$ itself? In [199, Prop. 4.3.8], Voevodsky shows that there is an obstruction for his category as soon as $cd_2(k) > 1$. On the other hand, he has an étale version $DM_{\text{gm}, \text{ét}}(k)$ of $D(k)$ [199, §3.3]. I expect

that this category does have an integrally defined motivic t -structure: see the next paragraph on Nori's category for more details. Note that the two categories coincide after tensoring morphisms by $\mathbf{Q}$ by [199, Th. 3.3.2].⁵

Besides the triangulated approach, there have been attempts to construct directly something like $\mathrm{MMot}(k, \mathbf{Q})$. The first idea was to use absolute Hodge cycles à la Deligne (Deligne [36, §1], Jannsen [78, Part I]). This is using cohomology classes rather than algebraic cycles. Another one is to try and construct at least a simpler subcategory of $\mathrm{MMot}(k, \mathbf{Q})$, like the full abelian subcategory generated by Tate motives, using algebraic cycles. This was performed by Bloch–Kriz [23]. On the other hand, Levine proved [115] that Conjecture 5 (for the motivic cohomology of k !) is sufficient to provide a motivic t -structure on the thick triangulated category of $D(k) \otimes \mathbf{Q}$ generated by the $\mathbf{Q}(n)$ (this works for any version of $D(k)$, or more generally for any tensor triangulated category satisfying suitable axioms). Quite different is Nori's approach, which is described in the next section.

4.4.4 Nori's Category

For considerably more details on this subsection, see §3.3 in Levine's chapter.

Using Betti cohomology, Madhav Nori constructs for any subfield k of $\mathbf{C}$ an abelian, $\mathbf{Z}$ -linear category of mixed motives. This exposition is based on notes from a series of lectures he gave in Bombay in 2000: any misunderstanding is of course my own responsibility.

The two fundamental building blocks of Nori's construction are:

11

Theorem 11: (*Basic lemma*) Let X be an affine k -variety of dimension d , and let Z be a closed subset of X of dimension $< d$. Then there exists a closed subset $Z' \supseteq Z$ of dimension $< d$ such that the relative cohomology groups $H^i(X, Z', \mathbf{Z})$ are 0 for $i \neq d$.

This theorem is stated and proven in [150], but Nori points out that it also follows from earlier work of Beilinson [13, Lemma 3.3], whose proof also works in positive characteristic.

For the next theorem we need the notion of a *diagram*, also called *pre-category* or *quiver*: this is the same as a category, except that composition of morphisms is not defined. Any category defines a diagram by forgetting composition, and any functor defines a morphism of diagrams. Let R be a commutative ring, D be a diagram and $T : D \rightarrow R\text{-Mod}$ a *representation* of D , that is, a morphism from D to the diagram underlying $R\text{-Mod}$. Following Nori, one enriches T as follows: if D is finite, T obviously lifts to a representation $\tilde{T} : D \rightarrow \mathrm{End}_R(T)\text{-Mod}$, where $\mathrm{End}_R(T)$ is the

⁵ Actually, in *loc. cit.* a category $DM_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}(k)$ analogous to $DM^{\mathrm{eff}}(k)$ is defined: this is a “big” category. One should probably define $DM_{\mathrm{gm}, \mathrm{ét}}^{\mathrm{eff}}(k)$ as follows: take the full subcategory of $DM_{-}^{\mathrm{eff}}(k)$ generated by the image of $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$, invert $\mathbf{Z}(1)$ and finally take the pseudo-abelian hull.

ring of R -linear natural transformations from T to itself. In general one may write D as a union of its finite subdiagrams D' , and define $\mathcal{C}(T) = \lim_{\substack{\longrightarrow \\ D'}} \text{End}_R(T|_{D'})\text{-Mod}$. Then there is an obvious forgetful functor $\omega : \mathcal{C}(T) \rightarrow R\text{-Mod}$ and T lifts through ω to

$$\tilde{T} : D \rightarrow \mathcal{C}(T). \quad (4.16)$$

The following is a universal property of (4.16):

Theorem 12: (*Nori's tannakian lemma*) Suppose R Noetherian. Let $\mathcal{A}$ be an R -linear abelian category, $\ell : \mathcal{A} \rightarrow R\text{-Mod}$ an R -linear additive faithful exact functor and $S : D \rightarrow \mathcal{A}$ a representation such that $T = \ell S$. Then there exists an R -linear exact functor $S' : \mathcal{C}(T) \rightarrow \mathcal{A}$, unique up to unique isomorphism, making the following diagram commutative up to natural isomorphism:

$$\begin{array}{ccc} & \mathcal{C}(T) & \\ \tilde{T} \nearrow & \downarrow S' & \searrow \omega \\ & \mathcal{A} & \\ S \nearrow & & \searrow \ell \\ D & \xrightarrow{T} & R\text{-Mod} \end{array}$$

Note that S' is automatically faithful since ω is.

For a proof of this theorem, see Bruguières [29].

(As Srinivas pointed out, the uniqueness statement is not completely correct. For it to be one needs at least ℓ to be “totally faithful”, a notion introduced by Nori: if an arrow goes to an identity arrow then it is already an identity arrow. This condition is basically sufficient.)

Nori then takes for D the diagram whose objects are triples (X, Y, i) where X is affine of finite type over k , Y is a closed subset and $i \geq 0$, and morphisms are (I) morphisms of triples (same i) and (II) to any chain $Z \subset Y \subset X$ and integer $i > 0$ corresponds a morphism $(X, Y, i) \rightarrow (Y, Z, i-1)$. He takes $T(X, Y, i) = H_i(X, Y, \mathbb{Z})$, and also $T^*(X, Y, i) = H^i(X, Y, \mathbb{Z})$ on the dual diagram D^* . The corresponding categories $\mathcal{C}(T)$ and $\mathcal{C}(T^*)$ are respectively called $EHM(k)$ (*effective homological motives*) and $ECM(k)$ (*effective cohomological motives*).

These categories are independent from the embedding of k into $\mathbb{C}$; $EHM(k)$ is a tensor category and enjoys exact faithful tensor functors to Galois representations, mixed Hodge structures and a category of “periods” (a *period* is a triple (M, W, ϕ) where M is a $\mathbb{Z}$ -module, W a k -vector space and ϕ an isomorphism $\mathbb{C} \otimes_{\mathbb{Z}} M \rightarrow \mathbb{C} \otimes_k W$). There is a tensor triangulated functor

$$DM_{\text{gm}}^{\text{eff}}(k) \rightarrow D^b(EHM(k)).$$

One may then define the category $M(k)$ of motives by inverting $\mathbf{Z}(1) := H_1(\mathbb{G}_m)$, exactly as for pure motives. Starting from $ECM(k)$ and $H^1(\mathbb{G}_m)$ yields the same result. This category is “neutral tannakian” in the sense that it is equivalent to the category of comodules of finite type over a certain pro-Hopf algebra... There is also a weight filtration, all this being integrally defined!

I expect $M(k)$ to be eventually equivalent to the heart of a motivic t -structure on $DM_{\text{gm}, \text{ét}}(k)$, compare the end of Sect. 4.4.3. This is evidenced by the fact that Betti cohomology compares nicely with étale cohomology with finite coefficients. Of course this issue is closely related to the Hodge conjecture.

4.4.5 $\mathbf{A}^1$ -Homotopy and Stable Homotopy Categories

Before Fabien Morel and Vladimir Voevodsky started constructing it in the early nineties, first independently and then together, nobody had thought of developing a “homotopy theory of schemes” just as one develops a homotopy theory of (simplicial) sets. In this subsection we shall give a brief outline of this theory and its stable counterpart, referring the reader to [144], [147] and [202] for details; see also the few words in [89, §§5 to 7], [200], the exposition of Joël Riou [159], [145] and the programmatic [146].

Homotopy of Schemes

There are two constructions of the $\mathbf{A}^1$ -homotopy category of k -schemes $\mathcal{H}(k)$, which can be considered as an algebro-geometric generalisation of the classical homotopy category $\mathcal{H}$: [144] and [147]. It can be shown that they are equivalent. We shall describe the second with its features, as it is the best-known anyway.

We start with the category $(Sm/k)_{\text{Nis}}$ of smooth k -schemes of finite type endowed with the Nisnevich topology. We first introduce the category $\mathcal{H}_s(k)$. A map $f : \mathcal{F} \rightarrow \mathcal{G}$ of Nisnevich sheaves of simplicial sets on $(Sm/k)_{\text{Nis}}$ is a *simplicial weak equivalence* if $f_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is a weak equivalence of simplicial sets for any point x of the site; $\mathcal{H}_s(k)$ is the localisation of $\Delta^{\text{op}} Shv((Sm/k)_{\text{Nis}})$ with respect to simplicial weak equivalences. Next, a simplicial sheaf $\mathcal{F}$ is $\mathbf{A}^1$ -local if for any other simplicial sheaf $\mathcal{G}$, the map

$$\text{Hom}_{\mathcal{H}_s(k)}(\mathcal{G}, \mathcal{F}) \rightarrow \text{Hom}_{\mathcal{H}_s(k)}(\mathcal{G} \times \mathbf{A}^1, \mathcal{F})$$

induced by the projection $\mathbf{A}^1 \rightarrow \text{Spec } k$ is a bijection, and a morphism $f : \mathcal{G} \rightarrow \mathcal{H}$ is an $\mathbf{A}^1$ -weak equivalence if for any $\mathbf{A}^1$ -local $\mathcal{F}$ the corresponding map

$$f^* : \text{Hom}_{\mathcal{H}_s(k)}(\mathcal{H}, \mathcal{F}) \rightarrow \text{Hom}_{\mathcal{H}_s(k)}(\mathcal{G}, \mathcal{F})$$

is a bijection. Then $\mathcal{H}(k, \mathbf{A}^1) = \mathcal{H}(k)$ is the localisation of $\Delta^{\text{op}} Shv((Sm/k)_{\text{Nis}})$ with respect to $\mathbf{A}^1$ -weak equivalences.

Morel and Voevodsky provide $\Delta^{\text{op}} Shv((Sm/k)_{\text{Nis}})$ with a closed model structure of which $\mathcal{H}(k)$ is the homotopy category. By construction, any k -scheme of finite type can be viewed in $\mathcal{H}(k)$. For smooth schemes, this transforms elementary Nisnevich covers (for example open covers by two Zariski open sets) into cocartesian

squares, the affine line is contractible, one has a homotopy cocartesian blow-up square (up to a suspension) and a homotopy purity theorem for a smooth pair, expressible in terms of the Thom space of the normal bundle.

Moreover, most of the important cohomology theories are representable in $\mathcal{H}(k)$: this is the case in particular for algebraic K-theory and for motivic cohomology.

When $k \subseteq \mathbf{C}$, there is a *realisation functor* $\mathcal{H}(k) \rightarrow \mathcal{H}$; when $k \subseteq \mathbf{R}$ there is another one, quite different from the first.

In contrast to the classical case of $\mathcal{H}$, there are *two* circles in $\mathcal{H}(k)$: the simplicial circle S_s^1 and the A^1 -circle S_t^1 . They account for the fact that motivic cohomology is a bigraded theory; the fact that algebraic K-theory is single-indexed can be interpreted as an algebraic analogue to Bott periodicity.

Stable Homotopy of Schemes

The construction of this category is outlined in [202]; see also [89, 5.4 ff], [145], [146] and [159]. Briefly, one considers T -spectra, where $T = S_s^1 \wedge S_t^1$ and one constructs $\mathcal{SH}(k)$ “as in topology”, except that “as in” hides not inconsiderable technical difficulties. This is a tensor triangulated category, just as the classical stable homotopy category $\mathcal{SH}$. There exists an infinite version of $DM_{\text{gm}}(k)$, denoted by $DM(k)$ ([180], [211], [146, §5.2]), to which $\mathcal{SH}(k)$ bears the same relationship as $\mathcal{SH}$ bears to the derived category of abelian groups: there is a “motive” functor

$$M : \mathcal{SH}(k) \rightarrow DM(k) ,$$

which extends the functor $M : Sm/k \rightarrow DM_{\text{gm}}^{\text{eff}}(k)$ (an analogue to the “chain complex” functor in topology), and which has as a right adjoint an “Eilenberg–MacLane” functor

$$H : DM(k) \rightarrow \mathcal{SH}(k) .$$

Moreover, if one tensors morphisms by $\mathbf{Q}$, then H is a right inverse to M ; if -1 is a sum of squares in k , H is even an inverse to M .

An important theorem is that motivic cohomology is representable by a Ω_T -spectrum [202, Th. 6.2]: this rests on Voevodsky’s cancellation theorem [203]. Similarly, K-theory is representable by a Ω_T -spectrum KGL . This has allowed Hopkins and Morel to construct an Atiyah–Hirzebruch spectral sequence having the same form as (4.6), by using a tower on KGL rather than the “skeleta” approach leading to (4.6) [75]. Unlike the topological case, it is not clear that the two spectral sequences coincide⁶.

⁶ In the topological case, one can easily prove that the two ways to produce the Atiyah–Hirzebruch spectral sequence for a generalised cohomology theory (cell filtration on the space or Postnikov tower on the spectrum) yield the same spectral sequence by reducing to the case where the space is a sphere and where the spectrum is an Eilenberg–Mac Lane spectrum, because the cohomology of spheres is so basic. In the scheme-homotopy theoretic case, even if one had the suitable generality of construction of the two spectral sequences, one would have to find the right analogues: for example, motivic cohomology of fields is quite complicated.

To prove Theorem 21 below, Voevodsky makes an essential use of the categories $\mathcal{H}(k)$ and $\mathcal{SH}(k)$, and in particular of motivic Steenrod operations.

4.5 Comparisons

4.5.1 Étale topology

Instead of computing higher Chow groups for the Zariski topology, we may use étale topology. For X smooth and $n \geq 0$, we thus get groups

$$H_{\text{ét}}^i(X, \mathbf{Z}(n)) := \mathbb{H}_{\text{ét}}^{i-2n}(X, \alpha^* z^n(X, *))$$

where α is the projection $(Sm/k)_{\text{ét}} \rightarrow (Sm/k)_{\text{Zar}}$. There are canonical maps $H^i(X, \mathbf{Z}(n)) \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}(n))$. Similarly, replacing the complexes $z^n(X, *)$ by $z^n(X, *) \otimes \mathbf{Z}/m$, we may define motivic cohomology with finite coefficients, both in the Zariski and the étale topology:

$$H^i(X, \mathbf{Z}/m(n)) \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}/m(n)).$$

13 Theorem 13: (Geisser–Levine) Let X be smooth over k .

- [62, Th. 1.5] If m is invertible in k , there is a quasi-isomorphism $\underline{z}^n(-, *)_{\text{ét}} \otimes \mathbf{Z}/m \simeq \mu_m^{\otimes n}$.
- [61] If m is a power of $p = \text{char } k > 0$, say $m = p^s$, there is a quasi-isomorphism $\underline{z}^n(-, *)_{\text{ét}} \otimes \mathbf{Z}/m \simeq v_s(n)[-n]$, where $v_s(n)$ is the n -th logarithmic Hodge–Witt sheaf of level s .⁷

4.5.2 Zariski Topology

Keep the above notation. We first have an easy comparison with rational coefficients (e.g. [93, Prop. 1.18]):

14 Theorem 14 For X smooth over k , $H^i(X, \mathbf{Q}(n)) \xrightarrow{\sim} H_{\text{ét}}^i(X, \mathbf{Q}(n))$ for all i, n .

15 Theorem 15: (Geisser–Levine [61, Th. 8.4]) Let X be smooth over k . If $m = p^s$, $p = \text{char } k > 0$, $H_{\text{Zar}}^i(X, \mathbf{Z}/m(n)) \simeq H_{\text{Zar}}^{i-n}(X, \alpha_* v_s(n))$.

See §3.2 in Geisser’s chapter for more details on this theorem.

For m invertible in k , the situation is more conjectural. Let l be prime and invertible in k . Recall

⁷ This statement does not appear explicitly in [61]; however it can be deduced from Theorem 15 by étale localisation!

Conjecture 16: (*Bloch–Kato Conjecture*) For any finitely generated extension K/k and any $n \geq 0$, the norm residue homomorphism

16

$$K_n^M(K)/l \rightarrow H_{\text{ét}}^n(K, \mu_l^{\otimes n})$$

is bijective.

The references for this conjecture are [100, Conj. 1] and [19, Intr. and Lect. 5]. For $l = 2$, it is due to Milnor [141, p. 540].

Conjecture 17: (*Beilinson–Lichtenbaum Conjecture*) For any smooth X over k , the quasi-isomorphism of Theorem 13 a) induces a quasi-isomorphism $\underline{z}^n(-, *) \otimes \mathbb{Z}/m \simeq \tau_{\leq n} R\alpha_* \mu_m^{\otimes n}$.

17

The references for this conjecture are [12, 5.10 D (vi)] and [124, §5]. In view of Theorems 14 and 15, it may be reformulated as follows:

Lemma 18 Conjecture 17 is equivalent to the following statement: for any smooth k -variety X , the natural morphism

18

$$\mathbf{Z}(X, n) \rightarrow \tau_{\leq n+1} R\alpha_* \alpha^* \mathbf{Z}(X, n)$$

is an isomorphism in $D^-(X_{\text{Zar}})$, where $\mathbf{Z}(X, n)$ is the class of the Zariski sheafification of the shifted Bloch cycle complex $z^*(X, n)[-2n]$ and where α is the projection of the small étale site of X onto its small Zariski site. $\square$

(The vanishing of $R^{n+1} \alpha_* \alpha^* \mathbf{Z}(X, n)$ is called *Hilbert 90 in weight n* .)

The following theorem is due to Suslin–Voevodsky [187] in characteristic 0 and to Geisser–Levine [62] in general.

Theorem 19 Conjectures 16 and 17 are equivalent.

19

The arguments of [61] for the proof of Theorem 15 and of [187] and [62] for the proof of Theorem 19 can be abstracted [93] and give a *uniqueness theorem* for motivic cohomology. Let us explain this theorem. Define (in this explanation) a *cohomology theory* as a sequence of complexes of sheaves $(B(n))_{n \in \mathbb{Z}}$ over the category of smooth k -schemes endowed with the Zariski topology, enjoying two natural properties: homotopy invariance and Gysin exact sequences (purity). There is an obvious notion of morphisms of cohomology theories. For example, $(\mathbf{Z}(n))_{n \in \mathbb{Z}}$ defines a cohomology theory as soon as we modify Bloch’s cycle complexes suitably so as to make them strictly contravariant for morphisms between smooth vari-

eties [93, Th. 1.17]⁸. Suppose now that C is a bounded above complex of abelian groups (in applications to Theorems 15 and 19, C would be $\mathbb{Z}/m$ for some m) and that we are given a morphism of cohomology theories

$$\mathbf{Z}(n) \otimes^L C \rightarrow B(n). \quad (4.17)$$

Under what conditions is (4.17) an isomorphism? Two obvious necessary conditions are the following:

1. $B(n)$ is bounded above for all n .
 2. (4.17) is a quasi-isomorphism for $n \leq 0$; in particular, $B(n) = 0$ for $n < 0$.
- The third condition is very technical to state: it is called *malleability* and is enjoyed by $\mathbf{Z}(n)$ for $n > 0$ by a nontrivial theorem of Geisser and Levine ([61, Cor. 4.4], [93, Th. 2.28]). For $n = 1$ it is closely related to the fact that, if A is a semi-local ring, then for any ideal I in A the homomorphism $A^* \rightarrow (A/I)^*$ is surjective. It is stable under tensoring with C , so if (4.17) is an isomorphism then
3. $B(n)$ is malleable for all $n > 0$.

Conversely:

20

Theorem 20: ([93, Prop. 2.30]) If Conditions 1, 2 and 3 are satisfied, then (4.17) is an isomorphism of cohomology theories.

Finally, we have the celebrated theorem of Voevodsky:

21

Theorem 21: (Voevodsky [204]) Conjecture 16 is true for $l = 2$.

The reader can have a look at [50], [89], [143] and [184] for some insights in the proof.

For an odd prime l , Conjecture 16 is “proven” in the following sense: 1) A preprint of Voevodsky [206] gives a proof modulo two lemmas on mod l Steenrod operations (see *loc. cit.*, Lemmas 2.2 and 2.3) and two results of Rost (see *loc. cit.*, Th. 6.3). 2) Voevodsky plans to write up a proof of the Steenrod operation lemmas, but no such proof is available at the moment. 3) The two results of Rost have been announced by him in his address to the Beijing International Congress of Mathematicians in 2002 with proofs of special cases [162], but no complete proof is available at the moment.

⁸ One should be careful that the construction given in [93] is incomplete. The problem is that the claimed equality in *loc. cit.*, (1.4) is only an inclusion in general. As a consequence, the object defined in the proof of Theorem 1.17 is not a functor, but only a lax functor. This lax functor can be rectified e.g. by the methods of [122] (see also Vogt [207]). I am grateful to Marc Levine for pointing out this gap, and the way to fill it.

Corollary 22 For n a power of 2 (or more generally for all n with the caveat just above), the E_2 -terms of the spectral sequence (4.7) have the form

22

$$E_2^{p,q} = H_{\text{Zar}}^p(X, \tau_{\leq -q/2} R\alpha_* \mu_n^{\otimes(-q/2)}) .$$

In particular, if X is semi-local, then

$$E_2^{p,q} = \begin{cases} H_{\text{ét}}^p(X, \mu_n^{\otimes(-q/2)}) & \text{if } p \leq -q/2 \\ 0 & \text{if } p > -q/2 . \end{cases}$$

Corollary 23: (Compare [120, proof of Cor 13.3]) Let X be a connected regular scheme essentially of finite type over a field or a Dedekind scheme S . Let δ be the étale cohomological 2-dimension of the function field of X and $d = \dim X$. Then, for any $n \geq 0$, $H^i(X, \mathbf{Z}/2^s(n)) = 0$ for $i > \delta + d$.

23

Proof We may assume $\delta < \infty$. Consider the hypercohomology spectral sequence

$$E_2^{p,q} = H_{\text{Zar}}^p(X, \mathcal{H}^q(\mathbf{Z}/2^s(n))) \Rightarrow H^{p+q}(X, \mathbf{Z}/2^s(n)) .$$

It is sufficient to show that $E_2^{p,q} = 0$ for $p + q > \delta + d$. We distinguish two cases:

- $q > n$. Then it follows from the definition of Bloch's higher Chow groups plus Gersten's Conjecture [32], which implies that the stalks of the sheaf $\mathcal{H}^q(\mathbf{Z}/2^s(n))$ inject into its stalk at the generic point.
- $q \leq n$. By Theorems 19 and 21, $\mathcal{H}^q(\mathbf{Z}/2^s(n)) \xrightarrow{\sim} \mathcal{H}_{\text{ét}}^q(\mathbf{Z}/2^s(n))$. But the right sheaf is 0 for $q > \delta$ by the argument in [91, Proof of Cor. 4.2].

Back to the Beilinson–Soulé Conjecture

4.5.3

Lemma 24 Under Conjecture 17, Conjecture 5 is equivalent to Conjecture 5 tensored by $\mathbf{Q}$.

24

Indeed, Conjecture 17 implies that $H^i(X, \mathbf{Z}/m(n)) = H_{\text{ét}}^i(X, \mu_m^{\otimes n}) = 0$ for $i < 0$ and any $m > 0$. This in turn implies that $H^i(X, \mathbf{Z}(n)) \rightarrow H^i(X, \mathbf{Q}(n))$ is an isomorphism for any $i < 0$ and is injective for $i = 0$. $\square$

Borel–Moore Étale Motivic Homology

4.5.4

For the sequel we shall need a Borel–Moore homology theory that has the same relationship to étale motivic cohomology as ordinary Borel–Moore motivic homology has to ordinary motivic cohomology. Ideally, we would like to associate

to any k -scheme of finite type X a collection of abelian groups $H_i^{BM, \text{ét}}(X, \mathbf{Z}(n))$ enjoying the following two properties:

1. **Poincaré duality:** if X is a closed subscheme of a smooth k -scheme M of dimension d , we have isomorphisms

$$H_i^{BM, \text{ét}}(X, \mathbf{Z}(n)) \simeq H_{X, \text{ét}}^{2d-i}(M, \mathbf{Z}(d-n)) \quad (4.18)$$

where the right hand side is étale hypercohomology of M with supports in X .

2. **Localisation:** if Z is a closed subset of X and $U = X - Z$, we have long exact sequences

$$\begin{aligned} \cdots \rightarrow H_i^{BM, \text{ét}}(Z, \mathbf{Z}(n)) \rightarrow H_i^{BM, \text{ét}}(X, \mathbf{Z}(n)) \\ \rightarrow H_i^{BM, \text{ét}}(U, \mathbf{Z}(n)) \rightarrow H_{i-1}^{BM, \text{ét}}(Z, \mathbf{Z}(n)) \rightarrow \cdots \end{aligned}$$

The problem is that 1) does not make sense a priori because if $q < 0$ and M is smooth the groups $H_{\text{ét}}^j(M, \mathbf{Z}(q))$ have not been defined. For $n \in \mathbf{Z}$, let

$$\begin{aligned} (\mathbf{Q}/\mathbf{Z})'(n) &:= \lim_{\substack{\rightarrow \\ (m, \text{char } k)=1}} \mu_m^{\otimes n} \\ \mathbf{Q}_p/\mathbf{Z}_p(n) &:= \begin{cases} \lim_{\rightarrow} v_s(n)[-n] & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \quad \text{if } \text{char } k = p > 0 \\ \mathbf{Q}/\mathbf{Z}(n) &:= (\mathbf{Q}/\mathbf{Z})'(n) \oplus \mathbf{Q}_p/\mathbf{Z}_p(n). \end{aligned}$$

Theorems 13 and 14 imply that for X smooth and $n \geq 0$ we have long exact sequences

$$\begin{aligned} \cdots \rightarrow H_{\text{ét}}^{i-1}(X, \mathbf{Q}/\mathbf{Z}(n)) \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}(n)) \\ \rightarrow H^i(X, \mathbf{Q}(n)) \rightarrow H_{\text{ét}}^i(X, \mathbf{Q}/\mathbf{Z}(n)) \rightarrow \cdots \quad (4.19) \end{aligned}$$

where so that we might try and define

$$H_{\text{ét}}^i(X, \mathbf{Z}(n)) := H_{\text{ét}}^{i-1}(X, \mathbf{Q}/\mathbf{Z}(n)) \quad \text{for } X \text{ smooth and } n < 0. \quad (4.20)$$

This is vindicated by the projective bundle formula

$$H_{\text{ét}}^i(\mathbf{P}_X^n, \mathbf{Z}(q)) \simeq \bigoplus_{j=0}^n H_{\text{ét}}^{i-2j}(X, \mathbf{Z}(q-j))$$

which may be proven as [86, Th. 5.1].

Suppose that in 1) X is smooth of dimension d . Then we may take $M = X$ and we get isomorphisms $H_i^{BM, \text{ét}}(X, \mathbf{Z}(n)) \simeq H_{\text{ét}}^{2d-i}(X, \mathbf{Z}(d-n))$. Suppose now that in 2) Z is also smooth, of codimension c . Then, after reindexing, the localisation exact

sequences translate into long exact “Gysin” sequences

$$\begin{aligned} \cdots \rightarrow H_{\text{ét}}^{j-2c}(Z, \mathbf{Z}(m-c)) \rightarrow H_{\text{ét}}^j(X, \mathbf{Z}(m)) \\ \rightarrow H_{\text{ét}}^j(U, \mathbf{Z}(m)) \rightarrow H_{\text{ét}}^{j-2c+1}(Z, \mathbf{Z}(m-c)) \rightarrow \cdots \end{aligned} \quad (4.21)$$

However, with Definition (4.20) the exact sequences (4.21) are wrong in characteristic $p > 0$: take for example k algebraically closed, $X = \mathbf{P}^1$, $U = \mathbf{A}^1$, $m = 0$ and $j = 2$. The exactness of (4.21) would imply that the homomorphism $H_{\text{ét}}^1(\mathbf{P}^1, \mathbf{Q}/\mathbf{Z}) \rightarrow H_{\text{ét}}^1(\mathbf{A}^1, \mathbf{Q}/\mathbf{Z})$ is surjective, which is false because $H_{\text{ét}}^1(\mathbf{P}^1, \mathbf{Q}_p/\mathbf{Z}_p) = 0$ while $H_{\text{ét}}^1(\mathbf{A}^1, \mathbf{Q}_p/\mathbf{Z}_p)$ is huge. Therefore the most we can hope for is to have exact sequences (4.21) after inverting the exponential characteristic p .

This turns out to be true: (4.21) $\otimes \mathbf{Z}[1/p]$ are obtained by gluing together the purity theorem for motivic cohomology (4.14) and the purity theorem for étale cohomology with finite coefficients, as in [86, Th. 4.2].

It also turns out that a Borel–Moore homology theory

$$X \mapsto (H_i^{BM, \text{ét}}(X, \mathbf{Z}[1/p](n)))_{(i,n) \in \mathbf{Z} \times \mathbf{Z}} \quad (4.22)$$

having properties 1) and 2) after inverting p does exist. These groups sit in long exact sequences analogous to (4.19)

$$\begin{aligned} \cdots \rightarrow H_{i+1}^{BM, \text{ét}}(X, (\mathbf{Q}/\mathbf{Z})'(n)) \rightarrow H_i^{BM, \text{ét}}(X, \mathbf{Z}[1/p](n)) \\ \rightarrow H_i^{BM}(X, \mathbf{Q}(n)) \rightarrow H_i^{BM, \text{ét}}(X, (\mathbf{Q}/\mathbf{Z})'(n)) \rightarrow \cdots \end{aligned} \quad (4.23)$$

where $H_{i+1}^{BM, \text{ét}}(X, (\mathbf{Q}/\mathbf{Z})'(n)) = H^{-i-1}(X, f_X^!(\mathbf{Q}/\mathbf{Z})'(-n))$ is étale Borel–Moore homology, with $f_X^!$ the extraordinary inverse image of [SGA 4, Exposé XVIII] associated to $f_X : X \rightarrow \text{Spec } k$. Supposing that they are constructed, they are characterised either by (4.18) (with p inverted) or by (4.23).

I know two techniques to construct the theory (4.22). The first is to proceed “naïvely” as in [62] and construct a homotopy version of the homological cycle class map of [SGA 4 1/2, Cycle, §2.3]: this had been done in [92, §1.3]. This yields functorial zig-zags of morphisms

$$\alpha^* z_n(X, *) \xrightarrow{\sim} f_X^! \mathbf{Z}/m(-n)[-2n]$$

for $(m, \text{char } k) = 1$, which are compatible when m varies; in the limit one gets zig-zags

$$\alpha^* z_n(X, *) \otimes \mathbf{Q} \xrightarrow{\sim} f_X^!(\mathbf{Q}/\mathbf{Z})'(-n)[-2n]$$

and one defines (4.22) as the homology groups of the homotopy fibre. The other method is much more expensive but also more enlightening. First, one proves that the cohomology theory on smooth schemes

$$X \mapsto (H_{\text{ét}}^i(X, \mathbf{Z}[1/p](n)))_{(i,n) \in \mathbf{Z} \times \mathbf{Z}}$$

is representable by a T -spectrum $H_{\mathbf{Z}[1/p]}^{\text{ét}}$ in $\mathcal{SH}(k)$: for this one may glue the T -spectrum $H_{\mathbf{Q}}$ with the T -spectrum $H_{(\mathbf{Q}/\mathbf{Z})'}^{\text{ét}}$ representing étale cohomology with $(\mathbf{Q}/\mathbf{Z})'$ coefficients (cf. [159, Cor. 8.53]) in the spirit above, except that it is much easier here. (As Joël Riou pointed out, it is even easier to apply the Dold–Kan construction to truncations of Godement resolutions representing $R\alpha_*\alpha^*\mathbf{Z}[1/p](n)$: the projective bundle formula and homotopy invariance imply that they yield an Ω_T -spectrum.) Then, according to Voevodsky’s formalism of cross functors (cf. [9, 205]), given a k -scheme of finite type X with structural morphism $f : X \rightarrow \text{Spec } k$, we have an “extraordinary direct image” functor

$$f_! : \mathcal{SH}(X) \rightarrow \mathcal{SH}(k) .$$

We set $BM(X) = f_!S^0$: this is the *Borel–Moore object* associated to X . For any T -spectrum $E \in \mathcal{SH}(k)$, we may then define

$$E_{p,q}^{BM}(X) := [\Sigma^{-p,-q}BM(X), E]$$

(I am indebted to Riou for discussions about this.) Applying this to $E = H_{\mathbf{Z}[1/p]}^{\text{ét}}$, we get the desired theory.

Note that, for a singular scheme X , one may also consider the groups

$$H_{\text{ét}}^{2n-i}(X, \alpha^*_{\mathbf{Z}_n}(X, *)[1/p])$$

obtained by sheafifying the Bloch cycle complexes for the étale topology. These groups map to $H_i^{BM, \text{ét}}(X, \mathbf{Z}[1/p](n))$, but these maps are not isomorphisms in general as one can see easily because of (4.18). So the isomorphism of [92, (1.6)] is wrong. (I am indebted to Geisser for pointing out this issue.) However they become isomorphisms after tensoring with $\mathbf{Q}$, and these groups then reduce to Bloch’s higher Chow groups tensored with $\mathbf{Q}$.

Finally, one can repeat the story above after tensoring the étalified Bloch cycle complexes (for smooth schemes) by a fixed complex of étale sheaves C on the small étale site of $\text{Spec } k$: this will be used in the sequel.

Applications: Local Structure of Algebraic K -Groups and Finiteness Theorems

Definition 25 Let X be a $\text{Spec } \mathbf{Z}[1/2]$ -scheme: it is *non-exceptional* if for any connected component X_α , the image of the cyclotomic character $\kappa_2 : \pi_1(X_\alpha) \rightarrow \mathbf{Z}_2^*$ does not contain -1 .

The first result says that, locally for the Zariski topology, algebraic K -theory with $\mathbf{Z}/2^v$ coefficients is canonically a direct sum of étale cohomology groups, at least in the nonexceptional case:

Theorem 26: ([91, Th. 1]) Let A be a semi-local non-exceptional $\mathbf{Z}[1/2]$ -algebra.

26

a) There are canonical isomorphisms ($n \geq 0, v \geq 2$)

$$\coprod_{0 \leq i \leq n} H_{\text{ét}}^{2i-n}(A, \mu_{2^v}^{\otimes i}) \xrightarrow{\sim} K_n(A, \mathbf{Z}/2^v).$$

b) If A is essentially smooth over field or a discrete valuation ring, the spectral sequence (4.5) with $\mathbf{Z}/2^v$ coefficients canonically degenerates.

c) If A is a field and $\mu_{2^v} \subset A$, the natural map

$$K_*^M(A) \otimes \mathbf{Z}/2^v[t] \rightarrow K_*(A, \mathbf{Z}/2^v)$$

given by mapping t to a “Bott element” is an isomorphism.

Note that the reason why Thomason disbelieved the Bloch–Kato conjecture was precisely that it would imply the vanishing of all differentials in the Atiyah–Hirzebruch spectral sequence for étale K -theory [195, p. 409]: similar results had been observed by Dwyer–Friedlander [45]. See §3 in Weibel’s chapter for details on the construction of the isomorphism a) in some special cases.

For X a scheme, define $d_2(X) := \sup\{cd_2(\eta)\}$, where η runs through the generic points of X .

Theorem 27: (*ibid.*, Th. 2) Let X be a finite-dimensional Noetherian non-exceptional $\mathbf{Z}[1/2]$ -scheme.

27

a) The natural map

$$K_n^{TT}(X, \mathbf{Z}/2^v) \rightarrow K_n^{TT}(X, \mathbf{Z}/2^v)[\beta^{-1}]$$

is injective for $n \geq \sup(d_2(X) - 2, 1)$ and bijective for $n \geq \sup(d_2(X) - 1, 1)$.

The 1 in the sup is not necessary if X is regular. (Recall that K^{TT} denotes Thomason–Trobauhg K -theory.)

b) The natural map

$$K'_n(X, \mathbf{Z}/2^v) \rightarrow K'_n(X, \mathbf{Z}/2^v)[\beta^{-1}]$$

is injective for $n \geq d_2(X) - 2$ and bijective for $n \geq d_2(X) - 1$.

c) If $cd_2(X) < +\infty$, there are isomorphisms

$$K_n^{TT}(X, \mathbf{Z}/2^v)[\beta^{-1}] \xrightarrow{\sim} K_n^{\text{ét}}(X, \mathbf{Z}/2^v)$$

for all $n \in \mathbf{Z}$.

Remark 28 For X regular over a field or a discrete valuation ring, one can directly use the spectral sequences (4.5) and (4.6) with finite coefficients, cf. Levine [120]; but this approach does not work for singular X and K^{TT} .

28

29 **Corollary 29:** (*ibid.*, Cor. 1) Let S be $\mathbf{Z}[1/2]$ -scheme and X a non-exceptional separated S -scheme of finite type. Assume that S is

- (i) $\mathrm{Spec} R[1/2]$, where R the ring of integers of a non-exceptional number field, or
- (ii) $\mathrm{Spec} \mathbf{F}_p$, $p > 2$, or
- (iii) $\mathrm{Spec} k$, k separably closed field of characteristic $\neq 2$, or
- (vi) $\mathrm{Spec} k$, k a higher local field in the sense of Kato.

Then $K_n^{TT}(X, \mathbf{Z}/2^v)$ and $K'_n(X, \mathbf{Z}/2^v)$ are finite for $n \geq \dim(X/S) + d_2(S) - 2$.

30 **Remark 30** For X regular, the map $H^i(X, \mathbf{Z}/2^v(n)) \rightarrow H_{\text{ét}}^i(X, \mu_{2^v}^{\otimes n})$ is injective for $i \leq n+1$ (even an isomorphism for $i \leq n$) by Conjecture 17 and Theorem 21, hence $H^i(X, \mathbf{Z}/2^v(n))$ is finite by Deligne's finiteness theorem for étale cohomology [SGA 4 1/2, th. finitude] plus arithmetic finiteness theorems. This remark yields Corollary 29 by applying Corollaries 22 and 23. The general case needs the methods of [91].

31 **Corollary 31:** (*ibid.*, Cor. 2) Let X be a variety of dimension d over $k = \mathbf{F}_p$ (resp. $\mathbf{Q}_p$), $p > 2$. Then $K_n^{TT}(X)\{2\}$ is finite and $\frac{K_{n+1}^{TT}(X)}{K_{n+1}^{TT}(X)\{2\}}$ is uniquely 2-divisible for $n \geq d$ (resp. $d+1$). The same statements hold with $K'_*(X)$.

32 **Theorem 32:** (*ibid.*, Th. 3) Let $d \geq 0$ and $n \geq 3$. There exists an effectively computable integer $N = N(d, n) > 0$ such that, for any Noetherian $\mathbf{Z}[1/2]$ -scheme X separated of Krull dimension $\leq d$ and all $v \geq 2$, the kernel and cokernel of the map

$$K_n^{TT}(X, \mathbf{Z}/2^v) \xrightarrow{(ic_{i,2i-n})} \prod_{i \geq 1} H_{\text{Zar}}^{2i-n}(X, B/2^v(i))$$

(given by Chern classes) are killed by N . If X is smooth over a field or a discrete valuation ring, this holds also for $n = 2$.

Let us come back to Conjecture 4 in the light of this theorem. If X is of finite type over a field k of characteristic 0, the construction of the Haesemeyer spectral sequence (4.13) yields a version with coefficients $\mathbf{Z}/2^v$. The abutment is $K_*^{TT}(X, \mathbf{Z}/2^v)$ by homotopy invariance of the latter theory. The E_2 -terms are of the form $H_{\text{cdh}}^p(X, \theta^* B/2^v(-q/2))$, where θ is the projection of the cdh site of k onto its big Zariski site. This looks closely like the spectral sequence in Conjecture 4. Is there any reason why the maps $H_{\text{Zar}}^p(X, B/2^v(-q/2)) \rightarrow H_{\text{cdh}}^p(X, \theta^* B/2^v(-q/2))$ should be isomorphisms? A moment of reflection suggests that there might be a base change theorem between étale and cdh topology (involving Geisser's éh topol-

ogy [60]) which should be closely related to Gabber's affine analogue of proper base change [55].

If $X = \operatorname{Spec} O_S$ where O_S is a localised ring of integers in a global field (with $1/2 \in O_S$) and O_S is not formally real, then $cd_2(O_S) = 2$ and the spectral sequence (4.7) degenerates for dimension reasons. Hence Corollary 22 directly yields isomorphisms

$$\begin{aligned} K_{2i-1}(O_S, \mathbf{Z}/2^v) &\simeq H_{\text{ét}}^1(O_S, \mu_{2^v}^{\otimes i}) \\ K_{2i-2}(O_S, \mathbf{Z}/2^v) &\simeq H_{\text{ét}}^2(O_S, \mu_{2^v}^{\otimes i}) \quad (i \geq 2). \end{aligned}$$

This is a finite coefficients version of the original Quillen conjecture (*cf.* Conjecture 1)

$$K_{2i-1}(O_S) \otimes \mathbf{Z}_l \simeq H_{\text{ét}}^1(O_S, \mathbf{Z}_l(i)) \quad (4.24)$$

$$K_{2i-2}(O_S) \otimes \mathbf{Z}_l \simeq H_{\text{ét}}^2(O_S, \mathbf{Z}_l(i)) \quad (i \geq 2). \quad (4.25)$$

The latter readily follows from the finite version by passing to the inverse limit, because of the finiteness of the étale cohomology groups and Quillen's finite generation theorem (see below).

When O_S is formally real, we have $cd_2(O_S) = +\infty$ and the above does not apply. In fact, the spectral sequence (4.7) does *not* degenerate at E_2 in this case, neither for O_S nor for its quotient field F . It can be shown however that it degenerates at E_4 as well as (4.6), see [88, Lemma 4.3] for the latter. (For coefficients $\mathbf{Z}/2^s$ with $s \geq 2$, see [83, Appendix]: this argument is detailed in Sect. 7 of Weibel's chapter for the real numbers. For coefficients $\mathbf{Z}/2$, see [152]. In [160] Rognes and Weibel avoid the use of a product structure by a clever reciprocity argument.) This yields the following version of Quillen's conjecture:

Theorem 33: ([88, Th. 1]) Let r_1 be the number of real places of F . Then there exist homomorphisms

33

$$K_{2i-j}(O_S) \otimes \mathbf{Z}_2 \xrightarrow{ch_{i,j}} H_{\text{ét}}^j(O_S, \mathbf{Z}_2(i)) \quad (j = 1, 2, i \geq j),$$

which are

- (i) bijective for $2i - j \equiv 0, 1, 2, 7 \pmod{8}$
- (ii) surjective with kernel isomorphic to $(\mathbf{Z}/2)^{r_1}$ for $2i - j \equiv 3 \pmod{8}$
- (iii) injective with cokernel isomorphic to $(\mathbf{Z}/2)^{r_1}$ for $2i - j \equiv 6 \pmod{8}$.

Moreover, for $i \equiv 3 \pmod{4}$ there is an exact sequence

$$\begin{aligned} 0 \rightarrow K_{2i-1}(O_S) \otimes \mathbf{Z}_2 &\rightarrow H_{\text{ét}}^1(O_S, \mathbf{Z}_2(i)) \rightarrow (\mathbf{Z}/2)^{r_1} \\ &\rightarrow K_{2i-2}(O_S) \otimes \mathbf{Z}_2 \rightarrow H_{\text{ét}}^2(O_S, \mathbf{Z}_2(i)) \rightarrow 0 \end{aligned}$$

in which $\operatorname{Im}(H_{\text{ét}}^1(O_S, \mathbf{Z}_2(i)) \rightarrow (\mathbf{Z}/2)^{r_1})$ has 2-rank $\varphi_i \geq 1$ if $r_1 \geq 1$.

The homomorphisms $ch_{i,j}$ are natural in O_S .

34 Remark 34 The above results hold modulo powers of an odd prime l (without the non-exceptional complications) as soon as the Bloch–Kato conjecture is proven mod l . See §6 in Weibel’s chapter.

For X regular, Theorem 27 has been generalised by P. A. Østvær and A. Rosenschon by removing the nonexceptional hypotheses [152]: they get essentially the same statements by replacing the étale cohomological 2-dimension by the virtual étale cohomological 2-dimension. However they do not deal with singular schemes.

J. Rognes and C. Weibel [160] used Theorem 21 and the version with divisible coefficients of the spectral sequence (4.7) to compute much of the 2-torsion in $K_*(O_F)$ where O_F is the ring of integers of a number field (see also [88, Cor. 3]): see Weibel’s chapter in this Handbook.

35 Question 35: *Open Question* Let X be regular of finite type over $\mathrm{Spec} \mathbf{Z}[1/m]$. Is $H^i(X, \mathbf{Z}/m(n))$ finite for all i ?

This is false over $\mathrm{Spec} \overline{\mathbf{Q}}$: by Schoen [166], there exists an elliptic curve E and a prime l with $CH^2(E^3)/l = H^4(E^3, \mathbf{Z}/l(2))$ infinite. I now tend to doubt whether this is true even over $\mathbf{Z}$: see the discussion in Sect. 4.9.1.

4.7

The Picture in Arithmetic Geometry

4.7.1

Finite Generation Theorems

A basic conjecture underlying all further conjectures is

36 Conjecture 36: (*Bass Conjecture*)

- a) For any scheme X of finite type over $\mathrm{Spec} \mathbf{Z}$, the groups $K'_i(X)$ are finitely generated.
- b) For any regular scheme X of finite type over $\mathrm{Spec} \mathbf{Z}$, the groups $K_i(X)$ are finitely generated.

By Poincaré duality for K' and K -theory, a) evidently implies b). But conversely, b) implies a) by the localisation exact sequence (if X is of finite type over $\mathrm{Spec} \mathbf{Z}$, its regular points form a dense open subset so we may argue by Noetherian induction).

In view of the spectral sequences (4.5) and (4.6), it is tempting to approach this conjecture via the stronger

Conjecture 37: (*Motivic Bass Conjecture*)

37

- a) For any scheme X of finite type over $\text{Spec } \mathbf{Z}$, the groups $H_i^{BM}(X, \mathbf{Z}(n))$ are finitely generated.
- b) For any regular scheme X of finite type over $\text{Spec } \mathbf{Z}$, the groups $H^i(X, \mathbf{Z}(n))$ are finitely generated.

Just as before, a) $\iff$ b).

I will explain in Sect. 4.9.1 why I now doubt that these versions of the Bass conjecture are true, and also why it does not matter too much. Nevertheless let us start with positive results:

Proposition 38

38

- a) Conjecture 37 is true for $n \leq 1$.
- b) (Quillen) Conjecture 36 is true for $\dim X \leq 1$.

Sketch of proofs

We may reduce to X regular and connected. First, a) may be deduced from a combination of

- Dirichlet's unit theorem: finite generation of units in the ring of integers of a number field, and the finiteness of the class group of such a ring.
- The Mordell–Weil theorem: for any abelian variety A over a number field K , the group $A(K)$ is finitely generated.
- The Néron-Severi theorem: for any smooth projective variety X over an algebraically closed field, the Néron-Severi group $NS(X)$ is finitely generated.

De Jong's alteration theorem also enters the proof: we skip details (see [97], and also [95, Lemma 4.1] for characteristic p).

b) Here Quillen's proofs go through a completely different path ([66, 153, 155]): homology of the general linear group. For any ring R , one has

$$K_i(R) = \pi_i(K_0(R) \times BGL(R)^+).$$

Since $BGL(R)^+$ is an H -space, by Hurewicz's theorem all K_i are finitely generated if and only if $K_0(R)$ is finitely generated and all $H_i(BGL(R)^+, \mathbf{Z}) = H_i(GL(R), \mathbf{Z})$ are finitely generated. At the time when Quillen proved the theorems, he needed to go through delicate arguments involving (in the dimension 1 case) homology of the Steinberg module. However, later stability theorems may be used to simplify the argument, except in the function field case: by van der Kallen and Maazen [98], $H_i(GL(R), \mathbf{Z}) = H_i(GL_N(R), \mathbf{Z})$ for N large (depending on i). If R is finite, this finishes the proof. If R is a localised number ring, finite generation depends on a theorem of Raghunathan [157] which ultimately uses the action of $SL_N(R)$ on certain symmetric spaces, hence Riemannian geometry... There is a similarity with Dirichlet's proof of his unit theorem ($N = 1$).

For curves over a finite field, Quillen's proof, passing through Steinberg modules, is mainly related to the fact that semi-stable vector bundles over a curve admit moduli. It would be useful to combine this idea with the van der Kallen–Maazen stability theorem in order to simplify the proof. We shall give a completely different proof in Remark 71 3).

The Beilinson–Soulé Conjecture Again

The following result was prefigured in [84]:

39

Theorem 39: ([87]) Conjecture 36 $\Rightarrow$ Conjecture 5.

Sketch

We shall actually sketch a proof of the slightly weaker result that Conjecture 37 $\Rightarrow$ Conjecture 5 for X regular of finite type over $\mathbf{Z}[1/2]$. There are long exact sequences

$$\cdots \rightarrow H^i(X, \mathbf{Z}_{(2)}(n)) \xrightarrow{2} H^i(X, \mathbf{Z}_{(2)}(n)) \rightarrow H^i(X, \mathbf{Z}/2(n)) \rightarrow \cdots$$

For $i < 0$, Theorem 21 + Theorem 19 $\Rightarrow H^i(X, \mathbf{Z}/2(n)) = 0$. Since the $H^i(X, \mathbf{Z}_{(2)}(n))$ are finitely generated over $\mathbf{Z}_{(2)}$, this does the proof for $i < 0$. For $i = 0$, we need a little more: after reducing to a finitely generated field K , a dyadic argument using that K contains only finitely many roots of unity. With even more effort one can catch the Merkurjev–Suslin Conjecture 8.

To get the actual statement of the theorem, one has to check that in the spectral sequence (4.6), the appropriate E_∞ terms are uniquely 2-divisible as subquotients of motivic cohomology groups, and that then the corresponding K -groups are also almost uniquely divisible, hence vanish up to a group of finite exponent, and therefore the motivic groups too. This back and forth uses the degeneration of (4.6) up to small torsion and is a bit messy; the arguments in [84] give a good idea of it. (Note that the quasi-degeneration of the spectral sequence implies that a given E_∞ -term is equal to the corresponding E_2 -term up to groups of finite exponent.) It may not be extremely interesting to make this proof completely explicit.

Motivic Cohomology of Finite and Global Fields

In this subsection we want to indicate a proof of

40

Theorem 40 Conjecture 37 holds for $\dim X \leq 1$.

Sketch

As in the proof of Proposition 38 we may restrict to X regular connected. In view of Proposition 38, we may try and deduce it from Conjecture 36 via the spectral sequence (4.6).

If one tries the crude approach via Adams operations, one runs into the problem indicated at the end of Sect. 4.2: we only get that the groups $H^i(X, \mathbb{Z}(n))$ are finitely generated up to some group of finite exponent (bounded in terms of i and n). We are going to get by by granting the Beilinson–Lichtenbaum Conjecture 17. The main point is:

Lemma 41 The Beilinson–Soulé Conjecture 5 is true for $\dim X \leq 1$; moreover $H^i(X, \mathbb{Z}(n)) = 0$ for $i \geq \dim X + 2$ (up to a finite 2-group if the function field of X is formally real).

41

There are three very different proofs of this lemma. The first combines the rank computations of Borel [24] with the results of Soulé [173, 174], cf. [77, p. 327, Ex. 3]. The second uses the proof of the rank conjecture for number fields (the rank filtration is opposite to the gamma filtration) by Borel and Yang [27]. The third is to apply Theorem 39 in this special case: see [88, proof of Th. 4.1].

Given Lemma 41, (4.6) degenerates at E_2 for dimension reasons, except in the formally real case. When it degenerates at E_2 the finite generation conclusion is immediate; in the formally real case one gets relationships between K -theory and motivic cohomology similar to those of Theorem 33 and the conclusion follows again.

To get this finite generation result for motivic cohomology, we have used a very circuitous and quite mathematically expensive route: Quillen’s finite generation theorems for K -theory (involving the homology of GL_n and Riemannian geometry), the Bloch–Lichtenbaum spectral sequence and finally the Bloch–Kato conjecture! In characteristic 0 this seems to be the only available route at the moment. In characteristic p , however, we shall see in Remark 71 3) that Frobenius provides a shortcut allowing us to avoid the passage through K -theory.

Ranks, Torsion and Zeta Functions

4.7.2

The primeval formula in this subject is certainly Dedekind’s analytic class number formula: let K be a number field, ζ_K its Dedekind zeta function, (r_1, r_2) its signature, h its class number, w the number of its roots of unity and R its regulator. Then

$$\lim_{s \rightarrow 0} s^{-r_1 - r_2 + 1} \zeta_K(s) = -\frac{hR}{w}. \quad (4.26)$$

So we recover analytically the rank $r_1 + r_2 - 1$ of the units O_K^* as well as a number involving h, R and w . Up to the rational number h/w , the special value of $\zeta_K(s)$ at $s = 0$ is the regulator. Deligne, Lichtenbaum, Soulé and Beilinson have formulated

conjectures generalising this formula. These conjectures really are for two very different types of zeta or L -functions:

- The zeta function of an arithmetic scheme (Lichtenbaum, Soulé).
- The “Hasse–Weil” L -functions associated to H^i of a smooth projective variety over a number field K , or more generally to a K -motive for absolute Hodge cycles (Deligne, Beilinson, Bloch–Kato...)

They have shaped the development of algebraic K -theory and later motivic cohomology and the theory of motives ever since they were formulated. Here I am only going to discuss the first case: the second one is much harder to even state and completely beyond the scope of these notes.

4.7.3 Soulé’s Conjecture

Lichtenbaum formulated very precise conjectures, at least in special cases, while Soulé formulated a general conjecture but only for orders of poles. Let me start with this one. Recall that an *arithmetic scheme* is a scheme of finite type over $\mathbf{Z}$. If X is an arithmetic scheme, its *zeta function*⁹ is

$$\zeta(X, s) = \prod_{x \in X_{(0)}} (1 - N(x)^{-s})^{-1},$$

where $X_{(0)}$ is the set of closed points of X and, for $x \in X_{(0)}$, $N(x) = |\kappa(x)|$, the cardinality of the residue field at x . This formal expression has some obvious properties:

1. $\zeta(X, s)$ only depends on the reduced structure of X .
2. If Z is closed in X with open complement U , then

$$\zeta(X, s) = \zeta(U, s) \zeta(Z, s). \quad (4.27)$$

3. $\zeta(X \times \mathbf{A}^1, s) = \zeta(X, s - 1).$ (4.28)

4. If $f : X \rightarrow Y$ is a morphism, then

$$\zeta(X, s) = \prod_{y \in Y_{(0)}} \zeta(X_y, s), \quad (4.29)$$

where X_y is the fibre of f at y .

Using this, one easily proves that $\zeta(X, s)$ converges absolutely for $\operatorname{Re}(s) > \dim X$ by reducing to Riemann’s zeta function (see [169, Proof of Theorem 1] for details), hence is analytic in this domain as a Dirichlet series. It is conjectured to have a meromorphic continuation to the whole complex plane: this is known at least in the half-plane $\operatorname{Re}(s) > \dim X - 1/2$ [169, Th. 2].

⁹ This notion goes back to Artin, Hasse and Weil. To the best of my knowledge, the place where it is first defined in this generality is Serre [168].

Finally, if X is defined over a finite field k with q elements, then one has the famous formula (Grothendieck–Artin–Verdier):

$$\zeta(X, s) = \prod_{i=0}^{2d} \det(1 - F_X q^{-s} \mid H_c^i(\overline{X}, \mathbf{Q}_l))^{(-1)^{i+1}}, \quad (4.30)$$

where $H_c^*(\overline{X}, \mathbf{Q}_l)$ is the $\mathbf{Q}_l$ -adic cohomology with compact supports of the geometric fibre $\overline{X}$ [SGA 5, Exp. XV]. In particular $\zeta(X, s)$ is a rational function in q^{-s} (a result originally proven by Dwork [42]) and the meromorphic continuation is obvious.

If $X = \text{Spec } O_K$ for a number field K , we recover the Dedekind zeta function of K .

Conjecture 42: (*Soulé Conjecture [175, Conj.2.2]*) For any $n \in \mathbf{Z}$, we have

42

$$\text{ord}_{s=n} \zeta(X, s) = \sum_{i \in \mathbf{Z}} (-1)^{i+1} \dim_{\mathbf{Q}} K'_i(X)_{(n)},$$

where $K'_i(X)_{(n)}$ is the part of weight n of $K'_i(X)$ under the homological Adams operations.

Remark 43

43

- 1) This is a conjecture built over conjectures! First, it presupposes the meromorphic continuation of $\zeta(X, s)$. Then, implicitly, the dimensions involved in this formula are finite and almost all 0: this would be a consequence of Conjecture 36, via Theorem 39.
- 2) Using (4.11), we may now rewrite the right hand side as

$$\sum_{i \in \mathbf{Z}} (-1)^{i+1} \dim_{\mathbf{Q}} H_{2n-i}^{BM}(X, \mathbf{Q}(n)) = \sum_{j \in \mathbf{Z}} (-1)^{j+1} \dim_{\mathbf{Q}} H_j^{BM}(X, \mathbf{Q}(n)),$$

which looks much more like an Euler–Poincaré characteristic.

For a conjecture on the special values of this zeta function, see Theorem 72.

Example 44. Let $X = \text{Spec } O_K$, where K is a number field. It is known that $K_i(X)_{(n)} = K_i(X)^{(1-n)} = 0$ for $i \neq 2(1-n) - 1 = 1 - 2n$, except that $K_i(X)^{(0)} = 0$ for $i \neq 0$; moreover $K_0(X)^{(0)} = K_0(X) \otimes \mathbf{Q}$ and $K_{1-2n}(X)^{(1-n)} = K_{1-2n}(X) \otimes \mathbf{Q}$. Hence the conjecture reads, replacing n by $1-n$:

$$\text{ord}_{s=1-n} \zeta(X, s) = \begin{cases} -\dim_{\mathbf{Q}} K_0(X) \otimes \mathbf{Q} & \text{for } n = 0 \\ \dim_{\mathbf{Q}} K_{2n-1}(X) \otimes \mathbf{Q} & \text{for } n > 0. \end{cases} \quad (4.31)$$

For $n = 0$, this says that $\zeta(X, s)$ has a pole of order 1 at $s = 1$, which is classical. For $n = 1$, it follows from (4.26). For $n > 1$, it is easy to compute the left hand side via the functional equation: one finds

$$g_n := \text{ord}_{s=1-n} \zeta(X, s) = \begin{cases} r_1 + r_2 & \text{if } n \text{ is odd} \\ r_2 & \text{if } n \text{ is even} . \end{cases}$$

It is a theorem of Borel [24] that the right hand side has the same value, so that (4.31) is true (of course, Soulé's conjecture was only formulated much later); this prompted Lichtenbaum's conjecture (or question) 46 below, which in turn prompted further work of Borel in this direction [25], see Theorem 47.

Let us toy with the Soulé conjecture as we toyed with the Beilinson–Soulé conjecture. From (4.27) and (4.4) (or the easier localisation theorem of Quillen for K' -theory), one deduces that if $X = U \cup Z$ and the conjecture is true for two among X, U, Z , then it is true for the third. From this follows easily:

45

Lemma 45 The following conditions are equivalent:

- (i) Conjecture 42 is true for all X .
- (ii) Conjecture 42 is true for all X affine and regular.
- (iii) Conjecture 42 is true for all X projective over $\mathbb{Z}$.

One would like to refine this lemma further, reducing to X as in Lemma 7 c) (or d) if we restrict to X s of positive characteristic). Unfortunately I don't see how to do this: unlike an abelian group, a number does not have direct summands! This reduction will work however if we know some strong form of resolution of singularities (e.g. for $\dim X \leq 2$, by Abyankhar). This approach is probably too crude, see Theorem 58.

Soulé's conjecture is true for $n > d = \dim X$ because both sides of the equality are then 0. For $n = d$ it is true by [169, Th. 6]. For $n = d - 1$ and X regular and irreducible, it was formulated by Tate in [189] and implies the Birch–Swinnerton–Dyer conjecture (in fact, is equivalent to it under some strong enough form of resolution of singularities, see above); more generally, it is compatible with the Beilinson conjectures in a suitable sense. For details on all this, see Soulé [175, 2.3 and 4.1]. Finally, it is equivalent to a part of Lichtenbaum's second conjecture if X is smooth projective over a finite field, see Sect. 4.7.6.

We shall now state the two conjectures of Lichtenbaum on special values of zeta functions: the first concerns Dedekind zeta functions and the second those of smooth projective varieties over a finite field.

Lichtenbaum's First Conjecture: Rings of Algebraic Integers

4.7.4

(See Goncharov's chapter for many more details on this subsection, including the relationship with polylogarithms.)

To state this conjecture, recall that $K_{2n}(O_K)$ is finite because it is finitely generated (Proposition 38) and has rank 0 (Borel [24]). Next, Borel defined a regulator

$$\varphi_n : K_{2n-1}(O_K) \otimes \mathbf{R} \rightarrow \mathbf{R}^{g_n}$$

which is an isomorphism by [24]. Let $R_n(K)$ be the absolute value of the determinant of φ_n with respect of a basis of $K_{2n-1}(O_K)/\text{tors}$ and the canonical basis of $\mathbf{R}^{g_n}$. Lichtenbaum asks prudently:

Conjecture 46: (*Lichtenbaum [123, Question 4.2]*) When is it true that

46

$$\lim_{s \rightarrow 1-n} (s+n-1)^{-g_n} \zeta_K(s) = \pm \frac{|K_{2n-2}(O_K)|}{|K_{2n-1}(O_K)_{\text{tors}}|} R_n(K) ? \quad (4.32)$$

Note that the sign is not mysterious at all: it is easy to get as follows. If we restrict $\zeta_K(s)$ to s real, it takes real values. For $s > 0$ it is positive. Since it has a single pole at $s = 1$, it is negative for $s < 1$ in the neighbourhood of $s = 1$. Between $s = 0$ and $s = 1$, its only possible zero is for $s = 1/2$. But the functional equation shows that this possible zero has even order (I am indebted to Pierre Colmez for this trick). Therefore its value for $s > 0$ near 0 is still negative and the sign at $s = 0$ is -1 . Then it is known that the only zeroes or poles are at negative integers, and the above reasoning gives the sign in (4.32) immediately. One finds

$$(-1)^{g_{n-1}+g_{n-2}+\dots+g_1+1} = \begin{cases} (-1)^{\frac{n}{2}r_1+r_2} & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}r_1} & \text{if } n \text{ is odd } > 1. \end{cases} \quad (4.33)$$

This computation also appears in Kolster [111] (using the functional equation).

In [25], Borel gave the following partial answer:

Theorem 47

47

$$\lim_{s \rightarrow 1-n} (s+n-1)^{-g_n} \zeta_K(s) = R_n(K)$$

up to a nonzero rational number.

Here are some comments on Conjecture 46. Besides (4.26) and Borel's computation of the ranks of K -groups in [24], it was inspired by an earlier conjecture of Birch and Tate (the case $n = 2$) when K is totally real [17], [191] and by a conjecture of Serre [171, p. 164] that, still for K totally real and n even, $\zeta_K(1-n)|H^0(K, \mathbf{Q}/\mathbf{Z}(n))|$ should be an integer; this conjecture was proven later by Deligne–Ribet [40]. This

case is much simpler because then $g_n = 0$ and it had been proven long ago by Siegel that the left hand side of (4.32) was a rational number. In this special case, Lichtenbaum initially conjectured the following equality:

$$\zeta_K(1-n) = \pm \prod_{l \text{ prime}} \frac{|H^1(O_K[1/l], \mathbf{Q}_l/\mathbf{Z}_l(n))|}{|H^0(O_K[1/l], \mathbf{Q}_l/\mathbf{Z}_l(n))|}.$$

Under this form, the conjecture was proven by Mazur–Wiles for K abelian over $\mathbf{Q}$ [131], and then by Wiles in general, except perhaps for $l = 2$ [212], as a consequence of their proofs of Iwasawa’s Main Conjecture. Still in this special case, Quillen’s conjectures (4.24) and (4.25) prompted the K -theoretic formulation (4.32), up to a power of 2 since (4.24) and (4.25) were formulated only for l odd and computations showed that the 2-primary part of the formula was false. This is now explained, for example, by Theorem 33: the correct formula, still in the case where K is totally real and n is even, is (cf. [88, Cor. 1], [160])

$$\zeta_K(1-n) = \pm 2^{r_1} \frac{|K_{2n-2}(O_K)|}{|K_{2n-1}(O_K)|} = \pm \frac{|H^2(O_K, \mathbf{Z}(n))|}{|H^1(O_K, \mathbf{Z}(n))|}.$$

We could say that this conjecture is essentially proven now if one believes that the proof of the Bloch–Kato conjecture (for Milnor’s K -theory) is complete.

How about the general conjecture? First there was an issue on the correct normalisation of the Borel regulator, as Borel’s original definition does not give Theorem 47, but the same formula with the right hand side multiplied by π^{g_n} . The normalisation issue is basically accounted for by the difference between the Hodge structures $\mathbf{Z}$ and $\mathbf{Z}(1) = 2\pi i \mathbf{Z}$: we refer to [30, Ch. 9] for a very clear discussion (see also [26]). Then Beilinson formulated his general conjectures which should have Borel’s theorem as a special case: there was therefore the issue of comparing the Borel and the Beilinson regulators.¹⁰ This was done by Beilinson himself up to a nonzero rational number (see [158]), and finally Burgos [30] showed that the Beilinson and Borel regulator *maps* differ by a factor 2, hence the corresponding determinants differ by 2^{g_n} .

Let me give what I believe is the correct formulation in terms of motivic cohomology and a version of Beilinson’s regulator (see also for example [110, 111]). This will be the only allusion to Beilinson’s point of view in this survey. We define $H^i(O_K, \mathbf{Z}(n))$ as Levine does in [119] and [121], using a suitable version of Bloch’s cycle complexes for schemes over $\mathbf{Z}$. Then the construction of a motivic cycle class map yields “regulator” maps to Deligne’s cohomology (see §6.1 in Levine’s chapter; note that $H^1(O_K, \mathbf{Q}(n)) \xrightarrow{\sim} H^1(K, \mathbf{Q}(n))$ for $n \geq 2$ and that the regulator is just Dirichlet’s regulator for $n = 1$)

$$\varrho'_n : H^1(O_K, \mathbf{Z}(n)) \otimes \mathbf{R} \rightarrow H_D^1(O_K \otimes_{\mathbf{Z}} \mathbf{R}, \mathbf{R}(n)),$$

¹⁰ The great superiorities of the Beilinson regulator over the Borel regulator are its conceptual definition, its functoriality and its computability in certain cases. On the other hand, no proof of Theorem 47 directly in terms of the Beilinson regulator is known at present.

which can be compared to Beilinson's regulator, the latter being essentially a Chern character. The Lichtenbaum conjecture should then read

$$\lim_{s \rightarrow 1-n} (s+n)^{-g_n} \zeta_K(s) = \varepsilon_n \frac{|H^2(O_K, \mathbf{Z}(n))|}{|H^1(O_K, \mathbf{Z}(n))_{\text{tors}}|} R'_n(K), \quad (4.34)$$

where $R'_n(K)$ is the absolute value of the determinant of ϕ'_n with respect to integral bases and ε_n is as in (4.33).

The best formulation would be in terms of étale motivic cohomology with compact supports à la Kato–Milne [137, p. 203], but this would lead us too far. (Lichtenbaum is currently working on a conjectural formula involving cohomology groups defined by means of the Weil groups, in the spirit of his Weil–étale topology in characteristic p which we shall explain in Sect. 4.7.8 [127].)

As for the general case of the Lichtenbaum conjecture, it is now proven with the same caveat (Bloch–Kato conjecture) for K abelian over $\mathbf{Q}$, by the work of Fleckinger–Kolster–Nguyen Quang Do [47] (see also [15] and [16, appendix]). For nonabelian K we are still far from a proof.

Note that, if one is only interested in totally real K and even n , one may reformulate Conjecture 46 purely in terms of étale cohomology, and if one is only interested in Theorem 47 one may reformulate things in terms of the homology of $GL_n(O_K)$. In both cases one can get rid of algebraic K -theory and motivic cohomology. However, if one wants the general case, there is no way to avoid them. This encapsulates the beauty and the depth of this conjecture!

Remark 48 In the sequel we shall amply discuss varieties over finite fields. Let us make here a few comments on the 1-dimensional case. Let X be a smooth projective curve over $\mathbf{F}_p$. By the already mentioned theorem of Quillen [66], the groups $K_i(X)$ are finitely generated. On the other hand, their rank was computed by the work of Harder [73]: for $i > 0$ it is 0, hence $K_i(X)$ is finite. Harder computes the rank of the homology of $SL_n(A)$, where A is the coordinate ring of an affine open subset of X , very much in the style of Borel [24], hence using Riemannian geometry.

There are two completely different proofs of this rank computation. The first one is due to Soulé [176, 2.3.4] and uses motivic methods: see Sect. 4.8.2 below. The second one uses the Milnor conjecture (Theorem 21): by the spectral sequence (4.6) (or the isomorphism (4.9)) it is enough to show that $H^i(X, \mathbf{Q}(n)) = 0$ for $i \neq 2n$. This can be done as for the proof of Theorem 39. This argument relies on knowing the finite generation of the $K_i(X)$ while Harder's and Soulé's proofs do not. On the other hand, we shall get the finite generation of $H^i(X, \mathbf{Z}(n))$ directly in Sect. 4.8, without appealing to Quillen's theorem but using the Bloch–Kato conjecture.

The Tate, Beilinson and Parshin Conjectures

4.7.5

For the rest of this section, k is a finite field and X is a smooth projective k -variety. We also give ourselves a nonnegative integer n . Before introducing the

second conjecture of Lichtenbaum, it is appropriate to recall two closely-related conjectures. The first one is the famous Tate conjecture:

49 **Conjecture 49:** (*Tate Conjecture*) $\text{ord}_{s=n} \zeta(X, s) = -\dim_{\mathbf{Q}} A_{\text{num}}^n(X, \mathbf{Q})$.

The second one, due to Beilinson, is a special case of his conjectures on filtrations on Chow groups [81].

50 **Conjecture 50:** (*Beilinson Conjecture*) $A_{\text{rat}}^n(X, \mathbf{Q}) = A_{\text{num}}^n(X, \mathbf{Q})$.

There is a third related conjecture, due to Beilinson and Parshin:

51 **Conjecture 51:** (*Beilinson–Parshin Conjecture*) $K_i(X)$ is torsion for $i > 0$.

In view of (4.10), the Beilinson–Parshin conjecture may be reformulated in terms of motivic cohomology as follows: $H^i(X, \mathbf{Q}(n)) = 0$ for $i \neq 2n$. In particular, this conjecture is a strong reinforcement of the Beilinson–Soulé conjecture for schemes of characteristic p (compare Lemma 7).

Geisser has proven:

52 **Theorem 52:** ([56, Th. 3.3]) Conjecture 49 + Conjecture 50 $\Rightarrow$ Conjecture 51.

Similarly, the Bass Conjecture 36 implies Conjecture 51, just as it implies Conjecture 5 [87].

Let us compare these conjectures with Soulé’s Conjecture 42 restricted to smooth projective varieties over $\mathbf{F}_p$. Using the functional equation, Conjecture 49 may be reformulated as follows: $\text{ord}_{s=n} \zeta(X, s) = -\dim_{\mathbf{Q}} A_{\text{num}}^n(X, \mathbf{Q})$. On the other hand, Conjecture 42 predicts that the value of the left hand side should be $\sum_{i \in \mathbf{Z}} (-1)^{i+1} \dim_{\mathbf{Q}} H_i^{BM}(X, \mathbf{Q}(n))$ (see Remark 43 2)). Under Conjecture 51, this reduces to $-\dim_{\mathbf{Q}} H_{2n}^{BM}(X, \mathbf{Q}(n)) = -\dim_{\mathbf{Q}} CH_n(X) \otimes \mathbf{Q}$. Hence, assuming the Parshin conjecture, among the Soulé, the Tate and the Beilinson conjecture, any two imply the third. In particular, the Tate conjecture plus the Beilinson conjecture imply the Soulé conjecture for smooth projective varieties – but see in fact Theorem 58 below. Alternatively, we may replace the use of the functional equation by the observation that $\dim_{\mathbf{Q}} A_{\text{num}}^n(X, \mathbf{Q}) = \dim_{\mathbf{Q}} A_{\text{num}}^{d-n}(X, \mathbf{Q})$, where $d = \dim X$.

The Tate conjecture is known in codimension 1 for abelian varieties, by Tate’s theorem [190]; it is trivial in dimension 0. Besides this it is known in many scattered cases, all being either abelian varieties or varieties “of abelian type”, see examples 75 below. In particular, Soulé deduced it from [190] for products of 3 curves by a very simple motivic argument, and then for all varieties of abelian type of dimension ≤ 3 (in a slightly restricted sense compared to the one of [95], see Example 75 1)) by a dévissage argument from the former case [176, Th. 4 (i)].

The Beilinson conjecture is trivial in codimension 1; in dimension 0 it is true for any variety by a theorem of Kato and Saito [102]. Soulé proved it in the same cases as the Tate conjecture (*loc. cit.*), and in particular his Conjecture 42 is true for this type of varieties. Besides this, it was unknown except for trivial cases like projective homogeneous varieties until Theorem 76 below, which proves new cases of it.

Finally, let us give a consequence of Conjecture 51 for fields, using de Jong's alteration theorem (*cf.* [56, Th. 3.4]):

Lemma 53 If Conjecture 51 holds for all smooth projective varieties over $\mathbf{F}_p$, then for any field K of characteristic p and any $n \geq 0$, **53**

- (i) $H^i(K, \mathbf{Q}(n)) = 0$ for $i \neq n$.
- (ii) $K_n^M(K)$ is torsion as soon as $n > \text{trdeg}(K/\mathbf{F}_p)$ (Bass–Tate conjecture).

The proof goes exactly as in that of Lemma 7 (recall that $H^i(K, \mathbf{Z}(n)) = 0$ for $i > n$ anyway). As for the consequence on Milnor's K -theory, one uses Theorem 2 and the fact that $H^{2n}(X, \mathbf{Z}(n)) = CH^n(X) = 0$ for $n > \dim X$.

Lichtenbaum's Second Conjecture: Varieties over Finite Fields

4.7.6

This conjecture, which appears in [124], was formulated in two steps, in terms of a not yet constructed “arithmetic cohomology theory”, later rechristened “motivic cohomology”. It is important to notice that Lichtenbaum formulated it for the *étale* hypercohomology of certain complexes. Here it is:

1. $H_{\text{ét}}^i(X, \mathbf{Z}(n)) = 0$ for i large.
2. $H_{\text{ét}}^{2n}(X, \mathbf{Z}(n))$ is a finitely generated abelian group.
3. $H_{\text{ét}}^i(X, \mathbf{Z}(n))$ is finite for $i \neq 2n, 2n+2, 0$ for $i \leq 0$ when $n > 0$.
4. $H_{\text{ét}}^{2d+2}(X, \mathbf{Z}(d))$ is canonically isomorphic to $\mathbf{Q}/\mathbf{Z}$, where $d = \dim X$.
5. The pairing

$$H_{\text{ét}}^i(X, \mathbf{Z}(n)) \times H_{\text{ét}}^{2d+2-i}(X, \mathbf{Z}(d-n)) \rightarrow H_{\text{ét}}^{2d+2}(X, \mathbf{Z}(d)) \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}$$

is “perfect” in the sense that it defines a perfect duality of finite groups for $i \neq 2n$ and a perfect duality between a finitely generated group and a group of finite cotype pour $i = 2n$. In particular, $\text{rg } H_{\text{ét}}^{2d}(X, \mathbf{Z}(d)) = 1$.

6. The groups $H_{\text{ét}}^{2n}(X, \mathbf{Z}(n))$ and $H_{\text{ét}}^{2d-2n}(X, \mathbf{Z}(d-n))$ have the same rank $m(n)$.
7. $m(n)$ is the order of the pole of $\zeta(X, s)$ at $s = n$.
8. $\lim_{s \rightarrow n} (1 - q^{n-s})^{m(n)} \zeta(X, s) = \pm q^{\chi(X, \mathcal{O}_X(n))} \chi(X, \mathbf{Z}(n))$, with

$$\chi(X, \mathbf{Z}(n)) =$$

$$\prod_{i \neq 2n, 2n+2} |H_{\text{ét}}^i(X, \mathbf{Z}(n))|^{(-1)^i} \cdot \frac{|H_{\text{ét}}^{2n}(X, \mathbf{Z}(n))_{\text{tors}}| \cdot |H_{\text{ét}}^{2n+2}(X, \mathbf{Z}(n))_{\text{cotors}}|}{R_n(X)}$$

where $R_n(X)$ is the absolute value of the determinant of the pairing

$$H_{\text{ét}}^{2n}(X, \mathbf{Z}(n))/\text{tors} \times H_{\text{ét}}^{2d-2n}(X, \mathbf{Z}(d-n))/\text{tors} \rightarrow H_{\text{ét}}^{2d}(X, \mathbf{Z}(d))/\text{tors} \xrightarrow{\sim} \mathbf{Z}$$

relatively to arbitrary bases of $\frac{H_{\text{ét}}^{2n}(X, \mathbf{Z}(n))}{\text{tors}}$ and $\frac{H_{\text{ét}}^{2d-2n}(X, \mathbf{Z}(d-n))}{\text{tors}}$, and

$$\chi(X, \mathcal{O}_X, n) = \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq d}} (-1)^{i+j} (n-i) h_{ij}, \quad h_{ij} = \dim H^i(X, \Omega^j).$$

Let us examine these predictions in terms of the present state of knowledge. Statement 1 is known: for $i > 2n+1$, $H_{\text{ét}}^{i-1}(X, \mathbf{Q}/\mathbf{Z}(n)) \xrightarrow{\sim} H_{\text{ét}}^i(X, \mathbf{Z}(n))$ by Theorem 14 and the discussion before Theorem 2; but $cd(X) = 2d+1$ since a finite field has étale cohomological dimension 1. So we may take $i > 2d+2$. Similarly, Statement 4 is known, as well as the fact that $\text{rk} H_{\text{ét}}^{2d}(X, \mathbf{Z}(d)) = 1$.

For the other statements, the following remarks are in order. Statement 6 is a formal consequence of the part of 5 which predicts a nondegenerate pairing between the two groups. In view of Theorem 14 and the fact that $H^{2n}(X, \mathbf{Z}(n)) = CH^n(X)$, Statement 7 follows from the conjunction of Conjectures 49 and 50; given 3 and 5 it is equivalent to Soulé's conjecture 42. Finally, statements 2, 3 and 5 are striking in that they predict finite generation properties of étale motivic cohomology, but in a rather scattered way. This will be corrected (by an idea of Lichtenbaum!) in Sect. 4.7.8.

Motivic Reformulation of the Tate and Beilinson Conjectures

4.7.7

One major point of this whole story is that Conjectures 49 and 50 really have to be considered together. Then they have a very nice and very powerful reformulation: this was the subject of [90]. I wrote it using Voevodsky's version of motivic cohomology, which made a rather simple construction but necessitated some undesirable assumptions on resolution of singularities in characteristic p . The version with Bloch's higher Chow groups, developed in [93], involves more technicality but is free of resolution of singularities assumptions. Let me explain it now.

For $l \neq p$, define

$$\mathbf{Z}_l(n)^c = R \lim_{\leftarrow} \mu_{\mathbb{F}_l}^{\otimes n}.$$

This is an object of $D^+((Sm/\mathbb{F}_p)_{\text{ét}})$, whose hypercohomology computes Jannsen's *continuous étale cohomology* $H_{\text{cont}}^*(X, \mathbf{Z}_l(n))$ [76] for smooth varieties X over $\mathbb{F}_p$. Naturally, by Deligne's finiteness theorem for étale cohomology [SGA 4 1/2, Th. Finitude], we have

$$H_{\text{cont}}^i(X, \mathbf{Z}_l(n)) = \lim_{\leftarrow} H_{\text{ét}}^i(X, \mu_{\mathbb{F}_l}^{\otimes n})$$

but this is a theorem rather than a definition. In any case, the quasi-isomorphisms of Theorem 13 a) can now be assembled into a morphism in $D((Sm/\mathbf{F}_p)_{\text{ét}})$ [93, §1.4]:

$$\alpha^* \mathbf{Z}(n) \otimes \mathbf{Z}_l \rightarrow \mathbf{Z}_l(n)^c .$$

If one is looking for a quasi-isomorphism then this morphism is not yet quite right: for example, for $n = 0$ the left hand side is $\mathbf{Z}_l$ while the isomorphisms $H_{\text{cont}}^1(\mathbf{F}_q, \mathbf{Z}_l) \xrightarrow{\sim} \mathbf{Z}_l$ for all q yield $\mathcal{H}^1(\mathbf{Z}_l(0)^c) \xrightarrow{\sim} \mathbf{Q}_l$ [90, §4 and Th. 6.3]. Using cup-product, let us perform the minimal modification correcting this: we get a morphism

$$\alpha^* \mathbf{Z}(n) \otimes^L \mathbf{Z}_l(0)^c \rightarrow \mathbf{Z}_l(n)^c . \quad (4.35)$$

Theorem 54: ([93, Th. 3.4]) The following statements are equivalent:

54

- (i) Conjectures 49 and 50 are true for all X, n .
- (ii) (4.35) is an isomorphism for any n .
- (iii) $\mathbf{Z}_l(n)^c$ is malleable for any $n > 0$ (see p. 378).

For $l = p$ one can define a morphism analogous to (4.35), using instead of $\mathbf{Z}_l(n)^c$ the object

$$\mathbf{Z}_p(n)^c := R \lim_{\leftarrow} \nu_s(n)[-n] ,$$

where $\nu_s(n)$ is the sheaf of logarithmic de Rham–Witt differential forms:

$$\alpha^* \mathbf{Z}(n) \otimes^L \mathbf{Z}_p(0)^c \rightarrow \mathbf{Z}_p(n)^c . \quad (4.36)$$

see [95, §3.5]. Then an equivalent condition to the above is (cf. [95, §3.6], [58]):

- (ii) bis (4.36) yields an isomorphism on the hypercohomology of any smooth projective X .

Note that (i) involves only algebraic cycles, (iii) involves only cohomology and (ii) is a comparison between them. Also, (i) does not involve l , hence (ii) and (iii) are independent of l .

In fact, in [90, §4] we construct a complex of length 1 of $G_{\mathbf{F}_p}$ -modules $\mathbf{Z}^c$ such that for all l (including $l = p$) there is a canonical isomorphism

$$\mathbf{Z}_l(0)^c \simeq \pi^* \mathbf{Z}^c \otimes \mathbf{Z}_l , \quad (4.37)$$

where π is the projection of the big étale site of $\text{Spec } \mathbf{F}_p$ onto its small étale site (*ibid.*, Th. 4.6 b) and 6.3). So, strikingly, (ii) predicts the existence of a canonical integral structure on arithmetic l -adic cohomology, independent of l . (One should not confuse this prediction with the “independence of p ” conjectures for *geometric* l -adic cohomology.)

Sketch

The equivalence between (ii) and (iii) follows from Theorem 20: the fact that Condition 2 in it is satisfied follows from the results of [94]. The proof of the equivalence between (i) and (ii) is not really difficult: first, by Theorem 13 a), $(ii) \otimes^L \mathbb{Z}/l^v$ is true, so (ii) and $(ii) \otimes \mathbb{Q}$ are equivalent. Using de Jong, we get as in Lemma 7 that $(ii) \otimes \mathbb{Q}$ holds if and only if it holds for every smooth projective variety X . Then one examines the two sides of the maps

$$H_{\text{ét}}^i(X, \mathbb{Q}(n) \otimes \mathbb{Q}_l(0)^c) \rightarrow H_{\text{cont}}^i(X, \mathbb{Q}_l(n))$$

and one deduces via the “Riemann hypothesis” (Weil conjecture) and some of the folklore in [192] that isomorphism for all i and n is equivalent to the conjunction of conjectures 49, 50 and 51. One concludes by Theorem 52. See [93] for details.

55

Definition 55: (*Tate–Beilinson Conjecture*) For simplicity, we call the equivalent conjectures of Theorem 54 the *Tate–Beilinson conjecture*.

Some Consequences

Besides being clearly of a motivic nature, the main point of the Tate–Beilinson conjecture under the form (ii) in Theorem 54 is that it allows one to pass easily from smooth projective varieties to general smooth varieties, or even to arbitrary schemes of finite type over $\mathbb{F}_p$. It has remarkable consequences: one could say that it implies almost everything that one expects for varieties over finite fields. We have already seen that it implies the Beilinson–Parshin conjecture (via Geisser’s Theorem 52), hence the Beilinson–Soulé conjecture in characteristic p . But there is much more. Let us first give some motivic consequences.

By [139, Remark 2.7 and Theorem 2.49], Conjecture 49 implies:

- For any finite field k , $\text{Mot}_{\text{num}}(k, \mathbb{Q})$ is generated by motives of abelian varieties and Artin motives.
- Every mixed motive over a finite field is a direct sum of pure motives.

The last statement is a bit vague as long as one does not have a precise definition of a mixed motive, as was the case when Milne wrote his article. Since now we have at least triangulated categories of motives at our disposal, let me give a precise theorem.

56

Theorem 56 Suppose that the Tate–Beilinson Conjecture 55 holds. Then, for any finite field k :

- (i) Voevodsky’s triangulated category $DM_{\text{gm}}(k, \mathbb{Q})$ is semi-simple in the sense that any exact triangle is a direct sum of split exact triangles.

(ii) The functor δ of (4.15) induces an equivalence of categories

$$\begin{aligned}\Delta : \text{Mot}_{\text{num}}(k, \mathbf{Q})^{(\mathbf{Z})} &\xrightarrow{\sim} DM_{\text{gm}}(k, \mathbf{Q}) \\ (M_i) &\mapsto \bigoplus_{i \in \mathbf{Z}} \delta(M_i)[i] .\end{aligned}$$

(iii) Equivalently, (4.15) induces an equivalence of categories

$$\Delta : D^b(\text{Mot}_{\text{num}}(k, \mathbf{Q})) \xrightarrow{\sim} DM_{\text{gm}}(k, \mathbf{Q}) .$$

Proof First we check that, for $M, N \in \text{Mot}_{\text{rat}}(k, \mathbf{Q}) = \text{Mot}_{\text{num}}(k, \mathbf{Q})$

$$\text{Hom}_{DM}(\delta(M), \delta(N)[i]) = \begin{cases} 0 & \text{for } i \neq 0 \\ \text{Hom}_{\text{Mot}}(M, N) & \text{for } i = 0 . \end{cases}$$

For this, we reduce by duality to the case where $M = 1$, and then to the case where N is of the form $h(X)(n)$ for X smooth projective. Then the left hand side is $H^{i+2n}(X, \mathbf{Q}(n))$ by [199, Cor. 3.2.7] and the cancellation theorem of Voevodsky [203], and the conclusion follows from Theorem 52.

This implies that Δ is fully faithful. To see that it is essentially surjective, using de Jong it now suffices to show that its essential image is thick, *i.e.* stable under exact triangles and direct summands. This follows from the following trivial but very useful lemma (*cf.* [6, Lemma A.2.13]): in a semi-simple abelian category, any morphism is the direct sum of an isomorphism and a 0 morphism. The same lemma implies that $DM_{\text{gm}}(k, \mathbf{Q})$ is semi-simple. Finally, (iii) is equivalent to (ii) because $\text{Mot}_{\text{num}}(k, \mathbf{Q})$ is semi-simple.

Remark 57

- 1) This implies trivially a number of conjectures: the existence of a motivic t -structure on $DM_{\text{gm}}(k, \mathbf{Q})$, the semi-simplicity of Galois action, independence of $l \dots$
 - 2) Theorem 56 (i) extends to $DM(k, \mathbf{Q})$, hence to $\mathcal{SH}(k, \mathbf{Q})$ (see 4.4.5).
-

Next, the Tate–Beilinson conjecture implies the Lichtenbaum conjecture of the previous section. This was proven in [90] after localising at l and under resolution of singularities, but using the higher Chow groups version of $\mathbf{Z}(n)$ we can get rid of the last assumption. Localising at l can also be got rid of. In fact one can get a version of the Lichtenbaum conjecture for arbitrary, not just smooth projective, schemes of finite type over $\mathbf{F}_p$: (ii) is especially well-adapted to this. We shall give details on this in Theorem 72. For the moment, let us sketch a proof of:

58 Theorem 58 The Tate–Beilinson Conjecture 55 implies the Soulé Conjecture 42.

Sketch

We shall derive the form of Remark 43 2). In view of (4.30), it is sufficient to prove the stronger equality

$$\text{ord}_{s=n} \det(1 - F_X q^{-s} | H_c^i(\overline{X}, \mathbf{Q}_l)) = \dim_{\mathbf{Q}} H_i^{BM}(X, \mathbf{Q}(n))$$

for all X, i, n . Now the morphism (4.35) has a homological version

$$H_i^{BM, \text{ét}}(X, \mathbf{Z}(n) \otimes \mathbf{Z}_l(0)^c) \rightarrow H_i^{c, \text{cont}}(X, \mathbf{Z}_l(n)), \quad (4.38)$$

where the left hand side is Borel–Moore étale motivic homology as explained in Sect. 4.5.4 (see the end for the coefficients $\mathbf{Z}_l(0)^c$) and the right hand side is the continuous version of Borel–Moore étale homology (relative to $\mathbf{F}_p$).

59 Lemma 59 Under the Tate–Beilinson Conjecture 55, (4.38) is an isomorphism for any X, i, n .

This is easily proven by a dévissage using the localisation exact sequence plus Poincaré duality for both sides. $\square$

From this one deduces isomorphisms

$$H_i^{BM}(X, \mathbf{Q}(n)) \otimes \mathbf{Q}_l \xrightarrow{\sim} H_i^{c, \text{cont}}(\overline{X}, \mathbf{Q}_l(n))^G$$

with $G = G_{\mathbf{F}_p}$, and moreover that G acts semi-simply on $H_{c, \text{cont}}^i(\overline{X}, \mathbf{Q}_l(n))$ at the eigenvalue 1 (this means that the characteristic subspace corresponding to the eigenvalue 1 is semi-simple). Since $H_{c, \text{cont}}^i(\overline{X}, \mathbf{Q}_l(n))$ is dual to $H_i^{c, \text{cont}}(\overline{X}, \mathbf{Q}_l(-n))$, the result follows.

The first instance I know of this dévissage argument is [78, Th. 12.7]. Jannsen assumed resolution of singularities there but this is now unnecessary thanks to de Jong’s theorem.

Since any finitely generated field K over $\mathbf{F}_p$ is a filtering direct limit of finitely generated smooth $\mathbf{F}_p$ -algebras and any smooth variety over K is a filtering inverse limit of smooth varieties over $\mathbf{F}_p$, one also gets consequences of the Tate–Beilinson conjecture for such varieties. Typically:

60 Theorem 60: ([90, Th. 8.32]) The Tate–Beilinson Conjecture 55 implies the following for any finitely generated field $K/\mathbf{F}_p$ and any smooth projective variety X/K :

- (i) (Tate conjecture) The map $CH^n(X) \otimes \mathbf{Q}_l \rightarrow H_{\text{cont}}^{2n}(\overline{X}, \mathbf{Q}_l(n))^{G_K}$ is surjective, where $G_K = \text{Gal}(\overline{K}/K)$.

- (ii) The action of G_K on $H_{\text{cont}}^*(\overline{X}, \mathbf{Q}_l)$ is semi-simple.
 (iii) The cycle map $CH^i(X) \otimes \mathbf{Q}_l \rightarrow \widetilde{H}_{\text{cont}}^{2i}(X/\mathbf{F}_p, \mathbf{Q}_l(i))$ is injective for all i .

In (iii), the group $\widetilde{H}_{\text{cont}}^{2i}(X/\mathbf{F}_p, \mathbf{Q}_l(i))$ is by definition the direct limit of the $H_{\text{cont}}^{2i}(\mathcal{X}, \mathbf{Q}_l(i))$, where $\mathcal{X}$ runs through the smooth models of X of finite type over $\mathbf{F}_p$.

Sketch

Extend X to a smooth, projective morphism $f : \mathcal{X} \rightarrow U$ over a suitable smooth model U of K . By Hard Lefschetz and Deligne's degeneration criterion [38], the Leray spectral sequence

$$E_2^{pq} = H_{\text{cont}}^p(U, R^q f_*^{l\text{-adic}} \mathbf{Q}_l(n)) \Rightarrow H_{\text{cont}}^{p+q}(\mathcal{X}, \mathbf{Q}_l(n))$$

degenerates. (i) follows rather easily from this and the conjecture. The semi-simplicity statement (ii) is only proven in [90] at the eigenvalue 1: the proof consists roughly of “hooking” the geometric semi-simplicity theorem of Deligne [34, Cor. 3.4.13] on the arithmetic semi-simplicity (special case $K = k$). One can however prove it in general by using some folklore ([80, 105] and the argument in [46, pp. 212–213]¹¹), cf. [96]. The proof of (iii) is a simple direct limit argument.

Here are two other nice consequences (the proofs are the same as for [95, Cor. 2.6 and Th. 4.6], using Theorem 26 a)):

Theorem 61 Assume the Tate–Beilinson and the Bloch–Kato conjectures 55 and 16. Then

61

- a) Gersten's conjecture for algebraic K -theory holds for any discrete valuation ring (hence for any local ring of a scheme smooth over a discrete valuation ring by Gillet–Levine [64]).
 b) For any field K of characteristic p one has canonical isomorphisms

$$K_n^M(K) \oplus \bigoplus_{0 \leq i \leq n-1} H^{2i-n-1}(K, (\mathbf{Q}/\mathbf{Z})'(i)) \xrightarrow{\sim} K_n(K),$$

where $(\mathbf{Q}/\mathbf{Z})'(i) = \lim_{\rightarrow (m, \text{char } k)=1} \mu_m^{\otimes i}$. The spectral sequence (4.6) canonically degenerates.

By [192, Prop. 2.6 and Th. 3.1]), Theorem 60 (i) and (ii) imply the standard conjecture HN on page 368 (hence the other ones: B and C). On the other hand, by an argument similar to that in [81, Lemma 2.7], (iii) implies the filtration conjecture of Bloch–Beilinson–Murre. As was proved by Peter O'Sullivan, the latter

¹¹ Which I am grateful to Yves André for explaining me.

implies Voevodsky's conjecture 10 (see [3, Th. 11.5.3.1]). Also, under the Bloch–Beilinson–Murre conjecture, Hanamura's vanishing conjecture (Van) in [71, III] may be reformulated as follows: for any smooth projective X/K , one has

$$H^q(h^{\geq i}X, \mathbf{Q}(n)) = 0 \quad \text{for} \quad \begin{cases} q \leq i & \text{if } q \neq 2n \\ q < i & \text{if } q = 2n, \end{cases}$$

where $h^{\geq i}(X)$ denotes the part of weight $\geq i$ of $h(X) \in \text{Mot}_{\text{rat}}(K, \mathbf{Q})$ under the Bloch–Beilinson–Murre filtration. It would be sufficient to have this vanishing in order to get a motivic t -structure on his category, but I have not derived it from the Tate–Beilinson conjecture. Presumably one should first prove a version of (Van) relative to a smooth model of K and then pass to the limit as we did for the Beilinson–Soulé conjecture: this looks feasible but fairly technical.

Another conjecture I don't know how to derive from the Tate–Beilinson conjecture is the Hodge index standard conjecture, see [109, §5].

Lichtenbaum's Weil-étale Topology; Reformulation of his Second Conjecture

4.7.8

In [126], Lichtenbaum introduced a new Grothendieck topology on schemes of characteristic p : he christened it *Weil-étale topology*. This leads to a fundamental clarification of the formulation of his previous conjectures, and of what should be true or not in terms of finite generation conjectures.

Roughly, Lichtenbaum replaces the Galois group $\text{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q) \simeq \widehat{\mathbf{Z}}$ by its dense subgroup generated by Frobenius ($\simeq \mathbf{Z}$) and extends this idea (which of course goes back to Weil) to schemes of higher dimension. The corresponding cohomology theory should be called *Weil-étale cohomology*. I find this terminology awkward because it can create confusion with a “Weil cohomology”, especially as most known Weil cohomology theories in characteristic p are based on étale cohomology! For this reason, and also as a tribute to Lichtenbaum's paternity, I prefer to rechristen it *Lichtenbaum cohomology*, while keeping his notation $H_W^*(X, \mathcal{F})$ which recalls Weil's contribution.

We may take the hypercohomology of Bloch's cycle complexes (or the Suslin–Voevodsky complexes) in the Weil-étale topology and get *Lichtenbaum motivic cohomology*¹² $H_W^i(X, \mathbf{Z}(n))$. The various motivic cohomology groups map to each other as follows:

$$H^i(X, \mathbf{Z}(n)) \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}(n)) \rightarrow H_W^i(X, \mathbf{Z}(n)) .$$

Lichtenbaum's cohomology has been developed by Geisser in [58]. His main results are the following:

¹² Thereby conflicting with a terminology briefly introduced by Voevodsky for étale motivic cohomology...

Theorem 62: (*Geisser*) Let ε be the projection of the Weil-étale site onto the (usual) étale site. Then for any complex of étale sheaves C ,

62

(i) There is a quasi-isomorphism

$$R\varepsilon_*\varepsilon^*C \simeq C \otimes^L R\varepsilon_*\varepsilon^*\mathbf{Z}.$$

(ii) There are long exact sequences

$$\begin{aligned} \cdots \rightarrow H_{\text{ét}}^i(X, C) \rightarrow H_W^i(X, \varepsilon^*C) \\ \rightarrow H_{\text{ét}}^{i-1}(X, C) \otimes \mathbf{Q} \xrightarrow{\partial} H_{\text{ét}}^{i+1}(X, C) \rightarrow \cdots \end{aligned} \quad (4.39)$$

Moreover $R\varepsilon_*\varepsilon^*\mathbf{Z} \simeq \mathbf{Z}^c$, where $\mathbf{Z}^c$ is the complex alluded to in (4.37).

As important special cases, which give a feel of Lichtenbaum cohomology, we get:

Corollary 63

63

- (i) $H_{\text{ét}}^i(X, C) \xrightarrow{\sim} H_W^i(X, \varepsilon^*C)$ if the cohomology sheaves of C are torsion.
- (ii) $H_W^i(X, \varepsilon^*C) \simeq H_{\text{ét}}^i(X, C) \oplus H_{\text{ét}}^{i-1}(X, C)$ if the cohomology sheaves of C are $\mathbf{Q}$ -vector spaces.

In the isomorphism of (ii), a very important element shows up: the generator e of $H_W^1(\mathbf{F}_p, \mathbf{Z}) \simeq \mathbf{Z}$ (normalised, say, by sending the geometric Frobenius to 1).

The sequence (4.39) is completely similar to one derived from the Tate–Beilinson conjecture in [90, Prop. 9.12] – except that it is not conjectural. With this and the last result of Theorem 62, everything falls into place and we are able to give a much more understandable reformulation of Conjecture (ii) in Theorem 54:

Theorem 64 The Tate–Beilinson conjecture is also equivalent to the following one: the map

64

$$(\varepsilon\alpha)^*\mathbf{Z}(n) \otimes \mathbf{Z}_l \rightarrow \varepsilon^*\mathbf{Z}_l(n)^c$$

induces isomorphisms on Weil-étale cohomology groups

$$H_W^*(X, \mathbf{Z}(n)) \otimes \mathbf{Z}_l \xrightarrow{\sim} H_{\text{cont}}^*(X, \mathbf{Z}_l(n)) \quad (4.40)$$

for all smooth X if $l \neq p$ (*resp.* for all smooth projective X if $l = p$).

Geisser’s theorem also allows us to reformulate Lichtenbaum’s conjectures in terms of Lichtenbaum motivic cohomology (*cf.* [95, Cor. 3.8]):

65

Conjecture 65 Let X be a smooth projective variety over $\mathbf{F}_p$, and $d = \dim X$.

a) The pairing

$$H_W^{2n}(X, \mathbf{Z}(n)) \times H_W^{2d-2n}(X, \mathbf{Z}(d-n)) \rightarrow H_W^{2d}(X, \mathbf{Z}(d)) \rightarrow \mathbf{Z} \quad (4.41)$$

is nondegenerate modulo torsion for all n .

b) For any (i, n) , the pairing

$$H_W^i(X, \mathbf{Z}(n))_{\text{tors}} \times H_W^{2d+1-i}(X, \mathbf{Q}/\mathbf{Z}(d-n)) \rightarrow H_W^{2d+1}(X, \mathbf{Q}/\mathbf{Z}(d)) \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}$$

induces a perfect pairing of finite groups

$$H_W^i(X, \mathbf{Z}(n))_{\text{tors}} \times H_W^{2d+2-i}(X, \mathbf{Z}(d-n))_{\text{tors}} \rightarrow \mathbf{Q}/\mathbf{Z}.$$

c) $H_W^i(X, \mathbf{Z}(n))$ is finitely generated, finite for $i \notin \{2n, 2n+1\}$ and 0 for $i \leq 0$ (if $n > 0$).

d) The kernel and cokernel of cup-product by e (generator of $H_W^1(\mathbf{F}_p, \mathbf{Z})$)

$$H_W^{2n}(X, \mathbf{Z}(n)) \rightarrow H_W^{2n+1}(X, \mathbf{Z}(n))$$

are finite.

e) The canonical homomorphism

$$H_{\text{ét}}^i(X, \mathbf{Z}(n)) \rightarrow H_W^i(X, \mathbf{Z}(n))$$

is an isomorphism for $i \leq 2n$.

Concerning the zeta function $\zeta(X, s)$, the following much nicer reformulation is due to Geisser (op. cit.):

66

Conjecture 66

1. $\text{ord}_{s=n} \zeta(X, s) = -\text{rk } CH^n(X) := -m(n)$.
2. $\lim_{s \rightarrow n} (1 - q^{n-s})^{m(n)} \zeta(X, s) = \pm q^{\chi(X, \mathcal{O}_X, n)} \chi(X, \mathbf{Z}(n))$, where

$$\chi(X, \mathbf{Z}(n)) = \prod_i |H_W^i(X, \mathbf{Z}(n))_{\text{tors}}|^{(-1)^i} \cdot R_n(X)^{-1},$$

$$\chi(X, \mathcal{O}_X, n) = \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq d}} (-1)^{i+j} (n-i) h_{ij}, \quad h_{ij} = \dim H^j(X, \Omega^i)$$

and $R_n(X)$ is the absolute value of the determinant of the pairing (4.41) (modulo torsion) with respect to arbitrary bases of $H_W^{2n}(X, \mathbf{Z}(n))/\text{tors}$ and $H_W^{2d-2n}(X, \mathbf{Z}(d-n))/\text{tors}$.

Here I would like to correct a mistake in [95, Remark 3.11] about the sign. It is stated there that this sign is $(-1)^{\sum_{a>n} m(a)}$. However, the Weil conjecture only says that

the real zeroes of $\zeta(X, s)$ are half integers, so the correct formula is $(-1)^{\sum_{a/2 > n} m(a/2)}$, where a is an integer. By semi-simplicity, the value of $m(a/2)$ is the multiplicity of the eigenvalue $q^{a/2}$ (positive square root) for the action of Frobenius on $H^a(\overline{X}, \mathbf{Q}_l)$. This multiplicity may well be nonzero, for example if X is a supersingular elliptic curve: I am grateful to A. Chambert-Loir for raising this issue. I have no idea how to relate $m(a/2)$ to cycle-theoretic invariants: there are no half Tate twists or half-dimensional Chow groups...

Theorem 67 The Tate–Beilinson Conjecture 55 implies Conjectures 65 and 66 (hence Lichtenbaum’s conjectures in Sect. 4.7.6).

67

Sketch

(For details, see [95].) By the finiteness results on étale cohomology, the right hand side, hence the left hand side of (4.40) is a finitely generated $\mathbf{Z}_l$ -module. Hence, by faithful flatness, $H_W^*(X, \mathbf{Z}(n)) \otimes \mathbf{Z}_{(l)}$ is a finitely generated $\mathbf{Z}_{(l)}$ -module. From there it is tempting to descend directly to $\mathbf{Z}$, but this is wrong as Lichtenbaum pointed out several years ago: for example, the $\mathbf{Z}$ -module $M = \bigoplus \mathbf{Z}/l$ is such that $M \otimes_{\mathbf{Z}} \mathbf{Z}_{(l)}$ is finitely generated over $\mathbf{Z}_{(l)}$ for all l , while it is certainly not finitely generated. For a torsion-free example, take the subgroup of $\mathbf{Q}$ formed of all fractions with square-free denominator.¹³ A correct proof uses a duality argument, which is encapsulated in Lemma 68 below. (Arithmetic) Poincaré duality for continuous étale cohomology allows us to apply this duality argument. This basically explains the proof of a), b), c) and d); as for e), it follows from (4.39) and the Beilinson–Parshin Conjecture 51. Finally, the deduction of Conjecture 66 is not especially new and goes back to Milne [138, Th. 4.3 and Cor. 5.5] (see also [90, Cor. 7.10 and Th. 9.20] and [58, Proof of Th. 8.1]).

Lemma 68: [95, Lemma 3.9] Let R be a commutative ring and $A \times B \rightarrow R$ a pairing of two flat R -modules A, B .

68

- a) Suppose that this pairing becomes non-degenerate after tensoring by R_l for some prime ideal l of R , where R_l denotes the completion of R at l . Then it is non-degenerate.
- b) Suppose that R is a noetherian domain and let K be its field of fractions. If, moreover, $\dim_K A \otimes K < \infty$ or $\dim_K B \otimes K < \infty$, then A and B are finitely generated. $\square$

¹³ One should be careful that this mistake can be found in the literature, e.g. see in [136, Proof of Th. 12.5] the proof that the group of morphisms between two abelian varieties is finitely generated; the corresponding proof in [148, p. 177] is completely correct. Lemma 68 will also justify the proof of [135, Ch. VI, Th. 11.7].

69

Corollary 69 Under the Tate–Beilinson Conjecture 55:

- a) $\dim_{\mathbf{Q}} H^i(X, \mathbf{Q}(n)) < \infty$ for all i, n .
- b) Assuming further the Beilinson–Lichtenbaum Conjecture 17, $H^i(X, \mathbf{Z}(n))$ is finitely generated for any $i \leq n + 2$.

70

Corollary 70 Under the Tate–Beilinson conjecture 55 and the Bloch–Kato Conjecture 16, the Bass and motivic Bass conjectures 36 and 37 are true in the following cases for smooth projective varieties X over $\mathbf{F}_p$:

- (i) $d = \dim X \leq 3$.
- (ii) (for Conjecture 37 b):) $n \leq 2$.

Proof

- (i) It suffices to prove Conjecture 37 (for Conjecture 36, use the spectral sequence (4.6)). Independently of any conjecture one has $H^i(X, \mathbf{Z}(n)) = 0$ for $i > n + d$. For $i = n + d$, by the coniveau spectral sequence for motivic cohomology, this a group is a quotient of

$$\bigoplus_{x \in X_{(0)}} H^{n-d}(k(x), \mathbf{Z}(n-d)) .$$

The latter group is 0 for $n < d$ and also for $n \geq d + 2$ by Theorem 2, since Milnor’s K -groups of finite fields vanish in degree ≥ 2 . So far we have only used that X is smooth. Suppose now X smooth projective: for $n = d$, $H^{n+d}(X, \mathbf{Z}(n)) = CH_0(X)$ is finitely generated by Bloch [18] (see also Kato–Saito [102]), and for $n = d + 1$ it is isomorphic to k^* by Akhtar [7]. If $d \leq 3$, this plus Corollary 69 b) covers all motivic cohomology.

- (ii) Same argument, noting that for $n \leq 2$ Corollary 69 b) again covers all the motivic cohomology of X .

71

Remark 71

- 1) Trying to extend Corollary 70 to open varieties via de Jong’s theorem is a little delicate: we can apply part 2 of the argument in the proof of Lemma 7 provided we have an a priori control of the torsion of the motivic cohomology groups involved. By the Beilinson–Lichtenbaum conjecture, the group $H^i(X, \mathbf{Z}/m(n))$ is finite for any smooth variety X as long as $i \leq n + 1$. This implies that ${}_m H^i(X, \mathbf{Z}(n))$ is finite as long as $i \leq n + 2$, so that Corollary 70 goes through for arbitrary smooth varieties as long as $d \leq 2$ or $n \leq 2$, because then this finiteness covers all motivic cohomology groups. For $d = 3$ we have a problem with $H^6(X, \mathbf{Z}(3))$ and $H^7(X, \mathbf{Z}(4))$, however. Unfortunately, Abyankhar’s resolution

of singularities for 3-folds in characteristic > 5 [1, Th. (13.1)] only works over an algebraically closed field.

- 2) For singular schemes X , we may introduce Lichtenbaum Borel–Moore motivic homology groups as in Sect. 4.5.4, using the Lichtenbaum topology rather than the étale topology, or define them as

$$H_i^{BM,W}(X, \mathbb{Z}[1/p](n)) = H_i^{BM,\acute{e}t}(X, \mathbb{Z}[1/p](n) \overset{L}{\otimes} \mathbb{Z}^c)$$

cf. (4.37) and (4.38). Then, under the Tate–Beilinson conjecture, the groups $H_i^{W,BM}(X, \mathbb{Z}[1/p](n))$ are all finitely generated $\mathbb{Z}[1/p]$ -modules. This follows by dévissage from the smooth projective case.

- 3) Corollary 70 applies trivially when X is a curve. Hence we get (under the Bloch–Kato conjecture) that all the motivic cohomology of X is finitely generated, by a method totally different from that in the proof of Theorem 40! Using the spectral sequence (4.6) we can then recover Quillen’s finite generation theorem for algebraic K -theory...

Finally, let us give a version of Conjecture 66 for an arbitrary $\mathbb{F}_p$ -scheme of finite type, and explain that it follows from the Tate–Beilinson conjecture. It rests on Remark 71 2).

Theorem 72 Let X be a scheme of finite type over $\mathbb{F}_p$. If the Tate–Beilinson Conjecture 55 holds, then, for any $n \in \mathbb{Z}$:

72

- (i) $\text{ord}_{s=n} \zeta(X, s) = \sum_{i \in \mathbb{Z}} (-1)^i i \text{rk} H_i^{W,BM}(X, \mathbb{Z}[1/p](n)) := -m(n)$.
 (ii) The cohomology groups of the complex

$$\dots \rightarrow H_i^{W,BM}(X, \mathbb{Z}[1/p](n)) \xrightarrow{\cdot e} H_{i-1}^{W,BM}(X, \mathbb{Z}[1/p](n)) \rightarrow \dots$$

are finite, where e is the canonical generator of $H_W^1(\mathbb{F}_p, \mathbb{Z}) \simeq \mathbb{Z}$.

- (iii) Up to $\pm$ a power of p , one has

$$\lim_{s \rightarrow n} (1 - q^{n-s})^{m(n)} \zeta(X, s) = \chi(H_*^{W,BM}(X, \mathbb{Z}[1/p](n)), \cdot e) .$$

Sketch

A version of this for l -adic cohomology with compact supports was proven in [90, Th. 7.8] under the assumption that Galois acts semi-simply at the eigenvalue 1 (cf. Remark 57 1). One passes from there to Borel–Moore l -adic homology by arithmetic duality (cf. loc. cit., Th. 3.17). It is actually simpler to redo the proof of [90, Th. 7.8] with Borel–Moore l -adic homology by using a description of $\zeta(X, s)$ in these terms, which only involves duality for the geometric groups with $\mathbb{Q}_l$ coefficients. One then concludes thanks to Lemma 59.

73

Remark 73

- 1) This approach does not handle the missing power of p . This has recently been achieved by Geisser [60]: his point of view is to define a compactly supported version of Lichtenbaum's cohomology. To get the right groups he refines Lichtenbaum's topology by adding cdh coverings to it, which unfortunately forces him to assume resolution of singularities. Presumably, the corresponding non-compactly supported cohomology (for smooth schemes) involves the logarithmic part of Mokrane's de Rham–Witt cohomology with logarithmic poles at infinity [142] (whose definition unfortunately also assumes resolution of singularities), glued to motivic cohomology in a similar way as (4.19). Can one give a direct definition of this motivic cohomology?
- 2) In characteristic 0, Lichtenbaum has an exactly parallel formulation of an integral conjecture for the special values of the zeta function, in terms of his cohomology still under development [127].

Unconditional Results: Varieties of Abelian Type over Finite Fields

4.8

4.8.1

Main Result

We shall give cases in which we can prove the Tate–Beilinson Conjecture 55. Namely, let $\mathcal{A} = \text{Mot}_{\text{rat}}(k, \mathbb{Q})$.

74

Definition 74

- a) Let $\mathcal{A}_{\text{ab}}$ be the thick rigid subcategory of $\mathcal{A}$ generated by Artin motives and motives of abelian varieties.
- b) $B(k) = \{X \mid h(X) \in \mathcal{A}_{\text{ab}}\}$.
- c) $B_{\text{tate}}(k) = \{X \in B(k) \mid \text{the Tate conjecture holds for the } l\text{-adic cohomology of } X \text{ for some } l \neq \text{char } k\}$.

(In c), this does not depend on l because Frobenius acts semi-simply on $H^*(\overline{X}, \mathbb{Q}_l)$.)

Example 75 .

- 1) $X \in B(k)$ and $\dim X \leq 3 \Rightarrow X \in B_{\text{tate}}(k)$. This is a slight strengthening of Soulé [176, Th. 4 i)]: the problem is that Soulé works with a collection $A(k)$ of varieties such that, clearly, $A(k) \subseteq B(k)$, but I don't know if equality holds, so that this claim unfortunately does not follow from [176], contrary to what was indicated in [95, Example 1 b)]. For this reason I shall justify it in Sect. 4.8.5.
- 2) Products of elliptic curves are in $B_{\text{tate}}(k)$ (Spieß [179]).

- 3) There are many examples of abelian varieties in $B_{\text{tate}}(k)$ (Zarhin, Lenstra, Milne): powers of simple abelian varieties “of K3 type” or “of ordinary type”, etc. [112, 140, 213].
- 4) Certain Fermat hypersurfaces (Tate, Katsura–Shioda [103, 189]).

The main result of [95] is:

Theorem 76 Conjectures 49 and 50 are true for $X \in B_{\text{tate}}(k)$.

76

Of course, it is not difficult to get Conjecture 49: indeed, Galois action on the l -adic cohomology of X is semi-simple (reduce to an abelian variety A and use the fact that the arithmetic Frobenius is the inverse of the geometric Frobenius, which is central in the semi-simple algebra $\text{End}(A) \otimes \mathbb{Q}$). By [192], this plus the cohomological Tate conjecture imply Conjecture 49. What is new is to obtain Conjecture 50. We shall explain in the sequel of this section how this follows from the Kimura–O’Sullivan theory of “finite dimensional” Chow motives.

In the previous sections, we referred to [95] for proofs or details of proofs on some consequences of the Tate–Beilinson conjecture. In *loc. cit.*, the corresponding proofs are given for varieties in $B_{\text{tate}}(k)$, and yield unconditional theorems.

The Soulé–Geisser Argument

4.8.2

This argument is first found in Soulé’s paper [176] and was amplified by Geisser in [56]¹⁴. It is really a weight argument and is very simple to explain: suppose that Frobenius acts on some group H and that

- For one reason we know that it acts by multiplication by some power of p , say p^n .
- For another reason we know that it is killed by some polynomial P with integral coefficients.

If we can prove that $P(p^n) \neq 0$, then we get that $H \otimes \mathbb{Q} = 0$ (more precisely, that H is torsion of exponent dividing $P(p^n)$).

Typically, H will be a Hom group between a certain motive M and a Tate motive (M might also be a shift of a pure motive in $DM_{\text{gm}}(k)$). The issue is then to show that the characteristic polynomial of the Frobenius endomorphism of M , assuming that this polynomial exists, is not divisible by $T - p^n$. A nilpotence theorem will allow us to prove this below.

¹⁴ The reader should also look at Coombes’ paper [33] where the author uses Soulé’s work to get a K -cohomological variant of Lichtenbaum’s conjecture for the zeta function of a rational surface over a finite field: I am grateful to the referee for pointing out this paper.

4.8.3 The Kimura–O’Sullivan Theory

This theory was developed independently by S. I. Kimura [106] and P. O’Sullivan [151]. An abstract version (which is also most of O’Sullivan’s point of view) is developed in [6, §9]. See also André’s recent Bourbaki talk [4].

77 **Definition 77** Let $\mathcal{A}$ be a $\mathbf{Q}$ -linear tensor category. An object $M \in \mathcal{A}$ is *even* if some exterior power of M vanishes, *odd* if some symmetric power of M vanishes, *finite dimensional* if it is a direct sum of a even and an odd object.

(Kimura says evenly and oddly finite dimensional; O’Sullivan says positive and negative, and semi-positive instead of finite-dimensional.)

There are two reasons why finite dimensionality is an important notion: first its remarkable stability properties, and second Kimura’s nilpotence theorem.

78 **Theorem 78:** (Kimura [106, Cor. 5.11, Prop. 6.9], O’Sullivan) Suppose $\mathcal{A}$ rigid. Then the full subcategory $\mathcal{A}_{\text{kim}}$ of $\mathcal{A}$ formed of finite dimensional objects is thick and rigid, *i.e.* stable under direct sums, direct summands, tensor products and duals.

Kimura developed his theory for $\mathcal{A} = \text{Mot}_{\text{rat}}(K, \mathbf{Q})$ (K a field) and proved a nilpotence theorem for correspondences on a finite dimensional motive which are homologically equivalent to 0. This theorem was slightly strengthened in [6], replacing homological by numerical equivalence (and nil by nilpotent). See [6, Prop. 9.1.14] for an abstract statement. In the case of Chow motives, this gives:

79 **Theorem 79** Let $M \in \mathcal{A} = \text{Mot}_{\text{rat}}(K, \mathbf{Q})$ and $\overline{M}$ its image in $\overline{\mathcal{A}} = \text{Mot}_{\text{num}}(k, \mathbf{Q})$. Then the kernel of $\mathcal{A}(M, M) \rightarrow \overline{\mathcal{A}}(\overline{M}, \overline{M})$ is a nilpotent ideal.

All this theory would be nice but rather formal if one had no examples of finite dimensional motives. Fortunately, there are quite a few:

80 **Theorem 80:** (Kimura [106, Th. 4.2], O’Sullivan) For $\mathcal{A} = \text{Mot}_{\text{rat}}(K, \mathbf{Q})$, $\mathcal{A}_{\text{kim}} \supset \mathcal{A}_{\text{ab}}$.

The proof is essentially a reformulation of Šermenev’s proof of the Künneth decomposition of the Chow motive of an abelian variety [181].

81 **Conjecture 81:** (Kimura–O’Sullivan Conjecture) Let K be a field and $\mathcal{A} = \text{Mot}_{\text{rat}}(K, \mathbf{Q})$. Then $\mathcal{A} = \mathcal{A}_{\text{kim}}$.

By [6, Ex. 9.2.4], this conjecture follows from the standard conjecture on Künneth projectors and the existence of the Bloch–Beilinson–Murre filtration.

The Proof

4.8.4

The proof of Conjecture 50 in Theorem 76 is fairly simple: the nilpotence theorem 79 is used three times. First, decompose the numerical motive $\overline{h}(X)$ into a direct sum of simple motives by Jannsen's Theorem 9. By nilpotence, this decomposition lifts to Chow motives, hence we may replace $h(X)$ by a Chow motive S whose numerical image $\overline{S}$ is simple. We need to show that

$$\mathcal{A}(S, L^n) \xrightarrow{\sim} \overline{\mathcal{A}}(\overline{S}, \overline{L}^n) \quad (4.42)$$

for any n , where L is the Lefschetz motive. Then we have the usual dichotomy:

- a) $\overline{S} \simeq \overline{L}^n$. Then, by nilpotence, $S \simeq L^n$ and this is obvious.
- b) $\overline{S} \not\simeq \overline{L}^n$. Then the right hand side of (4.42) is 0 and we have to show that the left hand side is also 0. By [139, Prop. 2.6], the characteristic polynomial P of the Frobenius endomorphism $F_{\overline{S}}$ of $\overline{S}$ is not $T - q^n$. But, by nilpotence, there is an $N > 0$ such that $P(F_S)^N = 0$. The conclusion now follows by the Soulé–Geisser argument. $\square$

Justification of Example 75 1)

4.8.5

We shall actually prove directly:

Theorem 82 If $X \in B(k)$ and $d = \dim X \leq 3$, then the Tate–Beilinson conjecture holds for X .

82

Proof In general, let $M \in \mathcal{A}_{\text{kim}}$ and $\overline{M}$ be its image in $\overline{\mathcal{A}}$. By Theorem 79, the weight grading $\overline{M} = \bigoplus \overline{M}^{(i)}$ (cf. p. 368) lifts to a grading $M = \bigoplus M^{(i)}$.¹⁵ For simplicity, we shall say that an object $M \in \mathcal{A}$ is of weight i if $\overline{M}$ is of weight i , so that $M^{(i)}$ is of weight i . Also, if X is smooth projective and $h(X) \in \mathcal{A}_{\text{kim}}$, we simply write $h^i(X)$ for $h(X)^{(i)}$.

Let M be of weight $2n$; consider the following property:

(*) The natural homomorphism

$$\mathcal{A}(M, L^{2n}) \otimes \mathbf{Q}_l \rightarrow (H_l(M)(n))^G$$

is an isomorphism.

¹⁵ This grading is not necessarily unique, but the idempotents defining it are unique up to conjugation, hence the $M^{(i)}$ are unique up to isomorphism.

We need a lemma:

- 83** **Lemma 83** Let $\mathcal{A}_{\text{ab}}^{(2)}$ be the full subcategory of $\mathcal{A}_{\text{ab}}$ formed of motives of weight 2. Then
- a) Property (*) holds for all objects of $\mathcal{A}_{\text{ab}}^{(2)}$.
 - b) If $M \in \mathcal{A}_{\text{ab}}^{(2)}$, then $M^\vee(1) \in \mathcal{A}_{\text{ab}}^{(2)}$.

Proof

- a) We immediately reduce to the case where M is of the form $h^2(A \otimes_k L)$ for A an abelian variety and L a finite extension of k ; then it follows from Tate's theorem [190] and the finiteness of $\text{Pic}^0(A) = A^\vee(k)$.
- b) We reduce to the same case as in a). Thanks to Lieberman [128], any polarisation of A induces via Poincaré duality an isomorphism $\bar{h}^2(A \otimes_k L)^\vee \simeq \bar{h}^2(A \otimes_k L)(-1)$; by Theorem 79 this lifts to an isomorphism $h^2(A \otimes_k L)^\vee \simeq h^2(A \otimes_k L)(-1)$. $\square$

Let now X be as in Theorem 82. We must prove that, for all n , $h^{2n}(X)$ verifies (*). For $n = 0$ it is trivial. For $n = 1$, it follows from Lemma 83 a). For $n = d - 1$, it follows from Poincaré duality (lifted to $\mathcal{A}$ by Theorem 79) and Lemma 83 b). For $n = d$, it also follows by Poincaré duality from the case $d = 0$ by the same argument as in the proof of Lemma 83 b). If $d \leq 3$, this covers all values of n .

- 84** **Corollary 84** If $X \in B(k)$ and $\dim X \leq 3$, under the Bloch–Kato Conjecture 16 all motivic cohomology groups of X are finitely generated.

This follows from Theorem 82 and Corollary 70 (i). $\square$

- 85** **Definition 85** A finitely generated field $K/\mathbb{F}_p$ is of *abelian type* if it is the function field of a smooth projective variety of abelian type.

- 86** **Corollary 86** Let X be an $\mathbb{F}_p$ -scheme of finite type. Assume that $\dim X \leq 2$ and that the function fields of all its irreducible components of dimension 2 are of abelian type. Then the conclusions of Theorem 72 hold for X .

Proof This is just an effective case of Theorem 72. The point is that in the dévissage, the closed subvarieties one encounters are all of dimension ≤ 1 and all smooth projective curves are of abelian type.

Questions and Speculations

4.9

The Finite Generation Issue

4.9.1

Recall that, by Theorem 67, the Tate–Beilinson conjecture implies the finite generation of the Lichtenbaum cohomology groups $H_W^i(X, \mathbf{Z}(n))$ for any smooth projective variety $X/\mathbf{F}_p$ and any i, n , and that by Remark 71.2) this in turn implies the finite generation of $H_i^{W, BM}(X, \mathbf{Z}(n)) \otimes \mathbf{Z}[1/p]$ over $\mathbf{Z}[1/p]$ for any scheme X of finite type over $\mathbf{F}_p$. In particular, $H_*^{W, BM}(X, \mathbf{Q}(n))$ is a finite-dimensional $\mathbf{Q}$ -vector space which implies by an analogue of Corollary 63 (ii) (or by dévissage from the smooth case) the same result for usual Borel–Moore motivic homology $H_*^{BM}(X, \mathbf{Q}(n))$.

On the other hand, Corollary 70 and Remark 71.1) show that under the Beilinson–Lichtenbaum and the Tate–Beilinson conjecture, $H^i(X, \mathbf{Z}(n))$ is finitely generated for X smooth in a certain range. The first case not reached is $CH^3(X)$ for X a smooth projective 4-fold. It is explained in [95, Remark 4.10] that, under the two conjectures, the following conditions are equivalent:

1. $CH^3(X)$ is finitely generated.
2. $CH^3(X)_{\text{tors}}$ is finite.
3. $H_{\text{Zar}}^0(X, \mathcal{H}_{\text{ét}}^4((\mathbf{Q}/\mathbf{Z})'(3)))$ is finite (it is a priori of finite exponent).

I don't see any argument allowing one to deduce finite generation in this case from known conjectures. The only one I can think of is Kato's conjecture:

Conjecture 87: (*Kato Conjecture [101, Conj. (0.3)]*) For any smooth projective variety X of dimension d over $\mathbf{F}_p$ and any $m \geq 1$, the homology in degree i of the Gersten complex

87

$$0 \rightarrow \bigoplus_{x \in X_{(d)}} H_{\text{ét}}^{d+1}(k(x), \mathbf{Z}/m(d)) \rightarrow \dots \rightarrow \bigoplus_{x \in X_{(0)}} H_{\text{ét}}^1(k(x), \mathbf{Z}/m) \rightarrow 0$$

is $\begin{cases} 0 & \text{if } i > 0 \\ \mathbf{Z}/m & \text{if } i = 0 \end{cases}$, the last isomorphism being induced by the trace map.

This conjecture is class field theory for $d = 1$; it has been proven by Kato for $d = 2$ [101], by Colliot-Thélène for general d and $i \geq d - 3$ if m is prime to p [31] and by Suwa under the same condition if m is a power of p [188].

However it does not seem to bear on the issue, except in limit cases (see below). I am therefore tempted to think that there is a counterexample to Conjecture 37. It might involve infinite 2-torsion; however if it involves infinite l -torsion for some odd prime l , it will yield an example where $K_0(X)$ is not finitely generated, disproving the original Bass conjecture (since the natural map $CH^3(X) \rightarrow \text{gr}^3 K_0(X)$ has kernel killed by $(3 - 1)! = 2$).

This also sheds some doubt in my mind on [146, Conj. 1.1.4] (a homotopy-theoretic Bass conjecture), which should however be correct if one replaces the Nisnevich topology by Lichtenbaum's topology.

What about this? As far as one is concerned by application to number-theoretic conjectures like the Soulé conjecture or the Lichtenbaum conjectures, this is not very serious: concerning the orders of zeroes, the ranks will still be finite and almost always 0, and the groups involved in special values are Lichtenbaum cohomology groups anyway. If one wants to get back a version of the Bass conjecture for algebraic K -theory, all one has to do is to define a “Lichtenbaum K -theory” similar to étale K -theory:

$$K^W(X) := \mathbb{H}_W(X, \mathcal{K})$$

where the notation means hypercohomology à la Thomason ([193]; see §5 in Geisser's chapter) for the Weil-étale topology.

On the other hand, it is quite amusing to remark that the étale topology, not the Zariski topology, shows up in the Lichtenbaum conjectures 4.7.6. In fact the Bloch–Kato or Beilinson–Lichtenbaum conjectures do not seem to play any rôle either in their formulation or in their (partial) proofs. (Even if we gave several examples where the Milnor conjecture gives vanishing or finiteness results, it was not used in the proofs of [95].) This also means that, in characteristic 0, the correct formulation (for, say, the zeta function) most certainly involves an étale-related version of motivic cohomology. In small Krull dimension it may be replaced by plain motivic cohomology but this will not work from dimension 3 onwards, as one already sees in characteristic p . For rings of integers of number fields, the original Lichtenbaum formulation 46 led to the Quillen–Lichtenbaum and the Beilinson–Lichtenbaum conjectures and a huge development of algebraic K -theory and motivic cohomology. The Bloch–Kato conjecture is needed to prove it (in the cases one can) under this form. If it is indeed étale motivic cohomology rather than ordinary motivic cohomology that is relevant, all this work will have been the result of a big misunderstanding!

Let me give one nice consequence of Kato's Conjecture 87, or rather of its partial proof by Colliot-Thélène–Suwa:

88 Theorem 88 Let X be smooth projective of dimension d over F_p . Then the map

$$CH^d(X) \rightarrow H_{\text{ét}}^{2d}(X, \mathbf{Z}(d))$$

is bijective.

Proof For $d = 2$, this follows from the short exact sequence [86]

$$0 \rightarrow CH^2(X) \rightarrow H_{\text{ét}}^4(X, \mathbf{Z}(2)) \rightarrow H^0(X, \mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2))) \rightarrow 0$$

and Kato's theorem. In general, consider the coniveau spectral sequence for étale motivic cohomology

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H_{\text{ét}}^{q-p}(k(x), \mathbf{Z}(d-p)) \Rightarrow H_{\text{ét}}^{p+q}(X, \mathbf{Z}(d)) .$$

We have $E_1^{p,q} = 0$ for $p > d$ for dimension reasons. For $q > d$ there are exact sequences

$$H_{\text{ét}}^{q-p-1}(k(x), \mathbf{Q}/\mathbf{Z}(d-p)) \rightarrow H_{\text{ét}}^{q-p}(k(x), \mathbf{Z}(d-p)) \rightarrow H_{\text{ét}}^{q-p}(k(x), \mathbf{Q}(d-p)) .$$

By Theorem 14, the last group is 0. The first is 0 pour $q-1 > d+1$ for cohomological dimension reasons. Hence $E_1^{p,q} = 0$ for $q \geq d+3$. Moreover, $E_1^{d-1,d+1} = E_1^{d-2,d+1} = 0$ by Hilbert 90 and Hilbert 90 in weight 2 (Merkurjev–Suslin theorem). Finally, $E_2^{d-2,d+2} = 0$ by the Colliot-Thélène–Suwa theorem. Hence $E_2^{p,2d-p} = 0$ except for $p = d$ and there are no differentials arriving to $E_2^{d,d} = CH^d(X)$. The proof is complete.

Characteristic 0

4.9.2

In characteristic 0, things are considerably more complicated. If we start with the Beilinson Conjecture 50, its analogue for smooth projective $\mathbf{Q}$ -varieties predicts a *two-layer* filtration on their Chow groups; cycles homologically (*i.e.*, conjecturally, numerically) equivalent to 0 should be detected by an *Abel–Jacobi* map to Deligne–Beilinson cohomology.

Concerning the Tate–Beilinson Conjecture 55, the only thing I can do is to conjecture that there is a conjecture.

Conjecture 89 There is a conjecture in the form of that in Theorem 64 in characteristic 0, where the left hand side is a form of motivic cohomology and the right hand side is a form of an absolute cohomology theory in the sense of Beilinson [11].

89

Presumably the left hand side would be motivic hypercohomology with respect to the “Weil topology in characteristic 0” that Lichtenbaum is currently developing [127]. As for the right hand side, I feel that it should probably be a mixture of the various (absolute counterparts of the) classical cohomology theories: l -adic, Betti, de Rham, p -adic, so as to involve the comparison isomorphisms. This formulation should be as powerful as in characteristic p and account for most motivic conjectures in characteristic 0.

This being said, there is a basic problem to start the construction: if we take the l -adic cohomology of a ring of integers $\mathcal{O}_S$ (in which l is invertible), it is nonzero even for negative Tate twists: by Tate and Schneider [165] we have

$$\sum_{i=0}^2 (-1)^{i+1} \dim_{\mathbf{Q}_l} H^i(\mathcal{O}_S, \mathbf{Q}_l(n)) = \begin{cases} r_2 & \text{if } n \text{ is even} \\ r_1 + r_2 & \text{if } n \text{ is odd} \end{cases}$$

hence there is no chance to compare it in the style of Theorem 20 with motivic cohomology, which vanishes in negative weights. The first thing to do would be to modify l -adic cohomology in order to correct this phenomenon: although this is clearly related to real and complex places, I have no idea how to do this. Note that Lichtenbaum's theory will be for Arakelov varieties.

One definitely needs new insights in order to follow this line of investigation!

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Mixed Motives

II.5

Marc Levine*

5.1	<i>Introduction</i>	431
	Mixed Motives.....	431
	Notations and Conventions	432
5.2	<i>Motivic Complexes</i>	433
	Weight-two Complexes.....	434
	Beilinson–Lichtenbaum Complexes	439
	Bloch’s Cycle Complexes.....	440
	Suslin Homology and Friedlander–Suslin Cohomology	446
	Cubical Versions	450
5.3	<i>Abelian Categories of Mixed Motives</i>	453
	Background and Conjectures	453
	Motives by Compatible Realizations.....	458
	Motives by Tannakian Formalism	462
5.4	<i>Triangulated Categories of Motives</i>	469
	The Structure of Motivic Categories	470
	Huber’s Construction	474
	Hanamura’s Construction	475
	Levine’s Construction	478
	Voevodsky’s Construction	483
5.5	<i>Mixed Tate Motives</i>	488
	The Triangulated Category of Mixed Tate Motives	489
	The Bloch Cycle Algebra and Lie Algebra	492
	Categories Arising from a cdga	494

* The author gratefully acknowledges the support of: the Humboldt Foundation through the Wolfgang Paul Program and the NSF via grant DMS 0140445.

	Categories of Mixed Tate Motives.....	500
	Spitzweck's Representation Theorem	502
5.6	<i>Cycle Classes, Regulators and Realizations</i>	506
	Cycle Classes	506
	Realizations.....	510
	<i>References</i>	515

Introduction

5.1

Mixed Motives

5.1.1

During the early and mid-eighties, Beilinson [2] and Deligne [24] independently described a conjectural abelian tensor category of *mixed motives* over a given base field k , $\mathcal{MM}_k$, which, in analogy to the category of mixed Hodge structures, should contain Grothendieck's category of pure (homological) motives as the full subcategory of semi-simple objects, but should have a rich enough structure of extensions to allow one to recover the weight-graded pieces of algebraic K -theory. More specifically, one should have, for each smooth scheme X of finite type over a given field k , an object $h(X)$ in the derived category $D^b(\mathcal{MM}_k)$, as well as Tate twists $h(X)(n)$, and natural isomorphisms

$$\mathrm{Hom}_{D^b(\mathcal{MM}_k)}(1, h(X)(n)[m]) \otimes \mathbb{Q} \cong K_{2n-m}(X)^{(n)},$$

where $K_p(X)^{(n)}$ is the weight n eigenspace for the Adams operations. The abelian groups

$$H_{\mathcal{M}}^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{D^b(\mathcal{MM}_k)}(1, h(X)(q)[p])$$

should form the universal Bloch–Ogus cohomology theory on smooth k -schemes of finite type; as this theory should arise from mixed motives, it is called *motivic cohomology*.

This category $\mathcal{MM}_k$ should on the one hand give a natural framework for Beilinson's unified conjectures on the relation of algebraic K -theory to values of L -functions, and on the other hand, give a direct relation of singular cohomology and the Chow ring. For this, conjectures of Beilinson, Bloch and Murre [74] suggest a decomposition (with $\mathbb{Q}$ -coefficients)

$$h(X)_{\mathbb{Q}} = \bigoplus_{i=0}^{2d} h^i(X)[-i]$$

for X a smooth projective variety of dimension d over k , with the $h^i(X)$ semi-simple objects in $\mathcal{MM}_k \otimes \mathbb{Q}$. This yields a decomposition

$$H^{2n}(X, \mathbb{Q}(n)) = \bigoplus_{i=0}^{2d} \mathrm{Ext}_{\mathcal{MM}_k \otimes \mathbb{Q}}^{2n-i}(1, h^i(X)(n));$$

since one expects $H^{2n}(X, \mathbb{Q}(n)) = K_0(X)^{(n)} = \mathrm{CH}^n(X)_{\mathbb{Q}}$, this would give an interesting decomposition of the Chow group $\mathrm{CH}^n(X)_{\mathbb{Q}}$. For instance, the expected properties of the $h^i(X)$ would lead to a proof of *Bloch's conjecture*:

Conjecture 1 Let X be a smooth projective surface over $\mathbb{C}$ such that $H^2(X, \mathcal{O}_X) = 0$. Let $A^2(X)$ be the kernel of the degree map $\mathrm{CH}^2(X) \rightarrow \mathbb{Z}$. Then the Albanese map $\alpha_X : A^2(X) \rightarrow \mathrm{Alb}(X)(\mathbb{C})$ is an isomorphism.

1

The relation of the conjectural category of mixed motives to various generalizations of Bloch's conjecture and other fascinating conjectures of a geometric nature,

as well as to values of L -functions, has been widely discussed in the literature and we will not discuss these topics in any detail in this article. For some more details on the conjectured properties of $\mathcal{MM}_k$ and applications, we refer the reader to [21, 52, 53, 75, 80–82, 103], as well as additional articles in [104] and the article of Goncharov [35] in this volume.

The category $\mathcal{MM}_k$ has yet to be constructed. However, in the nineties, progress was made toward the construction of the derived category $D^b(\mathcal{MM}_k)$, that is, the construction of a triangulated tensor category $DM(k)$ that has many of the structural properties expected of $D^b(\mathcal{MM}_k)$. In particular, we now have a very good candidate for motivic cohomology $H_{\mathcal{M}}^p(X, \mathbb{Z}(q))$, which, roughly speaking, satisfies all the expected properties which can be deduced from the existence of a triangulated tensor category of mixed motives, without assuming there is an underlying abelian category whose derived category is $DM(k)$, or even that $DM(k)$ has a reasonable t -structure.

In addition to the triangulated candidates for $D^b(\mathcal{MM}_k)$, there are also constructions of candidates for $\mathcal{MM}_k$; these however are not known to have all the desired properties, e.g., the correct relation to K -theory.

In this article, we will outline the constructions and basic properties of various versions of categories of mixed motives which are now available. We will also cover in some detail the known theory of the subcategory of *mixed Tate motives*, that is, the subcategory (either triangulated or abelian) generated by the rational Tate objects $\mathbb{Q}(n)$.

We will make some mention of the relevance of these construction for the mod n -theory, the Beilinson–Lichtenbaum conjectures and the Bloch–Kato conjectures, but as these themes have been amply explained elsewhere (see e.g. [32, 55]), we will not make more than passing reference to this topic.

The discussion of mixed Tate motivic categories in Sect. 5.5 is based in large part on a seminar on this topic that ran during the fall of 2002 at the University of Essen while I was visiting there. I would like to thank the participants of that seminar, and especially Sviataslav Archava, Najmuddin Fakhruddin, Marco Schlichting, Stefan Müller-Stach and Helena Verrill, and for their lectures and discussions; a more detailed discussion of mixed Tate motivic categories arising from this seminar is now in the process of being written. I would also like to thank the Mathematics Department at the University of Essen and especially my hosts, Hélène Esnault and Eckart Viehweg, for their hospitality and support, which helped so much in the writing of this article.

5.1.2 Notations and Conventions

If $A(-)$ is a simplicial abelian group $n \mapsto A(n)$, we have the associated (homological) complex $A(*)$, with $A(*)_n = A(n)$ and $d_n : A(n) \rightarrow A(n-1)$ the alternating sum

$$d_n := \sum_{j=0}^n (-1)^j A(\delta_j),$$

where the δ_j are the standard co-face maps.

We let $C^\pm(\mathbf{Ab})$ denote the category of cohomological complexes, bounded below (+) or bounded above (−). We let $C_\pm(\mathbf{Ab})$ denote the category of homological complexes, bounded below (+) or bounded above (−). In both categories, we have the suspension operation $C \mapsto C[1]$, and cone sequences

$$A \xrightarrow{f} B \rightarrow \text{Cone}(f) \rightarrow A[1] .$$

Thus, in the cohomological category $(A[1])^n = A^{n+1}$ and in the homological category $(A[1])_n = A_{n-1}$. We extend these notations to the respective derived categories.

For a scheme S , we let $\mathbf{Sch}_S$ denote the category of schemes of finite type over S , $\mathbf{Sm}_S$ the full sub-category of smooth quasi-projective S -schemes. If $S = \text{Spec } A$ for some ring A , we write $\mathbf{Sch}_A$ and $\mathbf{Sm}_A$ for $\mathbf{Sch}_{\text{Spec } A}$ and $\mathbf{Sm}_{\text{Spec } A}$.

For a noetherian commutative ring R , we let $R\text{-mod}$ denote the category of finitely generated R -modules; for a field F , we let $F\text{-Vec}$ be the category of vector spaces over F (not necessarily of finite dimension). If G is a pro-finite group, we let $\mathbb{Q}_p[G]\text{-mod}$ denote the category of finitely dimensional $\mathbb{Q}_p$ -vector spaces with a continuous G -action.

Motivic Complexes

5.2

In this first section, we begin with a discussion of Bloch's seminal work in the weight-two case. We then give an overview of the conjectures of Beilinson and Lichtenbaum on absolute cohomology, as a prelude to our discussion of mixed motives and motivic cohomology. After this, we describe two constructions of theories of absolute cohomology: Bloch's construction of the higher Chow groups, and the Friedlander–Suslin construction of motivic complexes. For later use, we also give some details on associated cubical versions of these complexes.

The relation of the Zariski cohomology of $\mathbb{G}_m$ to K_0 and K_1 was well-known from the very beginning: the Picard group $H^1(X_{\text{Zar}}, \mathbb{G}_m)$ appears as a quotient of the reduced K_0 and the group of global units $H^0(X_{\text{Zar}}, \mathbb{G}_m)$ is likewise a quotient of $K_1(X)$, both via a determinant mapping. Hilbert's theorem 90 says that $H^i(X_{\text{Zar}}, \mathbb{G}_m) \rightarrow H^i(X_{\text{ét}}, \mathbb{G}_m)$ is an isomorphism for $i = 0, 1$; the Kummer sequence

$$1 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \rightarrow 1$$

relates the torsion and cotorsion in $H^*(X_{\text{ét}}, \mathbb{G}_m)$ to $H^*(X, \mu_n)$. Rationally, $H^1(X_{\text{Zar}}, \mathbb{G}_m)$ and $H^0(X, \mathbb{G}_m)$ give the weight-one portion of $K_0(X)$ and $K_1(X)$, respectively (the weight-zero portion of K_0 is similarly given by $H^0(X_{\text{Zar}}, \mathbb{Z})$).

The idea behind motivic complexes is, rather than arranging K -theory by the K -theory degree, one can also collect together the pieces of the same weight (for the Adams operations), and by doing so, one should be able to construct the universal Bloch–Ogus cohomology theory with integral coefficients. For weight one, this is given by the cohomology of the single sheaf $\mathbb{G}_m$, but for weight $n > 1$,

one would need a complex of length at least $n - 1$. Later on, the complexes assumed a secondary role as explicit representatives for the total derived functors $R\mathrm{Hom}_{D^b(\mathcal{M}, \mathcal{M}_k)}(1, h^0(X)(n))$, where $\mathcal{M}_k$ is the conjectural category of mixed motives over k , see Sect. 5.3.1.

Our discussion is historically out of order, in that quasi-isomorphism of Bloch's complexes with the Friedlander–Suslin construction was only constructed after Voevodsky introduced the machinery of finite correspondences [100] and showed how to adapt Quillen's proof of Gersten's conjecture to this setting in the course of his construction of a triangulated category of mixed motives. However, it is now apparant that one can deduce the Mayer–Vietoris properties of the Friedlander–Suslin complexes from Bloch's complexes, and conversely, one can acheive a more natural functoriality for Bloch's complexes from the Friedlander–Suslin version, without giving any direct relation to categories of mixed motives.

5.2.1 Weight-two Complexes

Before a general framework emerged in the early '80's, there was a lively development of the weight-two case, starting with Bloch's Irvine notes [8], in which he related:

1. the relations defining K_2 of a field F
2. the indecomposable K_3 of F
3. the values of the dilogarithm function
4. the Borel regulator on K_3 of a number field.

These relations were made more precise by Suslin's introduction of the 5-term dilogarithm relation [87, 88], uniting Bloch's work with Dupont and Sah's study [30] of the homology of SL_2 and the scissors congruence group. Lichtenbaum [68], building on Bloch's introduction of the relative K_2 of the semi-local ring $F[t]_{t(1-t)}$, constructed a length-two complex which computed the weight-two portions of K_2 and K_3 , up to inverting small primes. These constructions formed the basis for the general picture, as conjectured by Beilinson and Lichtenbaum, as well as the later constructions of Goncharov [38, 39], Bloch [13] and Voevodsky–Suslin–Friedlander [100].

Bloch's Complexes

In [8], Bloch constructs 3 complexes:

- (1) Let F be a field. Let $R(F) = F[t]_{t(1-t)}$, i.e., the localization of the polynomial ring $F[t]$ formed by inverting all polynomials $P(t)$ with $P(0)P(1) \neq 0$. Let $I(F) = t(1-t)R$. We have the *relative K -groups* $K_n(R; I)$, which fit into a long exact sequence

$$\cdots \rightarrow K_{n+1}(R/I) \rightarrow K_n(R; I) \rightarrow K_n(R) \rightarrow K_n(R/I) \rightarrow \cdots$$

Using the localization sequence in K -theory, we have the boundary map

$$K_2(R) \rightarrow \bigoplus_{\substack{x \in \mathbb{A}_F^1 \setminus \{0,1\}, \\ x \text{ closed}}} k(x)^* ;$$

composing with $K_2(R; I) \rightarrow K_2(R)$ gives the length-one complex

$$K_2(R; I) \xrightarrow{\partial} \bigoplus_{\substack{x \in \mathbb{A}_F^1 \setminus \{0,1\} \\ x \text{ closed}}} k(x)^* . \quad (5.1)$$

Bloch shows

Proposition 2 There are canonical isomorphisms

2

$$\ker \partial \cong K_3^{\text{ind}}(F)$$

$$\text{coker} \partial \cong K_2(F)$$

Here $K_3^{\text{ind}}(F)$ is the quotient of $K_3(F)$ by the image of the cup-product map $K_1(F)^{\otimes 3} \rightarrow K_3(F)$.

To make the comparison with the other two complexes, one needs an extension of Matsumoto's presentation of K_2 of a field to the relative case: For a semi-local PID A with Jacobson radical J and quotient field L , there is an isomorphism (cf. [101])

$$K_2(A, J) \cong (1 + J)^* \otimes_{\mathbb{Z}} L^* / \langle f \otimes (1 - f) \mid f \in 1 + J \rangle .$$

In particular, $K_2(R, I)$ contains the subgroup of symbols $\{1 + I, F^*\}$; taking the quotient of (5.1) by this subgroup yields the exact sequence (assuming F algebraically closed)

$$0 \rightarrow \frac{K_3^{\text{ind}}(F)}{\text{Tor}_1(F^*, F^*)} \rightarrow \frac{K_2(R, I)}{\{1 + I, F^*\}} \rightarrow F^* \otimes F^* \rightarrow K_2(F) \rightarrow 0 .$$

(2) Let $\mathcal{A}(F)$ be the free abelian group on $F \setminus \{0, 1\}$, and form the complex

$$\mathcal{A}(F) \xrightarrow{\lambda} F^* \otimes F^*$$

by sending $x \in F \setminus \{0, 1\}$ to $\lambda := x \otimes (1 - x) \in F^* \otimes F^*$. By Matsumoto's theorem, $K_2(F) = \text{coker}(\lambda)$. Let $\mathcal{B}(F)$ be the kernel of λ , giving the exact sequence

$$0 \rightarrow \mathcal{B}(F) \rightarrow \mathcal{A}(F) \rightarrow F^* \otimes F^* \rightarrow K_2(F) \rightarrow 0 .$$

(3) Start with the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \xrightarrow{\exp 2\pi i} \mathbb{C}^* \rightarrow 1 .$$

Tensor with $\mathbb{C}^*$ (over $\mathbb{Z}$), giving the complex $\mathbb{C} \otimes \mathbb{C}^* \rightarrow \mathbb{C}^* \otimes \mathbb{C}^*$ and the exact sequence

$$0 \rightarrow \text{Tor}_1(\mathbb{C}^*, \mathbb{C}^*) \rightarrow \mathbb{C}^* \rightarrow \mathbb{C} \otimes \mathbb{C}^* \rightarrow \mathbb{C}^* \otimes \mathbb{C}^* \rightarrow 1 .$$

The image of $\text{Tor}_1(\mathbb{C}^*, \mathbb{C}^*)$ in $\mathbb{C}^*$ is the torsion subgroup; let $\widetilde{\mathbb{C}}^*$ be the torsion-free quotient, yielding the exact sequence

$$0 \rightarrow \widetilde{\mathbb{C}}^* \rightarrow \mathbb{C} \otimes \mathbb{C}^* \rightarrow \mathbb{C}^* \otimes \mathbb{C}^* \rightarrow 1.$$

To relate these three complexes, Bloch defines two maps on $\mathcal{A}(\mathbb{C})$. For $x \in \mathbb{C} \setminus \{0, 1\}$, let $\varepsilon(x) \in \mathbb{C} \otimes \mathbb{C}^*$ be defined by

$$\varepsilon(x) := \left[\frac{1}{2\pi i} \log(1-x) \otimes x \right] + \left[1 \otimes \exp \left(\frac{-1}{2\pi i} \int_0^x \log(1-t) \frac{dt}{t} \right) \right].$$

In this formula, define

$$\log(1-t) := - \int_0^t \frac{dt}{1-t},$$

and use the same path of integration for all the integrals. Bloch shows that $\varepsilon(x)$ is then well-defined and independent of the choice of path from 0 to x . Extending ε to $\mathcal{A}(\mathbb{C})$ by linearity gives the commutative triangle

$$\begin{array}{ccc} & \mathcal{A}(\mathbb{C}) & \\ \varepsilon \swarrow & & \searrow \lambda \\ \mathbb{C} \otimes \mathbb{C}^* & \xrightarrow{\exp 2\pi i \otimes \text{id}} & \mathbb{C}^* \otimes \mathbb{C}^* \end{array}$$

The second map $\eta : \mathcal{A}(\mathbb{C}) \rightarrow K_2(R(\mathbb{C}), I(\mathbb{C}))$ is defined explicitly by

$$\eta(x) := \left\{ 1 - \frac{xt^2}{(t-1)^3 - xt^2(t-1)}, \frac{t}{t-1} \right\}^6 \in K_2(R(\mathbb{C}), I(\mathbb{C})).$$

This all yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & \frac{K_3(\mathbb{C})^{\text{ind}}}{\text{Tor}_1(\mathbb{C}^*, \mathbb{C}^*)} & \rightarrow & \frac{K_2(R, I)}{\{1+I, \mathbb{C}^*\}} & \xrightarrow{\partial} & \mathbb{C}^* \otimes \mathbb{C}^* & \rightarrow & K_2(\mathbb{C}) & \rightarrow & 0 \\ & & \uparrow \Psi & & \uparrow \eta & & \parallel & & \parallel & & \\ 0 & \rightarrow & \mathcal{B}(\mathbb{C}) & \rightarrow & \mathcal{A}(\mathbb{C}) & \xrightarrow{\lambda} & \mathbb{C}^* \otimes \mathbb{C}^* & \rightarrow & K_2(\mathbb{C}) & \rightarrow & 0 \\ & & \downarrow \theta & & \downarrow \varepsilon & & \parallel & & \parallel & & \\ 0 & \rightarrow & \widetilde{\mathbb{C}}^* & \rightarrow & \mathbb{C} \otimes \mathbb{C}^* & \xrightarrow{\exp 2\pi i \otimes \text{id}} & \mathbb{C}^* \otimes \mathbb{C}^* & \rightarrow & K_2(\mathbb{C}) & \rightarrow & 0 \end{array}$$

where Ψ and θ are the maps induced by ε and η .

The Dilogarithm

Composing ε with

$$\mathbb{C} \otimes \mathbb{C}^* \xrightarrow{\text{real part} \otimes \text{id}} \mathbb{R} \otimes \mathbb{C}^* \xrightarrow{\text{id} \otimes \log|-|} \mathbb{R} \otimes \mathbb{R} \xrightarrow{\text{multiply}} \mathbb{R}$$

yields the map $D : \mathcal{A}(\mathbb{C}) \rightarrow \mathbb{R}$. On generators $x \in \mathbb{C} \setminus \{0, 1\}$, $D(x)$ is the *Bloch–Wigner dilogarithm*

$$D(x) = \arg(1-x) \log|x| - \text{Im} \left(\int_0^x \log(1-t) \frac{dt}{t} \right).$$

Bloch shows how to relate D to the Borel regulator on $K_3(\mathbb{C})$ via the map θ . If $F \subset \mathbb{C}$ is a number field and one has explicit elements in $\mathcal{B}(F)$ which form a basis for $K_3(F)^{\text{ind}}$, this gives an explicit formula for the value of the Borel regulator for $K_3(F)$.

Example 3. Let $F = \mathbb{Q}(\zeta)$, where $\zeta = \exp(\frac{2\pi i}{\ell})$ and ℓ is an odd prime. An easy calculations shows that $\ell[\zeta^i]$ is in $\mathcal{B}(F)$ for all i ; one shows that $\ell[\zeta^1], \dots, \ell[\zeta^{\ell-1/2}]$ maps to a basis of $K_3(F)_{\mathbb{Q}}$ under θ . Using the explicit formula

$$D(\zeta^i) = \text{Im} \left(\sum_{m=1}^{\infty} \frac{\zeta^{mi}}{m^2} \right),$$

Bloch computes: the lattice in $\mathbb{R}^{\ell-1/2}$ generated by the vectors

$$(D(\ell[\zeta^i]), \dots, D(\ell[\zeta^{ij}]), \dots, D(\ell[\zeta^{i(\ell-1)/2}])) ; \quad i = 1, \dots, \frac{\ell-1}{2},$$

has volume $2^{-(\ell-1)/2} \ell^{3(\ell-1)/4} \prod_{\chi \text{ odd}} |L(2, \chi)|$, where χ runs over the odd characters of $(\mathbb{Z}/\ell\mathbb{Z})^*$ and $L(s, \chi) := \sum \chi(n) n^{-s}$ is the Dirichlet L -function.

The Bloch–Suslin Complex

Suslin [87] refined Bloch’s construction of the complex $\mathcal{A}(F) \rightarrow F^* \otimes F^*$ by imposing the five-term relation satisfied by the dilogarithm function:

$$[x] - [y] + [y/x] - \left[\frac{y-1}{x-1} \right] + \left[\frac{y(x-1)}{x(y-1)} \right].$$

One checks that this element goes to zero in $F^* \wedge F^*$, giving the complex

$$A(F) \xrightarrow{\bar{\lambda}} \Lambda^2 F^* \tag{5.2}$$

with $A(F)$ being the above-mentioned quotient of $\mathcal{A}(F)$, and $\lambda(x) = x \wedge (1-x)$. Since $\{x, y\} = \{y, x\}^{-1}$ in $K_2(F)$, the cokernel of λ is still $K_2(F)$; Suslin shows

Proposition 4 Let F be an infinite field. There is a natural isomorphism

$$\ker \bar{\lambda} \cong K_3^{\text{ind}}(F) / \widetilde{\text{Tor}}_1(F^*, F^*),$$

where $\widetilde{\text{Tor}}_1(F^*, F^*)$ is an extension of $\text{Tor}_1(F^*, F^*)$ by $\mathbb{Z}/2$.

Higher Weight

The construction of the Bloch–Suslin complex (5.2) has been generalized by Goncharov [38, 39] to give complexes $C(n)$ of the form

$$\begin{aligned} A_F(n) \rightarrow A_F(n-1) \otimes F^* \rightarrow A_F(n-2) \otimes \Lambda^2 F^* \rightarrow \dots \\ \rightarrow A_F(2) \otimes \Lambda^{n-2} F^* \rightarrow \Lambda^n F^* \end{aligned}$$

These are homological complexes with $\Lambda^n F^*$ in degree n .

The groups $A_F(i)$ are defined inductively: Each $A_F(i)$ is a quotient of $\mathbb{Z}[F \cup \{\infty\}]$; denote the generator corresponding to $x \in F$ as $[x]_i$. For $i > 2$, the map

$$A_F(i) \otimes \Lambda^{n-i} F^* \rightarrow A_F(i-1) \otimes \Lambda^{n-i+1} F^*$$

sends $[x]_i \otimes \eta$ to $[x]_{i-1} \otimes x \wedge \eta$ for $x \neq 0, \infty$ and sends $[0]_i$, and $[\infty]_i$ to 0. $A_F(1) = F^*$, with $[x]_1$ mapping to $x \in F^*$ and $A_F(2)$ is the Bloch–Suslin construction $A(F)$ (set $[0]_i = [1]_i = [\infty]_i = 0$ for $i = 1, 2$). The map

$$A_F(2) \otimes \Lambda^{n-2} F^* \rightarrow \Lambda^n F^*$$

sends $[x]_2 \wedge \eta$ to $x \wedge (1-x) \wedge \eta$ for $x \neq 0, 1, \infty$.

To define $A_F(i)$ as a quotient of $\mathcal{A}_F(i) := \mathbb{Z}[F \cup \{\infty\}]$ for $i > 2$, Goncharov imposes “all rational relations”: Let $\mathcal{B}_F(i)$ be the kernel of

$$\begin{aligned} \mathcal{A}_F(i) &\rightarrow A_F(i-1) \otimes F^*, \\ [x] &\mapsto [x]_{i-1} \otimes x. \end{aligned}$$

For $\sum_j n_j [x_j(t)]$ in $\mathcal{B}_{F(t)}(i)$, t a variable, each $x_j(t)$ defines a morphism $x_j : \mathbb{P}_F^1 \rightarrow \mathbb{P}_F^1$, and so $x_j(a) \in F \cup \{\infty\}$ is well-defined for all $a \in F$. Let $\mathcal{R}_F(i) \subset \mathcal{A}_F(i)$ be the subgroup generated by $[0]$, $[\infty]$ and elements of the form

$$\sum_j n_j [x_j(1)] - \sum_j n_j [x_j(0)],$$

with $\sum_j n_j [x_j(t)] \in \mathcal{B}_{F(t)}(i)$, and set $A_F(i) := \mathcal{A}_F(i) / \mathcal{R}_F(i)$. One checks that this does indeed form a complex.

The role of these complexes and their applications to a number of conjectures is explained in detail in Goncharov’s article [35]. We will only mention that the homology $H_p(C(n))$ is conjectured to be the weight n K -group $K_p(F)^{(n)}$ for $n \leq p \leq 2n-1$.

5

Remark 5 In addition to inspiring later work on the construction of motivic complexes, Bloch’s introduction of the relative K_2 to study K_3^{ind} was later picked up by Merkurjev–Suslin [71] and Levine [66] in their computation of the torsion and co-torsion of K_3^{ind} of fields.

Beilinson–Lichtenbaum Complexes

5.2.2

In the early '80's Beilinson and Lichtenbaum gave conjectures for versions of universal cohomology which would arise as hypercohomology (in the Zariski, resp. étale topology) of certain complexes of sheaves. The conjectures describe sought-after properties of these representing complexes.

Beilinson's Conjectures

In [5], Beilinson gives a simultaneous generalization of a number of conjectures on values of L -functions (see Kahn's article [56] for details). A major part of this work involved generalizing the Borel regulator using Deligne cohomology and Gillet's Chern classes for higher K -theory. He also states:

"It is thought that for any schemes[*sic*] there exists a universal cohomology theory $H_{\mathcal{A}}^j(X, \mathbb{Z}(i))$ satisfying Poincaré duality and related to Quillen's K -theory in the same way as in topology the singular cohomology is related to K -theory. $H_{\mathcal{A}}^*$ must be closely related to the Milnor ring".

The reader should note that, at this point, Beilinson is speaking of a "universal" cohomology theory, but *not* "motivic" cohomology. In particular, one should expect that the rational version $H_{\mathcal{A}}^j(X, \mathbb{Q}(i))$ is weight-graded K -theory, and the integral version is related to Milnor K -theory, but there is as yet no direct connection to motives. In any case, here is a more precise formulation describing absolute cohomology:

Conjecture 6: *Beilinson [6]* For $X \in \mathbf{Sm}_k$ there are complexes $\Gamma_{\text{Zar}}(r)$, $r \geq 0$, in the derived category of sheaves of abelian groups on X_{Zar} , (functorial in X) with functorial graded product, and

- (0) $\Gamma_{\text{Zar}}(0) \cong \mathbb{Z}$, $\Gamma_{\text{Zar}}(1) \cong \mathbb{G}_m[-1]$
- (1) $\Gamma_{\text{Zar}}(r)$ is acyclic outside $[1, r]$ for $r \geq 1$.
- (2) $\Gamma_{\text{Zar}}(r) \otimes^L \mathbb{Z}/n \cong r_{\leq r} R\alpha \mu_n^{\otimes r}$ if n is invertible on X , where $\alpha : X_{\text{ét}} \rightarrow X_{\text{Zar}}$ is the change of topology morphism.
- (3) $\text{gr}_Y^r K_j(X) \otimes \mathbb{Q} \cong \mathbb{H}^{2r-i}(X_{\text{Zar}}, \Gamma_{\text{Zar}}(r))_{\mathbb{Q}}$ (or up to small primes)
- (4) $\mathcal{H}^r(\Gamma_{\text{Zar}}(r)) = \mathcal{K}_r^M$.

Here $\mathcal{K}_r^M$ is the sheaf of Milnor K -groups, where the stalk $\mathcal{K}_{r,x}^M$ for $x \in X$ is the kernel of the symbol map

$$K_r^M(k(X)) \rightarrow \bigoplus_{x \in X(1)} K_{r-1}^M(k(x)) .$$

Lichtenbaum's Conjectures

Lichtenbaum's conjectures seem to be motivated more by the search for an integral cohomology theory that would explain the values of L -functions. As the ℓ -part of these values was already seen to have a close connection with ℓ -adic étale cohomology, it is natural that these complexes would be based on the étale topology.

7

Conjecture 7: *Lichtenbaum* [67, 69] For $X \in \mathbf{Sm}_k$ there are complexes $\Gamma_{\text{ét}}(r)$, $r \geq 0$, in the derived category of sheaves of abelian groups on $X_{\text{ét}}$, (functorial in X) with functorial graded product, and

- (0) $\Gamma_{\text{ét}}(0) \cong \mathbb{Z}$, $\Gamma_{\text{ét}}(1) \cong \mathbb{G}_m[-1]$
- (1) $\Gamma_{\text{ét}}(r)$ is acyclic outside $[1, r]$ for $r \geq 1$.
- (2) $R^{r+1}\alpha\Gamma_{\text{ét}}(r) = 0$
- (3) $\Gamma_{\text{ét}}(r) \otimes^L \mathbb{Z}/n \cong \mu_n^{\otimes r}$ if n is invertible on X .
- (4) $\text{gr}_V^r \mathcal{K}_j^{\text{ét}} \cong \mathcal{H}^{2r-i}(\Gamma_{\text{ét}}(r))$ (up to small primes), where $\mathcal{K}_j^{\text{ét}}$ and $\mathcal{H}^{2r-i}(\Gamma_{\text{ét}}(r))$ are the respective Zariski sheaves.
- (5) For a field F , $H^r(\Gamma_{\text{ét}}(r)(F)) = K_r^M(F)$.

The two constructions should be related by

$$\tau_{\leq r} R\alpha_* \Gamma_{\text{ét}}(r) = \Gamma(r) ; \quad \Gamma_{\text{ét}}(r) = \alpha^* \Gamma_{\text{Zar}}(r) .$$

The relations (2) and (4) in Beilinson's conjectures and (2), (3) and (5) in Lichtenbaum's version are generalizations of the Merkurjev–Suslin theorem (the case $r = 2$); Lichtenbaum's condition (2) is a direct generalization of the classical Hilbert Theorem 90, and also the generalization for K_2 due to Merkurjev and Suslin [72]. These conjectures, somewhat reinterpreted for motivic cohomology, are now known as the Beilinson–Lichtenbaum conjectures (see [32] and also Sect. 5.2.4 for additional details).

5.2.3

Bloch's Cycle Complexes

In [13], Bloch gives a construction for complexes on X_{Zar} which satisfy some of the conjectured properties of Beilinson, and whose étale sheafification satisfies some of the properties conjectured by Lichtenbaum. The construction and basic properties of these complexes are discussed in [32]; we will use his notations here, but restrict ourselves mainly to the case of schemes of finite type over a field.

Cycle Complexes and Higher Chow Groups

Fix a field k . In [13], Bloch constructs, for each k -scheme X of finite type and equidimensional over k , and each integer $q \geq 0$, a simplicial abelian group $n \mapsto z^q(X, n)$. The associated homological complex $z^q(X, *)$ is called Bloch's *cycle complex* and the *higher Chow groups* $\text{CH}^q(X, n)$ are defined by

$$\text{CH}^q(X, n) := H_n(z^q(X, *)) .$$

We recall some details of this construction here for later use.

The *algebraic n -simplex* is the scheme

$$\Delta^n := \text{Spec } \mathbb{Z}[t_0, \dots, t_n] / \left(\sum_{i=0}^n t_i - 1 \right) .$$

The *vertex* v_i^n of Δ^n is the closed subscheme defined by $t_j = 0, j \neq i$. More generally, a *face* of Δ^n is a closed subscheme defined by equations of the form $t_{i_1} = \dots = t_{i_s} = 0$. We let $v(n)$ denote the set of vertices of Δ^n ; sending i to v_i^n defines a bijection $v_n : \underline{n} \rightarrow v(n)$. The choice of an index $i \in \underline{n}$ determines an isomorphism $\Delta^n \cong \mathbb{A}^n$ via the coordinates $t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_n$. Note that each face $F \subset \Delta^n$ is isomorphic to Δ^m for some $m \leq n$, we set $\dim F := m$. Let R_n denote the coordinate ring $\mathbb{Z}[t_0, \dots, t_n]/(\sum_{i=0}^n t_i - 1)$.

If $g : \underline{n} \rightarrow \underline{m}$ is a map of sets, let $g^* : R_m \rightarrow R_n$ be the map defined by $g^*(t_i) = \sum_{j \in g^{-1}(i)} t_j$ (so $g^*(t_i) = 0$ if i is not in the image of g). We thus have the map $\Delta(g) : \Delta^n \rightarrow \Delta^m$, and this forms the cosimplicial scheme $\Delta^* : \Delta \rightarrow \mathbf{Sch}$. More generally, if X is a k -scheme, we have the cosimplicial k -scheme $X \times \Delta^*$.

Definition 8 For a finite type k -scheme X and integer n , let $z_p(X, n) \subset z_{p+n}(X \times \Delta^n)$ be the subgroup generated by integral closed subscheme W of $X \times \Delta^n$ with

$$\dim_k (W \cap (X \times F)) \leq \dim F + p$$

for each face F of Δ^n .

If X is locally equi-dimensional over k , let $z^p(X, n) \subset z^p(X \times \Delta^n)$ be the subgroup generated by integral closed subscheme W of $X \times \Delta^n$ with

$$\mathrm{codim}_{X \times F} (W \cap (X \times F)) \geq p$$

for each face F of Δ^n .

For $g : \underline{n} \rightarrow \underline{m}$ a map in Δ , we let $g^* : z_p(X, m) \rightarrow z_p(X, n)$ be the map induced by

$$g^*(\overline{w}) := (\mathrm{id} \times g)^* : z_{p+m}(X \times \Delta^m)_{\mathrm{id} \times g} \rightarrow z_{p+n}(X \times \Delta^n) .$$

If X is locally equi-dimensional over k , the map $g^* : z^p(X, m) \rightarrow z^p(X, n)$ is defined similarly.

The assignment

$$n \mapsto z_p(X, n) ,$$

$$(g : \underline{n} \rightarrow \underline{m}) \mapsto (g^* : z_p(X, m) \rightarrow z_p(X, n))$$

forms the simplicial abelian group $z_p(X, -)$. We let $z_p(X, *)$ denote the associated complex of abelian groups. If X is locally equi-dimensional over k , we have the simplicial abelian group $z^p(X, -)$ and the complex $z^p(X, *)$; if X has pure dimension d over k , then $z^p(X, -) = z_{d-p}(X, -)$.

Definition 9 Let X be a k -scheme of finite type. Set

$$\mathrm{CH}_p(X, n) := \pi_n(z_p(X, -)) = H_n(z_p(X, *)) .$$

If X is locally equi-dimensionnal over k , we set

$$\mathrm{CH}^p(X, n) := \pi_n(z^p(X, -)) = H_n(z^p(X, *)) .$$

Elementary Functorialities

The complexes $z_p(X, *)$ and groups $\mathrm{CH}_p(X, n)$ satisfy the following functorialities:

1. Let $f : Y \rightarrow X$ be a proper map in $\mathbf{Sch}_k$. Then the maps $(f \times \mathrm{id}_{\Delta^n})_*$ give rise to the map of complexes

$$f_* : z_p(Y, *) \rightarrow z_p(X, *)$$

yielding $f_* : \mathrm{CH}_p(X, n) \rightarrow \mathrm{CH}_p(Y, n)$. The maps f_* satisfy the functoriality $(gf)_* = g_* \circ f_*$ for composable proper maps f, g .

2. Let $f : Y \rightarrow X$ be an equi-dimensional l.c.i. map in $\mathbf{Sch}_k$ with fiber dimension d . Then the maps $(f \times \mathrm{id}_{\Delta^n})^*$ give rise to the map of complexes

$$f^* : z_p(X, -) \rightarrow z_{p+d}(Y, -) ,$$

yielding $f^* : \mathrm{CH}_p(X, n) \rightarrow \mathrm{CH}_{p+d}(Y, n)$. The maps f^* satisfy the functoriality $(gf)^* = g^* \circ f^*$ for composable equi-dimensional l.c.i. maps f, g .

Classical Chow Groups

The groups $\mathrm{CH}_p(X, 0)$ are by definition the cokernel of the map

$$\delta_0^* - \delta_1^* : z_{p+1}(X, 1) \rightarrow z_p(X, 0) = z_p(X) .$$

From this, one has the identity $\mathrm{CH}_p(X, 0) = \mathrm{CH}_p(X)$.

10

Remark 10 All the above extends to schemes essentially of finite type over k by taking the evident direct limit over finite-type models. One can also extend the definitions to scheme of finite type over a regular base B of Krull dimension one: for $X \rightarrow B$ finite type and locally equi-dimensional, the definition of $z^p(X, *)$ is word-for-word the same. The definition of $z_p(X, -)$ for X a finite-type B -scheme requires only a reasonable notion of dimension to replace $\dim_k$. The choice made in [62] is as follows: Suppose that B is integral with generic point η . Let $p : W \rightarrow B$ be of finite type, with W integral. If the generic fiber W_η is non-empty, set $\dim W := \dim_{k(\eta)} W_\eta + 1$; if on the other hand $p(W) = x$ is a closed point of B , set $\dim W := \dim_{k(x)} W$. In particular, one has a good definition of the higher Chow groups $\mathrm{CH}_p(X, n)$ for X of finite type over the ring of integers $\mathcal{O}_F$ in a number field F .

Fundamental Properties and their Consequences

We now list the fundamental properties of the complexes $z_p(X, *)$.

Theorem 11: *Homotopy property [13]* Let X be in $\mathbf{Sch}_k$ and let $\pi : X \times \mathbb{A}^1 \rightarrow X$ be the projection. Then the map $\pi^* : z_p(X, *) \rightarrow z_{p+1}(X \times \mathbb{A}^1, *)$ is a quasi-isomorphism, i.e., the map $\pi^* : \mathrm{CH}_p(X, n) \rightarrow \mathrm{CH}_{p+1}(X \times \mathbb{A}^1, n)$ is an isomorphism for $n = 0, 1, \dots$

11

Theorem 12: *Localization [10]* Let X be in $\mathbf{Sch}_k$, let $i : W \rightarrow X$ be a closed subscheme and $j : U \rightarrow X$ the open complement $X \setminus W$. Then the sequence

12

$$z_p(W, *) \xrightarrow{i_*} z_p(X, *) \xrightarrow{j^*} z_p(U, *)$$

induces a quasi-isomorphism

$$z_p(W, *) \rightarrow \mathrm{Cone}(z_p(X, *) \xrightarrow{j^*} z_p(U, *))[-1] .$$

Definition 13 Let $f : Y \rightarrow X$ be a morphism in $\mathbf{Sch}_k$, with Y and X locally equi-dimensional over k . Let $z^p(X, n)_f \subset z^p(X, n)$ be the subgroup generated by irreducible $W \subset X \times \Delta^n$ with $1 \cdot W \in z^p(X, n)$ and $1 \cdot Z \in z^p(Y, n)$ for each irreducible component Z of $(\mathrm{id} \times f)^{-1}(W)$. This forms a subcomplex $z^p(X, *)_f$ of $z^p(X, *)$.

13

Theorem 14: *Moving Lemma [63, part I, chapt. II, §3.5]* Let $f : Y \rightarrow X$ be a morphism in $\mathbf{Sch}_k$ with X in $\mathbf{Sm}_k$. Suppose X is either affine or projective over k . Then the inclusion $z^p(X, *)_f \rightarrow z^p(X, *)$ is a quasi-isomorphism.

14

These results have the following consequences.

Mayer–Vietoris

Let X be in $\mathbf{Sch}_k$, $U, V \subset X$ open subschemes with $X = U \cup V$. Then the sequence (the maps are the evident restriction maps)

$$z_p(X, *) \rightarrow z_p(U, *) \oplus z_p(V, *) \rightarrow z_p(U \cap V, *)$$

gives a quasi-isomorphism

$$z_p(X, *) \rightarrow \mathrm{Cone}(z_p(U, *) \oplus z_p(V, *) \rightarrow z_p(U \cap V, *))[-1] .$$

This yields a long exact Mayer–Vietoris sequence for the higher Chow groups.

Functoriality

Let $f : Y \rightarrow X$ be a morphism in $\mathbf{Sch}_k$ with $X \in \mathbf{Sm}_k$ and Y locally equi-dimensional over k . Take an affine cover $\mathcal{U} = \{U_1, \dots, U_m\}$ of X , and let $\mathcal{V} := \{V_1, \dots, V_m\}$ be the

cover $V_j := f^{-1}(U_j)$ of Y . For $I \subset \{1, \dots, m\}$ let $U_I = \cap_{i \in I} U_i$, define V_I similarly, and let $f_I : U_I \rightarrow V_I$ be the morphism induced by f .

Form the Čech complex $z^p(\mathcal{U}, *)$ as the total complex of the evident double complex

$$\oplus_i z^p(U_i, *) \rightarrow \oplus_{i < j} z^p(U_i \cap U_j, *) \rightarrow \dots \rightarrow z^p(\cap_{i=1}^m U_i, *)$$

and define $z^p(\mathcal{V}, *)$ similarly. Replacing $z^p(U_I, *)$ with $z^p(U_I, *)_{f_I}$ yields the sub-complex $z^p(\mathcal{U}, *)_f$ of $z^p(\mathcal{U}, *)$; the pull-backs f_I^* yield the map of complexes

$$f^* : z^p(\mathcal{U}, *)_f \rightarrow z^p(\mathcal{V}, *) .$$

By the moving lemma (Theorem 14), the inclusion $z^p(\mathcal{U}, *)_f \rightarrow z^p(\mathcal{U}, *)$ is a quasi-isomorphism. We thus have the morphism $f^* : z^p(X, *) \rightarrow z^p(Y, *)$ in $D^-(\mathbf{Ab})$ defined by the zig-zag diagram

$$z^p(X, *) \rightarrow z^p(\mathcal{U}, *) \leftarrow z^p(\mathcal{U}, *)_f \xrightarrow{f^*} z^p(\mathcal{V}, *) \leftarrow z^p(Y, *) .$$

One shows that this makes the assignment $X \mapsto z^p(X, *) \in D^-(\mathbf{Ab})$ into a functor $z^p(-, *) : \mathbf{Sm}_k^{\text{op}} \rightarrow D^-(\mathbf{Ab})$. In particular, $X \mapsto \text{CH}^p(X, n)$ becomes a functor

$$\text{CH}^p(-, n) : \mathbf{Sm}_k^{\text{op}} \rightarrow \mathbf{Ab} .$$

Products

The cycle complexes admit natural associative and commutative external products

$$\cup_{X,Y} : z_p(X, *) \otimes z_q(Y, *) \rightarrow z_{p+q}(X \times Y, *)$$

in $D^-(\mathbf{Ab})$; for X smooth over k , one has natural cup products in $D^-(\mathbf{Ab})$

$$\cup_X := \delta^* \circ \cup_{X,X} :: z^p(X, *) \otimes z^q(X, *) \rightarrow z^{p+q}(X, *) ,$$

where $\delta : X \rightarrow X \times_k X$ is the diagonal. The cup products make $\oplus_p z^p(X, *)$ an associative commutative ring in the derived category, with unit the fundamental class $1 \cdot X \in z^0(X, 0)$. In particular, this makes $\oplus_{p,q} \text{CH}^p(X, q)$ into a bigraded ring (commutative in the p -grading, graded-commutative in the q -grading), functorial in X .

One easily verifies the projection formula

$$p_*(p^* \alpha \cup \beta) = \alpha \cup p_* \beta$$

for a proper map $p : Y \rightarrow X$ in $\mathbf{Sm}_k$.

The external products are essentially given by the usual external product of cycles. However, as the external product of cycle on $X \times \Delta^n$ and a cycle on $Y \times \Delta^m$ yields a cycle on $X \times Y \times \Delta^n \times \Delta^m$, not a cycle on $X \times Y \times \Delta^{n+m}$, the natural target of the

external product is the total complex of the double complex $z_{p+q}(X \times_k Y, *, *)$, where $z^p(X \times Y, n, m)$ is the subgroup of $z^p(X \times Y \times \Delta^n \times \Delta^m)$ of cycles in good position with respect to “bi-faces” $X \times Y \times F \times F'$. One then needs to map $\text{Tot } z_{p+q}(X \times_k Y, *, *)$ back to $z_{p+q}(X \times_k Y, *)$, in the derived category. There are two techniques for doing this:

1. Use the standard triangulation of $\Delta^n \times \Delta^m$ into $n + m$ -simplices
2. Show that the inclusion

$$z_{p+q}(X \times_k Y, *) = z_{p+q}(X \times_k Y, *, 0) \subset \text{Tot } z_{p+q}(X \times_k Y, *, *)$$

is a quasi-isomorphism

Both these techniques work and give the same product structure, see e.g. [64] or [33] for details.

Projective Bundle Formula

For an invertible sheaf $\mathcal{L}$ on $X \in \mathbf{Sm}_k$, we may choose a Cartier divisor D on X with $\mathcal{O}_X(D) \cong \mathcal{L}$. Sending $\mathcal{L}$ to the class of D in $\text{CH}^1(X, 0) = \text{CH}^1(X)$ gives a homomorphism

$$c_1 : \text{Pic}(X) \rightarrow \text{CH}^1(X, 0).$$

If $\mathcal{E} \rightarrow X$ is a locally free sheaf of rank $n + 1$, and $q : \mathbb{P}(\mathcal{E}) \rightarrow X$ the associated $\mathbb{P}^n$ -bundle $\text{Proj}_{\mathcal{O}_X}(\text{Sym}^* \mathcal{E})$, we have the tautological invertible (quotient) sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$; let $\xi := c_1(\mathcal{O}(1))$. $\text{CH}^*(\mathbb{P}(\mathcal{E}), *)$ is a $\text{CH}^*(X, *)$ -module via q^* ; in fact, $\text{CH}^*(\mathbb{P}(\mathcal{E}), *)$ is a free $\text{CH}^*(X, *)$ -module with basis $1, \xi, \dots, \xi^n$.

Relation with K -Theory

Once one has the projective bundle formula, one can apply the technique of Gillet [34] to give natural Chern class maps

$$c_{p,q} : K_{2q-p}(X) \rightarrow \text{CH}^q(X, 2q - p)$$

and a multiplicative Chern character

$$\text{ch}_* : K_*(X)_{\mathbb{Q}} \rightarrow \bigoplus_{p,q} \text{CH}^p(X, q)_{\mathbb{Q}}$$

We let $K_n(X)^{(p)}$ denote the weight p subspace of $K_n(X)_{\mathbb{Q}}$, i.e.

$$K_n(X)^{(p)} = \{x \in K_n(X)_{\mathbb{Q}} \mid \psi_k(x) = k^p \cdot x \text{ for all } k \geq 2\},$$

where ψ_k is the k th Adams operation on $K_n(X)$.

Theorem 15: [10, 64] Let X be in $\mathbf{Sm}_k$. The Chern character gives an isomorphism

$$K_n(X)^{(p)} \rightarrow \text{CH}^p(X, n)_{\mathbb{Q}}.$$

Milnor K -Theory

As a special case of Theorem 15, we have the isomorphism

$$\mathrm{CH}^n(F, n)_{\mathbb{Q}} \cong K_n(F)^{(n)}$$

for F a field. From work of Suslin [89], we know that the canonical map of Milnor K -theory to Quillen K theory identifies $K_n^M(F)_{\mathbb{Q}}$ with $K_n(F)^{(n)}$. In fact, one has

16 Theorem 16: *Nestorenko–Suslin [76], Totaro [92]* Let F be a field. There is a natural isomorphism

$$K_n^M(F) \cong \mathrm{CH}^n(F, n).$$

The case $n = 1$ is a special case of the result in [13]:

17 Proposition 17 Let X be in $\mathbf{Sm}_k$. Sending a unit $u \in H^0(X, \mathcal{O}_X^*)$ to the subscheme $(u - 1)t_1 = u$ of $X \times \Delta^1$ defines an isomorphism $H^0(X, \mathcal{O}_X^*) \cong \mathrm{CH}^1(X, 1)$. For $n \neq 1$, $\mathrm{CH}^1(X, n) = 0$.

5.2.4 Suslin Homology and Friedlander–Suslin Cohomology

We describe Suslin’s construction of “abstract homology” for algebraic varieties, and various modifications. For further details on this construction, we refer the reader to the article [42] in this volume.

Finite Cycles and Quasi-finite Cycles

18 Definition 18 Take Y in $\mathbf{Sm}_k$ and X in $\mathbf{Sch}_k$.

- (1) Let $z_{\mathrm{fin}}(X)(Y)$ be the subgroup of $z_*(Y \times_k X)$ generated by integral closed subschemes $W \subset Y \times_k X$ such that $p_1 : W \rightarrow Y$ is finite and dominant over an irreducible component of Y .
- (2) Let $z_{\mathrm{q.fin}}(X)(Y)$ be the subgroup of $z_*(Y \times_k X)$ generated by integral closed subschemes $W \subset Y \times_k X$ such that $p_1 : W \rightarrow Y$ is quasi-finite and dominant over an irreducible component of Y .

For a morphism $f : Y' \rightarrow Y$ in $\mathbf{Sm}_k$, the morphism $f \times \mathrm{id} : Y' \times_k X \rightarrow Y \times_k X$ is an l.c.i.-morphism; the finiteness, resp., quasi-finiteness conditions imply that cycle-pull-back gives well-defined homomorphisms

$$f^* : z_{\mathrm{fin}}(X)(Y) \rightarrow z_{\mathrm{fin}}(X)(Y') ; \quad f^* : z_{\mathrm{q.fin}}(X)(Y) \rightarrow z_{\mathrm{q.fin}}(X)(Y) ,$$

making $z_{\mathrm{fin}}(X)$ and $z_{\mathrm{q.fin}}(X)$ into presheaves of abelian groups on $\mathbf{Sm}_k$. It is easy to see that these are in fact sheaves for the étale topology on $\mathbf{Sm}_k$.

Let $\mathcal{F}$ be a presheaf of abelian groups on $\mathbf{Sm}_k$. For $Y \in \mathbf{Sm}_k$, we may apply $\mathcal{F}$ to the cosimplicial scheme $Y \times \Delta^*$, giving the simplicial abelian group $\mathcal{F}(Y \times \Delta^*)$.

Definition 19 Let $\mathcal{F}$ be a presheaf of abelian groups on $\mathbf{Sm}_k$.

(1) The *Suslin complex* $C_*^{\text{Sus}}(\mathcal{F})$ is the complex of presheaves

$$Y \mapsto C_*^{\text{Sus}}(\mathcal{F})(Y) ,$$

where $C_*^{\text{Sus}}(\mathcal{F})(Y)$ is the complex associated to the simplicial abelian group $\mathcal{F}(Y \times \Delta^*)$.

(2) For $Y \in \mathbf{Sm}_k$, write $\mathbb{Z}_{\text{FS},Y}(q)$ for the (cohomological) complex of sheaves on Y_{Zar}

$$U \mapsto C_{2q-*}(z_{q,\text{fin}}(\mathbb{A}^q))(U) .$$

and $\mathbb{Z}_{\text{FS}}(q)$ for the corresponding complex of sheaves on $\mathbf{Sm}_k^{\text{Nis}}$.

(3) For $X \in \mathbf{Sch}_k$, and abelian group A , the *Suslin homology* of X , $H_*^{\text{Sus}}(X, A)$ is defined by

$$H_n^{\text{Sus}}(X, A) := H_n(C_*^{\text{Sus}}(z_{\text{fin}}(X)(k) \otimes A)) .$$

(4) For $Y \in \mathbf{Sm}_k$, the *Friedlander–Suslin cohomology* $H_{\text{FS}}^p(Y, A(q))$ is defined by

$$H_{\text{FS}}^p(Y, A(q)) := \mathbb{H}^p(Y_{\text{Zar}}, \mathbb{Z}_{\text{FS},Y}(q) \otimes A) .$$

Remark 20 Let $\mathcal{F}$ be a presheaf on $\mathbf{Sm}_k$. The homology presheaf on $\mathbf{Sm}_k$

$$Y \mapsto H_n(C_*^{\text{Sus}}(\mathcal{F})(Y))$$

is homotopy invariant, i.e., the natural map

$$p^* : H_n(C_*^{\text{Sus}}(\mathcal{F})(Y)) \rightarrow H_n(C_*^{\text{Sus}}(\mathcal{F})(Y \times \mathbb{A}^1))$$

is an isomorphism. See e.g. [100, chapt. 3, prop. 3.6] for a proof.

Comparison with the Higher Chow Groups

For

$$W \in z_{q,\text{fin}}(\mathbb{A}^q)(Y \times \Delta^n) \subset z^q(Y \times \mathbb{A}^q \times \Delta^n) ,$$

and $F \subset \Delta^n$ a face, the intersection $W \cap (Y \times \mathbb{A}^q \times F)$ is quasi-finite over $Y \times F$, hence

$$\text{codim}_{Y \times \mathbb{A}^q \times F} W \cap (Y \times \mathbb{A}^q \times F) \geq q .$$

Thus, we have inclusions

$$z_{q,\text{fin}}(\mathbb{A}^q)(Y \times \Delta^n) \subset z^q(Y \times \mathbb{A}^q, n) \subset z^q(Y \times \mathbb{A}^q \times \Delta^n) ,$$

giving the inclusion of complexes

$$\alpha_Y^q : C_*(z_{q,\text{fin}}(\mathbb{A}^q))(Y) \rightarrow z^q(Y \times \mathbb{A}^q, *) .$$

Let $\mathbb{Z}_{\mathrm{Bl},Y}(q)$ be the sheaf of (cohomological) complexes on Y_{Zar} associated to the presheaf

$$U \mapsto z^q(U \times \mathbb{A}^q, 2q - *) .$$

The maps α_U^q thus give the map of sheaves of complexes

$$\alpha^q : \mathbb{Z}_{\mathrm{FS},Y}(q) \rightarrow \mathbb{Z}_{\mathrm{Bl},Y}(q)$$

The main result of this section is

21 Theorem 21: *Suslin [100, chapt. 6]* The map $\alpha^q : \mathbb{Z}_{\mathrm{FS},Y}(q) \rightarrow \mathbb{Z}_{\mathrm{Bl},Y}(q)$ is a quasi-isomorphism for all $Y \in \mathbf{Sm}_k$.

22 Corollary 22 Let Y be in $\mathbf{Sm}_k$. Then α^q induces an isomorphism

$$H_{\mathrm{FS}}^p(Y, \mathbb{Z}(q)) \rightarrow \mathrm{CH}^q(Y, 2q - p) .$$

Proof of the corollary

By the Mayer–Vietoris property for the complexes $z^q(U \times \mathbb{A}^q, *)$, the natural map

$$H_{2q-n}(z^q(Y \times \mathbb{A}^q, *)) \rightarrow \mathbb{H}^n(Y_{\mathrm{Zar}}, \mathbb{Z}_{\mathrm{Bl},Y}(q))$$

is an isomorphism for all n . By the homotopy property (Theorem 11), the pull-back map

$$p^* : z^q(\mathbb{A}^q, *) \rightarrow z^q(Y \times \mathbb{A}^q, *)$$

is a quasi-isomorphism, so we have the isomorphisms

$$\mathrm{CH}^q(Y, 2q - n) = H_{2q-n}(z^q(Y, *)) \cong H_{2q-n}(z^q(Y \times \mathbb{A}^q, *)) .$$

Thus, we have isomorphisms

$$H_{\mathrm{FS}}^p(Y, \mathbb{Z}(q)) = \mathbb{H}^p(Y_{\mathrm{Zar}}, \mathbb{Z}_{\mathrm{FS},Y}(q)) \cong \mathrm{CH}^q(Y, 2q - p) .$$

The proof of Theorem 21 goes in two steps: First one uses Suslin’s technique [100, chapt. 6, thm. 2.1] to show that $C_*(z_{\mathrm{q},\mathrm{fin}}(\mathbb{A}^q)(\mathrm{Spec} F)) \rightarrow z^q(\mathbb{A}_F^q, *)$ is a quasi-isomorphism for F a field. One may then use any one of several versions of a result of Voevodsky, that for $\mathcal{F}$ a homotopy invariant presheaf with transfers, $\mathcal{O}$ the local ring of a smooth point on a scheme of finite type over k with quotient field F , the map

$$\mathcal{F}(\mathcal{O}) \rightarrow \mathcal{F}(F)$$

is injective. One particular nice way to do this is the version due to Ojanguran–Panin [79], which allows one to use a fairly restricted form of transfers, namely:

1. For $f : X \rightarrow Y$ a finite separable morphism in $\mathbf{Sm}_k$, there is a homomorphism $f_* : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$.
2. Let $f : X \rightarrow Y$ be as in (1), let $i_D : D \rightarrow Y$ be the inclusion of a smooth divisor, and suppose that $\bar{f} : E := X \times_Y D \rightarrow D$ is étale. Let $i_E : E \rightarrow X$ be the inclusion. Then $\bar{f}_* \circ i_E^* = i_D^* \circ f_*$.

It is not hard, using the moving lemma (Theorem 14) to show that the presheaf

$$Y \mapsto H_n(\text{Cone}(\alpha_Y^q))$$

has the structure of a presheaf on $\mathbf{Sm}_k$ with Ojanguran–Panin transfers; as this presheaf vanishes on fields, it follows that α^q is a quasi-isomorphism.

Remark 23 Via Proposition 17 and Theorem 21, we have an isomorphism

23

$$u : \mathbb{G}_m[-1] \rightarrow \mathbb{Z}_{\text{FS}}(1)$$

in $D^-(\text{Sh}^{\text{Nis}}(\mathbf{Sm}_k))$.

The mod- n Theory

Suslin and Voevodsky show in [86] that, for an algebraically closed field k and n prime to the characteristic of k , and for A a regular Henselian local k algebra with residue field k , there is a natural isomorphism of complexes

$$\mathbb{Z}_{\text{FS}}(q)(\text{Spec } A) \otimes^L \mathbb{Z}/n \cong \mu_n^{\otimes q}(\text{Spec } A)$$

(actually, this is only shown for k of characteristic zero, but using de Jong’s theory of alterations, the same argument works in positive characteristics). This verifies part (3) of Lichtenbaum’s conjectures 7.

The analogous conjecture, part (2) of Beilinson’s conjectures 6, which essentially asserts the existence of a natural isomorphism

$$H^p(F, \mathbb{Z}/n(q)) \cong H_{\text{ét}}^p(F, \mu_n^{\otimes q})$$

for fields F finitely generated over a chosen base-field k (not necessarily algebraically closed) was shown in [90] and later in [33] to be equivalent to the *Bloch–Kato conjecture*:

Conjecture 24: [17] Let F be a field, n an integer prime to the characteristic of F . Then the Galois symbol

24

$$\theta : K_q^M(F)/n \rightarrow H_{\text{Gal}}^p(F, \mu_n^{\otimes q})$$

is an isomorphism.

Here, the Galois symbol is the map sending a symbol $\{a_1, \dots, a_q\} \in K_q^M(F)$ to the cup product $l(a_1) \cup \dots \cup l(a_q)$, where $l(a) \in H_{\text{Gal}}^1(F, \mu_n)$ is the image of $a \in F^*$ under the Kummer sequence

$$H_{\text{Gal}}^0(F, \mathbb{G}_m) \xrightarrow{\times n} H_{\text{Gal}}^0(F, \mathbb{G}_m) \rightarrow H_{\text{Gal}}^1(F, \mu_n).$$

One reduces directly to the case $n = \ell^\nu$, ℓ a prime. The case $\ell = 2$ is also known as part of the *Milnor conjecture* [73] proven by Voevodsky [98, 99]. The case of odd ℓ has been recently reduced by Voevodsky [93] to results of Rost (as yet unpublished) on the construction and properties of so-called “generic splitting varieties”.

5.2.5 Cubical Versions

One can also use cubes instead of simplices to define the various versions of the cycle complexes. The major advantage is that the product structure for the cubical complexes is easier to define, and with $\mathbb{Q}$ -coefficients, one can construct cycle complexes which have a strictly commutative and associative product. This approach is used by Hanamura in his construction of a category of mixed motives, as well as in the construction of categories of Tate motives by Bloch, Bloch–Kriz and Kriz–May.

Cubical Complexes

Let $(\square^1, \partial\square^1)$ denote the pair $(\mathbb{A}^1, \{0, 1\})$, and $(\square^n, \partial\square^n)$ the n -fold product of $(\square^1, \partial\square^1)$. Explicitly, $\square^n = \mathbb{A}^n$, and $\partial\square^n$ is the divisor $\sum_{i=1}^n (x_i = 0) + \sum_{i=1}^n (x_i = 1)$, where $x_1, \dots, x_n$ are the standard coordinates on $\mathbb{A}^n$. A *face* of $\square^n$ is a face of the normal crossing divisor $\partial\square^n$, i.e., a subscheme defined by equations of the form $t_{i_1} = \varepsilon_1, \dots, t_{i_s} = \varepsilon_s$, with the ε_j in $\{0, 1\}$. If a face F has codimension m in $\square^n$, we write $\dim F = n - m$.

For $\varepsilon \in \{0, 1\}$ and $j \in \{1, \dots, n\}$ we let $\iota_{j,\varepsilon} : \square^{n-1} \rightarrow \square^n$ be the closed embedding defined by inserting an ε in the j th coordinate. We let $\pi_j : \square^n \rightarrow \square^{n-1}$ be the projection which omits the j th factor.

25

Definition 25 Let X be in Sch_k . Let $\widehat{z}_p(X, n)^{\text{cb}}$ be the subgroup of $z_{p+n}(X \times \square^n)$ generated by integral subschemes $W \subset X \times \square^n$ such that

$$\dim_k W \cap (X \times F) \leq p + \dim F.$$

If X is equi-dimensional over k of dimension d , we write $\widehat{z}^p(X, n)^{\text{cb}}$ for $\widehat{z}_{d-p}(X, n)^{\text{cb}}$ and extend to locally equi-dimensional X by taking direct sums over the connected components of X .

Clearly the pull-back of cycles $\iota_{j,\varepsilon}^* : \widehat{z}_p(X, n)^{\text{cb}} \rightarrow \widehat{z}_p(X, n-1)^{\text{cb}}$ and $\pi_j^* : \widehat{z}_p(X, n-1)^{\text{cb}} \rightarrow \widehat{z}_p(X, n)^{\text{cb}}$ are defined. We let $z_p(X, n)^{\text{cb}}$ be the quotient

$$z_p(X, n)^{\text{cb}} := \widehat{z}_p(X, n)^{\text{cb}} / \sum_{j=1}^n \pi_j^* (\widehat{z}_p(X, n-1)^{\text{cb}}).$$

One easily checks that

$$\sum_{j=1}^n (-1)^{j-1} t_{j,1}^* - \sum_{j=1}^n (-1)^{j-1} t_{j,0}^* : \widehat{z}_p(X, n)^{\text{cb}} \rightarrow \widehat{z}_p(X, n-1)^{\text{cb}}$$

descends to

$$d_n : z_p(X, n)^{\text{cb}} \rightarrow z_p(X, n-1)^{\text{cb}}$$

and that $d_{n-1} \circ d_n = 0$. Thus, we have the complex $z_p(X, *)^{\text{cb}}$, and for X locally equi-dimensional over k the complex $z^p(X, *)^{\text{cb}}$.

We let $\mathbb{Z}_{\text{Bl}, X}(p)^{\text{cb}}$ denote the sheafification of the presheaf on X_{Zar} , $U \mapsto z^p(U \times \mathbb{A}^p, *)^{\text{cb}}$.

Replacing $z^p(X, n)^{\text{cb}}$ with $z_{\text{q.fin}}(\mathbb{A}^p)(X \times \square^n) / \sum_{j=1}^n \pi_j^* z_{\text{q.fin}}(\mathbb{A}^p)(X \times \square^{n-1})$ and using the similarly defined differential, we have the cubical version of Suslin's complex, $C_*^{\text{cb}}(z_{\text{q.fin}}(\mathbb{A}^p)(X))$ and the sheaf of complexes $\mathbb{Z}_{\text{FS}, X}(p)^{\text{cb}}$ on X_{Zar} .

Cubes and Simplices

The main comparison results are

Theorem 26 Let X be in Sch_k .

26

(1) There is an isomorphism in $D^-(\mathbf{Ab})$

$$z_p(X, *)^{\text{cb}} \cong z_p(X, *) ,$$

natural with respect to flat pull-back and proper push-forward.

(2) There is a natural (in the same sense as above) isomorphism in $D^-(\mathbf{Ab})$

$$C_*^{\text{cb}}(z_{\text{q.fin}}(\mathbb{A}^p))(X) \cong C_*(z_{\text{q.fin}}(\mathbb{A}^p)(X)) .$$

The proof of (1) is given in, e.g., [64, thm. 4.7]; the same argument (in fact somewhat easier) also proves (2).

This has as immediate corollary:

Corollary 27 For $X \in \mathbf{Sm}_k$, there is an isomorphism in the derived category of sheaves on X_{Zar}

27

$$\mathbb{Z}_{\text{Bl}, X}(p)^{\text{cb}} \cong \mathbb{Z}_{\text{FS}, X}(p)^{\text{cb}} ,$$

natural with respect to pull-back by maps in $\mathbf{Sm}_k$.

Remark 28 Let $f : Y \rightarrow X$ be a morphism in Sch_k , with X and Y locally equi-dimensional over k . One can define the subcomplex $z^p(X, *)_f^{\text{cb}} \subset z^p(X, *)^{\text{cb}}$ as the cubical version of the subcomplex $z_p(X, *)_f \subset z_p(X, *)$. The argument of [64,

28

thm. 4.7] mentioned above shows in addition that the isomorphism $z_p(X, *)^{\text{cb}} \cong z_p(X, *)$ induces an isomorphism $z^p(X, *)^{\text{cb}} \cong z^p(X, *)_f$, and thus, in case X is in $\mathbf{Sm}_k$ and is affine, the inclusion $z^p(X, *)^{\text{cb}} \rightarrow z^p(X, *)^{\text{cb}}$ is a quasi-isomorphism. Thus, sending X to $z^p(X, *)^{\text{cb}}$ extends to a functor

$$z^p(-, *)^{\text{cb}} : \mathbf{Sm}_k^{\text{op}} \rightarrow D^-(\mathbf{Ab}).$$

This explains the naturality assertion in the above corollary.

Products

As already mentioned, the cubical complexes are convenient for defining products. Indeed, the simple external product of cycles (after rearranging the terms in the product) defines the map of complexes

$$\cup_{X,Y} : z_p(X, *)^{\text{cb}} \otimes z_q(Y, *)^{\text{cb}} \rightarrow z_{p+q}(X \times_k Y, *)^{\text{cb}}$$

Thus, we have a cup product

$$\cup_X := \delta_X^* \circ \cup_{X,X} : z^p(X, *)^{\text{cb}} \otimes z^q(X, *)^{\text{cb}} \rightarrow z^{p+q}(X, *)^{\text{cb}}$$

in $D^-(\mathbf{Ab})$, and the isomorphism of Theorem 26 respects the two products.

Alternating Complexes

The symmetric group Σ_n acts on $z_p(X, n)^{\text{cb}}$ by permuting the factors of $\square^n$. Extending coefficients to $\mathbb{Q}$, we let $z_p(X, n)^{\text{Alt}}$ be the subspace of $z_p(X, n)_{\mathbb{Q}}^{\text{cb}}$ on which Σ^n acts by the sign representation, and let $\pi_n^{\text{Alt}} : z_p(X, n)_{\mathbb{Q}}^{\text{cb}} \rightarrow z_p(X, n)^{\text{Alt}}$ be the (Σ -equivariant) projection on this summand. One checks (see [11, lemma 1.1]) that the differential on $z_p(X, *)_{\mathbb{Q}}^{\text{cb}}$ descends to give a map

$$d_n^{\text{Alt}} : z_p(X, n)^{\text{Alt}} \rightarrow z_p(X, n-1)^{\text{Alt}}$$

forming the subcomplex $z_p(X, *)^{\text{Alt}}$ of $z_p(X, *)_{\mathbb{Q}}^{\text{cb}}$.

29

Lemma 29: [64, thm. 4.11] The inclusion $z_p(X, *)^{\text{Alt}} \rightarrow z_p(X, *)_{\mathbb{Q}}^{\text{cb}}$ is a quasi-isomorphism, with inverse the alternating projection $\pi^{\text{Alt}} : z_p(X, *)_{\mathbb{Q}}^{\text{cb}} \rightarrow z_p(X, *)^{\text{Alt}}$.

We may define an external product on the alternating complexes by

$$\cup_{X,Y}^{\text{Alt}} := \pi_{p+q}^{\text{Alt}} \circ \cup_{X,Y} : z^p(X, *)^{\text{Alt}} \otimes z^q(Y, *)^{\text{Alt}} \rightarrow z^{p+q}(X \times_k Y, *)^{\text{Alt}}.$$

This agrees (up to homotopy) with the product on $z^*(-, *)_{\mathbb{Q}}^{\text{cb}}$.

In particular, for $X = \text{Spec } k$, we have the commutative, associative product

$$\cup^{\text{Alt}} : z^p(k, *)^{\text{Alt}} \otimes_{\mathbb{Q}} z^q(k, *)^{\text{Alt}} \rightarrow z^{p+q}(k, *)^{\text{Alt}},$$

satisfying the Leibniz rule

$$d(a \cup^{\text{Alt}} b) = da \cup^{\text{Alt}} b + (-1)^{\deg a} a \cup^{\text{Alt}} b.$$

We will see in Sect. 5.5.2 how the complexes $z^q(k, *)^{\text{Alt}}$ form a (graded) commutative differential graded algebra over $\mathbb{Q}$, which may be used to give a concrete description of the category of mixed Tate motives over k .

Abelian Categories of Mixed Motives

5.3

We will now proceed to examine the framework proposed by Beilinson and Deligne for a category of mixed motives in somewhat more detail. Before doing so, however, we will fix some ideas concerning Bloch–Ogus cohomology and Tannakian categories. Having done this, we give a formulation of some of the hoped-for properties of the abelian category of mixed motives, and then describe two very different approaches to a construction. The first, following Jannsen and Deligne, attempts to define a “mixed motive” by its singular/étale/de Rham realizations. The second, due to Nori, first considers the ring of natural endomorphisms of the singular cohomology functor on pairs of schemes, and then defines a mixed motive as a module over this ring (roughly speaking). As we mentioned in the introduction, it is not at all clear what relation K -theory has to the cohomology theory arising from these constructions.

We will not discuss the theory of “pure” motives here at all. As a reference, we refer the reader to the relevant articles in [104], as well as [56, sect. 3] in this volume, where in addition some of the material in this section is handled in shorter form.

Background and Conjectures

5.3.1

We formulate a version of Bloch–Ogus cohomology, somewhat modified from the original definition in [19] to fit our purposes. We recall some notions from the theory of Tannakian categories, and then give a version of the properties one would like in an abelian category of mixed motives.

Bloch–Ogus Cohomology

Let $\text{Sh}^{\text{Zar}}(k)$ be the category of sheaves of abelian groups on the big Zariski site $\mathbf{Sm}_k$. Let $\Gamma(*) := \bigoplus_n \Gamma(n)$ be a graded object in $C(\text{Sh}^{\text{Zar}}(k))$ (with $\Gamma(n)$ in graded degree $2n$), together with a graded product $\mu : \Gamma(*) \otimes^L \Gamma(*) \rightarrow \Gamma(*)$ in $D(\text{Sh}^{\text{Zar}}(k))$. For $X \in \mathbf{Sm}_k$, we set

$$H_{\Gamma}^n(X, m) := \mathbb{H}^n(X_{\text{Zar}}, \Gamma(m))$$

and for $W \subset X$ a closed subset, set

$$H_{\Gamma, W}^n(X, m) := \mathbb{H}_W^n(X_{\text{Zar}}, \Gamma(m)).$$

We note that, if $W \subset W' \subset X$ are closed subsets of $X \in \mathbf{Sm}_k$, we have the natural map

$$H_{\Gamma, W}^n(X, m) \rightarrow H_{\Gamma, W'}^n(X, m) .$$

30

Definition 30 We say that $\Gamma(*)$ defines a *Bloch–Ogus cohomology theory* if

1. The product μ is associative and commutative in $D(\mathrm{Sh}^{\mathrm{Zar}}(k))$.
2. $\Gamma(*)$ is *homotopy invariant*: $p^* : H_{\Gamma}^*(X, m) \rightarrow H_{\Gamma}^*(X \times \mathbb{A}^1, m)$ is an isomorphism for all m .
3. $\Gamma(*)$ satisfies *purity*: Let $W \subset X$ be a closed subset, with $X \in \mathbf{Sm}_k$. If $\mathrm{codim}_X W \geq q$ for some integer q , then $H_{\Gamma, W}^p(X, q) = 0$ for $p < 2q$.
4. $\Gamma(*)$ admits natural *cycle classes*: Let $W \subset X$ be an irreducible closed codimension q subset with X in $\mathbf{Sm}_k$. Then there is a *fundamental class* $[W] \in H_{\Gamma, W}^{2q}(X, q)$ satisfying:
 - a) *Naturality*: Let $z := \sum_i n_i W_i$ be in $z^q(X)$, let W be the support of z , and set $\mathrm{cl}(z) = \sum_i n_i [W_i] \in H_{\Gamma, W}^{2q}(X, q)$. Let $f : Y \rightarrow X$ be a morphism in $\mathbf{Sm}_k$ such that $f^{-1}(W)$ has codimension q on Y . Then

$$f^*(\mathrm{cl}(z)) = \mathrm{cl}(f^*(z)) \in H_{\Gamma, f^{-1}(W)}^{2q}(Y, q) .$$

- b) *Gysin isomorphism*: Suppose that $W \subset X$ is a pure codimension q closed subset, with X and W in $\mathbf{Sm}_k$. Suppose that the inclusion $i : W \rightarrow X$ is split by a smooth morphism $p : X \rightarrow W$. Then the composition

$$H_{\Gamma}^n(W, m) \xrightarrow{p^*} H_{\Gamma}^n(X, m) \xrightarrow{\cup[W]} H_{\Gamma, W}^{n+2q}(X, m+q)$$

is an isomorphism.

- c) *Products*: For $X_i \in \mathbf{Sm}_k$, $z_i \in z^{q_i}(X_i)$ with support W_i , $i = 1, 2$, we have

$$\mathrm{cl}(z_1 \times z_2) = p_1^* \mathrm{cl}(z_1) \cup p_2^* \mathrm{cl}(z_2)$$

$$\text{in } H_{\Gamma, W_1 \times_k W_2}^{2q_1+2q_2}(X_1 \times_k X_2, q_1 + q_2) .$$

5. *Coefficients*: For $p : X \rightarrow \mathrm{Spec} k$ in $\mathbf{Sm}_k$, X irreducible, the map

$$p^* : H_{\Gamma}^0(\mathrm{Spec} k, 0) \rightarrow H_{\Gamma}^0(X, 0)$$

is an isomorphism.

The functor $X \mapsto \bigoplus_{p, q} H_{\Gamma}^p(X, q)$ is called the *Bloch–Ogus theory on $\mathbf{Sm}_k$ represented by $\Gamma(*)$* . The ring $H_{\Gamma}^0(\mathrm{Spec} k, 0)$ is called the *coefficient ring* of the theory Γ .

31

Remark 31 This notion of a Bloch–Ogus cohomology theory is somewhat more general than that considered by Gillet in [34], in that Gillet requires

1. A structure map $\mathbb{G}_m[-1] \rightarrow \Gamma(1)$ in $D(\mathrm{Sh}^{\mathrm{Zar}}(\mathbf{Sm}_k))$.
2. The complexes $\Gamma(n)$ should be in $C^+(\mathrm{Sh}^{\mathrm{Zar}}(k))$.

The existence of the structure map (1) follows from the cycle class map discussed in Sect. 5.6.1; see Remark 119 for a precise statement. Allowing the $\Gamma(q)$ to be unbounded forces one to take a bit more care in the definition of the universal Chern classes on the simplicial ind-scheme BGL , in that one needs to use the *extended* total complex to define $\Gamma(n)(BGL_N)$:

$$\Gamma(n)(BGL_N)^m := \prod_{p \geq 0} \Gamma(n)^{m-p}(BGL_N)_p ,$$

and then take

$$\Gamma(n)(BGL) := \lim_N \Gamma(n)(BGL_N) .$$

Having made this definition, Gillet's argument extends word-for-word to allow for $\Gamma(n) \in C(\mathrm{Sh}^{\mathrm{Zar}}(k))$, giving a good theory of Chern classes

$$c_T^{q,p} : K_{2q-p}(X) \rightarrow H_T^p(X, q)$$

for a Bloch–Ogus theory in our sense.

Example 32. The standard cohomology theories: singular cohomology, ℓ -adic étale cohomology, de Rham cohomology and Deligne cohomology, are all examples which can be fit into the framework of the above. Also, motivic cohomology, represented by $\Gamma(n) := \mathbb{Z}_{\mathrm{FS}}(n)$ (see Sect. 5.2.4), is an example.

Tannakian Formalism

We use [22, 28, 83] as references for this section.

Let F be a field. An F -linear abelian tensor category $\mathcal{A}$ is called *rigid* if there exists internal Homs in $\mathcal{A}$, i.e., for each pair of objects A, B of $\mathcal{A}$, there is an object $\mathcal{H}om(A, B)$ and a natural isomorphism of functors

$$(C \mapsto \mathrm{Hom}_{\mathcal{A}}(A \otimes C, B)) \cong (C \mapsto \mathrm{Hom}_{\mathcal{A}}(C, \mathcal{H}om(A, B))) .$$

For example, the abelian tensor category of finite-dimensional F -vector spaces, $F\text{-mod}$, has the internal Hom $\mathcal{H}om(V, W) := V^\vee \otimes W$.

An F -linear rigid abelian tensor category $\mathcal{A}$ is a *Tannakian category* if there exists an exact faithful F -linear tensor functor to F' -mod for some field extension F' of F ; such a functor is called a *fiber functor*. If a fiber functor to $F\text{-mod}$ exists, we call $\mathcal{A}$ a *neutral Tannakian category*.

The primary example of a neutral Tannakian category is the category $\mathrm{Rep}_F(G)$ of representations of an affine group scheme G over F in finite dimensional F -vector spaces; the forgetful functor $\mathrm{Rep}_F(G) \rightarrow F\text{-mod}$ is the evident fiber functor. Note that, if A is the Hopf algebra $\Gamma(G, \mathcal{O}_G)$, so that $G = \mathrm{Spec} A$, then $\mathrm{Rep}_F(G)$ is isomorphic to the category of co-representations of A in $F\text{-mod}$, $\mathrm{co}\text{-rep}_F(A)$.

Neutral Tannakian categories are of interest because they are all given as categories of representations: If $\mathcal{A}$ is a Tannakian category over F with fiber functor $\omega : \mathcal{A} \rightarrow F\text{-mod}$, then there is an affine group scheme G over F with a canonical isomorphism

$$G(F) \cong \text{Aut}(\omega)$$

and an equivalence of $\mathcal{A}$ with $\text{Rep}_F(G)$ with ω going over to the forgetful functor. G is canonically determined by $\mathcal{A}$ and ω , and different choices of ω lead to isomorphic G 's. G is called the *Galois group* of $\mathcal{A}$.

The Category of Mixed Motives

In [2] and [24], a framework for a category of mixed motives over a base field k is proposed. There are many articles describing the consequences of such a theory, e.g., [21, 53, 75]. We give here a quick description of the properties one should expect in this category, derived from [2] and [24].

33

Conjecture 33 Let k be a field. There is a rigid tensor category $\mathcal{MM}_k$ containing “Tate objects” $\mathbb{Z}(n)$, $n \in \mathbb{Z}$, and a functor

$$h : \mathbf{Sm}_k^{\text{op}} \rightarrow D^b(\mathcal{MM}_k)$$

such that

1. Setting $H_{\mathcal{M}}^p(X, \mathbb{Z}(q)) := \text{Ext}_{\mathcal{MM}_k}^p(1, h(X) \otimes \mathbb{Z}(q))$, the functor $X \mapsto \bigoplus_{p,q} H_{\mathcal{M}}^p(X, \mathbb{Z}(q))$ is the universal Bloch–Ogus cohomology theory on $\mathbf{Sm}_k$.
2. Let Γ be a Bloch–Ogus theory on $\mathbf{Sm}_k$, and $R\Gamma : \mathbf{Sm}_k^{\text{op}} \rightarrow D(\mathbf{Ab})$ the functor $X \mapsto \tilde{\Gamma}(q)(X)$, where $\tilde{\Gamma}(q)(X)$ is as in Sect. 5.3.1 the global sections on X of a functorial flasque model for $\Gamma(q)$. Then there is a “realization functor”

$$\mathfrak{R}_{\Gamma} : \mathcal{MM}_k \rightarrow D(\mathbf{Ab})$$

such that the induced map $D^b\mathfrak{R}_{\Gamma} : D^b(\mathcal{MM}_k) \rightarrow D(\mathbf{Ab})$ yields a factorization of $R\Gamma$ as $D^b\mathfrak{R}_{\Gamma} \circ R$. Applying H^p yields the canonical natural transformation

$$H_{\mathcal{M}}^p(X, \mathbb{Z}(q)) \rightarrow H_{\Gamma}^p(X, q)$$

given by (1).

3. In the $\mathbb{Q}$ -extension $\mathcal{MM}_k \otimes \mathbb{Q}$, the full subcategory of semi-simple objects is equivalent to the category $\mathcal{M}_k$ of homological motives over k , and for each $X \in \mathbf{Sm}_k$, the object $h^i(X)(q) := H^i(h(X)) \otimes \mathbb{Q}(q)$ is in $\mathcal{M}_k$.
4. For X smooth and projective over k , there is a decomposition (not necessarily unique) in $D^b(\mathcal{MM}_k \otimes \mathbb{Q})$

$$h(X)_{\mathbb{Q}} = \bigoplus_{i=0}^{2 \dim_k X} h^i(X)[-i] .$$

5. Let $\sigma : k \rightarrow \mathbb{C}$, and let $\mathfrak{R}_{\text{sing}, \sigma}$ be the realization functor corresponding to singular cohomology $H_{\text{sing}}^*(X^\sigma(\mathbb{C}), \mathbb{Z}(2\pi i)^j)$, where $X^\sigma(\mathbb{C})$ is the analytic manifold of $\mathbb{C}$ -points of $X \times_k \mathbb{C}$. Then the functor

$$H^0 \circ \mathfrak{R}_{\text{sing}, \sigma} : \mathcal{MM}_k \otimes \mathbb{Q} \rightarrow \mathbb{Q}\text{-mod}$$

is a fiber functor, making $\mathcal{MM}_k \otimes \mathbb{Q}$ a neutral Tannakian category over $\mathbb{Q}$. Also, if $\mathfrak{R}_{\text{ét}, \ell}$ is the realization functor corresponding to $X \mapsto H_{\text{ét}}^*(X \times_k \bar{k}, \mathbb{Q}_\ell(*))$, then

$$H^0 \circ \mathfrak{R}_{\text{ét}, \ell} : \mathcal{MM}_k \otimes \mathbb{Q}_\ell \rightarrow \mathbb{Q}_\ell\text{-mod}$$

is a fiber functor, making $\mathcal{MM}_k \otimes \mathbb{Q}_\ell$ a neutral Tannakian category over $\mathbb{Q}_\ell$.

6. For each object M in $\mathcal{MM}_k$, there is a natural finite *weight filtration*

$$0 = W_{n-1}M \subset W_n M \subset \dots \subset W_m M = M.$$

The graded quotients $\text{gr}_W^* M$ are in $\mathcal{M}_k$ (after passing to the $\mathbb{Q}$ -extension). For $M = h^i(X)$, the weight filtration is sent to the weight filtration for singular cohomology, respectively étale cohomology, under the respective realization functor.

7. There are natural isomorphisms

$$H_{\mathcal{M}}^p(X, \mathbb{Z}(q)) \otimes \mathbb{Q} \cong K_{2q-p}(X)^{(q)}.$$

These should arise from a natural spectral sequence of Atiyah–Hirzebruch type

$$E_2^{p,q} = H_{\mathcal{M}}^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X),$$

which degenerates at E_2 after tensoring with $\mathbb{Q}$.

Remark 34 Rather than limiting oneself to motives over a field k , Beilinson suggests in [2] that one should look for a theory of “mixed motivic sheaves” $\mathcal{MM}/S$ over a base-scheme S , analogous to the category of say sheaves of abelian groups or perverse sheaves or constructible étale sheaves, or mixed Hodge modules. In any case, one would want to have the Grothendieck–Verdier formalism of four functors f_* , f^* , $f_!$ and $f^!$, as well as a relation with K -theory and the realization properties analogous to $D^b(\mathcal{MM}_k)$. However, as suggested by Deligne [21], one might ask rather for a triangulated tensor category $\mathcal{D}(S)$ with a t -structure whose heart is the abelian category $\mathcal{MM}/S$, but without necessarily requiring that $D^b(\mathcal{MM}/S) = \mathcal{D}(S)$. Voevodsky [95] has axiomatized the situation in his theory of “crossed functors”, and has announced a construction of a category of mixed motives over S which satisfies the necessary conditions. As the theory is still in its beginning stages, we will not discuss these results further.

The Motivic Galois Groups

Suppose k admits an embedding $\sigma : k \rightarrow \mathbb{C}$, giving us the fiber functor $F_\sigma := H^0 \circ \mathfrak{N}_{\text{sing}}$ over $\mathbb{Q}$ corresponding to singular cohomology. Let MotGal_k be the Galois group of the Tannakian category $\mathcal{M}\mathcal{M}_k \otimes \mathbb{Q}$, and let $\text{MotGal}_k^{\text{ss}}$ be the Galois group for the semi-simple subcategory $\mathcal{M}_k$. Taking the associated graded for the weight filtration defines a functor $\mathcal{M}\mathcal{M}_k \rightarrow \mathcal{M}_k$, and hence a homomorphism $\text{MotGal}_k^{\text{ss}} \rightarrow \text{MotGal}_k$ splitting the map induced by the restriction functor $\text{MotGal}_k \rightarrow \text{MotGal}_k^{\text{ss}}$. The map $\text{MotGal}_k^{\text{ss}} \rightarrow \text{MotGal}_k$ is thought of as an analog of the map on the algebraic π_1 :

$$\pi_1(\overline{X}, *) \rightarrow \pi_1(X, *)$$

corresponding to the projection $\overline{X} := X \times_k \overline{k} \rightarrow X$ for a scheme X over k . The split surjection $\text{MotGal}_k \rightarrow \text{MotGal}_k^{\text{ss}}$ yields the exact sequence

$$1 \rightarrow \widehat{\mathcal{U}}_k \rightarrow \text{MotGal}_k \rightarrow \text{MotGal}_k^{\text{ss}} \rightarrow 1 ;$$

$\widehat{\mathcal{U}}_k$ is a connected pro-unipotent algebraic group scheme over $\mathbb{Q}$, encoding the extension information in $\mathcal{M}\mathcal{M}_k$.

One can restrict to the category of *mixed Tate motives* TM_k , i.e., the full abelian subcategory (of $\mathcal{M}\mathcal{M}_k \otimes \mathbb{Q}$) closed under extensions and generated by the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$. The abelian subcategory $\overline{\text{TM}}_k$ of TM_k generated by the $\mathbb{Q}(n)$'s is equivalent to the category of graded finite dimensional $\mathbb{Q}$ -vector spaces, i.e., the category of representations of $\mathbb{G}_m$ in $\mathbb{Q}\text{-mod}$. As taking the associated graded for the weight filtration defines an exact tensor functor $\text{TM}_k \rightarrow \overline{\text{TM}}_k$ splitting the inclusion, we have the split surjection

$$\text{Gal TM}_k \rightarrow \mathbb{G}_m \rightarrow 1$$

with kernel $\mathcal{U}_k$ a pro-unipotent algebraic group with $\mathbb{G}_m$ -action. Since the action of $\mathbb{G}_m$ just gives the information of a grading, we thus have an equivalence of TM_k with the category of graded representations of $\mathcal{U}_k$ on finite dimensional $\mathbb{Q}$ -vector spaces. More about this in Sect. 5.5 on Tate motives.

5.3.2 Motives by Compatible Realizations

Building on Deligne's theory of absolute Hodge cycles [28], Jannsen [54] constructs an abelian category of "simultaneous realizations", as an attempt to capture the idea of a mixed motive by looking at structures modeled on singular, de Rham and étale cohomology, together with comparison isomorphisms between these structures. The known comparisons between singular, de Rham and étale cohomology of a scheme X yields objects $H^n(X)$ in this category, and a reasonable approximation to a good category of motives is the sub-abelian category generated by these and their duals. Deligne has also given a construction from this point of view in [23], adding a crystalline component to the collection of realizations. The viewpoint of compatible realizations has also been used in the setting of triangulated categories by Huber [48], see Sect. 5.4.2 for some details of this construction.

The Category of Realizations

Let k be a field finitely generated over $\mathbb{Q}$, $\bar{k}$ the algebraic closure of k . Let $G_k = \text{Gal}(\bar{k}.k)$. Form the category of *mixed realizations* MR_k , with objects tuples of the form $H := (H_{\text{DR}}, H_\ell, H_\sigma, I_{\infty, \sigma}, I_{\ell, \bar{\sigma}})$, with ℓ running over prime numbers, σ over embeddings $k \rightarrow \mathbb{C}$ and $\bar{\sigma}$ over embeddings $\bar{k} \rightarrow \mathbb{C}$, where

- (a) H_{DR} is a finite dimensional k -vector space with an exhaustive decreasing filtration $F^n H_{\text{DR}}$, and an exhaustive increasing filtration $W_n H_{\text{DR}}$.
- (b) H_ℓ is a finite-dimensional $\mathbb{Q}_\ell$ -vector space with a continuous G_k -action, and an exhaustive increasing G_k -stable filtration $W_n H_\ell$.
- (c) H_σ is a $\mathbb{Q}$ -mixed Hodge structure: H_σ is a finite dimensional $\mathbb{Q}$ -vector space with an exhaustive decreasing filtration F^n on $H_\sigma \otimes \mathbb{C}$, and an exhaustive increasing filtration W_n on H_σ inducing a pure $\mathbb{Q}$ -Hodge structure of weight m on $\text{gr}_n^W H_\sigma$, i.e., there is a direct sum decomposition

$$(\text{gr}_n^W H_\sigma) \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q}$$

with $H^{q,p} = \overline{H^{p,q}}$ and with

$$\text{gr}_n^W F^a H_\sigma \otimes \mathbb{C} = \bigoplus_{p \geq a} H^{p,q}.$$

- (d) $I_{\infty, \sigma} : H_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow H_{\text{DR}} \otimes_k \mathbb{C}$ is an isomorphism, identifying the F - and W -filtrations.
- (e) $I_{\ell, \bar{\sigma}} : H_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \rightarrow H_\ell$ is an isomorphism identifying the W -filtrations. In addition, for each $\varphi \in G_k$, the diagram

$$\begin{array}{ccc}
 & & H_\ell \\
 & \nearrow I_{\ell, \bar{\sigma}} & \uparrow \varphi \\
 H_\sigma \otimes \mathbb{Q}_\ell & & \\
 & \searrow I_{\ell, \bar{\sigma}\varphi} & \downarrow \\
 & & H_\ell
 \end{array}$$

commutes.

The various W -filtrations (resp. F -filtrations) are called *weight filtrations* (resp. *Hodge filtrations*) and the isomorphisms I are called *comparison isomorphisms*.

Morphisms $H \rightarrow H'$ in MR_k are $(k, \mathbb{Q}_\ell, \mathbb{Q})$ -linear maps

$$(H_{\text{DR}}, H_\ell, H_\sigma) \rightarrow (H'_{\text{DR}}, H'_\ell, H'_\sigma)$$

respecting the various filtrations and comparison isomorphisms. Defining the operations componentwise, one has a tensor product, dual and internal Hom.

In addition, for $H = (H_{\text{DR}}, H_\ell, H_\sigma)$, the weight-filtrations on $H_{\text{DR}}, H_\ell, H_\sigma$ are all compatible via the comparison isomorphisms, so we have the functor

$$W_n : \text{MR}_k \rightarrow \text{MR}_k$$

$$W_n((H_{\text{DR}}, H_\ell, H_\sigma)) := (W_n H_{\text{DR}}, W_n H_\ell, W_n H_\sigma) .$$

From the exactness of W_n in the category of $\mathbb{Q}$ -mixed Hodge structures, it follows that W_n is exact.

Tannakian Structure

The main structural result for MR_k is

35 Theorem 35: [54, theorem 2.13] MR_k is a neutral Tannakian category over $\mathbb{Q}$.

In fact, the functor $H \mapsto H_\sigma$ for a single choice of σ gives the fiber functor.

The Category of Mixed Motives

Let (X, Y) be a pair consisting of a finite type k -scheme X and a closed subscheme Y . For an embedding $\sigma : k \rightarrow \mathbb{C}$, let (X_σ, Y_σ) be the pair of topological spaces given by the $\mathbb{C}$ -points of $(X \times_k \mathbb{C}, Y \times_k \mathbb{C})$, with the $\mathbb{C}$ -topology. Let $H^n(X_\sigma, Y_\sigma; \mathbb{Q})$ be the singular cohomology of the pair (X_σ, Y_σ) . Let $H_{\text{ét}}^n(X, Y; \mathbb{Q}_\ell)$ be the $\mathbb{Q}_\ell$ -étale cohomology of the pair and let $H_{\text{DR}}^n(X, Y)$ be the deRham cohomology.

Let X be a smooth quasi-projective k -scheme and take $Y = \emptyset$. Give $H^n(X_\sigma; \mathbb{Q})$ the mixed Hodge structure of Deligne [26]. Give $H_{\text{DR}}^n(X)$ the analogous weight and Hodge filtration: Take a smooth projective variety $\overline{X}$ containing X as a dense open subscheme with normal crossing divisor $D := \overline{X} \setminus X$ at infinity. One then has

$$H_{\text{DR}}^n(X) = \mathbb{H}^n(\overline{X}, \Omega_{\overline{X}/k}^*(\log D)) .$$

The stupid filtration on the deRham complex $\Omega_{\overline{X}/k}^*(\log D)$ gives the Hodge filtration on $H_{\text{DR}}^n(X)$ and the weight-filtration on $\Omega_{\overline{X}/k}^*(\log D)$ by number of components in the polar locus of a form induces (after shift by n) the weight-filtration on $H_{\text{DR}}^n(X)$. Similarly, we identify the dual of the relative cohomology $H_{\text{ét}}^{2 \dim X - n}(\overline{X}_k, D_k; \mathbb{Q}_\ell)$ with $H_{\text{ét}}^n(X_k; \mathbb{Q}_\ell)$; the skeletal filtration on D induces the weight-filtration on $H_{\text{ét}}^{2 \dim X - n}(\overline{X}_k, D_k; \mathbb{Q}_\ell)$ and thus on $H_{\text{ét}}^n(X_k; \mathbb{Q}_\ell)$.

The classical deRham theorem gives comparison isomorphisms

$$I_{\infty, \sigma} : H^n(X_\sigma, \mathbb{Q}) \otimes \mathbb{C} \rightarrow H_{\text{DR}}^n(X) \otimes_k \mathbb{C}$$

and Artin's comparison isomorphism yields

$$I_{\ell, \overline{\sigma}} :: H^n(X_\sigma, \mathbb{Q}) \otimes \mathbb{Q}_\ell \rightarrow H_{\text{ét}}^n(X_k, \mathbb{Q}_\ell) .$$

Jannsen shows that setting

$$H_\sigma := H^n(X_\sigma, \mathbb{Q})$$

$$H_{\text{DR}} := H_{\text{DR}}^n(X)$$

$$H_\ell := H_{\text{ét}}^n(X_{\bar{k}}, \mathbb{Q}_\ell)$$

with the above filtrations and comparison isomorphisms defines an object $H^n(X)$ in MR_k , functorial in X , giving the functor

$$H^n : \mathbf{Sm}_k^{\text{op}} \rightarrow \text{MR}_k$$

Definition 36 Jannsen's category of *mixed motives by realizations over k* , JMM_k , is the smallest full Tannakian subcategory of MR_k containing all the objects $H^n(X)$ for X smooth and quasi-projective over k . The objects of JMM_k are called *mixed motives*. The smallest Tannakian subcategory M_k of JMM_k containing all objects $H^n(X)$ for X smooth and projective over k , and closed under taking direct summands is called the subcategory of *pure motives*.

36

Remark 37 As mentioned above, Deligne [23] has also described a category of motives over $\mathbb{Q}$ by compatible realizations, adding a crystalline component to the list of possible realizations. This yields a category analogous to the category $\text{MR}_{\mathbb{Q}}$. However, Deligne gives no precise definition of the subcategory analogous to $\text{JMM}_{\mathbb{Q}}$, saying that the objects should be those systems of compatible realizations of *geometric origin* but explicitly leaving the definition of this term open.

37

Remark 38

38

- (1) Jannsen shows that the objects of JMM_k are exactly the subquotients of $H^n(U) \otimes H^m(V)^\vee$ for smooth, quasi-projective U and V over k . In addition, JMM_k is stable under the functors W_n and gr_n^W .
- (2) For each $M \in \text{JMM}_k$, the weight-filtration W_*M is finite and exhaustive, and the graded pieces $\text{gr}_n^W M$ are all pure motives. Thus each mixed motive is a successive extension of pure motives. The category of pure motives is semi-simple.
- (3) The method of cubical hyperresolutions of Guillen and Navarro-Aznar [41] extends H^n to a functor on arbitrary pairs of finite type k -schemes sending (X, Y) to the deRham/étale/singular cohomology

$$H^n(X, Y) := (H_{\text{DR}}^n(X, Y), H_{\text{ét}}^n(X, Y, \mathbb{Q}_\ell), H_\sigma^n(X_\sigma, Y_\sigma, \mathbb{Q}))$$

with the canonical mixed Hodge over $\mathbb{Q}$ /weight filtration/ $\mathbb{Q}$ -mixed Hodge structure gives an object in JMM_k . Also, for a triple (X, Y, Z) , the connecting morphism $H^n(X, Y) \rightarrow H^{n+1}(Y, Z)$ is a morphism in JMM_k .

5.3.3 Motives by Tannakian Formalism

Let k be a subfield of $\mathbb{C}$. For a pair consisting of a finite type k -scheme X and a closed subscheme Y , one has the singular homology $H_*(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$, which we denote by $H_*(X, Y)$. Nori constructs an abelian category of mixed motives by considering the ring of all natural endomorphisms α of the functor $(X, Y) \mapsto H_*(X, Y)$, with the additional requirement that α should commute with all boundary maps $\partial_i : H_i(X, Y) \rightarrow H_{i-1}(Y, Z)$ for all triples $X \supset Y \supset Z$. The external products in homology make this ring into a bi-algebra; dualizing and inverting the resulting character corresponding to the Tate object $H^2(\mathbb{P}^1, \mathbb{Z})$ yields a Hopf algebra χ_{mot} . The category of co-modules of χ_{mot} in finitely generated abelian groups is then Nori's abelian category of mixed motives. In this section, we give some details regarding this construction. Some of these results involve relations with the triangulated categories of motives constructed by Voevodsky; for the notations involved, we refer the reader to Sect. 5.4.5.

39

Remark 39 Unfortunately, there are at present no public manuscripts detailing Nori's construction. We have relied mainly on [31], with some additional detail coming from [77]. Hopefully, one of these will soon be available to the public.

A Universal Construction

A *small diagram* D consists of a set of objects $O(D)$ and for each pair of objects (p, q) a set of morphisms $M(p, q)$ (but no composition law). If $\mathcal{C}$ is a category, a *representation* of D in $\mathcal{C}$, $F : D \rightarrow \mathcal{C}$ is given by assigning an object Fp of $\mathcal{C}$ for each $p \in O(D)$, and a morphism $Fm : Fp \rightarrow Fq$ in $\mathcal{C}$ for each $m \in M(p, q)$. For a noetherian commutative ring R , we let $R\text{-mod}$ denote the abelian category of finite R -modules.

Example 40. Let $H_*\text{Sch}_k$ be the diagram with objects the triples (X, Y, i) , where X is a k -scheme of finite type, Y a closed subscheme of X and i an integer. There are two types of morphisms: for $f : X \rightarrow X'$ a morphism of k -schemes which restricts to a morphism of closed subschemes $Y \rightarrow Y'$ (i.e. a morphism of pairs $f : (X, Y) \rightarrow (X', Y')$), we have the morphism $f_* : (X, Y, i) \rightarrow (X', Y', i)$. For a triple (X, Y, Z) of closed subschemes $X \supset Y \supset Z$, we have the morphism $d : (X, Y, i) \rightarrow (Y, Z, i - 1)$.

Sending (X, Y, i) to $H_i(X, Y) := H_i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$, $f_* : (X, Y, i) \rightarrow (X', Y', i)$ to $f_* : H_i(X, Y) \rightarrow H_i(X', Y')$ and $d : (X, Y, i) \rightarrow (Y, Z, i - 1)$ to the boundary map $\partial_i : H_i(X, Y) \rightarrow H_{i-1}(Y, Z)$ in the long exact homology sequence of the triple $(X(\mathbb{C}), Y(\mathbb{C}), Z(\mathbb{C}))$ defines a representation

$$H_* : H_*\text{Sch}_k \rightarrow \mathbf{Ab}.$$

Reversing the arrow f_* to $f^* : (X', Y', i) \rightarrow (X, Y, i)$ and changing d to $d : (X, Y, i) \rightarrow (Y, Z, i + 1)$ gives the cohomological version $H^* \mathbf{Sch}_k$ and the representation

$$H^* : H^* \mathbf{Sch}_k \rightarrow \mathbf{Ab} ,$$

$$H^*((X, Y, i)) = H^i(X, Y) := H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z}).$$

The main theorem regarding representations of diagrams is

Theorem 41 Let $T : D \rightarrow R\text{-mod}$ be a representation of a small diagram D . Then there is an R -linear abelian category $\mathcal{C}(T)$, a faithful exact R -linear functor $\text{ff}_T : \mathcal{C}(T) \rightarrow R\text{-mod}$ and a representation $\tilde{T} : D \rightarrow \mathcal{C}(T)$ such that

41

1. $\text{ff}_T \circ \tilde{T} = T$
2. $\tilde{T}$ is universal: if $\mathcal{A}$ is an R -linear abelian category with a faithful exact R -linear functor $f : \mathcal{A} \rightarrow R\text{-mod}$ and $F : D \rightarrow \mathcal{A}$ is a representation such that $f \circ F = T$, then there is a unique R -linear functor $L(F) : \mathcal{C}(T) \rightarrow \mathcal{A}$ such that the diagram

$$\begin{array}{ccccc}
 & & \mathcal{C}(T) & & \\
 & \nearrow \tilde{T} & \downarrow L(F) & \searrow \text{ff}_T & \\
 D & & & & R\text{-mod} \\
 & \searrow F & & \nearrow f & \\
 & & \mathcal{A} & &
 \end{array}$$

commutes.

The construction of $\tilde{T} : D \rightarrow \mathcal{C}(T)$ follows the Tannakian pattern: Suppose first that D is a finite set. Let $\text{End}(T)$ be ring of (left) endomorphisms of T , that is, the subset of $\prod_{p \in O(D)} \text{End}_R(Tp)$ consisting of all tuples $e = \prod_p e_p$ such that, for all $m \in M(p, q)$, the diagram

$$\begin{array}{ccc}
 Tp & \xrightarrow{Tm} & Tq \\
 e_p \downarrow & & \downarrow e_q \\
 Tp & \xrightarrow{Tm} & Tq
 \end{array}$$

commutes. It is clear that $\text{End}(T)$ is a sub- R -algebra of the product algebra $\prod_{p \in O(D)} \text{End}_R(Tp)$; since each Tp is a finite R -module and D is finite, $\text{End}(T)$ is an R -algebra, finite as an R -module. We let $\mathcal{C}(T)$ be the category of finitely generated $\text{End}(T)$ -modules, and $\text{ff}_T : \mathcal{C}(T) \rightarrow R\text{-mod}$ the forgetful functor. By construction, each Tp is a left $\text{End}(T)$ -module by the projection $\text{End}(T) \rightarrow \text{End}_R(Tp)$, and each map $Tm : Tp \rightarrow Tq$ is $\text{End}(T)$ -linear. This yields the lifting $\tilde{T} : D \rightarrow \mathcal{C}(T)$.

In general, we apply the above construction to all finite subsets $O(F)$ of $O(D)$, i.e., to all “finite, full” subdiagrams F of D (where we use the same sets of morphisms $M(p, q)$ for all F). If $F \subset F' \subset D$ are two such finite full subdiagrams, the projection

$$\prod_{p \in O(F')} \text{End}_R(Tp) \rightarrow \prod_{p \in O(F)} \text{End}_R(Tp)$$

yields a homomorphism $\text{End}(T|_{F'}) \rightarrow \text{End}(T|_F)$, and hence an exact faithful functor $\mathcal{C}(T|_F) \rightarrow \mathcal{C}(T|_{F'})$. Define

$$\mathcal{C}(T) := \varinjlim_{\text{finite } F \subset D} \mathcal{C}(T|_F);$$

the forgetful functors $\mathcal{C}(T|_F) \rightarrow R\text{-mod}$ and the liftings $\tilde{T}|_F$ fit together to give $\text{ff}_T : \mathcal{C}(T) \rightarrow R\text{-mod}$ and $\tilde{T} : D \rightarrow \mathcal{C}(T)$.

To prove the universality, it suffices to consider the case of a small abelian category $\mathcal{A}$ with a faithful exact functor $f : \mathcal{A} \rightarrow R\text{-mod}$. Let $D(\mathcal{A})$ be the diagram associated to $\mathcal{A}$, i.e., the objects and morphisms are the same as $\mathcal{A}$, just forget the composition law. The above construction is obviously natural in D , so we have the commutative diagram

$$\begin{array}{ccccc} D & \xrightarrow{D(F)} & D(\mathcal{A}) & \xrightarrow{\text{id}} & \mathcal{A} \\ \tilde{T} \downarrow & & \tilde{F} \downarrow & \swarrow \hat{F} & \\ \mathcal{C}(T) & \xrightarrow{\mathcal{C}(D(F))} & \mathcal{C}(D(\mathcal{A})) & & \end{array}$$

with $\hat{F}$ an exact R -linear functor. Nori shows that $\hat{F}$ is an equivalence; an inverse to $\hat{F}$ yields the desired functor $\mathcal{C}(T) \rightarrow \mathcal{A}$.

Abelian Categories of Effective Motives

We apply the universal construction to the representations H_* and H^* .

Definition 42 Let k be a subfield of $\mathbb{C}$. Let $\text{EHM}(k) = \mathcal{C}(H_*)$ and $\text{ECM}(k) = \mathcal{C}(H^*)$.

Nori shows that these categories are independent of the choice of embedding $k \subset \mathbb{C}$. The universal property of the $\mathcal{C}$ -construction yields faithful exact functors

$$\mathrm{ECM}(k) \rightarrow \mathrm{MHS} \quad (5.3)$$

$$\mathrm{ECM}(k) \rightarrow \mathrm{Gal}(k)\text{-Rep}$$

$$\mathrm{ECM}(k) \rightarrow \mathrm{Period}(k)$$

Here MHS is the category of mixed Hodge structures, $\mathrm{Gal}(k)\text{-Rep}$ is the category of representations of $\mathrm{Gal}(\bar{k}/k)$ on finitely generated abelian groups, and $\mathrm{Period}(k)$ is the category of tuples (L, V, ϕ, ∇) , where L is a finitely generated abelian group, V a finite-dimensional k -vector space, $\phi : L \otimes \mathbb{C} \rightarrow V \otimes_k \mathbb{C}$ an isomorphism of $\mathbb{C}$ -vector spaces and $\nabla : V \rightarrow \Omega_k^1 \otimes V$ the Gauß-Manin connection, i.e. a $\mathbb{Q}$ -linear connection with $\nabla^2 = 0$ and with regular singular points at infinity. Similarly, using Remark 38(3), the universal property yields an exact faithful functor

$$\mathrm{ECM}(k) \rightarrow \mathrm{JMM}_k$$

The Basic Lemma and Applications

We have the functors H_i from pairs of finite-type k -schemes to $\mathrm{EHM}(k)$; in order to define the total derived functor

$$m : \mathbf{Sch}_k \rightarrow D_b(\mathrm{EHM}(k)) ,$$

Nori shows that affine finite type k -schemes have a type of “cellular decomposition” which, from the point of cohomology, looks like the usual cellular decomposition of a CW-complex. Specifically, the basic result is

Theorem 43: [78] Let X be a finite type affine k -scheme of dimension n over $k \subset \mathbb{C}$. Let $Z \subset X$ be a closed subset with $\dim Z \leq n - 1$. Then there exists a closed subset Y of X containing Z such that

1. $\dim Y \leq n - 1$
2. $H_i(X, Y) = 0$ for $i \neq n$
3. $H_n(X, Y)$ is a finitely generated abelian group.

43

Remark 44 Nori has mentioned to me that at the time of his proof of Theorem 43, he was unaware that Beilinson had already proven this result (actually, a stronger result, as Beilinson proves the above in characteristic $p > 0$ as well) in [3, lemma 3.3], by a different argument. He has also remarked that the same method was used by Kari Vilonen in his Harvard University Masters’ thesis to prove Artin’s comparison theorem.

44

To construct m , let X be an affine k -scheme of finite type. Applying Theorem 43 repeatedly, there is a filtration X_* of X by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset \dots \subset X_{n-1} \subset X_n = X$$

such that $H_i(X_j, X_{j-1}) = 0$ for $i \neq j$ and $H_j(X_j, X_{j-1})$ is a finitely generated abelian group for all j . Call such a filtration a *good* filtration of X . Form the complex $C_*(X_*)$ with $C_j = H_j(X_j, X_{j-1})$ and with differential the boundary map $H_j(X_j, X_{j-1}) \rightarrow H_{j-1}(X_{j-1}, X_{j-2})$. This is clearly a complex in $\text{EHM}(k)$, and is natural in $\text{EHM}(k)$ with respect to morphisms $f : X \rightarrow X'$ which are compatible with the chosen filtrations X_* and X'_* .

Let $\lim_{\rightarrow} \text{EHM}(k)$ be the category of ind-systems in $\text{EHM}(k)$, and let $\text{Ch}(\lim_{\rightarrow} \text{EHM}(k))$ be the category of chain complexes in $\lim_{\rightarrow} \text{EHM}(k)$ which have bounded homology in $\text{EHM}(k)$. Taking the system of good filtrations X_* of X (or equivalently, all filtrations) yields the functor

$$C_* : \mathbf{Aff}(k) \rightarrow \text{Ch}(\lim_{\rightarrow} \text{EHM}(k)) .$$

Here $\mathbf{Aff}(k)$ is the category of affine k -schemes of finite type.

Passing to the derived categories

$$D(\text{Ch}(\lim_{\rightarrow} \text{EHM}(k))) \sim D_b(\text{EHM}(k))$$

and using a Čech construction yields the functor

$$m : \mathbf{Sch}_k \rightarrow D_b(\text{EHM}(k)) .$$

As a second application, replace the diagram category $H_*\mathbf{Sch}_k$ with the diagram category $H_*\mathbf{Sch}'_k$ of “good triples” (X, Y, i) , i.e., those having $H_j(X, Y) = 0$ for $j \neq i$, and let $\text{EHM}(k)' = \mathcal{C}(H'_*)$, where $H'_* : H_*\mathbf{Sch}'_k \rightarrow \mathbf{Ab}$ is the restriction of H_* . Nori shows

45 Proposition 45 The natural map $\text{EHM}(k)' \rightarrow \text{EHM}(k)$ is an equivalence of abelian categories.

As applications of this result, Nori defines a tensor structure on $\text{EHM}(k)$ by considering the map of diagrams:

$$\begin{aligned} \times : H_*\mathbf{Sch}'_k \times H_*\mathbf{Sch}'_k &\rightarrow H_*\mathbf{Sch}'_k \\ (X, Y) \times (X', Y') &= (X \times_k X', X \times_k Y' \cup Y \times_k X') . \end{aligned}$$

and the representation $H_* \times H_* : H_*\mathbf{Sch}'_k \times H_*\mathbf{Sch}'_k \rightarrow \mathbf{Ab} \times \mathbf{Ab}$. This gives the commutative diagram

$$\begin{array}{ccc} H_*\mathbf{Sch}'_k \times H_*\mathbf{Sch}'_k & \xrightarrow{\quad \times \quad} & H_*\mathbf{Sch}'_k \\ \downarrow H_* \times H_* & & \downarrow H_* \\ \mathbf{Ab} \times \mathbf{Ab} & \xrightarrow{\quad \otimes \quad} & \mathbf{Ab} \end{array}$$

Noting that $\mathcal{C}(H_* \times H_*) = \mathrm{EHM}(k)' \times \mathrm{EHM}(k)'$, the universal property of $\mathcal{C}$ yields the exact functor

$$\otimes : \mathrm{EHM}(k)' \times \mathrm{EHM}(k)' \rightarrow \mathrm{EHM}(k)' ;$$

via Proposition 45, this gives the tensor product operation

$$\otimes : \mathrm{EHM}(k) \times \mathrm{EHM}(k) \rightarrow \mathrm{EHM}(k) .$$

Nori constructs a duality functor

$$\vee : \mathrm{EHM}(k)' \rightarrow \mathrm{ECM}(k)^{\mathrm{op}}$$

respecting the representations H_* and H^* via the usual duality

$$\mathrm{Hom}(-, \mathbb{Z}) : \mathbf{Ab} \rightarrow \mathbf{Ab} ,$$

by sending a good pair (X, Y, n) to (X, Y, n) , noting that

$$H^n(X, Y) = H_n(X, Y)^\vee$$

for a good pair (X, Y) . This induces an equivalence on the derived categories

$$\vee : D_b(\mathrm{EHM}(k)) \rightarrow D^b(\mathrm{ECM}(k))^{\mathrm{op}} .$$

Finally, using Theorem 43, Nori shows that the restriction of C_* to $\mathbf{Sm}_k$ factors through the embedding $\Gamma : \mathbf{Sm}_k \rightarrow \mathrm{Cor}(k)$ (see Sect. 5.4.5 for the notation):

Proposition 46 Let $W \subset X \times_k Y$ be an effective finite correspondence, X, Y in $\mathbf{Sm}_k$ and affine. Then there is a map $W_* : C_*(X) \rightarrow C_*(Y)$, satisfying

46

1. For a morphism $f : X \rightarrow Y$ with graph $\Gamma, f_* = \Gamma_*$.
2. $(W \circ W')_* = W_* \circ W'_*$.

Using this result, Nori shows that the restriction of m to $\mathbf{Sm}_k$ extends to a functor

$$\Pi : DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow D^b(\mathrm{EHM}(k)) .$$

making

$$\begin{array}{ccc} & & DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \\ & \nearrow & \downarrow \Pi \\ \mathrm{Cor}(k) & & \\ & \searrow & \\ & & D^b(\mathrm{EHM}(k)) \end{array}$$

commute.

Motives

For a finite subdiagram F of $H_*\mathbf{Sch}_k$, let $A(F)$ be the dual of $\mathrm{End}(H_{*|F})$:

$$A(F) := \mathrm{Hom}(\mathrm{End}(H_{*|F}), \mathbb{Z}) .$$

Let A be the limit

$$A := \varinjlim_{\vec{F}} A(F) .$$

The ring structure on $\mathrm{End}(H_{*|F})$ makes $A(F)$ and A into a co-algebra (over $\mathbb{Z}$). Nori shows

47 **Lemma 47** $\mathrm{EHM}(k)'$ is equivalent to the category of left comodules M over A , which are finitely generated as abelian groups.

The tensor product on $\mathrm{EHM}(k)'$ induces a comultiplication

$$\mathrm{End}(H_{*|F}) \otimes \mathrm{End}(H_{*|F'}) \rightarrow \mathrm{End}(H_{*|F \cdot F'}) ,$$

where $O(F \cdot F')$ is the set of triples of the form $(X, Y, i) \times (X', Y', i')$ for $(X, Y, i) \in F$, $(X', Y', i') \in F'$. This yields a commutative, associative multiplication $A \otimes A \rightarrow A$, making A into a bi-algebra over $\mathbb{Z}$.

Let $\mathbb{Z}(1) = H_1(\mathbb{G}_m)$, as an object of $\mathrm{EHM}(k)'$. As a $\mathbb{Z}$ -module, $\mathbb{Z}(1) = \mathbb{Z}$. If F is a finite diagram containing $(\mathbb{G}_m, \emptyset, 1)$, then $\mathbb{Z}(1)$ is an $\mathrm{End}(H_{*|F})$ -module; sending $a \in \mathrm{End}(H_{*|F})$ to $a \cdot 1 \in \mathbb{Z}(1) = \mathbb{Z}$ determines an element

$$\chi_F \in A(F) = \mathrm{Hom}(\mathrm{End}(H_{*|F}), \mathbb{Z}) .$$

The image of χ_F in A is independent of the choice of F , giving the element $\chi \in A$. Let A_χ be the localization of A by inverting χ .

48 **Theorem 48** A_χ is a Hopf algebra.

Let $G_{\mathrm{mot}}(k)$ be the corresponding affine group-scheme $\mathrm{Spec} A_\chi$.

49 **Definition 49** Nori's category of mixed motives over k , $\mathrm{NMM}(k)$, is the category of representations of $G_{\mathrm{mot}}(k)$ in finitely generated $\mathbb{Z}$ -modules, i.e., the category of co-modules over A_χ which are finitely generated as an abelian group.

Since $\otimes \mathbb{Z}(1) : \mathrm{NMM}(k) \rightarrow \mathrm{NMM}(k)$ is invertible, the functor $\Pi : DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow D^b(\mathrm{EHM}(k))$ extends to

$$\Pi : DM_{\mathrm{gm}}(k) \rightarrow D^b(\mathrm{NMM}(k)) . \quad (5.4)$$

Similarly, the functors (5.3) and the functor $\mathrm{ECM}(k) \rightarrow \mathrm{JMM}_k$ extend to functors on NMM_k .

There are a number of classical conjectures one can restate or generalize using this formalism. For example, Beilinson's conjectures on the existence of an abelian category of mixed motives over k with the desired properties can be restated as

Conjecture 50 The functor $\Pi_{\mathbb{Q}} : DM_{\text{gm}}^{\text{eff}}(k)_{\mathbb{Q}} \rightarrow D^b(\text{EHM}(k)_{\mathbb{Q}})$ is fully faithful.

50

The Hodge conjecture can be generalized as

Conjecture 51 The functor $hs : \text{ECM}(k) \rightarrow \text{MHS}$ induces a fully faithful functor $\text{NMM}(k)_{\mathbb{Q}} \rightarrow \text{MHS}_{\mathbb{Q}}$. Equivalently, the map from the Mumford–Tate group MT to $G_{\text{mot}}(k)$ corresponding to hs gives a surjective map $\text{MT} \rightarrow G_{\text{mot}}(k)_{\mathbb{Q}}$. Equivalently, for all V in $\text{NMM}(k)$, the map

51

$$\text{Hom}_{\text{NMM}(k)}(1, V)_{\mathbb{Q}} \rightarrow \text{Hom}_{\text{MHS}_{\mathbb{Q}}}(1, hs(V))$$

is surjective.

Suppose that k is finitely generated over $\mathbb{Q}$. Using the universal property of $\text{NMM}(k)$ with respect to p -adic étale cohomology, one has an exact functor

$$\text{NMM}(k) \rightarrow \mathbb{Q}_p[\text{Gal}(\bar{k}/k)]\text{-mod},$$

equivalently, a homomorphism of $\text{Gal}(\bar{k}/k)$ to the $\mathbb{Q}_p$ -points of $G_{\text{mot}}(k)$. The Tate conjecture generalizes to

Conjecture 52 The image of $\text{Gal}(\bar{k}/k) \rightarrow G_{\text{mot}}(k)(\mathbb{Q}_p)$ is Zariski dense in $G_{\text{mot}}(k)_{\mathbb{Q}_p}$. Equivalently, let $\text{NMM}(k)_{\mathbb{Q}_p}$ be the $\mathbb{Q}_p$ -extension of $\text{NMM}(k)$. Then the functor

52

$$\text{NMM}(k)_{\mathbb{Q}_p} \rightarrow \mathbb{Q}_p[\text{Gal}(\bar{k}/k)]\text{-mod}$$

induced by $\text{NMM}(k) \rightarrow \mathbb{Q}_p[\text{Gal}(\bar{k}/k)]\text{-mod}$ is fully faithful.

Triangulated Categories of Motives

5.4

One can attempt a construction of a *triangulated* category of motives, which should ideally have the properties expected by the derived category of Beilinson's conjectural abelian category of mixed motives. In this direction there are two essentially different approaches: One, due to Huber, is via simultaneous realizations, and the second (the approach used by Hanamura, Levine and Voevodsky) builds a category out of some form of algebraic cycles or correspondences.

The main problem in the second approach is that the composition of arbitrary correspondences is not defined, unless one passes to a suitable equivalence relation.

If however, one imposes the equivalence relation first (as in Grothendieck's construction) one would lose most of the extension data that the category of motives is supposed to capture. Thus, one is forced to modify the notion of correspondence in some way so that all compositions are defined. Hanamura, Levine and Voevodsky all use different approaches to solving this problem. The constructions of Levine and Voevodsky both lead to equivalent categories; while it is not at present clear that Hanamura's construction is also equivalent to the other two (at least with $\mathbb{Q}$ -coefficients) the resulting $\mathbb{Q}$ -motivic cohomology is the same, and so one expects that this category is equivalent as well.

5.4.1 The Structure of Motivic Categories

All the constructions of triangulated categories of motives enjoy some basic structural properties, which we formulate in this section. We give both a cohomological as well as a homological formulation.

Let A be a subring of $\mathbb{R}$ which is a Dedekind domain. By a *cohomological triangulated category of motives* over a field k with A -coefficients, we mean an A -linear triangulated tensor category $\mathcal{D}$, equipped with a functor

$$h : \mathbf{Sm}_k^{\text{op}} \rightarrow \mathcal{D}$$

and Tate objects $A(n)$, $n \in \mathbb{Z}$, with the following properties (we write f^* for $h(f)$):

1. *Additivity.* $h(X \amalg Y) = h(X) \oplus h(Y)$.
2. *Homotopy.* The map $p^* : h(X) \rightarrow h(X \times \mathbb{A}^1)$ is an isomorphism.
3. *Mayer–Vietoris.* Let $U \cup V$ be a Zariski open cover of $X \in \mathbf{Sm}_k$, $i_U : U \cap V \rightarrow U$, $i_V : U \cap V \rightarrow V$, $j_U : U \rightarrow X$ and $j_V : V \rightarrow X$ the inclusions. Then the sequence

$$h(X) \xrightarrow{(j_U^*, j_V^*)} h(U) \oplus h(V) \xrightarrow{(i_U^*, -i_V^*)} h(U \cap V)$$

extends canonically and functorially to a distinguished triangle.

4. *Künneth isomorphism.* For $\alpha \in \mathcal{D}$, write $\alpha(n)$ for $\alpha \otimes A(n)$. There are associative, commutative external products

$$\cup_{X,Y} : h(X)(n) \otimes h(Y)(m) \rightarrow h(X \times_k Y)(n+m)$$

which are isomorphisms. $A(0)$ is the unit for the tensor structure. We let

$$\cup_X : h(X) \otimes h(X) \rightarrow h(X)$$

be the composition $\delta_X^* \circ \cup_{X,X}$, where $\delta_X : X \rightarrow X \times_k X$ is the diagonal.

5. *Gysin distinguished triangle.* For each closed codimension q embedding in $\mathbf{Sm}_k$, $i : W \rightarrow X$, there is a distinguished triangle

$$h(W)(-d)[-2d] \xrightarrow{i_*} h(X) \xrightarrow{j^*} h(X \setminus W) \rightarrow h(W)(-d)[1-2d]$$

which is natural in the pair (W, X) . Here $j : X \setminus W \rightarrow X$ is the inclusion, and “natural” means with respect to both to morphisms of pairs $f : (W', X') \rightarrow (W, X)$ such that W' is the pull-back of W , as well as the functoriality $(i_1 \circ i_2)_* = i_{1*} \circ i_{2*}$ for a composition of closed embeddings in $\mathbf{Sm}_k$. Also, if $i : W \rightarrow X$ is an open component of X , then i_* is the inclusion of the summand $h(W)$ of $h(X) = h(W) \oplus h(X \setminus W)$.

6. *Cycle classes.* For $X \in \mathbf{Sm}_k$, there are homomorphisms

$$\mathrm{cl}^q : \mathrm{CH}^q(X) \rightarrow \mathrm{Hom}_{\mathcal{D}}(A(0), h(X)(q)[2q]) .$$

The maps cl^q are compatible with external products, and pull-back morphisms. If $i : W \rightarrow X$ is a codimension d closed embedding in $\mathbf{Sm}_k$, and Z is in $\mathrm{CH}^{q-d}(W)$, then $\mathrm{cl}^q(i_*(Z)) = i_* \circ \mathrm{cl}^{q-d}(Z)$.

7. *Unit.* The map $\mathrm{cl}^0([\mathrm{Spec} k]) : A(0) \rightarrow h(\mathrm{Spec} k)$ is an isomorphism.
 8. *Motivic cohomology.* For $X \in \mathbf{Sm}_k$, set

$$H^p(X, A(q)) := \mathrm{Hom}_{\mathcal{D}}(A(0), h(X)(q)[p]) .$$

As a consequence of the above axioms, the bi-graded group $\bigoplus_{p,q} H^p(X, A(q))$ becomes a bi-graded commutative ring (with product $\cup_X$), with $H^p(X, A(q))$ in bi-degree $(p, 2q)$. The element $\mathrm{cl}^0(1 \cdot X)$ is the unit.

9. *Projective bundle formula.* Let $\mathcal{E}$ be a rank $n+1$ locally free sheaf on $X \in \mathbf{Sm}_k$ with associated $\mathbb{P}^n$ -bundle $\mathbb{P}(\mathcal{E}) \rightarrow X$ and invertible quotient tautological sheaf $\mathcal{O}(1)$. Let $c_1(\mathcal{O}(1)) \in \mathrm{CH}^1(\mathbb{P}(\mathcal{E}))$ be the 1st Chern class of $\mathcal{O}(1)$, and set

$$\xi := \mathrm{cl}^1(c_1(\mathcal{O}(1))) \in H^2(\mathbb{P}(\mathcal{E}), A(1)) .$$

Letting $\alpha_i : h(X)(-i)[-2i] \rightarrow h(\mathbb{P}(\mathcal{E}))$ be the map $(-\cup_{\mathbb{P}(\mathcal{E})} \xi^i) \circ q^*$, the sum

$$\sum_{i=0}^n \alpha_i : \bigoplus_{i=0}^n h(X)(-i)[-2i] \rightarrow h(\mathbb{P}(\mathcal{E}))$$

is an isomorphism.

Remark 54 It follows from (4) and (7) that $A(n) \otimes A(m)$ is canonically isomorphic to $A(n+m)$, and thus we have isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(\alpha, \beta) \cong \mathrm{Hom}_{\mathcal{D}}(\alpha(n), \beta(n))$$

for all α, β in $\mathcal{D}$ and all $n \in \mathbb{Z}$.

All the properties of $h(X)$ induce related properties for $H^*(X, A(*))$ by taking long exact sequences associated to $\mathrm{Hom}_{\mathcal{D}}(A(0), -)$.

Using (5) and (9), one can define a push-forward map

$$f_* : h(Y)(-d)[-2d] \rightarrow h(X)$$

for a projective morphism $f : Y \rightarrow X$ of relative dimension d . For this, one factors f as $q \circ i$, with $i : Y \rightarrow \mathbb{P}^n \times X$ a closed embedding and $q : \mathbb{P}^n \times X \rightarrow X$ the projection. We use (5) to define i_* and let

$$q_* : h(\mathbb{P}^n \times X) \rightarrow h(X)(-n)[-2n]$$

be the inverse of the isomorphism $\sum_{i=0}^n \alpha_i$ of (9) (with $\mathcal{E} = \mathcal{O}_X^{n+1}$) followed by the projection onto $h(X)(-n)[-2n]$. One sets $f_* := q_* \circ i_*$, shows that f_* is independent of the choices made and that $(fg)_* = f_* g_*$. For details on this construction, see, e.g., [63, part 1, chapt. III, §2].

55 Remark 55 To define a *homological* triangulated category $\mathcal{D}$ of motives over a field k with A -coefficients, one replaces the functor h with an additive functor

$$m : \mathbf{Sm}_k \rightarrow \mathcal{D}$$

and denote $m(f)$ by f_* . The properties (1)-(4) remain the same, reversing the arrows in (2) and (3). The Gysin map i_* in (5) becomes $i^* : m(X) \rightarrow m(W)(d)[2d]$.

We define $H^p(X, A(q)) := \mathrm{Hom}_{\mathcal{D}}(m(X)(-q)[-p], A(0))$. The cycle classes in (6) become maps $\mathrm{cl}^q : \mathrm{CH}^q(X) \rightarrow H^{2q}(X, A(q))$, with the same functoriality and properties as in (6) and (8). The projective bundle formula (9) becomes the isomorphism

$$\sum_{i=0}^n \alpha_i : \oplus_{i=0}^n m(X)(i)[2i] \rightarrow m(\mathbb{P}(\mathcal{E})) .$$

One uses the projective bundle formula to define a pull-back map

$$q^* : m(X)(n)[2n] \rightarrow m(\mathbb{P}(\mathcal{E}))$$

by setting $q^* := \alpha_n$. This allows one to define a functorial Gysin map $f^* : m(X)(d)[2d] \rightarrow m(Y)$, for $f : Y \rightarrow X$ projective of relative dimension d , as we defined f_* in the cohomological setting.

In short, the opposite of a cohomological category of motives is a homological category of motives, after changing the signs in the Tate objects.

56 Definition 56 Let $\mathcal{D}$ be a cohomological triangulated category of motives over k with A -coefficients. A *duality* on $\mathcal{D}$ is an exact pseudo-tensor functor $\vee : \mathcal{D} \rightarrow \mathcal{D}^{\mathrm{op}}$, together with maps $\delta_\alpha : A(0) \rightarrow \alpha^\vee \otimes \alpha$, $\varepsilon_\alpha : \alpha \otimes \alpha^\vee \rightarrow A(0)$ for each α in $\mathcal{D}$, such that

1. For each α in $\mathcal{D}$, $(\varepsilon_\alpha \otimes \text{id}_\alpha) \circ (\text{id}_\alpha \otimes \delta_\alpha) = \text{id}_\alpha$ and $(\text{id}_\alpha \otimes \varepsilon_\alpha) \circ (\delta_\alpha \otimes \text{id}_\alpha) = \text{id}_{\alpha^\vee}$.
2. For each smooth projective X of dimension d over k , $h(X)^\vee = h(X)(d)[2d]$ and $\delta_{h(X)}$ and $\varepsilon_{h(X)}$ are the compositions of pull-back and push-forward for

$$\text{Spec}(k) \leftarrow X \xrightarrow{\Delta_X} X \times_k X$$

In the homological case, we just change (2) to

$$m(X)^\vee = m(X)(-d)[-2d].$$

Remark 57 A duality on $\mathcal{D}$ is a duality in the usual sense of tensor categories, that is, for each α, β in $\mathcal{D}$, $\alpha^\vee \otimes \beta$ is an internal Hom object in $\mathcal{D}$. In fact, the map

57

$$\text{Hom}_{\mathcal{D}}(\alpha \otimes \gamma, \beta) \rightarrow \text{Hom}_{\mathcal{D}}(\gamma, \alpha^\vee \otimes \beta)$$

induced by sending $f : \alpha \otimes \gamma \rightarrow \beta$ to the composition

$$\gamma \cong A(0) \otimes \gamma \xrightarrow{\delta \otimes \text{id}} \alpha^\vee \otimes \alpha \otimes \gamma \xrightarrow{\text{id} \otimes f} \alpha^\vee \otimes \beta$$

is an isomorphism for all α, β and γ , with inverse similarly constructed using ε_α instead of δ_α . For details, see [63, part 1, chapt. IV, §1]

Definition 58 Let $\mathcal{D}$ be a (co)homological triangulated category of motives over k , with coefficients in A . We say that $\mathcal{D}$ is a *fine* category of motives if, for each $X \in \mathbf{Sm}_k$ there are homomorphisms

58

$$\text{cl}^{p,q} : \text{CH}^q(X, 2q - p) \rightarrow H^p(X, A(q))$$

satisfying:

1. $\text{cl}^{2q,q} = \text{cl}^q$
2. The maps $\text{cl}^{p,q}$ are functorial with respect to pull-back, products and push-forward for closed embeddings in $\mathbf{Sm}_k$.
3. The maps $\text{cl}^{*,q}$ commute with the boundary maps in the Mayer–Vietoris sequences for $H^*(-, A(q))$ and $\text{CH}^q(-, 2q - *)$.
4. The A -linear extension of $\text{cl}^{p,q}$,

$$\text{cl}_A^{p,q} : \text{CH}^q(X, 2q - p) \otimes A \rightarrow H^p(X, A(q))$$

is an isomorphism for all $X \in \mathbf{Sm}_k$ and all p, q .

An Overview

We give below sketches of four constructions of triangulated categories of motives, due to Huber, Hanamura, Levine and Voevodsky. Huber's construction yields a co-

homological triangulated category of motives over k with $\mathbb{Q}$ -coefficients, with duality. Hanamura's construction, assuming k admits resolution of singularities, yields a fine cohomological triangulated category of motives over k with $\mathbb{Q}$ -coefficients, with duality. Levine's construction yields a fine cohomological triangulated category of motives over k with $\mathbb{Z}$ -coefficients; the category has duality if k admits resolution of singularities. Voevodsky's construction yields a fine homological triangulated category of motives over k with $\mathbb{Z}$ -coefficients; the category has duality if k admits resolution of singularities. In addition, if k admits resolution of singularities, Levine's category is equivalent to Voevodsky's category.

5.4.2 Huber's Construction

Let k be a field finitely generated over $\mathbb{Q}$. Huber's construction of a triangulated category of mixed motives over k [48] is, roughly speaking, a combination of Jannsen's abelian category MR_k of compatible realizations, and Beilinson's category of mixed Hodge complexes [4]. In somewhat more detail, Huber considers compatible systems of bounded below complexes and comparison maps

$$((C_{\mathrm{DR}}, W_*, F^*)(C_\ell, W_*), (C_\sigma, W_*), (C'_\sigma, W_*), (C'_\ell, W_*))$$

$$I_\sigma : C_{\mathrm{DR}} \otimes_k \mathbb{C} \rightarrow C'_\sigma$$

$$I'_\sigma : C_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow C'_\sigma$$

$$I'_{\ell, \sigma} : C_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \rightarrow C'_\ell$$

$$I_{\ell, \sigma} : C_\ell \rightarrow C'_\ell$$

where

1. σ runs over embedding $k \rightarrow \mathbb{C}$ and ℓ runs over prime numbers.
2. C_{DR} is a bounded below complex of finite dimensional bi-filtered k -vector spaces, with strict differentials. W_* is an increasing filtration and F^* is a decreasing filtration
3. C_σ (resp. C'_σ) is a bounded below complex of finite dimension $\mathbb{Q}$ -vector spaces (resp. $\mathbb{C}$ -vector spaces), with decreasing filtration W_* and with strict differentials.
4. C'_ℓ (resp. C_ℓ) is a bounded below complex of finite dimension $\mathbb{Q}_\ell$ -vector spaces (resp. with continuous G_k -action), with decreasing filtration W_* and with strict differentials.
5. $I_\sigma, I'_\sigma, I'_{\ell, \sigma}$ and $I_{\ell, \sigma}$ are filtered quasi-isomorphisms of complexes (with respect to the W -filtrations).
6. For each n , the tuple of cohomologies $(H^n(C_{\mathrm{DR}}), H^n(C_\ell), H^n(C_\sigma))$ with the induced filtrations is an object in MR_k , where we give $H^n(C_\sigma) \otimes \mathbb{C}$ the Hodge filtration induced from the F -filtration on $H^n(C_{\mathrm{DR}})$.
7. The G_k -module $H^n(C_\ell)$ is *mixed* (we don't define this term here, see [48, definition 9.1.4] for a precise definition. Roughly speaking, the G_k -action

should arise from an inverse system of actions on finitely generated $\mathbb{Z}/\ell^v$ -modules and $\mathrm{gr}_W^m H^n(C_\ell)$) should be pure of weight m for almost all Frobenius elements in G_k .

Inverting quasi-isomorphisms of tuples yields Huber's triangulated tensor category of mixed motives, $\mathcal{D}_{\mathrm{MR}}(k)$.

The category $\mathcal{D}_{\mathrm{MR}}(k)$ has the structural properties given in Sect. 5.4.1 for a cohomological triangulated category of motives over k with $\mathbb{Q}$ -coefficients, and with duality. In addition, the functor $h : \mathbf{Sm}_k^p \rightarrow \mathcal{D}_{\mathrm{MR}}(k)$ extends to smooth simplicial schemes over k . This extension is important in the applications given by Huber and Wildeshaus [51] to the Tamagawa number conjecture of Bloch and Kato.

Hanamura's Construction

5.4.3

We give a sketch of Hanamura's construction of the category $\mathcal{D}(k)$ as the pseudo-abelianization of a subcategory $\mathcal{D}_{\mathrm{fin}}(k)$; in [46] $\mathcal{D}(k)$ is constructed as a subcategory of a larger category $\mathcal{D}_{\mathrm{inf}}(k)$, which we will not describe here.

The basic object is a *higher correspondence*: Let X and Y be irreducible smooth projective varieties over k . Let

$$\mathrm{HCor}((X, n), (Y, m))^a := z^{m-n+\dim_k X}(X \times Y, -a)^{\mathrm{Alt}}.$$

For irreducible $W \in \mathrm{HCor}((X, n), (Y, m))^a$, $W' \in \mathrm{HCor}((Y, m), (Z, l))^b$ we say that $W' \circ W$ is defined if the external product $W \cup_{X \times Y, Y \times Z} W'$ is in the subcomplex

$$\begin{aligned} z^{l-n+\dim_k X+\dim_k Y}(X \times Y \times Y \times Z, *)_{\mathrm{id}_X \times \delta_Y \times \mathrm{id}_Z}^{\mathrm{Alt}} \\ \subset z^{l-n+\dim_k X+\dim_k Y}(X \times Y \times Y \times Z, *)^{\mathrm{Alt}}. \end{aligned}$$

In general, if $W = \sum_i n_i W_i$, $W' = \sum_j m_j W'_j$, we say $W' \circ W$ is defined if $W'_j \circ W_i$ is defined for all i, j .

If $W' \circ W$ is defined, we set

$$W' \circ W := p_{X \times Z*}((\mathrm{id}_X \times \delta_Y \times \mathrm{id}_Z)^*(W \cup_{X \times Y, Y \times Z} W')).$$

The definition of the complex HCor is extended to *formal symbols*, i.e. finite formal sums $\oplus_\alpha (X_\alpha, n_\alpha)$, by the formula

$$\mathrm{HCor}(\oplus_\alpha (X_\alpha, n_\alpha), \oplus_\beta (Y_\beta, m_\beta)) := \prod_\alpha \oplus_\beta \mathrm{HCor}((X_\alpha, n_\alpha), (Y_\beta, m_\beta)).$$

0 is the empty sum. We let 1 denote the formal symbol $(\mathrm{Spec} k, 0)$.

If $K = \oplus_\alpha (X_\alpha, n_\alpha)$ is a formal symbol, we set $z^0(K) := \mathrm{HCor}(1, K)$. Thus, $z^0((X, n)^*) = z^n(X, -*)$. We set $(X, n)^\vee := (X, \dim_k X - n)$ and extend to formal symbols by linearity. Similarly, we define a tensor product operation $K \otimes L$ as the bilinear extension of $(X, n) \otimes (Y, m) := (X \times_k Y, n + m)$.

Definition 59 A *diagram* $K := (K^m, f^{m,n})$ consists of formal symbols K^m , $m \in \mathbb{Z}$, together with elements $f^{m,n} \in \mathrm{HCor}(K^n, K^m)^{n-m+1}$, $n < m$ such that:

1. For all but finitely many m , $K^m = 0$.
2. For all sequences $m_1 < \dots < m_s$ the composition $f^{m_s, m_{s-1}} \circ \dots \circ f^{m_2, m_1}$ is defined.
3. For all $n < m$ we have the identity

$$(-1)^m \partial f^{m, n} + \sum_l f^{m, l} \circ f^{l, n} = 0.$$

Here ∂ is the differential in the complex $\mathrm{HCor}(K^n, K^m)^*$.

The yoga of duals and tensor products of usual complexes in an additive category extend to give operations $K \mapsto K^\vee$ and $(K, L) \mapsto K \otimes L$ for diagrams; we refer the reader to [63, part 2, chapt. II, §1] or [46] for detailed formulas.

The diagrams (resp. finite diagrams) are objects in a triangulated category $\mathcal{D}_{\mathrm{fin}}(k)$. In order to describe the morphisms $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(K, L)$, we need the notion of a *distinguished subcomplex* of $z^p(X, *)^{\mathrm{Alt}}$.

60

Definition 60 Let X be a smooth projective variety. A distinguished subcomplex of $z^p(X, *)^{\mathrm{Alt}}$ is a subcomplex of the form $z^p(X, *)_f^{\mathrm{Alt}}$ for some projective map $f : Y \rightarrow X$ in Sch_k , with Y locally equi-dimensional over k . If $K = \oplus_\alpha (X_\alpha, n_\alpha)$ is a formal symbol, a distinguished subcomplex of $z^0(K)$ is a subcomplex of the form $\oplus_\alpha z^{n_\alpha}(X_\alpha, -*)_{f_\alpha}$, with $f_\alpha : Y_\alpha \rightarrow X_\alpha$ as above.

For $f : (X, n) \rightarrow (Y, m)$ in $\mathrm{HCor}((X, n), (Y, m))^*$, we say that f is defined on a distinguished subcomplex $z^0((X, n))' := z^n(X, -*)'$ if $f \circ \eta$ is defined for all $\eta \in z^n(X, -*)'$ (where we identify $z^n(X, -*)'$ with $\mathrm{HCor}(1, (X, n))$). This notion extends in the evident manner to $f \in \mathrm{HCor}(K, K')$ for formal symbols K and K' .

Let $K = (K^m, f^{m, n})$ be a diagram. A collection of distinguished subcomplexes $m \mapsto z^0(K^m)'$ is *admissible* for K if, for each sequence $m_1 < \dots < m_s$, the correspondence $f_*^{m_s, m_{s-1}} \circ \dots \circ f_*^{m_2, m_1}$ is defined on $z^0(K^{m_1})'$ and maps $z^0(K^{m_1})'$ to $z^0(K^{m_s})'$. If a collection $m \mapsto z^0(K^m)'$ is admissible for K , we define the corresponding cycle complex for K , $z^0(K)'$, by

$$z^0(K)^j := \oplus_i z^0(K^i)^{j+i}$$

with differential $d^j : z^0(K)^j \rightarrow z^0(K)^{j+1}$ given by

$$d^j := \sum_i (-1)^i \partial_i + \sum_{i, i'} f_{*, i'}^{i', i}.$$

Here ∂_i is the differential in $z^0(K^i)^*$.

61

Lemma 61 For each formal symbol K , there is an admissible collection of distinguished complexes, and two different choices of such admissible collections, $m \mapsto z^0(K^m)'$ and $m \mapsto z^0(K^m)''$, result in canonically quasi-isomorphic cycle complexes $z^0(K)'$ and $z^0(K)''$.

Thus, we may denote by $z^0(K)$ the image of a $z^0(K)'$ in $D(\mathbf{Ab})$.

Definition 62 Let K and L be diagrams. Set

62

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(K, L) := H^0(z^0(K^\vee \otimes L)) .$$

Unwinding this definition, we see that the complex $z^0(K^\vee \otimes L)$ is built out of the complexes $\mathrm{HCor}(K^m, L^n)$, and so a morphism $\phi : K \rightarrow L$ is built out of higher correspondences $\phi^{n,m} : K^m \rightarrow L^n$, which satisfy some additional compatibility conditions. In particular, the composition of higher correspondences induces an associative composition

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(L, M) \otimes \mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(K, L) \rightarrow \mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(K, M) .$$

One mimics the definition of the translation operator and cone operator of complexes (this type of extension was first considered by Kapranov in the construction of the category of complexes over a DG-category, see [58, 63] or [46] for details).

Theorem 63: *Hanamura, [46, 47]* The category $\mathcal{D}_{\mathrm{fin}}(k)$ with the above structures of shift, cone sequence, dual and tensor product is a rigid triangulated tensor category.

63

“Rigid” means that, setting $\mathcal{H}om(K, L) := K^\vee \otimes L$, the objects $\mathcal{H}om(K, L)$ form an internal Hom object in $\mathcal{D}_{\mathrm{fin}}(k)$.

Definition 64 The triangulated tensor category $\mathcal{D}(k)$ is the pseudo-abelian hull of $\mathcal{D}_{\mathrm{fin}}(k)$.

64

By the results of [1], $\mathcal{D}(k)$ has a canonical structure of a triangulated tensor category.

For X a smooth projective k -scheme, set $\mathbb{Q}_X(n) := (X, n)[-2n]$; we write $\mathbb{Q}(n)$ for $\mathbb{Q}_{\mathrm{Spec} k}(n)$. More or less by construction we have

$$\mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Q}(0), \mathbb{Q}_X(n)[m]) = H_{2n-m}(z^n(X, *)^{\mathrm{Alt}}) = \mathrm{CH}^n(X, 2n - m)_{\mathbb{Q}} . \quad (5.5)$$

Sending X to $\mathbb{Q}_X(0) := h(X)$ defines a functor

$$h : \mathbf{SmProj}_k^{\mathrm{op}} \rightarrow \mathcal{D}(k) ,$$

where $\mathbf{SmProj}_k$ is the full subcategory of $\mathbf{Sch}_k$ with objects the smooth projective k -schemes.

65

Remark 65 Suppose that k admits resolution of singularities, and let X be a smooth irreducible quasi-projective k -scheme of dimension n . Let $\overline{X} \supset X$ be a smooth projective k -scheme containing X as a dense open subscheme, such that the complement $D := \overline{X} \setminus X$ is a strict normal crossing divisor.

Write $D = \sum_{i=1}^m D_i$, with the D_i irreducible. For $I \subset \{1, \dots, m\}$ let $D_I := \cap_{i \in I} D_i$, and let $D^{(i)} = \coprod_{|I|=i} D_I$ (so $D^{(0)} = \overline{X}$).

Consider the diagram $(X, \overline{X}) :=$

$$(D^{(n)}, -n) \rightarrow (D^{(n-1)}, -n+1) \rightarrow \dots \rightarrow (D^{(1)}, -1) \rightarrow (D^{(0)}, 0),$$

where the correspondence $(D^{(i)}, -i) \rightarrow (D^{(i-1)}, -i+1)$ is the signed sum of inclusions $i_{I,j} : D_{I \cup \{j\}} \rightarrow D_I$, $|I| = i-1$, $j \notin I$, and the sign is $(-1)^r$ if there are exactly r elements $i \in I$ with $i < j$. Hanamura [44] shows that $(X, \overline{X})$ in $\mathcal{D}(k)$ is independent of the choice of $\overline{X}$ (up to canonical isomorphism) and that sending X to $\mathbb{Q}_X(0) := (X, \overline{X})$ extends the functor h on $\mathbf{SmProj}_k$ to

$$h : \mathbf{Sm}_k^{\mathrm{op}} \rightarrow \mathcal{D}(k).$$

The identification (5.5) extends to a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Q}(0), h(X)(n)[m]) \cong \mathrm{CH}^n(X, 2n-m)$$

for $X \in \mathbf{Sm}_k$.

Using the method of cubical hyperresolutions [41], Hanamura [44] extends h further to a functor

$$h : \mathbf{Sch}_k^{\mathrm{op}} \rightarrow \mathcal{D}(k).$$

In any case, assuming resolution of singularities for k , the category $\mathcal{D}(k)$ is a fine cohomological triangulated category of motives over k , with $\mathbb{Q}$ coefficients and with duality.

66

Remark 66 In [45], Hanamura shows that, assuming the standard conjectures of Grothendieck along with extensions by Murre and Soulé–Beilinson, there is t -structure on $\mathcal{D}(k)$ whose heart $\mathcal{H}$ is a good candidate for $\mathcal{MM}_k$. It is not clear what relation $\mathcal{H}$ has to say Nori’s category NMM_k .

5.4.4 Levine’s Construction

Rather than using the moving lemma for the complexes $z^p(X, *)$ as above, Levine adds extra data to the category $\mathbf{Sm}_k$ so that pull-back of cycles becomes a well-defined operation.

The Category $\mathcal{L}(k)$

Definition 67 Let $\mathcal{L}(k)$ be the category of pairs $(X, f : X' \rightarrow X)$ where

67

1. f is a morphism in $\mathbf{Sm}_k$.
2. f admits a section $s : X \rightarrow X'$, with s a smooth morphism.

The choice of the section s is not part of the data. For $(X, f : X' \rightarrow X)$ and $(Y, g : Y' \rightarrow Y)$ in $\mathcal{L}(k)$, $\mathrm{Hom}_{\mathcal{L}(k)}((X, f), (Y, g))$ is the subset of $\mathrm{Hom}_{\mathbf{Sm}_k}(X, Y)$ consisting of those maps $h : X \rightarrow Y$ such that there exists a *smooth* morphism $q : X' \rightarrow Y'$ making

$$\begin{array}{ccc} X' & \xrightarrow{q} & Y' \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{h} & Y \end{array}$$

commute. Composition is induced by the composition in $\mathbf{Sm}_k$.

The condition that $f : X' \rightarrow X$ admit a smooth section is just saying that X' admits a decomposition as a disjoint union $X' = X'_0 \coprod X'_1$ where f restricted to X'_0 is an isomorphism $X'_0 \cong X$.

Definition 68 For $(X, f : X' \rightarrow X) \in \mathbf{Sm}_k$, let $z^q(X)_f$ be the subgroup of $z^q(X)$ generated by integral codimension q closed subschemes $W \subset X$ such that

68

$$\mathrm{codim}_{X'} f^{-1}(W) \geq q.$$

The basic fact that makes things work is

Lemma 69 Let $h : (X, f) \rightarrow (Y, g)$ be a morphism in $\mathcal{L}(k)$. Then

69

1. h^* is defined on $z^q(X)_f$, i.e. for all $W \in z^q(X)_f$,

$$\mathrm{codim}_Y h^{-1}(\mathrm{supp}(W)) \geq q.$$

2. h^* maps $z^q(X)_f$ to $z^q(Y)_g$.

The proof is elementary.

The Category $\overline{\mathcal{A}}_{\mathrm{mot}}(k)$

We use the cycle groups $z^q(X)_f$ to construct a graded tensor category $\overline{\mathcal{A}}_{\mathrm{mot}}(k)$ in a series of steps.

- (i) $\mathcal{A}_1(k)$ has objects $\mathbb{Z}_X(n)_f[m]$ for $(X, f) \in \mathcal{L}(k)$, X irreducible, and $n, m \in \mathbb{Z}$. The morphism-groups are given by

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}_1(k)}(\mathbb{Z}_X(n)_f[m], \mathbb{Z}_Y(n')_g[m']) \\ = \begin{cases} \mathbb{Z}[\operatorname{Hom}_{\mathcal{L}(k)}((Y, g), (X, f))] & \text{if } n = n', m = m' \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

We also allow finite formal direct sums, with the Hom-groups defined for such sums in the evident manner. The composition is induced from the composition in $\mathcal{L}(k)$. We write $\mathbb{Z}_X(n)_f$ for $\mathbb{Z}_X(n)_f[0]$, $\mathbb{Z}_X(n)$ for $\mathbb{Z}_X(n)_{\operatorname{id}}$ and $\mathbb{Z}(n)$ for $\mathbb{Z}_{\operatorname{Spec} k}(n)$. For $p : (Y, m) \rightarrow (X, n)$ in $\mathcal{L}(k)$, we write the corresponding morphism in $\mathcal{A}_1(k)$ as $p^* : \mathbb{Z}_X(n)_f \rightarrow \mathbb{Z}_Y(m)_g$.

Setting $\mathbb{Z}_X(n)_f \otimes \mathbb{Z}_Y(m)_g := \mathbb{Z}_{X \times_k Y}(n + m)_{f \times g}$ extends to give $\mathcal{A}_1(k)$ the structure of a tensor category, graded with respect to the shift operator $\mathbb{Z}_X(n)_f[m] \mapsto \mathbb{Z}_X(n)_f[m + 1]$.

- (ii) $\mathcal{A}_2(k)$ is formed from $\mathcal{A}_1(k)$ by adjoining (as a graded tensor category) an object $*$ and morphisms

$$[Z] : * \rightarrow \mathbb{Z}_X(n)_f[2n]$$

for each $Z \in z^n(X)_f$, with the relations:

1. $[aZ + bW] = a[Z] + b[W]$; $Z, W \in z^n(X)_f$, $a, b \in \mathbb{Z}$.
2. $p^* \circ [Z] = [p^*(Z)]$ for $p : (Y, g) \rightarrow (X, f)$ in $\mathcal{L}(k)$ and $Z \in z^n(X)_f$.
3. The exchange involution $\tau_{*,*} : * \otimes * \rightarrow * \otimes *$ is the identity.
4. For $Z \in z^n(X)_f$, $W \in z^m(Y)_g$, $[Z] \otimes [W] = [\operatorname{Spec} k] \otimes [Z \times W] = [Z \times W] \otimes [\operatorname{Spec} k]$ as maps $* \otimes * \rightarrow \mathbb{Z}_{X \times_k Y}(n + m)_{f \times g}[2n + 2m]$.

- (iii) $\overline{\mathcal{A}}_{\operatorname{mot}}(k)$ is the full additive subcategory of $\mathcal{A}_2(k)$ with objects sums of $*^{\otimes m} \otimes \mathbb{Z}_X(n)_f$, $m \geq 0$.

The Categories $D_{\operatorname{mot}}^b(k)$ and $\mathcal{DM}(k)$

Let $C^b(\overline{\mathcal{A}}_{\operatorname{mot}}(k))$ be the category of bounded complexes over $\overline{\mathcal{A}}_{\operatorname{mot}}(k)$, and $K_{\operatorname{mot}}^b(k)$ the homotopy category $K^b(\overline{\mathcal{A}}_{\operatorname{mot}}(k))$. $K_{\operatorname{mot}}^b(k)$ is a triangulated tensor category, where the shift operator and distinguished triangles are the usual ones. Note that one needs to modify the definition of morphisms in $C^b(\overline{\mathcal{A}}_{\operatorname{mot}}(k))$ slightly to allow one to identify the shift in $\overline{\mathcal{A}}_{\operatorname{mot}}(k)$ with the usual shift of complexes (see [63, part 2, chapt. II, §1.2] for details). The tensor product in $\overline{\mathcal{A}}_{\operatorname{mot}}(k)$ makes $K_{\operatorname{mot}}^b(k)$ a triangulated tensor category.

To form $D_{\operatorname{mot}}^b(k)$, we localize $K_{\operatorname{mot}}^b(k)$; we first need to introduce some notation.

Let $(X, f : X' \rightarrow X)$ be in $\mathcal{L}(k)$, let $W \subset X$ be a closed subset, and $j : U \rightarrow X$ be the open complement. Define $\mathbb{Z}_X^W(n)_f$ by

$$\mathbb{Z}_X^W(n)_f := \operatorname{Cone}(j^* : \mathbb{Z}_X(n)_f \rightarrow \mathbb{Z}_U(n)_{f_U})[-1] ,$$

where $f_U : U' \rightarrow U$ is the projection $U \times_X X' \rightarrow U$. If Z is in $z^n(X)_f$ and is supported in W , then $j^*Z = 0$, so the morphism $[Z] : * \rightarrow \mathbb{Z}_X(n)_f[2n]$ lifts canonically to the morphism

$$[Z]^W : * \rightarrow \mathbb{Z}_X^W(n)_f[2n]$$

(in $C^b(\mathcal{A}_2(k))$).

If $\mathcal{C}$ is a triangulated category, and S a collection of morphisms, we let $\mathcal{C}[S^{-1}]$ be the localization of $\mathcal{C}$ with respect to the thick subcategory generated by objects $\text{Cone}(f)$, $f \in S$. If $\mathcal{C}$ is a triangulated tensor category, let $\mathcal{C}[S^{-1}]_{\otimes}$ be the triangulated tensor category formed by localizing $\mathcal{C}$ with respect to the small thick subcategory containing the object $\text{Cone}(f)$, $f \in S$, and closed under $\otimes X$ for X in $\mathcal{C}$; $\mathcal{C}[S^{-1}]_{\otimes}$ is a triangulated tensor category, called the triangulated tensor category formed from $\mathcal{C}$ by inverting the morphisms in S . We can extend these notions to inverting finite zig-zag diagrams by taking the cone of the direct sum of the sources mapping to the direct sum of the targets.

Definition 70 Let $D_{\text{mot}}^b(k)$ be the triangulated tensor category formed from $K_{\text{mot}}^b(k)$ by localizing as a triangulated tensor category:

70

1. *Homotopy.* For all X in $\mathbf{Sm}_k$, invert the map $p^* : \mathbb{Z}_X(n) \rightarrow \mathbb{Z}_{X \times \mathbb{A}^1}(n)$.
2. *Nisnevich excision.* Let $(X, f : X' \rightarrow X)$ be in $\mathcal{L}(k)$, and let $p : Y \rightarrow X$ be an étale map, $W \subset X$ a closed subset such that $p : p^{-1}(W) \rightarrow W$ is an isomorphism. Then invert the map $p^* : \mathbb{Z}_X^W(n) \rightarrow \mathbb{Z}_Y^{p^{-1}(W)}(n)$.
3. *Unit.* Invert the map $[\text{Spec } k] \otimes \text{id}^* : * \otimes 1 \rightarrow 1 \otimes 1 = 1$.
4. *Moving lemma.* For all (X, f) in $\mathcal{L}(k)$, invert the map $\text{id}^* : \mathbb{Z}_X(n)_f \rightarrow \mathbb{Z}_X(n)$.
5. *Gysin isomorphism.* Let $q : P \rightarrow X$ be a smooth morphism in $\mathbf{Sm}_k$ with section s and let $W = s(X) \subset P$. Let $d = \dim_X P$. Invert the zig-zag diagram:

$$\begin{aligned} \mathbb{Z}_X(-d)[-2d] &\xrightarrow{q^*} \mathbb{Z}_P(-d)[-2d] = 1 \otimes \mathbb{Z}_P(-d)[-2d] \\ &\xleftarrow{[\text{Spec } k] \otimes \text{id}} * \otimes \mathbb{Z}_P(-d)[-2d] \xrightarrow{[W]^W \otimes \text{id}} \mathbb{Z}_P^W(d)[2d] \otimes \mathbb{Z}_P(-d)[-2d] \\ &= \mathbb{Z}_{P \times P}^{W \times P}(0) \xleftarrow{\text{id}^*} \mathbb{Z}_{P \times P}^{W \times P}(0)_{\delta_P} \xrightarrow{\delta_P^*} \mathbb{Z}_P^W(0), \end{aligned}$$

where $\delta_P : P \rightarrow P \times P$ is the diagonal.

Remark 71 Our description of $D_{\text{mot}}^b(k)$ is slightly different than that given in [63], but yields an equivalent triangulated tensor category $D_{\text{mot}}^b(k)$. The category $\overline{\mathcal{A}}_{\text{mot}}(k)$ described here is denoted $\mathcal{A}_{\text{mot}}^0(\mathcal{V})^*$ in [63].

71

The category $\mathcal{DM}(k)$ is now defined as the pseudo-abelianization of $D_{\text{mot}}^b(k)$. By [1], $\mathcal{DM}(k)$ inherits the structure of a triangulated tensor category from $D_{\text{mot}}^b(k)$.

Gysin Isomorphism

Let $i : W \rightarrow X$ be a codimension d closed embedding in $\mathbf{Sm}_k$. If i is split by a smooth morphism $p : X \rightarrow W$, one uses the Gysin isomorphism of Definition 70(5) to define $i_* : \mathbb{Z}_W(-d)[-2d] \rightarrow \mathbb{Z}_X(0)$; in general one uses the standard method of deformation to the normal bundle to reduce to this case.

Duality

Assuming that k admits resolution of singularities, the category $\mathcal{DM}(k)$ has an exact pseudo-tensor duality involution $\vee : \mathcal{DM}(k) \rightarrow \mathcal{DM}(k)^{\text{op}}$; for smooth projective X of dimension d over k one has

$$(\mathbb{Z}_X(n))^\vee = \mathbb{Z}_X(d-n)[2d].$$

To construct $\vee$, the method of [63] is to note that, in a tensor category $\mathcal{C}$, the dual of an object X can be viewed as a triple $(X^\vee, \delta, \varepsilon)$ with $\delta : 1 \rightarrow X \otimes X^\vee$, $\varepsilon : X^\vee \otimes X \rightarrow 1$, and with

$$(\varepsilon \otimes \text{id}_X) \circ (\text{id}_X \otimes \delta) = \text{id}_X.$$

In $\mathcal{DM}(k)$, for X smooth and projective of dimension d over k , the diagonal $[\Delta] : 1 \rightarrow \mathbb{Z}_{X \times X}(d)[2d]$ gives δ , and ε is the composition $p_{X*} \circ \delta_X^*$, where $\delta_X : X \rightarrow X \times X$ is the diagonal inclusion and $p_X : X \rightarrow \text{Spec } k$ is the structure morphism. One then shows

72

Lemma 72: [63, part 1, chapt. IV, lemma 1.2.3] Let D be a triangulated tensor category, S a collection of objects of D . Suppose that

1. There is a tensor category $\mathcal{C}$ such that D is the localization of $K^b(\mathcal{C})$ (as a triangulated tensor category).
2. Each $X \in S$ admits a dual $(X^\vee, \delta, \varepsilon)$.

Then the smallest triangulated tensor subcategory of D containing S , $D(S)$, admits a duality involution $\vee : D(S) \rightarrow D(S)^{\text{op}}$, extending the given duality on S .

If k satisfies resolution of singularities, the motives $\mathbb{Z}_X(n)$, X smooth and projective over k , $n \in \mathbb{Z}$, generate $D_{\text{mot}}^b(k)$ as a tensor triangulated category, so the given duality extends to $D_{\text{mot}}^b(k)$, and then to the pseudo-abelianization $\mathcal{DM}(k)$. As for Hanamura's category, the functor $h : \mathbf{Sm}_k^{\text{op}} \rightarrow \mathcal{DM}(k)$, $h(X) := \mathbb{Z}_X(0)$, extends to

$$h : \mathbf{Sch}_k^{\text{op}} \rightarrow \mathcal{DM}(k),$$

assuming k satisfies resolution of singularities.

Summing up, the category $\mathcal{DM}(k)$ is fine cohomological triangulated category of motives over k with $\mathbb{Z}$ -coefficients. $\mathcal{DM}(k)$ has duality if k admits resolution of singularities.

Voevodsky's Construction

5.4.5

Voevodsky constructs a number of categories: the category of geometric motives $DM_{\text{gm}}(k)$ with its effective subcategory $DM_{\text{gm}}^{\text{eff}}(k)$, as well as a sheaf-theoretic construction DM_{-}^{eff} , containing $DM_{\text{gm}}^{\text{eff}}(k)$ as a full dense subcategory. In contrast to almost all other constructions, these are based on *homology* rather than cohomology as the starting point, in particular, the motives functor from $\mathbf{Sm}_k$ to these categories is covariant.

To solve the problem of the partially defined composition of correspondences, Voevodsky introduces the notion of *finite* correspondences, for which all compositions are defined.

Finite Correspondences and Geometric Motives

Definition 73 Let X and Y be in $\mathbf{Sm}_k$. The group $c(X, Y)$ is the subgroup of $z^{\dim_k X}(X \times_k Y)$ generated by integral closed subschemes $W \subset X \times_k Y$ such that

73

1. the projection $p_1 : W \rightarrow X$ is finite
2. the image $p_1(W) \subset X$ is an irreducible component of X .

The elements of $c(X, Y)$ are called the *finite* correspondences from X to Y .

The following basic lemma is easy to prove:

Lemma 74 Let X, Y and Z be in $\mathbf{Sm}_k$, $W \in c(X, Y)$, $W' \in c(Y, Z)$. Suppose that X and Y are irreducible. Then each irreducible component C of $|W| \times Z \cap X \times |W'|$ is finite over X (via the projection p_1) and $p_1(C) = X$.

74

It follows from this lemma that, for $W \in c(X, Y)$, $W' \in c(Y, Z)$, we may define the composition $W' \circ W \in c(X, Z)$ by

$$W' \circ W := p_*(p_3^*(W) \cdot p_1^*(W')),$$

where $p_1 : X \times_k Y \times_k Z \rightarrow X$ and $p_3 : X \times_k Y \times_k Z \rightarrow Z$ are the projections, and $p : |W| \times Z \cap X \times |W'| \rightarrow X \times_k Z$ is the map induced by the projection $p_{13} : X \times_k Y \times_k Z \rightarrow X \times_k Z$. The associativity of cycle-intersection implies that this operation yields an associative bilinear composition law

$$\circ : c(Y, Z) \times c(X, Y) \rightarrow c(X, Z).$$

Definition 75 The category $\text{Cor}(k)$ is the category with the same objects as $\mathbf{Sm}_k$, with

75

$$\text{Hom}_{\text{Cor}(k)}(X, Y) := c(X, Y),$$

and with the composition as defined above.

For a morphism $f : X \rightarrow Y$ in $\mathbf{Sm}_k$, the graph $\Gamma_f \subset X \times_k Y$ is in $c(X, Y)$, so sending f to Γ_f defines a faithful functor

$$\mathbf{Sm}_k \rightarrow \mathrm{Cor}(k) .$$

We write the morphism corresponding to Γ_f as f_* , and the object corresponding to $X \in \mathbf{Sm}_k$ as $[X]$.

The operation $\times_k$ (on smooth k -schemes and on cycles) makes $\mathrm{Cor}(k)$ a tensor category. Thus, the bounded homotopy category $K^b(\mathrm{Cor}(k))$ is a triangulated tensor category.

- 76** **Definition 76** The category $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ of *effective geometric motives* is the localization of $K^b(\mathrm{Cor}(k))$, as a triangulated tensor category, by
1. *Homotopy.* For $X \in \mathbf{Sm}_k$, invert $p_* : X \times \mathbb{A}^1 \rightarrow X$
 2. *Mayer–Vietoris.* Let X be in $\mathbf{Sm}_k$, with Zariski open subschemes U, V such that $X = U \cup V$. Let $i_U : U \cap V \rightarrow U$, $i_V : U \cap V \rightarrow V$, $j_U : U \rightarrow X$ and $j_V : V \rightarrow X$ be the inclusions. Since $(j_{U*} + j_{V*}) \circ (i_{U*}, -i_{V*}) = 0$, we have the canonical map

$$\mathrm{Cone}([U \cap V] \xrightarrow{(i_{U*}, -i_{V*})} [U] \oplus [V]) \xrightarrow{(j_{U*} + j_{V*})} [X] .$$

Invert this map.

To define the category of geometric motives, $DM_{\mathrm{gm}}(k)$, we invert the Lefschetz motive. For $X \in \mathbf{Sm}_k$, the reduced motive $\widetilde{[X]}$ is defined as

$$\widetilde{[X]} := \mathrm{Cone}(p_* : [X] \rightarrow [\mathrm{Spec} k]) .$$

Set $\mathbb{Z}(1) := [\mathbb{P}^1][2]$, and set $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ for $n \geq 0$.

- 77** **Definition 77** The category $DM_{\mathrm{gm}}(k)$ is defined by inverting the functor $\otimes \mathbb{Z}(1)$ on $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$, i.e.,

$$\mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(X, Y) := \lim_n \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(X \otimes \mathbb{Z}(n), Y \otimes \mathbb{Z}(n)) .$$

- 78** **Remark 78** In order that $DM_{\mathrm{gm}}(k)$ be again a triangulated category, it suffices that the commutativity involution $\mathbb{Z}(1) \otimes \mathbb{Z}(1) \rightarrow \mathbb{Z}(1) \otimes \mathbb{Z}(1)$ be the identity, which is in fact the case.

Of course, there arises the question of the behavior of the evident functor $DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow DM_{\mathrm{gm}}(k)$. Here we have

- 79** **Theorem 79:** *Voevodsky [97]* The functor $DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow DM_{\mathrm{gm}}(k)$ is a fully faithful embedding.

The first proof of this result (in [100]) used resolution of singularities, but the later proof in [97] does not, and is valid in all characteristics.

Sheaves with Transfer

The sheaf-theoretic construction of mixed motives is based on the notion of a Nisnevich sheaf with transfer.

Let X be a k -scheme of finite type. A *Nisnevich cover* $\mathcal{U} \rightarrow X$ is an étale morphism of finite type such that, for each finitely generated field extension F of k , the map on F -valued points $\mathcal{U}(F) \rightarrow X(F)$ is surjective. The small Nisnevich site of X , X_{Nis} has underlying category finite type étale X -schemes with covering families finite families $U_i \rightarrow X$ such that $\coprod_i U_i \rightarrow X$ is a Nisnevich cover. The big Nisnevich site over k , with underlying category $\mathbf{Sm}_k$, is defined similarly. We let $\text{Sh}^{\text{Nis}}(k)$ denote the categories of Nisnevich sheaves of abelian groups on $\mathbf{Sm}_k$, and $\text{Sh}^{\text{Nis}}(X)$ the category of Nisnevich sheaves on X . For a presheaf $\mathcal{F}$ on $\mathbf{Sm}_k$ or X_{Nis} , we let $\mathcal{F}_{\text{Nis}}$ denote the associated sheaf. We often denote $H^*(X_{\text{Nis}}, \mathcal{F}_{\text{Nis}})$ by $H^*(X_{\text{Nis}}, \mathcal{F})$.

For a category $\mathcal{C}$ with finite coproducts, we have the category of presheaves of abelian groups on $\mathcal{C}$, i.e., the category of additive functors $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Ab}$.

Definition 80

- (1) The category $\text{PST}(k)$ of presheaves with transfer is the category of presheaves of abelian groups on $\text{Cor}(k)$. The category of Nisnevich sheaves with transfer on $\mathbf{Sm}_k$, $\text{Sh}^{\text{Nis}}(\text{Cor}(k))$, is the full subcategory of $\text{PST}(k)$ with objects those F such that, for each $X \in \mathbf{Sm}_k$, the restriction of F to X_{Nis} is a sheaf.
- (2) Let F be a presheaf of abelian groups on $\mathbf{Sm}_k$. We call F *homotopy invariant* if for all $X \in \mathbf{Sm}_k$, the map

$$p^* : F(X) \rightarrow F(X \times \mathbb{A}^1)$$

is an isomorphism.

- (3) Let F be a presheaf of abelian groups on $\mathbf{Sm}_k$. We call F *strictly homotopy invariant* if for all $q \geq 0$, the cohomology presheaf $X \mapsto H^q(X_{\text{Nis}}, F_{\text{Nis}})$ is homotopy invariant.

The category $\text{Sh}^{\text{Nis}}(\text{Cor}(k))$ is an abelian category with enough injectives, and we have the derived category $D^-(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$. For

$$F^* \in D^-(\text{Sh}^{\text{Nis}}(\text{Cor}(k))),$$

we have the cohomology sheaf, $\mathcal{H}^q(F^*)$, i.e., the Nisnevich sheaf with transfer associated to the presheaf

$$X \mapsto (\ker d^q : F^q(X) \rightarrow F^{q+1}(X)) / d^{q-1}(F^{q-1}(X)).$$

81 **Definition 81** The category $DM_{-}^{\text{eff}}(k)$ is the full subcategory of

$$D^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$$

consisting of those F^* whose cohomology sheaves are strictly homotopy invariant.

The Localization Theorem

The category $DM_{-}^{\text{eff}}(k)$ is a localization of $D^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$. To show this, one uses the *Suslin complex* of a sheaf with transfers.

82 **Definition 82** Let F be in $\text{Sh}^{\text{Nis}}(\text{Cor}(k))$. Define $C^{\text{Sus}}(F)$ to be the sheafification of the complex of presheaves

$$X \mapsto (\cdots \rightarrow F(X \times \Delta^n) \rightarrow F(X \times \Delta^{n-1}) \rightarrow \cdots \rightarrow F(X)) ,$$

where the differentials are the usual alternating sum of restriction maps, and $F(X \times \Delta^n)$ is in degree $-n$. For $F^* \in D^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$, define $L_{\mathbb{A}^1}(F^*)$ in $D^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$ as the total derived functor of $F \mapsto C^{\text{Sus}}(F)$.

For $F \in C^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$, we let $F^{\mathbb{A}^1}$ be the sheafification of the complex of presheaves $X \mapsto F(X \times \mathbb{A}^1)$; the projection $X \times \mathbb{A}^1 \rightarrow X$ defines the natural map $p^* : F \rightarrow F^{\mathbb{A}^1}$.

83 **Definition 83** Let $D_{\mathbb{A}^1}^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$ be the localization of the triangulated category $D^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$ with respect to the localizing subcategory generated by objects of the form $\text{Cone}(p^* : F \rightarrow F^{\mathbb{A}^1})$.

84 **Theorem 84:** [100, chapt. 5, prop. 3.2.3]

(1) For each $F \in D^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$, $L_{\mathbb{A}^1}(F)$ is in $DM_{-}^{\text{eff}}(k)$. The resulting functor

$$L_{\mathbb{A}^1} : D^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k))) \rightarrow DM_{-}^{\text{eff}}(k)$$

is exact and is left-adjoint to the inclusion

$$DM_{-}^{\text{eff}}(k) \rightarrow D^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k))) .$$

(2) The functor $L_{\mathbb{A}^1}$ descends to an equivalence of triangulated categories

$$L_{\mathbb{A}^1} : D_{\mathbb{A}^1}^{-}(\text{Sh}^{\text{Nis}}(\text{Cor}(k))) \rightarrow DM_{-}^{\text{eff}}(k) .$$

This result enables one to make $DM_{-}^{\text{eff}}(k)$ into a tensor category as follows. Let $\mathbb{Z}_{\text{tr}}(X)$ denote the representable Nisnevich sheaf with transfers $Y \mapsto c(Y, X)$. Define $\mathbb{Z}_{\text{tr}}(X) \otimes \mathbb{Z}_{\text{tr}}(Y) := \mathbb{Z}_{\text{tr}}(X \times_k Y)$. One shows that this operation extends

to give $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ the structure of a triangulated tensor category; the localizing functor $L_{\mathbb{A}^1}$ then induces a tensor operation on $D_{\mathbb{A}^1}^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$, making $D_{\mathbb{A}^1}^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ a triangulated tensor category.

Explicitly, in $DM_{-}^{\mathrm{eff}}(k)$, this gives us the functor

$$m : \mathbf{Sm}_k \rightarrow DM_{-}^{\mathrm{eff}}(k),$$

defined by $m(X) := C^{\mathrm{Sus}}(\mathbb{Z}_{\mathrm{tr}}(X))$, and the formula

$$m(X \times_k Y) = m(X) \otimes m(Y).$$

The Embedding Theorem

We now have the two functors

$$\begin{array}{ccc} \mathbf{Sm}_k & \xrightarrow{[-]} & DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \\ & \searrow m & \\ & & DM_{-}^{\mathrm{eff}}(k). \end{array} \quad (5.6)$$

Theorem 85 There is a unique exact functor $i : DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow DM_{-}^{\mathrm{eff}}(k)$ filling in the diagram (5.6). i is a fully faithful embedding and a tensor functor. In addition, $DM_{\mathrm{gm}}^{\mathrm{eff}}$ is dense in $DM_{-}^{\mathrm{eff}}(k)$.

85

Here “dense” means that every object X in $DM_{-}^{\mathrm{eff}}(k)$ fits in a distinguished triangle

$$\oplus_{\alpha} i(A_{\alpha}) \rightarrow \oplus_{\beta} i(B_{\beta}) \rightarrow X \rightarrow \oplus_{\alpha} i(A_{\alpha})[1],$$

where the A_{α} and the B_{β} are in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$, and the direct sums exist in $DM_{-}^{\mathrm{eff}}(k)$.

Applications of the Embedding Theorem

The embedding theorem allows one to apply sheaf-theoretic constructions to $DM_{\mathrm{gm}}(k)$, with some restrictions. As an example, the bi-functor $R\mathcal{H}om(-, -)$ on $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ induces an internal Hom in $DM_{-}^{\mathrm{eff}}(k)$ by restriction. One then gets an internal Hom in $DM_{\mathrm{gm}}(k)$ (assuming resolution of singularities) by setting

$$\mathcal{H}om_{DM_{\mathrm{gm}}(k)}(A, B) := \mathcal{H}om_{DM_{-}^{\mathrm{eff}}(k)}(A \otimes \mathbb{Z}(n), B \otimes \mathbb{Z}(n+m)) \otimes \mathbb{Z}(-m)$$

for n, m sufficiently large. Also, using the embedding theorem, one has

Theorem 86 For $X \in \mathbf{Sm}_k$, there is a natural isomorphism

86

$$\mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(m(X), \mathbb{Z}(q)[p]) \cong H_{\mathrm{FS}}^p(X, \mathbb{Z}(q)) := \mathbb{H}^p(X_{\mathrm{Nis}}, \mathbb{Z}_{\mathrm{FS}}(q)).$$

From this, Theorem 79 and Corollary 22, we have

87 **Corollary 87** For $X \in \mathbf{Sm}_k$, there is a natural isomorphism

$$\mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(m(X), \mathbb{Z}(q)[p]) \cong \mathrm{CH}^q(X, 2q - p) .$$

Once one has this description of the morphisms in $DM_{\mathrm{gm}}(k)$, it follows easily that $DM_{\mathrm{gm}}(k)$ is a fine homological triangulated category of motives over k with $\mathbb{Z}$ -coefficients, and that $DM_{\mathrm{gm}}(k)$ has duality if k admits resolution of singularities.

Comparison Results

We state the main comparison theorem relating Levine's $\mathcal{DM}(k)$ and Voevodsky's $DM_{\mathrm{gm}}(k)$. We note that replacing the functor $m : \mathbf{Sm}_k \rightarrow DM_{\mathrm{gm}}(k)$ with

$$h : \mathbf{Sm}_k^{\mathrm{op}} \rightarrow DM_{\mathrm{gm}}(k)$$

$$h(X) := \mathcal{H}om_{DM_{\mathrm{gm}}(k)}(m(X), \mathbb{Z})$$

changes $DM_{\mathrm{gm}}(k)$ from a homological category of motives to a cohomological category of motives.

88 **Theorem 88:** [63, part 1, chapt. VI, theorem 2.5.5] Let k be a field admitting resolution of singularities. Sending $\mathbb{Z}_X(n)$ in $\mathcal{DM}(k)$ to

$$\mathcal{H}om_{DM_{\mathrm{gm}}(k)}(m(X), \mathbb{Z}(n))$$

in $DM_{\mathrm{gm}}(k)$, for $X \in \mathbf{Sm}_k$, extends to a pseudo-tensor equivalence of cohomological categories over motives over k

$$\mathcal{DM}(k) \rightarrow DM_{\mathrm{gm}}(k) ,$$

i.e., an equivalence of the underlying triangulated tensor categories, compatible with the respective functors h on $\mathbf{Sm}_k^{\mathrm{op}}$.

5.5 Mixed Tate Motives

Let $G = \mathrm{Gal}(\bar{k}/k)$ for some field k , and let ℓ be a prime not dividing the characteristic. In the category of continuous representations of G in finite dimensional $\mathbb{Q}_\ell$ -vector spaces, one has the Tate objects $\mathbb{Q}_\ell(n)$; the subcategory formed by the successive extension of the Tate objects turns out to contain a surprising amount of information. Analogously, one has the Tate Hodge structure $\mathbb{Q}(n)$ and the subcategory of the category of admissible variations of mixed Hodge structures over

a base-scheme B consisting of successive extensions of the $\mathbb{Q}(n)$; this subcategory gives rise, for example, to all the multiple polylogarithm functions. In this section, we consider the motivic version of these constructions, looking at categories of mixed motives generated by Tate objects.

We begin with an abstract approach by considering the triangulated subcategory of $DM_{\text{gm}}(k) \otimes \mathbb{Q}$ generated by the Tate objects $\mathbb{Q}(n)$. Via the Tannakian formalism, this quickly leads to the search for a concrete description of this category as a category of representations of the *motivic Lie algebra* or dually, co-representations of the motivic co-Lie algebra. We outline constructions in this direction due to Bloch [11], Bloch–Kriz [18] and Kriz–May [61], in which the *motivic cycle algebra*, built out of the alternating version of Bloch cycle complex described in Sect. 5.2.5, plays a central role. The work of Kriz and May shows how all the representation-theoretic constructions are related and a theorem of Spitzweck [85] allows us to relate all these to the abstract construction inside DM_{gm} .

The Triangulated Category of Mixed Tate Motives

5.5.1

Since we now have a reasonable definition of “the” triangulated category of mixed motives over a field k , especially with $\mathbb{Q}$ -coefficients, it makes sense to define the triangulated category of mixed Tate motives as the full triangulated subcategory generated by the (rational) Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$. Since the cohomological formulation has been used most often in the literature, we will do so as well. We will assume in this section that the base field k admits resolution of singularities, for simplicity.

Concretely, this means we replace the functor $m : \mathbf{Sm}_k \rightarrow DM_{\text{gm}}(k)$ with the functor $h : \mathbf{Sm}_k^{\text{op}} \rightarrow DM_{\text{gm}}(k)$,

$$h(X) := \mathcal{H}om(m(X), \mathbb{Z}) .$$

To give an example to fix ideas, the projective bundle formula gives the isomorphism

$$h(\mathbb{P}^n) \cong \bigoplus_{j=0}^n \mathbb{Z}(-j)[-2j] .$$

Definition 89 Let k be a field. The triangulated category of Tate motives, $DTM(k)$, is the full triangulated subcategory of $DM_{\text{gm}}(k)^{\text{op}} \otimes \mathbb{Q}$ generated by the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.

89

As the duality involution $\vee : DM_{\text{gm}}(k) \rightarrow DM_{\text{gm}}(k)^{\text{op}}$ is an equivalence of triangulated tensor categories, and $\mathbb{Q}(n)^{\vee} = \mathbb{Q}(-n)$, we have a duality involution on $DTM(k)$, giving an equivalence

$$\vee : DTM(k) \rightarrow DTM(k)^{\vee} .$$

The Weight-structure

The category $\mathrm{DTM}(k)$ admits a natural weight-filtration: Let $\mathrm{DTM}(k)_{\leq n}$ be the full triangulated subcategory of $\mathrm{DTM}(k)$ generated by the Tate objects $\mathbb{Z}(-m)$ with $m \leq n$. This gives the tower of subcategories

$$\dots \subset \mathrm{DTM}(k)_{\leq n} \subset \mathrm{DTM}(k)_{\leq n+1} \subset \dots \subset \mathrm{DTM}(k)$$

Dually, let $\mathrm{DTM}(k)^{>n}$ be the full triangulated subcategory of $\mathrm{DTM}(k)$ generated by the Tate objects $\mathbb{Z}(-m)$ with $m > n$.

The basic fact upon which the subsequent construction rests is:

90

Lemma 90 For $X \in \mathrm{DTM}(k)_{\leq n}$, $Y \in \mathrm{DTM}(k)^{>n}$, we have

$$\mathrm{Hom}_{\mathrm{DTM}(k)}(X, Y) = 0.$$

Proof For generators $X = \mathbb{Q}(-a)[s]$, $Y = \mathbb{Q}(-b)[t]$, $a \leq n < b$, this follows from

$$\begin{aligned} \mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}(-a)[s], \mathbb{Q}(-b)[t]) &= \mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}, \mathbb{Q}(a-b)[t-s]) \\ &= \mathrm{CH}^{a-b}(\mathrm{Spec} k, 2(a-b) - t + s) \otimes \mathbb{Q} \end{aligned}$$

which is zero since $a - b < 0$. The general result follows easily from this.

By various methods (see, e.g., [65] or [57]), one can use the lemma to show that the inclusion $i_n : \mathrm{DTM}(k)_{\leq n} \rightarrow \mathrm{DTM}(k)$ admits a right adjoint $r_n : \mathrm{DTM}(k) \rightarrow \mathrm{DTM}(k)_{\leq n}$. This gives us the functor

$$W_n : \mathrm{DTM}(k) \rightarrow \mathrm{DTM}(k),$$

$W_n := i_n \circ r_n$, and the canonical map $\iota_n : W_n X \rightarrow X$ for X in $\mathrm{DTM}(k)$. One shows as well that $\mathrm{Cone}(\iota_n)$ is in $\mathrm{DTM}(k)^{>n}$, giving the canonical distinguished triangle

$$W_n X \rightarrow X \rightarrow W^{>n} X \rightarrow W_n X[1],$$

where $W^{>n} X := \mathrm{Cone}(\iota_n)$.

91

Remark 91 As pointed out in [57], one can perfectly well define an integral version $\mathrm{DTM}(k)_{\mathbb{Z}}$ of $\mathrm{DTM}(k)$ as the triangulated subcategory of $DM_{\mathrm{gm}}(k)$ generated by the Tate objects $\mathbb{Z}(n)$. The argument for weight filtration in $\mathrm{DTM}(k)$ works perfectly well to give a weight filtration in $\mathrm{DTM}(k)_{\mathbb{Z}}$.

The t -structure and Vanishing Conjecture

One can ask if the Beilinson formulation for mixed motives holds at least for mixed Tate motives. The first obstruction is the so-called *Beilinson–Soulé vanishing conjecture* (see [84]):

Conjecture 92 Let F be a field. Then $K_p(F)^{(q)} = 0$ if $2q \leq p$ and $p > 0$.

92

Thus, the Beilinson–Soulé vanishing conjecture is just the rational version of the acyclicity portion of the Beilinson–Lichtenbaum conjectures.

Translating to motivic cohomology, this says

Conjecture 93 Let F be a field. Then $H^p(F, \mathbb{Q}(q)) = 0$ if $p \leq 0$ and $q > 0$.

93

Since we have

$$\mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}, \mathbb{Q}(q)[p]) = H^p(k, \mathbb{Q}(q)) ,$$

we find a relation between the vanishing conjecture and the structure of the triangulated Tate category.

For example, if there were an abelian category $\mathrm{TM}(k)$ with $\mathrm{DTM}(k)$ equivalent to the derived category $D^b(\mathrm{TM}(k))$, in such a way that the Tate objects $\mathbb{Q}(n)$ were all in $\mathrm{TM}(k)$, then we would have

$$\mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}, \mathbb{Q}(q)[p]) = \mathrm{Ext}_{\mathrm{TM}(k)}^p(\mathbb{Q}, \mathbb{Q}(q)) ,$$

which would thus be zero for $p < 0$.

Suppose further that $\mathrm{TM}(k)$ is a rigid tensor category, inducing the tensor and duality on $\mathrm{DTM}(k)$, with functorial exact weight filtration W_* , inducing the functors W_n on $\mathrm{DTM}(k)$, and that taking the associated graded with respect to W_* induces a faithful exact functor to $\mathbb{Q}$ -vector-spaces

$$\mathrm{gr}_*^W : \mathrm{TM}(k) \rightarrow \mathrm{Vec}_{\mathbb{Q}} .$$

Then, as each map $f : \mathbb{Q} \rightarrow \mathbb{Q}(a)$, $a \neq 0$ has $\mathrm{gr}^* f = 0$, it follows that

$$\mathrm{Hom}_{\mathrm{TM}(k)}(\mathbb{Q}, \mathbb{Q}(q)) = 0$$

for $q \neq 0$ as well.

In short, the existence of an abelian category of mixed Tate motives $\mathrm{TM}(k)$ with good properties implies the vanishing conjectures of Beilinson and Soulé.

There is a partial converse to this, namely,

Theorem 94: [65] Let k be a field, and suppose that the Beilinson–Soulé vanishing conjectures hold for k . Then there is a t -structure on $\mathrm{DTM}(k)$ with heart $\mathrm{TM}(k)$ satisfying:

94

1. $\mathrm{TM}(k)$ contains all the Tate objects $\mathbb{Q}(n)$. The $\mathbb{Q}(n)$ generate $\mathrm{TM}(k)$ as an abelian category, closed under extensions in $\mathrm{DTM}(k)$.

2. The tensor operation and duality on $\mathrm{DTM}(k)$ restrict to $\mathrm{TM}(k)$, making $\mathrm{TM}(k)$ a rigid tensor category
3. The functors W_n map $\mathrm{TM}(k)$ into itself, giving each object X of $\mathrm{TM}(k)$ a functorial finite weight-filtration

$$0 = W_{M+1}X \subset W_M(X) \subset \dots \subset W_N(X) = X.$$

4. Taking associated graded for the weight-filtration gives a faithful exact tensor functor

$$\oplus_n \mathrm{gr}_n^W : \mathrm{TM}(k) \rightarrow \mathbb{Q}\text{-mod}$$

to finite dimensional $\mathbb{Q}$ -vector spaces, making $\mathrm{TM}(k)$ a neutral $\mathbb{Q}$ -Tannakian category.

5. There are canonical natural maps

$$\phi_p(X, Y) : \mathrm{Ext}_{\mathrm{TM}(k)}^p(X, Y) \rightarrow \mathrm{Hom}_{\mathrm{DTM}(k)}(X, Y[p])$$

for X, Y in $\mathrm{TM}(k)$. $\phi_p(X, Y)$ is an isomorphism for $p = 0, 1$, and an injection for $p = 2$.

We will describe below a criterion for $\mathrm{DTM}(k)$ to be the derived category $D^b(\mathrm{TM}(k))$.

Looking at part (4) above, the Tannakian formalism as explained in Sect. 5.3.1 gives an identification of $\mathrm{TM}(k)$ with the category of graded representations of a graded pro-unipotent affine algebraic group over $\mathbb{Q}$, or what amounts to the same thing, a graded pro-nilpotent Lie algebra over $\mathbb{Q}$, called the *motivic Lie algebra*. There have been a number of constructions of candidates for the motivic Lie algebra, or the associated Hopf algebra, which we will discuss below.

95

Remark 95 The works of Terasoma [91], Deligne–Goncharov [27], Goncharov [36, 37], Goncharov–Manin [40] and others has drawn a close connection between the mixed Tate category and values of the Riemann zeta function, polylogarithms and multi-zeta functions; due to lack of space, we will not discuss these works here. See the article [35] in this volume for further details.

5.5.2 The Bloch Cycle Algebra and Lie Algebra

96

Definition 96

- (1) Let F be a field. A *cdga* $(A^*, d, \cdot)$ over F consists of a unital, graded-commutative F -algebra $(A^* := \bigoplus_{n \in \mathbb{Z}} A^n, \cdot)$ together with a graded homomorphism $d = \bigoplus_n d^n$, $d^n : A^n \rightarrow A^{n+1}$, such that
 1. $d^{n+1} \circ d^n = 0$.
 2. $d^{n+m}(a \cdot b) = d^n a \cdot b + (-1)^n a \cdot d^m b$; $a \in A^n, b \in A^m$.

A^* is called *connected* if $A^n = 0$ for $n < 0$ and $A^0 = F \cdot 1$, *cohomologically connected* if $H^n(A^*) = 0$ for $n < 0$ and $H^0(A^*) = F \cdot 1$.

- (2) An *Adams-graded cdga* is a cdga A together with a direct sum decomposition into subcomplexes $A^* := \bigoplus_{r \geq 0} A^*(r)$ such that $A^*(r) \cdot A^*(s) \subset A^*(r+s)$. An Adams-graded cdga is (cohomologically) connected if the underlying cdga is (cohomologically) connected.

For $x \in A^n(r)$, we called n the *cohomological degree* of x , $n := \deg x$, and r the *Adams degree* of x , $r := |x|$.

Example 97. Let k be a field. Recall from Sect. 5.2.5 the alternating cycle complexes $z^p(k, *)^{\text{Alt}}$ with commutative associative product

$$\cup^{\text{Alt}} : z^p(k, *) \otimes z^q(k, *) \rightarrow z^{p+q}(k, *) .$$

Bloch [11] has defined the *motivic cdga* over k , $\mathcal{N}_k^*(*)$, as the Adams-graded cdga over $\mathbb{Q}$ with

$$\mathcal{N}_k^m(r) := \begin{cases} z^r(k, 2r-m)^{\text{Alt}} & \text{for } r > 0 \\ z^0(k, 0)^{\text{Alt}} (= \mathbb{Q} \cdot [\text{Spec } k]) & \text{for } r = 0 . \end{cases}$$

and product

$$\cdot : \mathcal{N}_k^m(r) \otimes \mathcal{N}_k^n(s) \rightarrow \mathcal{N}_k^{m+n}(r+s)$$

given by $\cup^{\text{Alt}}$. The unit is $1 \cdot [\text{Spec } k] \in \mathcal{N}^0(0)$.

Remark 98 The Beilinson–Soulé vanishing conjecture for the field k is exactly the statement that $\mathcal{N}_k^*(*)$ is cohomologically connected.

98

Bloch defines a graded co-Lie algebra $\mathcal{M}(*) = \bigoplus_{r \geq 0} \mathcal{M}(r)$ as follows: Start with the cycle cdga $\mathcal{N}$. Let $\mathcal{N}_+^0 := \bigoplus_{r \geq 0} \mathcal{N}^0(r)$, and let $\mathcal{J} \subset \mathcal{N}$ be the differential ideal generated by $\bigoplus_{n < 0, r} \mathcal{N}^n(r) \oplus \mathcal{N}_+^0$. Let $\overline{\mathcal{N}}$ be the quotient cdga $\mathcal{N}/\mathcal{J}$. Bloch shows

Lemma 99

99

- (1) The product $\Lambda^2 \overline{\mathcal{N}}^1 \rightarrow \overline{\mathcal{N}}^2$ is injective
 (2) Let $\mathcal{M}_k = \{x \in \overline{\mathcal{N}}^1 \mid dx \text{ is in } \Lambda^2 \overline{\mathcal{N}}^1 \subset \overline{\mathcal{N}}^2\}$. Then

$$d\mathcal{M}_k \subset \Lambda^2 \mathcal{M}_k \subset \Lambda^2 \overline{\mathcal{N}}^1 \subset \overline{\mathcal{N}}^2 .$$

- (3) The map $d : \mathcal{M}_k \rightarrow \Lambda^2 \mathcal{M}_k$ makes $\mathcal{M}_k$ into an Adams graded co-Lie algebra over $\mathbb{Q}$.

100

Definition 100 The category of Bloch–Tate mixed motives over k , BTM_k , is the category of graded co-representations of $\mathcal{M}_k$ in $\mathbb{Q}\text{-mod}$, that is, the category of finite-dimensional graded $\mathbb{Q}$ -vector spaces $V(*) = \bigoplus_r V(r)$ together with a graded, degree zero $\mathbb{Q}$ -linear map

$$\varrho : V(*) \rightarrow \mathcal{M}_k \otimes_{\mathbb{Q}} V(*)$$

satisfying the co-associativity condition $(\text{id} \wedge \varrho) \circ \varrho = (\partial \otimes \text{id}) \circ \varrho$ as maps $V(*) \rightarrow \Lambda^2 \mathcal{M}_k \otimes V(*)$.

BTM_k contains the Tate-objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, where $\mathbb{Q}(n)$ is the vector space $\mathbb{Q}$ supported in degree $-n$, with zero co-action ϱ . There is a map $H^1(\mathcal{N}^*(r)) \rightarrow \text{Ext}_{\text{BTM}_k}^1(\mathbb{Q}(0), \mathbb{Q}(r))$ by sending $\eta \in Z^1(\mathcal{N}^*(r))$ to the class of the extension

$$0 \rightarrow \mathbb{Q}(r) \rightarrow V_\eta \rightarrow \mathbb{Q}(0) \rightarrow 0,$$

where $V_\eta = \mathbb{Q}(0) \oplus \mathbb{Q}(r)$ as a graded $\mathbb{Q}$ -vector space, with co-action ϱ given by

$$\varrho(a, b) = \eta \otimes (0, a).$$

One checks that changing η by a co-boundary does not affect the extension class.

5.5.3 Categories Arising from a cdga

As we will see below, the construction of mixed Tate motives via co-representations of the Bloch co-Lie algebra $\mathcal{M}_k$ is reasonable only under the so-called 1-minimal conjecture. Bloch and Kriz [18] have given another construction of a co-Lie algebra, and at the same time the associated Hopf algebra, by using the bar construction of $\mathcal{N}_k$. Kriz and May [61] have given a construction of a triangulated category, which derives more directly from the cdga $\mathcal{N}_k$; in case $\mathcal{N}_k$ is cohomologically connected, this triangulated category has a heart which turns out to be equivalent to the category of graded co-modules over the Bloch–Kriz co-Lie algebra.

Before we go into this, we discuss some of the general theory of the bar construction of a cdga and related constructions. We have taken this material from [61].

The Bar Construction

We recall the definition of the reduced bar construction of an augmented cdga $\varepsilon : A^* \rightarrow k$ over a field F of characteristic zero. Let $\bar{A}^*$ be the kernel of ε , and form the tensor algebra

$$T_k^* \bar{A}^* = \bigoplus_{(n_1, \dots, n_m)} \bar{A}^{n_1} \otimes_F \dots \otimes_F \bar{A}^{n_m}$$

with $\bar{A}^{n_1} \otimes_F \dots \otimes_F \bar{A}^{n_m}$ in total degree $\sum_j n_j - m$, together with a copy of F in degree 0, corresponding to the empty tensor product, which we write as $F \cdot 1$.

Denote a decomposable element of $\overline{A}^{n_1} \otimes_F \dots \otimes_F \overline{A}^{n_m}$ in $T_k^* \overline{A}^*$ as $[x_1 | \dots | x_m]$, $x_j \in A^{n_j}$, and define the map d by

$$d([x_1 | \dots | x_m]) = \sum_j (-1)^{\sum_{i=1}^{j-1} \deg(x_i)} [x_1 | \dots | dx_j | \dots | x_m] \\ + \sum_j (-1)^j [x_1 | \dots | x_j x_{j+1} | \dots | x_m] .$$

Set $d(F \cdot 1) = 0$. This forms the complex $(\overline{B}(A), d)$.

The shuffle product

$$[x_1 | \dots | x_m] \cup [x_{n+1} | \dots | x_{m+n}] := \frac{m!n!}{(m+n)!} \sum_{\sigma} \text{sgn}(\sigma) [x_{\sigma(1)} | \dots | x_{\sigma(n+m)}] ,$$

where $\sigma \in \Sigma_{n+m}$ ranges over all permutations with $\sigma(1) < \dots < \sigma(n)$ and $\sigma(n+1) < \dots < \sigma(n+m)$, defines a product on $\overline{B}(A)$, satisfying the Leibniz rule with respect to d . The map

$$\delta : \overline{B}(A) \rightarrow \overline{B}(A) \otimes \overline{B}(A)$$

$$\delta([x_1 | \dots | x_m]) := \sum_{i=0}^m [x_1 | \dots | x_i] \otimes [x_{i+1} | \dots | x_m]$$

(the empty tensor being 1) defines a coproduct on $\overline{B}(A)$.

This all makes $(\overline{B}(A), d, \cup, \delta)$ into a differential graded Hopf algebra over k , which is graded-commutative with respect to the product $\cup$. The cohomology $H^*(\overline{B}(A))$ is thus a graded Hopf algebra over k , in particular $H^0(\overline{B}(A))$ is a commutative Hopf algebra over k .

Let $\mathcal{I}(A)$ be the kernel of the augmentation $H^0(\overline{B}(A)) \rightarrow k$. The coproduct δ on $H^0(\overline{B}(A))$ induces the structure of a co-Lie algebra on $\gamma_A := \mathcal{I}(A)/\mathcal{I}(A)^2$.

Suppose $A = \oplus_{r \geq 0} A^*(r)$ is an Adams-graded cdga over k . We give $\overline{B}(A)$ the Adams grading $\overline{B}(A) = \oplus_{r \geq 0} \overline{B}(A)(r)$ where the Adams degree of $[x_1 | \dots | x_m]$ is

$$|[x_1 | \dots | x_m]| := \sum_j |x_j| .$$

Thus $H^0(\overline{B}(A)) = \oplus_{r \geq 0} H^0(\overline{B}(A)(r))$ becomes a graded Hopf algebra over k , commutative as a k -algebra. We also have the Adams-graded co-Lie algebra $\gamma_A = \oplus_{r > 0} \gamma_A(r)$.

Remark 101 Let A be an Adams-graded cdga over a field F of characteristic zero. The Adams grading makes $G := \text{Spec } H^0(\overline{B}(A))$ into a graded pro-unipotent affine group-scheme (i.e., G comes equipped with an action of $\mathbb{G}_m$). Thus γ_A is a graded nilpotent co-Lie algebra, and there is an equivalence of categories between the graded co-representations of $H^0(\overline{B}(A))$ in finite dimensional graded

F -vector spaces, $\text{co-rep}_F(H^0(\overline{B}(A)))$, and the graded co-representations of γ_A in finite dimensional graded F -vector spaces, $\text{co-rep}_F(\gamma_A)$.

Weight Filtrations and Tate Objects

Let A be an Adams-graded cdga over a field F of characteristic zero, and let $M = \bigoplus_r M(r)$ be a graded co-module for γ_A , finite dimensional as an F -vector space. Let $W_n M = \bigoplus_{r \leq n} M(r)$. As γ_A is positively graded, $W_n M$ is a γ_A -sub-comodule of M . Thus, each M has a finite functorial weight filtration, and the functor gr_W^n is exact. We say that M has *pure weight* n if $W_n M = M$ and $W_{n-1} M = 0$.

We have the Tate object $F(n)$, being the 1-dimensional F -vector space, concentrated in Adams degree $-n$, and with trivial (i.e. 0) co-action $F(n) \rightarrow \gamma_A \otimes F(n)$. Clearly $\text{gr}_W^n M \cong F(n)^a$ for some a , so all objects in $\text{co-rep}_F(\gamma_A)$ are successive extensions of Tate objects. The full subcategory of objects of pure weight n is equivalent to $F\text{-mod}$.

Sending M to $\text{gr}_W^* M := \bigoplus_n \text{gr}_W^n M$ defines a fiber functor

$$\text{gr}_W^* : \text{co-rep}_F(\gamma_A) \rightarrow F\text{-mod}$$

making $\text{co-rep}_F(\gamma_A)$ a neutral F -Tannakian category.

The Category of Cell-modules

The approach of Kriz and May [61] is to define a triangulated category directly from the Adams graded cdga $\mathcal{N}$ without passing to the bar construction or using a co-Lie algebra, by considering a certain type of dg-modules over $\mathcal{N}$. We recall some of their work here.

Let A^* be a graded algebra over a field F . We let $A[n]$ be the left A^* -module which is A^{m+n} in degree m , with the A^* -action given by left multiplication. If $A^*(*) = \bigoplus_{n,r} A^n(r)$ is a bi-graded F -algebra, we let $A\langle r \rangle[n]$ be the left $A^*(*)$ -module which is $A^{m+n}(r+s)$ in bi-degree (m, s) , with action given by left multiplication.

102

Definition 102 Let A be a cdga over a field F of characteristic zero.

- (1) A dg- A -module (M^*, d) consists of a complex $M^* = \bigoplus_n M^n$ with differential d , together with a graded, degree zero map $\rho : A^* \otimes_F M^* \rightarrow M^*$ which makes M^* into a graded A^* -module, and satisfies the Leibniz rule

$$d(a \cdot m) = da \cdot m + (-1)^{\deg a} a \cdot m ; a \in A^*, m \in M^* ,$$

where we write $a \cdot m$ for $\rho(a \otimes m)$.

- (2) If $A = \bigoplus_{r \geq 0} A^*(r)$ is an Adams-graded cdga, an Adams-graded dg- A -module is a dg- A -module M^* together with a decomposition into subcomplexes $M^* = \bigoplus_s M^*(s)$ such that $A^*(r) \cdot M^*(s) \subset M^*(r+s)$.
- (3) An Adams-graded dg- A -module M is a *cell module* if M is free and finitely generated as a bi-graded A -module, where we forget the differential structure.

That is, there are elements $b_j \in M^{n_j}(r_j)$, $j = 1, \dots, s$, such that the maps $a \mapsto a \cdot b_j$ induces an isomorphism of bi-graded A -modules

$$\bigoplus_{j=0}^s A(r_j)[n_j] \rightarrow M.$$

The Derived Category

Let A be an Adams-graded cdga over a field F , and let M and N be Adams-graded dg- A -modules. Let $\mathcal{H}om(M, N)$ be the Adams-graded dg- A -module with $\mathcal{H}om(M, N)^n(r)$ the A -module maps $f : M \rightarrow N$ with $f(M^a(s)) \subset N^{a+n}(r+s)$, and differential d defined by $df(m) = d(f(m)) + (-1)^{n+1}f(dm)$ for $f \in \mathcal{H}om(M, N)^n(r)$. Similarly, let $M \otimes N$ be the Adams-graded dg- A -module

$$(M \otimes N)^n(r) = \bigoplus_{a+b=n, s+t=r} M^a(s) \otimes_F N^b(t),$$

with differential $d(m \otimes n) = dm \otimes n + (-1)^{\deg m} m \otimes dn$.

For $f : M \rightarrow N$ a morphism of Adams-graded dg- A -modules, we let $\text{Cone}(f)$ be the Adams-graded dg- A -module with

$$\text{Cone}(f)^n(r) := N^n(r) \oplus M^{n+1}(r)$$

and differential $d(n, m) = (dn + f(m), -dm)$. Let $M[1]$ be the Adams-graded dg- A -module with $M[1]^n(r) := M^{n+1}(r)$ and differential $-d$, where d is the differential of M . A sequence of the form

$$M \xrightarrow{f} N \xrightarrow{i} \text{Cone}(f) \xrightarrow{j} M[1],$$

where i and j are the evident inclusion and projection, is called a *cone sequence*.

Definition 103

- (1) The category $\text{KCM}(A)$ is the F -linear triangulated category with objects the cell- A -modules M , morphisms

$$\text{Hom}_K(M, N) := H^0(\mathcal{H}om(M, N))$$

with evident composition law, translation $M \mapsto M[1]$ and distinguished triangles those sequences isomorphic to a cone sequence.

- (2) The category $\text{DCM}(A)$ is the localization of $\text{KCM}(A)$ with respect to quasi-isomorphisms, that is, invert the maps $f : M \rightarrow N$ which induce an isomorphism on cohomology $H^*(M) \rightarrow H^*(N)$.

We note that the tensor product and internal Hom of cell modules gives $\text{KCM}(A)$ and $\text{DCM}(A)$ the structure of rigid triangulated tensor categories.

The following is a useful result (see [61, proposition 4.2]):

Proposition 104 Let $\phi : A \rightarrow A'$ be a quasi-isomorphism of Adams graded cdgas. Then ϕ induces an equivalence of triangulated tensor categories $\mathrm{DCM}(A) \rightarrow \mathrm{DCM}(A')$.

The Weight and t -structures

It is easy to describe the weight filtration in $\mathrm{DCM}(A)$. Indeed, let $M = \bigoplus_j A\langle r_j \rangle [n_j]$ be a cell A -module with basis $\{b_j\}$. The differential d is determined by

$$db_j = \sum_i a_{ij} b_i.$$

As $A^*(r) = 0$ if $r < 0$, and d is of weight 0 with respect to the Adams grading, it follows that $|b_i| \leq |b_j|$ if $a_{ij} \neq 0$. We may thus set

$$W_n M := \bigoplus_{r_j \leq n} A\langle r_j \rangle [n_j] \subset M,$$

with differential the restriction of d . One shows that this gives a well-defined exact functor $W_n : \mathrm{DCM}(A) \rightarrow \mathrm{DCM}(A)$, and a natural finite weight filtration

$$0 = W_{n-1} \rightarrow W_n M \rightarrow \cdots \rightarrow W_{m-1} M \rightarrow W_m M = M$$

for M in $\mathrm{DCM}(A)$. Let $F(n)$ be the “Tate object” $A\langle -n \rangle$.

For the t -structure, one needs to assume that A is cohomologically connected; by Proposition 104 we may assume that A is connected. Let $\varepsilon : A \rightarrow k$ be the augmentation given by projection on $A^0(0)$, and define

$$\mathrm{DCM}(A)_{\leq 0} := \{M \mid H^n(M \otimes_A k) = 0 \text{ for } n > 0\}$$

$$\mathrm{DCM}(A)_{\geq 0} := \{M \mid H^n(M \otimes_A k) = 0 \text{ for } n < 0\}$$

$$\mathcal{H}(A) := \{M \mid H^n(M \otimes_A k) = 0 \text{ for } n \neq 0\}$$

One shows that this defines a t -structure $(\mathrm{DCM}(A)_{\leq 0}, \mathrm{DCM}(A)_{\geq 0})$ on $\mathrm{DCM}(A)$ with heart $\mathcal{H}(A)$. Also, the $F(n)$ are in $\mathcal{H}(A)$ and these generate $\mathcal{H}(A)$, in that the smallest full abelian subcategory of $\mathcal{H}(A)$ containing all the $F(n)$ and closed under subquotients and extensions is all of $\mathcal{H}(A)$.

The subcategory of $\mathcal{H}(A)$ consisting of objects of pure weight n is equivalent to F -mod with generator the Tate object $F(-n)$, giving us the fiber functor

$$\mathrm{gr}_W^* : \mathcal{H}(A) \rightarrow F\text{-mod}$$

which makes $\mathcal{H}(A)$ a neutral F -Tannakian category.

Minimal Models

A cdga A over a field F of characteristic zero is said to be *generalized nilpotent* if

1. As a graded F -algebra, $A = \mathrm{Sym}^* E$ for some $\mathbb{Z}$ -graded F -vector space E , i.e., $A = A^* E_{\mathrm{odd}} \otimes \mathrm{Sym}^* E_{\mathrm{ev}}$. In addition, $E_n = 0$ for $n \leq 0$.

2. E is an increasing union of graded subspaces

$$0 = E^{-1} \subset E^0 \subset \dots \subset E^n \subset \dots \subset E$$

with $dE^n \subset \text{Sym}^* E^{n-1}$.

Note that a generalized nilpotent cdga is automatically connected.

Let A be a cohomologically connected cdga. An n -minimal model of A is a map of cdgas

$$s : \mathcal{M}\{n\} \rightarrow A,$$

with $\mathcal{M}\{n\}$ generalized nilpotent and generated (as an algebra) in degrees $\leq n$, such that s induces an isomorphism on H^m for $m \leq n$ and an injection on H^{n+1} . One shows that this characterizes $s : \mathcal{M}\{n\} \rightarrow A$, up to unique isomorphism, so we may speak of the n -minimal model of A . Similarly, the minimal model of A is a quasi-isomorphism $\mathcal{M}\{\infty\} \rightarrow A$ with $\mathcal{M}\{\infty\}$ generalized nilpotent; we can recover $\mathcal{M}\{n\}$ as the sub-cdga of $\mathcal{M}\{\infty\}$ generated by $\bigoplus_{0 \leq i \leq n} \mathcal{M}\{\infty\}^i$. We call A n -minimal if $\mathcal{M}\{n\} = \mathcal{M}\{\infty\}$. With the obvious changes, we have all these notions in the Adams-graded setting.

Remark 105 In rational homotopy theory, the rational homotopy type corresponding to a cdga A is a $K(\pi, 1)$ if and only if A is 1-minimal, so a 1-minimal cdga is often called a $K(\pi, 1)$.

105

Let A be a cohomologically connected cdga with 1-minimal model $\mathcal{M}\{1\}$, let $QA = \mathcal{M}\{1\}^1$ with map $\partial : QA \rightarrow \Lambda^2 QA$ the differential $d : \mathcal{M}\{1\}^1 \rightarrow \Lambda^2 \mathcal{M}\{1\}^1 = \mathcal{M}\{1\}^2$. Then (QA, ∂) is a co-Lie algebra over F . If A is an Adams-graded cdga, then QA becomes an Adams-graded co-Lie algebra. We can also form the co-Lie algebra γ_A as in §5.5.3.

Putting it all Together

In [61] the relations between the various constructions we have presented above are discussed. We recall the main points here.

Theorem 106 Let A be an Adams graded cdga over a field F of characteristic zero. Suppose the A is cohomologically connected.

106

(1) There is a functor $\varphi : D^b(\text{co-rep}_F(H^0(\overline{B}(A)))) \rightarrow \text{DCM}(A)$. φ respects the weight filtrations and sends Tate objects to Tate objects. φ induces a functor on the hearts

$$\mathcal{H}(\varphi) : \text{co-rep}_F(H^0(\overline{B}(A))) \rightarrow \mathcal{H}(A)$$

which is an equivalence of filtered Tannakian categories, respecting the fiber functors gr_W^* .

- (2) Let $\mathcal{M}_A\{1\}$ be the 1-minimal model of A . Then there are isomorphisms of graded Hopf algebras $H^0(\overline{B}(A)) \cong H^0(\overline{B}(\mathcal{M}_A\{1\}))$ and graded co-Lie algebras

$$QA \cong \gamma_{\mathcal{M}_A\{1\}} \cong \gamma_A.$$

- (3) The functor φ is an equivalence of triangulated categories if and only if A is a $K(\pi, 1)$ (i.e., A is 1-minimal).

5.5.4 Categories of Mixed Tate Motives

We are now ready to apply the machinery of Sect. 5.5.3.

Tate Motives as Modules

107

Definition 107 Let k be a field.

- (1) The Bloch–Kriz category of mixed Tate motives over k , BKTM_k , is the category $\text{co-rep}_{\mathbb{Q}}(H^0(\overline{B}(\mathcal{N}_k)))$ of graded co-representations of the Hopf algebra $H^0(\overline{B}(\mathcal{N}_k))$ in finite dimensional graded $\mathbb{Q}$ -vector spaces, equivalently, the category $\text{co-rep}_{\mathbb{Q}}(\gamma_{\mathcal{N}_k})$ of graded co-representations of the co-Lie algebra $\gamma_{\mathcal{N}_k}$ in finite dimensional graded $\mathbb{Q}$ -vector spaces.
- (2) The Kriz–May triangulated category of mixed Tate motives over k , DT_k , is the derived category $\text{DCM}(\mathcal{N}_k)$ of cell modules over $\mathcal{N}_k$.

108

Remark 108 One can show that, assuming $\mathcal{N}_k$ is 1-minimal, the co-Lie algebra $\mathcal{M}_k$ is $Q\overline{\mathcal{N}}_k$. In general, there is a map of co-Lie algebras

$$\phi : \gamma_{\mathcal{N}_k} \rightarrow \mathcal{M}_k,$$

and hence a functor

$$\phi_* : \text{BKTM}_k = \text{co-rep}(\gamma_{\mathcal{N}_k}) \rightarrow \text{co-rep}(\mathcal{M}_k) = \text{BTM}_k.$$

Applying Theorem 106 to the situation at hand, we have

109

Theorem 109 Let k be a field. Suppose $\mathcal{N}_k$ is cohomologically connected, i.e., the Beilinson–Soulé vanishing conjecture holds for k .

1. There is an exact tensor functor $\varphi : D^b(\text{BKTM}_k) \rightarrow \text{DT}_k$, preserving the weight-filtrations and sending Tate objects to Tate objects.
2. The functor φ induces an equivalence of filtered $\mathbb{Q}$ -Tannakian categories $\text{BKTM}_k \rightarrow \mathcal{H}(\mathcal{N}_k)$, respecting the fiber functors gr_W^* .

3. The functor φ is an equivalence of triangulated categories if and only if $\mathcal{N}_k$ is a $K(\pi, 1)$. In particular, if $\mathcal{N}_k$ is a $K(\pi, 1)$, then

$$\mathrm{Ext}_{\mathrm{BKT}\mathcal{M}_k}^p(\mathbb{Q}, \mathbb{Q}(q)) = H^p(k, \mathbb{Q}(q)) = K_{2q-p}(k)^{(q)}$$

for all p and q .

The very last assertion follows from the identities (assuming φ an equivalence)

$$\begin{aligned} \mathrm{Ext}_{\mathrm{BKT}\mathcal{M}_k}^p(\mathbb{Q}, \mathbb{Q}(q)) &= \mathrm{Hom}_{\mathrm{DT}_k}(\mathbb{Q}, \mathbb{Q}(q)[p]) \\ &= H^p(\mathcal{N}^*(q)) \\ &= H^p(k, \mathbb{Q}(q)) . \end{aligned}$$

Remark 110 If the Beilinson–Soulé vanishing conjecture fails to hold for k , then there is no hope of an equivalence of triangulated categories $D^b(\mathrm{BKT}\mathcal{M}_k) \rightarrow \mathrm{DT}_k$, as the lack of cohomological connectness for $\mathcal{N}_k$ is equivalent to having $\mathrm{Hom}_{\mathrm{DT}_k}(\mathbb{Q}, \mathbb{Q}(q)[p]) \neq 0$ for some $q > 0$ and $p < 0$.

It is not clear if the lack of cohomological connectness of $\mathcal{N}_k$ gives an obstruction to the existence of a reasonable functor $\varphi : D^b(\mathrm{BKT}\mathcal{M}_k) \rightarrow \mathrm{DT}_k$ (say, with $\varphi(\mathbb{Q}(n)) = \mathbb{Q}(n)$).

110

Tate Motives as Voevodsky Motives

The following result, extracted from [85, theorem 2], shows how the Kriz–May triangulated category serves as a bridge between the Bloch–Kriz category of co-modules, and the more natural, but also more abstract, category of Tate motives sitting inside of Voevodsky’s category DM_{gm} .

Theorem 111 Let k be a field. There is a natural exact tensor functor

$$\phi : \mathrm{DT}_k \rightarrow DM_{\mathrm{gm}}(k)$$

which induces an equivalence of triangulated tensor categories $\mathrm{DT}_k \rightarrow \mathrm{DTM}(k)$. The functor ϕ is compatible with the weight filtrations in DT_k and $\mathrm{DTM}(k)$. If $\mathcal{N}_k$ is cohomologically connected, then ϕ induces an equivalence of abelian categories

$$\mathcal{H}(\mathcal{N}_k) \rightarrow \mathrm{TM}(k) .$$

Note that this gives a module-theoretic description of $\mathrm{DTM}(k)$ for all fields k , without assuming any conjectures. This result also gives a context for the $K(\pi, 1)$ -conjecture:

111

112 **Conjecture 112** Let k be a field. Then the cycle cdga $\mathcal{N}_k$ is a $K(\pi, 1)$.

Indeed the conjecture would imply that all the different candidates for an abelian category of mixed Tate motives over k agree: If $\mathcal{N}_k$ is 1-minimal, the $\mathcal{N}_k$ is cohomologically connected. By Theorem 111, the abelian categories $\mathcal{H}(\mathcal{N}_k)$ and $\mathrm{TM}(k)$ are equivalent, as well as the triangulated categories $\mathrm{DT}_k = \mathrm{DCM}(\mathcal{N}_k)$ and $\mathrm{DTM}(k)$. By Theorem 109, we have an equivalence of triangulated categories $D^b(\mathcal{H}(\mathcal{N}_k))$ and DT_k , and $\mathcal{H}(\mathcal{N}_k)$ is equivalent to the Bloch–Kriz category $\mathrm{BKT}\mathrm{M}_k$. By Remark 108, the graded co-Lie algebras $Q\mathcal{N}_k$ and $\mathcal{M}_k$ agree, so we have equivalences of abelian categories

$$\mathrm{BTM}(k) \sim \mathrm{BKT}\mathrm{M}(k) \sim \mathrm{TM}(k) \sim \mathcal{H}\mathrm{DT}_k$$

and triangulated categories

$$D^b(\mathrm{TM}(k)) \sim \mathrm{DTM}(k) .$$

All these equivalences respect the tensor structure, the weight filtrations and duality.

5.5.5 Spitzweck's Representation Theorem

We sketch a proof of Theorem 111.

Cubical Complexes in $DM_{-}^{\mathrm{eff}}(k)$

To give a representation of DT_k into DM_{gm} , it is convenient to use a cubical version of the Suslin-complex C_* .

113 **Definition 113** Let $\mathcal{F}$ be presheaf on $\mathbf{Sm}_k$. Let $C_n^{\mathrm{cb}}(\mathcal{F})$ be the presheaf

$$C_n^{\mathrm{cb}}(\mathcal{F})(X) := \mathcal{F}(X \times \square^n) / \sum_{j=1}^n \pi_j^*(\mathcal{F}(X \times \square^{n-1})) .$$

and let $C_*^{\mathrm{cb}}(\mathcal{F})$ be the complex with differential

$$d_n = \sum_{j=1}^n (-1)^{j-1} F(i_{j,1}) - \sum_{j=1}^n (-1)^{j-1} F(i_{j,0}) .$$

If $\mathcal{F}$ is a Nisnevich sheaf, then $C_*^{\mathrm{cb}}(\mathcal{F})$ is a complex of Nisnevich sheaves, and if $\mathcal{F}$ is a Nisnevich sheaf with transfers, then $C_*^{\mathrm{cb}}(\mathcal{F})$ is a complex of Nisnevich sheaves with transfers. We extend the construction to bounded above complexes of sheaves (with transfers) by taking the total complex of the evident double complex.

For a presheaf $\mathcal{F}$, let $C_n^{\mathrm{Alt}}(\mathcal{F}) \subset C_n^{\mathrm{cb}}(\mathcal{F})_{\mathbb{Q}}$ denote as above the subspace of alternating elements with respect to the action of Σ_n on $\square^n$, forming the subcomplex

$C_*^{\text{Alt}}(\mathcal{F}) \subset C_*^{\text{cb}}(\mathcal{F})_{\mathbb{Q}}$. We extend this to bounded above complexes of presheaves as well.

The arguments used in §5.2.5 to compare Bloch's cycle complex with the cubical version show

Lemma 114 Let $\mathcal{F}$ be a bounded above complex of presheaves on $\mathbf{Sm}_k$.

114

1. There is a natural isomorphism $C_*^{\text{Sus}}(\mathcal{F}) \cong C_*^{\text{cb}}(\mathcal{F})$ in the derived category of presheaves on $\mathbf{Sm}_k$. If $\mathcal{F}$ is a presheaf with transfer, we have an isomorphism $C_*^{\text{Sus}}(\mathcal{F}) \cong C_*^{\text{cb}}(\mathcal{F})$ in the derived category $D^-(\text{PST}(k))$.
2. The inclusion $C_*^{\text{Alt}}(\mathcal{F})(Y) \subset C_*^{\text{cb}}(\mathcal{F})_{\mathbb{Q}}(Y)$ is a quasi-isomorphism for all $Y \in \mathbf{Sm}_k$.

In particular, $C_*^{\text{cb}}(\mathcal{F})$ has homotopy invariant cohomology sheaves, so we have the functors

$$\begin{aligned} C_*^{\text{cb}} : C^-(\text{Sh}^{\text{Nis}}(k)) &\rightarrow DM_{-}^{\text{eff}}(k) . \\ C_*^{\text{Alt}} : C^-(\text{Sh}^{\text{Nis}}(k)) &\rightarrow DM_{-}^{\text{eff}}(k) \otimes \mathbb{Q} . \end{aligned}$$

Taking the usual Suslin complex also gives us a functor

$$C_*^{\text{Sus}} : C^-(\text{Sh}^{\text{Nis}}(k)) \rightarrow DM_{-}^{\text{eff}}(k) .$$

and we thus have the isomorphism of functors $C_*^{\text{Sus}} \rightarrow C_*^{\text{cb}}$ and $C_*^{\text{Alt}} \rightarrow (C_*^{\text{cb}})_{\mathbb{Q}}$.

The Cycle cdga in $DM_{-}^{\text{eff}}(k)$

We apply this construction to $\mathcal{F} = \mathbb{Z}_{q,\text{fin}}(\mathbb{A}^q)$. The symmetric group Σ_q acts on this sheaf by permuting the coordinates in $\mathbb{A}^q$, we let $\mathcal{N}_k^{\text{gm}}(q) \subset C_*^{\text{Alt}}(\mathbb{Z}_{q,\text{fin}}(\mathbb{A}^q))$ be the subsheaf of *symmetric* sections with respect to this action.

Lemma 115 The inclusion $\mathcal{N}_k^{\text{gm}}(q) \subset C_*^{\text{Alt}}(\mathbb{Z}_{q,\text{fin}}(\mathbb{A}^q))$ is an isomorphism in $DM_{-}^{\text{eff}}(k)$

115

Proof Roughly speaking, it follows from Theorem 26, Lemma 29 and Lemma 114 that the inclusion

$$C_*^{\text{Alt}}(\mathbb{Z}_{q,\text{fin}}(\mathbb{A}^q))(Y) \rightarrow z^q(Y \times \mathbb{A}^q, *)^{\text{Alt}}$$

is a quasi-isomorphism for each $Y \in \mathbf{Sm}_k$. As the pull-back

$$z^q(Y \times \mathbb{A}^q, *)^{\text{Alt}} \rightarrow z^q(Y \times \mathbb{A}^q, *)^{\text{Alt}}$$

is also a quasi-isomorphism by the homotopy property, Σ_q acts trivially on $z^q(Y \times \mathbb{A}^q, *)^{\text{Alt}}$, in $D^-(\mathbf{Ab})$.

For $X, Y \in \mathbf{Sm}_k$, the external product of correspondences gives the associative external product

$$C_n^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^q))(X) \otimes C_m^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^p))(Y) \rightarrow C_{n+m}^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^{p+q}))(X \times_k Y)$$

Taking $X = Y$ and pulling back by the diagonal $X \rightarrow X \times_k X$ gives the cup product of complexes of sheaves

$$\cup : C_*^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^p)) \otimes C_*^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^q)) \rightarrow C_*^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^{p+q})) .$$

Taking the alternating projection with respect to the $\square^*$ and symmetric projection with respect to the $\mathbb{A}^*$ yields the associative, commutative product

$$\cdot : \mathcal{N}_k^{\text{gm}}(p) \otimes \mathcal{N}_k^{\text{gm}}(q) \rightarrow \mathcal{N}_k^{\text{gm}}(p+q) ,$$

which makes $\mathcal{N}_k^{\text{gm}} := \bigoplus_{r \geq 0} \mathcal{N}_k^{\text{gm}}(r)$ into an Adams-graded cdga object in $C^-(\text{Sh}^{\text{Nis}}(k))$.

In particular, if a is in $\mathcal{N}_k^{\text{gm}}$, multiplication by a gives an endomorphism $a \cdot - : \mathcal{N}_k^{\text{gm}} \rightarrow \mathcal{N}_k^{\text{gm}}$.

A Replacement for $\mathcal{N}_k$

Let

$$\mathcal{N}_k(\mathbb{A}^*)^n(r) := (z^r(\mathbb{A}^r, 2r-n)^{\text{Alt}})^{\text{sym}} ,$$

where sym means the symmetric subspace with respect to the Σ_r -action on $\mathbb{A}^r$ by permuting the coordinates. Taking the external product and the alternating and symmetric projections defines an Adams-graded cdga $\mathcal{N}_k(\mathbb{A}^*)$. We have the evident inclusion

$$i : \mathcal{N}_k^{\text{gm}} \rightarrow \mathcal{N}_k(\mathbb{A}^*) ,$$

and the pull-back via the maps $\pi_r : \mathbb{A}^r \rightarrow \text{Spec } k$ defines

$$\pi^* : \mathcal{N}_k \rightarrow \mathcal{N}_k(\mathbb{A}^*) .$$

As above, i and π^* are both quasi-isomorphisms of cdgas. Thus, we have the equivalence of triangulated tensor categories

$$\text{DT}_k := \text{DCM}(\mathcal{N}_k) \sim \text{DCM}(\mathcal{N}_k(\mathbb{A}^*)) \sim \text{DCM}(\mathcal{N}_k^{\text{gm}}) .$$

The Functor $\text{DT}_k \rightarrow DM_{\text{gm}}(k)_{\mathbb{Q}}$

We are now ready to define our representation of $\text{DT}_k := \text{DCM}(\mathcal{N}_k)$ into $DM_{\text{gm}}(k)_{\mathbb{Q}}$. Let $\mathcal{N} = \mathcal{N}_k^{\text{gm}}$. We actually define a functor

$$\phi : \text{DCM}(\mathcal{N})' \rightarrow DM_{\text{gm}}(k)_{\mathbb{Q}}$$

where $\mathrm{DCM}(\mathcal{N})'$ is the category of cell- $\mathcal{N}$ -modules with a choice of basis. As this is equivalent to $\mathrm{DCM}(\mathcal{N})$, which in turn is equivalent to DT_k , the functor ϕ suffices for our purposes.

Let $M = \bigoplus_j \mathcal{N} m_j$ be a cell $\mathcal{N}$ -module, with basis $\{m_j\}$ and differential d given by

$$dm_j = \sum_i a_{ij} m_i .$$

Encoding d as the matrix (a_{ij}) , the condition $d^2 = 0$ translates as

$$(a_{ij}) \cdot (a_{ij}) = (da_{ij}) , \quad (5.7)$$

where da_{ij} is the differential in $\mathcal{N}$. Let $\phi(M, d)$ be the complex of sheaves $\bigoplus_j \mathcal{N}_k^{\mathrm{gm}}(r_j)[n_j] \mu_j$, where μ_j is a formal basis element. The differential δ in $\phi(M, d)$ characterized by

$$\delta(\mu_j) := \sum_{ij} a_{ij} \mu_i ,$$

and the requirement that δ satisfy the Leibniz rule

$$\delta(a \cdot \mu_j) = da \cdot \mu_j + (-1)^{\deg(a)} a \cdot \delta(\mu_j)$$

for a a local section of $\mathcal{N}_k^{\mathrm{gm}}(r_j)[n_j]$. The matrix equation (5.7) ensures that $\delta^2 = 0$, giving a well-defined object of $DM^{\mathrm{eff}}(k)$.

If $f : M \rightarrow N$ is a morphism of cell $\mathcal{N}$ -modules, we choose bases $\{m_j\}$ for M and $\{n_j\}$ for N , let $\{\mu_j\}$ and $\{\nu_j\}$ be the corresponding bases for $\phi(M)$ and $\phi(N)$. If $f(m_j) = \sum_i f_{ij} n_i$, then define $\phi(f)$ by $\phi(f)(\mu_j) = \sum_i f_{ij} \nu_i$.

One easily checks that ϕ respects tensor products, the translation functor and cone sequences, so yields a well-defined exact tensor functor

$$\phi : \mathrm{DCM}(\mathcal{N}_k^{\mathrm{gm}})' \rightarrow DM_{\mathrm{gm}}(k)_{\mathbb{Q}} .$$

By construction, $\phi(\mathbb{Q}(n))$ is the object $\mathcal{N}_k^{\mathrm{gm}}(n)$ of $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$, which by Lemma 114 and Lemma 115 is isomorphic to $C_*^{\mathrm{Sus}}(\mathbb{Z}_{\mathrm{q.fin}}(\mathbb{A}^n))_{\mathbb{Q}} \cong \mathbb{Q}(n)$ in $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$. Furthermore, we have

$$\mathrm{Hom}_{\mathrm{DT}_k}(\mathbb{Q}(0), \mathbb{Q}(n)[m]) = H^m(\mathcal{N}_k(n)) \cong \mathrm{CH}^n(k, 2n - m) ,$$

which agrees with $\mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)[m])$; it is not hard to see that ϕ induces the identity maps between these two Hom-groups. Since the $\mathbb{Q}(n)$'s are generators of DT_k , it follows that ϕ is fully faithful; since $\mathrm{DTM}(k)$ is generated by the $\mathbb{Q}(n)$'s, ϕ is therefore an equivalence. This completes the proof of the representation Theorem 111.

5.6

Cycle Classes, Regulators and Realizations

If one uses the axioms of Sect. 5.3.1 for a Bloch–Ogus cohomology theory, motivic cohomology becomes the universal Bloch–Ogus theory on $\mathbf{Sm}_k$. The various regulators on higher K -theory can then be factored through the Chern classes with values in motivic cohomology. Pushing this approach a bit further gives rise to “realization functors” from the triangulated category of mixed motives to the category of sheaves of abelian groups on $\mathbf{Sm}_k^{\text{Zar}}$. In this section, we give a sketch of these constructions. See also the article of Goncharov [35] in this volume.

There are other methods available for defining realization functors which we will mention as well.

5.6.1

Cycle Classes

We fix a Bloch–Ogus cohomology theory Γ on $\mathbf{Sm}_k$. In this section, we describe how one constructs functorial cycle classes

$$\text{cl}_\Gamma^{q,p} : \text{CH}^q(X, 2q - p) \rightarrow H_\Gamma^p(X, q) ,$$

and describe some of their basic properties. We refer to Sect. 5.3.1 for the notation.

Relative Cycle Classes

The main point of the construction is to use the purity property of Γ to extend the cycle classes to the relative case. Let $D = \sum_{i=1}^m D_i$ be a strict normal crossing divisor on some $Y \in \mathbf{Sm}_k$, that is, for each subset $I \subset \{1, \dots, m\}$, the subscheme $D_I := \cap_{j \in I} D_j$ of Y is smooth over k and of pure codimension $|I|$ on Y . We include the case $I = \emptyset$ in the notation; explicitly $D_\emptyset = Y$.

Let $\tilde{\Gamma}(\ast)$ be a flasque model for $\Gamma(\ast)$, e.g., for each $X \in \mathbf{Sm}_k$, $\tilde{\Gamma}(n)(X)$ is the complex of global sections of the Godement resolution of the restriction of $\Gamma(n)$ to X_{Zar} ; in particular, we have

$$H_\Gamma^n(X, q) = H^n(\tilde{\Gamma}(q)(X)) .$$

Let $\tilde{\Gamma}(q)(X; D)$ be the iterated shifted cone of the restriction maps for the inclusions $D_i \rightarrow X$, that is, if $m = 1$, $D = D_1$, then

$$\tilde{\Gamma}(q)(X; D) := \text{Cone}(\tilde{\Gamma}_D^*(\tilde{\Gamma}(q)(X) \rightarrow \tilde{\Gamma}(q)(D)))[-1]$$

and in general, $\tilde{\Gamma}(q)(X; D)$ is defined inductively as

$$\tilde{\Gamma}(q)(X; D) := \text{Cone} \left(\tilde{\Gamma}(q) \left(X; \sum_{i=1}^{m-1} D_i \right) \xrightarrow{i_{D_1}^*} \tilde{\Gamma}(q) \left(D_1; \sum_{i=1}^{m-1} D_1 \cap D_i \right) \right) [-1] .$$

One can also define $\tilde{T}(q)(X; D)$ as the total complex associated to the m -cube of complexes

$$I \mapsto \tilde{T}(q)(D_I) ,$$

from which one sees that the definition of $\tilde{T}(q)(X; D)$ is independent of the ordering of the D_i . Define the relative cohomology by

$$H_r^*(X; D, q) := H^*(\tilde{T}(q)(X; D)) .$$

For $W \subset X$ a closed subset, we have relative cohomology with supports, defined as

$$H_{r,W}^*(X; D, q) := H^*(\tilde{T}^W(q)(X; D)) ,$$

where

$$\tilde{T}^W(q)(X; D) = \text{Cone}(j^* : \tilde{T}(q)(X; D) \rightarrow \tilde{T}(q)(X \setminus W; j^*D))[-1] ,$$

and $j : X \setminus W \rightarrow X$ is the inclusion.

Let $D' = \sum_{i=1}^r D'_i$ be a SNC divisor on X containing D . Let $z^q(X)_{D'}$ denote the subgroup of $z^q(X)$ generated by integral codimension q subschemes W such that

$$\text{codim}_{D'_I}(W \cap D'_I) \geq q$$

for all $I \subset \{1, \dots, r\}$, and let $z^q(X; D)_{D'}$ denote the kernel of the restriction map

$$z^q(X)_{D'} \xrightarrow{\sum_j i_j^*} \bigoplus_{j=1}^m z^q(D_j) .$$

If $W \subset X$ is a closed subset, let $z_W^q(X; D)_{D'}$ be the subgroup of $z^q(X; D)_{D'}$ consisting of cycles supported in W ; we write $z_W^q(X; D)$ for $z_W^q(X; D)_D$.

Lemma 116 Let $W \subset X$ be a closed subset, $D = \sum_{i=1}^m D_i$ a strict normal crossing divisor on $X \in \mathbf{Sm}_k$. Let A be the ring $H_r^0(\text{Spec } k, 0)$.

116

1. If $\text{codim}_{D_I}(W \cap D_I) > q$ for all I , then $H_{r,W}^*(Y; D, q) = 0$. If $\text{codim}_{D_I}(W \cap D_I) \geq q$ for all I , then $H_{r,W}^*(Y; D, q) = 0$ for all $p < 2q$.
2. Suppose that $\text{codim}_{D_I}(W \cap D_I) \geq q$ for all I . Then the cycle class map cl define an isomorphism

$$\text{cl} : z_W^q(X; D) \otimes A \rightarrow H_W^{2q}(X; D, q) .$$

Proof For $m = 0$, (1) is just the purity property of Definition 30(3). The property (5) and the Gysin isomorphism (4)(b) of 30 give the isomorphism of (2) for W smooth, and one uses purity again to extend to arbitrary W . In general, one uses the long exact cohomology sequences associated to a cone and induction on m .

Higher Chow Groups and Relative Chow Groups

Identifying the higher Chow groups with “relative Chow groups”, making a similar identification for Γ -cohomology, and using the relative cycle map completes the construction.

For $m \geq 0$, let ∂_X^n and Λ_X^n be the SNC divisors $\sum_{i=0}^n (t_i = 0)$ and $\sum_{i=0}^{n-1} (t_i = 0)$ on $X \times \Delta^n$, respectively. For a commutative ring A , we have the higher Chow groups with A -coefficients

$$\mathrm{CH}^q(X, n; A) := H_n(z^q(X, *) \otimes A) .$$

Define motivic cohomology with A -coefficients, $H^p(X, A(q))$, by

$$H^p(X, A(q)) := \mathrm{CH}^q(X, 2q - p; A) .$$

117

Lemma 117 There is an exact sequence

$$z^q(X \times \Delta^{n+1}, \Lambda_X^{n+1})_{\partial_X^{n+1}} \otimes A \xrightarrow{\mathrm{res}_{t_{n+1}=0}} z^q(X \times \Delta^n, \partial_X^n) \otimes A \rightarrow \mathrm{CH}^q(X, n; A) \rightarrow 0 .$$

Proof By the Dold–Kan theorem [29], the inclusion of the normalized subcomplex

$$Nz^q(X, *) \rightarrow z^q(X, *)$$

is a quasi-isomorphism. Since $Nz^q(X, n) = z^q(X \times \Delta^n, \Lambda_X^n)_{\partial_X^n}$ with differential

$$\mathrm{res}_{t_{n+1}=0} : z^q(X \times \Delta^{n+1}, \Lambda_X^{n+1})_{\partial_X^{n+1}} \rightarrow z^q(X \times \Delta^n, \Lambda_X^n)_{\partial_X^n}$$

the result follows.

118

Lemma 118 Let X be in $\mathbf{Sm}_k$. Then $H_\Gamma^*(X \times \Delta^n; \Lambda_X^n, q) = 0$ for all q and there is a natural isomorphism

$$H_\Gamma^p(X \times \Delta^n; \partial_X^n, q) \cong H_\Gamma^{p-n}(X, q)$$

Proof This follows from the homotopy property of Γ and induction on n .

We can now define the cycle class map

$$\mathrm{CH}^q(X, n; A) \xrightarrow{\mathrm{cl}^q(n)} H_\Gamma^{2q-n}(X, q) ,$$

where A is the coefficient ring $H_\Gamma^0(\mathrm{Spec} k, 0)$; we then have

$$\mathrm{cl}^{q,p} : H^p(X, A(q)) \rightarrow H_\Gamma^p(X, q)$$

by $\mathrm{cl}^{q,p} := \mathrm{cl}^q(2q - p)$.

Indeed, from Lemma 116, we have natural isomorphisms

$$z^q(X \times \Delta^n, \partial_X^n) \otimes A \cong \varinjlim_W H_{F,W}^{2q}(X \times \Delta^n; \partial_X^n, q)$$

$$z^q(X \times \Delta^{n+1}, \Lambda_X^{n+1})_{\partial_X^{n+1}} \otimes A \cong \varinjlim_{W'} H_{F,W'}^{2q}(X \times \Delta^{n+1}; \Lambda_X^{n+1}, q),$$

where W runs over codimension q closed subsets of $X \times \Delta^n$, “in good position” with respect to the faces of Δ^n , and W' runs over codimension q closed subsets of $X \times \Delta^{n+1}$, “in good position” with respect to the faces of Δ^{n+1} . “Forgetting the supports” gives maps

$$\varinjlim_W H_{F,W}^{2q}(X \times \Delta^n; \partial_X^n, q) \rightarrow H_F^{2q}(X \times \Delta^n; \partial_X^n, q)$$

$$\varinjlim_{W'} H_{F,W'}^{2q}(X \times \Delta^{n+1}; \Lambda_X^{n+1}, q) \rightarrow H_F^{2q}(X \times \Delta^{n+1}; \Lambda_X^{n+1}, q)$$

Putting these together and using Lemma 117 and Lemma 118 gives the desired cycle class maps

$$\mathrm{cl}^q(n) : \mathrm{CH}^q(X, n; A) \rightarrow H_F^{2q-n}(X, q).$$

Remark 119 With a bit more work, one can achieve the maps $\mathrm{cl}^{q,p}$ as maps

119

$$\mathrm{cl}^q : \mathbb{Z}_{\mathrm{FS}}(q) \otimes^L A \rightarrow \Gamma(q) \quad (5.8)$$

in $D(\mathrm{Sh}^{\mathrm{Zar}}(k))$, compatible with the multiplicative structure. Using Remark 23, we have the structure map

$$\mathrm{cl}^1 \circ u : \mathbb{G}_m[-1] \rightarrow \Gamma(1)$$

promised in Sect. 5.3.1.

For additional details, we refer the reader to [33] ([33] considers cl^q as a map from the cycle complexes $\mathbb{Z}_{\mathrm{Bl}}(q)$ instead, but one can easily recover the statements made above from this).

In any case, we have:

Theorem 120 Fix a coefficient ring A . Motivic cohomology with A -coefficients, $H^*(-, A(*))$, as the Bloch–Ogus theory on $\mathbf{Sm}_k$ represented by $\mathbb{Z}_{\mathrm{FS}}(*) \otimes^L A$, is the universal Bloch–Ogus cohomology theory with coefficient ring A , in the sense of Definition 30.

120

Remark 121 With minor changes, the cycle classes described here extend to the case of scheme smooth and quasi-projective over a Dedkind domain, for example,

121

over a localization of a ring of integers in a number field, using the extension of the cycle complexes described in Remark 10. For instance, we have cycle classes

$$\mathrm{cl}^{q,p} : H^p(X, \mathbb{Z}/n(q)) := \mathrm{CH}^q(X, 2q - p; \mathbb{Z}/n) \rightarrow H_{\mathrm{\acute{e}t}}^p(X, \mathbb{Z}/n(q))$$

for $X \rightarrow \mathrm{Spec}(\mathcal{O}_F[1/n])$ smooth and quasi-projective, F a number field.

Explicit Formulas

The abstract approach outlined above does not lend itself to easy computations in explicit examples, except perhaps for the case of units and Milnor K -theory. Goncharov explains in his article [35] how one can give a fairly explicit formula for the cycle class map to real Deligne cohomology; this has been refined recently in [59] and [60] to give formulas for the map to integral Deligne cohomology. Although this search for explicit formulas may at first seem to be merely a computational convenience, in fact such formulas lie at the heart of some important conjectures, for instance, Zagier's conjecture on relating values of L -functions to polylogarithms [102].

Regulators

The classical case of a regulator is the Dirichlet regulator, which is the co-volume of the lattice of units of a number field under the embedding given by the logarithm of the various absolute values. The term “regulator” now generally refers to a real-valued invariant of some K -group, especially if there is some link with the classical case.

The Dirichlet construction was first generalized to higher K -theory of number rings by Borel [20] using group cohomology, and was later reinterpreted by Beilinson [5] as a lattice co-volume arising from a Gillet-type Chern class to real Deligne cohomology. In the context of the cycle class maps described above, we only wish to remark that it is easy to show that Gillet's Chern class $c_T^{p,q} : K_{2q-p}(X) \rightarrow H_T^p(X, q)$ factors as

$$K_{2q-p}(X) \xrightarrow{c^{p,q}} H^p(X, \mathbb{Z}(q)) \xrightarrow{\mathrm{cl}_T^q} H_T^p(X, q) .$$

for $\Gamma(\ast)$ a Bloch–Ogus cohomology theory.

5.6.2

Realizations

Extending the Cycle Class Map

In this section, we describe the method used by Levine [63, part 1, chapt. V] for defining a realization functor on $\mathcal{DM}(k)$ associated to a Bloch–Ogus cohomology theory Γ (see [63, part 1, chapt. V, theorem 1.3.1] for a precise statement, but note the remark below). We retain the notation of Sect. 5.6.1.

Remark 122 There is an error in the statement of [63, part 1, chapt. V, definition 1.1.6 and theorem 1.3.1]: The graded complex of sheaves $\mathcal{F}$ should be of the form $\mathcal{F} = \bigoplus_{q \in \mathbb{Z}} \mathcal{F}(q)$, not $\bigoplus_{q \geq 0} \mathcal{F}(q)$, as it is stated in *loc. cit.*. In Definition 1.1.6, the axioms (ii) and (iii) are for $q \geq 0$, whereas the axiom (iv) is for all q_1, q_2 and axiom (v) is for all q . I am grateful to Bruno Kahn for pointing out this error.

One would at first like to extend the assignment $\mathbb{Z}_X(q) \mapsto \tilde{T}(q)(X)$ to a functor

$$\mathfrak{R}_\Gamma : \mathcal{DM}(k) \rightarrow D(\mathbf{Ab}) .$$

There are essentially two obstructions to doing this:

1. In $\mathcal{DM}(k)$, we have the isomorphism

$$\mathbb{Z}_X(q) \otimes \mathbb{Z}_Y(q') \cong \mathbb{Z}_{X \times Y}(q + q') ,$$

but there is no requirement that the external products for Γ induce an analogous isomorphism in $D(\mathbf{Ab})$,

$$\tilde{T}(q)(X) \otimes^L \tilde{T}(q')(Y) \cong \tilde{T}(q + q')(X \times Y) .$$

In fact, in many naturally occurring examples, the above map is *not* an isomorphism.

2. The object $\Gamma(*) = \bigoplus_q \Gamma(q)$ is indeed a commutative ring-object in the derived category of sheaves on $\mathbf{Sm}_k^{\text{Zar}}$, but the commutativity and associativity properties of the product may not lift to similar properties on the level of the representing complexes $\tilde{T}(q)(X)$.

To avoid these problems, one considers a refinement of a Bloch–Ogus theory, namely a *geometric* cohomology theory on $\mathbf{Sm}_k^?$ [63, part 1, chapt. V, def. 1.1.6], where $?$ is a Grothendieck topology, at least as fine as the Zariski topology, having enough points (e.g., the étale, Zariski or Nisnevich topologies). Let A be a commutative ring and let $\text{Sh}_A^?(\mathbf{Sm}_k)$ be the category of sheaves of A -modules on $\mathbf{Sm}_k^?$. Essentially, a geometric cohomology Γ is given by a graded commutative ring object $\widehat{\Gamma}(*) = \bigoplus_{q \in \mathbb{Z}} \widehat{\Gamma}(q)$ in $C(\text{Sh}_A^?(\mathbf{Sm}_k))$, such that

1. All stalks of the sheaves $\widehat{\Gamma}(q)^n$ are flat A -modules.
2. For X in $\mathbf{Sm}_k$, let $p_X : X \rightarrow \text{Spec } k$ denote the projection. Then for X and Y in $\mathbf{Sm}_k$, the product map

$$Rp_{X*}(\widehat{\Gamma}(q)|_X) \otimes^L Rp_{Y*}(\widehat{\Gamma}(q')|_Y) \rightarrow Rp_{X \times Y*}(\widehat{\Gamma}(q + q')|_{X \times Y})$$

is an isomorphism in $D(\text{Sh}_A^?(\text{Spec } k))$.

3. Let $\alpha : \mathbf{Sm}_k^? \rightarrow \mathbf{Sm}_k^{\text{Zar}}$ be the change of topology morphism, and let $\Gamma(n) := R\alpha_* \widehat{\Gamma}(n)$. Then $\Gamma(*) := \bigoplus_{n \geq 0} \Gamma(n)$ defines a Bloch–Ogus cohomology theory on $\mathbf{Sm}_k$, in the sense of Sect. 5.3.1.

4. Let 1 denote the unit in $\mathrm{Sh}_A^?(\mathrm{Spec} k)$ and $[\widehat{\mathrm{Spec} k}] : 1 \rightarrow \widehat{\Gamma}(0)_{|\mathrm{Spec} k}$ the map in $D(\mathrm{Sh}_A^?(\mathrm{Spec} k))$ corresponding to the cycle class $[\mathrm{Spec} k] \in H_\Gamma^0(\mathrm{Spec} k, 0)$. Then $[\mathrm{Spec} k]$ is an isomorphism.

Examples of such theories include: de Rham cohomology, singular cohomology, étale cohomology with mod n coefficients.

Having made this refinement, one is able to extend to assignment $\mathbb{Z}_X(q) \mapsto R p_{X*}(\widehat{\Gamma}(q)_{|X})$ to a good realization functor:

123

Theorem 123: [63, part 1, chapt. V, thm. 1.3.1] Let Γ be a geometric cohomology theory on $\mathbf{Sm}_k^?$, and let $A := H_\Gamma^0(\mathrm{Spec} k, 0)$. Then sending $\mathbb{Z}_X(q)$ to $R p_{X*}(\widehat{\Gamma}(q)_{|X})$ extends to an exact pseudo-tensor functor

$$\mathfrak{R}_\Gamma : \mathcal{DM}(k)_A \rightarrow D(\mathrm{Sh}_A^?(\mathrm{Spec} k)) .$$

Here $\mathrm{Sh}_A^?(\mathrm{Spec} k)$ is the category of sheaves of A -modules on $\mathrm{Spec} k$, for the $?$ -topology. $\mathcal{DM}(k)_A$ is the extension of $\mathcal{DM}(k)$ to an A -linear triangulated category formed by taking the A -extension of the additive category $\overline{\mathcal{A}}_{\mathrm{mot}}(k)$ and applying the construction used in Sect. 5.4.4 to form $\mathcal{DM}(k)$ (this is not the same as the standard A -extension $\mathcal{DM}(k) \otimes A$ if for instance A is not flat over $\mathbb{Z}$).

The rough idea is to first extend the assignment

$$\mathbb{Z}_X(q) \mapsto R p_{X*}(\widehat{\Gamma}(q)_{|X})$$

to the additive category $\overline{\mathcal{A}}_{\mathrm{mot}}(k) \otimes A$ (notation as in Sect. 5.4.4) by sending the cycle map $[Z] : * \rightarrow \mathbb{Z}_X(d)[2d]$ to a representative of the cycle class with supports in codimension q for the cohomology theory Γ . The lack of a canonical representative creates problems, so we replace $\overline{\mathcal{A}}_{\mathrm{mot}}(k)$ with a DG-category $\mathcal{A}_{\mathrm{mot}}(k)$ for which the relations among the cycle maps $[Z]$ are only satisfied up to homotopy and “all higher homotopies”. Proceeding along this line, one constructs a functor

$$\mathfrak{R}_\Gamma^{(*)} : K^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A) \rightarrow K(\mathrm{Sh}_A^?(\mathrm{Spec} k)) .$$

One then “forgets supports” in the theory Γ and passes to the derived category

$$\mathfrak{R}_\Gamma^K : K^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A) \rightarrow D(\mathrm{Sh}_A^?(\mathrm{Spec} k)) .$$

Now let $D^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A)$ be the localization of $K^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A)$ as a triangulated tensor category, formed by inverting the same generating set of maps we used to form $D^b(\overline{\mathcal{A}}_{\mathrm{mot}}(k))$ from $K^b(\overline{\mathcal{A}}_{\mathrm{mot}}(k))$. The Bloch–Ogus axioms for Γ imply that $\mathfrak{R}_\Gamma^K$ extends to a functor on $D^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A)$; one then extends to the pseudo-abelian hull of $D^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A)$ and proves that this pseudo-abelian hull is equivalent to our original category $\mathcal{DM}(k)_A$.

Remark 124 Although theories such as Beilinson's absolute Hodge cohomology, Deligne cohomology, or ℓ -adic étale cohomology do not fit into the framework of a geometric cohomology theory, the method of construction of the realization functor does go through to give realization functors for these theories as well. We refer the reader to [63, part 1, chapt. V, §2] for these constructions.

We would like to correct an error in our construction of the absolute Hodge realization, pointed out to us by Pierre Deligne: In diagram (2.3.8.1), pg. 279, defining the object $D[X, \bar{X}]$, the operation Dec is improperly applied, and the functor $p_{(X, \bar{X})*}$ (top of page 278) is incorrectly defined. To correct this, one changes $p_{(X, \bar{X})*}$ by first taking global sections as indicated in diagram (2.3.6.8), and then applying the operation Dec to all the induced W -filtrations on the global sections. One also deletes the operation Dec from all applications in the diagram (2.3.8.1) defining $D[X, \bar{X}]$. With these changes, the construction goes through as described in [63].

Huber's Method

Huber constructs realizations for the rational Voevodsky category $DM_{\text{gm}}(k)_{\mathbb{Q}}$ in ([49, 50]) using a method very similar to the construction used by Nori to prove Proposition 46. The idea is the following: Suppose the base field is $\mathbb{C}$. Let $W \rightarrow X$ be a finite dominant morphism, with $X \in \mathbf{Sm}_k$, and W and X irreducible. Let $W' \rightarrow X$ be the normalization of X in the Galois closure of $k(W)/k(X)$, let $G = \text{Gal}(k(W')/k(X))$, and let $C^*(X)$ denote the singular cochain complex of $X(\mathbb{C})$ with $\mathbb{Q}$ -coefficients. Then G acts on $C^*(W')$, and in fact the natural map $C^*(X) \rightarrow C^*(W')$ gives a quasi-isomorphism

$$C^*(X) \rightarrow C^*(W')^G,$$

where $C^*(W')^G$ is the subcomplex of $C^*(W')$ of G -invariant cochains. Also, since we have $\mathbb{Q}$ -coefficients, there is a projection $\pi : C^*(W') \rightarrow C^*(W')^G$. Thus, one can define the pushforward $\pi_{W/X*} : C^*(W) \rightarrow C^*(X)$ as the composition in $D^+(\mathbb{Q}\text{-mod})$

$$C^*(W) \xrightarrow{\frac{1}{d}p^*} C^*(W') \xrightarrow{\pi} C^*(W')^G \xleftarrow{\sim} C^*(X),$$

where $p : W' \rightarrow W$ is the projection and d is the degree of p . Now, if $W = \sum_i n_i W_i$ is in $\text{Cor}(X, Y)$, we have the map

$$W_* : C^*(Y) \rightarrow C^*(X)$$

in $D^+(\mathbb{Q}\text{-mod})$ defined as the sum $\sum_i n_i W_{i*}$, where W_{i*} is the composition

$$C^*(Y) \xrightarrow{\pi_{W_i/Y}^*} C^*(W_i) \xrightarrow{\pi_{W_i/X*}} C^*(X),$$

where $\pi_{W_i/Y} : W_i \rightarrow Y$ is the evident map.

Refining this to give maps on the level of complexes, the assignment $X \mapsto C^*(X)$ extends to a functor

$$R_{\text{sing}} : \text{Cor}(\mathbb{C})_{\mathbb{Q}}^{\text{op}} \rightarrow C^+(\mathbb{Q}\text{-Vec}) ;$$

the properties of singular cohomology as a Bloch–Ogus theory imply that R_{sing} extends to an exact functor

$$\mathfrak{R}_{\text{sing}} : DM_{\text{gm}}(\mathbb{C})_{\mathbb{Q}} \rightarrow D^+(\mathbb{Q}\text{-mod}) .$$

Two essential problems occur in this approach:

1. For many interesting theories Γ (e.g. de Rham cohomology), even though there are extensions of Γ to complexes on all reduced normal quasi-projective k -schemes, it is often not the case that $\Gamma(q)(X) \rightarrow \Gamma(q)(W')^G$ is a quasi-isomorphism, as was the case for singular cohomology.
2. It is not so easy (even in the case of singular cohomology) to refine the map $\pi_{W/X*}$ to give a functorial map on the level of complexes.

Huber overcomes these difficulties to give realizations for singular cohomology, as described above, as well as for $\mathbb{Q}_{\ell}$ -étale cohomology, and rational Deligne cohomology.

125

Remark 125 Deligne and Goncharov [27] have defined Hodge, Betti and ℓ -adic realizations on Voevodsky’s category $DM_{\text{gm}}(k)$. It seems that their method would define realizations of $DM_{\text{gm}}(k)$ for a wide range of cohomology theories, but at present there does not seem to be any such construction in the literature.

Nori’s Realizations

Using the functor (5.4) (see just below Definition 49)

$$\Pi : DM_{\text{gm}}(k) \rightarrow D^b(\text{NMM}(k)) ,$$

and the universal property of the category $\text{NMM}(k)$ (derived from the universal property of $\text{ECM}(k)$), one has integral realization functors from $DM_{\text{gm}}(k)$ for: singular cohomology, ℓ -adic étale cohomology, de Rham cohomology, and Beilinson’s absolute Hodge cohomology. These do not seem to have been used at all in the literature up to now, so we hope that a good version of Nori’s work will appear soon.

Bloch–Kriz Realizations

We conclude our overview of realizations by briefly discussing the method used in [18] for constructing realizations of the Tate category BKTM_k . Denote the motivic Hopf algebra $H^0(\overline{B}(\mathcal{N}_k))$ by $\chi_{\text{mot}}(k)$ (see Definition 107 for the notation).

One can consider for instance the category of continuous $G_k := \text{Gal}(\bar{k}/k)$ representations M in finite dimensional $\mathbb{Q}_\ell$ -vector spaces, such that M has a finite filtration W_*M with quotients $\text{gr}_n^W M$ being given by the n th power of the cyclotomic character. This forms a Tannakian $\mathbb{Q}_\ell$ -category, classified by an Adams-graded Hopf algebra $\chi_{\text{ét},\ell}(k)$. Thus, in order to define an étale realization of the category $\text{BKT}M_k = \text{co-rep}(\chi_{\text{mot}}(k))$, it suffices to give a homomorphism of Hopf algebras

$$\phi_{\text{ét}} : \chi_{\text{mot}}(k) \rightarrow \chi_{\text{ét},\ell}(k) .$$

Using a modification of the cycle-class method discussed in Sect. 5.6.1, they show that the cycle class map (5.8), for $\Gamma(*) = \mathbb{Q}_\ell$ -étale cohomology, can be refined to give rise to such a homomorphism $\phi_{\text{ét}}$, and hence a realization functor

$$\phi_{\text{ét}} : \text{BKT}M_k \rightarrow \text{co-rep}(\chi_{\text{ét},\ell}(k)) \rightarrow \mathbb{Q}_\ell[G_k]\text{-mod} .$$

It would be interesting to compare this realization functor with the one given by Spitzweck's representation theorem and Nori's realization functor.

A similar method yields a description of real mixed Hodge structures as the Tannakian category of co-representations of an Adams graded Hopf algebra χ_{Hdg} over $\mathbb{R}$, and a realization homomorphism

$$\phi_{\text{Hdg}} : \chi_{\text{mot}}(\mathbb{C}) \rightarrow \chi_{\text{Hdg}} .$$

Again, it would be interesting to compare this with Nori's approach, and to see if the refined cycle classes of [59, 60] allow one to give a more explicit description of ϕ_{Hdg} .

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Part III

***K*-Theory and Geometric Topology**

Witt Groups

III.1

Paul Balmer*

1.1	<i>Introduction</i>	540
1.2	<i>Usual Witt Groups: General Theory</i>	540
	Duality and Symmetric Spaces	540
	Exact Categories with Duality	542
	Lagrangians and Metabolic Spaces	543
	The Witt Group of an Exact Category with Duality	545
	The Sublagrangian Reduction	547
1.3	<i>Usual Witt Groups: Examples and Motivations</i>	549
	Schemes	549
	Motivation From Real Algebraic Geometry	553
	Rings with Involution, Polynomials and Laurent Rings	554
	Semi-local Rings and Fields	555
1.4	<i>A Glimpse at Other Theories</i>	556
1.5	<i>Triangular Witt Groups: General Theory</i>	558
	Basic Notions and Facts	558
	Agreement and Localization	561
	Products and Cofinality	562
1.6	<i>Witt Groups of Schemes Revisited</i>	563
	Witt Cohomology Theories	563
	Local to Global	566
	Computations	569
	Witt Groups and $\mathbb{A}^1$ -Homotopy Theory	571
	<i>References</i>	571

* Author supported by the Swiss National Science Foundation, grant 620-066065.

Introduction

In his 1937 paper [86], Ernst Witt introduced a group structure – and even a ring structure – on the set of isometry classes of anisotropic quadratic forms, over an arbitrary field k . This object is now called the *Witt group* $W(k)$ of k . Since then, Witt’s construction has been generalized from fields to rings with involution, to schemes, and to various types of categories with duality. For the sake of efficacy, we review these constructions in a non-chronological order. Indeed, in Sect. 1.2, we start with the now “classical framework” in its most general form, namely over *exact categories with duality*. This folklore material is a basically straightforward generalization of Knebusch’s scheme case [41], where the exact category was the one of vector bundles. Nevertheless, this level of generality is hard to find in the literature, like e.g. the “classical sublagrangian reduction” of Sect. 1.2.5. In Sect. 1.3, we specialize this classical material to the even more classical examples listed above: schemes, rings, fields. We include some motivations for the use of Witt groups.

This chapter focusses on the theory of Witt groups in parallel to Quillen’s K -theory and is not intended as a survey on quadratic forms. In particular, the immense theory of quadratic forms over fields is only alluded to in Sect. 1.3.4; see preferably the historical surveys of Pfister [66] and Scharlau [72]. Similarly, we do not enter the arithmetic garden of quadratic forms: lattices, codes, sphere packings and so on. In fact, even Witt-group-like objects have proliferated to such an extent that everything could not be included here. However, in the intermediate Sect. 1.4, we provide a very short guide to various sources for the connections between Witt groups and other theories.

The second part of this chapter, starting in Sect. 1.5, is dedicated to the Witt groups of *triangulated categories with duality*, and to the recent developments of this theory. In Sect. 1.6, we survey the applications of triangular Witt groups to the above described classical framework.

Usual Witt Groups: General Theory

Duality and Symmetric Spaces

Definition 1 A *category with duality* is a triple $(\mathcal{C}, *, \varpi)$ made of a category $\mathcal{C}$ and an involutive endo-functor $*$: $\mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ with given isomorphism

$$\varpi : \text{Id}_{\mathcal{C}} \xrightarrow{\sim} * \circ *$$

and subject to the condition below. Write as usual $M^* := *(M)$ for *the dual* of an object $M \in \mathcal{C}$ and similarly for morphisms. Then $M \mapsto M^*$ is a functor and $\varpi_M : M \xrightarrow{\sim} (M^*)^*$ is a natural isomorphism such that:

$$(\varpi_M)^* \circ \varpi_{M^*} = \text{id}_{M^*} \quad \text{for any object } M \in \mathcal{C}.$$

Definition 2 A *symmetric space* in $(\mathcal{C}, *, \varpi)$ – or simply in $\mathcal{C}$ – consists of a pair (P, φ) where P is an object of $\mathcal{C}$ and where $\varphi : P \xrightarrow{\sim} P^*$ is a symmetric isomorphism, called the *symmetric form* of the space (P, φ) . The symmetry of φ reads $\varphi^* \circ \varpi_P = \varphi$, i.e. $\varphi^* = \varphi$ when P is identified with P^{**} via ϖ_P :

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P^* \\ \varpi_P \downarrow \cong & & || \\ P^{**} & \xrightarrow{\varphi^*} & P^* \end{array}$$

Note that the notion of “symmetry” depends on the chosen identification ϖ of objects of $\mathcal{C}$ with their double dual. This allows us to treat *skew-symmetric* forms as symmetric forms in $(\mathcal{C}, *, -\varpi)$. Nevertheless, when clear from the context, we drop ϖ from the notations and identify P^{**} with P .

Remark 3 We shall focus here on “non-degenerate” or “unimodular” forms, that is, we almost always assume that φ is an isomorphism. In good cases, one can consider the non-unimodular forms as being unimodular in a different category (of morphisms). See Bayer-Fluckiger–Fainsilber [16].

Definition 4 Two symmetric spaces (P, φ) and (Q, ψ) are called *isometric* if there exists an *isometry* $h : (P, \varphi) \xrightarrow{\sim} (Q, \psi)$, that is an isomorphism $h : P \xrightarrow{\sim} Q$ in the category $\mathcal{C}$ respecting the symmetric forms, i.e. $h^* \psi h = \varphi$.

Definition 5 A *morphism of categories with duality*

$$(\mathcal{C}, *^{\mathcal{C}}, \varpi^{\mathcal{C}}) \rightarrow (\mathcal{D}, *^{\mathcal{D}}, \varpi^{\mathcal{D}})$$

consists of a pair (F, η) where $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor and $\eta : F \circ *^{\mathcal{C}} \xrightarrow{\sim} *^{\mathcal{D}} \circ F$ is an isomorphism respecting ϖ , i.e. for any object M of $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(M) & \xrightarrow{F(\varpi_M^{\mathcal{C}})} & F(M^{**}) \\ \varpi_{F(M)}^{\mathcal{D}} \downarrow & & \downarrow \eta_{M^*} \\ F(M)^{**} & \xrightarrow{(\eta_M)^*} & F(M^*)^* \end{array},$$

where $(-)^*$ is $(-)^{*^{\mathcal{C}}}$ or $(-)^{*^{\mathcal{D}}}$ depending on the context, in the obvious way.

Definition 6 An *additive category with duality* is a category with duality $(\mathcal{A}, *, \varpi)$ where $\mathcal{A}$ is additive and where $*$ is an additive functor, i.e. $(A \oplus B)^* = A^* \oplus B^*$ via the natural morphism.

7 Remark 7 The identification ϖ necessarily respects the additivity, i.e. $\varpi_{A \oplus B} = \varpi_A \oplus \varpi_B$. This is a general fact for natural transformations between additive functors. Similarly, we need not consider ϖ in the following:

8 Definition 8 A *morphism of additive categories with duality* is simply a morphism of categories with duality (F, η) in the sense of Def. 5, such that the functor F is additive.

Example 9. In an additive category with duality $(\mathcal{A}, *, \varpi)$, one can produce symmetric spaces (P, φ) as follows. Take any object $M \in \mathcal{A}$. Put $P := M \oplus M^*$ and

$$\varphi := \begin{pmatrix} 0 & \text{id}_{M^*} \\ \varpi_M & 0 \end{pmatrix}: \underbrace{M \oplus M^*}_{=P} \xrightarrow{\cong} \underbrace{M^* \oplus M^{**}}_{=P^*}.$$

Note that the symmetry of φ uses the assumption $(\varpi_M)^* = (\varpi_{M^*})^{-1}$. This space (P, φ) is called the *hyperbolic space* (over M) and is denoted by $H(M)$.

10 Definition 10 Let $(\mathcal{A}, *, \varpi)$ be an additive category with duality. Let (P, φ) and (Q, ψ) be symmetric spaces. We define *the orthogonal sum* of these spaces as being the symmetric space $(P, \varphi) \perp (Q, \psi) := \left(P \oplus Q, \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} \right)$.

11 Definition 11 Let $(F, \eta) : (\mathcal{C}, *^{\mathcal{C}}, \varpi^{\mathcal{C}}) \rightarrow (\mathcal{D}, *^{\mathcal{D}}, \varpi^{\mathcal{D}})$ be a morphism of categories with duality and let (P, φ) be a symmetric space in $\mathcal{C}$. Then

$$F(P, \varphi) := (F(P), \eta_P \circ F(\varphi))$$

is a symmetric space in $\mathcal{D}$, called *the image by F of the space (P, φ)* .

It is clear that two isometric symmetric spaces in $\mathcal{C}$ have isometric images by F . If we assume moreover that F is a morphism of *additive* categories with duality, it is also clear that the image of the orthogonal sum is isometric to the orthogonal sum of the images; similarly, the image of the hyperbolic space $H(M)$ over any $M \in \mathcal{C}$ is then isometric to $H(F(M))$.

1.2.2 Exact Categories with Duality

12 Remark 12 The reader is referred to the original Quillen [68] or to the minimal Keller [40, App. A] for the definition of an *exact category*. The basic example of such a category is the one of vector bundles over a scheme. We denote by $\rightarrowtail$ and

by $\rightarrow$ the *admissible* monomorphisms and epimorphisms, respectively. Note that being exact (unlike additive) is *not* an intrinsic property. By a *split exact category* we mean an additive category where the admissible exact sequences are exactly the split ones. The basic example of the latter is the split exact category of finitely generated projective modules over a ring.

Definition 13 An *exact category with duality* is an additive category with duality $(\mathcal{E}, *, \varpi)$ in the sense of Def. 6, where the category $\mathcal{E}$ is exact and such that the functor $*$ is exact. So, $\mathcal{E}$ is an exact category, $M \mapsto M^*$ is a contravariant endofunctor on $\mathcal{E}$, $\varpi_M : M \rightarrow M^{**}$ is a natural isomorphism such that $(\varpi_M)^* = (\varpi_{M^*})^{-1}$ and for any admissible exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\pi} C$, the following (necessarily exact) sequence is admissible:

$$C^* \xrightarrow{\pi^*} B^* \xrightarrow{\alpha^*} A^*$$

Example 14. The key example of an exact category with duality is the one of vector bundles over a scheme with the usual duality, see Sect. 1.3.1 below. Note also that any additive category with duality can be viewed as a (split) exact category with duality.

Definition 15 A *morphism of exact categories with duality* (F, η) is a morphism of categories with duality (Def. 5) such that F is exact, i.e. F sends admissible short exact sequences to admissible short exact sequences. Such a functor F is necessarily additive.

Lagrangians and Metabolic Spaces

Definition 16 Let $(\mathcal{E}, *, \varpi)$ be an exact category with duality (see Def. 13). Let (P, ϖ) be a symmetric space in $\mathcal{E}$. Let $\alpha : L \rightarrow P$ be an admissible monomorphism. The *orthogonal* in (P, φ) of the pair (L, α) is as usual

$$(L, \alpha)^\perp := \ker(\alpha^* \varphi : P \rightarrow L^*).$$

Explicitly, consider an admissible exact sequence $L \xrightarrow{\alpha} P \xrightarrow{\pi} M$ and dualize it to get the second line below:

$$\begin{array}{ccccc} & & P & & \\ & \nearrow & \downarrow \varphi & \searrow \alpha^* \varphi & \\ M^* & \xrightarrow{\pi^*} & P^* & \xrightarrow{\alpha^*} & L^* \end{array}$$

13

15

1.2.3

16

This describes $(L, \alpha)^\perp := \ker(\alpha^* \varphi)$ as the pair $(M^*, \varphi^{-1} \pi^* : M^* \rightarrowtail P)$. We shall write $L^\perp$ instead of M^* when the monomorphism α is understood.

17 Definition 17 An (admissible) *sublagrangian* of a symmetric space (P, φ) is an admissible monomorphism $\alpha : L \rightarrowtail P$ such that the following conditions are satisfied:

- (a) the form φ vanishes on L , that is $\alpha^* \varphi \alpha = 0 : L \rightarrow L^*$,
- (b) the induced monomorphism $\beta : L \rightarrow L^\perp$ is admissible; in the above notations β is the unique morphism such that $(\varphi^{-1} \pi^*) \circ \beta = \alpha$.

18 Remark 18 For condition (b), consider the diagram coming from above:

$$\begin{array}{ccccc}
 L & \xrightarrow{\alpha} & P & \xrightarrow{\pi} & (L^\perp)^* \\
 \beta \downarrow & & \simeq \downarrow \varphi & & \downarrow \beta^* \\
 L^\perp & \xrightarrow{\pi^*} & P^* & \xrightarrow{\alpha^*} & L^* .
 \end{array} \tag{1.1}$$

Since $\alpha^* \circ (\varphi \alpha) = 0$, there exists a unique $\beta : L \rightarrow L^\perp$ as claimed. Observe that β^* makes the right square commutative by symmetry (we drop the ϖ 's). This β is automatically a monomorphism since α is, and β^* is automatically an epimorphism. Condition (b) only requires them to be admissible. However, in many cases, it is in fact automatic, namely when the exact category $\mathcal{E}$ can be embedded into some abelian category $\iota : \mathcal{E} \hookrightarrow \mathcal{A}$ in such a way that a morphism q in $\mathcal{E}$ is an admissible epimorphism in $\mathcal{E}$ if and only if $\iota(q)$ is an epimorphism in $\mathcal{A}$. This can always be achieved if $\mathcal{E}$ is *semi-saturated*, i.e. if any split epimorphism is admissible, in particular if $\mathcal{E}$ is idempotent complete (see [79, App. A]). So, in real life, condition (b) is often dropped.

19 Definition 19 An (admissible) *lagrangian* of a symmetric space (P, φ) is an admissible sublagrangian (L, α) such that $L = L^\perp$, i.e. a sublagrangian as in Def. 17 such that the morphism $\beta : L \rightarrowtail L^\perp$ is an isomorphism.

Note that (L, α) is a lagrangian of the space (P, φ) if and only if the following is an admissible exact sequence – compare diagram (1.1):

$$L \xrightarrow{\alpha} P \xrightarrow{\alpha^* \varphi} L^* . \tag{1.2}$$

20 Definition 20 A symmetric space (P, φ) is called *metabolic* if it possesses an admissible lagrangian, i.e. if there exists an exact sequence as above.

Example 21. Assume that the exact sequence (1.2) is *split* exact. Such a metabolic symmetric space is usually called *split metabolic*. A symmetric space is split metabolic if and only if it is isometric to a space of the form

$$\left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \varpi_L & \xi \end{pmatrix} \right)$$

for some object L and some symmetric morphism $\xi = \xi^*$. In particular, any hyperbolic space $H(L)$ is split metabolic with $\xi = 0$. If we assume further that 2 is invertible in $\mathcal{E}$ (see Rem. 69), then any split metabolic space is isometric to a hyperbolic space $H(L)$ via the automorphism h of $L \oplus L^*$ defined by:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\xi & 1 \end{pmatrix}}_{h^*} \cdot \begin{pmatrix} 0 & 1 \\ \varpi_L & \xi \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & -\frac{1}{2}\xi \\ 0 & 1 \end{pmatrix}}_h = \begin{pmatrix} 0 & 1 \\ \varpi_L & 0 \end{pmatrix}.$$

Note that a symmetric space is split metabolic if and only if it is metabolic for the split exact structure of the additive category $\mathcal{E}$. See Ex. 39 below for an exact sequence like (1.2) which does not split (when the category $\mathcal{E}$ is not split) and Ex. 38 for a split-metabolic space which is not hyperbolic (when 2 is not invertible).

Example 22. Let $(\mathcal{A}, *, \varpi)$ be an additive category with duality and let (P, φ) be a symmetric space. Then the sequence

$$P \xrightarrow{\alpha := \begin{pmatrix} 1 \\ 1 \end{pmatrix}} P \oplus P \xrightarrow{(-\varphi \ \varphi)} P^*$$

is split exact and the second morphism is equal to $\alpha^* \circ \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix}$. This proves that the symmetric space $(P, \varphi) \perp (P, -\varphi)$ is split metabolic in $(\mathcal{A}, *, \varpi)$ and hence in any exact category.

Remark 23 It is easy to prove that the only symmetric space structure on the zero object is metabolic, that any symmetric space isometric to a metabolic one is also metabolic, that the orthogonal sum of metabolic spaces is again metabolic and that the image (see Def. 11) of a metabolic space by a morphism of exact categories with duality is again metabolic. For the latter, the image of a lagrangian is a lagrangian of the image.

23

The Witt Group of an Exact Category with Duality

1.2.4

We only consider additive categories which are *essentially small*, i.e. whose class of isomorphism classes of objects is a set.

24 **Definition 24** Let $(\mathcal{A}, *, \varpi)$ be an additive category with duality (6). Denote by $\text{MW}(\mathcal{A}, *, \varpi)$ the set of *isometry classes of symmetric spaces* in $\mathcal{A}$. The orthogonal sum gives a structure of abelian monoid on $\text{MW}(\mathcal{A}, *, \varpi)$.

25 **Definition 25** Let $(\mathcal{E}, *, \varpi)$ be an *exact* category with duality (13). Let $\text{NW}(\mathcal{E}, *, \varpi)$ be the subset of $\text{MW}(\mathcal{E}, *, \varpi)$ of the classes of metabolic spaces. This defines a submonoid of $\text{MW}(\mathcal{E}, *, \varpi)$ by Rem. 23.

26 **Remark 26** Let $(M, +)$ be an abelian monoid (i.e. “a group without inverses”) and let N be a submonoid of M (i.e. $0 \in N$ and $N + N \subset N$). Consider the equivalence relation: for $m_1, m_2 \in M$, define $m_1 \sim m_2$ if there exists $n_1, n_2 \in N$ such that $m_1 + n_1 = m_2 + n_2$. Then the set of equivalence classes $M/\sim$ inherits a structure of abelian monoid via $[m] + [m'] := [m + m']$. It is denoted by M/N . Assume that for any element $m \in M$ there is an element $m' \in M$ such that $m + m' \in N$, then M/N is an abelian group with $-[m] = [m']$. It is then canonically isomorphic to the quotient of the Grothendieck group of M by the subgroup generated by N .

27 **Definition 27:** (*Knebusch*) Let $(\mathcal{E}, *, \varpi)$ be an exact category with duality. The *Witt group* of $\mathcal{E}$ is the quotient of symmetric spaces modulo metabolic spaces, i.e.

$$W(\mathcal{E}, *, \varpi) := \frac{\text{MW}(\mathcal{E}, *, \varpi)}{\text{NW}(\mathcal{E}, *, \varpi)}.$$

This is an abelian group. We denote by $[P, \varphi]$ the class of a symmetric space (P, φ) in $W(\mathcal{E})$, sometimes called *the Witt class of the symmetric space* (P, φ) . We have $-[P, \varphi] = [P, -\varphi]$ by Ex. 22 and the above Remark.

28 **Definition 28** Two symmetric spaces (P, φ) and (Q, ψ) which define the same Witt class, $[P, \varphi] = [Q, \psi]$, are called *Witt equivalent*. This amounts to the existence of metabolic spaces (N_1, θ_1) and (N_2, θ_2) and of an isometry $(P, \varphi) \perp (N_1, \theta_1) \simeq (Q, \psi) \perp (N_2, \theta_2)$.

29 **Remark 29** A Witt class $[P, \varphi] = 0$ is trivial in $W(\mathcal{E})$ if and only if there exists a split metabolic space (N, θ) with $(P, \varphi) \perp (N, \theta)$ metabolic, or equivalently, if and only if there exists a metabolic space (N, θ) with $(P, \varphi) \perp (N, \theta)$ split metabolic. This follows easily from the definition, by stabilizing with suitable symmetric spaces inspired by Ex. 22. However, we will see in Ex. 40 below that a symmetric space

(P, φ) with $[P, \varphi] = 0$ needs not be metabolic itself, even when $\mathcal{E}$ is a split exact category.

Remark 30 It is easy to check that $W(-)$ is a covariant functor from exact categories with duality to abelian groups, via the construction of Def. 11.

30

The Sublagrangian Reduction

1.2.5

We now explain why the Witt-equivalence relation (Def. 28) is of interest for symmetric spaces. Two spaces are Witt equivalent in particular if we can obtain one of them by “chopping off” from the other one some subspace on which the symmetric form is trivial, i.e. by chopping off a sublagrangian.

Let $(\mathcal{E}, *, \varpi)$ be an exact category with duality. Let (P, φ) be a symmetric space in $\mathcal{E}$ and let (L, α) be an admissible sublagrangian (Def. 17) of the space (P, φ) . Recall from (1.1) that we have a commutative diagram

$$\begin{array}{ccccc}
 & & & Q^* & \\
 & & & \downarrow \mu^* & \\
 L & \xrightarrow{\alpha} & P & \xrightarrow{\pi} & (L^\perp)^* \\
 \beta \downarrow & & \downarrow \varphi & & \downarrow \beta^* \\
 L^\perp & \xrightarrow{\pi^*} & P^* & \xrightarrow{\alpha^*} & L^* \\
 \mu \downarrow & & & & \\
 Q := L^\perp/L & & & &
 \end{array} \tag{1.3}$$

where we also introduce the cokernel Q in $\mathcal{E}$ of the admissible monomorphism β , displayed in the first column. The third column is the dual of the first.

Now consider the morphism $s := \pi \varphi^{-1} \pi^* : L^\perp \rightarrow (L^\perp)^*$. This is nothing but the form φ “restricted” to the orthogonal $L^\perp$ via the monomorphism $\varphi^{-1} \pi^* : L^\perp \hookrightarrow P$ from Def. 16. Observe that the morphism s is symmetric: $s^* = s$, that $s\beta = 0$ and that $\beta^* s = 0$. From this, we deduce easily (in two steps) the existence of a *unique* morphism

$$\psi : Q \rightarrow Q^* \quad \text{such that} \quad s = \mu^* \psi \mu. \tag{1.4}$$

One checks that ψ^* also satisfies equation (1.4). Therefore ψ is *symmetric*: $\psi = \psi^*$. Below, we shall get for free that ψ is an isomorphism, and hence defines a form on $Q = L^\perp/L$, but note that we could deduce it immediately from the Snake Lemma in some “ambient abelian category”.

31

Lemma 31 The following left hand square is a push-out:

$$\begin{array}{ccccc}
 L & \xrightarrow{\alpha} & P & \xrightarrow{\pi} & (L^\perp)^* \\
 \beta \downarrow & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \parallel \\
 L^\perp & \xrightarrow{\begin{pmatrix} \varphi^{-1} \pi^* \\ \mu \end{pmatrix}} & P \oplus Q & \xrightarrow{\begin{pmatrix} \pi & -\mu^* \psi \end{pmatrix}} & (L^\perp)^*
 \end{array} \quad (1.5)$$

and the diagram commutes and has admissible exact lines.

Proof One checks directly that the left-hand square satisfies the universal property of the push-out: use that if two test-morphisms $x : P \rightarrow Z$ and $y : L^\perp \rightarrow Z$ are such that $x\alpha = y\beta$ then the auxiliary morphism $w := y - x\varphi^{-1}\pi^* : L^\perp \rightarrow Z$ factors uniquely as $w = \overline{w}\mu$ because of $w\beta = 0$ and hence $(x \ \overline{w}) : P \oplus Q \rightarrow Z$ is the wanted morphism. It follows from the axioms of an exact category that the morphism $\gamma := \begin{pmatrix} \varphi^{-1}\pi^* \\ \mu \end{pmatrix} : L^\perp \rightarrow P \oplus Q$ is an admissible monomorphism. It is a general fact that the two monomorphisms α and γ must then have the same cokernel, and it is easy to prove (using that μ is an epimorphism) that $\text{Coker}(\gamma)$ is as in the second line of (1.5).

Comparing that second line of (1.5) to its own dual and using symmetry of φ and ψ , we get the following commutative diagram with exact lines:

$$\begin{array}{ccccc}
 L^\perp & \xrightarrow{\begin{pmatrix} \varphi^{-1} \pi^* \\ \mu \end{pmatrix}} & P \oplus Q & \xrightarrow{\begin{pmatrix} \pi & -\mu^* \psi \end{pmatrix}} & (L^\perp)^* \\
 \parallel & & \downarrow \begin{pmatrix} \varphi & 0 \\ 0 & -\psi \end{pmatrix} & & \parallel \\
 L^\perp & \xrightarrow{\begin{pmatrix} \pi^* \\ -\psi \mu \end{pmatrix}} & P^* \oplus Q^* & \xrightarrow{\begin{pmatrix} \pi \varphi^{-1} & \mu^* \end{pmatrix}} & (L^\perp)^*
 \end{array}$$

This proves two things. First $\begin{pmatrix} \varphi & 0 \\ 0 & -\psi \end{pmatrix}$ is an isomorphism and hence ψ is an isomorphism, i.e. (Q, ψ) is a symmetric space, as announced. Secondly, our monomorphism $\gamma : L^\perp \rightarrow P \oplus Q$ is a lagrangian of the space $(P, \varphi) \perp (Q, -\psi)$. This means that the space $(P, \varphi) \perp (Q, -\psi)$ is metabolic, i.e. $[P, \varphi] = [Q, \psi]$ in the Witt group. So we have proven the following folklore result:

32

Theorem 32 Let $(\mathcal{E}, *, \overline{\omega})$ be an exact category with duality. Let (P, φ) be a symmetric space in $\mathcal{E}$ and let (L, α) be an admissible sublagrangian of the space (P, φ) . Consider the orthogonal $L^\perp$ and the quotient $L^\perp/L$. Then there is a unique form ψ on $L^\perp/L$ which is induced by the restriction of φ to $L^\perp$. Moreover, the symmetric space $(L^\perp/L, \psi)$ is Witt equivalent to (P, φ) .

Remark 33 A sort of converse holds: any two Witt equivalent symmetric spaces can be obtained from a common symmetric space by the above sublagrangian reduction, with respect to two different sublagrangians. This is obvious since a metabolic space with lagrangian L reduces to zero: $L^\perp/L = 0$.

33

Remark 34 Observe that $L^\perp/L$ is a *subquotient* of P and hence should be thought of as “smaller” than P . If $L^\perp/L$ still possesses an admissible sublagrangian, we can chop it out again. And so on. If the category $\mathcal{E}$ is reasonable, this process ends with a space possessing no admissible sublagrangian – this could be called (*admissibly*) *anisotropic*. Even then, such an admissibly anisotropic symmetric space needs not be unique up to isometry in the Witt class of the symmetric space (P, φ) that we start with. See more in Rem. 56.

34

Usual Witt Groups: Examples and Motivations

1.3

Still in a very anti-chronological order, we specialize the categorical definitions of the previous section to more classical examples.

Schemes

1.3.1

The origin is Knebusch [41]. The affine case is older: see the elegant Milnor–Husemoller [50]. A modern reference is Knus [42, Chap. VIII].

Let X be a scheme and let VB_X be the category of locally free coherent $\mathcal{O}_X$ -modules (i.e. vector bundles). Let $\mathcal{L}$ be a line bundle over X . One defines a duality $*$: $\mathrm{VB}_X \rightarrow \mathrm{VB}_X$ by $E^* := \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L}$, which is the *usual duality twisted by the line bundle $\mathcal{L}$* . One defines the natural identification $\varpi : E \xrightarrow{\sim} E^{**}$ in the usual way. For $\mathcal{L} = \mathcal{O}_X$, this E^* is of course the usual dual and ϖ is locally given by mapping an element e to the evaluation at e . The triple $(\mathrm{VB}_X, *, \varpi)$ is an exact category with duality in the sense of Def. 13. We can thus apply Def. 27 to get Knebusch’s original one [41]:

Definition 35 With the above notations, the *usual Witt group of a scheme X* with values in the line bundle $\mathcal{L}$ is the Witt group (Def. 27):

35

$$W(X, \mathcal{L}) := W(\mathrm{VB}_X, *, \varpi).$$

The special case $\mathcal{L} = \mathcal{O}_X$ is the usual usual Witt group $W(X)$.

36 Remark 36 Let R be a commutative ring. We define $W(R)$ as $W(\operatorname{Spec}(R))$; this common convention of dropping the “ $\operatorname{Spec}(-)$ ” applies everywhere below. The category $\mathbf{VB}_R$ is simply the category of finitely generated projective R -modules, which is a split exact category. So, here, metabolic spaces are the split-metabolic ones. If $\frac{1}{2} \in R$, these are simply the hyperbolic spaces, yielding the maybe better known definition of the Witt group of a commutative ring.

37 Remark 37 When $\mathcal{L} = \mathcal{O}_X$, then the group $W(X)$ is indeed a ring, with product induced by the tensor product: $(E, \varphi) \cdot (F, \psi) = (E \otimes_{\mathcal{O}_X} F, \varphi \otimes_{\mathcal{O}_X} \psi)$.

We now produce examples proving “strictness” of the trivial implications:
hyperbolic $\Rightarrow$ split metabolic $\Rightarrow$ metabolic $\Rightarrow$ trivial in the Witt group.

Example 38. Over the ring $R = \mathbb{Z}$, the symmetric space $(R^2, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix})$ is split metabolic but not hyperbolic (the hyperbolic space $H(R) = (R^2, \psi)$ has the property that $\psi(v, v) \in 2R$ for any $v \in R^2$ but the above form represents 1).

Example 39. An example of a metabolic space which is not split-metabolic cannot exist in the affine case. Choose an exact sequence $\mathcal{O}_X \hookrightarrow P \twoheadrightarrow \mathcal{O}_X$, say on an elliptic curve X , with P indecomposable. Then $\wedge^2 P$ is trivial and hence P has a structure of *skew-symmetric* space. It is metabolic with the (left) $\mathcal{O}_X$ as lagrangian but cannot be split metabolic since P itself is indecomposable as module. An example of a *symmetric* such space can be found in Knus–Ojanguren [44, last remark]. They produce a metabolic symmetric space, which is not split metabolic, as can be seen on its Clifford algebra.

Example 40. (*Ojanguren*) Let $A := \mathbb{R}[X, Y, Z]/X^2 + Y^2 + Z^2 - 1$ and let P be the indecomposable projective A -module of rank 2 corresponding to the tangent space of the sphere. The rank 4 projective module $E := \operatorname{End}_A(P)$ is equipped with the symmetric bilinear form $\varphi : E \xrightarrow{\sim} E^*$, where $\varphi(f)(g) = \frac{1}{2}(q(f + g) - q(f) - q(g))$, for any $f, g \in E$, is the form associated to the quadratic form $q(f) := \det(f)$. Then $[E, \varphi] = 0$ in $W(A)$ but the symmetric space (E, φ) is not metabolic.

If Q is the field of fractions of A , it is easy to write $\varphi \otimes_A Q$ and to check it is hyperbolic. Hence the class $[E, \varphi]$ belongs to the kernel of the homomorphism $W(A) \rightarrow W(Q)$, which is known to be injective (A is regular and $\dim(A) \leq 3$, see e.g. Thm. 95 below). Hence $[E, \varphi] = 0$ and (E, φ) is stably metabolic. To see that this symmetric space is not metabolic, assume the contrary. Here, a metabolic space is hyperbolic (affine case and $\frac{1}{2} \in A$). So, we would have $(E, \varphi) \simeq H(M)$ for some projective module M of rank 2. Using the hyperbolic form on $H(M)$ and the

presence of $\text{id}_P \in E$ with $q(\text{id}) = \det(\text{id}) = 1$, one can find an element $f \in E$ such that $q(f) = -1$, that is an endomorphism $f : P \rightarrow P$ with determinant -1 . Such an endomorphism cannot exist since it would yield a fibrewise decomposition of P into two eigenspaces, and hence would guarantee the unlikely triviality of the tangent space of the sphere.

(By the way, one can show that $W(A) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, see [42, § VIII.6.2].)

Remark 41 The simplest schemes are the points $X = \text{Spec}(k)$ for k a field, or k a local ring, containing $\frac{1}{2}$. In these cases, the Witt group allows a complete classification of quadratic forms – see Sect. 1.3.4. With this in mind, several people got interested in the map $W(A) \rightarrow W(Q)$ for A a domain with field of fractions Q . This is commented upon in Sect. 1.6.2.

41

Example 42. For elementary examples of Witt groups of affine schemes (i.e. commutative rings) like for $W(\mathbb{Z}) = W(\mathbb{R}) = \mathbb{Z}$, or, q being a power of a prime, for $W(\mathbb{F}_q) = \mathbb{Z}/2$ when q is even, $W(\mathbb{F}_q) = \mathbb{Z}/2[\varepsilon]/(\varepsilon^2)$ or $\mathbb{Z}/4$ when $q \equiv 1$ or $3 \pmod{4}$ respectively, or for $W(\mathbb{Q}) = W(\mathbb{Z}) \oplus \bigoplus_{p \in \mathbb{P}} W(\mathbb{F}_p)$, or for Witt groups of other fields, or of Dedekind domains, and so on, the reader is referred to the already mentioned [50] or to Scharlau [71].

Example 43. As a special case of Karoubi's Thm. 53, we see that for any commutative ring R containing $\frac{1}{2}$, for instance $R = k$ a field of odd characteristic, the Witt group of the affine space over R is canonically isomorphic to the one of R :

$$W(\mathbb{A}_R^n) = W(R) .$$

(See also Thm. 86 below.) The case of the projective space over a field is a celebrated result of Arason [1] (compare Walter's Thm. 104 below):

Theorem 44: (*Arason*) Let k be a field of characteristic not 2 and let $n \geq 1$. Then $W(\mathbb{P}_k^n) = W(k)$.

44

This has been extended to Brauer–Severi varieties:

Theorem 45: (*Pumplün*) Let k be a field of characteristic not 2. Let A be a central simple k -algebra and X the associated Brauer–Severi variety.

45

- (i) The natural morphism $W(k) \rightarrow W(X)$ is surjective.
- (ii) When A is of odd index, $W(k) \rightarrow W(X)$ is injective.

See [67], where further references and partial results for twisted dualities are to be found. Injectivity fails for algebras with even index, in general.

Example 46. Here is an example of the possible use of Witt groups in algebraic geometry. The problem of Lüroth, to decide whether a unirational variety is rational, is known to have a positive answer for curves over arbitrary fields (Lüroth), for complex surfaces (Castelnuovo), and to fail in general. Simple counterexamples, established by means of Witt groups, are given in Ojanguren [57], where an overview can be found. See also [20, Appendix].

Here is another connection between Witt groups and algebraic geometry.

47

Theorem 47: (*Parimala*) Let R be a regular finitely generated $\mathbb{R}$ - or $\mathbb{C}$ -algebra of Krull dimension 3. Then the Witt group $W(R)$ is finitely generated if and only if the Chow group $\mathrm{CH}^2(R)/2$ is finite.

See [64, Thm. 3.1] where examples are given; compare also Totaro's Thm. 99 below. This important paper of Parimala significantly contributed to the study of connections between Witt groups and étale cohomology. Abundant work resulted from this, among which the reader might want to consider Colliot-Thélène – Parimala [21], which relates to the subject of real connected components discussed below in Sect. 1.3.2. In this direction, see also Scheiderer [73].

We end this Section by a short guide to the literature for a selection of results in Krull dimension 1 and 2. The reader will find additional information in Knus [42, § VIII.2]. For Witt groups of fields ($\dim = 0$), see Sect. 1.3.4.

In dimension 1, we have:

Dedekind domains: If D is a Dedekind domain with field of fractions Q , there is an exact sequence: $0 \rightarrow W(D) \rightarrow W(Q) \xrightarrow{\partial} \bigoplus_p W(D/p)$ where the sum runs over the non-zero primes p of D and where ∂ is the classical *second residue homomorphism*, which depends on choices of local parameters. See [50, Chap. IV].

Elliptic curves: There is a series of articles by Arason, Elman and Jacob, describing the Witt group of an elliptic curve with generators and relations. See Arason–Elman–Jacob [2] for an overview and for further references. See also the work of Parimala–Sujatha [65].

Real curves: For curves over $\mathbb{R}$, the story stretches from the original work of Knebusch [41, § V.4] to the most recent work of Monnier [53]. Note that the latter gives a systematic overview including singular curves, which were already considered in Dietel [23]. See also Rem. 49 below.

In dimension 2, we have:

Complex surfaces: Fernández-Carmena [24, Thm. 3.4] proved among other things the following result: if X is a smooth complex quasi-projective surface then $W(X) \simeq$

$(\mathbb{Z}/2)^{1+s+q+b}$ where s , q and b are the number of copies of $\mathbb{Z}/2$ in $\mathcal{O}_X(X)^*/(\mathcal{O}_X(X)^*)^2$, in ${}_2\text{Pic}(X)$ and in ${}_2\text{Br}(X)$ respectively.

Real surfaces: The main results are due to Sujatha [75, Thms. 3.1 and 3.2] and look as follows. See also Sujatha - van Hamel [76] for further developments.

Theorem 48: (*Sujatha*) Let X be a smooth projective and integral surface over $\mathbb{R}$. 48

(i) Assume that $X(\mathbb{R}) \neq \emptyset$ and has s real connected components. Then

$$W(X) \simeq \mathbb{Z}^s \oplus (\mathbb{Z}/2)^m \oplus (\mathbb{Z}/4)^n.$$

(ii) Assume that $X(\mathbb{R}) = \emptyset$. Then

$$W(X) \simeq (\mathbb{Z}/2)^m \oplus (\mathbb{Z}/4)^n \oplus (\mathbb{Z}/8)^t.$$

Moreover, the integers m , n and t can be described in terms of 2-torsion of the Picard and Brauer groups of X , and of the level of $\mathbb{R}(X)$ in case (ii).

Remark 49 For an algebraic variety X over $\mathbb{R}$, the formulas describing $W(X)$, which can be found in the above literature, basically always look as follows: $W(X) = \mathbb{Z}^s \oplus (2\text{-primary torsion part})$, where s is the number of real connected components of $X(\mathbb{R})$ and where cohomological invariants are used to control the 2-primary torsion part. See Mahé's result 51. 49

Remark 50 Further results on Witt groups of schemes have been obtained by means of triangular Witt groups and are presented in Sect. 1.6. Even in low dimension, say up to 3, the situation is quite clarified by the corollaries of Thm. 91 below. 50

Motivation From Real Algebraic Geometry

1.3.2

There is a long lasting love-story between quadratic forms and real algebraic geometry, originating in their common passion for sums of squares. For a survey, see [18, Chap. 15]; early ideas are again in Knebusch [41, Chap. V].

A nice application of Witt groups to real geometry is the following problem, stated by Knebusch. Let X be an algebraic variety over $\mathbb{R}$. Consider the set of real points $X(\mathbb{R})$ with the real topology. Then its connected components are conjectured to be in one-to-one correspondence with *signatures* of $W(X)$, that are ring homomorphisms $W(X) \rightarrow \mathbb{Z}$. Basically, the construction goes as follows. Pick a closed point x in $X(\mathbb{R})$; its residue field $\mathbb{R}(x)$ is $\mathbb{R}$ and hence localization produces

a homomorphism $W(X) \rightarrow W(\mathbb{R}(x)) = \mathbb{Z}$ and one can show that this homomorphism only depends on the connected component C_x of $X(\mathbb{R})$ where x was chosen. In this way, one obtains the following pairing, where $CC(X(\mathbb{R}))$ denotes the above set of real connected components

$$\begin{aligned} \lambda : CC(X(\mathbb{R})) \times W(X) &\rightarrow \mathbb{Z} \\ C_x, \varphi &\mapsto \varphi(x) \in W(\mathbb{R}(x)) = \mathbb{Z}. \end{aligned}$$

This can be read either as a map $\Lambda : CC(X(\mathbb{R})) \rightarrow \text{Hom}_{\text{rings}}(W(X), \mathbb{Z})$ or as a ring homomorphism $\Lambda^* : W(X) \rightarrow \text{Cont}(X(\mathbb{R}), \mathbb{Z})$, the *total signature map*.

51 Theorem 51: (*Mahé*) Let $X = \text{Spec}(A)$ be an affine real algebraic variety. Then the map Λ is a bijection between the set of connected components of the real spectrum $\text{Spec}_r(A)$ and the set of signatures $W(A) \rightarrow \mathbb{Z}$.

See [47, Cor.3.3] for the above and see Houdebine–Mahé [31] for the extension to projective varieties. In fact, a key ingredient in the proof consists in showing that the cokernel of the total signature $\Lambda^* : W(X) \rightarrow \text{Cont}(X(\mathbb{R}), \mathbb{Z})$ is a 2-primary torsion group. Knowing this, it is interesting to try understanding the precise exponent of this 2-primary torsion group. Such exponents are obtained in another work of Mahé [48], and more recently by Monnier [52].

1.3.3 Rings with Involution, Polynomials and Laurent Rings

52 Definition 52 A *ring with involution* is a pair (R, σ) consisting of an associative ring R and an *involution* $\sigma : R \rightarrow R$, i.e. an additive homomorphism such that $\sigma(r \cdot s) = \sigma(s) \cdot \sigma(r)$, $\sigma(1) = 1$ and $\sigma^2 = \text{id}_R$.

For a left R -module M we can define its dual $M^* = \text{Hom}_R(M, R)$, which is naturally a right R -module via $(f \cdot r)(x) := f(x) \cdot r$ for all $x \in M, r \in R$ and $f \in M^*$. It inherits a left R -module structure via $r \cdot f := f \cdot \sigma(r)$, that is $(r \cdot f)(x) = f(x) \cdot \sigma(r)$. There is a natural R -homomorphism $\varpi_M : M \rightarrow M^{**}$ given by $(\varpi_M(m))(f) := \sigma(f(m))$. When $P \in R\text{-Proj}$ is a finitely generated projective left R -module, this homomorphism ϖ_P is an isomorphism. Hence the category $(R\text{-Proj}, *, \varpi)$ is an additive category with duality. The same holds for $(R\text{-Proj}, *, \varepsilon \cdot \varpi)$ for any central unit $\varepsilon \in R^\times$ such that $\sigma(\varepsilon) \cdot \varepsilon = 1$, like for instance $\varepsilon = -1$. The Witt group obtained this way is usually denoted

$$W^\varepsilon(R) := W(R\text{-Proj}, *, \varepsilon \cdot \varpi)$$

and is called the Witt group of ε -hermitian bilinear forms over R .

This part of Witt group theory is of course quite important, and the reader is referred to the very complete Knus [42] for more information. We mention here two big K -theory like results.

Theorem 53: (*Karoubi*) Let R be a ring with involution containing $\frac{1}{2}$. Then $W^\varepsilon(R[T]) = W^\varepsilon(R)$, where $R[T]$ has the obvious involution fixing T .

53

See Karoubi [37, Part II]. An elementary proof is given in Ojanguren–Panin [59, Thm. 3.1], who also prove a general theorem for the Witt group of the ring of Laurent polynomials, giving in particular:

Theorem 54: (*Ranicki*) Let R be a regular ring with involution containing $\frac{1}{2}$. Then the following homomorphism

54

$$W^\varepsilon(R) \oplus W^\varepsilon(R) \rightarrow W^\varepsilon(R[T, T^{-1}]) \quad (\alpha, \beta) \mapsto \alpha + \beta \cdot \langle T \rangle$$

is an isomorphism, where the involution on $R[T, T^{-1}]$ fixes the variable T .

See Ranicki [69] where regularity is not required (neither is it in [59]) and where suitable Nil-groups are considered. Compare Thm. 103 below.

Semi-local Rings and Fields

1.3.4

Recall that a commutative ring R is *semi-local* if it has only finitely many maximal ideals. Local rings and fields are semi-local.

Theorem 55: (*Witt Cancellation*) Let R be a commutative semi-local ring in which 2 is invertible. If (P_1, φ_1) , (P_2, φ_2) and (Q, ψ) are symmetric spaces such that $(P_1, \varphi_1) \perp (Q, \psi)$ is isometric to $(P_2, \varphi_2) \perp (Q, \psi)$, then (P_1, φ_1) and (P_2, φ_2) are isometric.

55

This was first proven for fields by Witt [86]. This result and much more information on these cancellation questions can be found in Knus [42, Chap. VI].

Remark 56 The above result is *wrong for non-commutative semi-local rings*, i.e. rings R such that $R/\text{rad}(R)$ is semi-simple. Keller [39] gives a very explicit counterexample, constructed as follows: let k be a field of odd characteristic; let A_0 be the semi-localization of $k[X, Y]/(X^2 + Y^2 - 1)$ at the maximal ideals $\xi = (0, 1)$ and $\eta = (0, -1)$; let $B \subset A_0$ be the subring of those $f \in A_0$ such that $f(\xi) = f(\eta)$; finally define the non-commutative semi-local ring to be $A = \left\{ \begin{pmatrix} b & r \\ s & a_0 \end{pmatrix} \mid b \in B, a_0 \in A_0, r, s \in \text{rad}(A_0) \right\}$ with transposition as involution. Then there are two symmetric forms on the same projective right A -module $N := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot A$ which are not isometric but become isometric after adding the rank one space $(A, \langle 1 \rangle)$. See more in [39] or in [42, VI.5.1].

56

Remark 57 Let R be a commutative semi-local ring containing $\frac{1}{2}$, with $\text{Spec}(R)$ connected (otherwise do everything component by component). Then any finitely

57

generated projective R -module if free. Using the sublagrangian reduction 32 and the above Witt cancellation, we know that any symmetric space (P, φ) over R can be written up to isometry as

$$(P, \varphi) \simeq (P_0, \varphi_0) \perp H(R^m)$$

for $m \in \mathbb{N}$ and for (P_0, φ_0) without admissible sublagrangian – let us say that the space (P_0, φ_0) is (admissibly) *anisotropic* – and we know that the number m and the isometry class of (P_0, φ_0) are *unique*. Moreover, the spaces (P, φ) and (P_0, φ_0) are Witt equivalent and by Witt cancellation again, there is exactly one (admissibly) anisotropic space in one Witt class. This establishes the following result, the original motivation for studying Witt groups:

58 Corollary 58 Let R be a commutative semi-local ring containing $\frac{1}{2}$ (for instance a field of characteristic not 2). The determination of the Witt group $W(R)$ allows the classification up to isometry of all quadratic forms over R .

59 Remark 59 Reading the above Corollary backwards, we avoid commenting the huge literature on Witt groups of fields, by referring the reader to the even bigger literature on quadratic forms at large. See in particular Lam [45], Scharlau [71] and Serre [74]. For instance, there exist so-called *structure theorems* for Witt groups of fields, due to Witt, Pfister, Scharlau and others, and revisited in Lewis [46], where further references can also be found. In fact, several results classically known for fields extend to (commutative) semi-local rings. See [41, Chap. II] again or Baeza [3].

1.4

A Glimpse at Other Theories

Our Chapter focusses on the internal theory of Witt groups but the reader might be interested in knowing which are the neighbor theories, more or less directly related to Witt groups. We give here a rapid overview with references.

Quadratic forms: When 2 is not a unit one must distinguish quadratic forms from the symmetric forms we mainly considered. See the classical references already given in Remark 59. The Witt group of quadratic forms can also be defined, see for instance Milnor–Husemoller [50, App. 1]. See also the recent Baeza [4] for quadratic forms over fields of characteristic two. The reader looking for a systematic treatise including quadratic forms and their connections with algebraic groups should consider *the* book [43].

Motivic approach: Techniques from algebraic geometry, Chow groups and motives, have been used to study quadratic forms over fields, by means of the corre-

sponding quadrics. See the work of Izhboldin, Kahn, Karpenko, Merkurjev, Rost, Sujatha, Vishik and others, which is still in development and for which we only give here a sample of references: [32–35, 38] among many more.

Topological Witt groups: Let M be a smooth paracompact manifold and let $R = C^\infty(M, \mathbb{R})$ be the ring of smooth real-valued functions on M . It is legitimate, having Swan–Serre’s equivalence in mind, to wonder if the Witt group of such a ring R can be interpreted in terms of the manifold M . The answer is that $W(R)$ is isomorphic to $KO(M)$ the real (topological) K^0 of M , that is the Grothendieck group of isomorphism classes of real vector bundles over M . This is due to Lusztig see [50, § V.2]. This should not be mistaken with the Witt group of real algebraic varieties discussed above.

Cohomological invariants: We already mentioned briefly in Rem. 49 the importance of cohomological invariants in the part dedicated to real algebraic geometry. For quadratic forms over fields, the relation between Witt groups and Galois cohomology groups is the essence of the famous Milnor Conjecture [49], now proven by Voevodsky, see e.g. Orlov, Vishik and Voevodsky [60]. See also Pfister’s historical survey [66].

For a scheme X , there is a homomorphism $\text{rk} : W(X) \rightarrow \text{Cont}(X, \mathbb{Z}/2)$, the *reduced rank*, to the continuous (hence locally constant) functions from X to $\mathbb{Z}/2$, which sends a symmetric space to its rank modulo 2 (metabolic spaces have even rank). The *fundamental ideal* $I(X)$ of $W(X)$ is the kernel of this homomorphism.

Following [42, § VIII.1], we denote by $\text{Disc}(X)$ the abelian group of isometry classes of symmetric line bundles, with $\otimes$ as product. We denote by $\delta : I(X) \rightarrow \text{Disc}(X)$ the *signed discriminant*, which sends the class of an even-rank symmetric space (E, φ) of rank $2m$ to the symmetric bundle $\langle (-1)^m \rangle \cdot (\wedge^m E, \wedge^m \varphi)$.

One can define further the *Witt invariant*, which takes values in the Brauer group, see Knus [42, § IV.8] and which is defined by means of Clifford algebras. See also Barge–Ojanguren [15] for the lift of the latter to K -theory. Higher invariants are not known in this framework. One can try to define general invariants into subquotients of K -theory groups, for arbitrary exact categories or in more general frameworks. This was started by Szyjewski in [77] and remains “in progress” for higher ones.

Grothendieck–Witt groups: One often considers also GW the Grothendieck–Witt group, which is defined by the same generators as the Witt group but with less relations; namely if a space (P, φ) is metabolic with lagrangian L then one sets the relation $(P, \varphi) - H(L) = 0$ in the Grothendieck–Witt group, instead of the relation $(P, \varphi) = 0 (= H(L))$ in the Witt group. There is a group homomorphism $K_0 \rightarrow \text{GW}$, induced by the hyperbolic functor $L \mapsto H(L)$ and whose cokernel is the Witt group. We intentionally do not specify what sort of categories we define $\text{GW}(-)$ for, because it applies whenever the Witt group is defined. For instance, in the triangular framework of the next two sections, it is also possible to define Grothendieck–Witt groups, as recently done by Walter [83].

Hermitian K -theory, Karoubi's Witt groups: The above Grothendieck -Witt group is equal to K_0^h , the 0-th group of Karoubi's hermitian K -theory. For a recent reference, see Hornbostel [29], where hermitian K -theory is extended to exact categories. There are higher and lower hermitian K -theory groups K_n^h and natural homomorphisms $Hyp : K_n \rightarrow K_n^h$ from K -theory towards hermitian K -theory, which fit in Karoubi's "Fundamental Theorem" long exact sequence. Karoubi's Witt groups are defined in a mixed way, namely as the cokernels of these homomorphisms $Hyp : K_n \rightarrow K_n^h$. For regular rings, these groups coincide with the triangular Witt groups, see more in [36] or in [30].

L -theory: We refer the reader to Williams [85] in this Handbook or to Ranicki [70] for the definition of the quadratic and symmetric L -theory groups of Wall–Mischenko–Ranicki and for further references. We shortly compare them to the triangular Witt groups to come. First, like triangular Witt groups, L -groups are algebraic, that is, their definition does not require the above hermitian K -theory. Secondly, *unlike* triangular Witt groups, L -groups also work when 2 is not assumed invertible and this is of central importance in surgery theory. Unfortunately, it does not seem unfair to say that the definition of these L -groups is rather *involved* and requires some heavy use of complexes.

The advantage of triangular Witt theory is two-fold: first, it applies to non-split exact categories and hence to schemes, and secondly, by its very definition, it factors via triangulated categories, freeing us from the burden of complexes.

Note that both theories coincide over split exact categories under the assumption that 2 is invertible and that even in the non-split case, the derived Witt groups of an exact category have a formation-like presentation by generators and relations (see Walter [83]). In the present stage of the author's understanding, the *triangular* theory of Witt groups, strictly speaking, does not exist without the "dividing by 2" assumption. Nevertheless, even when 2 is not assumed invertible, there are good reasons to believe that a sort of " L -theory of non-necessarily-split exact categories" should exist, unfolding the higher homotopies in a Waldhausen-category framework, using weak-equivalences, cofibrations and so on, but most probably renouncing the elegant simplicity of the triangular language...

1.5

Triangular Witt Groups: General Theory

The second half of this Chapter is dedicated to triangular Witt groups, i.e. Witt groups of triangulated categories with duality. The style is quite direct and a reader needing a more gentle introduction is referred to [10].

1.5.1

Basic Notions and Facts

All definitions and results of this section are to be found in [6].

For the definition of a *triangulated category* we refer to Verdier's original source [81], to Weibel [84, Chap. 10], or to [6, § 1], where the reader can find

Axiom (TR 4⁺), the *enriched version of the Octahedron Axiom*, due to Beilinson, Bernstein and Deligne [17]. All known triangulated categories and all triangulated categories considered below satisfy this enriched axiom. Note that a triangulation is an additional structure, not intrinsic, on an additive category $\mathcal{K}$, which consists of a *translation* or *suspension* $T : \mathcal{K} \rightarrow \mathcal{K}$ plus a collection of *distinguished triangles* satisfying some axioms. The fundamental idea is to replace admissible exact sequences by distinguished triangles.

Definition 60 Let $\delta = 1$ or -1 . A *triangulated category with δ -duality* is an additive category with duality $(\mathcal{K}, \#, \varpi)$ in the sense of Def. 6, where $\mathcal{K}$ is moreover triangulated, and satisfying the following conditions:

(a) The duality $\#$ is a δ -exact functor $\mathcal{K}^{\text{op}} \rightarrow \mathcal{K}$, which means that

$$T \circ \# \cong \# \circ T^{-1}$$

(we consider this isomorphism as an equality) and, more important, that for any distinguished triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ in $\mathcal{K}$, the following triangle is exact:

$$C^{\#} \xrightarrow{v^{\#}} B^{\#} \xrightarrow{u^{\#}} A^{\#} \xrightarrow{\delta \cdot T(w^{\#})} T(w^{\#}) .$$

(b) The identification ϖ between the identity and the double dual is compatible with the triangulation, which means $\varpi_{T(M)} = T(\varpi_M)$ for all $M \in \mathcal{K}$.

Note that all “additive notions” presented in Sect. 1.2.1 also make sense in this framework, as for instance the monoid $\text{MW}(\mathcal{K}, \#, \varpi)$ of symmetric spaces (Def. 24). We now explain how the other classical notions which depended on the exact category structure (absent here) can be replaced.

Definition 61 A symmetric space (P, φ) is called *neutral* (or *metabolic* if no confusion occurs) when it admits a *lagrangian*, i.e. a triple (L, α, β) such that $\alpha : L \rightarrow P$ is a morphism, such that the following triangle is distinguished

$$L \xrightarrow{\alpha} P \xrightarrow{\alpha^{\#} \varphi} L^{\#} \xrightarrow{\beta} T(L)$$

and such that $\beta : L^{\#} \rightarrow T(L)$ is δ -symmetric, which here means:

$$\delta \cdot T(\beta^{\#}) = \varpi_{T(L)} \circ \beta .$$

In short, the symmetric short exact sequence $L \rightarrowtail P \twoheadrightarrow L^*$ is replaced by the above symmetric distinguished triangle. Note that we still have $\alpha^{\#} \varphi \alpha = 0$, that is (L, α) is a sublagrangian. (By the way, there is a triangular partial analogue of the sublagrangian reduction, called the *sublagrangian construction*, which can be found in [6, § 4] or, in a simpler case, in [5, § 3].)

62 **Definition 62** Let $(\mathcal{K}, \#, \varpi)$ be a triangulated category with δ -duality. As before, its *Witt group* is the following quotient of abelian monoids:

$$W(\mathcal{K}, \#, \varpi) := \frac{MW(\mathcal{K}, \#, \varpi)}{NW(\mathcal{K}, \#, \varpi)},$$

where $NW(\mathcal{K}, \#, \varpi)$ is the submonoid of $MW(\mathcal{K}, \#, \varpi)$ consisting of the classes of neutral spaces.

63 **Definition 63** Let $(\mathcal{K}, \#, \varpi)$ be a triangulated category with δ -duality. Let $n \in \mathbb{Z}$ arbitrary. Then the square of the functor $T^n \circ \# : \mathcal{K}^{\text{op}} \rightarrow \mathcal{K}$ is again isomorphic to the identity, but this functor $T^n \#$ is only δ_n -exact, where $\delta_n := (-1)^n \cdot \delta$. We define the *n-th shifted duality on $\mathcal{K}$ to be*

$$T^n((\mathcal{K}, \#, \varpi)) := (\mathcal{K}, T^n \circ \#, \varepsilon_n \cdot \varpi),$$

where $\varepsilon_n := (-1)^{\frac{n(n+1)}{2}} \cdot \delta^n$.

It is easy to check that $T^m(T^n(\mathcal{K}, \#, \varpi)) = T^{m+n}(\mathcal{K}, \#, \varpi)$ for any $m, n \in \mathbb{Z}$, keeping in mind that the δ_n -exactness of $T^n \#$ is given by $\delta_n = (-1)^n \cdot \delta$.

64 **Definition 64** The *n-th shifted Witt group* of $(\mathcal{K}, \#, \varpi)$, or simply of $\mathcal{K}$, is defined as the Witt group of $T^n(\mathcal{K}, \#, \varpi)$:

$$W^n(\mathcal{K}, \#, \varpi) := W(T^n(\mathcal{K}, \#, \varpi)).$$

65 **Proposition 65** For any $n \in \mathbb{Z}$ we have a natural isomorphism, induced by $T : \mathcal{K} \rightarrow \mathcal{K}$, between $W^n(\mathcal{K}, \#, \varpi)$ and $W^{n+2}(\mathcal{K}, \#, -\varpi)$. In particular, we have the 4-periodicity: $W^n(\mathcal{K}, \#, \varpi) \cong W^{n+4}(\mathcal{K}, \#, \varpi)$.

See [6, Prop. 2.14]. In fact, these isomorphisms are induced by equivalences of the underlying triangulated categories with duality.

Example 66. Assume that $(\mathcal{K}, \#, \varpi)$ is a triangulated category with exact duality (that is $\delta = +1$). Then so is $T^2(\mathcal{K}, \#, \varpi)$ and the latter is isomorphic to $(\mathcal{K}, \#, -\varpi)$. The other two $T^1(\mathcal{K}, \#, \varpi)$ and $T^3(\mathcal{K}, \#, \varpi)$ are both categories with skew-exact duality (that is $\delta_1 = \delta_3 = -1$), respectively isomorphic to $(\mathcal{K}, T\#, -\varpi)$ and $(\mathcal{K}, T\#, \varpi)$.

67 **Definition 67** A *morphism of triangulated categories with duality* (F, η) is a morphism of categories with duality (Def. 5) such that F is an exact functor, i.e. F

sends distinguished triangles to distinguished triangles. More precise definitions are available in [14] or in [27]. With this notion of morphism, all the groups $W^n(-)$ constructed above become functorial.

The following very useful result [6, Thm. 3.5] contrasts with the classical framework (compare Ex. 40):

Theorem 68 Let $\mathcal{K}$ be a triangulated category with duality containing $\frac{1}{2}$ (see 69). Then a symmetric space (P, φ) which is Witt-equivalent to zero, i.e. such that $[P, \varphi] = 0 \in W(\mathcal{K})$, is necessarily neutral.

68

Agreement and Localization

1.5.2

Definition 69 Let $\mathcal{A}$ be an additive category (e.g. a triangulated category). We say that “ $\frac{1}{2} \in \mathcal{A}$ ” when the abelian groups $\text{Hom}_{\mathcal{A}}(M, N)$ are uniquely 2-divisible for all objects $M, N \in \mathcal{A}$, i.e. if $\mathcal{A}$ is a $\mathbb{Z}[\frac{1}{2}]$ -category.

69

The main result connecting usual Witt groups to the triangular Witt groups is the following.

Theorem 70 Let $(\mathcal{E}, *, \varpi)$ be an exact category with duality such that $\frac{1}{2} \in \mathcal{E}$. Equip the derived category $D^b(\mathcal{E})$ with the duality $\#$ derived from $*$. Then the obvious functor $\mathcal{E} \rightarrow D^b(\mathcal{E})$, sending everything in degree 0, induces an isomorphism

70

$$W(\mathcal{E}, *, \varpi) \xrightarrow{\cong} W(D^b(\mathcal{E}), \#, \varpi) .$$

This is the main result of [7, Thm. 4.3], under the mild assumption that $\mathcal{E}$ is semi-saturated. The general case is deduced from this in [14, after Thm. 1.4].

Example 71. The above Theorem provides us with lots of (classical) examples: all those described in Sect. 1.3, the most important being schemes. So, if X is a scheme “containing $\frac{1}{2}$ ” (i.e. a scheme over $\mathbb{Z}[\frac{1}{2}]$) and if the bounded derived category $\mathcal{K}(X) := D^b(\text{VB}_X)$ of vector bundles over X is equipped with the derived duality twisted by a line bundle $\mathcal{L}$ (e.g. $\mathcal{L} = \mathcal{O}_X$), then $W^0(\mathcal{K}(X))$ is the usual Witt group of Knebusch $W(X, \mathcal{L})$ and similarly $W^2(\mathcal{K}(X))$ is the usual Witt group of skew-symmetric forms $W^-(X, \mathcal{L})$. The Witt groups

$$W^n(D^b(\text{VB}_X))$$

are often called the n -th *derived Witt groups* of X . They are functorial (contravariant) for any morphism of scheme. Other triangulated categories with duality can be associated to a scheme X , see 78 below.

72

Remark 72 Let us stress that the definitions of Sect. 1.5.1 also make sense when 2 is not assumed invertible. The $\frac{1}{2}$ -assumption is used to prove results, like Thm. 70 for instance. As already mentioned, in the case of the derived category $(D^b(\mathcal{E}), \#, \varpi)$ of an exact category with duality $(\mathcal{E}, *, \varpi)$, Walter has a description of W^1 and of W^3 in terms of formations, generalizing the “split” L -theoretic definitions. See [83].

The key computational device in the triangular Witt group theory is the following localization theorem.

73

Theorem 73 Let $(\mathcal{K}, \#, \varpi)$ be a triangulated category with duality such that $\frac{1}{2} \in \mathcal{K}$. Consider a thick subcategory $\mathcal{J} \subset \mathcal{K}$ stable under the duality, meaning that $(\mathcal{J})^\# \subset \mathcal{J}$. Induce dualities from $\mathcal{K}$ to $\mathcal{J}$ and to $\mathcal{L} := \mathcal{K}/\mathcal{J}$. We have, so to speak, an exact sequence of triangulated categories with duality:

$$\mathcal{J} \twoheadrightarrow \mathcal{K} \rightarrow \mathcal{L}.$$

Then, there is a 12-term periodic long exact sequence of Witt groups:

$$\dots \rightarrow W^{n-1}(\mathcal{L}) \xrightarrow{\partial} W^n(\mathcal{J}) \rightarrow W^n(\mathcal{K}) \rightarrow W^n(\mathcal{L}) \xrightarrow{\partial} W^{n+1}(\mathcal{J}) \rightarrow \dots$$

where the connecting homomorphisms ∂ can be described explicitly.

This is [6, Thm. 6.2], with the easily removable extra hypothesis that $\mathcal{K}$ is “weakly cancellative” (see [14, Thm. 2.1] for how to remove it).

74

Remark 74 In applications, one often knows $\mathcal{K}$ and a localization $\mathcal{K} \twoheadrightarrow \mathcal{L}$, like in the case of the derived category of a regular scheme and of an open subscheme. Then the $\mathcal{J}$ is *defined* as the kernel of this localization and the *relative* Witt groups are defined to be the Witt groups of $\mathcal{J}$. See Sect. 1.6.1.

1.5.3

Products and Cofinality

The product structures on the groups W^n have been discussed in Gille–Nenashev [27]. Inspired by the situation of a triangulated category with duality *and compatible tensor product*, they consider the general notion of (*external*) *dualizing pairing* [27, Def. 1.11].

75

Theorem 75: (Gille–Nenashev) Let $\boxtimes : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{M}$ be a dualizing pairing. This induces naturally a left and a right pairing

$$W^r(\mathcal{K}) \times W^s(\mathcal{L}) \xrightarrow[\ast_r]{\ast_l} W^{r+s}(\mathcal{M})$$

differing by signs, having the following properties:

- i. When $\mathcal{K} = \mathcal{L} = \mathcal{M}$, both products turn $\bigoplus_{n \in \mathbb{Z}} W^n(\mathcal{K})$ into a graded ring.
- ii. The multiplicative structure is compatible with localization.
- iii. The multiplicative structure is compatible with 4-periodicity.

Points (i) and (ii) are in [27, Thm. 2.9 and 2.11], for (iii) see [11, App. 1].

The behaviour of Witt groups with respect to idempotent completion can be controlled with the following result of [30], whose proof uses the technicalities (and not only the front results) of [6]. See [30, App. I].

Theorem 76: (*Hornbostel–Schlichting*) Let $\mathcal{B}$ be a triangulated category with δ -duality ($\delta = \pm 1$) and $\mathcal{A}$ a full triangulated subcategory which is *cofinal* (i.e. any object $b \in \mathcal{B}$ is a direct summand of an object of $\mathcal{A}$). Then there is a 12-term periodic long exact sequence

76

$$\cdots \rightarrow W^n(\mathcal{A}) \rightarrow W^n(\mathcal{B}) \rightarrow \widehat{H}^n(\mathbb{Z}/2\mathbb{Z}, K_0(\mathcal{B})/K_0(\mathcal{A})) \rightarrow W^{n+1}(\mathcal{A}) \rightarrow \cdots$$

involving Tate cohomology groups of $\mathbb{Z}/2\mathbb{Z}$ with coefficients in $K_0(\mathcal{B})/K_0(\mathcal{A})$, on which $\mathbb{Z}/2\mathbb{Z}$ acts via the duality, and where K_0 is the 0-th K -theory group.

Witt Groups of Schemes Revisited

1.6

Witt Cohomology Theories

1.6.1

Consider a scheme X containing $\frac{1}{2}$. Consider a presheaf $(\mathcal{K}, \#, \varpi)$ of triangulated categories with duality on the scheme X . That is: for each Zariski-open $U \hookrightarrow X$, we give a triangulated category with duality $\mathcal{K}(U)$ and a restriction $q_{V,U} : \mathcal{K}(U) \rightarrow \mathcal{K}(V)$ for each inclusion $V \hookrightarrow U$, which is assumed to be a localization of triangulated categories, in a compatible way with the duality, and with the usual presheaf condition.

For each $U \subset X$ one can then consider the Witt groups of $\mathcal{K}(U)$, which we denote

$$W^n(\mathcal{K}(U)) .$$

Here is a list of such presheaves, with their presheaves of Witt groups.

Example 77. Assume that X is *regular* (that is here: noetherian, separated and the local rings $\mathcal{O}_{X,x}$ are regular for all $x \in X$). For each open $U \subset X$, put $\mathcal{K}(U) := D^b(\text{VB}_U)$ the bounded derived category of vector bundles over U . Regularity is used to insure that the restriction $\mathcal{K}(X) \rightarrow \mathcal{K}(U)$ is a localization. By 71, the 0-th and 2-nd Witt groups of $\mathcal{K}(U)$ are the usual Witt groups of U of

symmetric and skew-symmetric forms, respectively. The latter result remains true without regularity of course.

Example 78. Assume that X is Gorenstein of finite Krull dimension. For each open $U \subset X$, put $\mathcal{K}(U) := D_{\text{Coh}}^b(\text{QCoh}_U)$ the derived category of bounded complexes of quasi-coherent $\mathcal{O}_U$ -modules with coherent homology. The duality is the derived functor of $\text{Hom}_{\mathcal{O}_U}(-, \mathcal{O}_U)$. See details in Gille [25, § 2.5]. The Witt group obtained this way

$$\tilde{W}^n(U) := W^n(D_{\text{Coh}}^b(\text{QCoh}_U))$$

is called the n -th *coherent Witt groups* of U . The groups $\tilde{W}^n(-)$ are only functorial for flat morphisms of schemes. They do not agree with derived Witt groups of 71 in general but do in the regular case, since the defining triangulated categories are equivalent.

Example 79. Let X be a scheme containing $\frac{1}{2}$. One can equip the category of perfect complexes over X with a duality, essentially as above. The Witt group obtained this way could be called the *perfect* Witt groups. Nevertheless, the presheaf of triangulated categories $U \mapsto D^{\text{perf}}(U)$ would fit in the above approach only when $U \mapsto K_0(U)$ is flasque (it is not clear if this is really much more general than 77). Without this assumption, there will be a 2-torsion noise involved in the localization sequence below, by means of Thm. 76.

To any such data, we can associate relative Witt groups, as follows.

80

Definition 80 Let X be a scheme and $U \mapsto \mathcal{K}(U)$ a presheaf of triangulated categories with duality as above. Let $Z \subset X$ be a closed subset. Let us define W_Z^n , the *Witt groups with supports in Z* as the Witt groups of the kernel category $\mathcal{K}_Z(X) := \ker(\mathcal{K}(X) \rightarrow \mathcal{K}(X \setminus Z))$

$$W_Z^n(\mathcal{K}(X)) := W^n(\mathcal{K}_Z(X)).$$

More generally, for any $U \subset X$, one defines $W_Z^n(\mathcal{K}(U))$ as being $W_{Z \cap U}^n(\mathcal{K}(U))$.

We have the following cohomological behaviour.

81

Theorem 81 With the above notations, we have a 12-term periodic long exact sequence

$$\cdots \rightarrow W^{n-1}(U) \rightarrow W_Z^n(X) \rightarrow W^n(X) \rightarrow W^n(U) \rightarrow W_Z^{n+1}(X) \rightarrow \cdots$$

where $W^n(-)$ is a short for $W^n(\mathcal{K}(-))$, when the triangulated categories $\mathcal{K}(-)$ are clear from the context and similarly for $W_Z^n(-)$.

This follows readily from the Localization Theorem 73. This was considered for derived Witt groups in [8, Thm. 1.6], and for coherent Witt groups in [25, Thm. 2.19]. We turn below to the question of identifying the groups $W_Z^*(X)$ with some groups $W^*(Z)$, but we first obtain as usual the following:

Corollary 82 Assume that our presheaf $\mathcal{K}$ of triangulated categories is natural in X and *excisive* with respect to a class $\mathcal{C}$ of morphisms of schemes, i.e. for any morphism $f : Y \rightarrow X$ in $\mathcal{C}$ and for any closed subset $Z \subset X$ such that $f^{-1}(Z) \xrightarrow{\sim} Z$ (with reduced structures), then the induced functor

$$f^* : \mathcal{K}_Z(X) \rightarrow \mathcal{K}_{f^{-1}(Z)}(Y)$$

is an equivalence. (This is the case for $\mathcal{K}(X) = D^b(\text{VB}_X)$ from Ex. 77 and $\mathcal{C} =$ flat morphisms of regular schemes; it is also the case for $\mathcal{K}(X) = D_{\text{Coh}}^b(\text{QCoh}_X)$ from Ex. 78 and $\mathcal{C} =$ flat morphisms of Gorenstein schemes.) Then, for any such morphism $f : Y \rightarrow X$, any $Z \subset X$ such that $Z' := f^{-1}(Z) \xrightarrow{\sim} Z$, there is a *Mayer–Vietoris* long exact sequence:

$$\cdots \rightarrow W^{n-1}(Y \setminus Z') \rightarrow W^n(X) \rightarrow W^n(Y) \oplus W^n(X \setminus Z) \rightarrow W^n(Y \setminus Z') \rightarrow \cdots$$

where $W^n(-)$ is a short for $W^n(\mathcal{K}(-))$. (So this applies to derived Witt groups over regular schemes and to coherent Witt groups over Gorenstein schemes.)

Remark 83 This holds in particular in the usual situation where $Y := U$ is an open subset, where $f : U \hookrightarrow X$ is the inclusion and where $Z \subset U$. In this case, putting $V := X \setminus Z$, we have $X = U \cup V$ and $Y \setminus f^{-1}(Z) = U \cap V$, recovering in this way the usual Mayer–Vietoris long exact sequence. The above generality is useful though, since it applies to elementary distinguished squares in the Nisnevich topology for instance. Observe that this result is a direct consequence of the following three things: first, the very *definition* of relative Witt groups via triangulated categories; secondly, the excision property of triangulated categories themselves; thirdly, of course, the localization theorem.

We now turn to dévissage (in the affine case).

Theorem 84: (*Gille*) Let R be a Gorenstein $\mathbb{Z}[\frac{1}{2}]$ -algebra of finite Krull dimension n and let $J \subset R$ be an ideal generated by a regular sequence of length $l \leq n$. Then the closed immersion $\iota : \text{Spec}(R/J) \rightarrow \text{Spec}(R)$ induces an isomorphism:

$$\tilde{W}^i(R/J) \xrightarrow{\cong} \tilde{W}_J^{i+l}(R)$$

where, of course, $\tilde{W}_J^j(R) := \tilde{W}_Z^j(R)$ is the j -th coherent Witt group of R with supports in the closed subset $Z = V(J)$ of $\text{Spec}(R)$ defined by J .

82

83

84

This is [25, Thm. 4.1]. Since coherent and derived Witt groups agree in the regular case, one has the obvious and important:

85 **Corollary 85:** (*Gille*) Let R be a regular $\mathbb{Z}[\frac{1}{2}]$ -algebra of finite Krull dimension and let $J \subset R$ be an ideal generated by a regular sequence of length l . Assume moreover that R/J is itself regular. Then there is an isomorphism of derived Witt groups:

$$W^i(R/J) \xrightarrow{\cong} W_J^{i+l}(R).$$

It is natural to ask if the cohomology theory obtained by derived (and coherent) Witt groups is homotopy invariant. The following result is a generalization of Karoubi's Theorem 53.

86 **Theorem 86** Let X be a regular scheme containing $\frac{1}{2}$. Then the natural homomorphism of derived Witt groups $W^i(X) \rightarrow W^i(\mathbb{A}_X^1)$ is an isomorphism for all $i \in \mathbb{Z}$. (In particular, for $i = 0$, this is an isomorphism of classical Witt groups.)

This is [8, Cor. 3.3] and has then been generalized in [26] as follows (using coherent Witt group versions of the result):

87 **Theorem 87:** (*Gille*) Let X be a regular scheme containing $\frac{1}{2}$ and let $\mathcal{E} \rightarrow X$ be an affine bundle. Then the natural homomorphism $W^i(X) \rightarrow W^i(\mathcal{E})$ is an isomorphism for all $i \in \mathbb{Z}$.

1.6.2 Local to Global

Recall our convention: *regular* means regular, noetherian and separated.

Consider an integral scheme X , for instance (the spectrum of) a domain R , and consider its function field Q (the field of fractions of R). It is natural to study the homomorphism

$$W(X) \rightarrow W(Q)$$

e.g. because the Witt groups of fields are better understood (see Cor. 58). It is immediate that for $R = \mathbb{R}[X, Y]/(X^2 + Y^2)$, the map $\mathbb{Z} \simeq W(\mathbb{R}) \rightarrow W(R)$ is split injective but that $W(Q)$ is 2-torsion, since -1 is a square in Q and hence that $2 \cdot [P, \varphi] = [(P, \varphi) \perp (P, \varphi)] = [(P, \varphi) \perp (P, -\varphi)] = 0$. So the homomorphism $W(R) \rightarrow W(Q)$ is not injective in general.

For *regular* schemes of dimension up to 3, injectivity of $W(X) \rightarrow W(Q)$ holds: see Thm 95 below. It is well-known to fail in dimension 4 already, even for affine regular schemes. For an example of this, see Knus [42, Ex. VIII.2.5.3]. Nevertheless, injectivity remains true in the *affine complex* case, see Pardon [62] and Totaro [80]. In [9] it is proven that the kernel of $W(X) \rightarrow W(Q)$ is nilpotent with

explicit exponent, generalizing earlier results of Craven–Rosenberg–Ware [22] and Knebusch [41].

Theorem 88 Let X be a regular scheme containing $\frac{1}{2}$ and of finite Krull dimension d . Then there is an integer N , depending only on $\lfloor \frac{d}{4} \rfloor$ such that the N -th power of the kernel of $W(X) \rightarrow W(Q)$ is zero in $W(X)$.

88

One can always take $N = 2^{\lfloor \frac{d}{4} \rfloor}$ and one can take $N = \lfloor \frac{d}{4} \rfloor + 1$ if the conjectural injectivity $W(\mathcal{O}_{X,x}) \hookrightarrow W(Q)$ holds for all $x \in X$. This is indeed the case when X is defined over a field, as we discuss below. Moreover, Example [9, Cor. 5.3] show that $\lfloor \frac{d}{4} \rfloor + 1$ is the best exponent in all dimensions. We have alluded to the following conjecture of Knebusch, which is a special case of a general conjecture of Grothendieck:

Conjecture 89 Let R be a regular (semi-)local domain containing $\frac{1}{2}$ and let Q be its field of fractions. Then the natural homomorphism $W(R) \rightarrow W(Q)$ is injective.

89

The key result about this conjecture was obtained by Ojanguren in [56] and says that the conjecture holds if R is essential of finite type over some ground field. Conjecture 89 has been upgraded as follows by Pardon [61]:

Conjecture 90: (*Gersten Conjecture for Witt groups*) Let R be a regular (semi-)local ring containing $\frac{1}{2}$. There exists a complex

90

$$0 \rightarrow W(R) \rightarrow \bigoplus_{x \in X^{(0)}} W(\kappa(x)) \rightarrow \bigoplus_{x \in X^{(1)}} W(\kappa(x)) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(d)}} W(\kappa(x)) \rightarrow 0$$

and it is *exact*; where X is $\text{Spec}(R)$, $X^{(p)}$ are the primes of height p and $d = \dim(X)$. The complex is now admitted to be the one of 91 below.

For a long time, it remained embarrassing not even to *know* a complex as above (call this a *Gersten–Witt complex*), which one would then conjecture to be exact. In the case of K -theory, the complex is directly obtained from the coniveau filtration. Analogously, by means of triangular Witt groups and of the localization theorem, it became possible to construct Gersten–Witt complexes for all regular schemes [14, Thm. 7.2]:

Theorem 91: (*Balmer–Walter*) Let X be a regular scheme containing $\frac{1}{2}$ and of finite Krull dimension d . Then there is a convergent (cohomological) spectral sequence $E_1^{p,q} \Rightarrow W^{p+q}(X)$ whose first page is isomorphic to copies of a Gersten–Witt complex for X in each line $q \equiv 0$ modulo 4 and whose other lines are zero. These isomorphisms involve local choices but a canonical description of the first page is:

91

$$E_1^{p,q} := W^{p+q} \left(D^{(p)} / D^{(p+1)} \right)$$

where $D^{(p)} = D^{(p)}(X)$ is the full subcategory of $D^b(\text{VB}_X)$ of those complexes having support of their homology of codimension $\geq p$.

This was reproven and adapted to coherent Witt groups with supports in Gille [25, Thm. 3.14]. Using Thm. 91, the following Corollaries are immediate:

92 **Corollary 92** Let X be regular integral $\mathbb{Z}[\frac{1}{2}]$ -scheme of dimension 1. Let x_0 be the generic point and $Q = \kappa(x_0)$ be the function field of X . There is an exact sequence:

$$0 \rightarrow W(X) \rightarrow W(Q) \xrightarrow{\partial} \bigoplus_{x \in X \setminus \{x_0\}} W(\kappa(x)) \rightarrow W^1(X) \rightarrow 0$$

and we have $W^2(X) = W^3(X) = 0$.

Example 93. The above applies in particular to Dedekind domains containing $\frac{1}{2}$. For instance for $D := \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$, it follows from [50, Ex. IV.3.5] that $W^1(D) \simeq \mathbb{Z}$. Here, $W(D) = \mathbb{Z} \oplus \mathbb{Z}/2$, see [42, VIII.6.1].

94 **Corollary 94** Let X be a regular scheme containing $\frac{1}{2}$ and of Krull dimension $d \leq 4$. Let $W_{nr}(X)$ be the unramified Witt group of X . The homomorphism $W(X) \rightarrow W_{nr}(X)$ is surjective.

95 **Corollary 95** Let X be a regular integral $\mathbb{Z}[\frac{1}{2}]$ -scheme of Krull dimension 3 and of function field Q . Then, the above Gersten–Witt complex

$$0 \rightarrow W(X) \rightarrow W(Q) \rightarrow \bigoplus_{x \in X^{(1)}} W(\kappa(x)) \rightarrow \bigoplus_{x \in X^{(2)}} W(\kappa(x)) \rightarrow \bigoplus_{x \in X^{(3)}} W(\kappa(x)) \rightarrow 0$$

is exact at $W(X)$ and at $W(Q)$ and its homology in degree i (that is, where $X^{(i)}$ appears) is isomorphic to $W^i(X)$ for $i = 1, 2, 3$.

See [14, § 10] for detailed results and definitions as well as for the following:

96 **Corollary 96** Let X be regular scheme containing $\frac{1}{2}$ and of dimension at most 7. Then, with the notations of Thm. 91, there is an exact sequence:

$$\begin{array}{ccccccccccc} 0 & \rightarrow & E_{4,0}^2 & \rightarrow & W^0(X) & \rightarrow & E_{0,0}^2 & \rightarrow & E_{5,0}^2 & \rightarrow & W^1(X) & \rightarrow & E_{1,0}^2 \\ & & & & & & & & & & & & \downarrow \\ & & .0 & \leftarrow & E_{3,0}^2 & \leftarrow & W^3(X) & \leftarrow & E_{7,0}^2 & \leftarrow & E_{2,0}^2 & \leftarrow & W^2(X) & \leftarrow & E_{6,0}^2 \end{array}$$

Note that $E_{p,0}^2$ is the p -th homology group of the Gersten–Witt complex of X .

97 **Corollary 97** The Gersten Conjecture holds in low dimension up to 4.

This is [14, Thm. 10.4] in the local case. For *semi*-local, one needs [9, Cor. 3.6] plus the local vanishing of shifted Witt groups, which holds in any dimension:

Theorem 98: (*Balmer–Preeti*) Let R be a semi-local commutative ring containing $\frac{1}{2}$. Then $W^i(R) = 0$ for $i \not\equiv 0$ modulo 4. 98

This is [7, Thm. 5.6] for local rings and will appear in [13] in general.

Totaro [80] used the above spectral sequence 91 in combination with the Bloch–Ogus and the Pardon spectral sequences (see [63] for the latter) to bring several interesting computations. He provides an example of a smooth *affine complex* 5-fold U such that $W(U) \rightarrow W(\mathbb{C}(U))$ is not injective and also gives a global complex version of Parimala’s result 47 (note that being finitely generated over $W(\mathbb{C}) = \mathbb{Z}/2$ means being finite), see [80, Thm. 1.4]:

Theorem 99: (*Totaro*) Let X be a smooth complex 3-fold. Then the Witt group $W(X)$ is finite if and only if the Chow group $\mathrm{CH}^2(X)/2$ is finite. 99

We return to the Gersten Conjecture 90. In [58], Ojanguren and Panin established Purity, which is exactness of the complex at the first two places, for regular local rings containing a field. Using the general machinery of homotopy invariant excisive cohomology theories, as developed in Colliot-Thélène–Hoobler–Kahn [19], the author established the geometric case of the following result in [8]:

Theorem 100 The Gersten Conjecture 90 holds for semi-local regular k -algebras over any field k of characteristic different from 2. 100

Like for the original K -theoretic Gersten Conjecture, the geometric case [8, Thm. 4.3] is the crucial step. It can then be extended to regular local k -algebras via Popescu’s Theorem, by adapting to Witt groups ideas that Panin introduced in K -theory. This is done in [12]. Now that we have the vanishing of odd-indexed Witt groups for *semi*-local rings as well, see Thm. 98, this Panin–Popescu extension also applies to *semi*-local regular k -algebras, as announced in the statement. Details of this last step have been checked in [51].

Computations 1.6.3

Here are some computations using triangular Witt groups:

Theorem 101: (*Gille*) Let R be a Gorenstein $\mathbb{Z}[\frac{1}{2}]$ -algebra of finite Krull dimension and $n \geq 1$. Consider the hyperbolic affine $(2n - 1)$ -sphere 101

$$\Sigma_R^{2n-1} := \mathrm{Spec} \left(R[T_1, \dots, T_n, S_1, \dots, S_n] \Big/ \left(1 - \sum_{i=1}^n T_i S_i \right) \right).$$

Then its coherent Witt groups are $\widetilde{W}^i(\Sigma_R^{2n-1}) = \widetilde{W}^i(R) \oplus \widetilde{W}^{i+1-n}(R)$. In particular for R regular, these are derived Witt groups. In particular for $R = k$ a field, or a regular semi-local ring, the classical Witt groups of Σ_k^{2n-1} is

$$W(\Sigma_k^{2n-1}) = \begin{cases} W(k) \oplus W(k) & \text{if } n \equiv 1 \text{ modulo } 4 \\ W(k) & \text{if } n \not\equiv 1 \text{ modulo } 4. \end{cases}$$

102

Theorem 102: (*Balmer–Gille*) Let X be a regular scheme containing $\frac{1}{2}$ and let $n \geq 2$. Consider the usual punctured affine space $\mathbb{U}_X^n \subset \mathbb{A}_X^n$ defined by $\mathbb{U}_X^n = \bigcup_{i=1}^n \{T_i \neq 0\}$. Then its total graded Witt ring $W^{tot} := \bigoplus_{i \in \mathbb{Z}/4} W^i$ is:

$$W^{tot}(\mathbb{U}_X^n) \simeq W^{tot}(X)[\varepsilon] / \varepsilon^2 = W^{tot}(X) \oplus W^{tot}(X) \cdot \varepsilon$$

where $\varepsilon \in W^{n-1}(\mathbb{U}_X^n)$ is of degree $n-1$ and squares to zero: $\varepsilon^2 = 0$.

The element ε is given quite explicitly in [11] by means of Koszul complexes. The above hypothesis $n \geq 2$ is only needed for proving $\varepsilon^2 = 0$. For $n = 1$, the scheme $\mathbb{U}_X^1 = X \times \text{Spec}(\mathbb{Z}[T, T^{-1}])$ is the scheme of *Laurent “polynomials”* over X and one has the following generalization of Thm. 54:

103

Theorem 103 Let X be a regular scheme containing $\frac{1}{2}$. Consider the scheme of Laurent polynomials $X[T, T^{-1}] = \mathbb{U}_X^1$. There is an isomorphism:

$$W^i(X) \oplus W^i(X) \simeq W^i(X[T, T^{-1}])$$

given by $(\alpha, \beta) \mapsto \alpha + \beta \cdot \langle T \rangle$ where $\langle T \rangle$ is the rank one space with form T .

The most striking computation obtained by means of triangular Witt groups is probably the following generalization of Arason’s Theorem 44:

104

Theorem 104: (*Walter*) Let X be a scheme containing $\frac{1}{2}$ and $r \geq 1$. Let $\mathbb{P}_X^r$ be the projective space over X . Let $m \in \mathbb{Z}/2$. Consider $\mathcal{O}(m) \in \text{Pic}(\mathbb{P}_X^r)/2$.

$$\begin{aligned} \text{For } r \text{ even, } \quad W^i(\mathbb{P}_X^r, \mathcal{O}(m)) &= \begin{cases} W^i(X) & \text{for } m \text{ even} \\ W^{i-r}(X) & \text{for } m \text{ odd.} \end{cases} \\ \text{For } r \text{ odd, } \quad W^i(\mathbb{P}_X^r, \mathcal{O}(m)) &= \begin{cases} W^i(X) \oplus W^{i-r}(X) & \text{for } m \text{ even} \\ 0 & \text{for } m \text{ odd.} \end{cases} \end{aligned}$$

This is indeed a special case of a general projective bundle theorem, for Witt and Grothendieck–Witt groups, which is to appear in [82]. Walter has also announced results for (Grothendieck–)Witt groups of quadratics, which are in preparation. The case of Grassmannians was started by Szyjewski in [78] and might also follow.

Witt Groups and $\mathbb{A}^1$ -Homotopy Theory

1.6.4

Using the above cohomological behaviour of Witt groups, Hornbostel [28, Cor. 4.9 and Thm. 5.7] establishes the following representability result.

Theorem 105: (*Hornbostel*) Witt groups are representable both in the unstable and the stable $\mathbb{A}^1$ -homotopy categories of Morel and Voevodsky.

105

This is one ingredient in Morel's announced proof of the following:

Theorem 106: (*Morel*) Let k be a (perfect) field of characteristic not 2. Let $\mathcal{SH}_k$ be the stable $\mathbb{A}^1$ -homotopy category over k . Then the graded ring

106

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{SH}_k}(S^0, \mathbb{G}_m^{\wedge n})$$

is isomorphic to the Milnor–Witt K -theory of k . In particular, $\mathrm{Hom}_{\mathcal{SH}_k}(S^0, S^0)$ is isomorphic to the Grothendieck–Witt group of k and for all $n < 0$, $\mathrm{Hom}_{\mathcal{SH}_k}(S^0, \mathbb{G}_m^{\wedge n})$ is isomorphic to the Witt group of k .

Of course, this result requires further explanations (which can be found in [55, § 6] or in [54]) but the reader should at least close this Chapter remembering that Witt groups quite miraculously appear at the core of the stable homotopy category $\mathcal{SH}_k$, disguised as “motivic stable homotopy groups of spheres”, objects which, at first sight, do not involve any quadratic form.

Acknowledgements: It is a pleasure to thank Inta Bertuccioni, Ivo Dell’Ambrogio, Louis Mahé, Pell Mell, Manuel Ojanguren, Raman Parimala and Raman Preeti for providing me with numerous comments, references, examples or encouragements.

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K-Theory and Geometric Topology

Jonathan Rosenberg*

2.1	<i>The Wall Finiteness Obstruction and Its Variants</i>	578
2.2	<i>Flat Bundles and K-Theory</i>	581
2.3	<i>Whitehead and Reidemeister Torsion</i>	586
2.4	<i>Controlled K-Theory and Connections with Negative K-Theory...</i>	591
2.5	<i>Equivariant and Stratified Situations</i>	592
2.6	<i>Waldhausen's A-Theory</i>	597
2.7	<i>K-Theory and Pseudo-Isotopy</i>	599
2.8	<i>K-Theory and Symbolic Dynamics</i>	602
	<i>References</i>	604

* Partially supported by NSF Grant DMS-0103647. The author thanks John Rognes and Allen Hatcher for helpful comments on the first draft of this article.

Historically, one of the earliest motivations for the development of K -theory was the need to put on a firm algebraic foundation a number of invariants or obstructions that appear in topology. The primary purpose of this chapter is to examine many of these K -theoretic invariants, not from a historical point of view, but rather *a posteriori*, now that K -theory is a mature subject.

There are two reasons why this may be a useful exercise. First, it may help to show K -theorists brought up in the “algebraic school” how their subject is related to topology. And secondly, clarifying the relationship between K -theory and topology may help topologists to extract from the wide body of K -theoretic literature the things they need to know to solve geometric problems.

For purposes of this article, “geometric topology” will mean the study of the topology of manifolds and manifold-like spaces, of simplicial and CW-complexes, and of automorphisms of such objects. As such, it is a vast subject, and so it will be impossible to survey everything that might relate this subject to K -theory. I instead hope to hit enough of the interesting areas to give the reader a bit of a feel for the subject, and the desire to go off and explore more of the literature.

Unless stated otherwise, all topological spaces will be assumed to be Hausdorff and compactly generated. (A Hausdorff space X is *compactly generated* if a subset C is closed if and only if $C \cap K$ is closed, or equivalently, compact, for all compact subsets K of X . Sometimes compactly generated spaces are called k -spaces. The k stands both for the German *Kompakt* and for Kelley, who pointed out the advantages of these spaces.) This eliminates certain pathologies that cause trouble for the foundations of homotopy theory. “Map” will always mean “continuous map.” A map $f: X \rightarrow Y$ is called a *weak equivalence* if its image meets every path component of Y and if $f_*: \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ is an isomorphism for every $x \in X$.

The Wall Finiteness Obstruction and Its Variants

2.1

We begin this survey with the “Wall finiteness obstruction,” not because it came first historically (Whitehead torsion dates back much earlier) and not because it is most important (again, most geometric topologists would argue that Whitehead torsion is more fundamental) but because most algebraic treatments of K -theory usually begin with K_0 of a ring or a category.

The discussion here will be brief; for a more complete treatment, see [33].

A basic theorem of homotopy theory states that every space X has a *CW-approximation*; in other words, there is a CW-complex Y and a weak equivalence $Y \rightarrow X$. More is true; the Y is unique up to homotopy equivalence and can be chosen functorially in X . In fact one can take $Y = |S_\bullet(X)|$ to be the geometric realization of the simplicial set $S_\bullet(X)$ of singular n -simplices in X [59, Chaps. 10, 16].

One says a space X is *dominated* by a space Y if there are maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \simeq 1_X$, where the symbol $\simeq$ denotes “is homotopic to.” A corollary of the CW-approximation theorem is that if a space X is dominated by a CW-complex, then it is homotopy-equivalent to a CW-complex. We say X is *finitely dominated* if it is dominated by a *finite* CW-complex. Clearly this is a necessary condition for X to be homotopy-equivalent to a finite CW-complex. The condition of being finitely dominated is sometimes not so hard to check. For example, a famous theorem of Borsuk [9, p. 1093] implies that any compact, locally contractible, and finite dimensional metric space is a retract of a finite polyhedron, hence in particular is finitely dominated.

Theorem 1: (Wall [100, 101]) Let X be a path-connected and locally 1-connected space, and let $C_*(X)$ be its singular chain complex. Note that the singular chain complex $C_*(\tilde{X})$ of the universal cover $\tilde{X}$ can be regarded as a complex of free R -modules, where $R = \mathbb{Z}\pi_1(X)$, and that $C_*(X) = \mathbb{Z} \otimes_R C_*(\tilde{X})$. Then if X is finitely dominated, $\pi_1(X)$ is finitely presented and $C_*(\tilde{X})$ is chain homotopy-equivalent to a finite complex C_* of finitely generated projective R -modules. The “Euler characteristic” of this complex,

$$\chi(X) = \sum_i (-1)^i [C_i],$$

is well defined in $\tilde{K}_0(R)$ (the quotient of $K_0(R)$ by the copy of $\mathbb{Z}$ coming from the finitely generated free R -modules), and vanishing of $\chi(X)$ in $\tilde{K}_0(R)$ is necessary and sufficient for X to be homotopically finite (homotopy-equivalent to a finite CW-complex).

Proof We give a brief sketch. If X is finitely dominated, then $\pi_1(X)$ is an algebraic retract of a finitely presented group, hence is itself finitely presented. First we note that the Euler characteristic $\chi(X)$ is well defined. The key thing to prove is that if there is a chain equivalence $h: C_* \rightarrow C'_*$, then $\sum_i (-1)^i [C_i] = \sum_i (-1)^i [C'_i]$. But this is true even for the $K_0(R)$ -valued Euler characteristic (not only for its image in $\tilde{K}_0(R)$), by the Euler–Poincaré principle.

Clearly, if there is an equivalence $Z \rightarrow X$ with Z a finite CW-complex, then $C_*(\tilde{X})$ is chain homotopy-equivalent to $C_*(\tilde{Z})$, which in dimension j is a free R -module with one generator for each j -cell in Z . Thus $[C_i(\tilde{Z})]$ lies in the subgroup $\mathbb{Z}$ of $K_0(R)$ generated by the free modules, and maps to 0 in $\tilde{K}_0(R)$, so $\chi(X) = \chi(Z) = 0$.

Wall’s main contribution was to prove sufficiency of the condition. First one shows that if $\chi(X) = 0$, then $C_*(\tilde{X})$ is chain equivalent to a finite complex of finitely generated free R -modules. This is elementary; start with an equivalent finite complex C_* of projective modules, say of dimension n , and choose a finitely generated projective R -module Q_0 such that $C_0 \oplus Q_0$ is free. Then the direct sum of C_* with the complex

$$Q_0 \xrightarrow{\cong} Q_0 \leftarrow 0 \leftarrow \cdots$$

is still equivalent to $C_*(\tilde{X})$ and is free in degree 0. Proceed similarly by induction. Since $\chi(X) = 0$, once the $(n-1)$ -st module has been made free, the n -th module is stably free. So making Q_{n-1} larger if necessary, one can arrange that all the modules are now free (and still finitely generated).

The last step is to build a finite CW-complex Z modeling the free chain complex from the last step, and to construct the required homotopy equivalence h . The Z and the h are constructed simultaneously by starting with a 2-complex $Z^{(2)}$ with the correct fundamental group (recall $\pi_1(X)$ is finitely presented) and with the correct C_1 and C_2 , along with a map $h^{(2)}: Z^{(2)} \rightarrow X$ inducing an isomorphism on π_1 . Then one attaches cells and extends the map by induction on the dimension. This is an exercise in obstruction theory. Eventually one gets the desired complex Z and a map $h: Z \rightarrow X$ which is an isomorphism on π_1 and which induces a homology isomorphism $\tilde{Z} \rightarrow \tilde{X}$. By Whitehead's Theorem, this map is a homotopy equivalence.

One situation where the Wall finiteness obstruction comes into play is the *spherical space form problem*. This is the problem of determining what finite groups G can act freely on S^n . Of course, there are certain obvious examples, namely groups which act freely and *isometrically* on S^n with its standard metric. These are classified in [105]. The necessary and sufficient condition for G to act freely and isometrically on *some* S^n is that for all primes p and q , not necessarily distinct, all subgroups of G of order pq must be cyclic. But if one doesn't require the action to be isometric (or even smooth), there are many more examples. The one obvious necessary condition is a homological one. For if X is a connected CW-complex with finite fundamental group G and with universal cover $\tilde{X}$ homotopy-equivalent to S^n , then the spectral sequence

$$H^p(G, H^q(\tilde{X}, \mathbb{Z})) \Rightarrow H^{p+q}(X, \mathbb{Z})$$

of the homotopy fibration $\tilde{X} \rightarrow X \rightarrow BG$ implies that G has periodic cohomology of period $n+1$, and thus that the Sylow subgroups of G are all either cyclic or generalized quaternion [17, Ch. XVI, §9, Application 4]. Conversely, if G satisfies this condition, Swan [87] showed that there is a periodic resolution of the trivial G -module $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}G$ -modules. In effect, the finiteness obstruction of this resolution is an obstruction to G acting freely and cellularly on a *finite* homotopy n -sphere. (We are explaining this *a posteriori*; Swan's paper predated Wall's, but the principle is the same.) But since $\tilde{K}_0(\mathbb{Z}G)$ is finite for G finite, one can kill off the obstruction by replacing the period by a suitably large multiple. Thus the result of [87] is that, after replacing the period of G by a suitably large multiple if necessary, G acts freely and cellularly on a *finite* n -dimensional CW-complex $\tilde{X}$ homotopy-equivalent to S^n , n one less than this larger period. For an explanation of how one then checks if X can be chosen to be a smooth manifold, see [88], [58], and [26]. The result of the analysis is that there is a simple necessary and sufficient condition for G to act freely and smoothly on *some* sphere:

Theorem 2: (Madsen–Thomas–Wall [58]) A finite group G acts freely and smoothly on a sphere S^n for **some** n if and only if G has periodic cohomology, and if, in addition, every subgroup of G of order $2p$, p an odd prime, is cyclic.

2

However, it is not always easy to tell from knowledge of G what is the minimal value of n . The necessity of the “ $2p$ condition” is due to Milnor [60], and follows from the following geometric result:

Theorem 3: (Milnor [60]) Let $T: S^n \rightarrow S^n$ be a map of period 2 without fixed points, and let $f: S^n \rightarrow S^n$ be a map of odd degree. Then there is a point $x \in S^n$ with $Tf(x) = fT(x)$.

3

Proof of necessity of the Madsen–Thomas–Wall condition from Theorem 3

Suppose G acts freely on a sphere and there is some subgroup H of G of order $2p$ which is not cyclic. Then H is dihedral. Let T be the action of the generator of H of order 2, and let f be the action of the generator of H of order p . Then by Theorem 3, $TfT^{-1}f^{-1}$ has a fixed point. Since G acts freely, that means $TfT^{-1}f^{-1} = 1$, so the two generators of H commute with each other, a contradiction.

Another application is to the problem of when a non-compact manifold M is homeomorphic to the interior of a compact manifold W with boundary. Clearly this implies that M should be homologically finite. Since any compact topological manifold W , even with boundary, has the homotopy type of a finite CW-complex, an additional necessary condition is that the Wall obstruction of M should vanish. The highly influential thesis of Siebenmann [78] showed that in high dimensions, this condition and an obvious “tameness” condition are sufficient.

Flat Bundles and K-Theory

2.2

Another connection between geometric topology (or more precisely, geometry and topology of manifolds) and algebraic K -theory comes from the study of flat vector bundles. Suppose M is a smooth manifold and $E \rightarrow M$ is a smooth vector bundle over M . A *connection* on E is a way of differentiating sections of E . More precisely, a connection is a map

$$\nabla: \Gamma^\infty(E) \rightarrow \Gamma^\infty(E \otimes T^*M),$$

where Γ^∞ denotes “smooth sections,” which we also think of as a bilinear pairing $\Gamma^\infty(E) \times \Gamma^\infty(TM) \rightarrow \Gamma^\infty(E)$, $(s, X) \mapsto \nabla_X(s)$, satisfying the “Leibniz rule” $\nabla_X(fs) = X(f) \cdot s + f\nabla_X(s)$ for $f \in C^\infty(M)$. (See for example [29, pp. 56–60].) A connection

is *flat* if it satisfies the analogue of the identity $d^2 = 0$ for the exterior derivative, or in other words if $[\nabla_X, \nabla_Y] = \nabla_{[X, Y]}$ for all vector fields X and Y . This condition turns out to be equivalent [29, Cor. 3.22] to saying that there is a reduction of the structure group of the bundle from $G = GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ to a discrete group. Now isomorphism classes of ordinary vector bundles are determined by their “transition functions,” and are thus parameterized by the non-abelian sheaf cohomology group $H^1(M, \mathcal{G})$, $\mathcal{G}$ the sheaf of germs of G -valued functions on M . Equivalently, they are classified by homotopy classes of maps $M \rightarrow BG$. In the same way, isomorphism classes of flat vector bundles (where we keep track of the flat structure ∇) are parameterized by non-abelian sheaf cohomology of the *constant* sheaf, $H^1(M, G^\delta) = \text{Hom}(\pi_1(M), G^\delta)$ or by homotopy classes of maps $M \rightarrow BG^\delta$, where G^δ denotes G with the discrete topology. Via the plus construction $BGL(n, \mathbb{C})^\delta \rightarrow BGL(\infty, \mathbb{C})^\delta \rightarrow B(GL(\infty, \mathbb{C})^\delta)^+$, we see that flat complex vector bundles give classes in $H^0(X; \mathbb{K}(\mathbb{C}))$, the cohomology of X with coefficients in the (algebraic, not topological) K -theory spectrum of $\mathbb{C}$. In particular, flat complex vector bundles over homology n -spheres can be viewed as representing classes in $H^0(S^n; \mathbb{K}(\mathbb{C})) = \pi_n(\mathbb{K}(\mathbb{C})) = K_n(\mathbb{C})$, and it is easy to see that every class in $K_n(\mathbb{C})$ arises from some flat vector bundle over a homology n -sphere. In a similar vein, Hausmann and Vogel [40, Corollary 4.2] have shown that for any ring A and $n \geq 5$, $K_n(A)$ can be described as the “homology sphere bordism” of $BGL(A)$, i.e., as the group of equivalence classes of pairs (Σ^n, f) , where Σ^n is a (based) oriented n -dimensional PL manifold which is an integral homology sphere¹, $f: \Sigma^n \rightarrow BGL(A)$ (and sends basepoint to basepoint), and $(\Sigma_1^n, f_1) \simeq (\Sigma_2^n, f_2)$ if and only if there exists a compact manifold W^{n+1} with $\partial W = \Sigma_1 \amalg -\Sigma_2$, there exists a map $F: W \rightarrow BGL(A)$ extending f_1 and f_2 (sending a “base arc” joining the basepoints of the boundary components to the basepoint of $BGL(A)$), and the inclusions $\Sigma_j \hookrightarrow W$ are integral homology equivalences.

We return again to the study of flat real or complex vector bundles. Suslin has shown [85] that for k any infinite field (in particular for $k = \mathbb{R}$ or $\mathbb{C}$), the inclusion $GL(n, k)^\delta \hookrightarrow GL(\infty, k)^\delta$ induces an isomorphism on $H_j(\_; \mathbb{Z})$ for $j \leq n$. Thus for studying characteristic classes of flat vector bundles on n -dimensional spaces, it's enough to look at flat vector bundles of rank $\leq n$. There are stability theorems saying that the map $B(GL(n, k)^\delta)^+ \rightarrow B(GL(\infty, k)^\delta)^+$ is $(n/2)$ -connected, and it's plausible that this map is even n -connected. Hence for computing $K_n(\mathbb{R})$ or $K_n(\mathbb{C})$, it's enough to look at flat vector bundles of rank $\leq 2n$, and it may even be that every class in $K_n(\mathbb{R})$ or $K_n(\mathbb{C})$ is represented by a flat vector bundle of rank n . But while the map $\pi_n(B(GL(n, \mathbb{R})^\delta)^+) \rightarrow K_n(\mathbb{R})$ may be surjective, it is known not to be injective; we will see why in a moment.

Various natural geometric questions about flat bundles can now be reduced (at least in part) to K -theory, and *vice versa*. (However, if one is interested in bundles not in the stable range, e.g., with rank equal to the dimension of the base space, then unstable K -theory is required.) We give only a few representative examples.

¹ We use PL manifolds rather than smooth ones to avoid complications coming from the finite group Θ_n of exotic n -spheres.

First we should say something about characteristic classes. A basic fact about flat vector bundles is that since the real (or rational) Chern or Pontrjagin classes of a vector bundle can be computed from the curvature of a connection using Chern–Weil theory, and since a flat connection has (by definition) curvature zero, these classes for a flat vector bundle necessarily vanish [29, 9.1 and 9.2]. Hence the Chern or Pontrjagin classes of a flat vector bundle are torsion. Since $K_{2n}^{\text{top}}(\mathbb{C}) \cong \mathbb{Z}$ is determined by Chern classes, it follows that the natural map from algebraic to topological K -theory, $K_{2n}(\mathbb{C}) \rightarrow K_{2n}^{\text{top}}(\mathbb{C})$, coming from the obvious continuous map $GL(n, \mathbb{C})^{\delta} \rightarrow GL(n, \mathbb{C})$, vanishes for $n > 0$. (However the map of spectra $\mathbb{K} \rightarrow \mathbb{K}^{\text{top}}$ induces isomorphisms on homotopy groups with *finite* coefficients by a famous theorem of Suslin [86], which is related to the fact that the Chern classes of flat bundles can carry non-trivial torsion information.)

One might guess on the basis of the above that all rational invariants of flat vector bundles have to vanish, but celebrated work of Milnor [61] shows that this is not the case for the Euler class of an oriented real vector bundle. More precisely, Milnor showed that if M^2 is a closed oriented surface of genus $g \geq 2$, so that the oriented rank-two real vector bundles E over M are classified by $\langle e(E), [M] \rangle \in \mathbb{Z}$, where $e(E)$ is the Euler class in $H^2(M, \mathbb{Z})$, then E admits a flat connection if and only if $|\langle e(E), [M] \rangle| < g$. (See also [29, §9 and Corollary 9.18] for a nice exposition.) This theorem prompted a huge explosion of interest in characteristic classes of flat vector bundles. For example, Deligne and Sullivan [27] showed that every flat *complex* vector bundle over a finite CW-complex becomes trivial on some finite cover. Using some of the ideas of Milnor, Smillie [81] showed there are flat manifolds with non-zero Euler characteristic in all even dimensions greater than or equal to four. This in turn motivated a more complete study by Hausmann [39] of what manifolds can admit a flat structure, i.e., a flat connection on the tangent bundle. For example, he showed that (in dimension ≥ 5) a stably parallelizable closed manifold M^{2m} is semi- s -cobordant to a manifold M' with a $\mathbb{Z}$ -flat structure (coming from a map $\pi_1(M') \rightarrow BSL(2m, \mathbb{Z})$) if and only if it is parallelizable. Here M *semi- s -cobordant* to M' means that there is a compact manifold W^{2m+1} with boundary $\partial W = M \amalg M'$ such that the inclusion $M \hookrightarrow W$ is a simple homotopy equivalence. (If the same is true for $M' \hookrightarrow W$, then M and M' are called s -cobordant, hence diffeomorphic if they have dimension ≥ 5 ; see Sect. 2.3 below.) In particular, every parallelizable closed manifold is homology-equivalent to a closed manifold with a $\mathbb{Z}$ -flat structure. Hausmann's methods proved at the same time that the natural map $\pi_n \left(B(GL(n, \mathbb{R})^{\delta})^+ \right) \rightarrow K_n(\mathbb{R})$ cannot be injective for $n = 2m$ even, for the image of the Euler class e under the restriction map $H^n(BSL(n, \mathbb{R}), \mathbb{Q}) \rightarrow H^n(BSL(n, \mathbb{R})^{\delta}, \mathbb{Q})$ is non-zero on the image of the Hurewicz map $\pi_n \left((BGL(n, \mathbb{R})^{\delta})^+ \right) = \pi_n \left((BSL(n, \mathbb{R})^{\delta})^+ \right) \rightarrow H_n \left((BSL(n, \mathbb{R})^{\delta})^+, \mathbb{Z} \right) \cong H_n(SL(n, \mathbb{R})^{\delta}, \mathbb{Z})$, but does not lie in the image of the restriction map $H^n(BSL(\infty, \mathbb{R})^{\delta}, \mathbb{Q}) \rightarrow H^n(BSL(n, \mathbb{R})^{\delta}, \mathbb{Q})$. Note that this now implies that the map $B(GL(n, \mathbb{R})^{\delta})^+ \rightarrow B(GL(\infty, \mathbb{R})^{\delta})^+$ cannot be $(n+1)$ -connected.

The vanishing of rational characteristic classes of flat bundles makes it possible to define *secondary* characteristic classes, which can be used to detect some of the K -theory of fields. For simplicity we consider only complex vector bundles. From the long exact sequence

$$\cdots \rightarrow H^{2k-1}(X, \mathbb{C}^\times) \xrightarrow{\partial} H^{2k}(X, \mathbb{Z}) \rightarrow H^{2k}(X, \mathbb{C}) \rightarrow H^{2k}(X, \mathbb{C}^\times) \xrightarrow{\partial} \cdots, \quad (2.1)$$

it follows that any integral torsion cohomology class in degree $2k$ lifts to a class of degree $2k-1$ with coefficients in $\mathbb{C}^\times$. A choice of such a lifting for the k -th Chern class of a flat rank- n vector bundle (E, ∇) over X , defined using the flat connection ∇ , was (essentially) given by Chern and Simons [23], [24] and is called the *Chern–Simons class*. For example, a flat structure on a complex line bundle over X is given simply by a homomorphism $\pi_1(X) \rightarrow \mathbb{C}^\times$, and thus defines a class in $H^1(X, \mathbb{C}^\times)$. In general, Chern and Simons consider the case, which one can always reduce to, where X is a smooth manifold, and then they use the connection ∇ to construct a closed differential form on the principal $GL(n)$ -bundle associated to E , whose restriction to each fiber is integral. One can then view this form as defining a $(\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times)$ -valued class on the base. An alternative approach to the construction of the Chern–Simons classes for flat bundles may be found in [29, Exercise 3, pp. 163–164]. The approach there involves the space $F = GL(n, \mathbb{C})/GL(k-1, \mathbb{C})$, which is $(2k-2)$ -connected and satisfies $H^{2k-1}(F, \mathbb{Z}) \cong \mathbb{Z}$. For the reader's convenience, we fill in some of the missing details.

4

Proposition 4 The space $F = GL(n, \mathbb{C})/GL(k-1, \mathbb{C})$ is $(2k-2)$ -connected and satisfies $H^{2k-1}(F, \mathbb{Z}) \cong \mathbb{Z}$.

Proof There are deformation retractions from $GL(n, \mathbb{C})$ down to $U(n)$, and from $GL(k-1, \mathbb{C})$ down to $U(k-1)$. Since $U(k)$ acts transitively on the unit sphere S^{2k-1} in $\mathbb{C}^k$, with $U(k-1)$ the stabilizer of a point, F has the homotopy type of S^{2k-1} when $k = n$, and then the result is obvious. If $n > k$, we have a fibration

$$U(k)/U(k-1) \rightarrow U(n)/U(k-1) \rightarrow U(n)/U(k),$$

and since $U(n)/U(k)$ is at least $2k$ -connected, the result follows.

The fact that F is highly connected is then used as follows.

5

Proposition 5 Again let $F = GL(n, \mathbb{C})/GL(k-1, \mathbb{C})$. There is a “filling” σ of F by singular simplices up through dimension $2k-1$, or in other words a family of singular simplices

$$\sigma(g_1, \dots, g_q): \Delta^q \rightarrow F, \quad g_1, \dots, g_q \in GL(n, \mathbb{C}), \quad q \leq 2k-1,$$

which satisfy

$$\sigma(g_1, \dots, g_q) \circ \varepsilon^i = \begin{cases} g_1 \cdot \sigma(g_2, \dots, g_q), & i = 0, \\ \sigma(g_1, \dots, g_i g_{i+1}, \dots, g_q), & 0 < i < q, \\ \sigma(g_1, \dots, g_{q-1}), & i = q. \end{cases} \quad (2.2)$$

where the ε^i are the face maps.

Proof This is proved by induction on q . To start the induction, let $\sigma(\Delta^0)$ be the origin $o = e GL(k-1, \mathbb{C})$ in F . Assume σ is defined for smaller values of q ; then one can check that (2.2) defines $\sigma(g_1, \dots, g_q)$ on the boundary of Δ^q in a consistent way. (For example, we need to check that the formulas for $\sigma(g_1, \dots, g_q) \circ \varepsilon^0 = g_1 \cdot \sigma(g_2, \dots, g_q)$ and for $\sigma(g_1, \dots, g_q) \circ \varepsilon^1 = \sigma(g_1 g_2, \dots, g_q)$ agree on the intersection of the 0-th face and the 1-th face, which is a $(q-2)$ -simplex. So we need to check that $g_1 \cdot \sigma(g_2, \dots, g_q) \circ \varepsilon^0 = \sigma(g_1 g_2, \dots, g_q) \circ \varepsilon^0$; both are given by $g_1 g_2 \cdot \sigma(g_3, \dots, g_q)$. The other verifications are similar.) Thus we just need to fill in. But for $q \leq 2k-1$, $\pi_{q-1}(F) = 0$, and thus any map $S^{q-1} \cong \partial\Delta^q \rightarrow F$ extends continuously to Δ^q .

Proposition 6 There is a $GL(n, \mathbb{C})$ -invariant closed $(2k-1)$ -form ω on F , representing the de Rham class of a generator of $H^{2k-1}(F, \mathbb{Z})$. 6

Proof Since $H^{2k-1}(F, \mathbb{Z}) \cong \mathbb{Z}$ by Proposition 4, and in fact by the proof of that proposition there is a preferred generator (coming from the usual orientation of S^{2k-1}), there is a canonical de Rham class representing this generator in $H^{2k-1}(F, \mathbb{R})$. This de Rham class may be realized by a $U(n)$ -invariant closed real form, since $U(n)$ is compact. (Just “average” any closed form in the de Rham class with respect to Haar measure on the compact group.) Then since $GL(n, \mathbb{C})$ is the complexification of $U(n)$ and acts transitively on F , we may complexify to a $GL(n, \mathbb{C})$ -invariant complex closed form.

Proposition 7 Define a group cochain s on $GL(n, \mathbb{C})^\delta$ (with values in $\mathbb{C}/\mathbb{Z}$) by the formula 7

$$s(g_1, \dots, g_{2k-1}) = \int_{\Delta^{2k-1}} \sigma(g_1, \dots, g_{2k-1})^*(\omega) \quad (\text{reduced mod } \mathbb{Z}).$$

Then s is a cocycle and its cohomology class in $H^{2k-1}(BGL(n, \mathbb{C})^\delta, \mathbb{C}/\mathbb{Z})$ is a lifting of the k -th Chern class for flat bundles.

Proof Let $G = GL(n, \mathbb{C})$. By definition,

$$\begin{aligned}
 \delta s(g_1, \dots, g_{2k}) &= s(g_2, \dots, g_{2k}) \\
 &\quad + \sum_{0 < i < 2k} (-1)^i s(g_1, \dots, g_i g_{i+1}, \dots, g_{2k}) + s(g_1, \dots, g_{2k-1}) \\
 &= \int_{\Delta^{2k-1}} \left(\sigma(g_2, \dots, g_{2k})^*(\omega) \right. \\
 &\quad \left. + \sum_{0 < i < 2k} (-1)^i \sigma(g_1, \dots, g_i g_{i+1}, \dots, g_{2k})^*(\omega) \right. \\
 &\quad \left. + \sigma(g_1, \dots, g_{2k-1})^*(\omega) \right) \\
 &= \int_{C(g_1, \dots, g_{2k})} \omega,
 \end{aligned} \tag{2.3}$$

where $C(g_1, \dots, g_{2k})$ is the singular chain

$$g_1 \cdot \sigma(g_2, \dots, g_{2k}) + \sum_{0 < i < 2k} (-1)^i \sigma(g_1, \dots, g_i g_{i+1}, \dots, g_{2k}) + \sigma(g_1, \dots, g_{2k-1}).$$

(Note that we've used G -invariance of ω to replace $\sigma(g_2, \dots, g_{2k})$ by $g_1 \cdot \sigma(g_2, \dots, g_{2k})$ here.) By the defining property (2.2) of σ , $C(g_1, \dots, g_{2k})$ is a singular cycle. But ω represents an integral de Rham class, so its integral over $C(g_1, \dots, g_{2k})$ vanishes in $\mathbb{C}/\mathbb{Z}$. Thus s is a group cocycle.

It remains to show that $\partial[s] = c_k$ in the sequence (2.1). But by the calculation in (2.3), $\partial[s]$ is represented by the group cocycle whose value on $(g_1, \dots, g_{2k})$ is given by $\int_{C(g_1, \dots, g_{2k})} \omega$, $C(g_1, \dots, g_{2k})$ as above. We can see that this is the primary obstruction to triviality of the universal bundle over BG^δ with fiber F (associated to the universal principal G -bundle over BG^δ). Indeed, it was the homotopy group $\pi_{2k-1}(F)$ which in the proof of Proposition 5 gave the obstruction to extending the filling σ to dimension $2k$, and had we been able to do this, $C(g_1, \dots, g_{2k})$ would be the boundary of $\sigma(g_1, \dots, g_{2k})$ and thus $\int_{C(g_1, \dots, g_{2k})} \omega$ would have vanished. The definition of Chern classes by obstruction theory then gives the result.

2.3

Whitehead and Reidemeister Torsion

One of the early sources for the development of K -theory is the geometric invariant known as *Whitehead torsion*, for which convenient textbook treatments are [25] and [76]. However, the best condensed reference is still probably Milnor's classic survey article, [63]. Another good exposition is in [64]. In its essence, the

idea of Whitehead torsion is to measure the extent to which a given homotopy equivalence, say between finite polyhedra, is of the “trivial” sort. Here “trivial” homotopy equivalences are generated by three basic kinds of operations: simplicial homeomorphisms (possibly after subdivision of some simplices) and elementary *expansions* and *collapses*. Expansions and their duals, collapses, are best illustrated by a picture (Fig. 2.1).

In other words, we say X' collapses to X if $X' = X \cup \sigma^n$, where σ^n is an n -simplex attached to X along one of its codimension-one faces, and then clearly we can “squash” X' down to X , and this gives a homotopy equivalence from X' to X . A homotopy equivalence between finite polyhedra is called *simple* if it can be constructed out of a chain of simplicial homeomorphisms (after subdivision) and elementary collapses and expansions. There is a similar notion for finite CW-complexes as well: in the CW-context, X' collapses to X if X' is obtained from X by attaching first an $(n-1)$ -cell with a null-homotopic attaching map $S^{n-2} \rightarrow X$, and then an n -cell bounded in the obvious way by this $(n-1)$ -cell, the same way D^n is bounded by S^{n-1} . It is easy to see that a polyhedral collapse is a special case of a cellular collapse, since attaching σ^n to X as in Fig. 2.1 is the same as first attaching the boundary of σ^n and then filling in with an n -cell. Any homotopy equivalence $h: X \rightarrow X'$ of (connected) finite polyhedra or finite CW-complexes has an invariant $\tau(h) \in \text{Wh}(\pi_1(X))$, where $\text{Wh}(\pi)$ is a certain quotient of $K_1(\mathbb{Z}\pi)$, and this invariant is trivial exactly when h is simple. We will content ourselves with describing this invariant in the simplest case. If h is an inclusion map and (X', X) is a finite relative CW-complex, with all relative cells of dimensions $n-1$ and n (so that X' is obtained from X by attaching first $(n-1)$ -cells and then n -cells), then since h is a homotopy equivalence, the relative cellular chain complex of the universal covers, $C_*(\tilde{X}', \tilde{X})$, reduces simply to an isomorphism $\partial: C_n \rightarrow C_{n-1}$ of finitely generated free $\mathbb{Z}\pi_1(X)$ -modules. We have obvious bases for the chain modules C_n and C_{n-1} which only depend on a choice of orientation for each relative cell of (X', X) and a choice of an inverse image for this cell in $\tilde{X}'$. Since the cellular boundary map ∂ is an isomorphism, one can think of ∂ as defining an invertible matrix with entries in $\mathbb{Z}\pi_1(X)$. Now of course the matrix depends on the choice of bases for the free $\mathbb{Z}\pi_1(X)$ -modules involved, but the ambiguity in the choice only affects the K_1 -class of the matrix by at most a sign (coming from the choices of orientations) and an element of $\pi_1(X)$

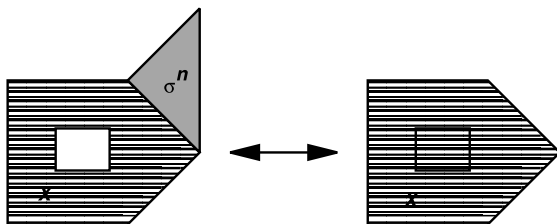


Figure 2.1. An elementary expansion (or collapse, depending on whether one reads the picture from right to left or left to right)

(coming from the choices of inverse images in $\tilde{X}'$ for the cells of (X', X)). Thus if we define $\text{Wh}(\pi_1(X))$ to be the quotient of $K_1(\mathbb{Z}\pi_1(X))$ by the subgroup generated by the canonical images of $\mathbb{Z}^\times \cong \{\pm 1\}$ and of $\pi_1(X)$, we obtain an invariant in this group independent of all choices.² One can show that this invariant vanishes if and only if the homotopy equivalence is simple. The “if” direction is rather straightforward from the definitions. The “only if” definition requires showing that that an elementary matrix corresponds geometrically to a collapse. (This requires “unhooking” one of the cells involved.)

It would appear that the construction of Whitehead torsion is highly dependent on a choice of simplicial or cellular structures for the spaces involved, but a deep and surprising theorem of Chapman says that this dependence is illusory.

8

Theorem 8: *Chapman [19]* If X and X' are connected finite polyhedra and $h: X \rightarrow X'$ is a (simplicial) homotopy equivalence, then $\tau(h)$ is a topological invariant. In other words, if we can fit h into a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{h'} & Y' \end{array},$$

where Y and Y' are also finite polyhedra, h' is a simplicial homotopy equivalence, and f and f' are homeomorphisms, then $\tau(h) = \tau(h')$.

This suggests that Whitehead torsion has some deeper significance, and in fact it plays a basic role in the classification of manifolds, for the following reason. If M and M' are compact connected n -manifolds (smooth, let's say), an $(n+1)$ -manifold with boundary W is called a *cobordism* between M and M' if $\partial W = M \sqcup M'$. (If W , M , and M' are oriented, then W should induce the opposite of the given orientation on M' , so that $M \times [0, 1]$ is an allowable cobordism from M to itself.) We call W an *h -cobordism* if the inclusions $M \hookrightarrow W$ and $M' \hookrightarrow W$ are homotopy equivalences, in which cases the torsions $\tau(W, M)$ and $\tau(W, M')$ are defined. We call W an

s -cobordism if the inclusions $M \hookrightarrow W$ and $M' \hookrightarrow W$ are simple homotopy equivalences, i.e., the torsions $\tau(W, M)$ and $\tau(W, M')$ both vanish. An h -cobordism is called *trivial* if it is diffeomorphic to $M \times [0, 1]$. When this is the case, note that M' is automatically diffeomorphic to M , and $\tau(W, M)$ and $\tau(W, M')$ are both trivial. Smale's famous h -cobordism theorem [62] asserts that every simply connected h -cobordism is trivial if $n \geq 5$. However, this cannot possibly be true in the non-simply connected case because of the Whitehead torsion obstruction, and the substitute is the s -cobordism theorem.

² Strictly speaking, $\tau(X', X)$ is this class multiplied by a sign depending on the parity of n ; that's because when there are relative cells of many dimensions, what we want is a kind of multiplicative analogue of the Euler characteristic.

Theorem 9: *s-cobordism theorem* [52] Suppose W^{n+1} is an h -cobordism between connected smooth manifolds M and M' , and suppose $n \geq 5$. If $\tau(W, M) = 0$, then W is trivial. Moreover, if $n \geq 5$, then every element of $\text{Wh}(\pi_1(M))$ can be realized by an h -cobordism from M to some homotopy-equivalent manifold M' .

9

The same statement holds in the PL category, for which a suitable reference is [76], and even (thanks to the work of Kirby–Siebenmann [54]) in the topological category.

The importance of Whitehead torsion for geometric topology makes it important to understand the Whitehead group $\text{Wh}(\pi)$ for various classes of groups π . It is not too hard to prove that $\text{Wh}(\pi) = 0$ for π of order ≤ 4 and that $\text{Wh}(\pi)$ is infinite cyclic for π of order 5. More generally, the most basic fact about the Whitehead group for finite groups is:

Theorem 10: Bass – see [66], *Theorems 2.5 and 2.6* Suppose π is a finite group. Then $\text{Wh}(\pi)$ is finitely generated, and $\text{rk}(\text{Wh}(\pi))$ is the difference between the number of irreducible representations of π over $\mathbb{R}$ and the number of irreducible representations of π over $\mathbb{Q}$.

10

Just as an example, if π is of order p , an odd prime, π has $(p-1)/2$ inequivalent two-dimensional irreducible representations over $\mathbb{R}$, but one $(p-1)$ -dimensional irreducible representation over $\mathbb{Q}$ (since $\mathbb{Q}\pi \cong \mathbb{Q} \times \mathbb{Q}(\zeta)$, ζ a primitive p -th root of unity, and $[\mathbb{Q}(\zeta) : \mathbb{Q}] = p-1$), so $\text{rk}(\text{Wh}(\pi)) = \frac{p-1}{2} + 1 - 2 = (p-3)/2$.

However, computing the exact structure of $\text{Wh}(\pi)$ for finite groups π is difficult, though in principle understood. The best survey on this is the book [66] by Oliver.

For infinite groups π , there is a widespread belief that $\text{Wh}(\pi)$ should be attributable to the torsion in π . (For an exact formulation of a conjecture to this effect, see the chapter by Lück and Reich.) But still open is the most fundamental version of this conjecture:

Conjecture 11 The Whitehead group $\text{Wh}(\pi)$ vanishes for any torsion-free group π .

11

There are many situations in geometric topology where Whitehead torsion is not well defined, but one can still define a torsion-like invariant called *Reidemeister torsion*. For example, if X is a finite connected CW-complex with fundamental group π , it may be that the reduced cellular chain complex $\overline{C}_*(\tilde{X})$ is not acyclic (i.e., $H_j(X, \mathbb{Z}\pi) \neq 0$ for some $j > 0$), so that $\tau(X, *)$ is undefined, and yet $C_*(X, V)$ may be acyclic for some local coefficient system V . In this case, we can define the Reidemeister torsion of X with coefficients in V . Roughly speaking, the difference between Whitehead and Reidemeister torsion is this. An $n \times n$ matrix a with entries in $\mathbb{Z}\pi$ defines a class in $\text{Wh}(\pi)$ if the matrix is invertible. However, it may

be that the matrix is not invertible, but its image under some representation of π is invertible. For example, suppose one has an orthogonal or unitary representation $\pi \rightarrow O(m)$ or $\pi \rightarrow U(m)$. Then this induces a ring homomorphism $\mathbb{Z}\pi \rightarrow M_m(\mathbb{R})$ or $M_m(\mathbb{C})$, under which the group of units $\mathbb{Z}^\times \times \pi$ maps to matrices with determinant of absolute value 1. So the absolute value of the determinant $|\det(a)| \in \mathbb{R}_+^\times$ is unchanged if we change a by an element of the image of $\mathbb{Z}^\times \times \pi \subseteq GL(1, \mathbb{Z}\pi) \hookrightarrow GL(n, \mathbb{Z}\pi)$. The simplest geometric example is the case of $X = S^1$ and the representation of $\pi_1(S^1) \cong \mathbb{Z}$ sending the generator to $e^{i\theta}$, $0 < \theta < 2\pi$. The cellular chain complex of S^1 with coefficients in the associated local system is $\mathbb{C} \xrightarrow{e^{i\theta}-1} \mathbb{C}$, so the complex is acyclic (under the assumption $0 < \theta < 2\pi$) and the torsion is $|e^{i\theta} - 1| = 2|\sin(\theta/2)|$.

There are two important classical examples of Reidemeister torsion. If X is the complement of a knot in S^3 and one takes the representation $\pi_1(X) \rightarrow \mathbb{C}^\times$ sending a generator of $H_1(X) = \pi_1(X)_{\text{ab}} \cong \mathbb{Z}$ to a transcendental number t , then the Reidemeister torsion becomes essentially (except for a trivial factor) the Alexander polynomial $\Delta(t)$ of the knot [63, Example 2, p. 387]. The second important case is where X is a lens space, the quotient of S^{2n-1} by a free linear action of a cyclic group $\pi = \mathbb{Z}/m$ on $\mathbb{C}^n$. In this case, the Reidemeister torsion is the essential invariant for classifying lens spaces with fixed dimension and fundamental group up to homeomorphism. More precisely (see [63, §12] for details), the lens spaces with fundamental group π and dimension $2n-1$ are classified by n elements $r_1, \dots, r_n \in (\mathbb{Z}/m)^\times$, modulo a certain equivalence relation, and the Reidemeister torsion (for the representation of π sending the generator to a primitive m -th root of unity t) turns out to be

$$\prod_{j=1}^n (t^{r_j} - 1),$$

modulo multiplication by factors of $\pm t^k$. The torsion is of course an invariant of the simple homotopy type, and by Chapman's Theorem (Theorem 8), even a homeomorphism invariant. From this one can prove that two lens spaces are homeomorphic if and only if they are isometric, which is certainly not obvious. (On the other hand, there are plenty of examples of lens spaces which are homotopy equivalent but not simple homotopy equivalent, and also plenty of examples of lens spaces with the same dimension and fundamental group which are not even homotopy equivalent.)

One of the most remarkable things about Reidemeister torsion is its relation to a global analytic invariant in Riemannian geometry, the *analytic torsion* of Ray and Singer [73]. Ray and Singer defined the analytic torsion by reformulating the definition of the Reidemeister torsion in terms of the “combinatorial Laplacian,” then replacing this operator in the definition by the Laplace–Beltrami operator of Riemannian geometry. They conjectured that the resulting invariant, given in terms of the spectrum of the Laplacian on differential forms, coincides with the Reidemeister torsion, and this conjecture was eventually proven by Cheeger [22] and Müller [65], working independently.

Various generalizations of the Cheeger–Müller theorem, for example, replacing ordinary determinants by the Kadison–Fuglede determinant³ on a finite von Neumann algebra (e.g., [12]), or allowing manifolds with boundary or non-compact manifolds, are a major topic of current research.

Controlled *K*-Theory and Connections with Negative *K*-Theory

2.4

One of the most interesting areas where algebraic *K*-theory and geometric topology come together is in the subject of controlled *K*-theory. In this theory, one studies not just projective modules over a ring and morphisms between them, but also the effect of imposing conditions on the “placement” or “support” of the modules or morphisms.

Probably the simplest example of controlled *K*-theory is an elegant description of negative *K*-theory by Pedersen [69], which led to a description by Pedersen and Weibel [68], [67] of the homology theory attached to the (nonconnective) *K*-theory spectrum $\mathbb{K}(R)$ of a ring *R*. These examples lead to what is often called *K*-theory with *bounded control*. Say one is given a ring *R* and a (non-empty) metric space (X, d) . One considers the category $\mathcal{C}_X(R)$ of “locally finitely generated” configurations of projective modules over *X*, i.e., maps $x \mapsto P_x$ from *X* to finitely generated projective *R*-modules, such that $\bigoplus_{x \in B} P_x$ is finitely generated for each set $B \subseteq X$ of finite diameter. Morphisms are *R*-module endomorphisms of $\bigoplus_{x \in X} P_x$ whose component $P_x \rightarrow P_y$ vanishes once $d(x, y)$ is sufficiently large. Applying the usual *K*-theoretic constructions gives a *K*-theory spectrum $\mathbb{K}(R; X)$ and thus groups $K_i(R; X) = \pi_i(\mathbb{K}(R; X))$. Here only the “large scale” geometry of *X* is relevant. For example, if *X* has finite diameter, $\mathbb{K}(R; X) \simeq \mathbb{K}(R; \text{pt}) = \mathbb{K}(R)$, and similarly $\mathbb{K}(R; \mathbb{R}^n) \simeq \mathbb{K}(R; \mathbb{Z}^n)$ (if $\mathbb{R}^n$ and $\mathbb{Z}^n$ are given the usual metrics). In this language, the main theorem of [69] asserts that $\mathbb{K}(R; \mathbb{Z}^n)$ is the usual non-connective *n*-fold delooping of $\mathbb{K}(R)$, and thus $K_0(R; \mathbb{Z}^n) \cong K_{-n}(R)$. Then the papers [68] and [67] go on to show that if $\mathcal{O}(Y)$ is the infinite open cone on a compact space *Y*, with the usual metric (so that if *Y* is embedded in $S^{n-1} \subset \mathbb{R}^n$, $\mathcal{O}(Y)$ is an $\mathbb{R}_+^\times$ -invariant subset of $\mathbb{R}^n$, from which it inherits the induced metric), then $K_i(R; \mathcal{O}(Y)) \cong \tilde{H}_{i-1}(Y; \mathbb{K}(R))$.

The boundedly controlled *K*-theory $\mathbb{K}(R; X)$ appears in many geometric applications, both directly and implicitly. Examples include the thin *h*-cobordism theorem of Quinn [70, Theorem 2.7] (this predated the above formulation of the theory, but involves some of the same ideas), the bounded *s*-cobordism theorem of Ferry and Pedersen [34, Theorem 2.17], and the work of Gunnar Carlsson [16] on the *K*-theoretic version of the Novikov conjecture. (See also Carlsson’s chapter in this volume for more details.)

For applications to geometric topology, sometimes *K*-theory with *epsilon control* is more relevant. The best motivation for this subject is the Chapman–Ferry

³ On a II_1 factor *A*, this “determinant” gives an isomorphism $K_1(A) \rightarrow \mathbb{R}_+^\times$ [57].

Theorem ([32], [21]), which asserts that a homotopy equivalence $h: M' \rightarrow M$ between closed manifolds M and M' of dimension $n \geq 5$ is homotopic to a homeomorphism once it is “sufficiently controlled.” To explain what this means, recall that the definition of a homotopy equivalence means that there is a map $h': M \rightarrow M'$ and there are homotopies $H_1: h \circ h' \simeq \text{id}_M, H_2: h' \circ h \simeq \text{id}_{M'}$. For h to be “sufficiently controlled” means that if we fix a metric d on M , $d(H_1(x, t), x) \leq \varepsilon$ and $d(h \circ H_2(y, t), h(y)) \leq \varepsilon$ for all $x \in M, y \in M'$, and all $t \in [0, 1]$. The theorem asserts that given M and d , there is an $\varepsilon > 0$ such that all ε -controlled homotopy equivalences $h: M' \rightarrow M$ are homotopic (even ε -homotopic) to homeomorphisms. While neither the statement nor the proof of the Chapman–Ferry Theorem involves K -theory directly, one can see that there has to be a connection. In fact, for the theorem to be true, it is clearly necessary (because of Theorem 8) for $\tau(h) = 0 \in \text{Wh}(\pi_1(M))$ once h is sufficiently controlled, which is not immediately obvious.

A treatment of ε -controlled Whitehead torsion and an associated controlled s -cobordism theorem [20, §14] may be found in [20]. Chapman also states and proves [20, §§6–8] an ε -controlled version of the Wall finiteness obstruction (Theorem 1). This concerns the situation where one has a space X with a reference map $p: X \rightarrow B$, B a metric space. We say X is ε -dominated by a space Y if there are maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \simeq_\varepsilon 1_X$, where the symbol $\simeq_\varepsilon$ denotes “is ε -homotopic to,” i.e., there is a homotopy whose composition with p doesn’t move points more than a distance ε . Chapman answers the question of when an ε -finitely dominated space is ε -homotopy equivalent to a finite polyhedron.

One can formulate many similar theorems that involve controlled versions of Whitehead torsion or similar K -theoretic obstructions. Examples are the thin h -cobordism theorem of Quinn ([70, Theorem 2.7] and [71, Theorem 2.1.1]).

2.5

Equivariant and Stratified Situations

So far, we have mostly discussed the topology of smooth, topological, or PL manifolds just by themselves. But K -theory also comes into play in the study of actions of groups (let’s say finite groups for simplicity) on such manifolds, or in the study of stratified spaces such as complex algebraic or analytic varieties. (Such a variety has a dense open subset which is smooth; the complement of this smooth set, the *singular set*, is of smaller dimension and itself contains a dense smooth set, etc.) The connection between these two topics may be seen in the fact that if a finite group G acts (smoothly, say) on a manifold M , then there is a dense open subset consisting of “principal orbits” (where the stabilizers are as small as possible), and once again the complement of this set is of smaller dimension and consists of more “singular” orbits.

The simplest example of a singular space is the one-point compactification $X = M_+$ of a non-compact manifold M , or equivalently, a compact space with exactly one singular point. Detailed study of this example can tell us much about

the general case. Just as an example, a natural question is how to formulate the s -cobordism theorem for such spaces. This problem is clearly equivalent to that of formulating a (proper) s -cobordism theorem for non-compact manifolds, which was done by Siebenmann in [79]:

Theorem 12: (*proper s -cobordism theorem [79]*) Suppose W^{n+1} is a proper h -cobordism between connected smooth non-compact manifolds M and M' , and suppose $n \geq 5$. (In other words, $\partial W = M \amalg M'$, and the inclusions $M \hookrightarrow W$ and $M' \hookrightarrow W$ are proper homotopy equivalences.) Then a Whitehead torsion invariant $\tau(W, M)$ is defined in a group $\text{Wh}^p(M)$, and if $\tau(W, M) = 0$, then W is isomorphic (in the appropriate category) to $M \times [0, 1]$. Moreover, if $n \geq 5$, then every element of $\text{Wh}^p(M)$ is realized by an h -cobordism. Assuming for simplicity that M has one end $\mathcal{E}$ and that $\mathcal{E}$ is tame, i.e., that for sufficiently large compact $C \subset M$, $M \setminus C$ is connected and its fundamental group $\pi_1(\mathcal{E})$ is independent of C , the group $\text{Wh}^p(M)$ fits into an exact sequence

$$\text{Wh}(\pi_1(\mathcal{E})) \rightarrow \text{Wh}(\pi_1(M)) \rightarrow \text{Wh}^p(M) \rightarrow \tilde{K}_0(\mathbb{Z}\pi_1(\mathcal{E})) \rightarrow \tilde{K}_0(\mathbb{Z}\pi_1(M)) .$$

A direct algebraic description of the obstruction group $\text{Wh}^p(M)$ is given in [31].

A non-obvious corollary of this theorem is that simple homotopy type has a geometrical meaning: two finite-dimensional CW-complexes have the same simple homotopy type if and only if they have piecewise linearly homeomorphic (closed) regular neighborhoods in some Euclidean spaces. (For finite CW-complexes this result is classical and is discussed in [103, pp. 22–23].) One direction is clear: if X and X' are finite-dimensional CW-complexes with piecewise linearly homeomorphic (closed) regular neighborhoods, then since a PL homeomorphism is simple, we obtain a simple homotopy equivalence from X to X' (via the intermediary of the regular neighborhoods). To prove the other direction, observe that a simple homotopy equivalence $X \approx X'$ can without loss of generality be taken to be the inclusion of one end of a mapping cylinder. Taking a regular neighborhood in a sufficiently large Euclidean space, one can convert this mapping cylinder into a proper h -cobordism, where the two ends of the cobordism are regular neighborhoods of X and X' . Then simplicity of $X \approx X'$ says by Theorem 12 that the h -cobordism is a product, and the result follows.

Next, we discuss some applications of algebraic K -theory to the study of actions of finite groups on complexes or manifolds. Some of this could be (and has been) generalized to actions of more general compact Lie groups or to proper actions of infinite discrete groups, but even the case of finite groups is too complicated to treat in detail here.

An easy place to begin is with the Wall finiteness obstruction. Let G be a finite group and let X be a G -CW-complex. The notions of finite domination and finiteness make sense in the equivariant world (we replace homotopies by G -homotopies, homotopy equivalences by G -homotopy equivalences). So it is natural to ask, assuming X is G -dominated by a finite G -CW-complex, whether X is G -homotopy

equivalent to a finite G -CW-complex. One case we have effectively already done – if X is connected and simply connected and the action of G on X is free, so $Y = X/G$ has fundamental group G , then this reduces to the question of whether Y is finitely dominated, which is the case if and only if the usual Wall obstruction in $\tilde{K}_0(\mathbb{Z}G)$ vanishes. The more general situation was first treated by Baglivo [6], who studied the case where X is connected in the equivariant sense, i.e., where X^H is connected and non-empty for every subgroup $H \subseteq G$. More general cases were treated by Lück [56] and others – see [3] for a survey of the many approaches.

The equivariant Wall obstruction appears in a number of problems about group actions, in combination with the *Swan homomorphism*

$$\sigma: (\mathbb{Z}/|G|)^\times \rightarrow \tilde{K}_0(\mathbb{Z}G),$$

the boundary map σ in the Mayer–Vietoris sequence in K -theory

$$\cdots \rightarrow K_1(\mathbb{Z}) \oplus K_1(\mathbb{Z}G/(n)) \rightarrow K_1(\mathbb{Z}/|G|) \xrightarrow{\sigma} \tilde{K}_0(\mathbb{Z}G) \rightarrow \cdots$$

associated to the pull-back square

$$\begin{array}{ccc} \mathbb{Z}G & \twoheadrightarrow & \mathbb{Z}G/(n) \\ \downarrow \varepsilon & & \downarrow \\ \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/|G|. \end{array}$$

Here $\varepsilon: \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ is the augmentation map (sending each element of G to 1) and $n = \sum_{g \in G} g$ is the “norm element” of $\mathbb{Z}G$. The relevance of the map σ in this context was first noticed in [5].

Let G be a finite group and let X be a G -CW-complex. Then X is called *Smith acyclic* if, for each subgroup H of G of prime power order p^r , $r \geq 1$, $\tilde{H}_*(X^H, \mathbb{F}_p) = 0$. A famous result of P. A. Smith [82] says that the singular set (the set of points with non-trivial stabilizer) of an action of G on a finite-dimensional contractible space is Smith acyclic, and it is natural to ask about the converse.

13

Theorem 13: ([5], Proposition 0.4) Let G a finite group, and let X be a Smith acyclic finite G -CW-complex for which every point has a non-trivial stabilizer. Then X is the singular set for an action of G on a contractible finite G -CW-complex if and only if

$$\sum_i (-1)^i \sigma(\tilde{H}_i(X, \mathbb{Z}/|G|)) = 0 \quad (2.4)$$

in $\tilde{K}_0(\mathbb{Z}G)$. (The Smith acyclicity of X implies that each $\tilde{H}_i(X, \mathbb{Z}/|G|)$ is of order prime to $|G|$, so that we can think of it as representing an element of $(\mathbb{Z}/|G|)^\times$, and thus (2.4) makes sense.)

The “only if” direction of this theorem follows from making precise the equivariant Wall obstruction. If “if” direction is proved by a direct inductive construction,

where we add equivariant cells of type $e^i \times G$ to X , analogous to the proof of Theorem 1.

Theorem 13 paved the way for the study of many problems about extension of group actions and “homology propagation.” The latter has to do with showing that, roughly speaking, if two manifolds have similar homology, then they carry similar group actions. Results of this type may be found in [5, 14, 102], and [15], just to cite a few sources.

Still another application of the equivariant Wall finiteness obstruction, but one requiring controlled topology also, may be found in a dramatic theorem of Steinberger and West [84]: a locally linear action of a finite group G on a manifold M , assuming all components of the fixed point sets of all subgroups have dimension ≥ 6 and none has codimension 1 or 2 in another, can be given an equivariant handle structure if and only if, for each $\varepsilon > 0$, M is equivariantly ε -homotopy equivalent to a finite G -CW-complex.

The equivariant Whitehead group $\text{Wh}_G(X)$ and its basic properties were defined by Illman [50]. Anderson [1], Hauschild [38], and Illman later [51] showed that the equivariant Whitehead group $\text{Wh}_G(X)$ can be expressed as a direct sum of ordinary Whitehead groups $\text{Wh}((WH)_\alpha^*)$. The sum is over equivalence classes of connected components X_α^H of fixed sets X^H , where H runs over the subgroups of G . The group $(WH)_\alpha$ is defined as $(WH)_\alpha = \{w \in WH : w \cdot X_\alpha^H = X_\alpha^H\}$, where $WH = N_G(H)/H$. Finally, the group $(WH)_\alpha^*$ is an extension of $(WH)_\alpha$ by $\pi_1(X_\alpha^H)$. As expected, the equivariant Whitehead group appears in the equivariant s -cobordism theorem in [83] and in [4].

Finally, we return to the case of more general stratified spaces. This case gets to be quite complicated, and the best place for the novice to begin is first with the survey article [42] and then with Weinberger’s book [103]. As explained in [42, §1], many definitions and categories of stratified sets have been proposed. In all cases, we want to consider locally finite partitions Σ of a locally compact, separable metric space X into pairwise disjoint, locally closed subsets X_i , called the (pure) *strata*, each of which is a topological manifold, with $\text{cl } X_i \cap X_j \neq \emptyset$ if and only if $X_j \subseteq \text{cl } X_i$. The index set is then partially ordered by $j \leq i$ if and only if $X_j \subseteq \text{cl } X_i$. The closed sets $\text{cl } X_i$ are often called the *closed strata*. The differences between the various categories of stratified spaces have to do with “gluing” conditions on how the strata are joined. Essentially all of the definitions apply to “good” stratified spaces, like projective algebraic varieties over $\mathbb{C}$, but they do not necessarily apply to orbit spaces of finite groups acting locally linearly on topological manifolds, where one needs a weak form of the definition.

For many purposes, the best theory of stratified spaces to use is that of Browder and Quinn [11] – see also [103, §§6–10]. In this theory one keeps track of *mapping cylinder neighborhoods*. In other words, if X_i is a stratum and

$$\Sigma X_i =_{\text{def}} (\text{cl } X_i \setminus X_i) = \bigcup \{X_j \mid j \not\leq i\},$$

we suppose there is a closed neighborhood N_i of ΣX_i in $X^i = \text{cl } X_i$ and a map $v_i: \partial N_i \rightarrow \Sigma X_i$ such that:

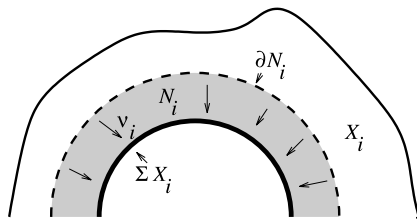


Figure 2.2. A mapping cylinder neighborhood

1. ∂N_i is a codimension-1 submanifold of X_i ,
2. N_i is the mapping cylinder of v_i (with ∂N_i and ΣX_i corresponding to the top and bottom of the cylinder),
3. if $j \not\geq i$ and $W_j = X_j \setminus \text{int } N_j$, then $v_i|_{v_i^{-1}(W_j)}: v_i^{-1}(W_j) \rightarrow W_j$ is a submersion (in the appropriate category).

(See Fig. 2.2). Such mapping cylinder neighborhoods do not always exist in the weakest types of stratified sets, but an obstruction theory for their existence was given in [72, Theorem 1.7].

In the PL Browder–Quinn theory, Whitehead torsion and the s -cobordism theorem take an especially nice form. The appropriate obstruction group for a PL stratified space X with strata X_i as above is simply

$$\text{Wh}^{\text{BQ}}(X) = \bigoplus_i \text{Wh}(\text{cl } X_i) .$$

An h -cobordism W of stratified spaces, based on X , is itself a stratified space with boundary $X \sqcup X'$, where the inclusions of X and X' into W are stratified homotopy equivalences, and the neighborhood data for the strata of Z are the pullbacks with respect to the retractions of the data for X (and of X').

Theorem 14: (*Stratified s -cobordism theorem [103, §6]*) Let X be a PL stratified space in the sense of Browder–Quinn above. Then assuming all strata have dimension ≥ 5 , PL h -cobordisms of PL stratified spaces based on X are in natural bijection with $\text{Wh}^{\text{BQ}}(X)$.

One thing to keep in mind, however, is that in the stratified (or equivariant) world, the parallelism between the three categories of topological, PL, and smooth manifolds breaks down. The stratified topological s -cobordism theorem is quite different from the PL one, and involves a rather different obstruction group $\text{Wh}^{\text{top}}(X)$. One can already see this in the case of the one-point compactification $X = M_+$ of a non-compact manifold M , say with M PL (or even smooth). The space X has two strata, M and a point, so $\text{Wh}^{\text{BQ}}(X) = \text{Wh}(\pi_1(X))$, whereas $\text{Wh}^{\text{top}}(X) = \text{Wh}^p(M)$, the proper Whitehead group that appears in Theorem 12 (see [103, pp. 131–132] for an explanation of why this is the case). Also note that since $\text{Wh}^{\text{top}}(X)$ is a kind

of “relative” Whitehead group, it can involve K_0 and lower K -groups of the strata, not just Whitehead groups of the closed strata as in the case of $\text{Wh}^{\text{BQ}}(X)$.

Waldhausen’s A -Theory

2.6

For some of the applications of K -theory to geometric topology, one needs a variant of algebraic K -theory called the *algebraic K -theory of spaces*, A -theory, or *Waldhausen K -theory*. There are several equivalent versions of the definition of Waldhausen’s $A(X)$, but all of them are somewhat involved. So it’s worth giving the informal definition first. If X is a pointed space, let $Q((\Omega X)_+) = \Omega^\infty \Sigma^\infty((\Omega X)_+)$ be the infinite loop space whose homotopy groups are the stable homotopy groups of $(\Omega X)_+$, the loop group of X with a disjoint basepoint attached. The space $Q((\Omega X)_+)$ can be viewed as a “ring up to homotopy,” the multiplication coming from concatenation of loops in ΩX . If X is path-connected, then $\pi_0(\Omega X) = \pi_1(X)$ is an actual group, and there is a map $Q((\Omega X)_+) \rightarrow \mathbb{Z}\pi_1(X)$ from $Q((\Omega X)_+)$ to a genuine ring, the group ring of $\pi_1(X)$. (The map $Q(*_+) = Q(S^0) \rightarrow \mathbb{Z}$ sends a stable map $S^0 \rightarrow S^0$, i.e., a map $S^n \rightarrow S^n$ for some n , to its degree.) Waldhausen’s $A(X)$ [93] is the K -theory space (the space whose homotopy groups are the K -groups) of the ring up to homotopy $Q((\Omega X)_+)$, and the map $Q((\Omega X)_+) \rightarrow \mathbb{Z}\pi_1(X)$ induces a “linearization map” $L: A(X) \rightarrow \mathbb{K}(\mathbb{Z}\pi_1(X))$ which is close to being an equivalence in “low degrees.” More precisely, the space $A(X)$ splits as $Q(X_+) \times \text{Wh}^{\text{diff}}(X)$ ([94], [95], and [99]) for a certain homotopy functor Wh^{diff} to be discussed further in Sect. 2.7 below, but related to the (higher) Whitehead groups of $\pi_1(X)$. The homotopy fiber of

$$L: A(*) \rightarrow \mathbb{K}(\mathbb{Z}) = \mathbb{Z} \times \text{BGL}(\mathbb{Z})^+$$

has finite homotopy groups, and localized at a prime p is known to be $(2p - 3)$ -connected, with its first homotopy group isomorphic to $\mathbb{Z}/p$ in degree $2p - 2$ ([97], [55, Theorem 1.2]).

The main foundational paper on $A(X)$, giving a rigorous definition and proving the key properties, is [98]. As this is a 100-page technical *tour de force*, there is no hope to explain it all here, so we will just quickly summarize some of the key points. The longest part of the paper explains a method for defining the K -theory of a *category with cofibrations and weak equivalences*. Such a category $\mathcal{C}$ has a zero object and satisfies certain axioms modeled on the properties of the category of finite pointed simplicial sets, where the cofibrations and weak equivalences are defined as usual in homotopy theory. Other examples of this structure are exact categories in the sense of Quillen, with the admissible monomorphisms as cofibrations and the isomorphisms as weak equivalences.

Given a category $\mathcal{C}$ with cofibrations and weak equivalences, Waldhausen introduces the simplicial category $w\mathcal{S}_\bullet \mathcal{C}$. The category $w\mathcal{S}_n \mathcal{C}$ in degree n of this simplicial category has as its objects the diagrams

$$Y_1 \rightrightarrows Y_2 \rightrightarrows \cdots \rightrightarrows Y_n,$$

with the arrows $\twoheadrightarrow$ denoting cofibrations, and as its morphisms the diagrams

$$\begin{array}{ccccccc} Y_1 & \twoheadrightarrow & Y_2 & \twoheadrightarrow & \cdots & \twoheadrightarrow & Y_n \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ Z_1 & \twoheadrightarrow & Z_2 & \twoheadrightarrow & \cdots & \twoheadrightarrow & Z_n \end{array}$$

with the vertical arrows weak equivalences. One also needs to specify choices of quotients Y_j/Y_i . Thus, for example, $w\mathcal{S}_0\mathcal{C}$ is the trivial category consisting only of the 0-object and the 0-morphism, and $w\mathcal{S}_1\mathcal{C}$ is (equivalent to) the category $w\mathcal{C}$ of weak equivalences in $\mathcal{C}$. The K -theory $\mathbb{K}(\mathcal{C})$ of the category $\mathcal{C}$ can then be defined to be $\Omega|w\mathcal{S}_\bullet\mathcal{C}|$. This turns out to be an infinite loop space [98, p. 342]. Also, Waldhausen shows that this definition is essentially equivalent to the usual definition (via the $+$ -construction or Q -construction) of Quillen K -theory (when both make sense). The equivalence $w\mathcal{C} \xrightarrow{\sim} w\mathcal{S}_1\mathcal{C}$ gives rise to a map $\Sigma|w\mathcal{C}| \hookrightarrow |w\mathcal{S}_\bullet\mathcal{C}|$, and thus to a dual map

$$|w\mathcal{C}| \rightarrow \Omega|w\mathcal{S}_\bullet\mathcal{C}| =_{\text{def}} \mathbb{K}(\mathcal{C}) .$$

The algebraic K -theory of a space X (which we think of as a simplicial set – passage from simplicial sets to spaces is given by the geometric realization functor $|\cdot|$) is then defined to be $\mathbb{K}(\mathcal{R}_f(X))$, where $\mathcal{R}_f(X)$ is the category of *finite retractive spaces over X* , or in other words, simplicial sets Y equipped with an inclusion $X \hookrightarrow Y$, plus a retraction $r: Y \rightarrow X$, so that Y consists of the union of X and finitely many additional simplices. The map $X \mapsto A(X)$ is a homotopy functor of X [98, Proposition 2.1.7], and there is a pairing $A(X) \wedge A(X') \rightarrow A(X \times X')$ [96, pp. 400–402]. The map $|w\mathcal{R}_f(*)| \rightarrow A(*)$ is characterized by a certain additivity property [96, Lemmas 1.1 and 1.2]; on the level of π_0 , it sends a homotopy equivalence class of finite spaces (or simplicial sets) to $\pi_0(A(*)) = \mathbb{Z}$, and turns out to be the Euler characteristic. Other applications of the algebraic K -theory of spaces will be mentioned in the following section, Sect. 2.7. But we just mention that $A(X)$ satisfies an analogue of the “fundamental theorem of K -theory” (the calculation of $K_*(R[t, t^{-1}])$ in terms of $K_*(R)$):

15

Theorem 15: ([44, 45, 77]) There is a splitting of $A(X \times S^1)$ as

$$A(X \times S^1) \simeq A(X) \times \Omega^{-1}A(X) \times \text{“Nil term”} \times \text{“Nil term”} .$$

(This notation isn’t completely precise but is meant to imply that the second factor is a delooping of $A(X)$. More details may be found in the original papers.) The two Nil terms are homeomorphic, and the “canonical involution” on $A(X \times S^1)$ (analogous to the involution on K -theory of rings coming from the conjugate transpose on matrices) interchanges the two Nil terms and restricts to the canonical involutions on the other two factors.

Given that the definition of A -theory involves so much abstract machinery, it is perhaps surprising that so much is known about how to calculate $A(X)$. One

of the key tools in this regard is the *cyclotomic trace* of Bökstedt, Hsiang, and Madsen [8], a functorial map $\mathrm{Trc}: A(X) \rightarrow TC(X; p)$ from A -theory to topological cyclic homology. (To define this map, it is necessary to first fix a prime p .) There is a beautiful theorem of Dundas about the fiber of this map after p -completion:

Theorem 16: *Dundas [28]* If X is connected, then the diagram

$$\begin{array}{ccc} A(X)_p^\wedge & \xrightarrow{\mathrm{Trc}} & TC(X; p)_p^\wedge \\ L \downarrow & & \downarrow \\ \mathbb{K}(\mathbb{Z}\pi_1(X))_p^\wedge & \xrightarrow{\mathrm{Trc}} & TC(\mathbb{Z}\pi_1(X); p)_p^\wedge \end{array}$$

is homotopy Cartesian (i.e., is a homotopy pullback square).

In particular, the fiber of the cyclotomic trace map (after p -completion) only depends on $\pi_1(X)$, and not on the rest of the homotopy type of X . (This was earlier proved in [7].) And after p -completion, the homotopy fiber of the linearization map from A -theory to K -theory of the group ring can be computed entirely from TC -theory.

***K*-Theory and Pseudo-Isotopy**

Let M be a compact smooth manifold (for now without boundary, but we will be forced to consider manifolds with boundary later). The space of *pseudo-isotopies* (or *concordances*) of M is defined to be

$$\mathcal{C}(M) = \mathrm{Diff}(M \times I \text{ rel } (M \times \{0\} \cup \partial M \times I)) ,$$

with the C^∞ topology. (See Fig. 2.3.) This is of course a topological group under composition of diffeomorphisms. A basic problem in manifold topology is to understand this space, and especially its set of connected components. This problem is closely related to computing $\pi_0(\mathrm{Diff}(M))$, the group of diffeomorphisms of M modulo isotopy. The reason is that, on the one hand, an isotopy of diffeomorphisms

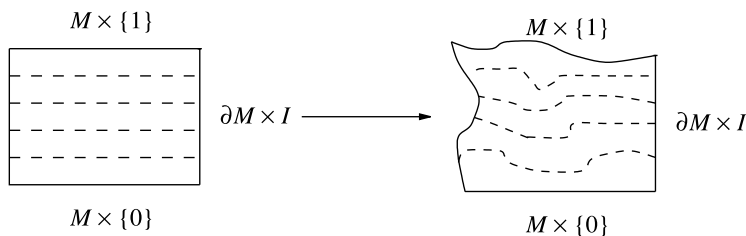


Figure 2.3. A pseudo-isotopy

of M clearly induces a pseudo-isotopy. But not every pseudo-isotopy comes from an isotopy, since the “level sets” $M \times \{t\}$ don’t have to be preserved for $t > 0$. (Again see Fig. 2.3.) But $\mathcal{C}(M)$ acts continuously on $\text{Diff}(M)$ by $h \cdot g = h|_{M \times \{1\}}g$, and the (open) orbit of the identity is the group of diffeomorphisms pseudo-isotopic to the identity. So if $\mathcal{C}(M)$ is path-connected, pseudo-isotopic diffeomorphisms are isotopic. The first major result about $\pi_0(\mathcal{C}(M))$ was a difficult theorem of Cerf [18]: $\mathcal{C}(M)$ is path-connected if M is simply connected and $\dim M \geq 6$.⁴ However, it was soon discovered that even in high dimensions, $\mathcal{C}(M)$ can be disconnected if $\pi_1(M)$ is non-trivial, and Hatcher and Wagoner [37] (originally working independently) discovered a remarkable connection between $\pi_0(\mathcal{C}(M))$ and the K -group $K_2(\mathbb{Z}\pi_1(M))$. This eventually led to an exact sequence for $\pi_0(\mathcal{C}(M))$:

$$\begin{aligned} K_3(\mathbb{Z}\pi_1(M)) \rightarrow \text{Wh}_1^+(\pi_1(M); \mathbb{Z}/2 \times \pi_2(M)) \\ \rightarrow \pi_0(\mathcal{C}(M)) \rightarrow \text{Wh}_2(\pi_1(M)) \rightarrow 0. \end{aligned} \quad (2.5)$$

Here $\text{Wh}_2(\pi_1(M))$ denotes the quotient of $K_2(\mathbb{Z}\pi_1(M))$ by its intersection (when we think of K_2 as a subgroup of the Steinberg group) with the subgroup of the Steinberg group $\text{St}(\mathbb{Z}\pi_1(M))$ generated by the special elements $w_{ij}(g)$, $g \in \pi_1(M)$. This insures that we divide K_2 by its trivial part (the image of $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$). (See [74, Definition 4.4.25].) The second term in (2.5) is to be interpreted using the definition

$$\text{Wh}_1^+(\pi; A) = H_0(\pi, A\pi)/H_0(\pi, A).$$

Note that we need to keep track of the action of $\pi_1(M)$ on $\pi_2(M)$ to compute this. Hatcher and Wagoner [37] constructed the surjection $\pi_0(\mathcal{C}(M)) \rightarrow \text{Wh}_2(\pi_1(M))$ in (2.5), Hatcher [37] extended the exact sequence to Wh_1^+ , and K. Igusa [47] corrected a mistake of Hatcher and extended the sequence to K_3 .

The exact sequence (2.5), along with Igusa’s work in [47] showing how the first Postnikov invariant $k_1(M) \in H^3(\pi_1(M), \pi_2(M))$ can affect $\pi_0(\mathcal{C}(M))$, makes it clear that calculation of the topology of $\mathcal{C}(M)$ must in general be quite complicated. Since this problem is hard and “unstable,” it is useful to “stabilize.” One can define a suspension map $\sigma: \mathcal{C}(M) \rightarrow \mathcal{C}(M \times I)$. (The subtlety here is that if M has a boundary, $M \times I$ is a manifold with corners, but still, there is no problem in suspending a pseudo-isotopy φ to $\varphi \times \text{id}_I$.) The inductive limit $\mathcal{P}(M) = \varinjlim \mathcal{C}(M \times I^n)$ turns out to be an infinite loop space whose structure can be calculated in many cases; more about this later. Then one can obtain results about $\mathcal{C}(M)$ itself thanks to a second result of Igusa (quite technical to prove):

Theorem 17: [48] The suspension map $\sigma: \mathcal{C}(M) \rightarrow \mathcal{C}(M \times I)$ is k -connected if $\dim M \geq \max(2k + 7, 3k + 4)$.

⁴ There is an obvious analogue of $\mathcal{C}(M)$, denoted $\mathcal{C}^{\text{PL}}(M)$, for PL manifolds, where we replace Diff by the group of PL automorphisms. The analogue of Cerf’s theorem holds for this as well [75].

Igusa's proof follows an outline in [35] of an analogous theorem for $\mathcal{C}^{\text{PL}}$ of a PL manifold, but there are problems with the PL proof given there, due to the fact that pushouts do not exist for most pairs of maps of polyhedra. However, concordance stability for smooth manifolds implies stability for PL or topological concordances, for manifolds that have a smooth structure, by a result of Burghlelea and Lashof [13].

Before proceeding to the more technical aspects of pseudo-isotopy, it might be worth explaining the rough idea of why $\pi_0(\mathcal{C}(M))$ is related to K_2 (and in fact surjects onto $\text{Wh}_2(\pi_1(M))$). The ideas here come from the papers of Cerf [18] and Hatcher–Wagoner [37] quoted above. The starting point of the proof is an observation of Cerf that $\mathcal{C}(M)$ is homotopy-equivalent to the space $\mathcal{E}(M)$ of functions $f: M \times [0, 1] \rightarrow [0, 1]$ which are smooth, have no critical points, and satisfy $f(x, 0) = 0$ and $f(x, 1) = 1$ for all $x \in M$. The homotopy equivalence is simply the map that sends $h \in \mathcal{C}(M)$ to $f: (x, t) \mapsto p_2 \circ h(x, t)$, where $p_2: M \times [0, 1] \rightarrow [0, 1]$ is projection onto the 2nd coordinate. A homotopy inverse $\mathcal{E}(M) \rightarrow \mathcal{C}(M)$ to this map is constructed by fixing a Riemannian metric on M and sending $f \in \mathcal{E}(M)$ to the pseudo-isotopy constructed from its gradient flow. So given $h \in \mathcal{C}(M)$, its obstruction in $\text{Wh}_2(\pi)$ will be constructed using a path f_t of smooth functions $M \times [0, 1] \rightarrow [0, 1]$ with $f_0 = p_2$ and $f_1 = p_2 \circ h$. If this path can be deformed to one with no critical points, then h must lie in the identity component of $\mathcal{C}(M)$. One starts by using the usual ideas of differential topology to deform f to a “generic” function with non-degenerate isolated critical points, and then analyzes what happens as one goes from one critical point to the next (so far this is like the start of the proof of the h -cobordism theorem). In the simplest case where all the critical points are either of index i or index $i + 1$, one gets for each t a realization of $M \times [0, 1]$ as being obtained from $M \times [0, 1]$ by attaching i -handles and $(i + 1)$ -handles. Since $M \times [0, 1]$ is topologically a product, these handles have to cancel as far as their effect on $(\pi_1(M)$ -equivariant) homology of the universal cover is concerned, so one gets an intersection matrix $A(t)$ in $GL(\mathbb{Z}\pi_1(M))$ measuring how the i -handles (coming from critical points of index i) are cancelled by the $(i + 1)$ -handles. The function $t \mapsto A(t)$ also has to be piecewise constant, with jumps just at the critical values of t . For t close to 0, $A(t)$ is the identity matrix; near $t = 1$ it is a product of a permutation matrix and a diagonal matrix with entries of the form $\pm g$, $g \in \pi_1(M)$; and in between it changes finitely many times by certain elementary matrices $e_{jk}(\pm g)$. So if one takes the Steinberg generators $x_{jk}(\pm g)$ corresponding to the $e_{jk}(\pm g)$, one finds that their product gives rise to an element of $\text{St}(\mathbb{Z}\pi_1(M))$ which lifts $A(1)$. However there is a canonical way to lift any product of a permutation matrix and a diagonal matrix, and in particular $A(1)$, as a product of the $w_{jk}(\pm g)$'s. Dividing, one gets an element of $K_2(\mathbb{Z}\pi)$ which is well-defined modulo the subgroup of $\text{St}(\mathbb{Z}\pi_1(M))$ generated by all $w_{ij}(g)$, $g \in \pi_1(M)$, i.e., an element of $\text{Wh}_2(\pi_1(M))$. One can show that this element doesn't change under smooth deformation, so it gives an obstruction to being able to deform f to a function without critical points.

A program for studying the stabilized pseudo-isotopy space $\mathcal{P}^{\text{PL}}(M)$ in the PL category, by relating it to more homotopy-theoretic objects, was sketched

in [36] and [35] without rigorous proofs. A vast generalization of the program was developed and carried out by Waldhausen. He introduced homotopy functors $\mathbf{Wh}^{\text{PL}}$ and $\mathbf{Wh}^{\text{diff}}$ with the properties that $\Omega^2 \mathbf{Wh}^{\text{diff}}(M) \simeq \mathcal{P}(M)$ for compact smooth manifolds and $\Omega^2 \mathbf{Wh}^{\text{PL}}(M) \simeq \mathcal{P}^{\text{PL}}(M)$ for compact PL manifolds. As we mentioned before (near the beginning of section 2.6), Waldhausen showed that $\mathbf{Wh}^{\text{diff}}(X)$ is one factor in $A(X)$. (The other factor is $Q(X_+)$.) There is also a map $A(X) \rightarrow \mathbf{Wh}^{\text{PL}}(X)$, and its homotopy fiber is a homology theory, but it's a little harder to understand. The correct analogue of the formula (2.5) for general X is an exact sequence [46, Theorem 13.1]:

$$\begin{aligned} \pi_3(A(X)) \rightarrow K_3(\mathbb{Z}\pi_1(X)) \rightarrow H_0(\pi_1(X), (\pi_2(X) \oplus \mathbb{Z}/2)\pi_1(X)) \\ \rightarrow \pi_2(A(X)) \rightarrow K_2(\mathbb{Z}\pi_1(X)) \rightarrow 0. \end{aligned}$$

The machinery that's known for computing $A(X)$ (at least rationally) in some circumstances thus implies quite a lot of information about pseudo-isotopies and groups of homeomorphisms and diffeomorphisms of manifolds. For example, Farrell and Jones [30, Corollaries 10.6 and 10.7] compute the rational homotopy groups $\pi_j(\text{Homeo}(M)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\pi_j(\text{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q}$ for M a real hyperbolic manifold of dimension $m > 10$ and j in a stable range ($\leq (m-4)/3$). The connection between $A(X)$ and pseudo-isotopies also makes it possible to study not only "higher" Whitehead torsion (as in [35]), but also higher Reidemeister torsion (as in [49]).

We should point out also that there are controlled versions of pseudo-isotopy theory, which are related to negative K -theory (e.g., [2], [43], and [41]).

2.8

***K*-Theory and Symbolic Dynamics**

Among the lesser known applications of K -theory to geometric topology are applications to *symbolic dynamics*, the study of invariant subspaces of the shift map acting on infinite sequences of letters from some alphabet. To fix notation, consider the *full n -shift*, $X_n^+ = \{0, 1, \dots, n-1\}^{\mathbb{N}}$ or $X_n = \{0, 1, \dots, n-1\}^{\mathbb{Z}}$ with the product topology. (Topologically, X_n^+ and X_n are both Cantor sets if $n > 1$.) Let $\sigma_n: X_n \rightarrow X_n$ and $\sigma_n^+: X_n^+ \rightarrow X_n^+$ be the shift map that shifts a sequence one unit to the left. The map σ_n is a self-homeomorphism of X_n , called the *two-sided n -shift*, and σ_n^+ is a surjective (but non-invertible) self-map of X_n^+ , called the *one-sided n -shift*.

A *subshift of finite type* is a pair (X_A, σ_A) , where A is an $n \times n$ matrix with entries in $\{0, 1\}$, where X_A is the closed σ_n -invariant subset of X_n consisting of sequences (x_k) with *allowable transitions*, i.e., with $A_{x_k x_{k+1}} = 1$ for all k , and where $\sigma_A = \sigma_n|_{X_A}$. The one-sided subshift of finite type (X_A^+, σ_A^+) is defined similarly from (X_n^+, σ_n^+) . The first basic problem of symbolic dynamics is to classify the pairs (X_A, σ_A) and (X_A^+, σ_A^+) up to topological conjugacy. Note that keeping track

of the shift structure is essential here, since all X_A 's and X_A^+ 's are Cantor sets⁵ and are thus homeomorphic to one another, regardless of the values of n and of the matrix A .

One might, of course, wonder why we are considering homeomorphisms of Cantor sets when we promised at the beginning of this article to restrict attention to "topology of manifolds and manifold-like spaces, of simplicial and CW-complexes, and of automorphisms of such objects." The reason is that as amply demonstrated by Smale [80], Bowen [10], and others, any attempt to study the dynamics of smooth self-maps of manifolds inevitably leads to problems of symbolic dynamics.

For purposes of studying the conjugacy problem for the pairs (X_A, σ_A) , it's convenient to allow A to be *any* square matrix with entries in $\mathbb{N}$, the non-negative integers. There is a canonical way to do this [91, pp. 272–273] without changing the definition of X_A in the case of a 0-1 matrix, and so that the 1×1 matrix (n) and the $n \times n$ matrix with all entries equal to 1 both give rise to X_n . However, any X_A can be rewritten as $X_{A^\#}$ for some 0-1 matrix $A^\#$ (usually of larger size than A).

The key initial work on the conjugacy problem for the pairs (X_A, σ_A) was done by Williams [104], who showed that σ_A and σ_B are topologically conjugate if there are rectangular (not necessarily square!) matrices R and S with entries in $\mathbb{N}$ such that $RS = A$, $SR = B$. This relation is called *elementary strong shift equivalence* over $\mathbb{N}$, but this is a slight misnomer: it is not an equivalence relation. The equivalence relation it generates (on square matrices of arbitrary size with entries in $\mathbb{N}$) is called *strong shift equivalence* over $\mathbb{N}$, and Williams proved that σ_A and σ_B are topologically conjugate if and only if the matrices A and B are strong shift equivalent over $\mathbb{N}$. Williams also gave a necessary and sufficient condition for topological conjugacy of the one-sided shifts σ_A^+ and σ_B^+ in terms of conjugacy of "total amalgamations," and this criterion is computable. However, strong shift equivalence is not especially computable – the problem is that there is no obvious way to bound the length of a chain of elementary strong shift equivalences. Thus Williams also introduced another equivalence relation. Two square matrices A and B with entries in $\mathbb{N}$ are called *shift equivalent* over $\mathbb{N}$ if there are rectangular matrices R and S with entries in $\mathbb{N}$ such that $AR = RB$, $SA = BS$, and for some $k \geq 1$, $A^k = RS$ and $B^k = SR$. It turns out that shift equivalence over $\mathbb{N}$ is computable and that the matrices A and B are shift equivalent over $\mathbb{N}$ if and only if σ_A^k and σ_B^k are topologically conjugate for all sufficiently large k . An unsolved problem for many years, called the *shift equivalence problem*, was whether shift equivalence implies strong shift equivalence (over $\mathbb{N}$), or equivalently, if conjugacy of σ_A^k and σ_B^k for all large k implies conjugacy of σ_A and σ_B .

The (negative) solution to the shift equivalence problem heavily involves K -theory. First of all, shift equivalence turns out to be connected with the ordering on K_0 of a ring, a certain C^* -algebra associated to the shift. As a result, one can for example prove:

⁵ This is assuming we are not in one of the uninteresting cases where X_A or X_A^+ contains an isolated point, as when $A = (1)$.

Theorem 18: [91, Corollary 2.13] If $A, B \in GL(n, \mathbb{Z}) \cap M_n(\mathbb{N})$, then A and B are shift equivalent over $\mathbb{N}$ if and only if A and B are conjugate in $GL(n, \mathbb{Z})$.

Also, if one drops the requirement that the matrices defining a shift equivalence have non-negative entries and thus considers shift equivalence and strong shift equivalence over $\mathbb{Z}$, then these two conditions are indeed equivalent [90].

However, over $\mathbb{N}$, Kim and Roush [53] showed that shift equivalence and strong shift equivalence are not equivalent, even for primitive matrices (the most important case). While their original construction did not directly involve K -theory, it was partially motivated by work of Wagoner [89] relating $\text{Aut}(\sigma_A)$ to K_2 , and in [92], Wagoner, Kim, and Roush showed that one can indeed construct a counterexample to the shift equivalence problem using an invariant based on $K_2(\mathbb{Z}[t]/(t^2))$. A good introduction to this work may be found in [91]. If one looks carefully, one can see the connection with the ideas of Cerf theory and the connection between pseudo-isotopy and K_2 .

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Quadratic K-Theory and Geometric Topology

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3.1	<i>Introduction</i>	613
3.2	<i>Basic Algebraic Definitions</i>	617
	K-Theory of Quadratic Forms	617
	L-Theory of Quadratic Forms	619
3.3	<i>Classification of Manifolds up to Homeomorphism</i>	620
3.4	<i>Higher Hermitian K-Theory</i>	625
	Homotopy Fixed Spectrum, Homotopy Orbit Spectrum, and the Norm Fibration Sequence	625
	Thomason's Homotopy Limit and Homotopy Colimit Problems	626
	Karoubi Periodicity	628
	Karoubi's Hyperbolic and Forgetful Tricks	629
	Twists and Dimension Shifting for Cohomology	631
	General Definition of L -Groups (when 2 is a Unit)	632
3.5	<i>Symmetric and Quadratic Structures on Chain Complexes</i>	634
	Symmetric Complexes	634
	Quadratic Chain Complexes	635
3.6	<i>Manifold Structures</i>	639
	Higher Whitehead Torsion	640
	Bounded Block Structure Spaces	641
	More About Torsion	642
	Poincaré Duality and Torsion	643

* Research partially supported by NSF.

Homotopy Cartesian for a Range	644
More on Connections Between Quadratic Forms and Manifold Structures.	644
Localize at Odd Primes	645
<i>References</i>	646

Introduction

Suppose R is a ring with an (anti)-involution $-: R \rightarrow R$, and with choice of central unit ε such that $\bar{\varepsilon}\varepsilon = 1$. Then one can ask for a computation of $\mathbb{K}Quad(R, -, \varepsilon)$, the K-theory of quadratic forms. Let $H: \mathbb{K}R \rightarrow \mathbb{K}Quad(R, -, \varepsilon)$ be the hyperbolic map, and let $F: \mathbb{K}Quad(R, -, \varepsilon) \rightarrow \mathbb{K}R$ be the forget map. Then the Witt groups

$$W_0(R, -, \varepsilon) = \text{coker} \left(K_0 R \xrightarrow{H} K_0 Quad(R, -, \varepsilon) \right)$$

$$W_1(R, -, \varepsilon) = \ker \left(F: KQuad_1(R, -, \varepsilon) \xrightarrow{F} K_1 R \right)$$

have been highly studied. See [6], [29, 32, 42, 46, 68], and [86]–[89]. However, the higher dimensional quadratic K-theory has received considerably less attention, than the higher K-theory of f.g. projective modules. (See however, [34, 35, 39], and [36].)

Suppose M is an oriented, closed topological manifold of dimension n . We let

$G(M)$ = simplicial monoid of homotopy automorphisms of M ,

$Top(M)$ = sub-simplicial monoid of self-homeomorphisms of M , and

$$\mathcal{S}(M) = \sqcup G(N)/Top(N),$$

where we take the disjoint union over homeomorphisms classes of manifolds homotopy equivalent to M . Then $\mathcal{S}(M)$ is called the *moduli space* of manifold structures on M .

In classical surgery theory (see Sect. 3.3) certain subquotients of $K_j Quad(R, -, \varepsilon)$ with $j = 0, 1$; $R = \mathbb{Z}\pi_1 M$; and $\varepsilon = \pm 1$ are used to compute $\pi_0 \mathcal{S}(M)$.

The main goals of this survey article are as follows:

1. Improve communication between algebraists and topologists concerning quadratic forms
2. Call attention to the central role of *periodicity*.
3. Call attention to the connections between $KQuad(R, -, \varepsilon)$ and $\mathcal{S}(M)$, not just $\pi_0 \mathcal{S}(M)$.
4. Stimulate interest in the higher dimensional quadratic K-groups.

The functor which sends a f.g. projective module P to $Hom_R(P, R)$ induces an involution T on $\mathbb{K}(R)$. For any $i, j \in \mathbb{Z}$ and any T -invariant subgroup $X \subset K_j(R)$, topologists (see Sect. 3.5) have defined groups $L_i^X(R)$. The subgroup X is called the *decoration* for the L-group. Here are a few properties:

1. *Periodicity* $L_i^X(R) \simeq L_{i+4}^X(R)$
2. $L_{2i}^{K_0}(R) \simeq W_0(R, -, (-1)^i)$
3. $L_{2i+1}^{K_2}(R) \simeq W_1(R, -, (-1)^i)$
4. $L_i^{K_j}(R) \simeq L_i^{O_{j-1}}(R)$, where O_{j-1} is the trivial subgroup of $K_{j-1}(R)$

5. *Rothenberg Sequences* If $X \subset Y \subset K_j(R)$, we get an exact sequence

$$\cdots \rightarrow L_i^X(R) \rightarrow L_i^Y(R) \rightarrow \hat{H}^i(\mathbb{Z}/2, Y/X) \rightarrow \cdots$$

6. *Shaneson Product Formula* For all $i, j \in \mathbb{Z}$,

$$L_{i+1}^{K_{j+1}}(R[t, t^{-1}]) \cong L_{i+1}^{K_{j+1}}(R) \oplus L_i^{K_j}(R),$$

where we extend the involution on R to the Laurent ring $R[t, t^{-1}]$ by $\bar{t} = t^{-1}$.

Notice that $L_i^X(R) \otimes \mathbb{Z}[\frac{1}{2}]$ is independent of X . Different choices for X are used to study various geometric questions. The classification of compact topological manifolds uses $X \subset K_1$ (See Sect. 3.3). The study of open manifolds involves $X \subset K_j$, with $j < 1$, (See [25, 31, 50, 54, 65] and [71]). The study of homeomorphisms of manifolds involves $X \subset K_j$, with $j > 1$, (see Sect. 3.6).

Localization Sequences: Suppose S is a multiplicative system in the ring R . Then we get an exact sequence

$$\cdots \rightarrow K_i(R, S) \rightarrow K_i(R) \rightarrow K_i(S^{-1}R) \rightarrow \cdots,$$

where $K_i(R, S)$ is the K -theory of the exact category of S -torsion R -modules of homological dimension 1. In the case of L -theory (with appropriate choice of decorations) one gets an analogous exact sequence using linking forms on torsion modules (see [61, 64], and [53]). However, what is striking about the L -theory localization is that it is gotten by splicing together two 6-term exact sequences. One of these involves $(R, -, +1)$ quadratic forms and the other involves $(R, -, -1)$ quadratic forms. The resulting sequence is then 12-fold *periodic*. In fact L -theory satisfies many other such periodic exact sequences (see [64]).

Let $L_i^{<-\infty>}(R)$ be the direct limit of $L_i^{K_0}(R) \rightarrow L_i^{K_{-1}}(R) \rightarrow \cdots$.

Let $\mathbb{K}(R)$ be the K -theory spectrum constructed by Wagoneer [78] where for all $i \in \mathbb{Z}$, $K_i(R) \simeq \pi_i(\mathbb{K}(R))$. Similarly, let $\mathbb{K}Quad(R, -, \epsilon)$ be the spectrum where for all $i \in \mathbb{Z}$, $KQuad_i(R, -, \epsilon) \simeq \pi_i(\mathbb{K}Quad(R, -, \epsilon))$. Similarly, let $\mathbb{K}Herm(R, -, \epsilon)$ be the K -theory spectrum for Hermitian forms (see Sects. 3.2 and 3.4). There is a functor $Quad(R, -, \epsilon) \rightarrow Herm(R, -, \epsilon)$ which induces a homotopy equivalence on K -theory when 2 is a unit in R .

Given a spectrum $\mathbb{K}$ equipped with an action by a finite group G we get the norm homotopy fibration sequence

$$\mathbb{H}_*(G, \mathbb{K}) \xrightarrow{N} \mathbb{H}^*(G, \mathbb{K}) \rightarrow \hat{\mathbb{H}}^*(G, \mathbb{K}),$$

where $\mathbb{H}_*(G, \mathbb{K})$ is the homotopy orbit spectrum of G acting on $\mathbb{K}$, $\mathbb{H}^*(G, \mathbb{K})$ is the homotopy fixed spectrum, and N is the *norm* map.

The key example for us is $\mathbb{K} = \mathbb{K}(R)$, $G = \mathbb{Z}/2$, and the action is given by the involution T .

Theorem 1: (*Hermitian K-theory Theorem*) There exists a homotopy cartesian diagram

$$\begin{array}{ccc} \mathbb{K}Herm(R, -, \varepsilon) & \rightarrow & \mathcal{L}(R, -, \varepsilon) \\ \tilde{F} \downarrow & & \downarrow \\ \mathbb{H}^*(\mathbb{Z}/2, \mathbb{K}(R)) & \rightarrow & \hat{\mathbb{H}}^*(\mathbb{Z}/2, \mathbb{K}(R)) \end{array}$$

with the following properties.

1. If 2 is a unit in R , then for $i = 0$ or 1 ,

$$\pi_i \mathcal{L}(R, -, +1) \simeq L_i^{<-\infty>}(R)$$

$$\pi_i \mathcal{L}(R, -, -1) \simeq L_{i+2}^{<-\infty>}(R)$$

2. **Periodicity:** If 2 is a unit in R , then $\Omega^2 \mathcal{L}(R, -, \varepsilon) \simeq \mathcal{L}(R, -, -\varepsilon)$.
3. The composition $\mathbb{K}Herm(R, -, \varepsilon) \xrightarrow{\tilde{F}} \mathbb{H}^*(\mathbb{Z}/2, \mathbb{K}(R)) \rightarrow \mathbb{K}(R)$ is the forgetful map F .
4. The homotopy fiber of $\mathbb{K}Herm(R, -, \varepsilon) \rightarrow \mathcal{L}(R, -, \varepsilon)$ is a map $\tilde{H}: \mathbb{H}_*(\mathbb{Z}/2, \mathbb{K}(R)) \rightarrow \mathbb{K}Herm(R, -, \varepsilon)$ such that the composition $\mathbb{K}(R) \rightarrow \mathbb{H}_*(\mathbb{Z}/2, \mathbb{K}(R)) \xrightarrow{\tilde{H}} \mathbb{K}Herm(R, -, \varepsilon)$ is the hyperbolic map.

We call $\tilde{F}$ the *enhanced forgetful map*, and $\tilde{H}$ the *enhanced hyperbolic map*.

Before we state an analogous theorem for $\mathcal{S}(M)$ we need to introduce some more background.

Let $hcob(M)$ be the simplicial set of h-cobordisms on M and let $hcob(M) \rightarrow \mathcal{S}(M)$ be the map which sends an h-cobordism $h: (W, \partial W) \rightarrow (M \times I, M \times \partial I)$ to $h|_{M_1}: M_1 \rightarrow M \times 1$ where $\partial W = M \sqcup M_1$. Let $HCOB(M)$ be the homotopy colimit of

$$hcob(M) \rightarrow hcob(M \times I) \rightarrow hcob(M \times I^2) \dots$$

Igusa has shown that if M is smoothable, then the map $hcob(M) \rightarrow HCOB(M)$ is at least $k + 1$ -connected where $n = \dim M \geq \max(2k + 7, 3k + 4)$.

Let $\Omega WH(\mathbb{Z}\pi_1(M))$ be the homotopy fiber of the assembly map $\mathbb{H}^*(M, \mathbb{K}\mathbb{Z}) \rightarrow \mathbb{K}\mathbb{Z}\pi_1(M)$. For $n > 4$, the s-cobordism theorem yields a bijection $\pi_0(hcob(M)) \rightarrow \pi_0(\Omega WH(\mathbb{Z}\pi_1(M)))$. Waldhausen [80] and Vogel [76, 77] has shown how in the definition of $\mathbb{K}\mathbb{Z}\pi_1(M)$ we can replace $\pi_1(M)$ with the loop space of M and $\mathbb{Z}$ with the sphere spectrum. This yields $\mathbb{A}(M)$, the K-theory of the space M . There exists a linearization map $\mathbb{A}(M) \rightarrow \mathbb{K}\mathbb{Z}\pi_1(M)$ which is 2-connected. Furthermore, there exists a homotopy equivalence $HCOB(M) \rightarrow \Omega WH(M)$, where $\Omega WH(M)$ is the homotopy fiber of the assembly map $\mathbb{H}_*(M, \mathbb{A}(*)) \rightarrow \mathbb{A}(M)$.

Constructions of Ranicki yield spectrum $\mathbb{L}^X(R)$ such that $\pi_i(\mathbb{L}^X(R)) \simeq L_i^X(R)$. Let $\mathbb{L}$ be the 1-connected cover of $\mathbb{L}^{K_0}(\mathbb{Z})$.

2

Theorem 2: (*Manifold Structure Theorem*) Poincaré duality yields an involution on $WH(M)$, and there exists a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{J}(M) & \rightarrow & \mathcal{J}^{-\infty}(M) \\ \downarrow & & \downarrow \\ \mathbb{H}^*(\mathbb{Z}/2, \Omega WH(M)) & \rightarrow & \hat{\mathbb{H}}(\mathbb{Z}/2, \Omega WH(M)) \end{array}$$

with the following properties.

1. There exists a homotopy equivalence between $\mathcal{J}^{-\infty}(M)$ and the -1 -connected cover of Ω^n of the homotopy fiber of the assembly map

$$\mathbb{H}_*(M, \mathbb{L}) \rightarrow \mathbb{L}^{<-\infty>}(\mathbb{Z}\pi_1(M)) .$$

(See [47] for background on assembly maps.)

2. **Periodicity:** There exists a map $\mathcal{J}^{-\infty}(M) \rightarrow \Omega^4 \mathcal{J}^{<-\infty>}(M)$ which induces an isomorphisms on π_i for $i > 0$.
3. The composition $\pi_0(\mathcal{J}(M)) \rightarrow \pi_0(\mathbb{H}^*(\mathbb{Z}/2, \Omega WH(M))) \rightarrow \pi_0(\Omega WH(M))$ sends a homotopy equivalence $h_1 : M_1 \rightarrow M$ to the Whitehead torsion of h_1 .
4. The map $hcob(M) \rightarrow S(M)$ factors thru the homotopy fiber of $\mathcal{J}(M) \rightarrow \mathcal{J}^{-\infty}(M)$.
5. If M is a smoothable manifold, then the above homotopy commutative diagram is homotopy cartesian thru dimension $k + 1$ where $\dim M \geq \max(2k + 7, 3k + 4)$.

Notice the strong analogy between these two theorems.

In Sect. 3.2 we use the hyperbolic and forgetful maps to define $L_i^{K_1(R)}(R)$ for $i \in \mathbb{Z}$. The product formula is then used to define $L_j^{K_j(R)}(R)$ for $j \leq 0$ and $(R, -, \epsilon)$ any Hermitian ring. In Sect. 3.3 we explain how $L_i^{K_1(R)}(R)$ is used to classify manifolds up to homeomorphism. In Sect. 3.4 we use constructions of Thomason and Karoubi to define $L_i^X(R)$ for X any $\mathbb{Z}/2$ -invariant subgroup of $K_j(R)$ for any $i, j \in \mathbb{Z}$ under the assumption that 2 is a unit in R . This assumption is needed in order to be able to use Karoubi Periodicity. In Sect. 3.4 we also discuss the Hermitian K-theory Theorem. In Sect. 3.5 we discuss Ranicki's approach to L -theory via structures on chain complexes. This yields a periodicity theorem where we do not have to assume 2 is a unit, and for any $i, j \in \mathbb{Z}$ and any T -invariant subgroup $X \subset K_j(R)$, we define $L^X(R)$ where $(R, -, \epsilon)$ is any Hermitian ring. In Sect. 3.6 we discuss the Manifold Structure Theorem. Also Ranicki's quadratic chain complexes are used to show that the relationship between these two theorems is much more than just an analogy.

Basic Algebraic Definitions

3.2

See [6, 57, 85, 88] and [40].

K-Theory of Quadratic Forms

3.2.1

A *hermitian ring*, denoted by $(R, -, \varepsilon)$ is a ring R equipped with an anti-involution $- : R \rightarrow R$ and a preferred element $\varepsilon \in \text{center}(R)$ such $\bar{\varepsilon} \cdot \varepsilon = 1$.

Let P be a finitely generated projective right R -module. A $(-, \varepsilon)$ -sesquilinear form on P is a biadditive map $\beta : P \times P \rightarrow R$ such that $\beta(p_1 r_1, p_2 r_2) = \bar{r}_1 \beta(p_1, p_2) r_2$ where $p_i \in P$ and $r_i \in R$, for $i = 1, 2$.

We let $\text{Sesq}(P) = \text{group of all } (-, \varepsilon)\text{-sesquilinear forms on } P$. Let $T_\varepsilon : \text{Sesq}(P) \rightarrow \text{Sesq}(P)$ be the involution: $T_\varepsilon(\beta)(p_1, p_2) = \beta(p_2, p_1)\varepsilon$. We make $P^* = \text{Hom}_R(P, R)$ a right R -module by $\alpha r(p) = \bar{r}\alpha(p)$, where $\alpha \in P^*$, $r \in R$ and, $p \in P$. If $f : P \rightarrow Q$ is a map of right R modules, then $f^* : Q^* \rightarrow P^*$ is the dual of f . A form $\beta \in \text{Sesq}(P)$ is *nonsingular* iff $\text{ad}(\beta) : P \rightarrow P^*$ is an isomorphism where $\text{ad}\beta(p) = \beta(p, -)$.

A $(-, \varepsilon)$ -hermitian form on P is an element $\beta \in \ker(1 - T_\varepsilon)$. A $(-, \varepsilon)$ -hermitian module is a pair (P, β) where P is a finitely generated projective R module and β is a hermitian form on P . A map from (P_1, β_1) to (P_2, β_2) is an R -linear map $f : P_1 \rightarrow P_2$ such that $\beta_1 = \beta_2 \circ (f \times f)$. The *sum* $(P_1, \beta_1) \perp (P_2, \beta_2) = (P_1 \oplus P_2, \beta_1 \perp \beta_2)$ of two hermitian modules is given by $(\beta_1 \perp \beta_2)(p_1 + p_2, p'_1 + p'_2) = \beta_1(p_1, p'_1) + \beta_2(p_2, p'_2)$.

We let $\text{Herm}(R, -, \varepsilon)$ denote the category of non-singular $(-, \varepsilon)$ hermitian modules. It is a symmetric monoidal category. We let $K\text{Herm}(R, \varepsilon)$ denote the infinite loop space gotten by applying the May-Segal machine [1]. For $i \geq 0$, we let $K\text{Herm}_i(R, \varepsilon)$ denote the i -th homotopy group of $K\text{Herm}(R, \varepsilon)$. Later in Sect. 3.4 we'll introduce the notion of the suspensions of a hermitian ring which will yield disconnected deloopings of $K\text{Herm}(R, -, \varepsilon)$ and the definition of $K\text{Herm}_i(R, -, \varepsilon)$ for $i < 0$.

A $(-, \varepsilon)$ -quadratic module is a pair (P, α) where P is a f.g. projective right R module and $\alpha \in \text{Sesq}(P)$. Notice that $\beta = (1 + T_\varepsilon)\alpha$ is a hermitian form, and we say that α is a quadratic form with associated pairing β . A map from (P_1, α_1) to (P_2, α_2) is a R -linear map $f : P_1 \rightarrow P_2$ such that $(\alpha_1 - (\alpha_2 \circ (f \times f))) \in \text{im}(1 - T_\varepsilon)$. We say (P, α) is *nonsingular* if $(1 + T_\varepsilon)\alpha$ is a nonsingular hermitian form. The *sum* $(P_1, \alpha_1) \perp (P_2, \alpha_2) = (P_1 \oplus P_2, \alpha_1 \perp \alpha_2)$ of two quadratic modules is given by $(\alpha_1 \perp \alpha_2)(p_1 + p_2, p'_1 + p'_2) = \alpha_1(p_1, p'_1) + \alpha_2(p_2, p'_2)$. (In [85] Wall shows that this way of viewing quadratic forms is equivalent to the classical definition.)

We let $\text{Quad}(R, -, \varepsilon)$ denote the category of nonsingular quadratic modules. We let $K\text{Quad}(R, -, \varepsilon)$ denote the infinite loop space gotten by applying the May-Segal machine [1]. For $i \geq 0$, we let $K\text{Quad}_i(R, -, \varepsilon)$ denote the i -th homotopy group of $K\text{Quad}(R, -, \varepsilon)$.

Theorem 3 If 2 is a unit in R , then the functor $\text{Quad}(R, -, \varepsilon) \rightarrow \text{Herm}(R, -, \varepsilon)$ which sends (P, α) to $(P, (1 + T_\varepsilon)\alpha)$ is an equivalence of categories and induces a homotopy equivalence $K\text{Quad}(R, -, \varepsilon) \rightarrow K\text{Herm}(R, -, \varepsilon)$.

3

Let $\text{iso}\mathcal{P}(R)$ be the category with objects finitely generated projective modules, and maps R -linear isomorphisms. The *hyperbolic functor* $H: \text{iso}\mathcal{P}(R) \rightarrow \text{Quad}(R, -, \varepsilon)$ is defined as follows:

$$\begin{aligned} \text{objects} : P &\mapsto \left(P \oplus P^*, \begin{bmatrix} 0 & 0 \\ \text{eval} & 0 \end{bmatrix} \right), \\ \text{maps} : (f: P \rightarrow Q) &\mapsto f \oplus (f^{-1})^*, \end{aligned}$$

where $\text{eval}(p, \alpha) = \alpha(p)$ for $P \in P$ and $\alpha \in P^*$.

We let $GQ_{2l}(R, -, \varepsilon) := \text{Aut}(H(R^l))$, and $GQ(R, -, \varepsilon)$ is the direct limit of the direct system $\{GQ_{2l}(R, -, \varepsilon), \theta_{2l}\}$ where $\theta_{2l}: GQ_{2l}(R, -, \varepsilon) \rightarrow GQ_{2l+2}(R, -, \varepsilon)$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Recall that if X is a connected topological space and $\pi_1(X)$ is a quasi-perfect group, i.e. $[\pi_1(X), \pi_1(X)]$ is perfect, then the Quillen plus construction is a map $X \rightarrow X^+$ which abelianizes π_1 and which induces an isomorphism on homology for all local coefficient systems on X^+ .

4

Theorem 4 The group $GQ(R, -, \varepsilon)$ is quasi-perfect and $K\text{Quad}(R, -, \varepsilon)$ is homotopy equivalent to $K\text{Quad}_0(R, -, \varepsilon) \times BGQ(R, -, \varepsilon)^+$.

See [6, 57], and [29] for information on the group $GQ(R, -, \varepsilon)$, in particular about generators for the commutator subgroup.

The hyperbolic functor induces a map of infinite loop spaces $H: KR \rightarrow K\text{Quad}(R, -, \varepsilon)$ and we let $K\text{Quad}^{(-1)}(R, -, \varepsilon)$ be the homotopy fiber of the de-loop of H . Thus we get a homotopy fibration sequence $KR \xrightarrow{H} K\text{Quad}(R, -, \varepsilon) \rightarrow K\text{Quad}^{(-1)}(R, -, \varepsilon)$.

The *forgetful functor* $F: \text{Quad}(R, -, \varepsilon) \rightarrow \text{iso}\mathcal{P}(R)$ is given as follows:

$$\begin{aligned} \text{objects} : (P, \alpha) &\mapsto P, \\ \text{maps} : [f: (P, \alpha) \rightarrow (P', \alpha')] &\mapsto f. \end{aligned}$$

This induces a map of infinite loop spaces $F: K\text{Quad}(R, -, \varepsilon) \rightarrow K(R)$ and we let $K\text{Quad}^{(1)}(R, -, \varepsilon)$ denote the homotopy fiber.

Let $T: \text{iso}\mathcal{P}(R) \rightarrow \text{iso}\mathcal{P}(R)$ be functor which sends an object P to P^* and which sends a map f to $(f^{-1})^*$. Then T induces a homotopy involution on $K_i R$ for all i . Also the composition $F \circ H: K_i(R) \rightarrow K_i(R)$ equals $1 + T$.

Theorem 5: (*Karoubi Periodicity*)

Assume 2 is a unit in R . Then the 2nd loop space of $KQuad^{(-1)}(R, -, \epsilon)$ is homotopy equivalent to $KQuad^{(1)}(R, -, -\epsilon)$.

In Sect. 3.3 we'll see that part (2) of the Hermitian K-theory Theorem follows from Karoubi Periodicity.

Giffen has suggested that there should be a version of Karoubi Periodicity without the assumption that 2 is a unit. Let $Even(R, -, \epsilon)$ be the category of even hermitian forms and let $Split(R, -, \epsilon)$ be the category of split quadratic forms (see [64] for definitions). Then we get forgetful functors

$$Split(R, -, \epsilon) \rightarrow Quad(R, -, \epsilon) \rightarrow Even(R, -, \epsilon) \rightarrow Herm(R, -, \epsilon),$$

which are equivalences when 2 is a unit in R . We can define analogues of $KQuad^{(1)}(R, -, -\epsilon)$ and $KQuad^{(-1)}(R, -, \epsilon)$ for each of these categories, Giffen's idea is that the 2nd loop space of $KQuad^{(-1)}(R, -, \epsilon)$ should be homotopy equivalent to $KEven^{(1)}(R, -, -\epsilon)$. There should also be a similar result for each adjacent pair of categories.

L-Theory of Quadratic Forms

3.2.2

When we are using only one involution on our ring R we'll write (R, ϵ) as short for $(R, -, \epsilon)$.

Let $F_j: KQuad_j(R, \epsilon) \rightarrow K_j R$ be the map induced by the forgetful functor F .

Let $H_j: K_j R \rightarrow KQuad_j(R, \epsilon)$ be the map induced by the hyperbolic functor.

Based L-Groups Following [88]

$$\begin{aligned} L_{2i}^S(R) &= L_{2i}^{K_2}(R) := \ker \left(\text{disc} : \pi_0 \left(KQuad^{(1)}(R, (-1)^i \right) \rightarrow K_1 R \right) \\ L_{2i+1}^S(R) &= L_{2i+1}^{K_2}(R) := \ker \left(F_1 : KQuad_1(R, (-1)^i) \rightarrow K_1 R \right), \end{aligned}$$

where $\pi_0(KQuad^{(1)}(R, (-1)^i))$ can be identified as the K_0 of the category of based, even rank quadratic forms and “disc” is the discriminate map.

Free L-Groups

$$\begin{aligned} L_{2i}^{K_1}(R) &:= \ker \left(F_0 : KQuad_0(R, (-1)^i) \rightarrow K_0 R \right) \\ L_{2i+1}^{K_1}(R) &:= \text{coker} \left(H_1 : K_1 R \rightarrow KQuad_1(R, (-1)^i) \right) \end{aligned}$$

Remarks: In the next section we'll explain how these free L-groups are used to classify compact manifolds.

Projective L-Groups

For $i \in \mathbb{Z}$, $L_{2i}^{K_0}(R) = L_{2i}^P(R) := \text{coker} \left(K_0 R \xrightarrow{H_0} KQuad_0(R, (-1)^i) \right)$. If 2 is a unit in R ,

then $L_0^p(R)$ is often denoted by $W(R)$. If R is also commutative, then tensor product of forms makes $W(R)$ into a ring called the *Witt ring*. (See [7, 46, 51], and [68].)

Remarks: the letter “p” stands for “projective”.

See [57] where Ranicki defined $L_{2i-1}^p(R)$ in terms of “formations”.

Also he proved the following *Shaneson Product Formula*

$$L_i^{K_j}(R) \cong \operatorname{coker} \left(L_{i+1}^{K_{j+1}}(R) \rightarrow L_{i+1}^{K_{j+1}}(R[t, t^{-1}]) \right)$$

for all $i, j = 0$ or 1 and where we extend the involution on R to the Laurent ring $R[t, t^{-1}]$ by $\bar{t} = t^{-1}$.

Ranicki also constructed the Rothenberg Sequence

$$\cdots \rightarrow L_i^{K_{j+1}}(R) \rightarrow L_i^{K_j}(R) \rightarrow \hat{H}^i(\mathbb{Z}/2, K_j(R)) \rightarrow \cdots$$

for all $i \in \mathbb{Z}$, and $j = 0$ or 1 .

“Lower” L-Groups

This product formula suggests the following downward inductive definition:

For $j < 0$ and any $i \in \mathbb{Z}$,

$$L_i^{K_j}(R) := \operatorname{coker} \left(L_{i+1}^{K_{j+1}}(R) \rightarrow L_{i+1}^{K_{j+1}}(R[t, t^{-1}]) \right)$$

Notice how this is analogous to Bass’s definition of $K_j(R)$ for $j < 0$ (see[8]).

By using the fundamental theorem of algebraic K-theory, and the fact that the involution T interchanges the two Nil terms in $K_j(R[t, t^{-1}])$ it is to easy that

$$\hat{H}^{i+1}(\mathbb{Z}/2, K_{j+1}(R[t, t^{-1}])) \cong \hat{H}^{i+1}(\mathbb{Z}/2, K_{j+1}(R)) \oplus \hat{H}^i(\mathbb{Z}/2, K_j(R))$$

for all $i, i \in \mathbb{Z}$.

Then one can deduce the following Rothenberg exact sequence

$$\cdots \rightarrow L^{K_{j+1}}(R) \rightarrow L^{K_j}R \rightarrow \hat{H}(\mathbb{Z}/2, K_j(R)) \rightarrow \cdots$$

for all $i \in \mathbb{Z}$ and all $j < 0$.

Recall that it was much harder to find the “correct” definition for high dimensional K-theory than for low dimensional K-theory. Similarly, the definition of $L_i^{K_j}(R)$ for $j \geq 1$ is harder than for $j \leq 1$. See Sect. 3.5 for the definition of L -groups with “higher” decorations for all Hermitian rings. In Sect. 3.4 we use Karoubi periodicity to give another description when 2 is a unit in the Hermitian ring.

Classification of Manifolds up to Homeomorphism

3.3

Surgery theory was invented by Kervaire–Milnor, Browder, Novikov, Sullivan, and Wall [87]. The reader is encouraged to look at the following new introductions to the subject [48, 66], and [30].

Poincaré Complexes

We first introduce the homotopy theoretic analogue of a closed manifold. A connected finite CW complex X is an (oriented) n -dimensional *Poincaré complex* with fundamental class $[X] \in H_n(X)$ if $[X] \cap -: H^*(X; \Lambda) \rightarrow H_{n-*}(X; \Lambda)$ is an isomorphism for every $\mathbb{Z}\pi$ -module Λ , where $\pi = \pi_1(X)$. Assume $q \gg n$, then X has a preferred S^{q-1} spherical fibration S_X such that $Thom(S_X)$ has a *reduction*, i.e. a map $c_X: S^{n+q} \rightarrow Thom(S_X)$ which induces an isomorphism on H_{n+q} . We call S_X the *Spivak fibration* for X , and given any S^{q-1} -fibration η equipped with a reduction c , there exists a map of spherical fibrations $\gamma: \eta \rightarrow S_X$ which sends c to c_X . The map γ is unique up to fiber homotopy. Notice that if X is a closed manifold embedded in S^{n+q} with normal bundle ν_X , then the map $c_X: S^{n+q} \rightarrow (S^{n+q}/(S^{n+q} - \text{tubular nhd})) \simeq Thom(\nu_X)$ is a reduction.

Manifold Structures

If X is an (oriented) n -dimensional Poincaré complex, we let $\mathcal{S}(X)$ denote the simplicial set of *topological manifold structures* on X . An element in $\pi_0(\mathcal{S}(X))$ is represented by a homotopy equivalence $h: M \rightarrow X$ where M is a closed topological manifold. A second homotopy equivalence $h_1: M_1 \rightarrow X$ represents the same element if there exist a homeomorphism $\alpha: M \rightarrow M_1$ such that h is homotopic to $h_1 \circ \alpha$. A k -simplex in $\mathcal{S}(X)$ is given by a fiber homotopy equivalence $M \times \Delta^k \rightarrow X \times \Delta^k$ over Δ^k .

Let's consider the following two questions.

(Existence) When is $\mathcal{S}(X)$ nonempty?

(Classification) Suppose $h: M \rightarrow X$ and $h_1: M_1 \rightarrow X$ represent $[h]$ and $[h_1]$ in $\pi_0(\mathcal{S}(X))$. How do we decide when $[h] = [h_1]$?

We break these two questions into a series of subquestions.

Existence Step I: Euclidean bundle structure on the Spivak fibration

Question 1E: (Homotopy Theory) Does there exist a topological $\mathbb{R}^q$ bundle η over X with a reduction $c: S^{n+q} \rightarrow Thom(\eta)$?

Notice the answer to 1E is yes iff the map $\overline{S_X}: X \rightarrow BG$ which classifies S_X factors thru $BTop$, the classifying space for stable Euclidean bundles. Also suppose $h: M \rightarrow X$ is a homotopy equivalence where M is a closed topological manifold. If g is a homotopy inverse to h , then $\eta_h = g^* \nu_M$ has a reduction $c_h: S^{n+q} \xrightarrow{c_M} Thom(\nu_M) \rightarrow Thom(\eta_h)$. We call the pair (η_h, c_h) the *normal invariant* of the manifold structure h .

This construction yields a map of simplicial sets $n: \mathcal{S}(X) \rightarrow Lift(S_X)$, where $Lift(S_X)$ is the simplicial set of lifts of $\overline{S_X}$ thru $BTop$.

Existence Step II: Surgery Problem

Suppose the answer to 1E is yes. Then by replacing c by a map transversal to the copy of X given by the zero section in $Thom(\eta)$, we get a pair $(f: M \rightarrow X, \hat{f})$, where $M = c^{-1}(\text{zero section})$, $f = c|_M$, and $\hat{f}: \nu_M \rightarrow \eta$ is the bundle map covering f given by transversality. The pair $(f, \hat{f})$ is an example of a *surgery map*. Notice that f might

not be a homotopy equivalence, but we can assume that f induces an isomorphism on H_n . A second surgery map $(f_1 : M_1 \rightarrow X, \hat{f}_1)$ is *normal cobordant* to $(f, \hat{f})$ iff there exists a manifold $(W, \partial W) \subset (S^{n+q} \times I, S^{n+q} \times \partial I)$ with $\partial W = M \sqcup M_1$, and maps $F : W \rightarrow X \times I, \hat{F} : \nu_W \rightarrow \eta \times I$ such that $F|_M = f, F|_{M_1} = f_1, \hat{F}|_{\nu_M} = \hat{f}$, and $\hat{F}|_{\nu_{M_1}} = \gamma \circ \hat{f}_1$, where $\gamma : \eta \rightarrow \eta_1$ is a bundle isomorphism. The pair $(\eta, c : S^{n+q} \rightarrow \text{Thom}(\eta))$ determines $(f, \hat{f})$ up to normal cobordism.

Question 2E: (Surgery Theory) Is $(f, \hat{f})$ normal cobordant to a homotopy equivalence?

Given a group ring $\mathbb{Z}\pi$ we let $- : \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi$ be the anti-involution

$$\sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g g^{-1}.$$

We need the following minor variation of the free L -groups. Let

$$L_{2i}^h(\mathbb{Z}\pi) := L_{2i}^{K_1}(\mathbb{Z}\pi), \text{ and}$$

$$L_{2i+1}^h(\mathbb{Z}\pi) := L_{2i}^{K_1}(\mathbb{Z}\pi) \text{ modulo the subgroup generated by } \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Here “h” stands for homotopy equivalence.

6

Theorem 6: (Surgery Theorem) Assume $n > 4$. An n -dimensional surgery problem $(f : M^n \rightarrow X, \hat{f})$ determines an element $\sigma(f, \hat{f}) \in L_n^h(\mathbb{Z}\pi)$ such that $\sigma(f, \hat{f}) = 0$ iff $(f, \hat{f})$ is normal cobordant to a homotopy equivalence.

There is also a relative version of this where the closed manifold M is replaced by a compact manifold with boundary $(M, \partial M)$, the Poincaré complex X is replaced by a Poincaré pair (X, Y) i.e. $[X, Y] \cap - : H^*(X; \Lambda) \rightarrow H_{n-*}(X, Y; \Lambda)$ is an isomorphism, and $f : (M, \partial M) \rightarrow (X, Y)$ is such that $f|_{\partial M} : \partial M \rightarrow Y$ is a homotopy equivalence. Then $(f, \hat{f})$ still determines an element in $L_n^h(\mathbb{Z}\pi_1(X))$ which is trivial iff $(f, \hat{f})$ is normal cobordant (rel the boundary) to a homotopy equivalence of pairs.

The first paragraph of the Surgery Theorem yields a map $\sigma : \pi_0(\text{Lift}(S_X)) \rightarrow L_n^h(\mathbb{Z}\pi)$ such that $\pi_0 \mathcal{S}(X) \xrightarrow{n} \pi_0 \text{Lift}(S_X) \xrightarrow{\sigma} L_n^h(\mathbb{Z}\pi)$ is exact at $\pi_0 \text{Lift}(S_X)$.

From Geometry to Quadratic Forms

A detailed explanation of the Surgery Theorem is given in [48] and [66]. Here we'll just give a brief outline.

First we'll introduce some terminology.

Suppose V^m is a cobordism from $\partial_- V$ to $\partial_+ V$, i.e. $\partial V = \partial_- V \sqcup \partial_+ V$. Given an embedding $g : \sqcup (S^{i-1} \times D^{m-i}) \rightarrow \partial_+ V$, we let V' be the result of using g to *attach handles* of index i to V , i.e. $V' = V \cup_g (\sqcup (D^i \times D^{m-i}))$. Then V' is a cobordism from

$\partial_- V$ to a new manifold $\partial_+ V'$, and we say that $\partial_+ V'$ is the result of *doing surgery* on g .

Suppose $(F: W \rightarrow X \times I, \hat{F})$ is a normal cobordism from $(f: M \rightarrow X, \hat{f})$ to some other surgery problem. Then W has a filtration $(M \times I) = W_0 \subset W_1 \subset \cdots \subset W_n = W$ where for each i , $W_{i+1} = W_i \cup$ (handles of index $i + 1$).

Our surgery problem $(f: M \rightarrow X, \hat{f})$ is a homotopy equivalence iff f induces an isomorphism on π_1 , and $\tilde{f}_*: H_j(\tilde{M}) \rightarrow H_j(\tilde{X})$ is an isomorphism for each j , where $\tilde{f}: \tilde{M} \rightarrow \tilde{X}$ is a π -equivariant map of universal covers over f .

Special case: $n = 2i > 4$

Then up to normal cobordism there is no obstruction to arranging that $(f, \hat{f})$ induces an isomorphism on π_1 , $\tilde{f}_*: H_j(\tilde{M}) \rightarrow H_j(\tilde{X})$ is an isomorphism for $j \neq i$, and $\ker(H_i(\tilde{M}) \rightarrow H_i(\tilde{X})) \simeq \pi_{i+1}(f)$ is a free $\mathbb{Z}\pi$ -module of even rank $2l$. Any element $a \in \pi_{i+1}(f)$ is presented by a continuous map $\partial a: S^i = \partial D^{i+1} \rightarrow M$ plus an extension of $f \circ \partial a$ to D^{i+1} . This extension plus the bundle map $\hat{f}$ determines a regular homotopy class of immersions $\hat{a}: S^i \times D^i \rightarrow M$. Notice that if this immersion is in fact an embedding then one can do surgery on $\hat{a}$. By considering transversal intersections of these immersions one gets a nonsingular $(-1)^i$ -Hermitian form $\beta: \pi_{i+1}(f) \times \pi_{i+1}(f) \rightarrow \mathbb{Z}\pi$. By considering transversal *self* intersections one gets a quadratic form α such that $(1 + T_{-1^i})\alpha = \beta$. (See [85] and [48].)

Suppose we have an isomorphism of quadratic forms $H((\mathbb{Z}\pi)^l) \rightarrow \pi_{i+1}(f)$. Let $(a_k, k = 1, \dots, l)$ be a basis for the image of $(\mathbb{Z}\pi)^l \subset H((\mathbb{Z}\pi)^l) \rightarrow \pi_{i+1}(f)$. Then there exists an embedding $g = \sqcup \hat{a}_k: \sqcup S^i \times D^i \rightarrow M$ such that if we do surgery on g we get a normal cobordism to a homotopy equivalence.

Special case: $n = 2i + 1 > 3$

Suppose one has a nonsingular $(-1)^i$ -quadratic form (P, α) , plus two isomorphisms $A_1, A_2: H((\mathbb{Z}\pi)^l) \rightarrow (P, \alpha)$. Then $A_2^{-1} \circ A_1$ is an element in $GQ_{2l}(\mathbb{Z}\pi, (-1)^i)$ which maps to $L_{2i+1}^h(\mathbb{Z}\pi)$. Roughly speaking this is what one gets from a $2i + 1$ -dimensional surgery problem after it is made highly connected. To make this precise it is best to introduce the notion of *formations*. See [57] and [66].

Classification:

Suppose $h: M \rightarrow X$ and $h_1: M_1 \rightarrow X$ represent $[h]$ and $[h_1]$ in $\pi_0(\mathcal{S}(X))$.

Classification Step I: Normal invariant

Question 1C: (Homotopy Theory) Are the normal invariants (η_h, c_h) and (η_{h_1}, c_{h_1}) equivalent?

(To simplify notation let $(\eta, c) := (\eta_h, c_h)$ and $(\eta_1, c_1) := (\eta_{h_1}, c_{h_1})$.)

In other words does there exist a bundle isomorphism $\gamma: \eta \rightarrow \eta_1$ and a homotopy $H: S^{n+q} \times I \rightarrow \text{Thom}(\eta_1)$ such that $H|_{S^{n+q} \times 0} = \text{Thom}(\gamma) \circ c$, and $H|_{S^{n+q} \times 1} = c_1$. If γ and H exist, then we can choose H so that it is transversal to $X \times I$. This then yields a normal cobordism $(F: W \rightarrow X \times I, \hat{F}: \nu_W \rightarrow \eta_1)$ from $h: M \rightarrow X$ to $h_1: M_1 \rightarrow X$.

Suppose $\overline{n(h)}, \overline{n(h_1)} : X \rightarrow BTop$ are the lifts of $\overline{S_X} : X \rightarrow BG$ which classify (η_h, c_h) and (η_{h_1}, c_{h_1}) respectively. Then $\overline{n(h)}$ and $\overline{n(h_1)}$ are homotopic as lifts iff (η_h, c_h) and (η_{h_1}, c_{h_1}) are equivalent. Furthermore, the group $[X, G/Top]$ acts simply transitively on the set of homotopy class of lifts of $\overline{S_X}$. See [49] for results of Sullivan and others on the space G/Top .

Classification Step II: Relative surgery problem

Question 2C: (Surgery Theory) Suppose $(F: W \rightarrow X \times I, \hat{F}: \nu_W \rightarrow \eta_1)$ is a solution to 1C. Is $(F, \hat{F})$ normal cobordant (rel boundary) to an h -cobordism?

Here W is an h -cobordism from M to M_1 iff the two inclusion maps $M \subset W$ and $M_1 \subset W$ are homotopy equivalences.

Notice that the second paragraph of the Surgery Theorem yields an element $\sigma(F, \hat{F}) \in L_{n+1}^h(\mathbb{Z}\pi)$ which is 0 iff the answer to 2C is yes.

Classification Step III: H-cobordism problem

Question 3C: (Product Structure on H-cobordisms) Suppose W is an h -cobordism from M to M_1 . When is W homeomorphic to $M \times I$?

Let $hcob(M) = \mathcal{S}(M \times I, M \times 0)$ be the simplicial set of topological manifold structures on $M \times I$ rel $M \times 0$. Thus an element in $\pi_0(hcob(M))$ is represented by an h -cobordism from M to some other manifold. Two such h -cobordism represent the same element iff there exists a diffeomorphisms between them which is the identity on M .

We let $Wh_1(\pi) := \text{coker}(\{\pm\pi\} \rightarrow Gl_1(\mathbb{Z}\pi) \rightarrow K_1(\mathbb{Z}\pi))$.

7

Theorem 7: (*S-Cobordism Theorem*) Assume $n > 4$. There exists a bijection $\tau: \pi_0(hcob(M)) \rightarrow Wh_1(\pi_1(M))$ such that the product h -cobordism $M \times I$ maps to the unit element.

Suppose R is a ring such that $R^n \simeq R^m$ implies that $n = m$. Let B be a nontrivial subgroup of $K_1(R)$. Let $g: R^n \xrightarrow{\sim} P$ and $g_1: R^n \xrightarrow{\sim} P$ be two bases for a f.g. R -module P . The bases g and g_1 are said to be B -equivalent iff the map $GL_n(R) \rightarrow K_1(R)$ sends $g^{-1} \circ g_1$ to an element in B . We say that P is B -based if it is equipped with an B -equivalence class of basis. Notice that an isomorphism between two f.g. B -based modules determines an element in $K_1(R)/B$. More generally an R -chain homotopy equivalence g between two f.g. B -based, R -chain complexes determines an element, $\tau(g) \in K_1(R)/B$ called the *torsion* of g , (See [48, 2.2]).

Geometric Example: Suppose $f: A_1 \rightarrow A_2$ is a homotopy equivalence between finite CW complexes with fundamental groups isomorphic to π . Then the universal covers $\tilde{A}_1$ and $\tilde{A}_2$ are also CW complexes, and the CW -chain complexes $C(\tilde{A}_1)$ are $C(\tilde{A}_2)$ are B -based, where $B = \text{im}(\{\pm\pi\} \rightarrow Gl_1(\mathbb{Z}\pi) \rightarrow K_1(\mathbb{Z}\pi))$. If $f: \tilde{A}_1 \rightarrow \tilde{A}_2$ is a π -equivariant map covering f , we let $\tau(f) = \tau(C(\tilde{f})) \in Wh_1(\pi)$.

The map τ in the s-cobordism theorem sends an h -cobordism $M \subset W \supset M_1$ to $\tau(M \subset W)$.

Let $S^h(M) = (\pi_0(\mathcal{J}(M)) / \text{h-cobordisms}) = \text{orbit set of the action of } Wh_1(\pi_1(M))$ on $\pi_0(\mathcal{J}(M))$.

Theorem 8: (*Wall's h-Realization Theorem*) Assume $n > 4$. There is an action of $L_{n+1}^h(\mathbb{Z}\pi)$ on $S^h(M)$ such that the normal invariant map $\pi_0\mathcal{J}(M) \rightarrow \pi_0\text{Lift}(S_M)$ factors thru an injection $S^h(M)/L_{n+1}^h(\mathbb{Z}\pi) \rightarrow \pi_0\text{Lift}(S_M)$.

8

Let $\tau: \pi_0\mathcal{J}(M) \rightarrow Wh_1(\pi)$ be the map which sends $h: M_1 \rightarrow M$ to $\tau(h)$. Let $S^s(M) = \ker(\tau: \pi_0\mathcal{J}(M) \rightarrow Wh_1(\pi))$.

Let $L_{2i}^s(\mathbb{Z}\pi) = L_{2i}^B(\mathbb{Z}\pi)$, and let $L_{2i+1}^s(\mathbb{Z}\pi) = L_{2i+1}^B(\mathbb{Z}\pi)$ modulo the subgroup generated by $\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$, where $B = \text{im}(\{\pm\pi\} \rightarrow Gl_1(\mathbb{Z}\pi) \rightarrow K_1(\mathbb{Z}\pi))$.

Theorem 9: (*Wall's s-Realization Theorem*) Assume $n > 4$. There is an action of $L_{n+1}^s(\mathbb{Z}\pi)$ on $S^s(M)$ such that the restriction of the normal invariant map $S^s(M) \subset \pi_0\mathcal{J}(M) \rightarrow \pi_0\text{Lift}(S_M)$ factors thru an injection $S^s(M)/L_{n+1}^s(\mathbb{Z}\pi) \rightarrow \pi_0\text{Lift}(S_M)$.

9

Higher Hermitian K-Theory

3.4

Homotopy Fixed Spectrum, Homotopy Orbit Spectrum, and the Norm Fibration Sequence

3.4.1

See [2, 28, 72–74], and [41]. Suppose $\mathbb{K}$ is an Ω -spectrum equipped with an action by a finite group G .

Classical Example: Suppose $\mathbb{K}$ is the Eilenberg–MacLane spectrum $\mathbb{H}(A)$ where $\pi_0(\mathbb{H}(A)) = A$, a G -module.

Let

$$\mathbb{H}^*(G; \mathbb{K}) = \mathbb{K}^{hG} = F_G(\Sigma^\infty EG_+, \mathbb{K}), \text{ and}$$

$$\mathbb{H}_*(G; \mathbb{K}) = \mathbb{K}_{hG} = \Omega^\infty(\Sigma^\infty EG_+ \wedge_G \mathbb{K});$$

where F_G is the function spectrum of G -equivariant maps, and where Ω^∞ is the functor which converts a spectrum to a homotopy equivalent Ω -spectrum.

Notice that

$$\pi_i(\mathbb{H}^*(G; \mathbb{H}(A))) = H^{-i}(G; A), \text{ and}$$

$$\pi_i(\mathbb{H}_*(G; \mathbb{H}(A))) = H_i(G; A).$$

For general $\mathbb{K}$, there exist spectral sequences which abut to $\pi_*(\mathbb{H}^*(G; \mathbb{K}))$ and to $\pi_*(\mathbb{H}^*(G; \mathbb{K}))$, where E_2 is $H^*(G; \pi_*\mathbb{K})$ and $H_*(G; \pi_*\mathbb{K})$ respectively.

The map $EG \rightarrow pt$ induces maps $\mathbb{H}^*(G; \mathbb{K}) \rightarrow \mathbb{K}$ and $\mathbb{K} \rightarrow \mathbb{H}_*(G; \mathbb{K})$. Let $n: \mathbb{K} \rightarrow \mathbb{K}$ be the map given by $\prod_{g \in G} g$.

Then Adem–Dwyer–Cohen [2], and May–Greenlees [28] have constructed a norm fibration sequence

$$\mathbb{H}_*(G; \mathbb{K}) \xrightarrow{N} \mathbb{H}^*(G; \mathbb{K}) \rightarrow \hat{\mathbb{H}}^*(G; \mathbb{K}),$$

where the following diagram is homotopy commutative

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{n} & \mathbb{K} \\ \downarrow & & \uparrow \\ \mathbb{H}_*(G; \mathbb{K}) & \xrightarrow{N} & \mathbb{H}^*(G; \mathbb{K}), \end{array}$$

see also [91]. Furthermore, $\pi_i(\hat{\mathbb{H}}^*(G; \mathbb{H}(A))) \simeq \hat{H}^i(G; A)$ in the sense of Tate, see [69]. Thus $\hat{\mathbb{H}}^*(G; \mathbb{K})$ is called the Tate spectrum for G acting on $\mathbb{K}$.

Thomason's Homotopy Limit and Homotopy Colimit Problems

3.4.2

If G is a finite group, then $\underline{G}$ is the category with a single object, and maps are elements of the group G . Composition of maps is given by multiplication in G . Let Cat be the category of small categories. An action of G on a category $\mathcal{C}$ is a functor $\underline{G} \rightarrow Cat$ which sends the single object in $\underline{G}$ to $\mathcal{C}$. Let $SymMon$ be the category with objects small symmetric monoidal categories, and maps symmetric monoidal functors. The category of G symmetric monoidal categories, $G - SymMon$, is then the category of functors from $\underline{G}$ into $SymMon$.

Suppose $\mathcal{C}$ is a G -symmetric monoidal category. Constructions of Thomason, then yield the following commutative diagram which commutes up to a preferred homotopy. (See [72–74], [43] and the next two page of this paper.)

$$\begin{array}{ccc} K(\mathcal{C}_{hG}) & \xrightarrow{\tilde{T}_r} & K(\mathcal{C}^{hG}) \\ \downarrow & & \tilde{F} \downarrow \\ \mathbb{H}_*(G, K\mathcal{C}) & \xrightarrow{N} & \mathbb{H}^*(G, K\mathcal{C}) \end{array}$$

In [74] Thomason showed that the left vertical map is a homotopy equivalence, and in [72] he observed that many fundamental questions can be viewed as asking when the right vertical map becomes an equivalence after some sort of completion.

Examples: We'll ignore the complication that each of the following categories should be replaced with equivalent small categories,

1. (Segal Conjecture) Let $\mathcal{C}$ be the category of finite sets, equipped with the trivial action by G . Then $K(\mathcal{C}^{hG}) \simeq K(\text{finite } G\text{-sets})$ is equivalent to $\vee \Sigma^\infty B(N_G H/H)_+$, where we wedge over the set of conjugacy classes of subgroups of G . Also $\mathbb{H}^*(G, K\mathcal{C})$ is equivalent to the function spectrum $F(\Sigma^\infty B G_+, \mathbb{S})$, where $\mathbb{S}$ is the sphere spectrum. The Segal Conjecture as proved by Carlsson states that in this case the map $K(\mathcal{C}^{hG}) \rightarrow \mathbb{H}^*(G, K\mathcal{C})$ becomes an equivalence after completion with respect to $I(G) = \ker K_0(\text{finite } G\text{-sets}) \xrightarrow{\text{rank}} \mathbb{Z}$. (See [16])
2. (Quillen–Lichtenbaum) Let $\mathcal{C} = \mathcal{P}(F)$ where the field F is a finite, Galois extension of a field f . Let $G = \text{Gal}(F/f)$. If $g \in G$ and V is a F -module with multiplication $m: F \times V \rightarrow V$, then $F \times V \xrightarrow{g \times \text{id}} F \times V \xrightarrow{m} V$ is a new F -module structure on V . This yields an action of G on $\mathcal{P}(F)$ such that $K(\mathcal{P}(F)^{hG}) \simeq K(f)$. Then Thomason [73] has shown that a version of the Quillen–Lichtenbaum Conjecture can be reduced to showing that the map $Kf \rightarrow \mathbb{H}^*(G, Kf)$ is an equivalence after profinite completion.
3. (Hermitian K-theory) Suppose $(R, -, \varepsilon)$ is a hermitian ring. Then T_ε is “almost” an involution on $\mathcal{P}(R)$ in that there exists a natural equivalence between T_ε^2 and id . We can *rectify* this to get an honest action by $\mathbb{Z}/2$ via the following construction. Let $\tilde{\mathcal{P}}(R, -, \varepsilon)$ be the category where an object is a triple $(P, Q, h: P \xrightarrow{\sim} Q^*)$, where P and Q are objects in $\mathcal{P}(R)$ and h is an isomorphism. A map from $(P, Q, P \xrightarrow{h} Q^*)$ to $(P_1, Q_1, P_1 \xrightarrow{h_1} Q_1^*)$ is given by a pair of R -module isomorphisms $f: P \rightarrow P_1$ and $g: Q \rightarrow Q_1$ such that $h = g^* \circ h_1 \circ f$. Then $\tilde{\mathcal{P}}(R, -, \varepsilon)$ is equivalent to $\text{iso}\mathcal{P}(R)$. Furthermore we get an involution $\tilde{T}_\varepsilon: \tilde{\mathcal{P}}(R, -, \varepsilon) \rightarrow \tilde{\mathcal{P}}(R, -, \varepsilon)$ that sends (P, Q, h) to $(Q, P, Q \xrightarrow{\eta_{-, \varepsilon}} Q^{**} \xrightarrow{h^*} P^*)$, where $Q \xrightarrow{\eta_{-, \varepsilon}} Q^{**}$ is the natural equivalence $\eta_{-, \varepsilon}(q)(f) = \varepsilon \overline{f(q)}$ for all $q \in Q$ and all $f \in Q^*$. Then $\tilde{\mathcal{P}}(R, -, \varepsilon)^{h\mathbb{Z}/2}$ is equivalent to $K\text{Herm}(R, -, \varepsilon)$.

Conjecture: The map $\tilde{F}: K\text{Herm}(R, -, \varepsilon) \rightarrow \mathbb{H}^*(\mathbb{Z}/2, KR)$ becomes an equivalence under profinite completion. (See [23] and [9].)

Let $\mathcal{E}G$ be the transport category for the group G . Thus $\text{Obj}(\mathcal{E}G) = G$, and $\text{Map}_{\mathcal{E}G}(g_1, g_2)$ has just one element for each pair ordered (g_1, g_2) . Then G acts on $\mathcal{E}G$ via multiplication in G . Notice that the classifying space $B\mathcal{E}G$ is contractible and the induced action of G on $B\mathcal{E}G$ is free. Thomason defines $\mathcal{C}^{hG}$ as $\text{Fun}_G(\mathcal{E}G, \mathcal{C})$, the category of G -equivariant functors from $\mathcal{E}G$ to $\mathcal{C}$. Notice that an object in $\mathcal{C}^{hG}$ can be viewed as a pair (x, α) where x is an object in $\mathcal{C}$ and α is a function assigning to each $g \in G$ an isomorphism $\alpha(g): x \rightarrow gx$. The function α must satisfy the identities $\alpha(1) = 1$ and $\alpha(gh) = g\alpha(h) \cdot \alpha(g)$. Then we get the *transfer functor*:

$$\begin{aligned} \text{Tr}: \mathcal{C} &\rightarrow \mathcal{C}^{hG} \\ x &\mapsto \left(\sum_{g \in G} gx, \alpha \right), \end{aligned}$$

where $\alpha(h): \sum gx \xrightarrow{\sim} h \sum gx$ is the obvious permutation isomorphism.

See [74] and [41] for the construction of $\mathcal{C}_{hG}$ and the factorization

$$Tr: \mathcal{C} \rightarrow \mathcal{C}_{hG} \xrightarrow{\tilde{Tr}} \mathcal{C}^{hG}.$$

When $\mathcal{C} = \mathcal{P}(R, -, \varepsilon)$, Tr is the hyperbolic functor.

3.4.3 Karoubi Periodicity

See [39] and [43–45].

Let $\tilde{H}$ be the composition

$$\mathbb{H}_*(\mathbb{Z}/2, KR) \simeq K(\tilde{\mathcal{P}}(R, -, \varepsilon))_{h\mathbb{Z}/2} \xrightarrow{\tilde{Tr}} KHerm(R, -, \varepsilon).$$

We want to improve the following homotopy commutative diagram in a couple of ways:

$$\begin{array}{ccc} \mathbb{H}_*(\mathbb{Z}/2, KR) & \xrightarrow{\tilde{H}} & KHerm(R, -, \varepsilon) \\ id \downarrow & & \tilde{F} \downarrow \\ \mathbb{H}_*(\mathbb{Z}/2, KR) & \xrightarrow{N} & \mathbb{H}^*(\mathbb{Z}/2, KR) \end{array}$$

1. We want to replace the (-1) -connective spectra KR and $KHerm(R, -, \varepsilon)$ with spectra $\mathbb{K}R$ and $\mathbb{K}Herm(R, -, \varepsilon)$ where for all $i \in \mathbb{Z}$, $K_i(R) = \pi_i(\mathbb{K}R)$ and $KHerm_i(R, -, \varepsilon) = \pi_i(\mathbb{K}Herm(R, -, \varepsilon))$.
2. We want to use Karoubi periodicity to show that when 2 is a unit in R , then $\Omega^2 \mathcal{L}(R, -, \varepsilon) \simeq \mathcal{L}(R, -, -\varepsilon)$ where $\mathcal{L}(R, -, \varepsilon)$ is the deloop of the homotopy fiber of the map $\mathbb{H}_*(\mathbb{Z}/2, \mathbb{K}R) \xrightarrow{\tilde{H}} \mathbb{K}Herm(R, -, \varepsilon)$.

Disconnected K-theory

For any ring R we let CR , the *cone* of R , be the ring of infinite matrices (a_{ij}) , $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that each row and each column has only a finite number of nonzero entries. let SR , the *suspension* of R , be CR modulo the ideal of matrices with only a finite number of nonzero rows and columns. Gersten and Wagoneer [78] have shown that $KCR \simeq *$ and that $\Omega KSR \simeq KR$. This yields a spectrum $\mathbb{K}R$ such that KR is the (-1) -connected cover of $\mathbb{K}R$ and for $i < 0$, $\pi_i(\mathbb{K}R) \simeq K_i R$ in the sense of Bass.

If $\phi: R_1 \rightarrow R_2$ is a ring homomorphism, we let

$$\Gamma(\phi) = \lim \left(SR_1 \xrightarrow{S\phi} SR_2 \leftarrow CR_2 \right)$$

and following Wagoneer [78] we get a homotopy fibration sequence

$$\mathbb{K}R_1 \rightarrow \mathbb{K}R_2 \rightarrow \mathbb{K}(\Gamma(\phi)).$$

Suppose $(R, -, \varepsilon)$ is a hermitian ring. We then get hermitian rings $C(R, -, \varepsilon)$ and $S(R, -, \varepsilon)$ with underlying rings CR and SR respectively. The anti-involution of the matrix rings CR and SR is given by $M \mapsto \overline{M}^t$, i.e. apply $-$ componentwise

and then take the matrix transpose. The choice of central unit is εI where I is the identity matrix. Then Karoubi has shown that $KHermC(R, -, \varepsilon) \simeq *$ and that $\Omega KHermS(R, -, \varepsilon) \simeq KHerm(R, -, \varepsilon)$. This yields the spectrum $\mathbb{K}Herm(R, -, \varepsilon)$ with (-1) -connected cover

$KHerm(R, -, \varepsilon/).$

If $\phi: (R_1, -, \varepsilon_1) \rightarrow (R_2, -, \varepsilon_2)$ is a map of hermitian rings, then $\Gamma(\phi)$ inherits hermitian structure and we get a homotopy fibration sequence

$$\mathbb{K}Herm(R_1, -, \varepsilon_1) \rightarrow \mathbb{K}Herm(R_2, -, \varepsilon_2) \rightarrow \mathbb{K}Herm(\Gamma(\phi)) .$$

Karoubi's Hyperbolic and Forgetful Tricks

3.4.4

For any hermitian ring $(R, -, \varepsilon)$ we let $(R \times R^{op}, s, \varepsilon \times \bar{\varepsilon})$ be the hermitian ring where $s(a, b) = (b, a)$.

Theorem 10: (*Forgetful Trick*) Let $d: R \rightarrow R \times R^{op}$ send r to $(r, \bar{r})$. Then we get the following commutative diagram

10

$$\begin{array}{ccc} \mathbb{K}Herm(R, -, \varepsilon) & \xrightarrow{d} & \mathbb{K}Herm(R \times R^{op}, s, \varepsilon \times \bar{\varepsilon}) \\ id \downarrow & & \simeq \downarrow \\ \mathbb{K}Herm(R, -, \varepsilon) & \xrightarrow{F} & \mathbb{K}R , \end{array}$$

where F is the forgetful map.

Thus if $V(R, -, \varepsilon) = \Gamma(d)$, we get a homotopy fibration

$$\mathbb{K}Herm(R, -, \varepsilon) \xrightarrow{F} \mathbb{K}R \rightarrow \mathbb{K}Herm(V(R, -, \varepsilon))$$

with connecting homomorphism $\partial: \Omega \mathbb{K}Herm(V(R, -, \varepsilon)) \rightarrow \mathbb{K}Herm(R, -, \varepsilon)$.

Let $\mathbb{K}Herm^{(1)}(R, -, \varepsilon) = \Omega \mathbb{K}Herm(V(R, -, \varepsilon))$.

We can iterate the construction of V and let

$$\mathbb{K}Herm^{(j)}(R, -, \varepsilon) = \Omega^j \mathbb{K}Herm(V^j(R, -, \varepsilon)) , \text{ for } j = 1, 2, \dots .$$

Also we let $\mathbb{K}Herm^{(\infty)}(R, -, \varepsilon)$ be the homotopy limit of the diagram

$$\dots \rightarrow \mathbb{K}Herm^{(j)}(R, -, \varepsilon) \rightarrow \mathbb{K}Herm^{(j-1)}(R, -, \varepsilon) \rightarrow \dots \mathbb{K}Herm(R, -, \varepsilon) .$$

Theorem 11: (*Kobal's Forgetful Theorem*) There exists a homotopy fibration

11

$$\mathbb{K}Herm^{(\infty)}(R, -, \varepsilon) \rightarrow \mathbb{K}Herm(R, -, \varepsilon) \xrightarrow{\tilde{F}} \mathbb{H}^*(\mathbb{Z}/2, \mathbb{K})$$

such that the following diagram commutes

$$\begin{array}{ccc} KHerm(R, -, \varepsilon) & \xrightarrow{\tilde{F}} & \mathbb{H}^*(\mathbb{Z}/2, K(R)) \\ \downarrow & & \downarrow \\ \mathbb{K}Herm(R, -, \varepsilon) & \xrightarrow{\tilde{F}} & \mathbb{H}^*(\mathbb{Z}/2, \mathbb{K}(R)) . \end{array}$$

For any hermitian ring $(R, -, \varepsilon)$ we let $(M_2(R), \gamma, \varepsilon I)$ be the hermitian ring where

$$\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}.$$

12 Theorem 12: (*Hyperbolic Trick*) If $e: (R \times R^{op}, s, \varepsilon \times \bar{\varepsilon}) \rightarrow (M_2(R), \gamma, \varepsilon I)$ is given by $e(a, b) = \begin{pmatrix} a & 0 \\ 0 & \bar{b} \end{pmatrix}$, then we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{K}Herm(R \times R^{op}, s, \varepsilon \times \bar{\varepsilon}) & \xrightarrow{e} & \mathbb{K}Herm(M_2(R), \gamma, \varepsilon I) \\ \simeq \downarrow & & \simeq \downarrow \\ \mathbb{K}R & \xrightarrow{H} & \mathbb{K}Herm(R, -, \varepsilon) \end{array}$$

where H is the hyperbolic map.

Thus if $U(R, -, \varepsilon) = \Gamma(e)$, we get a homotopy fibration

$$\mathbb{K}R \xrightarrow{H} \mathbb{K}Herm(R, -, \varepsilon) \rightarrow \mathbb{K}U(R, -, \varepsilon).$$

Let $\mathbb{K}Herm^{(-1)}(R, -, \varepsilon) = \mathbb{K}Herm(U(R, -, \varepsilon))$.

We can iterate the construction of U and let

$$\mathbb{K}Herm^{(-j)}(R, -, \varepsilon) = \mathbb{K}Herm(U^j(R, -, \varepsilon)), \quad \text{for } j = 1, 2, \dots.$$

Also we let $\mathcal{L}(R, -, \varepsilon)$ be the homotopy colimit of the diagram

$$\mathbb{K}Herm(R, -, \varepsilon) \rightarrow \mathbb{K}Herm^{(-1)}(R, -, \varepsilon) \rightarrow \dots \mathbb{K}Herm^{(-j)}(R, -, \varepsilon) \dots$$

13 Theorem 13: (*Kobal's Hyperbolic Theorem*) There exists a homotopy fibration sequence

$$\mathbb{H}_*(\mathbb{Z}/2, \mathbb{K}R) \xrightarrow{\tilde{H}} \mathbb{K}Herm(R, -, \varepsilon) \rightarrow \mathcal{L}(R, -, \varepsilon)$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{H}_*(\mathbb{Z}/2, KR) & \xrightarrow{\tilde{H}} & KHerm(R, -, \varepsilon) \\ \downarrow & & \downarrow \\ \mathbb{H}_*(\mathbb{Z}/2, \mathbb{K}R) & \xrightarrow{\tilde{H}} & \mathbb{K}Herm(R, -, \varepsilon) \end{array}$$

where the top horizontal map was described earlier using results of Thomason.

Let $\mathbb{K}Herm^{(0)}(R, -, \varepsilon) = \mathbb{K}Herm(R, -, \varepsilon)$, and for all $j \in \mathbb{Z}$ we let $\mathbb{K}H^{(j)} = \mathbb{K}Herm^{(j)}(R, -, \varepsilon)$.

Twists and Dimension Shifting for Cohomology

3.4.5

Consider the short exact sequence of $\mathbb{Z}[\mathbb{Z}/2]$ -modules

$$\mathbb{Z}^{<-1>} \rightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{\varepsilon} \mathbb{Z}$$

where $\varepsilon(n + mT) = n + m$. Here $T \in \mathbb{Z}/2$ acts trivially on $\mathbb{Z}$ and $\mathbb{Z}^{<-1>}$ is a copy of $\mathbb{Z}$ with the nontrivial action by $\mathbb{Z}/2$.

Let $J: \mathbb{Z}[\mathbb{Z}/2]\text{-modules} \rightarrow \mathbb{Z}[\mathbb{Z}/2]\text{-modules}$ be the functor which sends a module P to $P \otimes_{\mathbb{Z}} \mathbb{Z}^{<-1>}$ where $\mathbb{Z}/2$ acts diagonally. Then for $j = 1, 2, \dots$ we let $P^{<-j>} = J^j(P)$. We let $P^{<0>} = P$. Notice that $P^{<-2>} \cong P$ as $\mathbb{Z}[\mathbb{Z}/2]$ -modules.

If we apply $H^*(\mathbb{Z}/2; P \otimes_{\mathbb{Z}} ?)$ to the above sequence we get a long exact sequence with connecting homomorphism $\partial: H^*(\mathbb{Z}/2; P) \rightarrow H^{*-1}(\mathbb{Z}/2; P^{<-1>})$.

If $\mathbb{K}$ is a spectrum with an action by $\mathbb{Z}/2$ we can perform an analogous construction by replacing $\mathbb{Z}$ by the sphere spectrum. In particular we get a connecting homomorphisms $\mathbb{H}^*(\mathbb{Z}/2; \mathbb{K}) \rightarrow \mathbb{H}^*(\mathbb{Z}/2; \Omega^{-1}\mathbb{K}^{<-1>})$.

Warning: $\mathbb{K}^{<-2>}$ is not necessarily equivariantly equivalent to $\mathbb{K}$. Consider the special case when $\mathbb{K}$ is the sphere spectrum with the trivial action and compare homotopy orbits.

There is a homotopy equivalence between $\hat{\mathbb{H}}^*(\mathbb{Z}/2; \mathbb{K})$ and the homotopy colimit of the diagram

$$\mathbb{H}^*(\mathbb{Z}/2; \mathbb{K}) \rightarrow \mathbb{H}^*(\mathbb{Z}/2; \Omega^{-1}\mathbb{K}^{<-1>}) \rightarrow \dots \mathbb{H}^*(\mathbb{Z}/2; \Omega^{-j}\mathbb{K}^{<-j>}) \rightarrow \dots,$$

such that the map $\mathbb{H}^*(\mathbb{Z}/2; \mathbb{K}) \rightarrow \hat{\mathbb{H}}^*(\mathbb{Z}/2; \mathbb{K})$ from the norm fibration sequence gets identified with the map

$$\mathbb{H}^*(\mathbb{Z}/2; \mathbb{K}) \rightarrow \operatorname{hcolim}_j \mathbb{H}^*(\mathbb{Z}/2; \Omega^{-j}\mathbb{K}^{<-j>}).$$

Consider the following commutative diagram

$$\begin{array}{ccccc} KHerm^{(-j)}(R, -, \varepsilon) & \rightarrow & KHerm^{(-j-1)}(R, -, \varepsilon) & \rightarrow & \dots \\ \tilde{F} \downarrow & & \tilde{F} \downarrow & & \\ H^*(\mathbb{Z}/2, KU^j(R, -, \varepsilon)) & \rightarrow & H^*(\mathbb{Z}/2, KU^{j+1}(R, -, \varepsilon)) & \rightarrow & \dots \end{array}$$

Notice that each square in this diagram is homotopy cartesian (compare the horizontal homotopy fibers.) One can then conclude that the square in the Hermitian K-theory Theorem is homotopy cartesian by observing that it is equivalent to the following homotopy cartesian square.

$$\begin{array}{ccc} KHerm(U^0(R, -, \varepsilon)) & \rightarrow & \operatorname{hcolim}_j KHerm(U^j(R, -, \varepsilon)) \\ \downarrow & & \downarrow \\ H^*(\mathbb{Z}/2, KU^0(R, -, \varepsilon)) & \rightarrow & \operatorname{hcolim}_j H^*(\mathbb{Z}/2, KU^j(T, -, \varepsilon)) \end{array}$$

If we replace $(R, -, \varepsilon)$ by $U^j(R, -, \varepsilon)$ we get the same Karoubi tower, but shifted to the left j steps. Similarly, if we replace $(R, -, \varepsilon)$ by $V^j(R, -, \varepsilon)$ we get the same

Karoubi tower, but shifted to the right j steps. This observation plus the Karoubi Periodicity theorem in Sect. 3.2 yields the following.

14

Theorem 14: (*Generalized Karoubi Periodicity*) Assume 2 is a unit in R . Then the 2nd loop space of $KHerm^{(j)}(R, -, \epsilon)$ is homotopy equivalent to $KHerm^{(j+2)}(R, -, -\epsilon)$. Thus $\Omega^2 \mathcal{L}(R, -, \epsilon) \simeq \mathcal{L}(R, -, -\epsilon)$.

3.4.6

General Definition of L -Groups (when 2 is a Unit)

Let $\mathbb{K}Herm^{(0)}(R, -, \epsilon) = \mathbb{K}Herm(R, -, \epsilon)$, and for all $j \in \mathbb{Z}$ we let $\mathbb{K}H^{(j)} = \mathbb{K}Herm^{(j)}(R, -, \epsilon)$.

The following diagram is called the *Karoubi Tower*.

$$\begin{array}{ccccc}
 \Omega^{j+1} \mathbb{K}R & & \Omega^j \mathbb{K}R & & \\
 H^{(j+1)} \downarrow & & H^{(j)} \downarrow & & \\
 \dots \rightarrow \mathbb{K}H^{(j+1)} & \rightarrow & \mathbb{K}H^{(j)} & \rightarrow & \dots \\
 F^{(j+1)} \downarrow & & F^{(j)} \downarrow & & \\
 \Omega^{j+1} \mathbb{K}R & & \Omega^j \mathbb{K}R & & .
 \end{array}$$

where for each $j \in \mathbb{Z}$

$$\Omega^{(j+1)} \mathbb{K}R \xrightarrow{H^{(j+1)}} \mathbb{K}H^{(j+1)} \rightarrow \mathbb{K}H^{(j)} \xrightarrow{F^{(j)}} \Omega^j \mathbb{K}R$$

is a homotopy fibration sequence.

Furthermore, $\Omega^j \mathbb{K}R \xrightarrow{F^{(j)} \circ H^{(j)}} \Omega^j \mathbb{K}R$ is homotopic to Ω^j of $I \pm T_\epsilon$.

The $F^{(j)}$ for $j > 0$ can be viewed as *higher order forgetful maps*. The $H^{(j)}$ for $j < 0$ can be viewed as *higher order hyperbolic maps*.

Let $\pi_k F^{(j)}$ and $\pi_k H^{(j)}$ be the induced maps on the k -th homotopy groups.

For any T_ϵ -invariant subgroup $X \subset K_j(R)$ we let

$$\begin{aligned}
 \mathcal{L}_{2i}^X(R) &:= \frac{(\pi_0 F^{(j)})^{-1}(X)}{(\pi_0 H^{(j)})(X)}, \text{ where } \epsilon = (-1)^i \\
 \mathcal{L}_{2i+1}^X(R) &:= \frac{(\pi_1 F^{(j-1)})^{-1}(X)}{(\pi_1 H^{(j-1)})(X)}, \text{ where } \epsilon = (-1)^i.
 \end{aligned}$$

15

Proposition 15: (*Rothenberg Sequence*) (Assume 2 is a unit in R .) For any $i, j \in \mathbb{Z}$ we get an exact sequence

$$\dots \rightarrow \mathcal{L}_i^{K_{j+1}}(R) \rightarrow \mathcal{L}_i^{K_j}(R) \rightarrow \hat{H}^i(\mathbb{Z}/2, K_j) \rightarrow \dots$$

Following [88] the proof of this is an easy diagram chase except for exactness at the middle term of

$$\mathcal{L}_{4k+2}^{K_j}(R) \rightarrow \hat{H}^0(\mathbb{Z}/2, K_j) \rightarrow \mathcal{L}_{4k+1}^{K_{j+1}}(R).$$

The proof of this step uses the commutativity of the following diagram

$$\begin{array}{ccc} \Omega^2 K\mathcal{H}erm^{(j-2)}(R, \varepsilon) & \xrightarrow{\Omega^2 F^{(j-2)}} & \Omega^2 \Omega^{j-2} \mathbb{K}R \\ \text{Periodicity} \downarrow & & \cong \downarrow \\ K\mathcal{H}erm^{(j)}(R, -\varepsilon) & \xrightarrow{F^{(j)}} & \Omega^j \mathbb{K}R. \end{array}$$

It is fairly easy to see that when 2 is a unit in R , Theorem 1.1 implies that $L_i^{K_j}(R) \cong \mathcal{L}_i^{K_j}(R)$ for all $i \in \mathbb{Z}$ and $j = 1$ or 2 (see [40] for details).

Proposition 16: (*Shaneson Product Formula*) Assume 2 is a unit in R . For all $i \in \mathbb{Z}$, and $j \leq 1$

16

$$\mathcal{L}_{i+1}^{K_{j+1}}(R) \oplus \mathcal{L}_i^{K_j}(R) \cong \mathcal{L}_{i+1}^{K_{j+1}}(R[t, t^{-1}]),$$

The map $\mathcal{L}_{i+1}^{K_{j+1}}(R) \rightarrow \mathcal{L}_{i+1}^{K_{j+1}}(R[t, t^{-1}])$ is induced by a map of Hermitian rings.

Karoubi [39] has constructed pairings $\mathbb{K}H^{(j_1)}(R_1, \varepsilon_1) \times \mathbb{K}H^{(j_2)}(R_2, \varepsilon_2) \rightarrow \mathbb{K}H^{(j_1+j_2)}(R_1 \otimes R_2, \varepsilon_1 \otimes \varepsilon_2)$. There exists an element $\sigma \in KHerm_1(\mathbb{Z}[\frac{1}{2}][t, t^{-1}])$ such that when i is even, the map $\mathcal{L}_i^{K_j}(R) \rightarrow \mathcal{L}_{i+1}^{K_{j+1}}(R[t, t^{-1}])$ is induced by pairing with σ . When i is odd, the map uses periodicity plus pairing with σ . The element σ can be viewed as the “round” symmetric signature of the circle (see [65]).

When $j = 1$ one can see that the sum of these two maps is an isomorphism by using the Shaneson product formula from Sect. 3.2. One then does downward induction on j using the Rothenberg sequences.

Theorem 17 Assume 2 is a unit in R . Then

17

$$L_i^{K_j}(R) \cong \mathcal{L}_i^{K_j}(R)$$

for all $i \in \mathbb{Z}$ and $j \leq 1$.

We already noted this is true when $j = 1$. We then do downward induction of j by using the fact that both sides satisfy a Shaneson Product formula.

Symmetric and Quadratic Structures on Chain Complexes

3.5

See [62–64, 67], and [58]. Connections between geometric topology and algebra can be greatly enhanced by using chain complex descriptions of K-theory and L-theory. Also we want a version of periodicity without the assumption that 2 is a unit.

For example, a *parameterized version* of Whitehead torsion is gotten by applying Waldhausen's $S_\bullet$ construction to the category of f.g. projective R -chain complexes to get a more “geometric” model for KR . (See [79–84], and [22])

Our goal in this section is to give a quick introduction to some of the key ideas from the work of Ranicki on L-theory (see also [52]).

Let $(R, -, +1)$ be a hermitian ring. Recall that in Sect. 3.2, symmetric (i.e. hermitian) forms on a module P were defined using the group $Sesq(P)$ equipped with the involution T_ε . Quadratic forms were defined using the map $N_\varepsilon = I + T_\varepsilon: Sesq(P) \rightarrow Sesq(P)$.

3.5.1 Symmetric Complexes

Given a chain complex

$$\mathcal{C}: \cdots \rightarrow C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \rightarrow \cdots \rightarrow C_0 \rightarrow 0$$

of f.g. projective R modules write $C^r = (C_r)^*$. Let $\mathcal{C}^{n-*}$ be the chain complex with $\mathcal{C}_r^{n-*} = C^{n-r}$ and $d_{\mathcal{C}}^{n-*} = (-1)^r d_C: C^{n-r} \rightarrow C^{n-r+1}$.

The duality isomorphisms

$$T: Hom_R(C^p, C_q) \rightarrow Hom_R(C^q, C_p); \phi \mapsto (-1)^{pq} \phi^*$$

are involutions with the property that the dual of a chain map $f: \mathcal{C}^{n-*} \rightarrow \mathcal{C}$ is a chain map $Tf: \mathcal{C}^{n-*} \rightarrow \mathcal{C}$, with $T(Tf) = f$.

Let

$$W: \cdots \rightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2]$$

be the free $\mathbb{Z}[\mathbb{Z}/2]$ -module resolution of $\mathbb{Z}$.

A *n-dimensional symmetric chain complex* is a pair $(\mathcal{C}, \phi)$ where $\mathcal{C}$ is an n -dimensional f.g. projective chain complex and ϕ is an n -dim cycle in the $\mathbb{Z}$ -module chain complex

$$Hom_{\mathbb{Z}[\mathbb{Z}/2]}(W, Hom_R(\mathcal{C}^*, \mathcal{C})) .$$

The element ϕ can be viewed as a chain map $\phi_0: \mathcal{C}^{n-*} \rightarrow \mathcal{C}$, plus a chain homotopy ϕ_1 from ϕ_0 to $T\phi_0$, plus a second order homotopy from ϕ_1 to $T\phi_1$, etc.

The pair $(\mathcal{C}, \phi)$ is *Poincaré* i.e. nonsingular, if ϕ_0 is a homotopy equivalence.

Model Example: (Miščenko) Let X^n be an oriented Poincaré complex with universal cover $\tilde{X}$ and cellular $\mathbb{Z}\pi$ -chain complex $C(\tilde{X})$, where $\pi = \pi_1(X)$. Then capping with the fundamental class $[X]$ yields a chain homotopy equivalence $\phi_0: C(\tilde{X})^{n-*} \rightarrow C(\tilde{X})$. The higher chain homotopies $\phi_1, \phi_2, \dots$ are given by an analogue of the construction of the Steenrod squares.

By using Poincaré duality for a compact manifold with boundary $(W, \partial W)$ as a model, Ranicki also introduced the notion of *Poincaré symmetric pairs* of complexes and *bordism* of Poincaré symmetric complexes. Then the n -th (projective) symmetric L -group, $L_p^n(R)$, is defined as the group of bordism classes of n -dimensional Poincaré symmetric chain complexes. One also gets symmetric L -groups with other decorations such as $L_h^n(\mathbb{Z}\pi)$ and $L_s^n(\mathbb{Z}\pi)$ by using free or based chain complexes.

An oriented Poincaré complex X , then determines an element $\sigma_h^*(X) \in L_h^n(\mathbb{Z}\pi)$ called the *symmetric signature* of X . If $n = 4k$, then the image of $\sigma_h^*(X)$ under the map $L_s^{4k}(\mathbb{Z}\pi) \rightarrow L_s^{4k}(\mathbb{Z}) \simeq \mathbb{Z}$ is just the signature of the the pairing.

$$H^{2k}(X, \mathbb{R}) \times H^{2k}(X, \mathbb{R}) \rightarrow H^{4k}(X, \mathbb{R}) \simeq \mathbb{R}$$

given by cup products. If X is a manifold M , then $\sigma_h^*(X)$ has a preferred lifting to $\sigma_s^*(M) \in L_s^n(\mathbb{Z}\pi)$.

It is easy to see that $L_p^0(R, -, +1) \simeq K_0Herm(R, -, +1)/\text{metabolic forms}$, (see [64] p.66, [64] p.74, and [6] p.12).

Quadratic Chain Complexes

3.5.2

Recall from Sect. 3.4 that given a spectrum $\mathbb{K}$ with action by $\mathbb{Z}/2$ we get a norm map

$$N: \Omega^\infty(\Sigma^\infty E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \mathbb{K}) \rightarrow F_{\mathbb{Z}/2}(\Sigma^\infty E\mathbb{Z}/2_+, \mathbb{K}).$$

Similarly, we get a norm map for the $\mathbb{Z}[\mathbb{Z}/2]$ -chain complex $Hom_R(\mathcal{C}^*, \mathcal{C})$.

$$N: W \otimes_{\mathbb{Z}} [\mathbb{Z}/2] Hom_R(\mathcal{C}^*, \mathcal{C}) \rightarrow Hom_{\mathbb{Z}} [\mathbb{Z}/2] (W, Hom_r(\mathcal{C}^*, \mathcal{C})).$$

(Notice that W is the cellular chain complex for $E\mathbb{Z}/2$.)

An *n-dimensional quadratic chain complex* is a pair $(\mathcal{C}, \psi)$ where $\mathcal{C}$ is an n -dimensional f.g. projective chain complex and ψ is an n -cycle in $W \otimes_{\mathbb{Z}} [\mathbb{Z}/2] Hom_R(\mathcal{C}^*, \mathcal{C})$. Notice that then $(\mathcal{C}, N(\psi))$ is an n -dim symmetric chain complex. If $(\mathcal{C}, N(\psi))$ is Poincaré, we say $(\mathcal{C}, \psi)$ is Poincaré. Similarly there are notions of *quadratic pairs* and *quadratic bordism*. The n -th (projective) quadratic L -group, $L_n^p(R)$, is the the bordism group of n -dimensional Poincaré quadratic chain complexes. One also gets quadratic L -groups with other decorations such as $L_h^n(\mathbb{Z}\pi)$ and $L_s^n(\mathbb{Z}\pi)$ by using free or based chain complexes.

The norm map N induces a map $1 + T: L_n(R) \rightarrow L^n(R)$ for any choice of decoration. If 2 is a unit in R , then $1 + T$ is an isomorphism. Furthermore for all rings R , $1 + T: L_n(R) \otimes \mathbb{Z}[\frac{1}{2}] \rightarrow L^n(R) \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism.

Suppose ε is any central unit in R such that $\bar{\varepsilon}\varepsilon = 1$. If we replace T by $T_\varepsilon: \text{Hom}_R(C^p, C_q) \rightarrow \text{Hom}_R(C^q, C_p); \phi \mapsto (-1)^{pq}\varepsilon\phi^*$ we get the quadratic groups $L_n(R, \varepsilon)$.

It is easy to see that if $n = 0$ or 1 , then these quadratic chain complex descriptions of the quadratic L -groups are consistent with the definitions in Sect. 3.2. The following result implies consistency for all n .

18

Theorem 18: (*Ranicki Periodicity*) For all $n \geq 0$, and for all rings R , $L_n^p(R, -\varepsilon) \simeq L_{n+2}^p(R, \varepsilon)$.

Model Example: Suppose $(f: M^n \rightarrow X, \hat{f})$ is a surgery problem where f induces an isomorphism on π_1 . Let $C(f)$ be the mapping cone of $C(\tilde{M}) \rightarrow C(\tilde{X})$. It is easily seen that $C(f)$ admits Poincaré symmetric structure which represents $\sigma_h^*(X) - \sigma_h^*(M)$ in $L_h^n(\mathbb{Z}\pi_1(M))$. However, Ranicki [63] has shown that the bundle map $\hat{f}$ determines an element $\sigma_*^h(f, \hat{f}) \in L_h^n(\mathbb{Z}\pi_1(X))$ such that $N(\sigma_*(f, \hat{f})) = \sigma_h^*(X) - \sigma_h^*(M)$. Under Ranicki Periodicity, $\sigma_*^h(f, \hat{f})$ gets identified with the surgery obstruction discussed in Sect. 3.3.

There are operations on symmetric and quadratic chain complexes which are algebraic analogs of surgery on a manifold. This *algebraic surgery* is what is used to prove the Ranicki Periodicity Theorem. It would be good to have a better understanding of the relationship between Karoubi and Ranicki Periodicity (Also see Sharpe Periodicity [70] [40].)

Applications of Quadratic Chain Complexes

Besides bordism and surgery there are other geometric operations such as *transversality* which have quadratic chain complex analogues. Ranicki's chain complex description of L -theory has helped to yield many important results.

1. (Instant descriptions of the surgery obstruction)
Given a surgery problem $(f, \hat{f})$, $\sigma_*(f, \hat{f}) \in L_n(\mathbb{Z}\pi_1(X))$ is defined without first making f highly connected.

2. (Product Formula)

Suppose N^k is a k -dimensional manifold. There exists a pairing

$$\mu: L^k(\mathbb{Z}\pi_1(N)) \times L_n(\mathbb{Z}\pi_1(X)) \rightarrow L_{k+n}(\mathbb{Z}[\pi_1 N \times \pi_1 X])$$

such that the surgery obstruction for $id_N \times (f, \hat{f})$ is $\mu(\sigma^*(M), \sigma_*(f, \hat{f}))$.

3. (Relative L -groups)

Suppose $f: R_1 \rightarrow R_2$ is a map of rings with involution. Then there exist 4-periodic *relative* L -groups, $L_n(f)$ such that with appropriate choice of decorations there exists a long exact sequence

$$\cdots \rightarrow L_n(R_1) \rightarrow L_n(R_2) \rightarrow L_n(f) \rightarrow L_{n-1}(R_1) \rightarrow \cdots$$

Here $L_n(f)$ is defined in terms of n -Poincaré quadratic R_2 -pairs where the “boundary” is induced by f from a $(n - 1)$ -dim Poincaré quadratic R_1 -chain complex. When f is a localizing map, then $L_n(f)$ has a description in terms of quadratic linking pairings [53, 61, 64]. When a group G is the result of an amalgamated product or a HNN construction, one gets Mayer-Vietoris sequences for L -theory analogous to those given by Waldhausen for K -theory [15, 59, 60].

4. (L -theory Spectrum)

Quinn [55, 56] and Ranicki [58, 65] have constructed Ω -spectra $\mathbb{L}^X(\mathbb{Z}\pi)$ and $\mathbb{L}_X(\mathbb{Z}\pi)$ with decorations $X \subset K_j(\mathbb{Z}\pi), j < 2$ such that $\pi_n \mathbb{L}^X(\mathbb{Z}\pi) \simeq L_n^X(\mathbb{Z}\pi)$ and $\pi_n(\mathbb{L}_X(\mathbb{Z}\pi)) \simeq L_n^X(\mathbb{Z}\pi)$. A k -simplex in the infinite loop space associated to $\mathbb{L}^p(\mathbb{Z}\pi)$ is given by a pair $(\mathcal{C}, \phi)$ where $\mathcal{C}$ is a functor from the category of faces of the standard k -simplex Δ^k to the category of f.g. proj. chain complexes of $\mathbb{Z}\pi$ -modules, and where ϕ is a Poincaré symmetric structure on such a functor. Thus a 1-simplex is a symmetric bordism, a 2-simplex is a second order symmetric bordism, etc. The definitions of $\mathbb{L}^h(\mathbb{Z}\pi)$ and $\mathbb{L}^s(\mathbb{Z}\pi)$ are similar except projective is replaced by free and based respectively.

Suppose we let $\mathbb{L}^s = \mathbb{L}^{<2>}, \mathbb{L}^h = \mathbb{L}^{<1>}, \mathbb{L}^p = \mathbb{L}^{<0>}$, and $\mathbb{L}^{K_j} = \mathbb{L}^{<j>}$ for $j = -1, -2, -3, \dots$. Let C_∞ be the infinite cyclic group. Then for $j = 2, 1, 0, \dots$, we get that $\mathbb{L}^{<j>}(\mathbb{Z}\pi)$ is homotopy equivalent to the homotopy fiber of $\mathbb{L}^{<j+1>}(\mathbb{Z}\pi) \rightarrow \mathbb{L}^{<j+1>}(\mathbb{Z}[\pi \times C_\infty])$. Notice that the map $\mathbb{L}^h(\mathbb{Z}\pi) \rightarrow \mathbb{L}^p(\mathbb{Z}\pi)$ is induced by commutativity of the following diagram

$$\begin{array}{ccc} \mathbb{L}^s(\mathbb{Z}\pi) & \rightarrow & \mathbb{L}^s(\mathbb{Z}[\pi \times C_\infty]) \\ \downarrow & & \downarrow \\ \mathbb{L}^h(\mathbb{Z}\pi) & \rightarrow & \mathbb{L}^h(\mathbb{Z}[\pi \times C_\infty]) \end{array} .$$

Then by downward induction on j we get maps $\mathbb{L}^{<j>}(\mathbb{Z}\pi) \rightarrow \mathbb{L}^{<j-1>}(\mathbb{Z}[\pi])$ for $j = 2, 1, 0, \dots$.

Let $\mathbb{L}^{<-\infty>}(\mathbb{Z}\pi)$ be the homotopy colimit of

$$\mathbb{L}^p(\mathbb{Z}\pi) \rightarrow \mathbb{L}^{<-1>}(\mathbb{Z}\pi) \rightarrow \dots \mathbb{L}^{<-j>}(\mathbb{Z}\pi) \rightarrow \dots .$$

Open Question: Are $\mathbb{L}^{<-\infty>}(\mathbb{Z}[\frac{1}{2}]\pi)$ and $\mathcal{L}(\mathbb{Z}[\frac{1}{2}]\pi, -, +1)$ homotopy equivalent?

5. (Block Space of Homeomorphisms)

Suppose M is a compact manifold, and $Top(M)$ is the singular complex of the topological group of homeomorphisms of M . A k -simplex in $Top(M)$ is given by a homeomorphism $h: \Delta^k \times M \rightarrow \Delta^k \times M$ which commutes with projection to Δ^k . Classical surgery is not strong enough to determine $Top(M)$ itself so we introduce a pseudo or block version $\widehat{Top}(M)$; where a k -simplex is a homeomorphism $h: \Delta^k \times M \rightarrow \Delta^k \times M$ such that for any face $\tau \subset \Delta^k$, $h(\tau \times M) \subset (\tau \times M)$. Notice that we get an inclusion of simplicial groups $Top(M) \subset \widehat{Top}(M)$. If $G(M)$ is the simplicial monoid of homotopy automorphisms we get

a similar inclusion $G(M) \subset \widetilde{G}(M)$ but in this case the inclusion is a homotopy equivalence. For a Poincaré complex X we let

$$\widetilde{\mathcal{S}}(X) = \sqcup \widetilde{G}(N)/\widetilde{Top}(N),$$

where we take the disjoint union over homeomorphism classes of manifolds homotopy equivalent to X .

Notice that a component of $\widetilde{\mathcal{S}}(X)$ is represented by a homotopy equivalence $N \rightarrow X$.

If X is a manifold M , we let $\widetilde{\mathcal{S}}^s(M)$ be the union of the components of $\widetilde{\mathcal{S}}(M)$ represented by simple homotopy equivalences. The ideas described in Sect. 3.3 can be used to prove the following theorem. (See [5, 58], and [14].)

19

Theorem 19: (*Surgery Exact Sequence*) Assume $n > 4$. Suppose M is a n -dimensional closed oriented manifold. There exists a homotopy equivalence between $\widetilde{\mathcal{S}}^s(M)$ and the union of certain components of the -1 -connected cover of Ω^n of the homotopy fiber of the assembly map

$$\mathbb{H}_*(M, \mathbb{L}) \rightarrow \mathbb{L}^s(\mathbb{Z}\pi_1(M)),$$

where $\mathbb{L}$ is the 1-connected cover of $\mathbb{L}^p(\mathbb{Z})$.

The question of which components involves resolving homology manifolds (see [10] and [58]).

The homotopy fiber of $\mathcal{S}(M) \rightarrow \widetilde{\mathcal{S}}(M)$ over the “identity vertex” $id: M \rightarrow M$ is equivalent to $\widetilde{Top}(M)/Top(M)$. It is easy to see that there is an exact sequence $\pi_1 hcob(M) \rightarrow \pi_0 Top(M) \rightarrow \pi_0 \widetilde{Top}(M)$. Hatcher [33] has shown that here exists a spectral sequence which abuts to $\pi_*(\widetilde{Top}(M)/Top(M))$ and E_2 of the spectral sequence is given in terms of $\pi_*(hcob(M \times I^j))$. Recall from the introduction that $HCOB(M)$ is the homotopy colimit of

$$hcob(M) \rightarrow hcob(M \times I) \rightarrow hcob(M \times I^2) \cdots.$$

In [90] it is shown that there exists an involution on the infinite loop space $HCOB(M)$ such that if $HCOB_s(M)$ is the 0-connected cover of $HCOB(M)$ then there exists a map $\widetilde{Top}(M)/Top(M) \rightarrow \mathbb{H}_*(\mathbb{Z}/2, HCOB_s(M))$ which is at least $k+1$ connected where $\dim M \geq \max(2k+7, 3k+4)$ and M is smoothable.

6. (Map from L -theory to Tate of K -theory)

In order to study $S(M)$ instead of $\widetilde{\mathcal{S}}(M)$ in the next section we need to understand how to “glue together” L -theory with higher K -theory. Suppose R is any ring with involution, and $X \subset K_j(R)$ is an involution invariant subgroup. We let $c_X: \mathbb{K}^X(R) \rightarrow \mathbb{K}(R)$ be such that $\pi_i(\mathbb{K}^X(R)) = 0$, for $i < j$, $\pi_j(\mathbb{K}^X(R)) = X$, and c_X induces an isomorphism on π_i for $i > j$. Then one can construct the following homotopy cartesian square,

$$\begin{array}{ccc} \mathbb{L}^X(R) & \rightarrow & \mathbb{L}^{<-\infty>}(R) \\ \varepsilon^X \downarrow & & \varepsilon \downarrow \\ \hat{\mathbb{H}}(\mathbb{Z}/2, \mathbb{K}^X(R)) & \rightarrow & \hat{\mathbb{H}}(\mathbb{Z}/2, \mathbb{K}(R)). \end{array}$$

It is then very easy to see that we get the Rothenberg sequences and Shaneson formulae mentioned in the introduction. If $j < 2$, this $\mathbb{L}^X(R)$ is consistent with the one constructed by Quinn and Ranicki.

The map $\mathcal{E}$ is constructed by using the Thomason homotopy limit problem map $K(\mathcal{C}^{hG}) \rightarrow \mathbb{H}^*(G, K\mathcal{C})$ plus a “bordism-like” model for $\hat{H}(\mathbb{Z}, \mathbb{K}(R))$, see [91, 92]. It would be good to have a better understanding of the relationship between $\mathcal{E}$ and the right vertical map in the Hermitian K-theory Theorem.

Manifold Structures

3.6

Let M^n be a connected, oriented, closed manifold.

Our first goal is to explain the following tower of simplicial sets

$$\mathcal{S}(M) \rightarrow \mathcal{S}^b(M \times \mathbb{R}^1) \rightarrow \cdots \rightarrow \mathcal{S}^b(M \times \mathbb{R}^j) \rightarrow \cdots .$$

Given two spaces over $\mathbb{R}^j$, $X \xrightarrow{p} \mathbb{R}^j$ and $Y \xrightarrow{q} \mathbb{R}^j$, we say that a continuous map $f: X \rightarrow Y$ is bounded if there exists $K \in \mathbb{R}$ such that for all $x \in X$, $|p(x) - q(h(x))| < K$. When we write $M \times \mathbb{R}^j$ we mean the space over $\mathbb{R}^j$ given by the projection map $M \times \mathbb{R}^j \rightarrow \mathbb{R}^j$.

Given $p: X \rightarrow \mathbb{R}^j$ we get the following diagram of simplicial monoids

$$\begin{array}{ccc} \text{Top}^b(p) & \rightarrow & G^b(p) \\ \downarrow & & \downarrow \\ \widetilde{\text{Top}}^b(p) & \rightarrow & \widetilde{G}^b(p) \end{array} ,$$

where the superscript “b” denotes the fact we are using bounded versions of the simplicial monoids defined in previous sections. The map $G^b(p) \rightarrow \widetilde{G}^b(p)$ is a homotopy equivalence. Furthermore, the map $G(M) \rightarrow G^b(M \times \mathbb{R}^j)$ gotten by crossing with $id_{\mathbb{R}^j}$ is a homotopy equivalence.

We say that $p: V^m \rightarrow \mathbb{R}^j$ is an *m-dimensional manifold approximate fibration* if V is an m-dimensional manifold, p is proper, and p satisfies the ε -homotopy lifting property for all $\varepsilon \geq 0$, (see [37]).

Key Example: (Siebenmann and Hughes–Ranicki [37, Chap.16]) Assume $n > 4$. Let W^n be a manifold with a tame end ε , Then ε has a neighborhood which is the total space of a manifold approximate fibration over $\mathbb{R}^1$.

Let

$$\mathcal{S}^b(M^n \times \mathbb{R}^j) := \sqcup G^b(p: V^{n+j} \rightarrow \mathbb{R}^j) / \text{Top}(p: V^{n+j} \rightarrow \mathbb{R}^j) ,$$

where we take the disjoint union over bounded homeomorphism classes of $(n+j)$ -dimensional manifold approximate fibrations homotopy equivalent to M . Notice that if $j = 0$, then $\mathcal{S}^b(M \times \mathbb{R}^j) = \mathcal{S}(M)$.

Crossing with the identity map on $\mathbb{R}^1$ gives maps $\mathcal{S}^b(M \times \mathbb{R}^j) \rightarrow \mathcal{S}^b(M \times \mathbb{R}^{j+1})$, and $Top^b(M \times \mathbb{R}^j) \rightarrow Top^b(M \times \mathbb{R}^{j+1})$. Let $\mathcal{S}^{-\infty}(M) = hcolim_j \mathcal{S}^b(M \times \mathbb{R}^j)$.

Let $hcob^b(M \times \mathbb{R}^j)$ be the simplicial set of bounded h-cobordisms on $M \times \mathbb{R}^j$. Then there exists a homotopy fibration (see [3, 4])

$$hcob^b(M \times \mathbb{R}^j) \xrightarrow{\mathcal{P}} \mathcal{S}^b(M \times \mathbb{R}^j) \rightarrow \mathcal{S}^b(M \times \mathbb{R}^{j+1}).$$

Furthermore $\Omega hcob^b(M \times \mathbb{R}^j) \simeq hcob^b(M \times I \times \mathbb{R}^{j-1})$. This makes $HCOB(M) = hcolim_i hcob^b(M \times I^i)$ into the 0-th space of an Ω -spectrum with j -th delooping given by $HCOB(M \times \mathbb{R}^j) = hcolim_i hcob^b(M \times I^i \times \mathbb{R}^j)$.

Let $\pi = \pi_1(M)$, then (see [3])

$$\pi_k(HCOB(M)) = \begin{cases} Wh_1(\pi), & \text{for } k = 0 \\ \tilde{K}_0(\mathbb{Z}\pi), & \text{for } k = -1 \\ K_{k+1}(\mathbb{Z}\pi), & \text{for } k < -1. \end{cases}$$

Anderson and Hsiang have conjectured that for $k < 1$, $K_k(\mathbb{Z}\pi)$ is trivial. Carter [17] has proved this for finite groups. Farrell and Jones [24] have proved this for virtually infinite cyclic groups

Let $A(X)$ be Waldhausen's algebraic K-theory of the connected space X , see [80]. Let $\mathbb{A}(X)$ be the disconnected Ω -spectrum constructed by Vogel [76] [77] such that $A(X) \rightarrow \mathbb{A}(X)$ induces an isomorphism on homotopy groups in positive dimensions. Also there exists a linearization map $\mathbb{A}(X) \rightarrow \mathbb{K}(\mathbb{Z}\pi_1(X))$ which is 1-connected. Let $\Omega WH(X)$ be the homotopy fiber of the assembly map $\mathbb{H}^*(X, \mathbb{A}(*)) \rightarrow \mathbb{A}(X)$. Then we get

(1:) There exists a homotopy equivalence $HCOB(M) \rightarrow \Omega WH(M)$. (See [79–84], and [22, §9].)

(2:) There exists a homotopy fibration sequence

$$Top^{-\infty}(M)/Top(M) \rightarrow \mathcal{S}(M) \rightarrow \mathcal{S}^{-\infty}(M),$$

where $Top^{-\infty}(M) = hcolim_j Top^b(M \times \mathbb{R}^j)$.

(3:) There exists an involution T on $\Omega WH(M)$ and a map

$$\psi : Top^{-\infty}(M)/Top(M) \rightarrow \mathbb{H}_*(\mathbb{Z}/2, \Omega WH(M))$$

which is at least $k+1$ connected where k satisfies $\dim M \geq \max(2k+7, 3k+4)$ and M is smoothable. (See [90].)

3.6.1 Higher Whitehead Torsion

Recall the Whitehead torsion map $\tau: \pi_0(\mathcal{S}(M)) \rightarrow Wh_1(M)$. We want to promote τ to a map of spaces $\mathcal{S}(M) \rightarrow \Omega WH(M)$ and analogous maps for $\mathcal{S}(M \times \mathbb{R}^j)$. (See [18–21] and [38].)

Let Q be the Hilbert cube, and let $\mathcal{S}_Q^b(M \times \mathbb{R}^j)$ be the same as $\mathcal{S}^b(M \times \mathbb{R}^j)$ but instead of using finite dimensional manifolds we use Hilbert cube manifolds. Then $\Omega \mathcal{S}_Q^b(M \times \mathbb{R}^j) \simeq \mathcal{S}_Q^b(M \times \mathbb{R}^{j-1})$. Thus $\mathcal{S}_Q^b(M \times \mathbb{R}^j)$ is an infinite loop space for all j . We'll abuse notation and let $\mathcal{S}_Q^b(M \times \mathbb{R}^j)$ also denote the associated Ω -spectrum. We get the following properties.

- (1:) $\mathcal{S}_Q^b(M \times \mathbb{R}^j) \simeq HCOB(M \times \mathbb{R}^j) \simeq$ the j -th delooping of $\Omega WH(M)$
- (2:) The map $\mathcal{S}(M) \xrightarrow{\times Q} S_Q(M)$ induces the torsion map τ when we apply π_0 .
- (3:) The map $\mathcal{S}^b(M \times \mathbb{R}^{j-1}) \rightarrow \mathcal{S}^b(M \times \mathbb{R}^j)$ has a lifting to the homotopy fiber of $\mathcal{S}^b(M \times \mathbb{R}^j) \xrightarrow{\times Q} \mathcal{S}_Q^b(M \times \mathbb{R}^j)$ which is at least $j + k + 1$ connected where $\dim M \geq \max(2k + 7, 3k + 4)$ and M is smoothable.
- (4:) There exists a homotopy commutative diagram

$$\begin{array}{ccc} hcob(M \times \mathbb{R}^j) & \xrightarrow{\mathcal{P}} & \mathcal{S}^b(M \times \mathbb{R}^j) \\ \downarrow & & \downarrow \\ \Omega^{1-j}WH(M) & \xrightarrow{1+(-1)^jT} & \Omega^{1-j}WH(M) ; \end{array}$$

where the left vertical map is the composition $hcob(M \times \mathbb{R}^j) \rightarrow HCOB(M \times \mathbb{R}^j) \simeq \Omega^{1-j}WH(M)$, and the right vertical map is the composition $\mathcal{T}: \mathcal{S}^b(M \times \mathbb{R}^j) \xrightarrow{\times Q} \mathcal{S}_Q^b(M \times \mathbb{R}^j) \simeq \Omega^{1-j}WH(M)$.

Notice the analogy between the following tower and the right half of the Karoubi Tower described in Sect. 3.4.

$$\begin{array}{ccccc} hcob(M) & & hcob^b(M \times \mathbb{R}^1) & & hcob^b(M \times \mathbb{R}^j) \\ \mathcal{P} \downarrow & & \mathcal{P} \downarrow & & \mathcal{P} \downarrow \\ \mathcal{S}(M) & \rightarrow & \mathcal{S}^b(M \times \mathbb{R}^1) & \rightarrow \cdots & \mathcal{S}^b(M \times \mathbb{R}^j) \rightarrow \cdots \\ \tau \downarrow & & \tau \downarrow & & \tau \downarrow \\ \Omega WH(M) & & WH(M) & & \Omega^{1-j}WH(M) \end{array}$$

Bounded Block Structure Spaces

3.6.2

In order to use surgery theory to compute $\mathcal{S}^{-\infty}(M)$ we need to introduce the block or pseudo version of $\mathcal{S}^b(M \times \mathbb{R}^j)$.

Let

$$\widetilde{\mathcal{S}}^b(M^n \times \mathbb{R}^j) := \sqcup \widetilde{G}^b(p: V^{n+j} \rightarrow \mathbb{R}^j) / \widetilde{Top}(p: V^{n+j} \rightarrow \mathbb{R}^j),$$

where we take the disjoint union over bounded homeomorphism classes of $(n+j)$ -dimensional manifold approximate fibrations homotopy equivalent to M . Notice that if $j = 0$, then $\widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j) = \widetilde{\mathcal{S}}(M)$.

Notice that crossing with the identity map on $\mathbb{R}^1$ gives a map $\widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j) \rightarrow \widetilde{\mathcal{S}}^b(M \times \mathbb{R}^{j+1})$. Let $\widetilde{\mathcal{S}}^{-\infty}(M) = hcolim_j \widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j)$.

20 **Theorem 20:** (*Stabilization Kills the Difference Between Honest and Pseudo*) The maps $\mathcal{S}^b(M \times \mathbb{R}^j) \rightarrow \tilde{\mathcal{S}}^b(M \times \mathbb{R}^j)$, for $j = 0, 1, \dots$ induce a homotopy equivalence $\mathcal{S}^{-\infty}(M) \simeq \tilde{\mathcal{S}}^{-\infty}(M)$. (See [90].)

Since $\pi_0(\mathcal{S}^b(M \times \mathbb{R}^j)) \simeq \pi_0(\tilde{\mathcal{S}}^b(M \times \mathbb{R}^j))$ we get a “torsion map” $\pi_0(\tilde{\mathcal{S}}^b(M \times \mathbb{R}^j)) \rightarrow \pi_{1-j}(\Omega WH(M))$. Let $\tilde{\mathcal{S}}^{b,s}(M \times \mathbb{R}^j)$ be the union of the components of $\tilde{\mathcal{S}}^b(M \times \mathbb{R}^j)$ with trivial torsion.

21 **Theorem 21:** (*Bounded Surgery Exact Sequence*) Assume $n + j > 4$. Suppose M is a n -dimensional closed oriented manifold. There exists a homotopy equivalence between $\tilde{\mathcal{S}}^{b,s}(M \times \mathbb{R}^j)$ and the -1 -connected cover of Ω^n of the homotopy fiber of the assembly map

$$\mathbb{H}^*(M, \mathbb{L}) \rightarrow \mathbb{L}^{<2-j>}(\mathbb{Z}\pi_1(M)) ,$$

where $\mathbb{L}$ is the 1-connected cover of $\mathbb{L}^p(\mathbb{Z})$.

Thus we get that $\mathcal{S}^{-\infty}(M) \simeq \tilde{\mathcal{S}}^{-\infty}(M)$ is homotopy equivalent to the -1 -connected cover of Ω^n of the homotopy fiber of the assembly map

$$\mathbb{H}_*(M, \mathbb{L}) \rightarrow \mathbb{L}^{<-\infty>}(\mathbb{Z}\pi_1(M)) .$$

Notice that so far we have explained the top horizontal map in the Manifold Structure Theorem from the introduction. Also we have outlined the proofs of the following parts of that theorem: (1), (2), (3), and (4).

The diagram in the Manifold Structure Theorem is then a consequence of constructing an involution T on $\Omega WH(M)$ and factorizations $\tilde{\mathcal{T}}$ of $\mathcal{T}: \mathcal{S}(M \times \mathbb{R}^j) \rightarrow \Omega^{1-j}WH(M)$ thru $\mathbb{H}^*(\mathbb{Z}/2, \Omega^{1-j}WH(M))$ for $j = 0, 1, 2, \dots$ such that we get commutative diagrams

$$\begin{array}{ccc} \mathcal{S}(M \times \mathbb{R}^j) & \rightarrow & \mathcal{S}(M \times \mathbb{R}^{j+1}) \rightarrow \dots \\ \tilde{\mathcal{T}} \downarrow & & \tilde{\mathcal{T}} \downarrow \\ \mathbb{H}^*(\mathbb{Z}/2, \Omega^{1-j}WH(M)^{<j>}) & \rightarrow & \mathbb{H}^*(\mathbb{Z}/2, \Omega^{-j}WH(M)^{<j+1>}) \rightarrow \dots \end{array}$$

3.6.3 More About Torsion

First we'll recall more about the construction of the map $\mathcal{T}: \mathcal{S}(M) \rightarrow \Omega WH(M)$.

Recall that $\Omega Wh(M)$ is the homotopy fiber of the assembly map $\mathbb{H}_*(M; A(*)) \rightarrow A(M)$, and that $\Omega Wh(M)$ is the (-1) -connected cover of $\Omega WH(M)$.

Suppose G is a simplicial monoid and A is a simplicial G -set. Then $A^{hG} = \text{Map}_G(EG, A) = \text{Sec}(EG \times_G A \rightarrow BG)$, where $\text{Sec}(\)$ denotes the simplicial set of sections.

Notice that $\Omega Wh(M)$ is also the homotopy fiber of

$$EG(M) \times_{G(M)} \mathbb{H}_*(M; A(*)) \rightarrow EG(M) \times_{G(M)} A(M),$$

where $G(M)$ is the simplicial monoid of homotopy automorphisms of M . The map $\mathcal{T}$ is constructed by first constructing $\chi \in A(X)^{hG(X)}$ and then a lifting $\chi^{\%}: BTop(M) \rightarrow EG(M) \times_{G(M)} \mathbb{H}_*(M; A(*))$ of the composition $BTop(M) \rightarrow BG(M) \xrightarrow{\chi} EG(M) \times_{G(M)} A(M)$.

Thus $\chi^{\%} \in H^*(M; A(*))^{hTop(M)}$.

Construction of χ :

For any space X , $\mathcal{R}(X)$ is the category of retractive spaces over the topological space X . Thus an object in $\mathcal{R}(X)$ is a diagram of topological spaces $W \begin{smallmatrix} \xrightarrow{r} \\ \xleftarrow{s} \end{smallmatrix} X$ such that $rs = id_X$ and s is a closed embedding having the homotopy extension property. The morphisms in $\mathcal{R}(X)$ are continuous maps over and relative to X . A morphism is a *cofibration* if the underlying map of spaces is a closed embedding having the homotopy extension property. A morphism is a *weak equivalence* if the underlying map of spaces is a homotopy equivalence.

Let $\mathcal{R}^{fd}(X)$ be the full subcategory of homotopy finitely dominated retractive spaces over X (see [22, +II, Sec.6] for details). Then $\mathcal{R}^{fd}(X)$ is a category with cofibrations and weak equivalences, i.e. a Waldhausen category, and $A(X)$ is the K-theory of $\mathcal{R}^{fd}(X)$.

If X is a finitely dominated CW complex we let $\chi(X)$ be the vertex in $A(X)$ represented by the retraction space $X \sqcup X \begin{smallmatrix} \xrightarrow{r} \\ \xleftarrow{s} \end{smallmatrix} X$ where r is the identity on each copy of X , and s is the inclusion into the first copy of X . Suppose $p: E \rightarrow B$ is a fibration with finitely dominated fibers, then it is shown in [22] that the rule $b \mapsto \chi(p^{-1}(b)) \in A(p^{-1}(b))$ for each $b \in B$ is continuous. If we apply this to the universal M -fibration over $BG(M)$, this continuous rule is the desired map $\chi: BG(M) \rightarrow EG(M) \times_{G(M)} A(M)$.

The lifting $\chi^{\%}$ is constructed using controlled topology in [22].

The construction of $\mathcal{T}: \mathcal{S}(M \times \mathbb{R}^j) \rightarrow \Omega^{1-j}WH(M)$ for $j > 0$ is similar except $A(X)$ is replaced by Vogell's $A^b(X \times \mathbb{R}^j)$ where $\Omega^j A^b(X \times \mathbb{R}^j) \simeq A(X)$.

Poincaré Duality and Torsion

3.6.4

Recall Thomason's map $K(\mathcal{C}^{h\mathbb{Z}/2}) \rightarrow \mathbb{H}^*(\mathbb{Z}/2; K(\mathcal{C}))$ where $\mathcal{C}$ is a $\mathbb{Z}/2$ -symmetric monoidal category.

Recall that Waldhausen used his $S_{\bullet}$ construction to define K-theory for Waldhausen, i.e. categories with cofibrations and weak equivalences. In [92] axioms are given for the notion of duality D in a Waldhausen category $\mathcal{C}$. Duality in $\mathcal{C}$ can be used to define non-singular pairings in $\mathcal{C}$ and an involution so that one gets a map $K(\text{non-singular } D\text{-pairings in } \mathcal{C}) \rightarrow \mathbb{H}^*(\mathbb{Z}/2; K(\mathcal{C}))$

Examples:

(1): Suppose $\mathcal{C} = Ch(R)$ is the category of f.g. projective chain complexes over R which is equipped with an (anti)-involution. The weak equivalences are the chain homotopy equivalences. The cofibrations are the chain maps which are split mono in each dimensional. For each $n = 0, 1, 2, \dots$ there exists a duality D_n -such that the non-singular D_n -pairings are n -dimensional Poincaré symmetric complexes in the sense of Ranicki.

(2): Suppose $\mathcal{C} = \mathcal{R}^{fd}(X)$, where X is equipped with a spherical fibration η . Then by essentially just following Vogel [75, 91, 92, 94] one gets dualities D_n for $n = 0, 1, 2, \dots$ such that if X is an n -dimensional Poincaré complex and η is the Spivak fibration

of X , then the retractive space $X \sqcup X \xrightleftharpoons[s]{r} X$ has a preferred non-singular self D_n -pairing. The homotopy invariance of the Spivak fibration and this preferred pairing implies that we have a lifting of χ to $\mathbb{H}^*(\mathbb{Z}/2; A(X))^{hG(X)}$.

(3): Same as (2) except weak equivalence are controlled (see [22, §2 and §7]) and $X = M$ is a closed manifold. Then we get the desired $\chi^{90} \in \mathbb{H}^*(\mathbb{Z}/2; \mathbb{H}_*(M; A(*))^{hTop(M)})$.

The construction of $\tilde{\mathcal{T}}: \mathcal{S}(M \times \mathbb{R}^j) \rightarrow \mathbb{H}^*(\mathbb{Z}/2; \Omega^{1-j}WH(M))$ for $j > 0$ is similar.

With the exception of showing that the square in the Manifold Structure Theorem is homotopy cartesian for a certain range, we are now done.

3.6.5 Homotopy Cartesian for a Range

For $j = 0, 1, 2, \dots$ we get the following homotopy commutative diagram.

$$\begin{array}{ccc} \mathcal{S}(M \times \mathbb{R}^j) & \rightarrow & \mathcal{S}(M \times \mathbb{R}^{j+1}) \\ \tilde{\mathcal{T}} \downarrow & & \tilde{\mathcal{T}} \downarrow \\ \mathbb{H}^*(\mathbb{Z}/2, \Omega^{1-j}WH(M)^{<j>}) & \rightarrow & \mathbb{H}^*(\mathbb{Z}/2, \Omega^{1-j}WH(M)^{<j+1>}) . \end{array}$$

The top horizontal homotopy fiber is $hcob(M \times \mathbb{R}^j)$, the bottom horizontal homotopy fiber is $\Omega^{1-j}WH(M)^{<j>}$. The induced map Σ between them is the composition of the stabilization map $hcob(M \times \mathbb{R}^j) \rightarrow HCOB(M \times \mathbb{R}^j)$ and the equivalence $HCOB(M \times \mathbb{R}^j) \simeq \Omega^{1-j}WH(M)$. By Anderson–Hsiang [3] $hcob(M \times \mathbb{R}^j) \rightarrow HCOB(M \times \mathbb{R}^j)$ induces an isomorphism on π_k for $k \leq j$. Also if we loop this map j times we get the stabilization map $hcob(M) \rightarrow HCOB(M)$ which Igusa has shown is at least $k + 1$ connected where k satisfies $\dim M \geq \max(2k + 7, 3k + 4)$ and M is smoothable. The connectivity of $hcob(M) \rightarrow HCOB(M)$ is called the *h-cobordism stable range*.

By combining this with Sect. 3.4.5 we get part (5) of the Manifold Structure Theorem.

More on Connections Between Quadratic Forms and Manifold Structures

3.6.6

Let $\mathcal{S}^s(M)$ be the components of $\mathcal{S}(M)$ with trivial torsion. Let $\Omega Wh_s(M)$ be the 0-connected cover of $\Omega WH(M)$. Then we get the following homotopy commutative

diagram which is homotopy cartesian for the same range as the diagram in the Manifold Structure Theorem.

$$\begin{array}{ccc} \mathcal{S}^s(M) & \rightarrow & \widetilde{\mathcal{S}}^s(M) \\ \downarrow & & \downarrow \\ \mathbb{H}^*(\mathbb{Z}/2; \Omega Wh_s(M)) & \rightarrow & \hat{\mathbb{H}}^*(\mathbb{Z}/2; \Omega Wh_s(M)) \end{array} .$$

It is natural to ask for a so-called “super simple” form of surgery theory such that its assembly map determines $\mathcal{S}^s(M)$ (at least in the h-cobordism stable range) in the same way that $\mathbb{L}^s$ determines $\widetilde{\mathcal{S}}^s(M)$ via the surgery exact sequence. This leads one to ask for an algebraic description of the the right vertical map in the above diagram. In particular one might ask how this map is related to the right vertical map in the Hermitian K-theory Theorem, or the map $\Xi^p : \mathbb{L}^p(\mathbb{Z}\pi_1(M)) \rightarrow \hat{\mathbb{H}}^*(\mathbb{Z}/2, K(\mathbb{Z}\pi_1(M)))$ from Sect. 3.5.

Suppose $(\mathcal{C}, D)$ is a Waldhausen category with duality such as examples (1), (2), and (3). Then we get a quadratic L-theory spectrum $\mathbb{L}_*(\mathcal{C}, D)$, a symmetric L-theory spectrum $\mathbb{L}^*(\mathcal{C}, D)$, a $1 + T$ map $\mathbb{L}_*(\mathcal{C}, D) \rightarrow \mathbb{L}^*(\mathcal{C}, D)$, an involution on $K\mathcal{C}$, and a map $\Xi : \mathbb{L}^*(\mathcal{C}, D) \rightarrow \hat{\mathbb{H}}^*(\mathbb{Z}; K(\mathcal{C}))$. (It might be interesting to compare this L-theory of Waldhausen categories with duality with Balmer’s notion of Witt groups for triangulated categories [7].) Examples:

(1): Suppose $\mathcal{C} = Ch_R$ and $D = D_n$, then $\mathbb{L}_*(Ch_R C, D_n) = \mathbb{L}_n^p(R) = \Omega^n \mathbb{L}^p(R)$. Similarly $\mathbb{L}^*(Ch_R C, D_n) = \Omega^n \mathbb{L}_p(R)$.

(2): Suppose $\mathcal{C} = \mathcal{R}^{fd}(X)$ where X is equipped with the oriented spherical fibration η . Then for $n = 0, 1, \dots$ we get a homotopy equivalence $\mathbb{L}_*(\mathcal{R}^{fd}(X), D_n) \rightarrow \mathbb{L}_n^p(\mathbb{Z}\pi_1(X))$, but the analogous map for symmetric L-theory is not an equivalence. Thus we get a map $\Xi : \mathbb{L}_n^p(\mathbb{Z}\pi_1(X)) \rightarrow \hat{\mathbb{H}}^*(\mathbb{Z}; A(X))$.

(3): By using the controlled version of example (2) we get that Ξ is natural with respect to assembly maps, i.e. we get the following diagram which commutes up to a preferred homotopy.

$$\begin{array}{ccc} \mathbb{H}_*(M; \mathbb{L}_n^p(\mathbb{Z})) & \rightarrow & \mathbb{L}^p(\mathbb{Z}\pi_1(X)) \\ \downarrow & & \downarrow \\ \hat{\mathbb{H}}^*(\mathbb{Z}/2; \mathbb{H}_*(M; A(*))) & \rightarrow & \hat{\mathbb{H}}^*(\mathbb{Z}/2; A(M)) \end{array} .$$

There is an analogous s-version of this diagram. By the Surgery Exact Sequence the induced map on the horizontal homotopy fibers of the s-version is a map

$$\widetilde{\mathcal{S}}^s(M) \rightarrow \hat{\mathbb{H}}^*(\mathbb{Z}/2; \Omega Wh_s(M))$$

which can be identified with the right vertical map in the previous diagram.

Localize at Odd Primes

3.6.7

If we localize at odd primes, it is easy to see that $\mathcal{S}^s(M) \simeq \widetilde{\mathcal{S}}^s(M) \times \widetilde{Top}(M)/Top(M)$, see [12, 13], or [93, 1.5.2]. Burghel and Fiedorowicz [11, 12] have used this to

show that in the h -cobordism stable range $\mathcal{J}^s(M)$ can be rationally computed using $KHerm(\mathbb{Z}\Omega M)$, where ΩM is the simplicial group gotten by applying Kan's G -functor to the singular complex of M . In order to get a similar result at odd primes one needs to replace $\mathbb{Z}$ with the sphere spectrum, see [26] and [27].

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Part IV

***K*-Theory and Operator Algebras**

Bivariant K - and Cyclic Theories* IV.1

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1.1	<i>Introduction</i>	657
1.2	<i>Topological K-Theory and K-Homology</i>	659
	Topological K -Theory	659
	The Dual Theory: <i>Ext</i> and K -Homology	660
1.3	<i>KK-Theory and E-Theory</i>	661
1.4	<i>Other Bivariant Theories on Categories Related to C^*-Algebras</i> ...	667
	Equivariant KK - and E -Theory	667
	KK -Theory for C^* -Algebras over a Topological Space	668
	KK -Theory for Projective Systems of C^* -Algebras	668
	Bivariant Theories as Triangulated Categories	669
1.5	<i>Applications</i>	670
	Index Theorems	670
	K -Theory of Group-algebras, Novikov Conjecture, Baum–Connes Conjecture	673
	Existence of Positive Scalar Curvature Metrics	676
	Applications in the Classification of Nuclear C^* -Algebras	676
	Classification of Topological Dynamical Systems	677
1.6	<i>Bivariant K-Theory for Locally Convex Algebras</i>	679
1.7	<i>Bivariant Cyclic Theories</i>	685
	Operators on Differential Forms	685
	The Periodic Theory	688
	Local Cyclic Cohomology and Bivariant Local Theory	692

* Research supported by the Deutsche Forschungsgemeinschaft.

1.8	<i>Bivariant Chern Characters</i>	694
	The Bivariant Chern–Connes Character for Locally Convex Algebras	695
	The Chern Character for C^* -Algebras	698
	<i>References</i>	699

Summary. Bivariant K -theories generalize K -theory and its dual, often called K -homology, at the same time. They are a powerful tool for the computation of K -theoretic invariants, for the formulation and proof of index theorems, for classification results and in many other instances. The bivariant K -theories are paralleled by different versions of cyclic theories which have similar formal properties. The two different kinds of theories are connected by characters that generalize the classical Chern character. We give a survey of such bivariant theories on different categories of algebras and sketch some of the applications.

Introduction

1.1

Topological K -theory was introduced in the sixties, [2]. On the category of compact topological spaces it gives a generalized cohomology theory. It was used in the solution of the vector field problem on spheres [1] and in the study of immersion and embedding problems. A major motivating area of applications was the study of Riemann–Roch type and index theorems [4].

Soon it became clear that K -theory could be generalized without extra cost and keeping all the properties, including Bott periodicity, from commutative algebras of continuous functions on locally compact spaces to arbitrary Banach algebras.

Kasparov himself was initially motivated by the study of the Novikov conjecture. He used his equivariant theory to prove powerful results establishing the conjecture in important cases, [44, 45]. Another application of the theory to geometry is the work of Rosenberg on obstructions to the existence of metrics with positive scalar curvature, [70].

In the classification of nuclear C^* -algebras, the theory has been used to obtain a classification by K -theory that went beyond all expectations, [48, 49, 61].

There are many different descriptions of the elements of bivariant KK -theory and E -theory. They can be defined using Kasparov-modules, asymptotic morphisms or classifying maps for extensions or in still other ways. Each picture has its own virtues. The point of view that elements of KK are described by n -step extensions of A by $\mathcal{K} \otimes B$, with Kasparov product corresponding to the Yoneda concatenation product of extensions, was for the first time developed by Zekri in [81]. Here $\mathcal{K}$ denotes the algebra of compact operators on a Hilbert space and $\otimes$ denotes the C^* -tensor product.

Already from the start it was clear that one did not have to restrict to the case where the objects of the category are just C^* -algebras. In fact, already the first paper by Kasparov on the subject treated the case of C^* -algebras with several additional structures, namely the action of a fixed compact group, a $\mathbb{Z}/2$ -grading and a complex conjugation. In the following years, versions of KK -theory were introduced for C^* -algebras with the action of a locally compact group, for C^* -algebras fibered over a locally compact space, for projective systems of C^* -algebras and for C^* -algebras with a specified fixed primitive ideal space. A general framework that covers all the latter three cases has been described in [9].

Cyclic homology and cohomology was developed as an algebraic pendant to algebraic and topological K -theory [14, 75]. It can be used to accommodate characteristic classes for certain elements of K -theory and K -homology [14, 39]. Motivated by the apparent parallelism to K -theory, bivariant cyclic theories were introduced in [30, 37, 54, 65]. For instance, the periodic bivariant cyclic theory can be viewed also as an additive (here even linear) category HP_* whose objects are algebras (over a field of characteristic 0) and whose morphism sets are the $\mathbb{Z}/2$ -graded vector spaces $HP_*(A, B)$. This category has formally exactly the same properties as KK or E . One way of describing these properties is to say that KK , E and HP_* all form triangulated categories, [74]. The formalism of triangulated categories allows one to form easily quotient categories which are again triangulated and thus have the relevant properties, in order to enforce certain isomorphisms in the category. In [74], this technique is used to introduce and study bivariant theories for C^* -algebras whose restrictions to the category of locally compact spaces give connective K -theory and singular homology.

It certainly seems possible to construct bivariant versions of algebraic (Quillen) K -theory – a very promising attempt is in [59]. However, if one wants to have the important structural element of long exact sequences associated with an extension, one has to make the theory periodic by stabilizing with a Bott extension. Since this extension is not algebraic, it appears that one cannot avoid the assumption of some kind of topology on the class of algebras considered. This assumption can be weakened to a large extent.

Since the construction of KK -theory and of E -theory used techniques which are quite specific to C^* -algebras (in particular the existence of central approximate units) it seemed for many years that similar theories for other topological algebras, such as Banach algebras or Fréchet algebras, would be impossible. However, in [28] a bivariant theory kk with all the desired properties was constructed on the category of locally convex algebras whose topology is described by a family of submultiplicative seminorms (“ m -algebras”). The definition of $kk_*(A, B)$ is based on “classifying maps” for n -step extensions of A by $\mathcal{K} \otimes B$, where this time $\mathcal{K}$ is a Fréchet algebra version of the algebra of compact operators on a Hilbert space and $\otimes$ denotes the projective tensor product. The product of the bivariant theory kk_* corresponds to the Yoneda product of such extensions. This theory allows to carry over the results and techniques from C^* -algebras to this much more general category. It allows the construction of a bivariant multiplicative character into cyclic homology, i.e. of a functor from the additive category kk to the linear category HP which is compatible with all the structure elements. Since ordinary cyclic theory gives only pathological results for C^* -algebras such a character say from KK_* or from E_* to HP_* cannot make sense.

Another bivariant cyclic theory HE^{loc} , the local theory, which does give good results for C^* -algebras, was developed by Puschnigg [65]. The local theory is a far reaching refinement of Connes’ entire cyclic theory [15]. As a consequence, Puschnigg defines a bivariant character from KK to HE^{loc} which even is a rational isomorphism for a natural class of C^* -algebras.

The general picture that emerges shows that the fundamental structure in bivariant K -theory is the extension category consisting of equivalence classes of n -step extensions of the form

$$0 \rightarrow B \rightarrow E_1 \rightarrow \dots \rightarrow E_n \rightarrow A \rightarrow 0$$

with the product given by Yoneda product. The crucial point is of course to be able to compare extensions of different length. As a general rule, K -theoretic invariants can be understood as obstructions to lifting problems in extensions.

A very good source for the first two chapters is [8]. The choice of topics and emphasis on certain results is based on the preferences and the own work of the author. I am indebted to R. Meyer for his contribution to the chapter on local cyclic theory.

Topological K -Theory and K -Homology

1.2

Topological K -Theory

1.2.1

Topological K -theory was introduced, following earlier work of Grothendieck and Bott, by Atiyah and Hirzebruch in connection with Riemann–Roch type theorems [2]. For a compact space X , the abelian group $K^0(X)$ can be defined as the enveloping group for the abelian semigroup defined by isomorphism classes of complex vector bundles over X with direct sum inducing addition.

The reduced K -theory group $\widetilde{K}^0(X)$ is defined as the K^0 -group of X , divided by the subgroup generated by the image in K^0 of the trivial line bundle on X . Groups $K^{-n}(X)$ can then be defined as the reduced K^0 -group of the reduced n -fold suspension $S^n X$ of X . As it turns out, using clutching functions for the gluing of vector bundles, $K^{-1}(X) = \widetilde{K}^0(SX)$ can be identified with the group of homotopy classes of continuous maps from X to the infinite unitary group U_∞ .

The Bott periodicity theorem then asserts that $K^{-n-2}(X) \cong K^{-n}(X)$. This periodicity shows that the family of groups $K^{-n}(X)$ consists only of two different groups, denoted by $K^0(X)$ and $K^1(X)$.

Given a compact subspace Y of X one denotes by $K^i(X, Y)$ the reduced K -theory groups of the quotient space X/Y .

Theorem 1 Let X, Y be as above. There is a periodic cohomology exact sequence of the following form

1

$$\begin{array}{ccccccc}
 & & \overset{i^*}{\longrightarrow} & & \overset{j^*}{\longrightarrow} & & \\
 K^0(X, Y) & & & K^0(X) & & & K^0(Y) \\
 & \uparrow & & & & & \downarrow \\
 & & \overset{j^*}{\longleftarrow} & & \overset{i^*}{\longleftarrow} & & \\
 K^1(Y) & & & K^1(X) & & & K^1(X, Y) .
 \end{array} \tag{1.1}$$

Now the Serre–Swan theorem shows that isomorphism classes of finite-dimensional complex vector bundles over X correspond to isomorphism classes of finitely generated projective modules over the algebra $\mathcal{C}(X)$ of continuous complex-valued functions on X . Therefore $K^0(X)$ can be equivalently defined as the enveloping group of the semigroup of isomorphism classes of such projective modules over the C^* -algebra $\mathcal{C}(X)$, i.e. as the algebraic K_0 -group $K_0(\mathcal{C}(X))$ of the unital ring $\mathcal{C}(X)$.

The reduced suspension of a topological space corresponds to the following operation on C^* -algebras or Banach algebras. Given such an algebra A , the suspension SA is defined as the (non-unital!) algebra $\mathcal{C}_0((0, 1), A)$ of continuous A -valued functions on the unit interval, vanishing at 0 and 1. Using n -fold suspensions we can, as for spaces, define higher groups $K_{-n}(A)$ as $\tilde{K}_0(S^n A)$. Of course, $S^n A$ is a non-unital algebra. Just as for non-compact spaces, we have to define $K_0(I)$ for a non-unital algebra I in an awkward way as $\tilde{K}_0(\tilde{I})$, where $\tilde{I}$ denotes I with unit adjoined.

Some of the standard proofs of Bott periodicity carry over immediately from locally compact spaces to Banach algebras. So does the proof of the K -theory exact sequence (1) associated to an extension of the form

$$0 \rightarrow I \xrightarrow{i} A \xrightarrow{q} B \rightarrow 0$$

of Banach algebras. It takes the following form

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{K(i)} & K_0(A) & \xrightarrow{K(q)} & KK_0(B) \\ \uparrow & & & & \downarrow \\ K_1(B) & \xleftarrow{K(q)} & K_1(A) & \xleftarrow{K(i)} & K_1(I) . \end{array} \quad (1.2)$$

This generalizes the sequence (1) if we take $A = \mathcal{C}(X)$, $B = \mathcal{C}(Y)$ and for I the ideal in $\mathcal{C}(X)$ consisting of functions vanishing on Y . Note that of course I is not unital in general, so again we have to define $K_0(I)$ in an artificial way as $\tilde{K}_0(\tilde{I})$.

1.2.2 The Dual Theory: *Ext* and K -Homology

One of the major motivations for the interest in topological K -theory was of course its use in the formulation and the proof of the celebrated Atiyah–Singer index theorem [4]. It is natural to try to interpret the index of an elliptic operator on a vector bundle as a pairing between a K -theory class and a class in a dual “ K -homology” theory (for instance between the K -theory class given by the symbol and the K -homology class given by the extension of pseudodifferential operators, or as the pairing between a K -homology class defined by an untwisted elliptic operator and the K -theory class of a vector bundle by which it is twisted). Atiyah proposed abstract elliptic operators over a space X as possible cycles for a dual theory $Ell(X)$. This proposal was taken up and developed by Kasparov in [40].

Independently and earlier however such a theory was discovered by Brown–Douglas–Fillmore in connection with the investigation of essentially normal operators. Their theory $\text{Ext}(X)$ is based on extensions of the form

$$0 \rightarrow \mathcal{K} \rightarrow E \rightarrow \mathcal{C}(X) \rightarrow 0,$$

where $\mathcal{K}$ is the standard algebra of compact operators on a separable infinite-dimensional Hilbert space H and E is a subalgebra of $\mathcal{L}(H)$ (necessarily a C^* -algebra). As equivalence relation for such extensions they used unitary equivalence. There is a natural direct sum operations on such extensions, based on the fact that the algebra $M_2(\mathcal{K})$ of 2×2 -matrices over $\mathcal{K}$ is isomorphic to $\mathcal{K}$.

Every essentially normal operator $T \in \mathcal{L}(H)$ (essentially normal means that $T^*T - TT^*$ is compact) defines such an extension by choosing for E the C^* -algebra generated by T together with $\mathcal{K}$ and taking for X the essential spectrum of T , i.e. the spectrum of the image of T in $\mathcal{L}(H)/\mathcal{K}$.

The theory $\text{Ext}(A)$ was also developed to some extent for more general C^* -algebras A in place of $\mathcal{C}(X)$. A very important result in that connection is Voiculescu's theorem. It asserts that, for separable A , any two trivial (i.e. admitting a splitting by a homomorphism) extensions are equivalent and shows that their class gives a neutral element in $\text{Ext}(A)$. Some important questions like the homotopy invariance of Ext remained open.

The pairing between an element e in $\text{Ext}(X)$ and an element ζ of $K^1(X)$, represented by an invertible element z in $M_n(\mathcal{C}(X))$ can be nicely described in terms of the Fredholm index. In fact, any preimage of z in the extension defining e is a Fredholm operator. The pairing (ζ, e) is exactly given by its index.

KK -Theory and E -Theory

1.3

It was Kasparov who revolutionized the subject by his fundamental work in [41] (independently, at about the same time, Pimsner–Popa–Voiculescu had started to develop a bivariant Ext -theory, [62]). Formally Kasparov's bivariant theory is based on a combination of the ingredients of K -theory and K -homology. The elements of his bivariant groups $KK(A, B)$, for C^* -algebras A and B are represented by a “virtual” finitely generated projective module over B (given as the “index” of an abstract elliptic operator) on which A acts by endomorphisms.

More specifically, Kasparov works with Hilbert B -modules. This is a straightforward generalization of an ordinary Hilbert space over $\mathbb{C}$ with $\mathbb{C}$ -valued scalar product to a space (i.e. module) over B with a B -valued inner product $(\cdot | \cdot)$. The axioms for this inner product are quite natural and we don't want to reproduce them here. One uses the notation $\mathcal{L}(H)$ to denote the algebra of all operators on H that admit an adjoint (such operators are automatically bounded and B -module maps). The closed subalgebra of $\mathcal{L}(H)$ generated by all rank 1 operators of the form $\theta_{x,y}: z \mapsto x(y | z)$ is denoted by $\mathcal{K}(H)$ (the algebra of “compact” operators on H). It is a closed ideal in $\mathcal{L}(H)$.

Kasparov then considers triples of the following form.

2

Definition 2 (a) An odd A - B Kasparov module is a triple (H, φ, F) consisting of a countably generated Hilbert B -module H , a $*$ -homomorphism $\varphi: A \rightarrow \mathcal{L}(H)$ and a selfadjoint $F \in \mathcal{L}(H)$ such that, for each $x \in A$ the following expressions are in $\mathcal{K}(H)$:

- $\varphi(x)(F - F^2)$ (KM1)
- $F\varphi(x) - \varphi(x)F$ (KM2)

(b) An even A - B Kasparov module is a triple (H, φ, F) satisfying exactly the same conditions as under (a) where however in addition $H = H_+ \oplus H_-$ is $\mathbb{Z}/2$ graded, φ is of degree 0 (i.e. $\varphi(x)$ respects the decomposition of H for each $x \in A$) and F is odd (i.e. maps H_+ to H_- and vice versa).

One denotes by $\mathcal{E}_0(A, B)$ and $\mathcal{E}_1(A, B)$ the sets of isomorphism classes of even, respectively odd A - B Kasparov modules.

Kasparov defines two equivalence relations on these sets of modules:

- compact perturbation of F together with stabilization by degenerate elements (i.e. for which the expressions in (KM1), (KM2) are exactly 0)
- homotopy

He shows the quite non-trivial result that both equivalence relations do in fact coincide. If we divide $\mathcal{E}_0(A, B)$ and $\mathcal{E}_1(A, B)$ with respect to these equivalence relations, we obtain abelian groups $KK_0(A, B)$ and $KK_1(A, B)$ (where the addition is induced by direct sum of Kasparov modules).

Specializing to the case where one of the variables of KK is $\mathbb{C}$, we obtain K -theory and K -homology:

$$KK_*(\mathbb{C}, A) = K_*(A) \quad KK_*(A, \mathbb{C}) = K^*(A).$$

To understand the connection with the usual definition of K_0 , as sketched in Chapter 1, assume that A is unital. An element of $K_0(A)$ is then represented by a finitely generated projective module M over A . Considering the Kasparov module $(M \oplus 0, \varphi, 0)$, where φ is the natural action of A on M , we obtain an element of $KK_0(A, B)$. The crucial point of Kasparov's theory is the existence of an intersection product (which of course generalizes the pairing between K -theory and K -homology).

3

Theorem 3 There is an associative product

$$KK_i(A, B) \times KK_j(B, C) \rightarrow KK_{i+j}(A, C)$$

($i, j \in \mathbb{Z}/2$; A, B and C C^* -algebras), which is additive in both variables.

Further basic properties are described in the following theorem.

Theorem 4 The bivariant theory KK_* has the following properties

(a) There is a bilinear, graded commutative, exterior product

$$KK_i(A_1, A_2) \times KK_j(B_1, B_2) \rightarrow KK_{i+j}(A_1 \otimes A_2, B_1 \otimes B_2)$$

(using the minimal or maximal tensor product of C^* -algebras).

(b) Each homomorphism $\varphi: A \rightarrow B$ defines an element $KK(\varphi)$ in the group $KK_0(A, B)$. If $\psi: B \rightarrow C$ is another homomorphism, then

$$KK(\psi \circ \varphi) = KK(\varphi) \cdot KK(\psi)$$

$KK_*(A, B)$ is a contravariant functor in A and a covariant functor in B . If $\alpha: A' \rightarrow A$ and $\beta: B \rightarrow B'$ are homomorphisms, then the induced maps, in the first and second variable of KK_* , are given by left multiplication by $KK(\alpha)$ and right multiplication by $KK(\beta)$.

(c) $KK_*(A, A)$ is, for each C^* -algebra A , a $\mathbb{Z}/2$ -graded ring with unit element $KK(\text{id}_A)$.

(d) The functor KK_* is invariant under homotopies in both variables.

(e) The canonical inclusion $\iota: A \rightarrow \mathcal{K} \widehat{\otimes} A$ defines an invertible element in $KK_0(A, \mathcal{K} \otimes A)$. In particular, $KK_*(A, B) \cong KK_*(\mathcal{K} \otimes A, B)$ and $KK_*(B, A) \cong KK_*(B, \mathcal{K} \otimes A)$ for each C^* -algebra B (recall that $\mathcal{K}$ denotes the standard algebra of compact operators on a separable infinite-dimensional Hilbert space).

(f) (Bott periodicity) There are canonical elements in $KK_1(A, SA)$ and in $KK_1(SA, A)$ which are inverse to each other (recall that the suspension SA of A is defined as the algebra $\mathcal{C}_0((0, 1), A)$ of continuous A -valued functions on $[0, 1]$ vanishing in 0 and 1).

Every odd Kasparov A - B module (H, φ, F) gives rise to an extension

$$0 \rightarrow \mathcal{K}(H) \rightarrow E \rightarrow A' \rightarrow 0$$

by putting $P = 1/2(F + 1)$, $E = P\varphi(A)P$ and taking A' to be the image of E in $\mathcal{L}(H)/\mathcal{K}(H)$. Using stabilization, i.e. adding a degenerate Kasparov module to (H, φ, F) , we can always arrange that $\mathcal{K}(H) \cong \mathcal{K} \otimes B$, φ is injective and $A' \cong A$. We thus get an extension

$$0 \rightarrow \mathcal{K} \otimes B \rightarrow E \rightarrow A \rightarrow 0$$

Conversely, it is easy to see using the Stinespring theorem that every extension admitting a completely positive splitting arises that way. In particular, every such extension

$$E: 0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

defines an element of $KK_1(B, I)$, which we denote by $KK(E)$.

For computations of KK and other K -theoretic invariants, the following long exact sequences associated to an extension are an indispensable tool.

5

Theorem 5 Let D be any separable C^* -algebra. Every extension of C^* -algebras admitting a completely positive linear splitting

$$E: 0 \rightarrow I \xrightarrow{i} A \xrightarrow{q} B \rightarrow 0$$

induces exact sequences in $KK_*(D, \cdot)$ and $KK_*(\cdot, D)$ of the following form:

$$\begin{array}{ccccc} & \cdot KK(i) & & \cdot KK(q) & \\ KK_0(D, I) & \longrightarrow & KK_0(D, A) & \longrightarrow & KK_0(D, B) \\ \uparrow & & & & \downarrow \\ & \cdot KK(q) & & \cdot KK(i) & \\ KK_1(D, B) & \longleftarrow & KK_1(D, A) & \longleftarrow & KK_1(D, I) \end{array} \quad (1.3)$$

and

$$\begin{array}{ccccc} & KK(i) \cdot & & KK(q) \cdot & \\ KK_0(I, D) & \longleftarrow & KK_0(A, D) & \longleftarrow & KK_0(B, D) \\ \downarrow & & & & \uparrow \\ & KK(q) \cdot & & KK(i) \cdot & \\ KK_1(B, D) & \longrightarrow & KK_1(A, D) & \longrightarrow & KK_1(I, D) . \end{array} \quad (1.4)$$

The vertical arrows in (1.3) and (1.4) are (up to a sign) given by right and left multiplication, respectively, by the class $KK(E)$ described above.

A standard strategy to establish these long exact sequences, used for the first time in [24], goes as follows. Establish first, for any star homomorphism $\alpha: A \rightarrow B$ mapping cone exact sequences of the form

$$KK_0(D, C_\alpha) \rightarrow KK_0(D, A) \xrightarrow{\cdot KK(\alpha)} KK_0(D, B)$$

and

$$KK_0(C_\alpha, D) \leftarrow KK_0(A, D) \xleftarrow{KK(\alpha) \cdot} KK_0(B, D) ,$$

where C_α denotes the mapping cone for α . Using suspensions, these sequences can be extended to long exact sequences. To prove the exact sequences in 5 it then remains to show that the natural inclusion map from the ideal I into the mapping cone C_q for the quotient map q in the given extension E , gives an isomorphism in KK .

Kasparov in fact treats his theory more generally in the setting of C^* -algebras with a $\mathbb{Z}/2$ -grading. Using Clifford algebras he gets a very efficient formalism leading for instance to an elegant proof of Bott periodicity which also admits useful generalizations. His original proof of the existence and associativity in this setting however is a technical tour de force which is very difficult to follow.

A simple construction of the product was based in [26] on a rather different description of KK which also revealed some of the abstract properties of the theory.

Given a C^* -algebra A , let QA denote the free product $A * A$. It is defined by the universal property that there are two inclusion maps $\iota, \bar{\iota}: A \rightarrow A * A$, such that, given any two homomorphisms $\alpha, \bar{\alpha}: A \rightarrow B$, there is a unique homomorphism $\alpha * \bar{\alpha}: A * A \rightarrow B$ such that $\alpha = (\alpha * \bar{\alpha}) \circ \iota$ and $\bar{\alpha} = (\alpha * \bar{\alpha}) \circ \bar{\iota}$.

Thus in particular there is a natural homomorphism $\pi = \text{id} * \text{id}: QA \rightarrow A$. We denote by qA the kernel of π . We obtain an extension

$$0 \rightarrow qA \rightarrow QA \rightarrow A \rightarrow 0,$$

which is trivial and in fact has two different natural homomorphism splittings given by ι and $\bar{\iota}$. The following theorem holds

Theorem 6 The group $KK_0(A, B)$ can be described as $[qA, \mathcal{K} \otimes B]$ (where $[X, Y]$ denotes the set of homotopy classes of homomorphisms from X to Y).

6

KK_1 can be obtained from this by taking suspensions in one of the two variables. The proof of the theorem uses the following observation. Given an even A - B Kasparov module (H, φ, F) , one can always arrange that $F^2 = 1$ and that $\mathcal{K}(H) \cong \mathcal{K} \otimes A$. Then, setting $\bar{\varphi} = AdF \circ \varphi$, we get a pair of homomorphisms $\varphi, \bar{\varphi}$ from A to $\mathcal{L}(H)$, therefore a unique homomorphism from the free product QA to $\mathcal{L}(H)$. Since by the condition on a Kasparov module, $\varphi(x) - \bar{\varphi}(x)$ is in $\mathcal{K}(H)$ for each $x \in A$, this homomorphism has to map the ideal qA to $\mathcal{K}(H) \cong \mathcal{K} \otimes A$.

The existence and associativity of the product follows from the following theorem which can be proved using standard C^* -algebra techniques.

Theorem 7 The natural map $\pi: q(qA) \rightarrow qA$ is a homotopy equivalence after stabilizing by 2×2 -matrices, i.e. there exists a homomorphism $\eta: qA \rightarrow M_2(q(qA))$ such that $\pi \circ \eta$ and $\eta \circ \pi$ are both homotopic to the natural inclusions of $qA, q(qA)$ into $M_2(qA), M_2(q(qA))$, respectively.

7

The product between $KK_0(A_1, A_2) = [qA_1, \mathcal{K} \otimes A_2]$ and $KK_0(A_2, A_3) = [qA_2, \mathcal{K} \otimes A_3]$ is then defined as follows:

Let $\varphi: qA_1 \rightarrow \mathcal{K} \otimes A_2$ and $\psi: qA_2 \rightarrow \mathcal{K} \otimes A_3$ represent elements of these two groups. The product is defined as the homotopy class of the following composition

$$q(qA_1) \xrightarrow{q(\varphi)} q(\mathcal{K} \otimes A_2) \rightarrow \mathcal{K} \otimes q(A_2) \xrightarrow{\text{id}_{\mathcal{K}} \otimes \psi} \mathcal{K} \otimes \mathcal{K} \otimes A_3.$$

The arrow in the middle is the natural map. If we now identify $\mathcal{K} \otimes \mathcal{K} \otimes A_3$ with $\mathcal{K} \otimes A_3$ and $q(qA_1)$ with qA_1 using Theorem 7 we obtain the desired element of $KK_0(A_1, A_3)$. The associativity of this product is more or less obvious.

Building on this construction, Zekri, [81], used an algebra εA (which in fact is isomorphic to the crossed product $qA \rtimes \mathbb{Z}/2$) to describe $KK_n(A, B)$ as $[\varepsilon^n A, \mathcal{K} \otimes A]$ (where $\varepsilon^n A = \varepsilon(\varepsilon(\dots \varepsilon A \dots))$). The algebra εA is the universal ideal in an extension of A admitting a completely positive splitting. Therefore every n -step extension

$$0 \rightarrow B \rightarrow E_1 \dots \rightarrow E_n \rightarrow A \rightarrow 0$$

with completely positive splittings has a classifying map $\varepsilon^n A \rightarrow B$ and gives an element in $KK_n(A, B)$. Zekri showed that the Kasparov product of such elements corresponds to the Yoneda product of the original extensions.

It was noted in [25] that KK is a functor which is universal with respect to three natural properties in the following way. Let E be a functor from the category of separable C^* -algebras to the category of abelian groups satisfying:

- E is homotopy invariant, i.e. two homotopic homomorphisms $A \rightarrow B$ induce the same map $E(A) \rightarrow E(B)$
- E is stable, i.e. the natural inclusion $A \rightarrow \mathcal{K} \otimes A$ induces an isomorphism $E(A) \rightarrow E(\mathcal{K} \otimes A)$
- E is split exact, i.e. every extension $0 \rightarrow I \rightarrow A \xrightarrow{q} B \rightarrow 0$ which splits in the sense that there is a homomorphism $A \rightarrow E$ which is a right inverse for q induces a split exact sequence $0 \rightarrow E(I) \rightarrow E(A) \rightarrow E(B) \rightarrow 0$

Then KK acts on E , i.e. every element of $KK(A, B)$ induces a natural map $E(A) \rightarrow E(B)$.

A more streamlined formulation of this result was given by Higson. He noted that KK defines an additive category (i.e. a category where the Hom-sets are abelian groups and the product of morphisms is bilinear) by taking separable C^* -algebras as objects and $KK_0(A, B)$ as set of morphisms between the objects A and B . Then KK is the universal functor into an additive category which is homotopy invariant, stable and split exact in both variables.

More importantly, using abstract ideas from category theory, Higson constructed a new theory, later called E -theory. One shortcoming of KK is the fact that only extensions with a completely positive linear splitting induce long exact sequences. In fact, an important counterexample has been constructed by G. Skandalis, [73], showing that there exist extensions that do not give rise to a long exact sequence in KK (this example also limits the range of validity for some other important properties of KK). Higson now takes the additive category KK and forms a category of fractions E with morphism sets $E(A, B)$ by inverting in KK all morphisms induced by an inclusion $I \rightarrow A$ of a closed ideal I into a C^* -algebra A , for which the quotient A/I is contractible. The category E is additive with a natural functor from the category of separable C^* -algebras into E (which factors over KK). In E , every extension of C^* -algebras (not necessarily admitting a completely positive splitting) induces long exact sequences in $E(\cdot, D)$ and $E(D, \cdot)$ as in (1.3) and (1.4). Moreover, E is the universal functor into an additive category which is homotopy invariant, stable and half-exact.

Later, a more concrete description of $E(A, B)$ was given by Connes and Higson in terms of what they call asymptotic morphisms from A to B . An asymptotic morphism from A to B is a family of maps $(\varphi_t, t \in \mathbb{R})$ from A to B such that the expressions $\varphi_t(x)\varphi_t(y) - \varphi_t(xy)$, $\varphi_t(x) + \lambda\varphi_t(y) - \varphi_t(x + \lambda y)$, $\varphi_t(x)^* - \varphi_t(x^*)$ all tend to 0 for $t \rightarrow \infty$ and $x, y \in A, \lambda \in \mathbb{C}$. Connes and Higson then define $E(A, B)$ as

$$E(A, B) = [[A \otimes \mathcal{C}_0(\mathbb{R}), \mathcal{K} \otimes B \otimes \mathcal{C}_0(\mathbb{R})]] ,$$

where $[[\cdot, \cdot]]$ denotes the set of homotopy classes of asymptotic morphisms. The category E has the properties of KK listed in 4, except that the exterior product in 4(a) only exists with respect to the maximal tensor product of C^* -algebras.

To understand the connection between the bivariant theories KK and E on the one hand, and (monovariant) K -theory on the other hand, the so-called universal coefficient theorem (UCT) is quite useful. Let $\mathcal{N}$ denote the class of C^* -algebras which are isomorphic in KK to an abelian C^* -algebra. This class is in fact quite large. Using some of the standard computations of KK -groups it can be shown that it is invariant under extensions, inductive limits, crossed products by $\mathbb{Z}$ or amenable groups etc. Therefore it contains many if not most of the algebras occurring in applications, since those are often constructed using operations under which $\mathcal{N}$ is stable. Moreover, the convolution C^* -algebra for every amenable groupoid is in $\mathcal{N}$, [76]. It is however an open problem, if every nuclear C^* -algebra is in $\mathcal{N}$.

The universal coefficient theorem (UCT) is the following formula

Theorem 8 ([71]) Let A and B be separable C^* -algebras with A in $\mathcal{N}$. Then there is an exact sequence

8

$$0 \rightarrow \text{Ext}_1(K_*A, K_*B) \rightarrow KK_0(A, B) \rightarrow \text{Hom}_0(K_*A, K_*B) \rightarrow 0,$$

where Ext_1 is odd, i.e. pairs K_0 with K_1 and K_1 with K_0 and Hom_0 is even.

Of course there is an analogous formula for KK_1 . If $A = B$, then the image of the Ext -term on the left is a nilpotent ideal (with square 0) in the ring $KK_0(A, A)$. The product of elements in the Ext -term with elements in the Hom term is obvious. Moreover, the extension splits (unnaturally) as an extension of rings.

If A is in $\mathcal{N}$, then $KK(A, B) = E(A, B)$, therefore the UCT holds in exactly the same generality for E -theory. The counterexample of Skandalis, [73], shows that there are C^* -algebras A for which the UCT for $KK(A, \cdot)$ fails and which therefore are not in $\mathcal{N}$. On the other hand, a separable C^* -algebra A satisfies the UCT for $KK(A, B)$ with arbitrary B if and only if it is in $\mathcal{N}$.

Other Bivariant Theories on Categories Related to C^* -Algebras

1.4

Equivariant KK - and E -Theory

1.4.1

A new element however was introduced by Kasparov in [44] where he introduced equivariant KK -theory with respect to the action of any locally compact group and applied this theory to prove important cases of the Novikov conjecture (for discrete subgroups of connected Lie groups). The equivariant theory for locally

compact groups is technically much more delicate than for compact groups (where for instance the operator F in a Kasparov module can be assumed to be invariant). Equivariant E -theory can be defined in a very natural way using equivariant asymptotic morphisms [33].

The equivariant theory plays an important role in the study (in fact already in the formulation) of the Baum–Connes and Novikov conjectures.

Equivariant KK -theory for the action of a Hopf C^* -algebra H has been studied in [5] where also the following elegant duality result is proved for crossed products by the action of the two Hopf algebras $H = C_{red}^*(G)$ and $\widehat{H} = \mathcal{C}_0(G)$ associated with a locally compact group G .

9

Theorem 9 Let H and $\widehat{H}$ be the two Hopf C^* -algebras associated with a locally compact group G and let A and B be C^* -algebras with an action of H . Then there is an isomorphism

$$KK_H(A, B) \cong KK_{\widehat{H}}(A \rtimes_r H, B \rtimes_r H).$$

The same holds if we interchange H and $\widehat{H}$ (and the action of $\widehat{H}$ is non-degenerate).

1.4.2

KK -Theory for C^* -Algebras over a Topological Space

In his work on the Novikov conjecture, Kasparov used, besides the equivariant KK -theory for the action of a locally compact group in addition a KK -theory on a category of C^* -algebras which are in a well defined technical sense bundles over a fixed locally compact space X . A generalization of this equivariant theory to T_0 -spaces was used by Kirchberg in his work on the classification of non-simple nuclear purely infinite C^* -algebras (the T_0 -space in question here being the ideal space of the given C^* -algebra).

The equivariant theories for the action of a group and the action of a space can be generalized simultaneously to a KK -theory which is equivariant for the action of a groupoid, [51].

1.4.3

KK -Theory for Projective Systems of C^* -Algebras

If X is a noncompact locally compact space, then the algebra $\mathcal{C}(X)$ is not a C^* -algebra, but an inductive limit of C^* -algebras (for instance it can be viewed as the projective limit of the projective system of C^* -algebras $(\mathcal{C}(K))_K$, where K ranges over all compact subsets of X). There are many other natural examples of projective limits of C^* -algebras. It is therefore natural to look for a definition of KK or E -theory to such algebras. This has been done first by Weidner and independently partially by Phillips in [60, 79].

Recently A. Bonkat has developed in his thesis [9] KK -theories on various categories of projective systems of C^* -algebras. The objects of the categories he considers, are projective systems of C^* -algebras admitting a cofinal countable sub-

system. For different choices of morphism sets he obtains as special cases the bivariant theories of Weidner and Phillips but also the theories of Kasparov and Kirchberg for C^* -algebras fibered over a T_0 -space mentioned in Section 1.4.2. Another interesting example where Bonkat's approach applies is the category of 1-step extensions of C^* -algebras. Bonkat proves a UCT for KK on this category which allows him to compute these groups quite explicitly in interesting cases.

Bivariant Theories as Triangulated Categories

1.4.4

Methods from category theory were first used by N. Higson, when he constructed E -theory as a category of fractions from KK -theory. It has turned out later that similar constructions of quotients of bivariant theories can be used in different instances. Usually one forms such quotients in order to enforce certain properties on a bivariant theory. This means that one inverts certain maps which one wants to induce isomorphisms in the theory. More or less equivalently (using mapping cones) one divides by a “null”-subcategory. The framework best suited for that purpose seems to be the one of triangulated categories. A triangulated category is an additive category with a suspension operation on objects and abstract mapping cone sequences (called “triangles”) satisfying a rather long list of compatibility relations. For triangulated categories there is a very smooth way to form quotient categories which are again triangulated. Technically, this is described as follows.

Definition 10 Let $F: \mathfrak{T}_1 \rightarrow \mathfrak{T}_2$ be a functor between triangulated categories preserving the triangulated structure. One denotes by $\ker(F)$ the full triangulated subcategory of $\mathfrak{T}_1$ whose objects map to objects isomorphic to 0 in $\mathfrak{T}_2$.

10

Theorem 11 Let $\mathfrak{T}$ be an essentially small triangulated category and $\mathfrak{R}$ a triangulated subcategory. Then there exists a triangulated quotient category (Verdier quotient) $\mathfrak{T}/\mathfrak{R}$ and a functor $F: \mathfrak{T} \rightarrow \mathfrak{T}/\mathfrak{R}$ preserving the triangulated structure, with the universal property that $\mathfrak{R} \subset \ker(F)$.

11

KK -theory and E -theory as well as some other variants of bivariant theories can be viewed as a triangulated category. The technology of triangulated categories and their quotient categories has been applied in this connection first by Puschnigg, [65] to construct his “local” cyclic homology as a quotient of the bivariant entire theory. Triangulated categories have also been used by Valqui [77] as a framework for bivariant periodic cyclic theory.

The method to use triangulated categories and their quotient categories in order to enforce certain properties on a bivariant theory has been used systematically recently also by A. Thom in his thesis, [74]. The basic triangulated category in his

approach is stable asymptotic homotopy as defined by Connes and Dadarlat [16, 32]. The set of morphisms between two C^* -algebras A and B is defined in this category as

$$\varinjlim_n [[S^n A, S^n B]] ,$$

where S^n denotes n -fold suspension and $[[\cdot, \cdot]]$ homotopy classes of asymptotic morphisms. In the stable asymptotic homotopy category the mapping cone C_q of the quotient map q in an extension $0 \rightarrow I \rightarrow E \xrightarrow{q} A \rightarrow 0$ is isomorphic to I . Thom constructs various bivariant theories as quotient categories of the fundamental stable homotopy category. In that way he obtains for instance bivariant connective K -theory and bivariant singular homology.

Using this approach one can also construct E -theory or similar theories as quotients of the stable asymptotic homotopy category. We mention also that the approach in [28] gives another method to construct bivariant theories for many categories of C^* -algebras or other algebras with specified properties.

1.5

Applications

Many computations of K -theoretic invariants for C^* -algebras can be greatly simplified and generalized using bivariant K -theory. This is true for many of the computations of the early days, e.g. [13, 21, 63, 64] which were based at the beginning on more concrete considerations involving idempotents or invertible elements in algebras. The general method to compute the K -theoretic invariants for a given algebra A consists in constructing an isomorphism in KK with an algebra B for which this computation is simpler.

On the other hand KK_* defines a novel recipient for many new invariants that could not be defined before, such as bivariant symbol classes, equivariant K -homology classes, classes classifying extensions or bivariant classes associated with bundles and many more.

1.5.1

Index Theorems

Every elliptic pseudodifferential operator T of order 0 from sections of a vector bundle E_1 over X to sections of another bundle E_2 determines, by the very definition of a Kasparov module, an element $[T] = (H, \varphi, F)$ in $KK_0(\mathcal{C}(X), \mathbb{C})$. In fact, taking $H = H_1 \oplus H_2$, where H_i denotes the Hilbert space of L^2 -sections in E_i , we may always assume that T is normalized so that $1 - T^*T$ and $1 - TT^*$ are compact. We let then act $\mathcal{C}(X)$ by multiplication on H and put

$$F = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} .$$

Kasparov proved an especially elegant and illuminating form of the index theorem which determines this K -homology class, [40, 42].

If X is a (not necessarily closed) manifold, then the cotangent bundle T^*X considered as a manifold carries an almost complex structure. Therefore there is the Dolbeault operator $D = \bar{\partial} + \bar{\partial}^*: V \rightarrow V$, where V denotes the space of smooth sections with compact support of the bundle of differential forms $\Lambda^{0,*}$ associated to the almost complex structure. D extends to a selfadjoint operator and we can define the bounded operator

$$F = \frac{D}{\sqrt{1 + D^2}}$$

on the Hilbert space H of L^2 -sections of $\Lambda^{0,*}$. In fact, H splits naturally as a direct sum $H = H_+ \oplus H_-$ of sections of even and odd forms. Moreover, $\mathcal{C}_0(X)$ acts on H by multiplication. We therefore get a natural Kasparov module (H, φ, F) and thus an element of $KK(\mathcal{C}_0(T^*X), \mathbb{C})$, denoted by $[\bar{\partial}_X]$ (this element and variants of it play an important role in the work of Kasparov on the Novikov conjecture as the so called Dirac element).

It is important to note that, in the case of $X = \mathbb{R}^n$, the product with this element induces the Bott periodicity map $K_*(\mathcal{C}_0(\mathbb{R}^{2n})) \rightarrow K_*(\mathbb{C})$. In this case there is a natural inverse η in $KK(\mathbb{C}, \mathcal{C}_0(T^*X))$ to $[\bar{\partial}_X] \in KK(\mathcal{C}_0(T^*X), \mathbb{C})$, called the dual Dirac element (the “Fourier transform” of $[\bar{\partial}_X]$).

Assume now that X is a closed manifold and T is an elliptic pseudodifferential operator of order 0 from L^2 -sections of a vector bundle $E_1 \rightarrow X$ to sections of $E_2 \rightarrow X$. Let $T^*X \xrightarrow{\pi} X$ be the projection map for the cotangent bundle. The (full) symbol $\sigma(T)$ of T can be viewed as a morphism of vector bundles from π^*E_1 to π^*E_2 . We obtain a Kasparov module (L, ψ, σ) , where $L = L_1 \oplus L_2$ denotes the space of L^2 -sections of $\pi^*(E_1 \oplus E_2)$, ψ denotes action by multiplication and σ is the sum of $\sigma(T): L_1 \rightarrow L_2$ and $\sigma(T)^*: L_2 \rightarrow L_1$. This gives an element denoted by $[\sigma(T)]$ in $KK_0(\mathcal{C}_0(X), \mathcal{C}_0(T^*X))$. The product with the class $[1]$ of the trivial line bundle in $KK(\mathbb{C}, \mathcal{C}_0(X))$ gives the usual K -theory symbol class $[\Sigma(T)]$ used in the formulation of the Atiyah–Singer theorem.

Kasparov now shows that the K -homology class $[T]$ in $KK_0(\mathcal{C}(X), \mathbb{C})$ determined by T is given by the following formula.

Theorem 12 We have $[T] = [\sigma(T)] \cdot [\bar{\partial}_X]$

12

Note that the usual index of T is simply obtained by pairing the K -homology class $[T]$ with the K_0 -class given by 1 (the trivial line bundle). Thus the standard form of the Atiyah–Singer theorem saying that the analytic index $\text{ind}_a(T)$ equals the topological index $\text{ind}_t(T)$ follows from 12.

In fact, the analytic index is given by $\text{ind}_a(T) = 1 \cdot [T]$ and therefore by Kasparovs theorem by

$$1 \cdot [\sigma(T)] \cdot [\bar{\partial}_X] = [\Sigma(T)] \cdot [\bar{\partial}_X].$$

Using an embedding of X in $\mathbb{R}^n$ and identifying the cotangent bundle of a tubular neighbourhood N of X in this embedding with a bundle over T^*X , we get the following diagram whose commutativity is easy to check:

$$\begin{array}{ccc} K^*(T^*X) \cong K^*(T^*N) & \rightarrow & K^*(\mathbb{R}^{2n}) \\ \downarrow \cdot [\bar{\partial}_X] & & \downarrow \cdot [\bar{\partial}_{\mathbb{R}^n}] \\ \mathbb{Z} & = & \mathbb{Z} \end{array}$$

(K^* here means K -theory with compact supports, e.g. $K^*(\mathbb{R}^n) = K_*(\mathcal{C}_0(\mathbb{R}^n))$). According to Kasparov's theorem 12, the first vertical arrow applied to $[\Sigma(T)]$ gives the analytic index, while the composition of the first horizontal and the second vertical arrow gives the usual definition of the topological index.

To make the connection with the formulation of the index formula using de Rham cohomology and differential forms, Kasparov notes that the Chern character $ch([\bar{\partial}_X])$ is Poincaré dual in de Rham cohomology for T^*X to the Todd class of the complexified cotangent bundle of X (viewed as a bundle over T^*X). Thus applying the Chern character one obtains the usual formula

$$\text{ind}_a(T) = \int_{T^*M} Td(T^*M \otimes \mathbb{C}) \wedge ch([\Sigma(T)]) .$$

From this theorem or at least from its proof, one obtains many other index theorems. Instances are the index theorem for families or the Miscenko–Fomenko index theorem for pseudodifferential operators with coefficients in a C^* -algebra.

Another important and typical index theorem using C^* -algebras is the longitudinal index theorem for foliated manifolds [18]. A foliated manifold is a smooth compact manifold M together with an integrable subbundle F of the tangent bundle for M . The algebra of longitudinal pseudodifferential operators on (M, F) (differentiation only in direction of the leaves) can be completed to a C^* -algebra Ψ_ℓ . The principal symbol of an element in Ψ_ℓ is a function on the dual bundle F^* . The kernel of the symbol map σ is the foliation C^* -algebra $C^*(M, F)$ which is something like a crossed product of $\mathcal{C}(M)$ by translation by $\mathbb{R}^k$ in the direction of F with holonomy resolved. One obtains an exact sequence

$$0 \rightarrow C^*(M, F) \rightarrow \Psi_\ell \xrightarrow{\sigma} \mathcal{C}_0(F^*) \rightarrow 0 . \quad (1.5)$$

A longitudinal pseudodifferential operator T is called elliptic if its image $\sigma(T)$ in (matrices over) $\mathcal{C}_0(F^*)$ (its principal symbol) is invertible. The analytic index of T is by definition the image under the boundary map in the K -theory long exact sequence, of the class in $K_1(F^*)$ defined by $\sigma(T)$, in $K_0(C^*(M, F))$. The index theorem computes this boundary map and thereby allows to obtain explicit formulas for $\text{ind}_a(T)$.

The computation is in terms of the topological index ind_t which is a map from $K_1(F^*)$ to $K_0(C^*(M, F))$ defined using an embedding procedure analogous to the classical case above. It is constructed as follows. Embed M into $\mathbb{R}^n$, define M' as $M \times \mathbb{R}^n$ and define a foliation F' on M' as the product of F with the trivial

foliation by points on $\mathbb{R}^n$. For this new foliated manifold we have that $C^*(M', F')$ is isomorphic to $C^*(M, F) \otimes \mathcal{C}(\mathbb{R}^n)$ and therefore has the same K -theory (with a dimension shift depending on the parity of n). It has the advantage however that M' admits a submanifold N which is transverse to the foliation F' . In fact, one can embed the bundle F in the trivial bundle $M \times \mathbb{R}^n$ and take for N the orthogonal complement to F . Now, for any submanifold N transverse to the foliation F' , the C^* -algebra $C^*(M', F')$ contains $\mathcal{K} \otimes \mathcal{C}_0(N)$ (crossed product of functions on a tubular neighbourhood of N by translation by $\mathbb{R}^k$ in leaf direction).

The topological index is defined as the composition of the following maps

$$K_*(F^*) \cong K_*(N) \rightarrow K_*(C^*(M', F')) \cong K_*(C^*(M, F)) .$$

The index theorem then states that the boundary map in the long exact K -theory sequences associated to the extension 1.5 is exactly this map ind_t .

Theorem 13: ([18]) For every longitudinally elliptic pseudodifferential operator on the foliated manifold (M, F) one has $\text{ind}_a(T) = \text{ind}_t(T)$.

13

In order to prove the theorem, Connes and Skandalis compute a specific Kasparov product.

K -Theory of Group-algebras, Novikov Conjecture, Baum–Connes Conjecture

1.5.2

Let M be a compact connected oriented smooth manifold. The signature $\sigma(M)$ of M can be written as $\langle L(M), [M] \rangle$, where $L(M)$ is the Hirzebruch polynomial in the Pontryagin classes. The signature is homotopy invariant: if two manifolds M and M' are homotopy equivalent via an orientation preserving map, then $\sigma(M) = \sigma(M')$. For simply connected manifolds the signature is the only characteristic number of M with this property.

For non-simply connected manifolds however the “higher signatures” are further candidates for homotopy invariants. If π denotes the fundamental group and $f: M \rightarrow B\pi$ the classifying map then for any $x \in H^*(B\pi, \mathbb{Q})$ we can define the twisted signature $\sigma_x(M) = \langle x, f_* L(M)' \rangle$, where $L(M)'$ is the Poincaré dual to $L(M)$. The Novikov conjecture asserts that the numbers $\sigma_x(M)$ are homotopy invariants for all x or, equivalently, that $f_* L(M)'$ is an oriented homotopy invariant.

Let $[d + \delta]$ denote the K -homology class in $K_0(M) = K^0(\mathcal{C}(M))$ defined by the signature operator $d + \delta$. Using the Chern character isomorphism between $H^*(B\pi, \mathbb{Q})$ and $RK_0(B\pi) \otimes \mathbb{Q} := \varinjlim K_0(X) \otimes \mathbb{Q}$, where the limit is taken over all compact subsets X of $B\pi$ and the fact that, by the Atiyah–Singer theorem, the index of $d + \delta$ is given by pairing with $L(M)$ (i.e. the image under the Chern character of the K -homology class defined by $d + \delta$ is $L(M)'$) the conjecture is equivalent to the fact that $f_*([d + \delta]) \in RK_0(B\pi) \otimes \mathbb{Q}$ is a homotopy invariant.

The K -theoretic approach to a proof of the Novikov conjecture now considers the reduced group C^* -algebra $C_{red}^*\pi$ (i.e. the closure of the algebra of operators generated by the elements of π in the left regular representation). There is a natural construction that associates to every element in $RK_0(B\pi)$ a projection in a matrix algebra over $C_{red}^*\pi$ and therefore an element of $K_0(C_{red}^*\pi)$. This defines a map $\beta: RK_0(B\pi) \rightarrow K_0(C_{red}^*\pi)$. A construction of Mischenko using algebraic surgery shows that $\beta(f_*(d + \delta))$ is always a homotopy invariant.

The Novikov conjecture for π therefore follows from the following “strong Novikov conjecture” (Rosenberg) for π .

14 Conjecture 14: (SNC) The map $\beta: RK_0(B\pi) \rightarrow K_0(C_{red}^*\pi)$ is rationally injective.

This strong Novikov conjecture was proved by Mischenko in the case that $B\pi$ is a closed manifold with non-positive sectional curvature and by Kasparov in the case that $B\pi$ is a (not necessarily compact) complete Riemannian manifold with non-positive sectional curvature (in both cases β itself is already injective). This covers the case where π is a closed torsion-free discrete subgroup of a connected Lie group G , since in this case one can take $\pi \backslash G/K$ for $B\pi$. Using the special structure of discrete subgroups of Lie groups one can reduce to the torsion-free case.

Kasparov’s proof of SNC for groups as above uses the following two theorems which are of independent interest.

Assume that G is a separable locally compact group acting on the complete Riemannian manifold X by isometries. The Dolbeault operator used in 1.5.1 is G -invariant and defines an element $[\partial_X]$ in $KK_0^G(\mathcal{C}(T^*X), \mathbb{C})$.

15 Theorem 15 Let X be simply connected with non-positive sectional curvature. Then the element $[\overline{\partial_X}]$ is right invertible, i.e. there exists a right inverse δ_X in $KK_0^G(\mathbb{C}, \mathcal{C}(T^*X))$ such that $[\overline{\partial_X}] \cdot \delta_X = 1$ in $KK_0^G(\mathcal{C}(T^*X), \mathcal{C}(T^*X))$.

In the presence of a $Spin^c$ -structure on X , $\overline{\partial_X}$ can also be viewed as a Dirac operator. The element δ_X is constructed using a “Fourier transform” of $\overline{\partial_X}$ and is therefore usually called the dual Dirac element.

16 Theorem 16 Let G be connected, K a maximal compact subgroup and $X = G/K$. Then there is a right inverse δ_X to $[\overline{\partial_X}]$ as in 15. The element $\gamma_G = \delta_X \cdot [\overline{\partial_X}]$ in $KK^G(\mathbb{C}, \mathbb{C})$ is an idempotent and does not depend on the choice of K or δ_X .

For all C^* -algebras A and B with an action of G by automorphisms, the natural restriction map

$$KK_*^G(A, B) \rightarrow KK_*^K(A; B)$$

is an isomorphism on $\gamma_G KK_*^G(A, B)$ and has kernel $(1 - \gamma_G)KK_*^G(A, B)$ (note that $KK^G(\mathbb{C}, \mathbb{C})$ acts on $KK_*^G(A, B)$ by tensoring).

It is clear from the considerations above that the K -theoretic proof of the Novikov conjecture for a given group π depends on a partial computation of the K -theory of the group C^* -algebra $C_{red}^*\pi$.

The Baum–Connes conjecture proposes a general formula for $K_*(C_{red}^*\pi)$ by refining the map $\beta: RK_0(B\pi) \rightarrow K_0(C_{red}^*\pi)$. For the Baum–Connes conjecture one uses a map whose construction is similar to the one of β , but one modifies the left-hand side [7]. There is a universal contractible space $\underline{E}_G$ on which G acts properly. An action of a discrete group G on a Hausdorff space X is called proper if any two points x, y , in X have neighbourhoods U and V such that only finitely many translates of U by elements in G intersect V (in particular all stabilizer groups are finite). The left hand side of the Baum–Connes conjecture then is the equivariant K -homology $KK_*^G(\underline{E}_G, \mathbb{C})$ (again defined using an inductive limit over all G -compact subspaces X of $\underline{E}_G$). The map analogous to β is called μ and the conjecture predicts that

$$KK_*^G(\underline{E}_G, \mathbb{C}) \xrightarrow{\mu} K_*(C_{red}^*G)$$

is always an isomorphism. Since the left hand side is an object involving only the equivariant theory of ordinary spaces it can be understood using methods from (“commutative”) topology and there are means to compute it, [6]. The construction also works for groups which are not discrete.

The Baum–Connes conjecture contains the Novikov conjecture and the generalized Kadison conjecture and plays an important motivating role in current research on topological K -theory. While counterexamples to the more general conjecture “with coefficients” have recently been announced by various authors (Higson, Lafforgue–Skandalis, Yu), it is known to hold in many cases of interest (see e.g. [12, 36, 38, 43, 44, 50, 56]).

A general strategy which is used in basically all proofs of the Baum–Connes conjecture for different classes of groups, has been distilled in [33, 76]. It uses actions on so-called proper algebras and abstract versions of “Dirac-” and “dual Dirac-” elements.

Definition 17 Let Γ be a discrete group and let A be a C^* -algebra with an action of Γ . We say that A is proper if there exists a locally compact proper Γ -space X and a Γ -equivariant homomorphism from $\mathcal{C}_0(X)$ into the center of the multiplier algebra of A such that $\mathcal{C}_0(X)A$ is dense in A .

17

Theorem 18 Let Γ be a countable group and let A be a proper Γ - C^* -algebra. Suppose that there are elements α in $KK_*^{\Gamma}(A, \mathbb{C})$ and β in $KK_*^{\Gamma}(\mathbb{C}, A)$ such that $\beta \cdot \alpha = 1$. Then the Baum–Connes conjecture holds for Γ .

18

Let X be a complete Riemannian manifold with non-positive sectional curvature on which Γ acts properly and isometrically (an important special case being Γ

a discrete subgroup of a connected Lie group and $X = G/K$). The elements constructed by Kasparov in 15 define elements $\alpha = [\bar{\partial}_X]$ and $\beta = \delta_X$ for the proper algebra $\mathcal{C}_0(X)$ such that $\alpha \cdot \beta = 1$. If for these elements the element $\gamma = \beta \cdot \alpha$ is also equal to 1, then the Baum–Connes conjecture holds for Γ .

Lafforgue has shown that in some cases where γ is different from 1, this element can still act as the identity on the corresponding K -groups by mapping KK_* to a Banach algebra version of bivariant K -theory which allows more homotopies and Morita equivalences using analogues of Kasparov modules involving Banach modules. He deduced from this the validity of the Baum–Connes conjecture for a class of groups that contains certain property T groups.

1.5.3 Existence of Positive Scalar Curvature Metrics

Let M be a closed smooth spin manifold. J. Rosenberg [70] used the Miscenko–Fomenko index theorem to prove necessary conditions on M for the existence of a Riemannian metric with positive scalar curvature on M . In particular he showed that, if SNC holds for the fundamental group π of M , then the higher $\hat{A}$ -genera of the form

$$\langle \hat{A}(M) \cup f^*(x), [M] \rangle$$

vanish for all $x \in H^*(B\pi, \mathbb{Q})$ (here $f: M \rightarrow B\pi$ is the natural classifying map). This necessary condition can easily be used to show that many manifolds (with fundamental group for which SNC is known to hold) cannot admit a metric with positive scalar curvature. Much more can be said, see [71].

1.5.4 Applications in the Classification of Nuclear C^* -Algebras

The interest in K -theoretic methods among operator algebraists was strongly motivated by the fact that K -theoretic invariants allowed to distinguish C^* -algebras which looked otherwise very similar. One of the first computations of that kind was the one by Pimsner–Popa and, independently, Paschke–Salinas of the Ext-groups for the algebras $\mathcal{O}_n$. The algebra $\mathcal{O}_n$ is defined as the C^* -algebra with generators $s_1, \dots, s_n$ and relations $s_i^* s_i = 1, \sum_{i=1}^n s_i s_i^* = 1$. The algebra $\mathcal{O}_\infty$ has generators $s_1, s_2, \dots$ and relations $s_i^* s_i = 1, s_i^* s_j = 0, j \neq i$, [20]. The result of Pimsner–Popa and Paschke–Salinas is that $\text{Ext}(\mathcal{O}_n) = \mathbb{Z}/(n-1)$, so that in particular, they are not isomorphic for different n . Another striking application of K -theory were the influential results of Pimsner–Voiculescu on the K -theory of noncommutative tori [63] and – later – of the reduced group C^* -algebras of free groups [64]. Again these computations showed that noncommutative tori with different twist or reduced group C^* -algebras of free groups with different number of generators could not be isomorphic.

The effectiveness of K -theory in the classification of nuclear C^* -algebras has however proved to go beyond all expectations of these early days. In fact it turned

out that up to a notion of stable isomorphism, nuclear simple C^* -algebras are in a sense completely classified by KK -theory.

A simple C^* -algebra A is called purely infinite if for all $x, y \in A$ with x nonzero, there are $a, b \in A$ such that $y = axb$. The most standard examples of purely infinite algebras are the algebras $\mathcal{O}_n$, $n = 2, 3, \dots, \infty$ mentioned above. They have $K_0(\mathcal{O}_n) = \mathbb{Z}/n$, $n = 2, 3, \dots$, $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_n) = 0$, $n = 2, 3, \dots, \infty$. If A is any simple C^* -algebra, then $K_*(A \otimes \mathcal{O}_\infty) = K_*(A)$. Moreover $A \otimes \mathcal{O}_\infty$ is automatically purely infinite and $A \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong A \otimes \mathcal{O}_\infty$. Thus $A \otimes \mathcal{O}_\infty$ may be viewed as a purely infinite stabilization of A . This shows the interest of the following astonishing and deep theorem obtained independently by Kirchberg and Phillips after groundbreaking work of Kirchberg.

Recall that a C^* -algebra is called stable if $A \cong \mathcal{K} \otimes A$.

Theorem 19: (cf. [69], 8.4.1) Let A and B be purely infinite simple nuclear algebras.

19

- (a) Assume that A and B are stable. Then A and B are isomorphic if and only if they are isomorphic in KK . Moreover, for each invertible element x in KK there exists an isomorphism $\varphi: A \rightarrow B$ with $KK(\varphi) = x$.
- (b) Assume that A and B are stable and belong to the UCT class $\mathcal{N}$. Then A is isomorphic to B if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$. Moreover, for each pair of isomorphisms $\alpha_i: K_i(A) \rightarrow K_i(B)$, $i = 1, 2$, there is an isomorphism $\varphi: A \rightarrow B$ with $K_i(\varphi) = \alpha_i$, $i = 1, 2$.
- (c) Assume that A and B are unital. Then A and B are isomorphic if and only if there exists an invertible element x in $KK(A, B)$ such that $[1_A] \cdot x = [1_B]$ (where $\cdot$ denotes Kasparov product and $[1_A], [1_B]$ the elements of $K_0 A = KK(\mathbb{C}, A)$, $K_0(B) = KK(\mathbb{C}, B)$ defined by the units of A and B). For each such element x , there is an isomorphism $\varphi: A \rightarrow B$ with $KK(\varphi) = x$.
- (d) Assume that A and B are unital and belong to the UCT class $\mathcal{N}$. Then A is isomorphic to B if and only if there exist isomorphisms $\alpha_i: K_i(A) \rightarrow K_i(B)$, $i = 1, 2$ such that $\alpha_0([1_A]) = [1_B]$. Moreover, for each such pair of isomorphisms, there is an isomorphism $\varphi: A \rightarrow B$ with $K_i(\varphi) = \alpha_i$, $i = 1, 2$.

There are many examples of purely infinite nuclear algebras in the UCT class with the same K -groups but constructed in completely different ways. By the theorem these algebras have to be isomorphic, but in general it is impossible to find an explicit isomorphism.

Classification of Topological Dynamical Systems

1.5.5

A topological dynamical system is an action of $\mathbb{Z}$ on a compact space X by homeomorphisms. With such systems one can associate various noncommutative C^* -algebras, the most obvious one being the crossed product $\mathcal{C}(X) \rtimes \mathbb{Z}$.

Interesting for applications are in particular systems where X is a Cantor space. The K -theory of the crossed product for such a system has been analyzed and been

used to obtain results on various notions of orbit equivalence for such systems in [34].

Besides the crossed product one can also associate other C^* -algebras constructed from groupoids associated with the system. Such a construction can be applied to subshifts of finite type. A subshift of finite type is defined by an $n \times n$ -matrix $A = (a(ij))$ with entries $a(ij)$ in $\{0, 1\}$. The shift space X_A consists of all families $(c_k)_{k \in \mathbb{Z}}$ with $c_k \in \{1, \dots, n\}$ and $a(c_k c_{k+1}) = 1$ for all k . The subshift is given by the shift transformation σ_A on X_A . A groupoid associated with (X_A, σ_A) gives the C^* -algebra $\mathcal{O}_A$ considered in [23]. It is a homeomorphism invariant for the suspension flow space associated to the transformation (X_A, σ_A) . The K -groups for $\mathcal{O}_A$ recover invariants of flow equivalence discovered by Bowen and Franks, [10]:

$$K_0(\mathcal{O}_A) = \mathbb{Z}^n / (1 - A)\mathbb{Z}^n$$

$$K_1(\mathcal{O}_A) = \text{Ker}(1 - A).$$

If a topological Markov chain (X_C, σ_C) is not minimal, then it can be decomposed into two components (X_A, σ_A) and (X_B, σ_B) . Correspondingly, the matrix C can be written in the form

$$C = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}.$$

The corresponding C^* -algebra $\mathcal{O}_C$ contains $\mathcal{K} \otimes \mathcal{O}_B$ as an ideal with $\mathcal{O}_A$ as quotient. The corresponding extension

$$0 \rightarrow \mathcal{K} \otimes \mathcal{O}_B \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_A \rightarrow 0$$

defines an element of $KK_1(\mathcal{O}_A, \mathcal{O}_B)$ which describes how the suspension spaces for (X_A, σ_A) and (X_B, σ_B) are glued to give the one for (X_C, σ_C) .

In [22], it was proved that

$$KK_1(\mathcal{O}_A, \mathcal{O}_B) \cong K^1(\mathcal{O}_A) \otimes K_0(\mathcal{O}_B) \oplus \text{Hom}(K_0(\mathcal{O}_A), K_1(\mathcal{O}_B)).$$

The two summands can be described as equivalence classes of $n \times m$ -matrices. In fact they are the cokernels and kernels, for right multiplication by B , in the cokernel for left multiplication by A , on the space $M_{n,m}(\mathbb{Z})$ of $n \times m$ -matrices.

It has been shown that, for the case of reducible topological Markov chains (X_C, σ_C) with two components, this extension invariant together with the flow equivalence invariants for the components give complete invariants of flow equivalence for (X_C, σ_C) .

We mention that in many similar cases such as the C^* -algebras associated with a non-minimal foliation the extension invariant for the corresponding extension of the associated C^* -algebras can in principle be used to describe the way that the big system is composed from its components.

Bivariant K -Theory for Locally Convex Algebras

1.6

KK -theory and E -theory both use techniques which are quite specific to the category of C^* -algebras (in particular, central approximate identities play a crucial role). Therefore similar bivariant theories for other categories of algebras seemed for many years out of reach. Since ordinary and periodic cyclic theory give only pathological results for C^* -algebras, this made it in particular impossible to compare bivariant K -theory with cyclic theory via a Chern character.

A bivariant K -theory was developed finally in [28] for a large category of locally convex algebras (“ m -algebras”). Since in this category one has less analytic tools at one’s disposal, the construction had to be based on a better understanding of the underlying algebraic structure in bivariant K -theory. The definition is formally similar to the qA -picture described briefly in chapter 1. The underlying idea is to represent elements of the bivariant theory by extensions of arbitrary length and the product by the Yoneda product of extensions.

A locally convex algebra A is, in general, an algebra with a locally convex topology for which the multiplication $A \times A \rightarrow A$ is (jointly) continuous. In the present survey we restrict our attention however to locally convex algebras that can be represented as projective limits of Banach algebras.

A locally convex algebra $\mathcal{A}$ that can be represented as a projective limit of Banach algebras can equivalently be defined as a complete locally convex algebra whose topology is determined by a family $\{p_\alpha\}$ of submultiplicative seminorms, [55]. Thus for each α we have $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$. The algebra $\mathcal{A}$ is then automatically a topological algebra, i.e. multiplication is (jointly) continuous. We call such algebras m -algebras. The unitization $\tilde{\mathcal{A}}$ of an m -algebra is again an m -algebra in a natural way. Also the completed projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{B}$ of two m -algebras is again an m -algebra.

Since cyclic theory is homotopy invariant only for differentiable homotopies (called diffotopies below), we have to set up the theory in such a way that it uses only diffotopies in place of general homotopies.

Definition 20 Two continuous homomorphisms $\alpha, \beta: \mathcal{A} \rightarrow \mathcal{B}$ between two m -algebras are called differentially homotopic or **diffotopic**, if there is a continuous homomorphism φ from $\mathcal{A}$ to the algebra $\mathcal{C}^\infty([0, 1], \mathcal{B})$ of infinitely differentiable $\mathcal{B}$ -valued functions on the unit interval $[0, 1]$, such that the composition of φ with the evaluation maps $\mathcal{C}^\infty([0, 1], \mathcal{B}) \rightarrow \mathcal{B}$ at 0 and 1 give α and β , respectively.

20

It is not hard to see (though not completely obvious) that diffotopy is an equivalence relation.

The m -algebra $\mathcal{K}$ of “smooth compact operators” consists of all $\mathbb{N} \times \mathbb{N}$ -matrices (a_{ij}) with rapidly decreasing matrix elements $a_{ij} \in \mathbb{C}$, $i, j = 0, 1, 2, \dots$. The topology on $\mathcal{K}$ is given by the family of norms p_n , $n = 0, 1, 2, \dots$, which are defined by

$$p_n((a_{ij})) = \sum_{i,j} |1 + i + j|^n |a_{ij}|$$

It is easily checked that the p_n are submultiplicative and that $\mathcal{K}$ is complete. Thus $\mathcal{K}$ is an m -algebra. As a locally convex vector space, $\mathcal{K}$ is isomorphic to the sequence space s and therefore is nuclear in the sense of Grothendieck. The algebra $\mathcal{K}$ of smooth compact operators is of course smaller than the C^* -algebra of compact operators on a separable Hilbert space which we used in the previous sections. It plays however exactly the same role in the theory. We hope that the use of the same symbol $\mathcal{K}$ will not lead to confusion.

The map that sends $(a_{ij}) \otimes (b_{kl})$ to the $\mathbb{N}^2 \times \mathbb{N}^2$ -matrix $(a_{ij}b_{kl})_{(i,k)(j,l) \in \mathbb{N}^2 \times \mathbb{N}^2}$ obviously gives an isomorphism Θ between $\mathcal{K} \widehat{\otimes} \mathcal{K}$ and $\mathcal{K}$.

21

Definition 21 Let $\mathcal{A}$ and $\mathcal{B}$ be m -algebras. For any homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ of m -algebras, we denote by $\langle \varphi \rangle$ the equivalence class of φ with respect to diffotopy and we set

$$\langle \mathcal{A}, \mathcal{B} \rangle = \{ \langle \varphi \rangle \mid \varphi \text{ is a continuous homomorphism } \mathcal{A} \rightarrow \mathcal{B} \}$$

For two continuous homomorphisms $\alpha, \beta: \mathcal{A} \rightarrow \mathcal{K} \widehat{\otimes} \mathcal{B}$ we define the direct sum $\alpha \oplus \beta$ as

$$\alpha \oplus \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}: \mathcal{A} \rightarrow M_2(\mathcal{K} \widehat{\otimes} \mathcal{B}) \cong \mathcal{K} \widehat{\otimes} \mathcal{B}.$$

With the addition defined by $\langle \alpha \rangle + \langle \beta \rangle = \langle \alpha \oplus \beta \rangle$ the set $\langle \mathcal{A}, \mathcal{K} \widehat{\otimes} \mathcal{B} \rangle$ of diffotopy classes of homomorphisms from $\mathcal{A}$ to $\mathcal{K} \widehat{\otimes} \mathcal{B}$ is an abelian semigroup with 0-element $\langle 0 \rangle$.

Let V be a complete locally convex space. We define the smooth tensor algebra $T^s V$ as the completion of the algebraic tensor algebra

$$TV = V \oplus V \otimes V \oplus V^{\otimes 3} \oplus \dots$$

with respect to the family $\{\widehat{p}\}$ of seminorms, which are given on this direct sum as

$$\widehat{p} = p \oplus p \otimes p \oplus p^{\otimes 3} \oplus \dots,$$

where p runs through all continuous seminorms on V . The seminorms $\widehat{p}$ are submultiplicative for the multiplication on TV . The completion $T^s V$ therefore is an m -algebra.

We denote by $\sigma: V \rightarrow T^s V$ the map, which maps V to the first summand in TV . The map σ has the following universal property: Let $s: V \rightarrow \mathcal{A}$ be any continuous linear map from V to an m -algebra $\mathcal{A}$. Then there is a unique continuous homomorphism $\tau_s: T^s V \rightarrow \mathcal{A}$ of m -algebras such that $\tau_s \circ \sigma = s$. It is given by

$$\tau_s(x_1 \otimes x_2 \otimes \dots \otimes x_n) = s(x_1)s(x_2) \dots s(x_n)$$

The smooth tensor algebra is differentially contractible, i.e., the identity map on $T^s V$ is diffotopic to 0. A differentiable family $\varphi_t: T^s V \rightarrow T^s V$ of homomorphisms for which $\varphi_0 = 0, \varphi_1 = \text{id}$ is given by $\varphi_t = \tau_{t\sigma}, t \in [0, 1]$.

If $\mathcal{A}$ is an m -algebra, by abuse of notation, we write $T\mathcal{A}$ (rather than $T^s\mathcal{A}$) for the smooth tensor algebra over $\mathcal{A}$. Thus $T\mathcal{A}$ is again an m -algebra. For any m -algebra $\mathcal{A}$ there exists a natural extension

$$0 \rightarrow J\mathcal{A} \rightarrow T\mathcal{A} \xrightarrow{\pi} \mathcal{A} \rightarrow 0. \quad (1.6)$$

Here π maps a tensor $x_1 \otimes x_2 \otimes \dots \otimes x_n$ to $x_1 x_2 \dots x_n \in \mathcal{A}$ and $J\mathcal{A} = \text{Ker } \pi$. This extension is universal in the sense that, given any extension $0 \rightarrow \mathcal{I} \rightarrow \mathcal{E} \rightarrow \mathcal{A} \rightarrow 0$ of $\mathcal{A}$, admitting a continuous linear splitting, there is a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J\mathcal{A} & \longrightarrow & T\mathcal{A} & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\ & & \downarrow \gamma & & \downarrow \tau & & \downarrow \text{id} & & \\ 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{A} & \longrightarrow & 0. \end{array}$$

The map $\tau: T\mathcal{A} \rightarrow \mathcal{E}$ is obtained by choosing a continuous linear splitting $s: \mathcal{A} \rightarrow \mathcal{E}$ in the given extension and mapping $x_1 \otimes x_2 \otimes \dots \otimes x_n$ to $s(x_1)s(x_2) \dots s(x_n) \in \mathcal{E}$. Then γ is the restriction of τ .

Definition 22 The map $\gamma: J\mathcal{A} \rightarrow \mathcal{I}$ in this commutative diagram is called the *classifying map* for the extension $0 \rightarrow \mathcal{I} \rightarrow \mathcal{E} \rightarrow \mathcal{A} \rightarrow 0$. 22

If s and s' are two different linear splittings, then for each $t \in [0, 1]$ the map $s_t = ts + (1 - t)s'$ is again a splitting. The corresponding maps γ_t associated with s_t form a diffotopy between the classifying map constructed from s and the one constructed from s' . The classifying map is therefore unique up to diffotopy.

For an extension admitting a homomorphism splitting the classifying map is diffotopic to 0.

An extension of $\mathcal{A}$ of length n is an exact complex of the form

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \dots \rightarrow \mathcal{E}_n \rightarrow \mathcal{A} \rightarrow 0,$$

where the arrows or boundary maps (which we denote by φ) are continuous homomorphisms between m -algebras. We call such an extension linearly split, if there is a continuous linear map s of degree -1 on this complex, such that $s\varphi + \varphi s = \text{id}$. This is the case if and only if the given extension is a Yoneda product (concatenation) of n linearly split extensions of length 1: $\mathcal{I} \rightarrow \mathcal{E}_1 \rightarrow \text{Im } \varphi_1, \text{Ker } \varphi_2 \rightarrow \mathcal{E}_2 \rightarrow \text{Im } \varphi_2, \dots$.

$J\mathcal{A}$ is, for each m -algebra $\mathcal{A}$, again an m -algebra. By iteration we can therefore form $J^2\mathcal{A} = J(J\mathcal{A}), \dots, J^n\mathcal{A} = J(J^{n-1}\mathcal{A})$.

23 **Proposition 23** For any linearly split extension

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \dots \rightarrow \mathcal{E}_n \rightarrow \mathcal{A} \rightarrow 0$$

of $\mathcal{A}$ of length n there is a classifying map $\gamma: J^n \mathcal{A} \rightarrow \mathcal{I}$ which is unique up to diffotopy.

Proof Compare with the free n -step extension

$$0 \rightarrow J^n \mathcal{A} \rightarrow T(J^{n-1} \mathcal{A}) \rightarrow T(J^{n-2} \mathcal{A}) \rightarrow \dots \rightarrow T \mathcal{A} \rightarrow \mathcal{A} \rightarrow 0.$$

24 **Proposition 24** J (and J^n) is a functor, i.e. each continuous homomorphism $\mathcal{A} \rightarrow \mathcal{B}$ between m -algebras induces a continuous homomorphism $J\mathcal{A} \rightarrow J\mathcal{B}$.

Consider now the set $H_k = \langle J^k \mathcal{A}, \mathcal{K} \widehat{\otimes} \mathcal{B} \rangle$, where $H_0 = \langle \mathcal{A}, \mathcal{K} \widehat{\otimes} \mathcal{B} \rangle$. Each H_k is an abelian semigroup with 0-element for the K -theory addition defined in 21. Morally, the elements of H_k are classifying maps for linearly split k -step extensions. In applications all elements arise that way.

To define a map $S: H_k \rightarrow H_{k+2}$, we use the classifying map ε for the two-step extension which is obtained by composing the so called Toeplitz extension

$$0 \rightarrow \mathcal{K} \widehat{\otimes} \mathcal{A} \rightarrow \mathcal{T}_0 \widehat{\otimes} \mathcal{A} \rightarrow \mathcal{A}(0, 1) \rightarrow 0$$

with the cone or suspension extension

$$0 \rightarrow \mathcal{A}(0, 1) \rightarrow \mathcal{A}[0, 1) \rightarrow \mathcal{A} \rightarrow 0.$$

Here, $\mathcal{A}(0, 1)$ and $\mathcal{A}[0, 1)$ denote the algebras of smooth $\mathcal{A}$ -valued functions on the interval $[0, 1]$, that vanish in 0 and 1, or only in 1, respectively, and whose derivatives all vanish in both endpoints. The smooth Toeplitz algebra $\mathcal{T}_0$ is a standard extension of $\mathbb{C}(0, 1)$ where a preimage of an element e^{ih} in $\mathbb{C}(0, 1)^\sim$ has Fredholm index 1, for each monotone function h in $\mathbb{C}(0, 1]$ such that $h(0) = 0$ and $h(1) = 1$.

25 **Definition 25** For each m -algebra $\mathcal{A}$, we define the periodicity map $\varepsilon_{\mathcal{A}}: J^2 \mathcal{A} \rightarrow \mathcal{K} \widehat{\otimes} \mathcal{A}$ as the classifying map for the standard two-step Bott extension

$$0 \rightarrow \mathcal{K} \widehat{\otimes} \mathcal{A} \rightarrow \mathcal{T}_0 \widehat{\otimes} \mathcal{A} \rightarrow \mathcal{A}[0, 1) \rightarrow \mathcal{A} \rightarrow 0.$$

We can now define the Bott map S . For $\langle \alpha \rangle \in H_k$, $\alpha: J^k \mathcal{A} \rightarrow \mathcal{K} \widehat{\otimes} \mathcal{B}$ we set $S\langle \alpha \rangle = \langle (\text{id}_{\mathcal{K}} \widehat{\otimes} \alpha) \circ \varepsilon \rangle$. Here $\varepsilon: J^{k+2} \mathcal{A} \rightarrow \mathcal{K} \otimes J^k \mathcal{A}$ is the ε -map for $J^k \mathcal{A}$.

If an element γ of H_k is given as a classifying map for an extension of length k , then $S\gamma$ is the classifying map for the Yoneda product of the given extension with the Bott extension.

Let $\varepsilon_-: J^{k+2}\mathcal{A} \rightarrow \mathcal{K} \otimes J^k\mathcal{A}$ be the map which is obtained by replacing, in the definition of ε , the Toeplitz extension by the inverse Toeplitz extension. The sum $\varepsilon \oplus \varepsilon_-$ is then diffotopic to 0. Therefore $S\langle\alpha\rangle + S_-\langle\alpha\rangle = 0$, putting $S_-\langle\alpha\rangle = \langle(\text{id}_{\mathcal{K}} \otimes \alpha) \circ \varepsilon_-\rangle$.

Definition 26 Let $\mathcal{A}$ and $\mathcal{B}$ be m -algebras and $*$ = 0 or 1. We define

26

$$kk_*(\mathcal{A}, \mathcal{B}) = \varinjlim_k H_{2k+*} = \varinjlim_k \langle J^{2k+*}\mathcal{A}, \mathcal{K} \widehat{\otimes} \mathcal{B} \rangle.$$

The preceding discussion shows that $kk_*(\mathcal{A}, \mathcal{B})$ is not only an abelian semigroup, but even an abelian group (every element admits an inverse). A typical element of $kk_*(\mathcal{A}, \mathcal{B})$ is given by a classifying map of a linearly split extension

$$0 \rightarrow \mathcal{K} \otimes \mathcal{B} \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \dots \rightarrow \mathcal{E}_n \rightarrow \mathcal{A} \rightarrow 0$$

In the inductive limit, we identify such an extension with its composition with the Bott extension for $\mathcal{A}$ on the right hand side.

As usual for bivariant theories, the decisive element, which is also the most difficult to establish, is the composition product.

Theorem 27 There is an associative product

27

$$kk_i(\mathcal{A}, \mathcal{B}) \times kk_j(\mathcal{B}, \mathcal{C}) \rightarrow kk_{i+j}(\mathcal{A}, \mathcal{C})$$

($i, j \in \mathbb{Z}/2$; $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ m -algebras), which is additive in both variables.

Neglecting the tensor product by $\mathcal{K}$, the product of an element represented by $\varphi \in \langle J^k\mathcal{A}, \mathcal{K} \widehat{\otimes} \mathcal{B} \rangle$ and an element represented by $\psi \in \langle J^l\mathcal{B}, \mathcal{K} \widehat{\otimes} \mathcal{C} \rangle$ is defined as $\langle \psi \circ J^l(\varphi) \rangle$. Thus for elements of kk which are given as classifying maps for higher length extensions, the product simply is the classifying map for the Yoneda product of the two extensions. The fact that this is well defined, i.e. compatible with the periodicity map S , demands a new idea, namely the “*basic lemma*” from [28].

Lemma 28 Assume given a commutative diagram of the form

28

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \rightarrow & \mathcal{I} & \rightarrow & \mathcal{A}_{01} & \rightarrow & \mathcal{A}_{02} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{A}_{10} & \rightarrow & \mathcal{A}_{11} & \rightarrow & \mathcal{A}_{12} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{A}_{20} & \rightarrow & \mathcal{A}_{21} & \rightarrow & \mathcal{B} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array},$$

where all the rows and columns represent extensions of m -algebras with continuous linear splittings.

Let γ_+ and γ_- denote the classifying maps $J^2\mathcal{B} \rightarrow \mathcal{I}$ for the two extensions of length 2

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{A}_{10} \rightarrow \mathcal{A}_{21} \rightarrow \mathcal{B} \rightarrow 0$$

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{A}_{01} \rightarrow \mathcal{A}_{12} \rightarrow \mathcal{B} \rightarrow 0$$

associated with the two edges of the diagram. Then $\gamma_+ \oplus \gamma_-$ is diffotopic to 0.

This lemma implies that, the classifying maps for the compositions of a linearly split extension

$$0 \rightarrow \mathcal{B} \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \dots \rightarrow \mathcal{E}_n \rightarrow \mathcal{A} \rightarrow 0$$

with the Bott extension for $\mathcal{B}$ on the left or with the Bott extension for $\mathcal{A}$ on the right hand side are diffotopic.

The usual properties of a bivariant K -theory as listed in 4 and in particular the long exact sequences 1.3 and 1.4 in 5 for linearly split extensions of m -algebras can then be deduced for kk in a rather standard fashion.

Moreover, the following important theorem holds.

29 **Theorem 29** For every Banach algebra $\mathcal{A}$, the groups $kk_*(\mathbb{C}, \mathcal{A})$ and $K_*\mathcal{A}$ are naturally isomorphic.

In particular $kk_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$ and $kk_1(\mathbb{C}, \mathbb{C}) = 0$. Phillips, [60], has developed a topological K -theory K_* for Fréchet m -algebras that extends the theory from Banach algebras. Also for that theory one finds that $kk_*(\mathbb{C}, \mathcal{A}) \cong K_*\mathcal{A}$ for each Fréchet m -algebra $\mathcal{A}$.

30 **Remark 30** A version of bivariant K -theory for general locally convex algebras – not just m -algebras – has been worked out in [29]. In [29] a slightly different (but basically equivalent) approach is used to define the bivariant theory which can also be used to construct the bivariant theory for m -algebras described above. It is motivated by the thesis of A. Thom [74]. The theory is constructed as noncommutative stable homotopy, i.e. as an inductive limit over suspensions of both variables (using a noncommutative suspension for the first variable), rather than as an inductive limit over the inverse Bott maps ε as above. This simplifies some of the arguments and also clarifies the fact that Bott periodicity becomes automatic once we stabilize by (smooth) compact operators in the second variable.

Bivariant Cyclic Theories

1.7

The Algebra ΩA of Abstract Differential Forms over A and its Operators

1.7.1

There are many different but essentially equivalent descriptions of the complexes used to define cyclic homology. The most standard choice is the cyclic bicomplex. For our purposes it is more convenient to use the (B, b) -bicomplex. B and b are operators on the algebra ΩA of abstract differential forms over A .

Given an algebra A , we denote by ΩA the universal algebra generated by all $x \in A$ with relations of A and all symbols $dx, x \in A$, where dx is linear in x and satisfies $d(xy) = xdy + d(x)y$. We do not impose $d1 = 0$, i.e., if A has a unit, $d1 \neq 0$. ΩA is a direct sum of subspaces $\Omega^n A$ generated by linear combinations of $x_0 dx_1 \dots dx_n$, and $dx_1 \dots dx_n, x_j \in A$. This decomposition makes ΩA into a graded algebra. As usual, we write $\deg(\omega) = n$ if $\omega \in \Omega^n A$.

As a vector space, for $n \geq 1$,

$$\Omega^n A \cong \tilde{A} \otimes A^{\otimes n} \cong A^{\otimes(n+1)} \oplus A^{\otimes n} \quad (1.7)$$

(where $\tilde{A}$ is A with a unit adjoined, and $1 \otimes x_1 \otimes \dots \otimes x_n$ corresponds to $dx_1 \dots dx_n$). The operator d is defined on ΩA by

$$\begin{aligned} d(x_0 dx_1 \dots dx_n) &= dx_0 dx_1 \dots dx_n \\ d(dx_1 \dots dx_n) &= 0 \end{aligned} \quad (1.8)$$

The operator b is defined by

$$\begin{aligned} b(\omega dx) &= (-1)^{\deg \omega} [\omega, x] \\ b(dx) &= 0, \quad b(x) = 0, \quad x \in A, \quad \omega \in \Omega A \end{aligned} \quad (1.9)$$

Then, by definition, $d^2 = 0$ and one easily computes that also $b^2 = 0$.

Under the isomorphism in Equation (1.7) d becomes

$$\begin{aligned} d(x_0 \otimes \dots \otimes x_n) &= 1 \otimes x_0 \otimes \dots \otimes x_n \quad x_0 \in A \\ d(1 \otimes x_1 \otimes \dots \otimes x_n) &= 0 \end{aligned}$$

while b corresponds to the usual Hochschild operator

$$\begin{aligned} b(\tilde{x}_0 \otimes x_1 \otimes \dots \otimes x_n) &= \\ \tilde{x}_0 x_1 \otimes \dots \otimes x_n &+ \sum_{j=2}^n (-1)^{j-1} \tilde{x}_0 \otimes \dots \otimes x_{j-1} x_j \otimes \dots \otimes x_n \\ &+ (-1)^n x_n \tilde{x}_0 \otimes x_1 \otimes \dots \otimes x_{n-1}, \quad \tilde{x}_0 \in \tilde{A}, \quad x_1, \dots, x_n \in A. \end{aligned}$$

Another important natural operator is the degree (or number) operator:

$$N(\omega) = \deg(\omega)\omega . \quad (1.10)$$

We also define the Karoubi operator κ on ΩA by

$$\kappa = 1 - (db + bd) . \quad (1.11)$$

Explicitly, κ is given by

$$\kappa(\omega dx) = (-1)^{\deg \omega} dx \omega .$$

The operator κ satisfies $(\kappa^n - 1)(\kappa^{n+1} - 1) = 0$ on Ω^n . Therefore, by linear algebra, there is a projection operator P on ΩA corresponding to the generalized eigenspace for 1 of the operator $\kappa = 1 - (db + bd)$.

31

Lemma 31 Let $L = (Nd)b + b(Nd)$. Then $\Omega A = \text{Ker } L \oplus \text{Im } L$ and P is exactly the projection onto $\text{Ker } L$ in this splitting.

Proof This follows from the identity

$$L = (\kappa - 1)^2(\kappa^{n-1} + 2\kappa^{n-2} + 3\kappa^{n-3} + \dots + (n-1)\kappa + n) .$$

The operator L thus behaves like a “selfadjoint” operator. It can be viewed as an abstract Laplace operator on the algebra of abstract differential forms ΩA . The elements in the image of P are then “abstract harmonic forms”.

By construction, P commutes with b , d , N . Thus setting $B = NPd$ one finds $Bb + bB = PL = 0$ and $B^2 = 0$.

Explicitly, B is given on $\omega \in \Omega$ by the formula

$$B(\omega) = \sum_{j=0}^n \kappa^j d\omega .$$

Under the isomorphism in Equation (1.7), this corresponds to:

$$\begin{aligned} B(x_0 dx_1 \dots dx_n) &= dx_0 dx_1 \dots dx_n + (-1)^n dx_n dx_0 \dots dx_{n-1} \\ &\quad + \dots + (-1)^{nm} dx_1 \dots dx_n dx_0 \end{aligned}$$

The preceding identities show that we obtain a bicomplex – the (B, b) -bicomplex – in the following way

$$\begin{array}{ccccccc} & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\ \Omega^3 A & \xleftarrow{B} & \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A \\ & \downarrow b & & \downarrow b & & \downarrow b & \\ \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & \\ & \downarrow b & & \downarrow b & & & \\ \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & & & \\ & \downarrow b & & & & & \\ \Omega^0 A & & & & & & \end{array} \quad (1.12)$$

One can rewrite the (B, b) -bicomplex (1.12) using the isomorphism $\Omega^n A \cong A^{\otimes(n+1)} \oplus A^{\otimes n}$ in Equation (1.7). An easy computation shows that the operator $b: \Omega^n A \rightarrow \Omega^{n-1} A$ corresponds under this isomorphism to the operator $A^{\otimes(n+1)} \oplus A^{\otimes n} \rightarrow A^{\otimes n} \oplus A^{\otimes(n-1)}$ which is given by the matrix

$$\begin{pmatrix} b & (1-\lambda) \\ 0 & -b' \end{pmatrix}$$

where b' , b and λ are the operators in the usual cyclic bicomplex.

Similarly, the operator $B: \Omega^n A \rightarrow \Omega^{n+1} A$ corresponds to the operator $A^{\otimes(n+1)} \oplus A^{\otimes n} \rightarrow A^{\otimes(n+2)} \oplus A^{\otimes(n+1)}$ given by the 2×2 -matrix

$$\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix},$$

where $Q = 1 + \lambda + \lambda^2 + \dots + \lambda^n$.

This shows that the (B, b) -bicomplex is just another way of writing the usual cyclic bicomplex which is based on the operators b, b', λ and Q . The total complex D^Ω for the (B, b) -bicomplex is exactly isomorphic to the total complex for the cyclic bicomplex. We define

Definition 32 The cyclic homology $HC_n A$ is the homology of the complex

32

$$\rightarrow D_n^\Omega \xrightarrow{B'-b} D_{n-1}^\Omega \xrightarrow{B'-b} \dots \xrightarrow{B'-b} D_1^\Omega \xrightarrow{B'-b} D_0^\Omega \rightarrow 0$$

where

$$D_{2n}^\Omega = \Omega^0 A \oplus \Omega^2 A \oplus \dots \oplus \Omega^{2n} A$$

$$D_{2n+1}^\Omega = \Omega^1 A \oplus \Omega^3 A \oplus \dots \oplus \Omega^{2n+1} A$$

and B' is the truncated B -operator, i.e., $B' = B$ on the components $\Omega^k A$ of D_n^Ω , except on the highest component $\Omega^n A$, where it is 0.

The Hochschild homology $HH_n(A)$ is the homology of the first column in 1.12, i.e. of the complex

$$\rightarrow \Omega^n A \xrightarrow{b} \Omega^{n-1} A \xrightarrow{b} \dots \xrightarrow{b} \Omega^1 A \xrightarrow{b} \Omega^0 A \rightarrow 0$$

Remark 33 Assume that A has a unit 1. We may introduce in ΩA the additional relation $d(1) = 0$, i.e., divide by the ideal M generated by $d(1)$ (this is equivalent to introducing the relation $1 \cdot \omega = \omega$ for all ω in ΩA). We denote the quotient by $\overline{\Omega A}$. Now M is a graded subspace, invariant under b and B , and its homology with respect to b is trivial. The preceding proposition is thus still valid if we use $\overline{\Omega A}$ in place of ΩA . The convention $d(1) = 0$ is often used (implicitly) in the literature. In some cases it simplifies computations considerably. There are however situations where one cannot reduce the computations to the unital situation (in particular this is true for the excision problem).

33

The Periodic Theory

The periodic theory is the one that has the really good properties like diffeotopy invariance, Morita invariance and excision. It generalizes the classical de Rham theory to the non-commutative setting.

Periodic Cyclic Homology

Let A be an algebra. We denote by $\widehat{\Omega}A$ the infinite product

$$\widehat{\Omega}A = \prod_n \Omega^n A$$

and by $\widehat{\Omega}^{ev}A, \widehat{\Omega}^{odd}A$ its even and odd part, respectively. $\widehat{\Omega}A$ may be viewed as the (periodic) total complex for the bicomplex

$$\begin{array}{ccccccc} & \downarrow -b & & \downarrow -b & & \downarrow -b & & \downarrow -b \\ \leftarrow & \Omega^3 A & \xleftarrow{B} & \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A \\ & \downarrow -b & & \downarrow -b & & \downarrow -b & & \\ \leftarrow & \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & \\ & \downarrow -b & & \downarrow -b & & & & \\ \leftarrow & \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & & & \\ & \downarrow -b & & & & & & \\ \leftarrow & \Omega^0 A & & & & & & \end{array}$$

Similarly, the (continuous for the filtration topology) dual $(\widehat{\Omega}A)'$ of $\widehat{\Omega}A$ is

$$(\widehat{\Omega}A)' = \bigoplus_n (\Omega^n A)'.$$

34

Definition 34 The periodic cyclic homology $HP_*(A)$, $*$ = 0, 1, is defined as the homology of the $\mathbb{Z}/2$ -graded complex

$$\widehat{\Omega}^{ev}A \begin{array}{c} \xleftarrow{B-b} \\ \xrightarrow{B-b} \end{array} \widehat{\Omega}^{odd}A$$

and the periodic cyclic cohomology $HP^*(A)$, $*$ = 0, 1, is defined as the homology of the $\mathbb{Z}/2$ -graded complex

$$(\widehat{\Omega}^{ev}A)' \begin{array}{c} \xleftarrow{B-b} \\ \xrightarrow{B-b} \end{array} (\widehat{\Omega}^{odd}A)'.$$

Now by definition, S is the projection

$$D_n^\Omega = \Omega^n \oplus \Omega^{n-2} \oplus \dots \rightarrow D_{n-2}^\Omega = \Omega^{n-2} \oplus \Omega^{n-4} \oplus \dots$$

where D_n^Ω is as in 32. Therefore we get

$$\widehat{\Omega}A = \varprojlim_S (D_{2n}^\Omega \oplus D_{2n+1}^\Omega)$$

and

$$(\widehat{\Omega}A)' = \varinjlim_{S'} (D_{2n}^\Omega \oplus D_{2n+1}^\Omega)'.$$

We deduce

Proposition 35 For any algebra A and $*$ = 0, 1 one has

35

$$HP^*(A) = \varinjlim_S HC^{2n+*}A$$

and an exact sequence

$$0 \rightarrow \varprojlim_S^1 HC_{2n+*+1}A \rightarrow HP_*A \rightarrow \varprojlim_S HC_{2n+*}A \rightarrow 0$$

(where as usual $\varprojlim_S^1 HC_{2n+*+1}A$ is defined as

$$\left(\prod_n HC_{2n+*+1}A \right) / (1-s) \left(\prod_n HC_{2n+*+1}A \right)$$

s being the shift on the infinite product).

The Bivariant Theory

Now $\widehat{\Omega}A$ is in a natural way a complete metric space (with the metric induced by the filtration on ΩA – the distance of families (x_n) and (y_n) in $\prod \Omega^n A$ is $\leq 2^{-k}$ if the k first x_i and y_i agree). We call this the filtration topology and denote by $\text{Hom}(\widehat{\Omega}A, \widehat{\Omega}B)$ the set of continous linear maps $\widehat{\Omega}A \rightarrow \widehat{\Omega}B$. It can also be described as

$$\text{Hom}(\widehat{\Omega}A, \widehat{\Omega}B) = \varprojlim_m \varinjlim_n \text{Hom} \left(\bigoplus_{i \leq n} \Omega^i A, \bigoplus_{j \leq m} \Omega^j B \right).$$

It is a $\mathbb{Z}/2$ -graded complex with boundary map

$$\partial \varphi = \varphi \circ \partial - (-1)^{\deg(\varphi)} \partial \circ \varphi$$

where $\partial = B - b$.

Definition 36 Let A and B be algebras. Then the bivariant periodic cyclic homology $HP_*(A, B)$ is defined as the homology of the Hom-complex

36

$$HP_*(A, B) = H_*(\text{Hom}(\widehat{\Omega}A, \widehat{\Omega}B)) \quad * = 0, 1.$$

It is not difficult to see that the $\mathbb{Z}/2$ -graded complex $\widehat{\mathbb{C}}$ is (continuously with respect to the filtration topology) homotopy equivalent to the trivial complex

$$\mathbb{C} \xrightarrow{\sim} 0.$$

Therefore

$$HP_*(\mathbb{C}, B) = HP_*(B) \quad \text{and} \quad HP_*(A, \mathbb{C}) = HP^*(A).$$

There is an obvious product

$$HP_i(A_1, A_2) \times HP_j(A_2, A_3) \rightarrow HP_{i+j}(A_1, A_3)$$

induced by the composition of elements in $\text{Hom}(\widehat{\Omega}A_1, \widehat{\Omega}A_2)$ and $\text{Hom}(\widehat{\Omega}A_2, \widehat{\Omega}A_3)$, which we denote by $(x, y) \mapsto x \cdot y$. In particular, $HP_0(A, A)$ is a unital ring with unit 1_A given by the identity map on $\widehat{\Omega}A$.

An element $\alpha \in HP_*(A, B)$ is called invertible if there exists $\beta \in HP_*(B, A)$ such that $\alpha \cdot \beta = 1_A \in HP_0(A, A)$ and $\beta \cdot \alpha = 1_B \in HP_0(B, B)$. An invertible element of degree 0, i.e., in $HP_0(A, B)$ will also be called an *HP-equivalence*. Such an *HP-equivalence* exists in $HP_0(A, B)$ if and only if the supercomplexes $\widehat{\Omega}A$ and $\widehat{\Omega}B$ are continuously homotopy equivalent. Multiplication by an invertible element α on the left or on the right induces natural isomorphisms $HP_*(B, D) \cong HP_*(A, D)$ and $HP_*(D, A) \cong HP_*(D, B)$ for any algebra D .

37

Remark 37 One can also define a $\mathbb{Z}$ -graded version of the bivariate cyclic theory, [37], as follows. Say that a linear map $\alpha: \widehat{\Omega}A \rightarrow \widehat{\Omega}B$, continuous for the filtration topology, is of order $\leq k$ if $\alpha(F^n \widehat{\Omega}A) \subset F^{n-k} \widehat{\Omega}B$ for $n \geq k$, where $F^n \widehat{\Omega}A$ is the infinite product of $b(\Omega^{n+1}), \Omega^{n+1}, \Omega^{n+2}, \dots$. We denote by $\text{Hom}^k(\widehat{\Omega}A, \widehat{\Omega}B)$ the set of all maps of order $\leq k$. This is, for each k , a subcomplex of the $\mathbb{Z}/2$ -graded complex $\text{Hom}(\widehat{\Omega}A, \widehat{\Omega}B)$. We can define

$$HC_n(A, B) = H_i(\text{Hom}^n(\widehat{\Omega}A, \widehat{\Omega}B)) \quad \text{where } i \in \{0, 1\}, \quad i \equiv n \pmod{2}.$$

The bivariate theory $HC_n(A, B)$ has a product $HC_n(A, B) \times HC_m(B, C) \rightarrow HC_{n+m}(A, C)$ and satisfies

$$HC_n(\mathbb{C}, B) = HC_n(B) \quad \text{and} \quad HC_n(A, \mathbb{C}) = HC^n(A).$$

In general, there exist elements in $HP_*(A, B)$ which are not in the range of the natural map $HC_{2n+*}(A, B) \rightarrow HP_*(A, B)$ for any n , [30].

The bivariate periodic theory HP_* defines a linear category that has formally exactly the same properties (cf. 4) as the bivariate K -theories described above.

Theorem 38 HP_* has the following properties

(a) There is an associative product

$$HP_i(A, B) \times HP_j(B, C) \rightarrow HP_{i+j}(A, C)$$

($i, j \in \mathbb{Z}/2$), which is additive in both variables.

(b) There is a bilinear, graded commutative, exterior product

$$HP_i(A_1, A_2) \times HP_j(B_1, B_2) \rightarrow HP_{i+j}(A_1 \widehat{\otimes} A_2, B_1 \widehat{\otimes} B_2).$$

(c) Each homomorphism $\varphi: A \rightarrow B$ defines an element $HP(\varphi)$ in $HP_0(A, B)$. If $\psi: B \rightarrow C$, is another homomorphism, then

$$HP(\psi \circ \varphi) = HP(\varphi) \cdot HP(\psi)$$

$HP_*(A, B)$ is a contravariant functor in A and a covariant functor in B . If $\alpha: A' \rightarrow A$ and $\beta: B \rightarrow B'$ are homomorphisms, then the induced maps, in the first and second variable of HP_* , are given by left multiplication by $HP(\alpha)$ and right multiplication by $HP(\beta)$.

(d) $HP_*(A, A)$ is, for each algebra A , a $\mathbb{Z}/2$ -graded ring with unit element $HP(\text{id}_A)$.

(e) The functor HP_* is invariant under diffeotopies in both variables.

(f) The canonical inclusion $\iota: A \rightarrow \mathcal{K} \otimes A$ defines an invertible element in $HP_0(A, \mathcal{K} \otimes A)$, where $\mathcal{K}$ denotes the algebra of smooth compact operators considered in the previous. In particular, $HP_*(A, B) \cong HP_*(\mathcal{K} \otimes A, B)$ and $HP_*(B, A) \cong HP_*(B, \mathcal{K} \otimes A)$ for each algebra B .

(g) Let $0 \rightarrow S \rightarrow P \rightarrow Q \rightarrow 0$ be an extension of algebras and A an algebra. There are two natural six-term exact sequences

$$\begin{array}{ccccc} HP_0(A, S) & \longrightarrow & HP_0(A, P) & \longrightarrow & HP_0(A, Q) \\ \uparrow & & & & \downarrow \\ HP_1(A, Q) & \longleftarrow & HP_1(A, P) & \longleftarrow & HP_1(A, S) \end{array}$$

and

$$\begin{array}{ccccc} HP_0(S, A) & \longleftarrow & HP_0(P, A) & \longleftarrow & HP_0(Q, A) \\ \downarrow & & & & \uparrow \\ HP_1(Q, A) & \longrightarrow & HP_1(P, A) & \longrightarrow & HP_1(S, A), \end{array}$$

where the horizontal arrows are induced by the maps in the given extension and the vertical arrows are products by a canonical element in $HP_1(Q, S)$ associated with the extension.

The bivariant cyclic theory HP_* makes sense without any basic modifications for arbitrary locally convex algebras. One only has to impose continuity on all maps and use completed projective tensor products instead of algebraic tensor products everywhere. The theorem above then remains valid (again replacing tensor products by completed projective tensor products). Only in the proof of the long exact sequences one has to be a little more careful than in the purely algebraic case, cf. [27, 77].

An equivariant version, for the action of a discrete group, of bivariant cyclic theory has been developed by C. Voigt, [78].

1.7.3

Local Cyclic Cohomology and Bivariant Local Theory

Connes used inductive limit topologies with respect to bounded (if A is e.g. a Banach algebra) or finite subsets of A on the algebra $\Omega(A)$ to define his entire cyclic cohomology. This idea can be considerably extended. R. Meyer developed an elegant framework of bivariant entire theory for bornological algebras (i.e. algebras with a system of “bounded” sets). Even more significantly, Puschnigg took Connes’ idea as a basis to establish the bivariant “local” theory which furnishes the “correct” homological invariants for Banach algebras and in particular for C^* -algebras.

If A is a Fréchet algebra, for each precompact subset C of A , we take on $\Omega(A)$ the seminorm (Minkowski functional) defined by the absolutely convex hull of the union of all sets of the form $CdC \dots dC$ or $dCdC \dots dC$ in ΩA . We denote the Banach space obtained as the completion of ΩA with respect to this seminorm (after first dividing by its nullspace) as $(\Omega A)_C$ and thus obtain an inductive system $((\Omega A)_C)$ of Banach spaces. It is well known that such inductive systems form a category, the category of “Ind-spaces”, with morphism sets between two such systems $(K_i)_I$ and $(L_j)_J$ defined by

$$\mathrm{Hom}((K_i), (L_j)) = \lim_{\leftarrow I} \lim_{\rightarrow J} \mathrm{Hom}(K_i, L_j).$$

On ΩA we now have to consider the boundary operator $\frac{2}{N}B - \frac{N}{2}b$, where N denotes the degree operator on ΩA (see (1.10)), rather than just $B - b$. In the algebraic setting this makes no difference, since the complex $(\widehat{\Omega}A, B - b)$ is isomorphic to $(\widehat{\Omega}A, \frac{2}{N}B - \frac{N}{2}b)$. In the topological setting however the difference is crucial. The conceptual explanation for the choice of $\frac{2}{N}B - \frac{N}{2}b$ is the description of the cyclic complex in terms of the “X-complex” for a quasifree resolution of A , [30].

The operator $\frac{2}{N}B - \frac{N}{2}b$ defines maps of Ind-spaces

$$((\Omega^{ev} A)_C) \rightleftarrows ((\Omega^{odd} A)_C)$$

and thus a $\mathbb{Z}/2$ -graded Ind-complex.

To define H_{loc}^* we observe the following. The category $\mathrm{Ho}(\mathrm{Ind})$ of inductive systems of $\mathbb{Z}/2$ -graded complexes of normed spaces with homotopy classes of chain

maps as morphisms is a triangulated category. The subclass $\mathcal{N}$ of inductive systems isomorphic to a system of *contractible* complexes is a “null system” in $\text{Ho}(\text{Ind})$. Thus the corresponding quotient category $\text{Ho}(\text{Ind})/\mathcal{N}$ is again a triangulated category. In this quotient category all elements of $\mathcal{N}$ become isomorphic to the zero object.

In the following definition, we view $\mathbb{C}$ as a constant inductive system of complexes with zero boundary as usual.

Definition 39 Let $K = (K_i)_{i \in I}$ and $L = (L_j)_{j \in J}$ be two inductive systems of complexes. Define $H_0^{\text{loc}}(K, L)$ to be the space of morphisms $K \rightarrow L$ in the category $\text{Ho}(\text{Ind})/\mathcal{N}$ and $H_n^{\text{loc}}(K, L)$ the space of morphisms $K[n] \rightarrow L$. Let

$$H_{\text{loc}}^*(K) := H_*^{\text{loc}}(K, \mathbb{C}), \quad H_*^{\text{loc}}(L) := H_*^{\text{loc}}(\mathbb{C}, L).$$

One can compute $H_*^{\text{loc}}(K, L)$ via an appropriate projective resolution of K (see [65]). An analysis of this resolution yields that there is a spectral sequence whose E^2 -term involves the homologies $H_*(\mathcal{L}(K_i, L_j))$ and the derived functors of the projective limit functor $\varprojlim$, and which converges to $H_*^{\text{loc}}(K, L)$ under suitable assumptions. For countable $\varprojlim$ inductive systems, the derived functors $R^p \varprojlim$ with $p \geq 2$ vanish. Hence the spectral sequence degenerates to a Milnor $\varprojlim^1$ -exact sequence

$$\begin{aligned} 0 \rightarrow \varprojlim_I^1 \varinjlim_J H_{*-1}(\mathcal{L}(K_i, L_j)) &\rightarrow H_*^{\text{loc}}(K, L) \\ &\rightarrow \varprojlim_I \varinjlim_J H_*(\mathcal{L}(K_i, L_j)) \rightarrow 0 \end{aligned}$$

if I is countable. In the local theory we have this exact sequence for arbitrary countable inductive systems (K_i) . In particular, for $K = \mathbb{C}$ we obtain

$$H_*^{\text{loc}}(L) = \varinjlim H_*(L_j).$$

Since the inductive limit functor is exact, $H_*^{\text{loc}}(L)$ is equal to the homology of the inductive limit of L .

The completion of an inductive system of normed spaces $(X_i)_{i \in I}$ is defined entry-wise: $(X_i)^c := (X_i^c)_{i \in I}$.

Definition 40 Let A and B be Fréchet algebras. We define *local cyclic cohomology*, *local cyclic homology*, and *bivariant local cyclic homology* by

$$\begin{aligned} HE_{\text{loc}}^*(A) &:= H_{\text{loc}}^*(\Omega A), \\ HE_*^{\text{loc}}(B) &:= H_*^{\text{loc}}(\Omega B), \\ HE_*^{\text{loc}}(A, B) &:= H_*^{\text{loc}}(\Omega A, \Omega B). \end{aligned}$$

where ΩA denotes the *Ind*-space $((\Omega A)_C)$.

As usual, we have $HE_*^{\text{loc}}(A, \mathbb{C}) \cong HE_{\text{loc}}^*(A)$ and $HE_*^{\text{loc}}(\mathbb{C}, A) \cong HE_*^{\text{loc}}(A)$. The composition of morphisms gives rise to a product on HE_*^{loc} .

The main advantage of local cyclic cohomology is that it behaves well when passing to “smooth” subalgebras.

41 Definition 41 Let A be a Fréchet algebra, let B be a Banach algebra with closed unit ball U and let $j: A \rightarrow B$ be an injective continuous homomorphism with dense image. We call A a *smooth subalgebra* of B iff $S^\infty := \bigcup S^n$ is precompact whenever $S \subset A$ is precompact and $j(S) \subseteq rU$ for some $r < 1$.

Let $A \subseteq B$ be a smooth subalgebra. Then an element $a \in A$ that is invertible in B is already invertible in A . Hence A is closed under holomorphic functional calculus.

Whereas the inclusion of a smooth subalgebra $A \rightarrow B$ induces an isomorphism $K_*(A) \cong K_*(B)$ in K -theory, the periodic or entire cyclic theories of A and B may differ drastically. However, local theory behaves like K -theory in this situation:

42 Theorem 42 Let $j: A \rightarrow B$ be the inclusion and let $j_* \in HE_0^{\text{loc}}(A, B)$ be the corresponding element in the bivariant local cyclic homology.

If B has Grothendieck’s approximation property, then j_* is invertible.

Again HE_*^{loc} defines a linear category with objects Fréchet algebras and morphism sets $HE_*^{\text{loc}}(A, B)$. It has the properties listed in 3, 4, 5 for C^* -algebras but in a significantly more flexible form. In particular, it applies also to arbitrary Fréchet algebras. Moreover, tensor products by C^* -algebras and $\mathcal{C}_0(0, 1)$ can be replaced by smooth subalgebras of these tensor products, etc. In some important cases, such as the one of certain group C^* -algebras it can be computed directly [67]

1.8

Bivariant Chern Characters

The construction of characteristic classes in cyclic cohomology or homology associated to K -theory or K -homology elements has been one of the major guidelines for the development of cyclic theory, [14, 39].

After the development of a well understood machinery for cyclic homology and also of a corresponding bivariant theory with properties similar to those of bivariant K -theory, [31], the principal obstruction to the definition of a bivariant Chern character from bivariant K -theory to bivariant cyclic homology consisted in the fact that both theories were defined on different categories of algebras.

Bivariant K -theories were defined for categories of C^* -algebras. For C^* -algebras however the standard cyclic theory gives only trivial and pathological results. (One basic reason for that is the fact that cyclic theory is invariant only under differentiable homotopies, not under continuous ones. Thus for instance the algebra

$\mathcal{C}_0([0, 1])$ of continuous functions on the interval is not equivalent for cyclic theory to $\mathbb{C}$.) On the other hand, cyclic theory gives good results for many locally convex algebras where a bivariant K -theory was not available.

There are two ways out of this dilemma. The first one consists in defining bivariant K -theory with good properties for locally convex algebras, [28] and the second one in developing a cyclic theory which gives good results for C^* -algebras, [65]. In both cases, once the suitable theories are constructed, the existence of the bivariant character follows from the abstract properties of the theories.

The Bivariant Chern–Connes Character for Locally Convex Algebras

1.8.1

In this section we describe the construction of a bivariant multiplicative transformation from the bivariant theory kk_* described in Chapter 1.6 to the bivariant theory HP_* on the category of m -algebras. As a very special case it will furnish the correct frame for viewing the characters for idempotents, invertibles or Fredholm modules constructed by Connes, Karoubi and others.

Consider a covariant functor E from the category of m -algebras to the category of abelian groups which satisfies the following conditions:

- (E1) E is diffeotopy invariant, i.e., the evaluation map ev_t in any point $t \in [0, 1]$ induces an isomorphism $E(ev_t): E(\mathcal{A}[0, 1]) \rightarrow E(\mathcal{A})$.
- (E2) E is stable, i.e., the canonical inclusion $\iota: \mathcal{A} \rightarrow \mathcal{K} \widehat{\otimes} \mathcal{A}$ induces an isomorphism $E(\iota): E(\mathcal{A}) \rightarrow E(\mathcal{K} \widehat{\otimes} \mathcal{A})$.
- (E3) E is half-exact, i.e., each extension $0 \rightarrow \mathcal{I} \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow 0$ admitting a continuous linear splitting induces a short exact sequence $E(\mathcal{I}) \rightarrow E(\mathcal{A}) \rightarrow E(\mathcal{B})$.

(The same conditions can of course be formulated analogously for a contravariant functor E .) In (E1), $\mathcal{A}[0, 1]$ denotes, as in section 1.6, the algebra of smooth $\mathcal{A}$ -valued functions on $[0, 1]$ whose derivatives vanish in 0 and 1. Similarly, $\mathcal{A}(0, 1)$ consists of functions that, in addition, vanish at the endpoints. We note that a standard construction from algebraic topology, using property (E1) and mapping cones, permits to extend the short exact sequence in (E3) to an infinite long exact sequence of the form

$$\begin{aligned} \dots \rightarrow E(\mathcal{B}(0, 1)^2) \rightarrow E(\mathcal{I}(0, 1)) \rightarrow E(\mathcal{A}(0, 1)) \\ \rightarrow E(\mathcal{B}(0, 1)) \rightarrow E(\mathcal{I}) \rightarrow E(\mathcal{A}) \rightarrow E(\mathcal{B}) \end{aligned}$$

see, e.g., [41] or [8].

Theorem 43 Let E be a covariant functor with the properties (E1), (E2), (E3). Then we can associate in a unique way with each $h \in kk_0(\mathcal{A}, \mathcal{B})$ a morphism of abelian groups $E(h): E(\mathcal{A}) \rightarrow E(\mathcal{B})$, such that $E(h_1 \cdot h_2) = E(h_2) \circ E(h_1)$ for the product

43

$h_1 \cdot h_2$ of $h_1 \in kk_0(\mathcal{A}, \mathcal{B})$ and $h_2 \in kk_0(\mathcal{B}, \mathcal{C})$ and such that $E(kk(\alpha)) = E(\alpha)$ for each morphism $\alpha: \mathcal{A} \rightarrow \mathcal{B}$ of m -algebras (recall that $kk(\alpha)$ denotes the element of $kk_0(\mathcal{A}, \mathcal{B})$ induced by α).

An analogous statement holds for contravariant functors.

Proof Let h be represented by $\eta: J^{2n}\mathcal{A} \rightarrow \mathcal{K} \widehat{\otimes} \mathcal{B}$. We set

$$E(h) = E(\iota)^{-1}E(\eta)E(\epsilon^n)^{-1}E(\iota)$$

where ϵ^n is the classifying map for the iterated Bott extension used in the definition of kk_* and ι denotes the inclusion of an algebra into its tensor product by $\mathcal{K}$. It is clear that $E(h)$ is well-defined and that $E(kk(\alpha)) = E(\alpha)$.

The preceding result can be interpreted differently, see also [8,35]. For this, consider again kk_0 as an additive category, whose objects are the m -algebras, and where the morphism set between $\mathcal{A}$ and $\mathcal{B}$ is given by $kk_0(\mathcal{A}, \mathcal{B})$. This category is additive in the sense that the morphism set between two objects forms an abelian group and that the product of morphisms is bilinear.

We denote the natural functor from the category of m -algebras to the category kk_0 , which is the identity on objects, by kk .

44 Corollary 44 Let F be a functor from the category of m -algebras to an additive category C , such that $F(\beta \circ \alpha) = F(\alpha) \cdot F(\beta)$, for any two homomorphisms $\alpha: A_1 \rightarrow A_2$ and $\beta: A_2 \rightarrow A_3$ between m -algebras.

We assume that for each $\mathcal{B}$, the contravariant functor $C(F(\cdot), F(\mathcal{B}))$ and the covariant functor $C(F(\mathcal{B}), F(\cdot))$ on the category of m -algebras satisfy the properties (E1), (E2), (E3). Then there is a unique covariant functor F' from the category kk_0 to C , such that $F = F' \circ kk$.

45 Remark 45 Property (E3) implies that any such functor F' is automatically additive:

$$F'(h + g) = F'(h) + F'(g)$$

As a consequence of the preceding corollary we get a bilinear multiplicative transformation from kk_0 to HP_0 – the bivariant Chern–Connes character.

46 Corollary 46 There is a unique (covariant) functor $ch: kk_0 \rightarrow HP_0$, such that $ch(kk(\alpha)) = HP_0(\alpha) \in HP_0(\mathcal{A}, \mathcal{B})$ for every morphism $\alpha: \mathcal{A} \rightarrow \mathcal{B}$ of m -algebras.

Proof This follows from 44, since HP_0 satisfies conditions (E1), (E2) and (E3) in both variables.

The Chern–Connes-character ch is by construction compatible with the composition product on kk_0 and HP_0 . It also is compatible with the exterior product on kk_0 (as in 4(a)), and the corresponding product on HP_0 , see [31], p.86.

It remains to extend ch to a multiplicative transformation from the $\mathbb{Z}/2$ -graded theory kk_* to HP_* and to study the compatibility with the boundary maps in the long exact sequences associated to an extension for kk and HP .

The natural route to the definition of ch in the odd case is the use of the identity $kk_1(\mathcal{A}, \mathcal{B}) = kk_0(J\mathcal{A}, \mathcal{B})$.

Since HP satisfies excision in the first variable and since $HP_*(T\mathcal{A}, \mathcal{B}) = 0$ for all $\mathcal{B}$ ($T\mathcal{A}$ is contractible), we find that

$$HP_0(J\mathcal{A}, \mathcal{B}) \cong HP_1(\mathcal{A}, \mathcal{B}) . \quad (1.13)$$

However, for the product $kk_1 \times kk_1 \rightarrow kk_0$ we have to use the identification

$$kk_0(J^2\mathcal{A}, \mathcal{B}) \cong kk_0(\mathcal{A}, \mathcal{B}) ,$$

which is induced by the ε -map $J^2\mathcal{A} \rightarrow \mathcal{K} \widehat{\otimes} \mathcal{A}$. This identification is different from the identification $HP_0(J^2\mathcal{A}, \mathcal{B}) \cong HP_0(\mathcal{A}, \mathcal{B})$ which we obtain by applying (1.13) twice. In fact, we have

Proposition 47 Under the natural identification

47

$$HP_0(J^2\mathcal{A}, \mathcal{K} \widehat{\otimes} \mathcal{A}) \cong HP_0(\mathcal{A}, \mathcal{A})$$

from (1.13), the element $ch(\varepsilon)$ corresponds to $(2\pi i)^{-1}$.

We are therefore lead to the following definition.

Definition 48 Let u be an element in $kk_1(\mathcal{A}, \mathcal{B})$ and let u_0 be the corresponding element in $kk_0(J\mathcal{A}, \mathcal{B})$. We set

48

$$ch(u) = \sqrt{2\pi i} ch(u_0) \in HP_1(\mathcal{A}, \mathcal{B}) \cong HP_0(J\mathcal{A}, \mathcal{B}) .$$

Theorem 49 The thus defined Chern–Connes character $ch: kk_* \rightarrow HP_*$ is multiplicative, i.e., for $u \in kk_i(\mathcal{A}, \mathcal{B})$ and $v \in kk_j(\mathcal{B}, \mathcal{C})$ we have

49

$$ch(u \cdot v) = ch(u) \cdot ch(v) .$$

It follows that the character is also compatible with the boundary maps in the six-term exact sequences induced by an extension in both variables of kk_* and HP_* .

For m -algebras, the bivariant character ch constructed here is a far reaching generalization of the Chern characters from K -Theory and K -homology considered by Connes, Karoubi and many others.

1.8.2 The Chern Character for C^* -Algebras

Bivariant local cyclic homology is exact for extensions with a bounded linear section, homotopy invariant for smooth homotopies and stable with respect to tensor products with the trace class operators $\ell^1(H)$. Using Theorem 42, we can strengthen these properties considerably:

50

Theorem 50 Let A be a C^* -algebra. The functors $B \mapsto HE_*^{\text{loc}}(A, B)$ and $B \mapsto HE_*^{\text{loc}}(B, A)$ are split exact, stable homotopy functors on the category of C^* -algebras.

For separable C^* -algebras, there is a natural bivariant Chern character

$$ch: KK_*(A, B) \rightarrow HE_*^{\text{loc}}(A, B) .$$

The Chern character is multiplicative with respect to the Kasparov product on the left and the composition product on the right hand side.

If both A and B satisfy the universal coefficient theorem in Kasparov theory, then there is a natural isomorphism

$$HE_*^{\text{loc}}(A, B) \cong \text{Hom} \left(K_*(A) \otimes_{\mathbb{Z}} \mathbb{C}, K_*(B) \otimes_{\mathbb{Z}} \mathbb{C} \right) .$$

Proof Since $C^\infty([0, 1], A) \subseteq C([0, 1], A)$ is a smooth subalgebra, Theorem 42 and smooth homotopy invariance imply continuous homotopy invariance. The projective tensor product of A by the algebra $\mathcal{K}$ of smooth compact operators is a smooth subalgebra of the C^* -algebraic stabilization of A . Hence Theorem 42 and stability with respect to $\mathcal{K}$ imply C^* -algebraic stability (i.e. invariance under C^* -tensor product by the C^* -version of $\mathcal{K}$).

The existence of the bivariant Chern character follows from these homological properties by the universal property of Kasparov's KK -theory as in the previous section.

The last assertion is trivial for $A = B = \mathbb{C}$. The class of C^* -algebras for which it holds is closed under KK -equivalence, inductive limits and extensions with completely positive section. Hence it contains all C^* -algebras satisfying the universal coefficient theorem (see [8]).

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The Baum–Connes and the Farrell–Jones Conjectures in K - and L -Theory

Wolfgang Lück and Holger Reich

2.1	<i>Introduction</i>	705
2.2	<i>The Conjectures in the Torsion Free Case</i>	707
	Algebraic K -Theory – Low Dimensions	708
	Applications I	710
	Algebraic K -Theory – All Dimensions	714
	Applications II	717
	L -Theory	719
	Applications III	722
	The Baum–Connes Conjecture in the Torsion Free Case	724
	Applications IV	727
	Applications V	731
2.3	<i>The Conjectures in the General Case</i>	734
	Formulation of the Conjectures	734
	Varying the Family of Subgroups	741
2.4	<i>More Applications</i>	748
	Applications VI	748
	Applications VII	757
	Applications VIII	760
2.5	<i>Generalizations and Related Conjectures</i>	763
	Variants of the Baum–Connes Conjecture	763
	Variants of the Farrell–Jones Conjecture	768

2.6	<i>Status of the Conjectures</i>	773
	Status of the Baum–Connes–Conjecture	773
	Status of the Farrell–Jones Conjecture	779
	List of Groups Satisfying the Conjecture	785
	Inheritance Properties.....	789
2.7	<i>Equivariant Homology Theories</i>	790
	The Definition of an Equivariant Homology Theory	790
	Constructing Equivariant Homology Theories	793
	K - and L -Theory Spectra over Groupoids.....	797
	Assembly Maps in Terms of Homotopy Colimits	799
	Naturality under Induction	800
2.8	<i>Methods of Proof</i>	802
	Analytic Equivariant K -Homology	802
	The Analytic Assembly Map.....	803
	Equivariant KK -Theory	804
	The Dirac–Dual Dirac Method	805
	An Example of a Dirac Element	806
	Banach KK -Theory	806
	Controlled Topology and Algebra	807
	Assembly as Forget Control	808
	Methods to Improve Control	809
	The Descent Principle	811
	Comparing to Other Theories	812
2.9	<i>Computations</i>	812
	K - and L -Groups for Finite Groups	813
	Rational Computations for Infinite Groups.....	816
	Integral Computations for Infinite Groups	819
	Techniques for Computations.....	822
	<i>References</i>	824

Summary. We give a survey of the meaning, status and applications of the Baum–Connes Conjecture about the topological K -theory of the reduced group C^* -algebra and the Farrell–Jones Conjecture about the algebraic K - and L -theory of the group ring of a (discrete) group G .

Keywords: K - and L -groups of group rings and group C^* -algebras, Baum–Connes Conjecture, Farrell–Jones Conjecture.

Mathematics subject classification 2000: 19A31, 19B28, 19D99, 19G24, 19K99, 46L80.

Introduction

2.1

This survey article is devoted to the Baum–Connes Conjecture about the topological K -theory of the reduced group C^* -algebra and the Farrell–Jones Conjecture about the algebraic K - and L -theory of the group ring of a discrete group G . We will present a unified approach to these conjectures hoping that it will stimulate further interactions and exchange of methods and ideas between algebraic and geometric topology on the one side and non-commutative geometry on the other.

Each of the above mentioned conjectures has already been proven for astonishingly large classes of groups using a variety of different methods coming from operator theory, controlled topology and homotopy theory. Methods have been developed for this purpose which turned out to be fruitful in other contexts. The conjectures imply many other well-known and important conjectures. Examples are the Borel Conjecture about the topological rigidity of closed aspherical manifolds, the Novikov Conjecture about the homotopy invariance of higher signatures, the stable Gromov–Lawson–Rosenberg Conjecture about the existence of Riemannian metrics with positive scalar curvature and the Kadison Conjecture about idempotents in the reduced C^* -algebra of a torsionfree discrete group G .

Formulation of the Conjectures

The Baum–Connes and Farrell–Jones Conjectures predict that for every discrete group G the following so called “assembly maps” are isomorphisms.

$$\begin{aligned} K_n^G(E_{\mathcal{FIN}}(G)) &\rightarrow K_n(C_r^*(G)) ; \\ H_n^G(E_{\text{VCY}}(G); \mathbf{K}_R) &\rightarrow K_n(RG) ; \\ H_n^G(E_{\text{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) &\rightarrow L_n^{\langle -\infty \rangle}(RG) . \end{aligned}$$

Here the targets are the groups one would like to understand, namely the topological K -groups of the reduced group C^* -algebra in the Baum–Connes case and the algebraic K - or L -groups of the group ring RG for R an associative ring with unit.

In each case the source is a G -homology theory evaluated on a certain classifying space. In the Baum–Connes Conjecture the G -homology theory is equivariant topological K -theory and the classifying space $E_{\mathcal{F},1,N}(G)$ is the classifying space of the family of finite subgroups, which is often called the classifying space for proper G -actions and denoted $\underline{E}G$ in the literature. In the Farrell–Jones Conjecture the G -homology theory is given by a certain K - or L -theory spectrum over the orbit category, and the classifying space $E_{\mathcal{VCY}}(G)$ is the one associated to the family of virtually cyclic subgroups. The conjectures say that these assembly maps are isomorphisms.

These conjectures were stated in [28, conjecture 3.15 on page 254] and [111, 1.6 on page 257]. Our formulations differ from the original ones, but are equivalent. In the case of the Farrell–Jones Conjecture we slightly generalize the original conjecture by allowing arbitrary coefficient rings instead of $\mathbb{Z}$. At the time of writing no counterexample to the Baum–Connes Conjecture 59 or the Farrell–Jones Conjecture 58 is known to the authors.

One can apply methods from algebraic topology such as spectral sequences and Chern characters to the sources of the assembly maps. In this sense the sources are much more accessible than the targets. The conjectures hence lead to very concrete calculations. Probably even more important is the structural insight: to what extent do the target groups show a homological behaviour. These aspects can be treated completely analogously in the Baum–Connes and the Farrell–Jones setting.

However, the conjectures are not merely computational tools. Their importance comes from the fact that the assembly maps have geometric interpretations in terms of indices in the Baum–Connes case and in terms of surgery theory in the Farrell–Jones case. These interpretations are the key ingredient in applications and the reason that the Baum–Connes and Farrell–Jones Conjectures imply so many other conjectures in non-commutative geometry, geometric topology and algebra.

A User's Guide

We have tried to write the text in a way such that one can read small units independently from the rest. Moreover, a reader who may only be interested in the Baum–Connes Conjecture or only in the Farrell–Jones Conjecture for K -theory or for L -theory can ignore the other parts. But we emphasize again that one basic idea of this paper is to explain the parallel treatment of these conjectures.

A reader without much prior knowledge about the Baum–Connes Conjecture or the Farrell–Jones Conjecture should begin with Chapt. 2.2. There, the special case of a torsionfree group is treated, since the formulation of the conjectures is less technical in this case and there are already many interesting applications. The applications themselves however, are not needed later. A more experienced reader may pass directly to Chapt. 2.3.

Other (survey) articles on the Farrell–Jones Conjecture and the Baum–Connes Conjecture are [111, 128, 147, 225, 307].

Notations and Conventions

Here is a briefing on our main notational conventions. Details are of course discussed in the text. The columns in the following table contain our notation for: the spectra, their associated homology theory, the right hand side of the corresponding assembly maps, the functor from groupoids to spectra and finally the G -homology theory associated to these spectra valued functors.

BU	$K_n(X)$	$K_n(C_r^*G)$	$\mathbf{K}^{\text{top}}$	$H_n^G(X; \mathbf{K}^{\text{top}})$
$\mathbf{K}(R)$	$H_n(X; \mathbf{K}(R))$	$K_n(RG)$	$\mathbf{K}_R$	$H_n^G(X; \mathbf{K}_R)$
$\mathbf{L}^{(j)}(R)$	$H_n(X; \mathbf{L}^{(j)}(R))$	$L_n^{(j)}(RG)$	$\mathbf{L}_R^{(j)}$	$H_n^G(X; \mathbf{L}_R^{(j)})$

We would like to stress that $\mathbf{K}$ without any further decoration will always refer to the non-connective K -theory spectrum. $\mathbf{L}^{(j)}$ will always refer to quadratic L -theory with decoration j . For a C^* - or Banach algebra A the symbol $K_n(A)$ has two possible interpretations but we will mean the topological K -theory.

A ring is always an associative ring with unit, and ring homomorphisms are always unital. Modules are left modules. We will always work in the category of compactly generated spaces, compare [295] and [330, I.4]. For our conventions concerning spectra see Sect. 2.7.2. Spectra are denoted with boldface letters such as $\mathbf{E}$.

Acknowledgements We would like to thank Arthur Bartels, Indira Chatterji, Tom Farrell, Joachim Grunewald, Nigel Higson, Lowell Jones, Michel Matthey, Guido Mislin, Erik Pedersen, David Rosenthal, Marco Schmidt, Alain Valette, Marco Varisco, Julia Weber, Shmuel Weinberger, Bruce Williams, Guoliang Yu and the referee for useful comments and discussions. The first author thanks the Max-Planck Institute for Mathematics in Bonn for its hospitality during his visit in November and December 2002, and both authors thank the Department of Mathematics of the University of Chicago, in particular Shmuel Weinberger, for the pleasant stay in March 2003, when parts of this paper were written. We particularly enjoyed the “Windy City”-Martini respectively the “Earl Grey” on the 96th floor of the Hancock building, where this introduction was drafted.

Münster, August 2003

The Conjectures in the Torsion Free Case

2.2

In this chapter we discuss the Baum–Connes and Farrell–Jones Conjectures in the case of a torsion free group. Their formulation is less technical than in the general case, but already in the torsion free case there are many interesting and illuminating conclusions. In fact some of the most important consequences of the conjectures, like for example the Borel Conjecture (see Conjecture 27) or the Kadison Conjecture (see Conjecture 39), refer exclusively to the torsion free case. On the other hand in the long run the general case, involving groups with

torsion, seems to be unavoidable. The general formulation yields a clearer and more complete picture, and furthermore there are proofs of the conjectures for torsion free groups, where in intermediate steps of the proof it is essential to have the general formulation available (compare Sect. 2.8.9).

The statement of the general case and further applications will be presented in the next chapter. The reader may actually skip this chapter and start immediately with Chapt. 2.3.

We have put some effort into dealing with coefficient rings R other than the integers. A topologist may a priori be interested only in the case $R = \mathbb{Z}$ but other cases are interesting for algebraists and also do occur in computations for integral group rings.

2.2.1 Algebraic K -Theory – Low Dimensions

A ring R is always understood to be associative with unit. We denote by $K_n(R)$ the algebraic K -group of R for $n \in \mathbb{Z}$. In particular $K_0(R)$ is the Grothendieck group of finitely generated projective R -modules and elements in $K_1(R)$ can be represented by automorphisms of such modules. In this section we are mostly interested in the K -groups $K_n(R)$ with $n \leq 1$. For definitions of these groups we refer to [221, 266, 286, 299, 323] for $n = 0, 1$ and to [22] and [268] for $n \leq 1$.

For a ring R and a group G we denote by

$$A_0 = K_0(i): K_0(R) \rightarrow K_0(RG)$$

the map induced by the natural inclusion $i: R \rightarrow RG$. Sending $(g, [P]) \in G \times K_0(R)$ to the class of the RG -automorphism

$$R[G] \otimes_R P \rightarrow R[G] \otimes_R P, \quad u \otimes x \mapsto ug^{-1} \otimes x$$

defines a map $\Phi: G_{\text{ab}} \otimes_{\mathbb{Z}} K_0(R) \rightarrow K_1(RG)$, where G_{ab} denotes the abelianized group. We set

$$A_1 = \Phi \oplus K_1(i): G_{\text{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \rightarrow K_1(RG).$$

We recall the notion of a regular ring. We think of modules as left modules unless stated explicitly differently. Recall that R is *Noetherian* if any submodule of a finitely generated R -module is again finitely generated. It is called *regular* if it is Noetherian and any R -module has a finite-dimensional projective resolution. Any principal ideal domain such as $\mathbb{Z}$ or a field is regular.

The Farrell–Jones Conjecture about algebraic K -theory implies for a torsion free group the following conjecture about the low dimensional K -theory groups.

1

Conjecture 1: (*The Farrell–Jones Conjecture for Low Dimensional K -theory and Torsion Free Groups.*) Let G be a torsion free group and let R be a regular ring. Then

$$K_n(RG) = 0 \quad \text{for } n \leq -1$$

and the maps

$$K_0(R) \xrightarrow{A_0} K_0(RG) \quad \text{and}$$

$$G_{\text{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \xrightarrow{A_1} K_1(RG)$$

are both isomorphisms.

Every regular ring satisfies $K_n(R) = 0$ for $n \leq -1$ [268, 5.3.30 on page 295] and hence the first statement is equivalent to $K_n(i): K_n(R) \rightarrow K_n(RG)$ being an isomorphism for $n \leq -1$. In Remark 15 below we explain why we impose the regularity assumption on the ring R .

For a regular ring R and a group G we define $\text{Wh}_1^R(G)$ as the cokernel of the map A_1 and $\text{Wh}_0^R(G)$ as the cokernel of the map A_0 . In the important case where $R = \mathbb{Z}$ the group $\text{Wh}_1^{\mathbb{Z}}(G)$ coincides with the classical *Whitehead group* $\text{Wh}(G)$ which is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup consisting of the classes of the units $\pm g \in (\mathbb{Z}G)^{\text{inv}}$ for $g \in G$. Moreover for every ring R we define the *reduced algebraic K -groups* $\tilde{K}_n(R)$ as the cokernel of the natural map $K_n(\mathbb{Z}) \rightarrow K_n(R)$. Obviously $\text{Wh}_0^{\mathbb{Z}}(G) = \tilde{K}_0(\mathbb{Z}G)$.

Lemma 2 The map A_0 is always injective. If R is commutative and the natural map $\mathbb{Z} \rightarrow K_0(R)$, $1 \mapsto [R]$ is an isomorphism, then the map A_1 is injective. 2

Proof The augmentation $\varepsilon: RG \rightarrow R$, which maps each group element g to 1, yields a retraction for the inclusion $i: R \rightarrow RG$ and hence induces a retraction for A_0 . If the map $\mathbb{Z} \rightarrow K_0(R)$, $1 \mapsto [R]$ induces an isomorphism and R is commutative, then we have the commutative diagram

$$\begin{array}{ccc} G_{\text{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) & \xrightarrow{A_1} & K_1(RG) \\ \cong \downarrow & & \downarrow \\ & & K_1(RG_{\text{ab}}) \\ & & \downarrow (\det, K_1(\varepsilon)) \\ G_{\text{ab}} \oplus K_1(R) & \longrightarrow & RG_{\text{ab}}^{\text{inv}} \oplus K_1(R), \end{array}$$

where the upper vertical arrow on the right is induced from the map $G \rightarrow G_{\text{ab}}$ to the abelianization. Since RG_{ab} is a commutative ring we have the determinant $\det: K_1(RG_{\text{ab}}) \rightarrow (RG_{\text{ab}})^{\text{inv}}$. The lower horizontal arrow is induced from the obvious inclusion of G_{ab} into the invertible elements of the group ring RG_{ab} and in particular injective.

In the special case $R = \mathbb{Z}$ Conjecture 1 above is equivalent to the following conjecture.

3 **Conjecture 3:** (*Vanishing of Low Dimensional K-theory for Torsionfree Groups and Integral Coefficients.*) For every torsion free group G we have

$$K_n(\mathbb{Z}G) = 0 \quad \text{for } n \leq -1, \quad \tilde{K}_0(\mathbb{Z}G) = 0 \quad \text{and} \quad \text{Wh}(G) = 0.$$

4 **Remark 4:** (*Torsionfree is Necessary.*) In general $\tilde{K}_0(\mathbb{Z}G)$ and $\text{Wh}(G)$ do not vanish for finite groups. For example $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/23]) \cong \mathbb{Z}/3$ [221, pp. 29,30] and $\text{Wh}(\mathbb{Z}/p) \cong \mathbb{Z}^{\frac{p-3}{2}}$ for p an odd prime [70, 11.5 on page 45]. This shows that the assumption that G is torsion free is crucial in the formulation of Conjecture 1 above.

For more information on $\tilde{K}_0(\mathbb{Z}G)$ and Whitehead groups of finite groups see for instance [22, chapt. XI], [79, 220, 231, 232] and [299].

2.2.2 Applications I

We will now explain the geometric relevance of the groups whose vanishing is predicted by Conjecture 3.

The s-Cobordism Theorem and the Poincaré Conjecture

The Whitehead group $\text{Wh}(G)$ plays a key role if one studies manifolds because of the so called s-Cobordism Theorem. In order to state it, we explain the notion of an h-cobordism.

Manifold always means smooth manifold unless it is explicitly stated differently. We say that W or more precisely $(W; M^-, f^-, M^+, f^+)$ is an n -dimensional *cobordism* over M^- if W is a compact n -dimensional manifold together with the following: a disjoint decomposition of its boundary ∂W into two closed $(n-1)$ -dimensional manifolds $\partial^- W$ and $\partial^+ W$, two closed $(n-1)$ -dimensional manifolds M^- and M^+ and diffeomorphisms $f^-: M^- \rightarrow \partial^- W$ and $f^+: M^+ \rightarrow \partial^+ W$. The cobordism is called an *h-cobordism* if the inclusions $i^-: \partial^- W \rightarrow W$ and $i^+: \partial^+ W \rightarrow W$ are both homotopy equivalences. Two cobordisms $(W; M^-, f^-, M^+, f^+)$ and $(W'; M^-, f'^-, M'^+, f'^+)$ over M^- are *diffeomorphic relative M^-* if there is a diffeomorphism $F: W \rightarrow W'$ with $F \circ f^- = f'^-$. We call a cobordism over M^- *trivial*, if it is diffeomorphic relative M^- to the trivial h-cobordism given by the cylinder $M^- \times [0, 1]$ together with the obvious inclusions of $M^- \times \{0\}$ and $M^- \times \{1\}$. Note that “trivial” implies in particular that M^- and M^+ are diffeomorphic.

The question whether a given h-cobordism is trivial is decided by the Whitehead torsion $\tau(W; M^-) \in \text{Wh}(G)$ where $G = \pi_1(M^-)$. For the details of the definition of $\tau(W; M^-)$ the reader should consult [70, 220] or chapt. 2 in [200]. Compare also [266].

Theorem 5: (*s-Cobordism Theorem.*) Let M^- be a closed connected oriented manifold of dimension $n \geq 5$ with fundamental group $G = \pi_1(M^-)$. Then

5

- (i) An h -cobordism W over M^- is trivial if and only if its Whitehead torsion $\tau(W, M^-) \in \text{Wh}(G)$ vanishes.
- (ii) Assigning to an h -cobordism over M^- its Whitehead torsion yields a bijection from the diffeomorphism classes relative M^- of h -cobordisms over M^- to the Whitehead group $\text{Wh}(G)$.

The s -Cobordism Theorem is due to Barden, Mazur and Stallings. There are also topological and PL-versions. Proofs can be found for instance in [173], [176, Essay III], [200] and [272, pp. 87–90].

The s -Cobordism Theorem tells us that the vanishing of the Whitehead group (as predicted in Conjecture 3 for torsion free groups) has the following geometric interpretation.

Consequence 6 For a finitely presented group G the vanishing of the Whitehead group $\text{Wh}(G)$ is equivalent to the statement that each h -cobordism over a closed connected manifold M^- of dimension $\dim(M^-) \geq 5$ with fundamental group $\pi_1(M^-) \cong G$ is trivial.

6

Knowing that all h -cobordisms over a given manifold are trivial is a strong and useful statement. In order to illustrate this we would like to discuss the case where the fundamental group is trivial.

Since the ring $\mathbb{Z}$ has a Gaussian algorithm, the determinant induces an isomorphism $K_1(\mathbb{Z}) \xrightarrow{\cong} \{\pm 1\}$ (compare [268, theorem 2.3.2]) and the Whitehead group $\text{Wh}(\{1\})$ of the trivial group vanishes. Hence any h -cobordism over a simply connected closed manifold of dimension ≥ 5 is trivial. As a consequence one obtains the Poincaré Conjecture for high dimensional manifolds.

Theorem 7: (*Poincaré Conjecture.*) Suppose $n \geq 5$. If the closed manifold M is homotopy equivalent to the sphere S^n , then it is homeomorphic to S^n .

7

Proof We only give the proof for $\dim(M) \geq 6$. Let $f: M \rightarrow S^n$ be a homotopy equivalence. Let $D_-^n \subset M$ and $D_+^n \subset M$ be two disjoint embedded disks. Let W be the complement of the interior of the two disks in M . Then W turns out to be a simply connected h -cobordism over ∂D_-^n . Hence we can find a diffeomorphism

$$F: (\partial D_-^n \times [0, 1]; \partial D_-^n \times \{0\}, \partial D_-^n \times \{1\}) \rightarrow (W; \partial D_-^n, \partial D_+^n),$$

which is the identity on $\partial D_-^n = \partial D_-^n \times \{0\}$ and induces some (unknown) diffeomorphism $f^+: \partial D_-^n \times \{1\} \rightarrow \partial D_+^n$. By the *Alexander trick* one can extend

$f^+ : \partial D_-^n = \partial D_-^n \times \{1\} \rightarrow \partial D_+^n$ to a homeomorphism $\overline{f^+} : D_-^n \rightarrow D_+^n$. Namely, any homeomorphism $f : S^{n-1} \rightarrow S^{n-1}$ extends to a homeomorphism $\tilde{f} : D^n \rightarrow D^n$ by sending $t \cdot x$ for $t \in [0, 1]$ and $x \in S^{n-1}$ to $t \cdot f(x)$. Now define a homeomorphism $h : D_-^n \times \{0\} \cup_{i_-} \partial D_-^n \times [0, 1] \cup_{i_+} D_+^n \times \{1\} \rightarrow M$ for the canonical inclusions $i_k : \partial D_-^n \times \{k\} \rightarrow \partial D_-^n \times [0, 1]$ for $k = 0, 1$ by $h|_{D_-^n \times \{0\}} = \text{id}$, $h|_{\partial D_-^n \times [0, 1]} = F$ and $h|_{D_+^n \times \{1\}} = \overline{f^+}$. Since the source of h is obviously homeomorphic to S^n , Theorem 7 follows.

The Poincaré Conjecture (see Theorem 7) is at the time of writing known in all dimensions except dimension 3. It is essential in its formulation that one concludes M to be homeomorphic (as opposed to diffeomorphic) to S^n . The Alexander trick does not work differentiably. There are *exotic spheres*, i.e. smooth manifolds which are homeomorphic but not diffeomorphic to S^n [218].

More information about the Poincaré Conjecture, the Whitehead torsion and the s-Cobordism Theorem can be found for instance in [50, 70, 86, 131, 132, 141, 173, 200, 219, 220, 266] and [272].

Finiteness Obstructions

We now discuss the geometric relevance of $\tilde{K}_0(\mathbb{Z}G)$.

Let X be a CW-complex. It is called *finite* if it consists of finitely many cells. It is called *finitely dominated* if there is a finite CW-complex Y together with maps $i : X \rightarrow Y$ and $r : Y \rightarrow X$ such that $r \circ i$ is homotopic to the identity on X . The fundamental group of a finitely dominated CW-complex is always finitely presented.

While studying existence problems for spaces with prescribed properties (like for example group actions), it happens occasionally that it is relatively easy to construct a finitely dominated CW-complex within a given homotopy type, whereas it is not at all clear whether one can also find a homotopy equivalent *finite* CW-complex. Wall's *finiteness obstruction*, a certain obstruction element $\tilde{o}(X) \in \tilde{K}_0(\mathbb{Z}\pi_1(X))$, decides the question.

8

Theorem 8: (*Properties of the Finiteness Obstruction.*) Let X be a finitely dominated CW-complex with fundamental group $\pi = \pi_1(X)$.

- (i) The space X is homotopy equivalent to a finite CW-complex if and only if $\tilde{o}(X) = 0 \in \tilde{K}_0(\mathbb{Z}\pi)$.
- (ii) Every element in $\tilde{K}_0(\mathbb{Z}G)$ can be realized as the finiteness obstruction $\tilde{o}(X)$ of a finitely dominated CW-complex X with $G = \pi_1(X)$, provided that G is finitely presented.
- (iii) Let Z be a space such that $G = \pi_1(Z)$ is finitely presented. Then there is a bijection between $\tilde{K}_0(\mathbb{Z}G)$ and the set of equivalence classes of maps $f : X \rightarrow Z$ with X finitely dominated under the equivalence relation explained below.

The equivalence relation in (iii) is defined as follows: Two maps $f: X \rightarrow Z$ and $f': X' \rightarrow Z$ with X and X' finitely dominated are equivalent if there exists a commutative diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{j} & X_1 & \xrightarrow{h} & X_2 & \xleftarrow{h'} & X_3 & \xleftarrow{j'} & X' \\
 & \searrow f & \searrow f_1 & & \downarrow f_2 & \swarrow f_3 & & \swarrow f' & \\
 & & & & Z & & & &
 \end{array}$$

where h and h' are homotopy equivalences and j and j' are inclusions of subcomplexes for which X_1 , respectively X_3 , is obtained from X , respectively X' , by attaching a finite number of cells.

The vanishing of $\tilde{K}_0(\mathbb{Z}G)$ as predicted in Conjecture 3 for torsion free groups hence has the following interpretation.

Consequence 9 For a finitely presented group G the vanishing of $\tilde{K}_0(\mathbb{Z}G)$ is equivalent to the statement that any finitely dominated CW -complex X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW -complex.

9

For more information about the finiteness obstruction we refer for instance to [125, 126, 196, 224, 257, 266, 308, 317] and [318].

Negative K -Groups and Bounded h -Cobordisms

One possible geometric interpretation of negative K -groups is in terms of bounded h -cobordisms. Another interpretation will be explained in Sect. 2.2.4 below.

We consider *manifolds W parametrized over $\mathbb{R}^k$* , i.e. manifolds which are equipped with a surjective proper map $p: W \rightarrow \mathbb{R}^k$. We will always assume that the fundamental group(oid) is bounded, compare [239, definition 1.3]. A map $f: W \rightarrow W'$ between two manifolds parametrized over $\mathbb{R}^k$ is *bounded* if $\{p' \circ f(x) - p(x) \mid x \in W\}$ is a bounded subset of $\mathbb{R}^k$.

A *bounded cobordism* $(W; M^-, f^-, M^+, f^+)$ is defined just as in Sect. 2.2.2 but compact manifolds are replaced by manifolds parametrized over $\mathbb{R}^k$ and the parametrization for $M^\pm$ is given by $p_W \circ f^\pm$. If we assume that the inclusions $i^\pm: \partial^\pm W \rightarrow W$ are homotopy equivalences, then there exist deformations $r^\pm: W \times I \rightarrow W$, $(x, t) \mapsto r_t^\pm(x)$ such that $r_0^\pm = \text{id}_W$ and $r_1^\pm(W) \subset \partial^\pm W$.

A bounded cobordism is called a *bounded h -cobordism* if the inclusions $i^\pm$ are homotopy equivalences and additionally the deformations can be chosen such that the two sets

$$S^\pm = \{p_W \circ r_t^\pm(x) - p_W \circ r_1^\pm(x) \mid x \in W, t \in [0, 1]\}$$

are bounded subsets of $\mathbb{R}^k$.

The following theorem (compare [239] and [327, appendix]) contains the s-Cobordism Theorem 5 as a special case, gives another interpretation of elements in $\tilde{K}_0(\mathbb{Z}\pi)$ and explains one aspect of the geometric relevance of negative K -groups.

10

Theorem 10: (*Bounded h-Cobordism Theorem.*) Suppose that M^- is parametrized over $\mathbb{R}^k$ and satisfies $\dim M^- \geq 5$. Let π be its fundamental group(oid). Equivalence classes of bounded h-cobordisms over M^- modulo bounded diffeomorphism relative M^- correspond bijectively to elements in $\kappa_{1-k}(\pi)$, where

$$\kappa_{1-k}(\pi) = \begin{cases} \text{Wh}(\pi) & \text{if } k = 0, \\ \tilde{K}_0(\mathbb{Z}\pi) & \text{if } k = 1, \\ K_{1-k}(\mathbb{Z}\pi) & \text{if } k \geq 2. \end{cases}$$

More information about negative K -groups can be found for instance in [8, 22, 57, 58, 113, 213, 238, 239, 252, 259, 268] and [327, appendix].

2.2.3 Algebraic K -Theory – All Dimensions

So far we only considered the K -theory groups in dimensions ≤ 1 . We now want to explain how Conjecture 1 generalizes to higher algebraic K -theory. For the definition of higher algebraic K -theory groups and the (connective) K -theory spectrum see [35, 52, 158, 249, 268, 292, 315] and [323]. We would like to stress that for us $\mathbf{K}(R)$ will always denote the *non-connective algebraic K -theory spectrum* for which $K_n(R) = \pi_n(\mathbf{K}(R))$ holds for all $n \in \mathbb{Z}$. For its definition see [52, 194] and [237].

The Farrell–Jones Conjecture for algebraic K -theory reduces for a torsion free group to the following conjecture.

11

Conjecture 11: (*Farrell–Jones Conjecture for Torsion Free Groups and K -theory.*) Let G be a torsion free group. Let R be a regular ring. Then the assembly map

$$H_n(BG; \mathbf{K}(R)) \rightarrow K_n(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

Here $H_n(-; \mathbf{K}(R))$ denotes the homology theory which is associated to the spectrum $\mathbf{K}(R)$. It has the property that $H_n(\text{pt}; \mathbf{K}(R))$ is $K_n(R)$ for $n \in \mathbb{Z}$, where here and elsewhere pt denotes the space consisting of one point. The space BG is the *classifying space of the group G* , which up to homotopy is characterized by the property that it is a CW -complex with $\pi_1(BG) \cong G$ whose universal covering is contractible. The technical details of the construction of $H_n(-; \mathbf{K}(R))$ and the assembly map will be explained in a more general setting in Sect. 2.3.1.

The point of Conjecture 11 is that on the right-hand side of the assembly map we have the group $K_n(RG)$ we are interested in, whereas the left-hand side is

a homology theory and hence much easier to compute. For every homology theory associated to a spectrum we have the Atiyah–Hirzebruch spectral sequence, which in our case has $E_{p,q}^2 = H_p(BG; K_q(R))$ and converges to $H_{p+q}(BG; \mathbf{K}(R))$.

If R is regular, then the negative K -groups of R vanish and the spectral sequence lives in the first quadrant. Evaluating the spectral sequence for $n = p + q \leq 1$ shows that Conjecture 11 above implies Conjecture 1.

Remark 12: (*Rational Computation.*) Rationally an Atiyah–Hirzebruch spectral sequence collapses always and the homological Chern character gives an isomorphism

$$\text{ch}: \bigoplus_{p+q=n} H_p(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (K_q(R) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\cong} H_n(BG; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The Atiyah–Hirzebruch spectral sequence and the Chern character will be discussed in a much more general setting in Chapt. 2.9.

Remark 13: (*Separation of Variables.*) We see that the left-hand side of the isomorphism in the previous remark consists of a group homology part and a part which is the rationalized K -theory of R . (Something similar happens before we rationalize at the level of spectra: The left hand side of Conjecture 11 can be interpreted as the homotopy groups of the spectrum $BG_+ \wedge \mathbf{K}(R)$.) So essentially Conjecture 11 predicts that the K -theory of RG is built up out of two independent parts: the K -theory of R and the group homology of G . We call this principle *separation of variables*. This principle also applies to other theories such as algebraic L -theory or topological K -theory. See also Remark 179.

Remark 14: (*K -theory of the Coefficients.*) Note that Conjecture 11 can only help us to explicitly compute the K -groups of RG in cases where we know enough about the K -groups of R . We obtain no new information about the K -theory of R itself. However, already for very simple rings the computation of their algebraic K -theory groups is an extremely hard problem.

It is known that the groups $K_n(\mathbb{Z})$ are finitely generated abelian groups [248]. Due to Borel [39] we know that

$$K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } n = 0 ; \\ \mathbb{Q} & \text{if } n = 4k + 1 \text{ with } k \geq 1 ; \\ 0 & \text{otherwise .} \end{cases}$$

Since $\mathbb{Z}$ is regular we know that $K_n(\mathbb{Z})$ vanishes for $n \leq -1$. Moreover, $K_0(\mathbb{Z}) \cong \mathbb{Z}$ and $K_1(\mathbb{Z}) \cong \{\pm 1\}$, where the isomorphisms are given by the rank and the determinant. One also knows that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$, $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ [189] and $K_4(\mathbb{Z}) \cong$

12

13

14

0 [264]. Finite fields belong to the few rings where one has a complete and explicit knowledge of all K -theory groups [247]. We refer the reader for example to [177, 226, 265, 322] and Soulé's article in [193] for more information about the algebraic K -theory of the integers or more generally of rings of integers in number fields.

Because of Borel's calculation the left hand side of the isomorphism described in Remark 12 specializes for $R = \mathbb{Z}$ to

$$H_n(BG; \mathbb{Q}) \oplus \bigoplus_{k=1}^{\infty} H_{n-(4k+1)}(BG; \mathbb{Q}) \quad (2.1)$$

and Conjecture 11 predicts that this group is isomorphic to $K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Next we discuss the case where the group G is infinite cyclic.

15 Remark 15: (*Bass–Heller–Swan Decomposition.*) The so called *Bass–Heller–Swan-decomposition*, also known as the *Fundamental Theorem of algebraic K -theory*, computes the algebraic K -groups of $R[\mathbb{Z}]$ in terms of the algebraic K -groups and Nil-groups of R :

$$K_n(R[\mathbb{Z}]) \cong K_{n-1}(R) \oplus K_n(R) \oplus NK_n(R) \oplus NK_n(R) .$$

Here the group $NK_n(R)$ is defined as the cokernel of the split injection $K_n(R) \rightarrow K_n(R[t])$. It can be identified with the cokernel of the split injection $K_{n-1}(R) \rightarrow K_{n-1}(\text{Nil}(R))$. Here $K_n(\text{Nil}(R))$ denotes the K -theory of the exact category of nilpotent endomorphisms of finitely generated projective R -modules. For negative n it is defined with the help of Bass' notion of a contracting functor [22] (see also [57]). The groups are known as *Nil-groups* and often denoted $\text{Nil}_{n-1}(R)$.

For proofs of these facts and more information the reader should consult [22, chapt. XII], [25], [135, theorem on page 236], [249, corollary in §6 on page 38], [268, theorems 3.3.3 and 5.3.30], [292, theorem 9.8] and [300, theorem 10.1].

If we iterate and use $R[\mathbb{Z}^n] = R[\mathbb{Z}^{n-1}][\mathbb{Z}]$ we see that a computation of $K_n(RG)$ must in general take into account information about $K_i(R)$ for all $i \leq n$. In particular we see that it is important to formulate Conjecture 11 with the non-connective K -theory spectrum.

Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphism

$$H_n(B\mathbb{Z}; \mathbf{K}(R)) \cong K_{n-1}(R) \oplus K_n(R)$$

and hence Conjecture 11 predicts

$$K_n(R[\mathbb{Z}]) \cong K_{n-1}(R) \oplus K_n(R) .$$

This explains why in the formulation of Conjecture 11 the condition that R is regular appears. It guarantees that $NK_n(R) = 0$ [268, theorem 5.3.30 on page 295]. There are weaker conditions which imply that $NK_n(R) = 0$ but “regular” has the

advantage that R regular implies that $R[t]$ and $R[\mathbb{Z}] = R[t^{\pm 1}]$ are again regular, compare the discussion in sect. 2 in [23].

The Nil-terms $NK_n(R)$ seem to be hard to compute. For instance $NK_1(R)$ either vanishes or is infinitely generated as an abelian group [95]. In Sect. 2.5.2 we will discuss the Isomorphism Conjecture for NK -groups. For more information about Nil-groups see for instance [73, 74, 146, 324] and [325].

Applications II

2.2.4

The Relation to Pseudo-Isotopy Theory

Let I denote the unit interval $[0, 1]$. A topological *pseudoisotopy* of a compact manifold M is a homeomorphism $h: M \times I \rightarrow M \times I$, which restricted to $M \times \{0\} \cup \partial M \times I$ is the obvious inclusion. The space $P(M)$ of pseudoisotopies is the (simplicial) group of all such homeomorphisms. Pseudoisotopies play an important role if one tries to understand the homotopy type of the space $\text{Top}(M)$ of self-homeomorphisms of a manifold M . We will see below in Sect. 2.2.6 how the results about pseudoisotopies discussed in this section combined with surgery theory lead to quite explicit results about the homotopy groups of $\text{Top}(M)$.

There is a stabilization map $P(M) \rightarrow P(M \times I)$ given by crossing a pseudoisotopy with the identity on the interval I and the stable pseudoisotopy space is defined as $\mathcal{P}(M) = \text{colim}_k P(M \times I^k)$. In fact $\mathcal{P}(-)$ can be extended to a functor on all spaces [144]. The natural inclusion $P(M) \rightarrow \mathcal{P}(M)$ induces an isomorphism on the i -th homotopy group if the dimension of M is large compared to i , see [43] and [157].

Waldhausen [314, 315] defines the algebraic K -theory of spaces functor $\mathbf{A}(X)$ and the functor $\mathbf{Wh}^{PL}(X)$ from spaces to spectra (or infinite loop spaces) and a fibration sequence

$$X_+ \wedge \mathbf{A}(\text{pt}) \rightarrow \mathbf{A}(X) \rightarrow \mathbf{Wh}^{PL}(X).$$

Here $X_+ \wedge \mathbf{A}(\text{pt}) \rightarrow \mathbf{A}(X)$ is an assembly map, which can be compared to the algebraic K -theory assembly map that appears in Conjecture 11 via a commutative diagram

$$\begin{array}{ccc} H_n(X; \mathbf{A}(\text{pt})) & \longrightarrow & \pi_n(\mathbf{A}(X)) \\ \downarrow & & \downarrow \\ H_n(B\pi_1(X); \mathbf{K}(\mathbb{Z})) & \longrightarrow & K_n(\mathbb{Z}\pi_1(X)). \end{array}$$

In the case where $X \simeq BG$ is aspherical the vertical maps induce isomorphisms after rationalization for $n \geq 1$, compare [314, proposition 2.2]. Since $\Omega^2 \mathbf{Wh}^{PL}(X) \simeq \mathcal{P}(X)$ (a guided tour through the literature concerning this and related results can

be found in [91, sect. 9]), Conjecture 11 implies rational vanishing results for the groups $\pi_n(\mathcal{P}(M))$ if M is an aspherical manifold. Compare also Remark 111.

16 **Consequence 16** Suppose M is a closed aspherical manifold and Conjecture 11 holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$, then for all $n \geq 0$

$$\pi_n(\mathcal{P}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

Similarly as above one defines smooth pseudoisotopies and the space of stable smooth pseudoisotopies $\mathcal{P}^{\text{Diff}}(M)$. There is also a smooth version of the Whitehead space $\text{Wh}^{\text{Diff}}(X)$ and $\Omega^2 \text{Wh}^{\text{Diff}}(M) \simeq \mathcal{P}^{\text{Diff}}(M)$. Again there is a close relation to A -theory via the natural splitting $A(X) \simeq \Sigma^\infty(X_+) \vee \text{Wh}^{\text{Diff}}(X)$, see [316]. Here $\Sigma^\infty(X_+)$ denotes the suspension spectrum associated to X_+ . Using this one can split off an assembly map $H_n(X; \text{Wh}^{\text{Diff}}(\text{pt})) \rightarrow \pi_n(\text{Wh}^{\text{Diff}}(X))$ from the A -theory assembly map. Since for every space $\pi_n(\Sigma^\infty(X_+)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(X; \mathbb{Q})$ Conjecture 11 combined with the rational computation in (2.1) yields the following result.

17 **Consequence 17** Suppose M is a closed aspherical manifold and Conjecture 11 holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$. Then for $n \geq 0$ we have

$$\pi_n(\mathcal{P}^{\text{Diff}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{k=1}^{\infty} H_{n-4k+1}(M; \mathbb{Q}).$$

Observe that the fundamental difference between the smooth and the topological case occurs already when G is the trivial group.

Negative K -Groups and Bounded Pseudo-Isotopies

We briefly explain a further geometric interpretation of negative K -groups, which parallels the discussion of bounded h-cobordisms in Sect. 2.2.2.

Let $p: M \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ denote the natural projection. The space $P_b(M; \mathbb{R}^k)$ of bounded pseudoisotopies is the space of all self-homeomorphisms $h: M \times \mathbb{R}^k \times I \rightarrow M \times \mathbb{R}^k \times I$ such that restricted to $M \times \mathbb{R}^k \times \{0\}$ the map h is the inclusion and such that h is bounded, i.e. the set $\{p \circ h(y) - p(y) \mid y \in M \times \mathbb{R}^k \times I\}$ is a bounded subset of $\mathbb{R}^k$. There is again a stabilization map $P_b(M; \mathbb{R}^k) \rightarrow P_b(M \times I; \mathbb{R}^k)$ and a stable bounded pseudoisotopy space $\mathcal{P}_b(M; \mathbb{R}^k) = \text{colim}_j P_b(M \times I^j; \mathbb{R}^k)$. There is a homotopy equivalence $\mathcal{P}_b(M; \mathbb{R}^k) \rightarrow \Omega \mathcal{P}_b(M; \mathbb{R}^{k+1})$ [144, appendix II] and hence the sequence of spaces $\mathcal{P}_b(M; \mathbb{R}^k)$ for $k = 0, 1, \dots$ is an Ω -spectrum $\mathbf{P}(M)$. Analogously one defines the differentiable bounded pseudoisotopies $\mathcal{P}_b^{\text{diff}}(M; \mathbb{R}^k)$ and an Ω -spectrum $\mathbf{P}^{\text{diff}}(M)$. The negative homotopy groups of these spectra have an interpretation in terms of low and negative dimensional K -groups. In terms of unstable homotopy groups this is explained in the following theorem which is closely related to Theorem 10 about bounded h-cobordisms.

Theorem 18: (*Negative Homotopy Groups of Pseudoisotopies.*) Let $G = \pi_1(M)$. Suppose n and k are such that $n + k \geq 0$, then for $k \geq 1$ there are isomorphisms

18

$$\pi_{n+k}(\mathcal{P}_b(M; \mathbb{R}^k)) = \begin{cases} \text{Wh}(G) & \text{if } n = -1, \\ \tilde{K}_0(\mathbb{Z}G) & \text{if } n = -2, \\ K_{n+2}(\mathbb{Z}G) & \text{if } n < -2. \end{cases}$$

The same result holds in the differentiable case.

Note that Conjecture 11 predicts that these groups vanish if G is torsionfree. The result above is due to Anderson and Hsiang [8] and is also discussed in [327, appendix].

L -Theory

2.2.5

We now move on to the L -theoretic version of the Farrell–Jones Conjecture. We will still stick to the case where the group is torsion free. The conjecture is obtained by replacing K -theory and the K -theory spectrum in Conjecture 11 by 4-periodic L -theory and the L -theory spectrum $\mathbf{L}^{(-\infty)}(R)$. Explanations will follow below.

Conjecture 19: (*Farrell–Jones Conjecture for Torsion Free Groups and L -theory.*)

19

Let G be a torsion free group and let R be a ring with involution. Then the assembly map

$$H_n(BG; \mathbf{L}^{(-\infty)}(R)) \rightarrow L_n^{(-\infty)}(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

To a ring with involution one can associate (decorated) symmetric or quadratic algebraic L -groups, compare [44, 45, 256, 259] and [332]. We will exclusively deal with the quadratic algebraic L -groups and denote them by $L_n^{(j)}(R)$. Here $n \in \mathbb{Z}$ and $j \in \{-\infty\} \sqcup \{j \in \mathbb{Z} \mid j \leq 2\}$ is the so called *decoration*. The decorations $j = 0, 1$ correspond to the decorations p, h and $j = 2$ is related to the decoration s appearing in the literature. Decorations will be discussed in Remark 21 below. The L -groups $L_n^{(j)}(R)$ are 4-periodic, i.e. $L_n^{(j)}(R) \cong L_{n+4}^{(j)}(R)$ for $n \in \mathbb{Z}$.

If we are given an involution $r \mapsto \bar{r}$ on a ring R , we will always equip RG with the involution that extends the given one and satisfies $\bar{g} = g^{-1}$. On $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ we always use the trivial involution and on $\mathbb{C}$ the complex conjugation.

One can construct an L -theory spectrum $\mathbf{L}^{(j)}(R)$ such that $\pi_n(\mathbf{L}^{(j)}(R)) = L_n^{(j)}(R)$, compare [258, § 13]. Above and in the sequel $H_n(-; \mathbf{L}^{(j)}(R))$ denotes the homology theory which is associated to this spectrum. In particular we have $H_n(\text{pt}; \mathbf{L}^{(j)}(R)) = L_n^{(j)}(R)$. We postpone the discussion of the assembly map to Sect. 2.3.1 where we will construct it in greater generality.

20

Remark 20: (*The Coefficients in the L-theory Case.*) In contrast to K -theory (compare Remark 14) the L -theory of the most interesting coefficient ring $R = \mathbb{Z}$ is well known. The groups $L_n^{(j)}(\mathbb{Z})$ for fixed n and varying $j \in \{-\infty\} \sqcup \{j \in \mathbb{Z} \mid j \leq 2\}$ are all naturally isomorphic (compare Proposition 23 below) and we have $L_0^{(j)}(\mathbb{Z}) \cong \mathbb{Z}$ and $L_2^{(j)}(\mathbb{Z}) \cong \mathbb{Z}/2$, where the isomorphisms are given by the signature divided by 8 and the Arf invariant, and $L_1^{(j)}(\mathbb{Z}) = L_3^{(j)}(\mathbb{Z}) = 0$, see [41, chapt. III], [256, proposition 4.3.1 on page 419].

21

Remark 21: (*Decorations.*) L -groups are designed as obstruction groups for surgery problems. The decoration reflects what kind of surgery problem one is interested in. All L -groups can be described as cobordism classes of suitable quadratic Poincaré chain complexes. If one works with chain complexes of finitely generated free based R -modules and requires that the torsion of the Poincaré chain homotopy equivalence vanishes in $\tilde{K}_1(R)$, then the corresponding L -groups are denoted $L_n^{(2)}(R)$. If one drops the torsion condition, one obtains $L_n^{(1)}(R)$, which is usually denoted $L^h(R)$. If one works with finitely generated projective modules, one obtains $L^{(0)}(R)$, which is also known as $L^p(R)$.

The L -groups with negative decorations can be defined inductively via the Shaneson splitting, compare Remark 26 below. Assuming that the L -groups with decorations j have already been defined one sets

$$L_{n-1}^{<j-1>}(R) = \operatorname{coker}(L_n^{<j>}(R) \rightarrow L_n^{<j>}(R[\mathbb{Z}])) .$$

Compare [259, definition 17.1 on page 145]. Alternatively these groups can be obtained via a process which is in the spirit of Sects. 2.2.2 and 2.2.4. One can define them as L -theory groups of suitable categories of modules parametrized over $\mathbb{R}^k$. For details the reader could consult [55, sect. 4]. There are forgetful maps $L_n^{(j+1)}(R) \rightarrow L_n^{(j)}(R)$. The group $L_n^{(-\infty)}(R)$ is defined as the colimit over these maps. For more information see [254, 259].

For group rings we also define $L_n^s(RG)$ similar to $L_n^{(2)}(RG)$ but we require the torsion to lie in $\operatorname{im} A_1 \subset \tilde{K}_1(RG)$, where A_1 is the map defined in Sect. 2.2.1. Observe that $L_n^s(RG)$ really depends on the pair (R, G) and differs in general from $L_n^{(2)}(RG)$.

22

Remark 22: (*The Interplay of K - and L -theory.*) For $j \leq 1$ there are forgetful maps $L_n^{(j+1)}(R) \rightarrow L_n^{(j)}(R)$ which sit inside the following sequence, which is known as the *Rothenberg sequence* [256, proposition 1.10.1 on page 104], [259, 17.2].

$$\begin{aligned} \dots \rightarrow L_n^{(j+1)}(R) \rightarrow L_n^{(j)}(R) \rightarrow \widehat{H}^n(\mathbb{Z}/2; \tilde{K}_j(R)) \\ \rightarrow L_{n-1}^{(j+1)}(R) \rightarrow L_{n-1}^{(j)}(R) \rightarrow \dots \end{aligned} \quad (2.2)$$

Here $\widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(R))$ is the Tate-cohomology of the group $\mathbb{Z}/2$ with coefficients in the $\mathbb{Z}[\mathbb{Z}/2]$ -module $\widetilde{K}_j(R)$. The involution on $\widetilde{K}_j(R)$ comes from the involution on R . There is a similar sequence relating $L_n^s(RG)$ and $L_n^h(RG)$, where the third term is the $\mathbb{Z}/2$ -Tate-cohomology of $\text{Wh}_1^R(G)$. Note that Tate-cohomology groups of the group $\mathbb{Z}/2$ are always annihilated by multiplication with 2. In particular $L_n^{(j)}(R) \left[\frac{1}{2} \right] = L_n^{(j)}(R) \otimes_{\mathbb{Z}} \mathbb{Z} \left[\frac{1}{2} \right]$ is always independent of j .

Let us formulate explicitly what we obtain from the above sequences for $R = \mathbb{Z}G$.

Proposition 23 Let G be a torsion free group, then Conjecture 3 about the vanishing of $\text{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and $K_{-i}(\mathbb{Z}G)$ for $i \geq 1$ implies that for fixed n and varying $j \in \{-\infty\} \sqcup \{j \in \mathbb{Z} \mid j \leq 1\}$ the L -groups $L_n^{(j)}(\mathbb{Z}G)$ are all naturally isomorphic and moreover $L_n^{(1)}(\mathbb{Z}G) = L_n^h(\mathbb{Z}G) \cong L_n^s(\mathbb{Z}G)$.

23

Remark 24: (*Rational Computation.*) As in the K -theory case we have an Atiyah–Hirzebruch spectral sequence:

24

$$E_{p,q}^2 = H_p(BG; L_q^{(-\infty)}(R)) \Rightarrow H_{p+q}(BG; L^{(-\infty)}(R)) .$$

Rationally this spectral sequence collapses and the homological Chern character gives for $n \in \mathbb{Z}$ an isomorphism

$$\text{ch}: \bigoplus_{p+q=n} H_p(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (L_q^{(-\infty)}(R) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\cong} H_n(BG; L^{(-\infty)}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} . \quad (2.3)$$

In particular we obtain in the case $R = \mathbb{Z}$ from Remark 20 for all $n \in \mathbb{Z}$ and all decorations j an isomorphism

$$\text{ch}: \bigoplus_{k=0}^{\infty} H_{n-4k}(BG; \mathbb{Q}) \xrightarrow{\cong} H_n(BG; L^{(j)}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} . \quad (2.4)$$

This spectral sequence and the Chern character above will be discussed in a much more general setting in Chapt. 2.9.

Remark 25: (*Torsion Free is Necessary.*) If G is finite, $R = \mathbb{Z}$ and $n = 0$, then the rationalized left hand side of the assembly equals $\mathbb{Q}$, whereas the right hand side is isomorphic to the rationalization of the real representation ring. Since the group homology of a finite group vanishes rationally except in dimension 0, the previous remark shows that we need to assume the group to be torsion free in Conjecture 19.

25

26

Remark 26: (*Shaneson splitting.*) The Bass–Heller–Swan decomposition in K -theory (see Remark 15) has the following analogue for the algebraic L -groups, which is known as the *Shaneson splitting* [284]

$$L_n^{(j)}(R[\mathbb{Z}]) \cong L_{n-1}^{(j-1)}(R) \oplus L_n^{(j)}(R) . \quad (2.5)$$

Here for the decoration $j = -\infty$ one has to interpret $j-1$ as $-\infty$. Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphism

$$H_n(B\mathbb{Z}; L^{(j)}(R)) \cong L_{n-1}^{(j)}(R) \oplus L_n^{(j)}(R) .$$

This explains why in the formulation of the L -theoretic Farrell–Jones Conjecture for torsion free groups (see Conjecture 19) we use the decoration $j = -\infty$.

As long as one deals with torsion free groups and one believes in the low dimensional part of the K -theoretic Farrell–Jones Conjecture (predicting the vanishing of $\text{Wh}(G)$, $\tilde{K}_0(\mathbb{Z}G)$ and of the negative K -groups, see Conjecture 3) there is no difference between the various decorations j , compare Proposition 23. But as soon as one allows torsion in G , the decorations make a difference and it indeed turns out that if one replaces the decoration $j = -\infty$ by $j = s, h$ or p there are counterexamples for the L -theoretic version of Conjecture 58 even for $R = \mathbb{Z}$ [123].

Even though in the above Shaneson splitting (2.5) there are no terms analogous to the Nil-terms in Remark 15 such Nil-phenomena do also occur in L -theory, as soon as one considers amalgamated free products. The corresponding groups are the UNil-groups. They vanish if one inverts 2 [49]. For more information about the UNil-groups we refer to [15, 46, 47, 74, 77, 96, 260].

2.2.6

Applications III

The Borel Conjecture

One of the driving forces for the development of the Farrell–Jones Conjectures is still the following topological rigidity conjecture about closed aspherical manifolds, compare [107]. Recall that a manifold, or more generally a CW -complex, is called *aspherical* if its universal covering is contractible. An aspherical CW -complex X with $\pi_1(X) = G$ is a model for the classifying space BG . If X is an aspherical manifold and hence finite dimensional, then G is necessarily torsionfree.

27

Conjecture 27: (*Borel Conjecture.*) Let $f: M \rightarrow N$ be a homotopy equivalence of aspherical closed topological manifolds. Then f is homotopic to a homeomorphism. In particular two closed aspherical manifolds with isomorphic fundamental groups are homeomorphic.

Closely related to the Borel Conjecture is the conjecture that each aspherical finitely dominated Poincaré complex is homotopy equivalent to a closed topological man-

ifold. The Borel Conjecture 27 is false in the smooth category, i.e. if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism [106].

Using surgery theory one can show that in dimensions ≥ 5 the Borel Conjecture is implied by the K -theoretic vanishing Conjecture 3 combined with the L -theoretic Farrell–Jones Conjecture.

Theorem 28: (*The Farrell–Jones Conjecture Implies the Borel Conjecture.*) Let G be a torsion free group. If $\text{Wh}(G)$, $\tilde{K}_0(\mathbb{Z}G)$ and all the groups $K_{-i}(\mathbb{Z}G)$ with $i \geq 1$ vanish and if the assembly map

$$H_n(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z})) \rightarrow L_n^{(-\infty)}(\mathbb{Z}G)$$

is an isomorphism for all n , then the Borel Conjecture holds for all orientable manifolds of dimension ≥ 5 whose fundamental group is G .

The Borel Conjecture 27 can be reformulated in the language of surgery theory to the statement that the topological structure set $\mathcal{S}^{\text{top}}(M)$ of an aspherical closed topological manifold M consists of a single point. This set is the set of equivalence classes of homotopy equivalences $f: M' \rightarrow M$ with a topological closed manifold as source and M as target under the equivalence relation, for which $f_0: M_0 \rightarrow M$ and $f_1: M_1 \rightarrow M$ are equivalent if there is a homeomorphism $g: M_0 \rightarrow M_1$ such that $f_1 \circ g$ and f_0 are homotopic.

The surgery sequence of a closed orientable topological manifold M of dimension $n \geq 5$ is the exact sequence

$$\begin{aligned} \dots \rightarrow \mathcal{N}_{n+1}(M \times [0, 1], M \times \{0, 1\}) &\xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial} \mathcal{S}^{\text{top}}(M) \\ &\xrightarrow{\eta} \mathcal{N}_n(M) \xrightarrow{\sigma} L_n^s(\mathbb{Z}\pi_1(M)), \end{aligned}$$

which extends infinitely to the left. It is the basic tool for the classification of topological manifolds. (There is also a smooth version of it.) The map σ appearing in the sequence sends a normal map of degree one to its surgery obstruction. This map can be identified with the version of the L -theory assembly map where one works with the 1-connected cover $\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle$ of $\mathbf{L}^s(\mathbb{Z})$. The map $H_k(M; \mathbf{L}^s(\mathbb{Z})\langle 1 \rangle) \rightarrow H_k(M; \mathbf{L}^s(\mathbb{Z}))$ is injective for $k = n$ and an isomorphism for $k > n$. Because of the K -theoretic assumptions we can replace the s -decoration with the $\langle -\infty \rangle$ -decoration, compare Proposition 23. Therefore the Farrell–Jones Conjecture 19 implies that the maps $\sigma: \mathcal{N}_n(M) \rightarrow L_n^s(\mathbb{Z}\pi_1(M))$ and $\mathcal{N}_{n+1}(M \times [0, 1], M \times \{0, 1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M))$ are injective respectively bijective and thus by the surgery sequence that $\mathcal{S}^{\text{top}}(M)$ is a point and hence the Borel Conjecture 27 holds for M . More details can be found e.g. in [127, pages 17, 18, 28], [258, chapt. 18].

For more information about surgery theory we refer for instance to [41, 44, 45, 121, 122, 167, 178, 253, 293, 294], and [320].

Automorphisms of Manifolds

If one additionally also assumes the Farrell–Jones Conjectures for higher K -theory, one can combine the surgery theoretic results with the results about pseudoisotopies from Sect. 2.2.4 to obtain the following results.

29 **Theorem 29:** (*Homotopy Groups of $\text{Top}(M)$.*) Let M be an orientable closed aspherical manifold of dimension > 10 with fundamental group G . Suppose the L -theory assembly map

$$H_n(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z})) \rightarrow L_n^{(-\infty)}(\mathbb{Z}G)$$

is an isomorphism for all n and suppose the K -theory assembly map

$$H_n(BG; \mathbf{K}(\mathbb{Z})) \rightarrow K_n(\mathbb{Z}G)$$

is an isomorphism for $n \leq 1$ and a rational isomorphism for $n \geq 2$. Then for $1 \leq i \leq (\dim M - 7)/3$ one has

$$\pi_i(\text{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \text{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

In the differentiable case one additionally needs to study involutions on the higher K -theory groups. The corresponding result reads:

30 **Theorem 30:** (*Homotopy Groups of $\text{Diff}(M)$.*) Let M be an orientable closed aspherical differentiable manifold of dimension > 10 with fundamental group G . Then under the same assumptions as in Theorem 29 we have for $1 \leq i \leq (\dim M - 7)/3$

$$\pi_i(\text{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \text{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M; \mathbb{Q}) & \text{if } i > 1 \text{ and } \dim M \text{ odd}; \\ 0 & \text{if } i > 1 \text{ and } \dim M \text{ even}. \end{cases}$$

See for instance [97], [109, sect. 2] and [120, lecture 5]. For a modern survey on automorphisms of manifolds we refer to [329].

2.2.7 The Baum–Connes Conjecture in the Torsion Free Case

We denote by $K_*(Y)$ the complex K -homology of a topological space Y and by $K_*(C_r^*(G))$ the (topological) K -theory of the reduced group C^* -algebra. More explanations will follow below.

31 **Conjecture 31:** (*Baum–Connes Conjecture for Torsion Free Groups.*) Let G be a torsion free group. Then the Baum–Connes assembly map

$$K_n(BG) \rightarrow K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Complex K -homology $K_*(Y)$ is the homology theory associated to the topological (complex) K -theory spectrum $\mathbf{K}^{\text{top}}$ (which is often denoted $\mathbf{BU}$) and could also be written as $K_*(Y) = H_*(Y; \mathbf{K}^{\text{top}})$. The cohomology theory associated to the spectrum $\mathbf{K}^{\text{top}}$ is the well known complex K -theory defined in terms of complex vector bundles. Complex K -homology is a 2-periodic theory, i.e. $K_n(Y) \cong K_{n+2}(Y)$.

Also the *topological K -groups* $K_n(A)$ of a C^* -algebra A are 2-periodic. Whereas $K_0(A)$ coincides with the algebraically defined K_0 -group, the other groups $K_n(A)$ take the topology of the C^* -algebra A into account, for instance $K_n(A) = \pi_{n-1}(GL(A))$ for $n \geq 1$.

Let $\mathcal{B}(l^2(G))$ denote the bounded linear operators on the Hilbert space $l^2(G)$ whose orthonormal basis is G . The *reduced complex group C^* -algebra* $C_r^*(G)$ is the closure in the norm topology of the image of the regular representation $\mathbb{C}G \rightarrow \mathcal{B}(l^2(G))$, which sends an element $u \in \mathbb{C}G$ to the (left) G -equivariant bounded operator $l^2(G) \rightarrow l^2(G)$ given by right multiplication with u . In particular one has natural inclusions

$$\mathbb{C}G \subseteq C_r^*(G) \subseteq \mathcal{B}(l^2(G))^G \subseteq \mathcal{B}(l^2(G)).$$

It is essential to use the reduced group C^* -algebra in the Baum–Connes Conjecture, there are counterexamples for the version with the maximal group C^* -algebra, compare Sect. 2.5.1. For information about C^* -algebras and their topological K -theory we refer for instance to [37, 71, 80, 154, 188, 228, 279] and [321].

Remark 32: (*The Coefficients in the Case of Topological K -theory.*) If we specialize to the trivial group $G = \{1\}$, then the complex reduced group C^* -algebra reduces to $C_r^*(G) = \mathbb{C}$ and the topological K -theory is well known: by periodicity it suffices to know that $K_0(\mathbb{C}) \cong \mathbb{Z}$, where the homomorphism is given by the dimension, and $K_1(\mathbb{C}) = 0$. Correspondingly we have $K_q(\text{pt}) = \mathbb{Z}$ for q even and $K_q(\text{pt}) = 0$ for odd q .

32

Remark 33: (*Rational Computation.*) There is an Atiyah–Hirzebruch spectral sequence which converges to $K_{p+q}(BG)$ and whose E^2 -term is $E_{p,q}^2 = H_p(BG; K_q(\text{pt}))$. Rationally this spectral sequence collapses and the homological Chern character gives an isomorphism for $n \in \mathbb{Z}$

33

$$\begin{aligned} \text{ch}: \bigoplus_{k \in \mathbb{Z}} H_{n-2k}(BG; \mathbb{Q}) &= \bigoplus_{p+q=n} H_p(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (K_q(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &\xrightarrow{\cong} K_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned} \quad (2.6)$$

Remark 34: (*Torsionfree is Necessary.*) In the case where G is a finite group the reduced group C^* -algebra $C_r^*(G)$ coincides with the complex group ring $\mathbb{C}G$ and $K_0(C_r^*(G))$ coincides with the complex representation ring of G . Since the group

34

homology of a finite group vanishes rationally except in dimension 0, the previous remark shows that we need to assume the group to be torsion free in Conjecture 31.

35

Remark 35: (*Bass–Heller–Swan-Decomposition for Topological K -theory.*) There is an analogue of the Bass–Heller–Swan decomposition in algebraic K -theory (see Remark 15) or of the Shaneson splitting in L -theory (see Remark 26) for topological K -theory. Namely we have

$$K_n(C_r^*(G \times \mathbb{Z})) \cong K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)) ,$$

see [243, theorem 3.1 on page 151] or more generally [244, theorem 18 on page 632]. This is consistent with the obvious isomorphism

$$K_n(B(G \times \mathbb{Z})) = K_n(BG \times S^1) \cong K_{n-1}(BG) \oplus K_n(BG) .$$

Notice that here in contrast to the algebraic K -theory no Nil-terms occur (see Remark 15) and that there is no analogue of the complications in algebraic L -theory coming from the varying decorations (see Remark 26). This absence of Nil-terms or decorations is the reason why in the final formulation of the Baum–Connes Conjecture it suffices to deal with the family of finite subgroups, whereas in the algebraic K - and L -theory case one must consider the larger and harder to handle family of virtually cyclic subgroups. This in some sense makes the computation of topological K -theory of reduced group C^* -algebras easier than the computation of $K_n(\mathbb{Z}G)$ or $L_n(\mathbb{Z}G)$.

36

Remark 36: (*Real Version of the Baum–Connes Conjecture.*) There is an obvious *real version of the Baum–Connes Conjecture*. It says that for a torsion free group the real assembly map

$$KO_n(BG) \rightarrow KO_n(C_r^*(G; \mathbb{R}))$$

is bijective for $n \in \mathbb{Z}$. We will discuss in Sect. 2.5.1 below that this real version of the Baum–Connes Conjecture is implied by the complex version Conjecture 31.

Here $KO_n(C_r^*(G; \mathbb{R}))$ is the topological K -theory of the *real reduced group C^* -algebra* $C_r^*(G; \mathbb{R})$. We use KO instead of K as a reminder that we work with real C^* -algebras. The topological real K -theory $KO_*(Y)$ is the homology theory associated to the spectrum $\mathbf{BO}$, whose associated cohomology theory is given in terms of real vector bundles. Both, topological K -theory of a real C^* -algebra and KO -homology of a space are 8-periodic and $KO_n(\text{pt}) = K_n(\mathbb{R})$ is $\mathbb{Z}$, if $n = 0, 4$ (8), is $\mathbb{Z}/2$ if $n = 1, 2$ (8) and is 0 if $n = 3, 5, 6, 7$ (8).

More information about the K -theory of real C^* -algebras can be found in [281].

Applications IV

2.2.8

We now discuss some consequences of the Baum–Connes Conjecture for Torsion Free Groups 31.

The Trace Conjecture in the Torsion Free Case

The assembly map appearing in the Baum–Connes Conjecture has an interpretation in terms of index theory. This is important for geometric applications. It is of the same significance as the interpretation of the L -theoretic assembly map as the map σ appearing in the exact surgery sequence discussed in Sect. 2.2.5. We proceed to explain this.

An element $\eta \in K_0(BG)$ can be represented by a pair (M, P^*) consisting of a cocompact free proper smooth G -manifold M with Riemannian metric together with an elliptic G -complex P^* of differential operators of order 1 on M [29]. To such a pair one can assign an index $\text{ind}_{C_r^*(G)}(M, P^*)$ in $K_0(C_r^*(G))$ [223] which is the image of η under the assembly map $K_0(BG) \rightarrow K_0(C_r^*(G))$ appearing in Conjecture 31. With this interpretation the surjectivity of the assembly map for a torsion free group says that any element in $K_0(C_r^*(G))$ can be realized as an index. This allows to apply index theorems to get interesting information.

Here is a prototype of such an argument. The *standard trace*

$$\text{tr}_{C_r^*(G)}: C_r^*(G) \rightarrow \mathbb{C} \quad (2.7)$$

sends an element $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(1), 1 \rangle_{l^2(G)}$. Applying the trace to idempotent matrices yields a homomorphism

$$\text{tr}_{C_r^*(G)}: K_0(C_r^*(G)) \rightarrow \mathbb{R}.$$

Let $\text{pr}: BG \rightarrow \text{pt}$ be the projection. For a group G the following diagram commutes

$$\begin{array}{ccccc} K_0(BG) & \xrightarrow{A} & K_0(C_r^*(G)) & \xrightarrow{\text{tr}_{C_r^*(G)}} & \mathbb{R} \\ K_0(\text{pr}) \downarrow & & \cong & & \uparrow i \\ K_0(\text{pt}) & \xrightarrow{\cong} & K_0(\mathbb{C}) & \xrightarrow{\text{tr}_{\mathbb{C}}} & \mathbb{Z}. \end{array} \quad (2.8)$$

Here $i: \mathbb{Z} \rightarrow \mathbb{R}$ is the inclusion and A is the assembly map. This non-trivial statement follows from Atiyah's L^2 -index theorem [12]. Atiyah's theorem says that the L^2 -index $\text{tr}_{C_r^*(G)} \circ A(\eta)$ of an element $\eta \in K_0(BG)$, which is represented by a pair (M, P^*) , agrees with the ordinary index of $(G \backslash M; G \backslash P^*)$, which is $\text{tr}_{\mathbb{C}} \circ K_0(\text{pr})(\eta) \in \mathbb{Z}$.

The following conjecture is taken from [27, page 21].

37 **Conjecture 37:** (*Trace Conjecture for Torsion Free Groups.*) For a torsion free group G the image of

$$\mathrm{tr}_{C_r^*(G)}: K_0(C_r^*(G)) \rightarrow \mathbb{R}$$

consists of the integers.

The commutativity of diagram (2.8) above implies

38 **Consequence 38** The surjectivity of the Baum–Connes assembly map

$$K_0(BG) \rightarrow K_0(C_r^*(G))$$

implies Conjecture 37, the Trace Conjecture for Torsion Free Groups.

The Kadison Conjecture

39 **Conjecture 39:** (*Kadison Conjecture.*) If G is a torsion free group, then the only idempotent elements in $C_r^*(G)$ are 0 and 1.

40 **Lemma 40** The Trace Conjecture for Torsion Free Groups 37 implies the Kadison Conjecture 39.

Proof Assume that $p \in C_r^*(G)$ is an idempotent different from 0 or 1. From p one can construct a non-trivial projection $q \in C_r^*(G)$, i.e. $q^2 = q$, $q^* = q$, with $\mathrm{im}(p) = \mathrm{im}(q)$ and hence with $0 < q < 1$. Since the standard trace $\mathrm{tr}_{C_r^*(G)}$ is faithful, we conclude $\mathrm{tr}_{C_r^*(G)}(q) \in \mathbb{R}$ with $0 < \mathrm{tr}_{C_r^*(G)}(q) < 1$. Since $\mathrm{tr}_{C_r^*(G)}(q)$ is by definition the image of the element $[\mathrm{im}(q)] \in K_0(C_r^*(G))$ under $\mathrm{tr}_{C_r^*(G)}: K_0(C_r^*(G)) \rightarrow \mathbb{R}$, we get a contradiction to the assumption $\mathrm{im}(\mathrm{tr}_{C_r^*(G)}) \subseteq \mathbb{Z}$.

Recall that a ring R is called an *integral domain* if it has no non-trivial zero-divisors, i.e. if $r, s \in R$ satisfy $rs = 0$, then r or s is 0. Obviously the Kadison Conjecture 39 implies for $R \subseteq \mathbb{C}$ the following.

41 **Conjecture 41:** (*Idempotent Conjecture.*) Let R be an integral domain and let G be a torsion free group. Then the only idempotents in RG are 0 and 1.

The statement in the conjecture above is a purely algebraic statement. If $R = \mathbb{C}$, it is by the arguments above related to questions about operator algebras, and thus methods from operator algebras can be used to attack it.

Other Related Conjectures

We would now like to mention several conjectures which are not directly implied by the Baum–Connes or Farrell–Jones Conjectures, but which are closely related to the Kadison Conjecture and the Idempotent Conjecture mentioned above.

The next conjecture is also called the *Kaplansky Conjecture*.

Conjecture 42: (*Zero-Divisor-Conjecture.*) Let R be an integral domain and G be a torsion free group. Then RG is an integral domain. **42**

Obviously the Zero-Divisor-Conjecture 42 implies the Idempotent Conjecture 41. The Zero-Divisor-Conjecture for $R = \mathbb{Q}$ is implied by the following version of the Atiyah Conjecture (see [202, lemma 10.5 and lemma 10.15]).

Conjecture 43: (*Atiyah-Conjecture for Torsion Free Groups.*) Let G be a torsion free group and let M be a closed Riemannian manifold. Let $\overline{M} \rightarrow M$ be a regular covering with G as group of deck transformations. Then all L^2 -Betti numbers $b_p^{(2)}(\overline{M}; \mathcal{N}(G))$ are integers. **43**

For the precise definition and more information about L^2 -Betti numbers and the group von Neumann algebra $\mathcal{N}(G)$ we refer for instance to [202, 205].

This more geometric formulation of the Atiyah Conjecture is in fact implied by the following more operator theoretic version. (The two would be equivalent if one would work with rational instead of complex coefficients below.)

Conjecture 44: (*Strong Atiyah-Conjecture for Torsion Free Groups.*) Let G be a torsion free group. Then for all (m, n) -matrices A over $\mathbb{C}G$ the von Neumann dimension of the kernel of the induced G -equivariant bounded operator

$$r_A^{(2)}: \ell^2(G)^m \rightarrow \ell^2(G)^n$$

is an integer.

The Strong Atiyah-Conjecture for Torsion Free Groups implies both the Atiyah-Conjecture for Torsion Free Groups 43 [202, lemma 10.5 on page 371] and the Zero-Divisor-Conjecture 42 for $R = \mathbb{C}$ [202, lemma 10.15 on page 376].

Conjecture 45: (*Embedding Conjecture.*) Let G be a torsion free group. Then $\mathbb{C}G$ admits an embedding into a skewfield. **45**

Obviously the Embedding Conjecture implies the Zero-Divisor-Conjecture 42 for $R = \mathbb{C}$. If G is a torsion free amenable group, then the Strong Atiyah-Conjecture for Torsion Free Groups 44 and the Zero-Divisor-Conjecture 42 for $R = \mathbb{C}$ are equivalent [202, lemma 10.16 on page 376]. For more information about the Atiyah Conjecture we refer for instance to [192], [202, chapt. 10] and [261].

Finally we would like to mention the Unit Conjecture.

- 46** **Conjecture 46:** (*Unit-Conjecture.*) Let R be an integral domain and G be a torsion free group. Then every unit in RG is trivial, i.e. of the form $r \cdot g$ for some unit $r \in R^{\text{inv}}$ and $g \in G$.

The Unit Conjecture 46 implies the Zero-Divisor-Conjecture 42. For a proof of this fact and for more information we refer to [187, proposition 6.21 on page 95].

L^2 -Rho-Invariants and L^2 -Signatures

Let M be a closed connected orientable Riemannian manifold. Denote by $\eta(M) \in \mathbb{R}$ the *eta-invariant* of M and by $\eta^{(2)}(\tilde{M}) \in \mathbb{R}$ the L^2 -*eta-invariant* of the $\pi_1(M)$ -covering given by the universal covering $\tilde{M} \rightarrow M$. Let $\varphi^{(2)}(M) \in \mathbb{R}$ be the L^2 -*rho-invariant* which is defined to be the difference $\eta^{(2)}(\tilde{M}) - \eta(M)$. These invariants were studied by Cheeger and Gromov [64, 65]. They show that $\varphi^{(2)}(M)$ depends only on the diffeomorphism type of M and is in contrast to $\eta(M)$ and $\eta^{(2)}(\tilde{M})$ independent of the choice of Riemannian metric on M . The following conjecture is taken from Mathai [214].

- 47** **Conjecture 47:** (*Homotopy Invariance of the L^2 -Rho-Invariant for Torsionfree Groups.*) If $\pi_1(M)$ is torsionfree, then $\varphi^{(2)}(M)$ is a homotopy invariant.

Chang–Weinberger [62] assign to a closed connected oriented $(4k-1)$ -dimensional manifold M a Hirzebruch-type invariant $\tau^{(2)}(M) \in \mathbb{R}$ as follows. By a result of Hausmann [145] there is a closed connected oriented $4k$ -dimensional manifold W with $M = \partial W$ such that the inclusion $\partial W \rightarrow W$ induces an injection on the fundamental groups. Define $\tau^{(2)}(M)$ as the difference $\text{sign}^{(2)}(\tilde{W}) - \text{sign}(W)$ of the L^2 -signature of the $\pi_1(W)$ -covering given by the universal covering $\tilde{W} \rightarrow W$ and the signature of W . This is indeed independent of the choice of W . It is reasonable to believe that $\varphi^{(2)}(M) = \tau^{(2)}(M)$ is always true. Chang–Weinberger [62] use $\tau^{(2)}$ to prove that if $\pi_1(M)$ is not torsionfree there are infinitely many diffeomorphically distinct manifolds of dimension $4k+3$ with $k \geq 1$, which are tangentially simple homotopy equivalent to M .

- 48** **Theorem 48:** (*Homotopy Invariance of $\tau^{(2)}(M)$ and $\varphi^{(2)}(M)$.*) Let M be a closed connected oriented $(4k-1)$ -dimensional manifold M such that $G = \pi_1(M)$ is torsionfree.
- (i) If the assembly map $K_0(BG) \rightarrow K_0(C_{\max}^*(G))$ for the maximal group C^* -algebra is surjective (see Sect. 2.5.1), then $\varphi^{(2)}(M)$ is a homotopy invariant.
 - (ii) Suppose that the Farrell–Jones Conjecture for L -theory 19 is rationally true for $R = \mathbb{Z}$, i.e. the rationalized assembly map

$$H_n(BG; \mathbb{L}^{(-\infty)}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_n^{(-\infty)}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for $n \in \mathbb{Z}$. Then $\tau^{(2)}(M)$ is a homotopy invariant. If furthermore G is residually finite, then $\varrho^{(2)}(M)$ is a homotopy invariant.

Proof

- (i) This is proved by Keswani [174, 175].
- (ii) This is proved by Chang [61] and Chang–Weinberger [62] using [210].

Remark 49 Let X be a $4n$ -dimensional Poincaré space over $\mathbb{Q}$. Let $\bar{X} \rightarrow X$ be a normal covering with *torsion-free* covering group G . Suppose that the assembly map $K_0(BG) \rightarrow K_0(C_{\max}^*(G))$ for the maximal group C^* -algebra is surjective (see Sect. 2.5.1) or suppose that the rationalized assembly map for L -theory

49

$$H_{4n}(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_{4n}^{(-\infty)}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Then the following L^2 -signature theorem is proved in Lück–Schick [211, theorem 2.3]

$$\text{sign}^{(2)}(\bar{X}) = \text{sign}(X) .$$

If one drops the condition that G is torsionfree this equality becomes false. Namely, Wall has constructed a finite Poincaré space X with a finite G covering $\bar{X} \rightarrow X$ for which $\text{sign}(\bar{X}) \neq |G| \cdot \text{sign}(X)$ holds (see [258, example 22.28], [319, corollary 5.4.1]).

Remark 50 Cochran–Orr–Teichner give in [69] new obstructions for a knot to be slice which are sharper than the Casson–Gordon invariants. They use L^2 -signatures and the Baum–Connes Conjecture 59. We also refer to the survey article [68] about non-commutative geometry and knot theory.

50

Applications V

2.2.9

Novikov Conjectures

The Baum–Connes and Farrell–Jones Conjectures discussed so far imply obviously that for torsion free groups the rationalized assembly maps

$$H_*(BG; \mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$H_*(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_*^{(-\infty)}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are injective. For reasons that will be explained below these “rational injectivity conjectures” are known as “Novikov Conjectures”. In fact one expects these injectivity results also when the groups contain torsion. So there are the following conjectures.

51

Conjecture 51: (*K- and L-theoretic Novikov Conjectures.*) For every group G the assembly maps

$$H_*(BG; \mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$H_*(BG; \mathbf{L}^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_*^p(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are injective.

Observe that, since the $\mathbb{Z}/2$ -Tate cohomology groups vanish rationally, there is no difference between the various decorations in L -theory because of the Rothenberg sequence. We have chosen the p -decoration above.

The Original Novikov Conjecture

We now explain the Novikov Conjecture in its original formulation.

Let G be a (not necessarily torsion free) group and $u: M \rightarrow BG$ be a map from a closed oriented smooth manifold M to BG . Let $\mathcal{L}(M) \in \prod_{k \geq 0} H^k(M; \mathbb{Q})$ be the L -class of M , which is a certain polynomial in the Pontrjagin classes and hence depends a priori on the tangent bundle and hence on the differentiable structure of M . For $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ define the *higher signature of M associated to x and u* to be

$$\text{sign}_x(M, u) := \langle \mathcal{L}(M) \cup u^*x, [M] \rangle \in \mathbb{Q}. \quad (2.9)$$

The Hirzebruch signature formula says that for $x = 1$ the signature $\text{sign}_1(M, u)$ coincides with the ordinary signature $\text{sign}(M)$ of M , if $\dim(M) = 4n$, and is zero, if $\dim(M)$ is not divisible by four. Recall that for $\dim(M) = 4n$ the *signature* $\text{sign}(M)$ of M is the signature of the non-degenerate bilinear symmetric pairing on the middle cohomology $H^{2n}(M; \mathbb{R})$ given by the intersection pairing $(a, b) \mapsto \langle a \cup b, [M] \rangle$. Obviously $\text{sign}(M)$ depends only on the oriented homotopy type of M . We say that sign_x for $x \in H^*(BG; \mathbb{Q})$ is *homotopy invariant* if for two closed oriented smooth manifolds M and N with reference maps $u: M \rightarrow BG$ and $v: N \rightarrow BG$ we have

$$\text{sign}_x(M, u) = \text{sign}_x(N, v)$$

if there is an orientation preserving homotopy equivalence $f: M \rightarrow N$ such that $v \circ f$ and u are homotopic.

Conjecture 52: (*Novikov Conjecture.*) Let G be a group. Then sign_x is homotopy invariant for all $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$.

52

By Hirzebruch's signature formula the Novikov Conjecture 52 is true for $x = 1$.

Relations between the Novikov Conjectures

Using surgery theory one can show [260, proposition 6.6 on page 300] the following.

Proposition 53 For a group G the original Novikov Conjecture 52 is equivalent to the L -theoretic Novikov Conjecture, i.e. the injectivity of the assembly map

53

$$H_*(BG; L^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_*^p(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular for torsion free groups the L -theoretic Farrell–Jones Conjecture 19 implies the Novikov Conjecture 52. Later in Proposition 95 we will prove in particular the following statement.

Proposition 54 The Novikov Conjecture for topological K -theory, i.e. the injectivity of the assembly map

54

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the L -theoretic Novikov Conjecture and hence the original Novikov Conjecture.

For more information about the Novikov Conjectures we refer for instance to [38, 52, 55, 81, 120, 127, 179, 258] and [269].

The Zero-in-the-Spectrum Conjecture

The following Conjecture is due to Gromov [136, page 120].

Conjecture 55: (*Zero-in-the-spectrum Conjecture.*) Suppose that $\tilde{M}$ is the universal covering of an aspherical closed Riemannian manifold M (equipped with the lifted Riemannian metric). Then zero is in the spectrum of the minimal closure

55

$$(\Delta_p)_{\min}: L^2\Omega^p(\tilde{M}) \supset \text{dom}(\Delta_p)_{\min} \rightarrow L^2\Omega^p(\tilde{M}),$$

for some $p \in \{0, 1, \dots, \dim M\}$, where Δ_p denotes the Laplacian acting on smooth p -forms on $\tilde{M}$.

56

Proposition 56 Suppose that M is an aspherical closed Riemannian manifold with fundamental group G , then the injectivity of the assembly map

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Zero-in-the-spectrum Conjecture for $\tilde{M}$.

Proof We give a sketch of the proof. More details can be found in [195, corollary 4]. We only explain that the assumption that in every dimension zero is not in the spectrum of the Laplacian on $\tilde{M}$, yields a contradiction in the case that $n = \dim(M)$ is even. Namely, this assumption implies that the $C_r^*(G)$ -valued index of the signature operator twisted with the flat bundle $\tilde{M} \times_G C_r^*(G) \rightarrow M$ in $K_0(C_r^*(G))$ is zero, where $G = \pi_1(M)$. This index is the image of the class $[S]$ defined by the signature operator in $K_0(BG)$ under the assembly map $K_0(BG) \rightarrow K_0(C_r^*(G))$. Since by assumption the assembly map is rationally injective, this implies $[S] = 0$ in $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$. Notice that M is aspherical by assumption and hence $M = BG$. The homological Chern character defines an isomorphism

$$K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \geq 0} H_{2p}(M; \mathbb{Q}),$$

which sends $[S]$ to the Poincaré dual $\mathcal{L}(M) \cap [M]$ of the Hirzebruch L -class $\mathcal{L}(M) \in \bigoplus_{p \geq 0} H^{2p}(M; \mathbb{Q})$. This implies that $\mathcal{L}(M) \cap [M] = 0$ and hence $\mathcal{L}(M) = 0$. This contradicts the fact that the component of $\mathcal{L}(M)$ in $H^0(M; \mathbb{Q})$ is 1.

More information about the Zero-in-the-spectrum Conjecture 55 can be found for instance in [195] and [202, sect. 12].

2.3

The Conjectures in the General Case

In this chapter we will formulate the Baum–Connes and Farrell–Jones Conjectures. We try to emphasize the unifying principle that underlies these conjectures. The point of view taken in this chapter is that all three conjectures are conjectures about specific equivariant homology theories. Some of the technical details concerning the actual construction of these homology theories are deferred to Chapt. 2.7.

2.3.1 Formulation of the Conjectures

Suppose we are given

- A discrete group G ;
- A family $\mathcal{F}$ of subgroups of G , i.e. a set of subgroups which is closed under conjugation with elements of G and under taking finite intersections;
- A G -homology theory $\mathcal{H}_*^G(-)$.

Then one can formulate the following Meta-Conjecture.

Meta-Conjecture 57 The assembly map

57

$$A_{\mathcal{F}}: \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(\text{pt})$$

which is the map induced by the projection $E_{\mathcal{F}}(G) \rightarrow \text{pt}$, is an isomorphism for $n \in \mathbb{Z}$.

Here $E_{\mathcal{F}}(G)$ is the classifying space of the family $\mathcal{F}$, a certain G -space which specializes to the universal free G -space EG if the family contains only the trivial subgroup. A G -homology theory is the “obvious” G -equivariant generalization of the concept of a homology theory to a suitable category of G -spaces, in particular it is a functor on such spaces and the map $A_{\mathcal{F}}$ is simply the map induced by the projection $E_{\mathcal{F}}(G) \rightarrow \text{pt}$. We devote the Sects. 2.3.1 to 2.3.1 below to a discussion of G -homology theories, classifying spaces for families of subgroups and related things. The reader who never encountered these concepts should maybe first take a look at these subsections.

Of course the conjecture above is not true for arbitrary G , $\mathcal{F}$ and $\mathcal{H}_*^G(-)$, but the Farrell–Jones and Baum–Connes Conjectures state that for specific G -homology theories there is a natural choice of a family $\mathcal{F} = \mathcal{F}(G)$ of subgroups for every group G such that $A_{\mathcal{F}(G)}$ becomes an isomorphism for all groups G .

Let R be a ring (with involution). In Proposition 156 we will describe the construction of G -homology theories which will be denoted

$$H_n^G(-; \mathbf{K}_R), \quad H_n^G(-; \mathbf{L}_R^{(-\infty)}) \quad \text{and} \quad H_n^G(-; \mathbf{K}^{\text{top}}).$$

If G is the trivial group, these homology theories specialize to the (non-equivariant) homology theories with similar names that appeared in Chapt. 2.2, namely to

$$H_n(-; \mathbf{K}(R)), \quad H_n(-; \mathbf{L}^{(-\infty)}(R)) \quad \text{and} \quad K_n(-).$$

Another main feature of these G -homology theories is that evaluated on the one point space pt (considered as a trivial G -space) we obtain the K - and L -theory of the group ring RG , respectively the topological K -theory of the reduced C^* -algebra (see Proposition 156 and Theorem 158 (iii))

$$K_n(RG) \cong H_n^G(\text{pt}; \mathbf{K}_R),$$

$$L_n^{(-\infty)}(RG) \cong H_n^G(\text{pt}; \mathbf{L}_R^{(-\infty)}) \quad \text{and}$$

$$K_n(C_r^*(G)) \cong H_n^G(\text{pt}; \mathbf{K}^{\text{top}}).$$

We are now prepared to formulate the conjectures around which this article is centered. Let $\mathcal{FIN}$ be the family of finite subgroups and let $\mathcal{V}\mathcal{C}\mathcal{Y}$ be the family of virtually cyclic subgroups.

58 **Conjecture 58:** (*Farrell–Jones Conjecture for K - and L -theory.*) Let R be a ring (with involution) and let G be a group. Then for all $n \in \mathbb{Z}$ the maps

$$\begin{aligned} A_{\text{vCY}}: H_n^G(E_{\text{vCY}}(G); \mathbf{K}_R) &\rightarrow H_n^G(\text{pt}; \mathbf{K}_R) \cong K_n(RG); \\ A_{\text{vCY}}: H_n^G(E_{\text{vCY}}(G); \mathbf{L}_R^{(-\infty)}) &\rightarrow H_n^G(\text{pt}; \mathbf{L}_R^{(-\infty)}) \cong L_n^{(-\infty)}(RG), \end{aligned}$$

which are induced by the projection $E_{\text{vCY}}(G) \rightarrow \text{pt}$, are isomorphisms.

The conjecture for the topological K -theory of C^* -algebras is known as the Baum–Connes Conjecture and reads as follows.

59 **Conjecture 59:** (*Baum–Connes Conjecture.*) Let G be a group. Then for all $n \in \mathbb{Z}$ the map

$$A_{\mathcal{F}\mathcal{I}\mathcal{N}}: H_n^G(E_{\mathcal{F}\mathcal{I}\mathcal{N}}(G); \mathbf{K}^{\text{top}}) \rightarrow H_n^G(\text{pt}; \mathbf{K}^{\text{top}}) \cong K_n(C_r^*(G))$$

induced by the projection $E_{\mathcal{F}\mathcal{I}\mathcal{N}}(G) \rightarrow \text{pt}$ is an isomorphism.

We will explain the analytic assembly map $\text{ind}_G: K_n^G(X) \rightarrow K_n(C_r^*(G))$, which can be identified with the assembly map appearing in the Baum–Connes Conjecture 59 in Sect. 2.8.2.

60 **Remark 60** Of course the conjectures really come to life only if the abstract point of view taken in this chapter is connected up with more concrete descriptions of the assembly maps. We have already discussed a surgery theoretic description in Theorem 28 and an interpretation in terms index theory in Sect. 2.2.8. More information about alternative interpretations of assembly maps can be found in Sects. 2.8.2 and 2.8.8. These concrete interpretations of the assembly maps lead to applications. We already discussed many such applications in Chapt. 2.2 and encourage the reader to go ahead and browse through Chapt. 2.4 in order to get further ideas about these more concrete aspects.

61 **Remark 61:** (*Relation to the “classical” assembly maps.*) The value of an equivariant homology theory $\mathcal{H}_*^G(-)$ on the universal free G -space $EG = E_{[\emptyset]}(G)$ (a free G -CW-complex whose quotient EG/G is a model for BG) can be identified with the corresponding non-equivariant homology theory evaluated on BG , if we assume that $\mathcal{H}_*^G$ is the special value of an equivariant homology theory $\mathcal{H}_*^?$ at $? = G$. This means that there exists an induction structure (a mild condition satisfied in our examples, compare Sect. 2.7.1), which yields an identification

$$\mathcal{H}_n^G(EG) \cong \mathcal{H}_n^{[\emptyset]}(BG) = \mathcal{H}_n(BG).$$

Using these identifications the “classical” assembly maps, which appeared in Chapt. 2.2 in the versions of the Farrell–Jones and Baum–Connes Conjectures for torsion free groups (see Conjectures 11, 19 and 31),

$$\begin{aligned} H_n(BG; \mathbf{K}(R)) &\cong H_n^G(EG; \mathbf{K}_R) \rightarrow H_n^G(\mathrm{pt}; \mathbf{K}_R) \cong K_n(RG) ; \\ H_n(BG; \mathbf{L}^{(-\infty)}(R)) &\cong H_n^G(EG; \mathbf{L}_R^{(-\infty)}) \rightarrow H_n^G(\mathrm{pt}; \mathbf{L}_R^{(-\infty)}) \cong L_n^{(-\infty)}(RG) ; \\ \text{and } K_n(BG) &\cong H_n^G(EG; \mathbf{K}^{\mathrm{top}}) \rightarrow H_n^G(\mathrm{pt}; \mathbf{K}^{\mathrm{top}}) \cong K_n(C_r^*(G)) , \end{aligned}$$

correspond to the assembly maps for the family $\mathcal{F} = \{\}$ consisting only of the trivial group and are simply the maps induced by the projection $EG \rightarrow \mathrm{pt}$.

Remark 62: (*The choice of the right family.*) As explained above the Farrell–Jones and Baum–Connes Conjectures 58 and 59 can be considered as special cases of the Meta-Conjecture 57. In all three cases we are interested in a computation of the right hand side $\mathcal{H}_n^G(\mathrm{pt})$ of the assembly map, which can be identified with $K_n(RG)$, $L_n^{(-\infty)}(RG)$ or $K_n(C_r^*(G))$. The left hand side $\mathcal{H}_n^G(E_{\mathcal{F}}(G))$ of such an assembly map is much more accessible and the smaller $\mathcal{F}$ is, the easier it is to compute $\mathcal{H}_n^G(E_{\mathcal{F}}(G))$ using homological methods like spectral sequences, Mayer–Vietoris arguments and Chern characters.

In the extreme case where $\mathcal{F} = \mathcal{AL}\mathcal{L}$ is the family of all subgroups the assembly map $A_{\mathcal{AL}\mathcal{L}}: \mathcal{H}_n^G(E_{\mathcal{AL}\mathcal{L}}(G)) \rightarrow \mathcal{H}_n^G(\mathrm{pt})$ is always an isomorphism for the trivial reason that the one point space pt is a model for $E_{\mathcal{AL}\mathcal{L}}(G)$ (compare Sect. 2.3.1) and hence the assembly map is the identity. The goal however is to have an isomorphism for a family which is as small as possible.

We have already seen in Remark 4, Remark 25 and Remark 34 that in all three cases the classical assembly map, which corresponds to the trivial family, is not surjective for finite groups. This forces one to include at least the family $\mathcal{FIN}$ of finite groups. The K - or L -theory of the finite subgroups of the given group G will then enter in a computation of the left hand side of the assembly map similar as the K - and L -theory of the trivial subgroup appeared on the left hand side in the classical case, compare e.g. Remark 14. In the Baum–Connes case the family $\mathcal{FIN}$ seems to suffice. However in the case of algebraic K -theory we saw in Remark 15 that already the simplest torsion free group, the infinite cyclic group, causes problems because of the Nil-terms that appear in the Bass–Heller–Swan formula. The infinite dihedral group is a “minimal counterexample” which shows that the family $\mathcal{FIN}$ is not sufficient in the $L_{\mathbb{Z}}^{(-\infty)}$ -case. There are non-vanishing UNil-terms, compare Proposition 75 and its proof. Also the version of the L -theoretic Farrell–Jones Conjecture with the decoration s , $h = \langle 1 \rangle$ or $p = \langle 0 \rangle$ instead of $\langle -\infty \rangle$ is definitely false. Counterexamples are given in [123]. Recall that there were no Nil-terms in the topological K -theory context, compare Remark 35.

The choice of the family $\mathcal{VCY}$ of virtually cyclic subgroups in the Farrell–Jones conjectures pushes all the Nil-problems appearing in algebraic K - and L -theory

into the source of the assembly map so that they do not occur if one tries to prove the Farrell–Jones Conjecture 58. Of course they do come up again when one wants to compute the source of the assembly map.

We now take up the promised detailed discussion of some notions like equivariant homology theories and classifying spaces for families we used above. The reader who is familiar with these concepts may of course skip the following subsections.

***G*-CW-Complexes**

A *G*-CW-complex *X* is a *G*-space *X* together with a filtration $X_{-1} = \emptyset \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X$ such that $X = \operatorname{colim}_{n \rightarrow \infty} X_n$ and for each *n* there is a *G*-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

This definition makes also sense for topological groups. The following alternative definition only applies to discrete groups. A *G*-CW-complex is a CW-complex with a *G*-action by cellular maps such that for each open cell *e* and each *g* ∈ *G* with *ge* ∩ *e* ≠ ∅ we have *gx* = *x* for all *x* ∈ *e*. There is an obvious notion of a *G*-CW-pair.

A *G*-CW-complex *X* is called *finite* if it is built out of finitely many *G*-cells $G/H_i \times D^n$. This is the case if and only if it is *cocompact*, i.e. the quotient space $G \backslash X$ is compact. More information about *G*-CW-complexes can be found for instance in [197, sects. 1 and 2], [304, sects. II.1 and II.2].

Families of Subgroups

A *family* $\mathcal{F}$ of subgroups of *G* is a set of subgroups of *G* closed under conjugation, i.e. $H \in \mathcal{F}, g \in G$ implies $g^{-1}Hg \in \mathcal{F}$, and finite intersections, i.e. $H, K \in \mathcal{F}$ implies $H \cap K \in \mathcal{F}$. Throughout the text we will use the notations

$$\{\}, \quad \mathcal{FCY}, \quad \mathcal{FLN}, \quad \mathcal{CYC}, \quad \mathcal{VCY}_I, \quad \mathcal{VCY} \quad \text{and} \quad \mathcal{ALL}$$

for the families consisting of the trivial, all finite cyclic, all finite, all (possibly infinite) cyclic, all virtually cyclic of the first kind, all virtually cyclic, respectively all subgroups of a given group *G*. Recall that a group is called *virtually cyclic* if it is finite or contains an infinite cyclic subgroup of finite index. A group is *virtually cyclic of the first kind* if it admits a surjection onto an infinite cyclic group with finite kernel, compare Lemma 71.

Classifying Spaces for Families

Let $\mathcal{F}$ be a family of subgroups of G . A G -CW-complex, all whose isotropy groups belong to $\mathcal{F}$ and whose H -fixed point sets are contractible for all $H \in \mathcal{F}$, is called a *classifying space for the family $\mathcal{F}$* and will be denoted $E_{\mathcal{F}}(G)$. Such a space is unique up to G -homotopy because it is characterized by the property that for any G -CW-complex X , all whose isotropy groups belong to $\mathcal{F}$, there is up to G -homotopy precisely one G -map from X to $E_{\mathcal{F}}(G)$. These spaces were introduced by tom Dieck [303], [304, I.6].

A functorial “bar-type” construction is given in [82, sect. 7].

If $\mathcal{F} \subset \mathcal{G}$ are families of subgroups for G , then by the universal property there is up to G -homotopy precisely one G -map $E_{\mathcal{F}}(G) \rightarrow E_{\mathcal{G}}(G)$.

The space $E_{\{1\}}(G)$ is the same as the space EG which is by definition the total space of the universal G -principal bundle $G \rightarrow EG \rightarrow BG$, or, equivalently, the universal covering of BG . A model for $E_{\mathcal{A}\mathcal{L}\mathcal{L}}(G)$ is given by the space $G/G = \text{pt}$ consisting of one point.

The space $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ is also known as *the classifying space for proper G -actions* and denoted by $\underline{E}G$ in the literature. Recall that a G -CW-complex X is proper if and only if all its isotropy groups are finite (see for instance [197, theorem 1.23 on page 18]). There are often nice models for $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$. If G is word hyperbolic in the sense of Gromov, then the Rips-complex is a finite model [216, 217].

If G is a discrete subgroup of a Lie group L with finitely many path components, then for any maximal compact subgroup $K \subseteq L$ the space L/K with its left G -action is a model for $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ [2, corollary 4.14]. More information about $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ can be found for instance in [28, sect. 2], [180, 199, 206, 207] and [282].

G -Homology Theories

Fix a group G and an associative commutative ring Λ with unit. A *G -homology theory $\mathcal{H}_*^G$ with values in Λ -modules* is a collection of covariant functors $\mathcal{H}_n^G$ from the category of G -CW-pairs to the category of Λ -modules indexed by $n \in \mathbb{Z}$ together with natural transformations

$$\partial_n^G(X, A): \mathcal{H}_n^G(X, A) \rightarrow \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A, \emptyset)$$

for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- (i) G -homotopy invariance

If f_0 and f_1 are G -homotopic maps $(X, A) \rightarrow (Y, B)$ of G -CW-pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$.

- (ii) Long exact sequence of a pair

Given a pair (X, A) of G -CW-complexes, there is a long exact sequence

$$\begin{aligned} \dots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X, A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \\ \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X, A) \xrightarrow{\partial_n^G} \mathcal{H}_{n-1}^G(A) \xrightarrow{\mathcal{H}_{n-1}^G(i)} \dots, \end{aligned}$$

where $i: A \rightarrow X$ and $j: X \rightarrow (X, A)$ are the inclusions.

(iii) Excision

Let (X, A) be a G -CW-pair and let $f: A \rightarrow B$ be a cellular G -map of G -CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G -CW-pair. Then the canonical map $(F, f): (X, A) \rightarrow (X \cup_f B, B)$ induces for each $n \in \mathbb{Z}$ an isomorphism

$$\mathcal{H}_n^G(F, f): \mathcal{H}_n^G(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B).$$

(vi) Disjoint union axiom

Let $\{X_i \mid i \in I\}$ be a family of G -CW-complexes. Denote by $j_i: X_i \rightarrow \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\bigoplus_{i \in I} \mathcal{H}_n^G(j_i): \bigoplus_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\left(\coprod_{i \in I} X_i\right)$$

is bijective for each $n \in \mathbb{Z}$.

Of course a G -homology theory for the trivial group $G = \{1\}$ is a homology theory (satisfying the disjoint union axiom) in the classical non-equivariant sense.

The disjoint union axiom ensures that we can pass from finite G -CW-complexes to arbitrary ones using the following lemma.

63

Lemma 63 Let $\mathcal{H}_*^G$ be a G -homology theory. Let X be a G -CW-complex and $\{X_i \mid i \in I\}$ be a directed system of G -CW-subcomplexes directed by inclusion such that $X = \cup_{i \in I} X_i$. Then for all $n \in \mathbb{Z}$ the natural map

$$\operatorname{colim}_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective.

Proof Compare for example with [301, proposition 7.53 on page 121], where the non-equivariant case for $I = \mathbb{N}$ is treated.

Example 64. (*Bredon Homology.*) The most basic G -homology theory is *Bredon homology*. The orbit category $\operatorname{Or}(G)$ has as objects the homogeneous spaces G/H and as morphisms G -maps. Let X be a G -CW-complex. It defines a contravariant functor from the orbit category $\operatorname{Or}(G)$ to the category of CW-complexes by sending G/H to $\operatorname{map}_G(G/H, X) = X^H$. Composing it with the functor cellular chain complex yields a contravariant functor

$$C_*^c(X): \operatorname{Or}(G) \rightarrow \mathbb{Z}\text{-CHCOM}$$

into the category of $\mathbb{Z}$ -chain complexes. Let Λ be a commutative ring and let

$$M: \operatorname{Or}(G) \rightarrow \Lambda\text{-MODULES}$$

be a covariant functor. Then one can form the tensor product over the orbit category (see for instance [197, 9.12 on page 166]) and obtains the Λ -chain complex $C_*^c(X) \otimes_{\mathbb{Z}\mathrm{Or}(G)} M$. Its homology is the Bredon homology of X with coefficients in M

$$H_*^G(X; M) = H_*(C_*^c(X) \otimes_{\mathbb{Z}\mathrm{Or}(G)} M) .$$

Thus we get a G -homology theory H_*^G with values in Λ -modules. For a trivial group G this reduces to the cellular homology of X with coefficients in the Λ -module M .

More information about equivariant homology theories will be given in Sect. 2.7.1.

Varying the Family of Subgroups

2.3.2

Suppose we are given a family of subgroups $\mathcal{F}'$ and a subfamily $\mathcal{F} \subset \mathcal{F}'$. Since all isotropy groups of $E_{\mathcal{F}}(G)$ lie in $\mathcal{F}'$ we know from the universal property of $E_{\mathcal{F}'}(G)$ (compare Sect. 2.3.1) that there is a G -map $E_{\mathcal{F}}(G) \rightarrow E_{\mathcal{F}'}(G)$ which is unique up to G -homotopy. For every G -homology theory $\mathcal{H}_*^G$ we hence obtain a *relative assembly map*

$$A_{\mathcal{F} \rightarrow \mathcal{F}'}: \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(E_{\mathcal{F}'}(G)) .$$

If $\mathcal{F}' = \mathcal{ALL}$, then $E_{\mathcal{F}'}(G) = \mathrm{pt}$ and $A_{\mathcal{F} \rightarrow \mathcal{F}'}$ specializes to the assembly map $A_{\mathcal{F}}$ we discussed in the previous section. If we now gradually increase the family, we obtain qa factorization of the classical assembly $A = A_{\{\} \rightarrow \mathcal{ALL}}$ into several relative assembly maps. We obtain for example from the inclusions

$$\{\} \subset \mathcal{FCY} \subset \mathcal{FIN} \subset \mathcal{VCY} \subset \mathcal{ALL}$$

for every G -homology theory $\mathcal{H}_n^G(-)$ the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{H}_n^G(EG) & \xrightarrow{A} & \mathcal{H}_n^G(\mathrm{pt}) & & \\ \downarrow & & \nearrow A_{\mathcal{FIN}} & & \uparrow A_{\mathcal{VCY}} \\ \mathcal{H}_n^G(E_{\mathcal{FCY}}(G)) & \longrightarrow & \mathcal{H}_n^G(E_{\mathcal{FIN}}(G)) & \longrightarrow & \mathcal{H}_n^G(E_{\mathcal{VCY}}(G)) . \end{array} \quad (2.10)$$

Here A is the “classical” assembly map and $A_{\mathcal{FIN}}$ and $A_{\mathcal{VCY}}$ are the assembly maps that for specific G -homology theories appear in the Baum–Connes and Farrell–Jones Conjectures.

Such a factorization is extremely useful because one can study the relative assembly map $A_{\mathcal{F} \rightarrow \mathcal{F}'}$ in terms of absolute assembly maps corresponding to groups in the bigger family. For example the relative assembly map

$$A_{\mathcal{FIN} \rightarrow \mathcal{VCY}}: \mathcal{H}_n^G(E_{\mathcal{FIN}}(G)) \rightarrow \mathcal{H}_n^G(E_{\mathcal{VCY}}(G))$$

is an isomorphism if for all virtually cyclic subgroups V of G the assembly map

$$A_{\mathcal{F}\mathbb{L}\mathcal{N}} = A_{\mathcal{F}\mathbb{L}\mathcal{N} \rightarrow \mathcal{A}\mathcal{L}\mathcal{L}}: \mathcal{H}_n^V(E_{\mathcal{F}\mathbb{L}\mathcal{N}}(V)) \rightarrow \mathcal{H}_n^V(\text{pt})$$

is an isomorphism. Of course here we need to assume that the G -homology theory $\mathcal{H}_*^G$ and the V -homology theory $\mathcal{H}_*^V$ are somehow related. In fact all the G -homology theories $\mathcal{H}_*^G$ we care about are defined simultaneously for all groups G and for varying G these G -homology theories are related via a so called “induction structure”. Induction structures will be discussed in detail in Sect. 2.7.1.

For a family $\mathcal{F}$ of subgroups of G and a subgroup $H \subset G$ we define a family of subgroups of H

$$\mathcal{F} \cap H = \{K \cap H \mid K \in \mathcal{F}\}.$$

The general statement about relative assembly maps reads now as follows.

65

Theorem 65: (*Transitivity Principle.*) Let $\mathcal{H}_*^{\ell^2}(-)$ be an equivariant homology theory in the sense of Sect. 2.7.1. Suppose $\mathcal{F} \subset \mathcal{F}'$ are two families of subgroups of G . Suppose that $K \cap H \in \mathcal{F}$ for each $K \in \mathcal{F}$ and $H \in \mathcal{F}'$ (this is automatic if $\mathcal{F}$ is closed under taking subgroups). Let N be an integer. If for every $H \in \mathcal{F}'$ and every $n \leq N$ the assembly map

$$A_{\mathcal{F} \cap H \rightarrow \mathcal{A}\mathcal{L}\mathcal{L}}: \mathcal{H}_n^H(E_{\mathcal{F} \cap H}(H)) \rightarrow \mathcal{H}_n^H(\text{pt})$$

is an isomorphism, then for every $n \leq N$ the relative assembly map

$$A_{\mathcal{F} \rightarrow \mathcal{F}'}: \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(E_{\mathcal{F}'}(G))$$

is an isomorphism.

Proof If we equip $E_{\mathcal{F}}(G) \times E_{\mathcal{F}'}(G)$ with the diagonal G -action, it is a model for $E_{\mathcal{F}}(G)$. Now apply Lemma 153 in the special case $Z = E_{\mathcal{F}'}(G)$.

This principle can be used in many ways. For example we will derive from it that the general versions of the Baum–Connes and Farrell–Jones Conjectures specialize to the conjectures we discussed in Chapt. 2.2 in the case where the group is torsion free. If we are willing to make compromises, e.g. to invert 2, to rationalize the theories or to restrict ourselves to small dimensions or special classes of groups, then it is often possible to get away with a smaller family, i.e. to conclude from the Baum–Connes or Farrell–Jones Conjectures that an assembly map with respect to a family smaller than the family of finite or virtually cyclic subgroups is an isomorphism. The left hand side becomes more computable and this leads to new corollaries of the Baum–Connes and Farrell–Jones Conjectures.

The General Versions Specialize to the Torsion Free Versions

If G is a torsion free group, then the family $\mathcal{FLN}$ obviously coincides with the trivial family $\{1\}$. Since a nontrivial torsion free virtually cyclic group is infinite cyclic we also know that the family $\mathcal{VCY}$ reduces to the family of all cyclic subgroups, denoted $\mathcal{CYC}$.

Proposition 66 Let G be a torsion free group.

66

(i) If R is a regular ring, then the relative assembly map

$$A_{\{1\} \rightarrow \mathcal{CYC}} : H_n^G(E_{\{1\}}(G); \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{CYC}}(G); \mathbf{K}_R)$$

is an isomorphism.

(ii) For every ring R the relative assembly map

$$A_{\{1\} \rightarrow \mathcal{CYC}} : H_n^G(E_{\{1\}}(G); \mathbf{L}_R^{(-\infty)}) \rightarrow H_n^G(E_{\mathcal{CYC}}(G); \mathbf{L}_R^{(-\infty)})$$

is an isomorphism.

Proof Because of the Transitivity Principle 65 it suffices in both cases to prove that the classical assembly map $A = A_{\{1\} \rightarrow \mathcal{ALL}}$ is an isomorphism in the case where G is an infinite cyclic group. For regular rings in the K -theory case and with the $-\infty$ -decoration in the L -theory case this is true as we discussed in Remark 15 respectively Remark 26.

As an immediate consequence we obtain.

Corollary 67

67

- (i) For a torsion free group the Baum–Connes Conjecture 59 is equivalent to its “torsion free version” Conjecture 31.
- (ii) For a torsion free group the Farrell–Jones Conjecture 58 for algebraic K - is equivalent to the “torsion free version” Conjecture 11, provided R is regular.
- (iii) For a torsion free group the Farrell–Jones Conjecture 58 for algebraic L -theory is equivalent to the “torsion free version” Conjecture 19.

The Baum–Connes Conjecture and the Family $\mathcal{VCY}$

Replacing the family $\mathcal{FLN}$ of finite subgroups by the family $\mathcal{VCY}$ of virtually cyclic subgroups would not make any difference in the Baum–Connes Conjecture 59. The Transitivity Principle 65 and the fact that the Baum–Connes Conjecture 59 is known for virtually cyclic groups implies the following.

68 **Proposition 68** For every group G and every $n \in \mathbb{Z}$ the relative assembly map for topological K -theory

$$A_{\mathcal{F}\mathcal{L}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}}: H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}^{\text{top}}) \rightarrow H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}^{\text{top}})$$

is an isomorphism.

The Baum–Connes Conjecture and the Family $\mathcal{F}\mathcal{C}\mathcal{Y}$

The following result is proven in [215].

69 **Proposition 69** For every group G and every $n \in \mathbb{Z}$ the relative assembly map for topological K -theory

$$A_{\mathcal{F}\mathcal{C}\mathcal{Y} \rightarrow \mathcal{F}\mathcal{L}\mathcal{N}}: H_n^G(E_{\mathcal{F}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}^{\text{top}}) \rightarrow H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}^{\text{top}})$$

is an isomorphism.

In particular the Baum–Connes Conjecture predicts that the $\mathcal{F}\mathcal{C}\mathcal{Y}$ -assembly map

$$A_{\mathcal{F}\mathcal{C}\mathcal{Y}}: H_n^G(E_{\mathcal{F}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}^{\text{top}}) \rightarrow K_n(C_r^*(G))$$

is always an isomorphism.

Algebraic K -Theory for Special Coefficient Rings

In the algebraic K -theory case we can reduce to the family of finite subgroups if we assume special coefficient rings.

70 **Proposition 70** Suppose R is a regular ring in which the orders of all finite subgroups of G are invertible. Then for every $n \in \mathbb{Z}$ the relative assembly map for algebraic K -theory

$$A_{\mathcal{F}\mathcal{L}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}}: H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}_R)$$

is an isomorphism. In particular if R is a regular ring which is a $\mathbb{Q}$ -algebra (for example a field of characteristic 0) the above applies to all groups G .

Proof We first show that RH is regular for a finite group H . Since R is Noetherian and H is finite, RH is Noetherian. It remains to show that every RH -module M has a finite dimensional projective resolution. By assumption M considered as an R -module has a finite dimensional projective resolution. If one applies $RH \otimes_R$ – this

yields a finite dimensional RH -resolution of $RH \otimes_R \text{res } M$. Since $|H|$ is invertible, the RH -module M is a direct summand of $RH \otimes_R \text{res } M$ and hence has a finite dimensional projective resolution.

Because of the Transitivity Principle 65 we need to prove that the $\mathcal{FLN}$ -assembly map $A_{\mathcal{FLN}}$ is an isomorphism for virtually cyclic groups V . Because of Lemma 71 we can assume that either $V \cong H \rtimes \mathbb{Z}$ or $V \cong K_1 *_H K_2$ with finite groups H , K_1 and K_2 . From [313] we obtain in both cases long exact sequences involving the algebraic K -theory of the constituents, the algebraic K -theory of V and also additional Nil-terms. However, in both cases the Nil-terms vanish if RH is a regular ring (compare theorem 4 on page 138 and the remark on page 216 in [313]). Thus we get long exact sequences

$$\dots \rightarrow K_n(RH) \rightarrow K_n(RH) \rightarrow K_n(RV) \rightarrow K_{n-1}(RH) \rightarrow K_{n-1}(RH) \rightarrow \dots$$

and

$$\begin{aligned} \dots \rightarrow K_n(RH) &\rightarrow K_n(RK_1) \oplus K_n(RK_2) \rightarrow K_n(RV) \\ &\rightarrow K_{n-1}(RH) \rightarrow K_{n-1}(RK_1) \oplus K_{n-1}(RK_2) \rightarrow \dots \end{aligned}$$

One obtains analogous exact sequences for the sources of the various assembly maps from the fact that the sources are equivariant homology theories and one can find specific models for $E_{\mathcal{FLN}}(V)$. These sequences are compatible with the assembly maps. The assembly maps for the finite groups H , K_1 and K_2 are bijective. Now a Five-Lemma argument shows that also the one for V is bijective.

In particular for regular coefficient rings R which are $\mathbb{Q}$ -algebras the K -theoretic Farrell–Jones Conjecture specializes to the conjecture that the assembly map

$$A_{\mathcal{FLN}}: H_n^G(E_{\mathcal{FLN}}(G); \mathbf{K}_R) \rightarrow H_n^G(\text{pt}; \mathbf{K}_R) \cong K_n(RG)$$

is an isomorphism.

In the proof above we used the following important fact about virtually cyclic groups.

Lemma 71 If G is an infinite virtually cyclic group then we have the following dichotomy. 71

- (I) Either G admits a surjection with finite kernel onto the infinite cyclic group $\mathbb{Z}$, or
- (II) G admits a surjection with finite kernel onto the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$.

Proof This is not difficult and proven as lemma 2.5 in [113].

Splitting off Nil-Terms and Rationalized Algebraic K -Theory

Recall that the Nil-terms, which prohibit the classical assembly map from being an isomorphism, are direct summands of the K -theory of the infinite cyclic group (see Remark 15). Something similar remains true in general [16].

72

Proposition 72

- (i) For every group G , every ring R and every $n \in \mathbb{Z}$ the relative assembly map

$$A_{\mathcal{FIN} \rightarrow \mathcal{VCY}} : H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

is split-injective.

- (ii) Suppose R is such that $K_{-i}(RV) = 0$ for all virtually cyclic subgroups V of G and for sufficiently large i (for example $R = \mathbb{Z}$ will do, compare Proposition 78). Then the relative assembly map

$$A_{\mathcal{FIN} \rightarrow \mathcal{VCY}} : H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{(-\infty)}) \rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{(-\infty)})$$

is split-injective.

Combined with the Farrell–Jones Conjectures we obtain that the homology group $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R)$ is a direct summand in $K_n(RG)$. It is much better understood (compare Chapt. 2.9) than the remaining summand which is isomorphic to $H_n^G(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G); \mathbf{K}_R)$. This remaining summand is the one which plays the role of the Nil-terms for a general group. It is known that for $R = \mathbb{Z}$ the negative dimensional Nil-groups which are responsible for virtually cyclic groups vanish [113]. They vanish rationally, in dimension 0 by [76] and in higher dimensions by [182]. For more information see also [75]. Analogously to the proof of Proposition 70 we obtain the following proposition.

73

Proposition 73 We have

$$H_n^G(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) = 0 \quad \text{for } n < 0 \quad \text{and}$$

$$H_n^G(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 \quad \text{for all } n \in \mathbb{Z}.$$

In particular the Farrell–Jones Conjecture for the algebraic K -theory of the integral group ring predicts that the map

$$A_{\mathcal{FIN}} : H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always an isomorphism.

Inverting 2 in L -Theory

Proposition 74 For every group G , every ring R with involution, every decoration j and all $n \in \mathbb{Z}$ the relative assembly map

74

$$A_{\mathcal{FIN} \rightarrow \mathcal{VCY}}: H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{(j)}) \left[\frac{1}{2} \right] \rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{(j)}) \left[\frac{1}{2} \right]$$

is an isomorphism.

Proof According to the Transitivity Principle it suffices to prove the claim for a virtually cyclic group. Now argue analogously to the proof of Proposition 70 using the exact sequences in [48] and the fact that the UNil -terms appearing there vanish after inverting two [48]. Also recall from Remark 22 that after inverting 2 there are no differences between the decorations.

In particular the L -theoretic Farrell–Jones Conjecture implies that for every decoration j the assembly map

$$A_{\mathcal{FIN}}: H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{(j)}) \left[\frac{1}{2} \right] \rightarrow L_n^{(j)}(RG) \left[\frac{1}{2} \right]$$

is an isomorphism.

L -Theory and Virtually Cyclic Subgroups of the First Kind

Recall that a group is virtually cyclic of the first kind if it admits a surjection with finite kernel onto the infinite cyclic group. The family of these groups is denoted $\mathcal{VCY}_I$.

Proposition 75 For all groups G , all rings R and all $n \in \mathbb{Z}$ the relative assembly map

75

$$A_{\mathcal{FIN} \rightarrow \mathcal{VCY}_I}: H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{(-\infty)}) \rightarrow H_n^G(E_{\mathcal{VCY}_I}(G); \mathbf{L}_R^{(-\infty)})$$

is an isomorphism.

Proof The point is that there are no UNil -terms for infinite virtually cyclic groups of the first kind. This follows essentially from [254] and [255] as carried out in [204].

Rationally $\mathcal{FIN}$ Reduces to $\mathcal{FCY}$

We will see later (compare Theorem 172, 173 and 178) that in all three cases, topological K -theory, algebraic K -theory and L -theory, the rationalized left hand side of the $\mathcal{FIN}$ -assembly map can be computed very explicitly using the equivariant Chern-Character. As a by-product these computations yield that after rationalizing the family $\mathcal{FIN}$ can be reduced to the family $\mathcal{FCY}$ of finite cyclic groups and that the rationalized relative assembly maps $A_{\{\}} \rightarrow \mathcal{FCY}$ are injective.

76

Proposition 76 For every ring R , every group G and all $n \in \mathbb{Z}$ the relative assembly maps

$$A_{\mathcal{FCY} \rightarrow \mathcal{FIN}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$A_{\mathcal{FCY} \rightarrow \mathcal{FIN}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{L}_R^{(-\infty)}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{(-\infty)}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$A_{\mathcal{FCY} \rightarrow \mathcal{FIN}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}^{\text{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\text{top}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are isomorphisms and the corresponding relative assembly maps $A_{\{\}} \rightarrow \mathcal{FCY}$ are all rationally injective.

Recall that the statement for topological K -theory is even known integrally, compare Proposition 69. Combining the above with Proposition 73 and Proposition 74 we see that the Farrell–Jones Conjecture predicts in particular that the $\mathcal{FCY}$ -assembly maps

$$A_{\mathcal{FCY}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{L}_R^{(-\infty)}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_n^{(-\infty)}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$A_{\mathcal{FCY}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are always isomorphisms.

2.4 More Applications

2.4.1 Applications VI

Low Dimensional Algebraic K -Theory

As opposed to topological K -theory and L -theory, which are periodic, the algebraic K -theory groups of coefficient rings such as $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$ are known to be bounded below. Using the spectral sequences for the left hand side of an assembly map that will be discussed in Sect. 2.9.4, this leads to vanishing results in negative dimensions and a concrete description of the groups in the first non-vanishing dimension.

The following conjecture is a consequence of the K -theoretic Farrell–Jones Conjecture in the case $R = \mathbb{Z}$. Note that by the results discussed in Proposition 72

we know that in negative dimensions we can reduce to the family of finite subgroups. Explanations about the colimit that appears follow below.

Conjecture 77: (*The Farrell–Jones Conjecture for $K_n(\mathbb{Z}G)$ for $n \leq -1$.*) For every group G we have

77

$$K_{-n}(\mathbb{Z}G) = 0 \quad \text{for } n \geq 2,$$

and the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{F}, N}(G)} K_{-1}(\mathbb{Z}H) \xrightarrow{\cong} K_{-1}(\mathbb{Z}G)$$

is an isomorphism.

We can consider a family $\mathcal{F}$ of subgroups of G as a category $\operatorname{Sub}_{\mathcal{F}}(G)$ as follows. The objects are the subgroups H with $H \in \mathcal{F}$. For $H, K \in \mathcal{F}$ let $\operatorname{conhom}_G(H, K)$ be the set of all group homomorphisms $f: H \rightarrow K$, for which there exists a group element $g \in G$ such that f is given by conjugation with g . The group of inner automorphism $\operatorname{inn}(K)$ consists of those automorphisms $K \rightarrow K$, which are given by conjugation with an element $k \in K$. It acts on $\operatorname{conhom}(H, K)$ from the left by composition. Define the set of morphisms in $\operatorname{Sub}_{\mathcal{F}}(G)$ from H to K to be $\operatorname{inn}(K) \backslash \operatorname{conhom}(H, K)$. Composition of group homomorphisms defines the composition of morphisms in $\operatorname{Sub}_{\mathcal{F}}(G)$. We mention that $\operatorname{Sub}_{\mathcal{F}}(G)$ is a quotient category of the orbit category $\operatorname{Or}_{\mathcal{F}}(G)$ which we will introduce in Sect. 2.7.4. Note that there is a morphism from H to K only if H is conjugate to a subgroup of K . Clearly $K_n(R(-))$ yields a functor from $\operatorname{Sub}_{\mathcal{F}}(G)$ to abelian groups since inner automorphisms on a group G induce the identity on $K_n(RG)$. Using the inclusions into G , one obtains a map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{F}}(G)} K_n(RH) \rightarrow K_n(RG).$$

The colimit can be interpreted as the 0-th Bredon homology group

$$H_0^G(E_{\mathcal{F}}(G); K_n(R(?)))$$

(compare Example 64) and the map is the edge homomorphism in the equivariant Atiyah–Hirzebruch spectral sequence discussed in Sect. 2.9.4. In Conjecture 77 we consider the first non-vanishing entry in the lower left hand corner of the E_2 -term because of the following vanishing result [113, theorem 2.1] which generalizes vanishing results for finite groups from [57].

Proposition 78 If V is a virtually cyclic group, then $K_{-n}(\mathbb{Z}V) = 0$ for $n \geq 2$.

78

If our coefficient ring R is a regular ring in which the orders of all finite subgroups of G are invertible, then we know already from Sect. 2.3.2 that we can reduce to the family of finite subgroups. In the proof of Proposition 70 we have seen that then RH is again regular if $H \subset G$ is finite. Since negative K -groups vanish for

regular rings [268, 5.3.30 on page 295], the following is implied by the Farrell–Jones Conjecture 58.

79 **Conjecture 79:** (*Farrell–Jones Conjecture for $K_0(\mathbb{Q}G)$.*) Suppose R is a regular ring in which the orders of all finite subgroups of G are invertible (for example a field of characteristic 0), then

$$K_{-n}(RG) = 0 \quad \text{for } n \geq 1$$

and the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \xrightarrow{\cong} K_0(RG)$$

is an isomorphism.

The conjecture above holds if G is virtually poly-cyclic. Surjectivity is proven in [227] (see also [67] and chapt. 8 in [235]), injectivity in [271]. We will show in Lemma 87 (i) that the map appearing in the conjecture is always rationally injective for $R = \mathbb{C}$.

The conjectures above describe the first non-vanishing term in the equivariant Atiyah–Hirzebruch spectral sequence. Already the next step is much harder to analyze in general because there are potentially non-vanishing differentials. We know however that after rationalizing the equivariant Atiyah–Hirzebruch spectral sequence for the left hand side of the $\mathcal{FIN}$ -assembly map collapses. As a consequence we obtain that the following conjecture follows from the K -theoretic Farrell–Jones Conjecture 58.

80 **Conjecture 80** For every group G , every ring R and every $n \in \mathbb{Z}$ the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_n(RH) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Note that for $K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ the conjecture above is always true but not very interesting, because for a finite group H it is known that $\tilde{K}_0(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, compare [298, proposition 9.1], and hence the left hand side reduces to $K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. However, the full answer for $K_0(\mathbb{Z}G)$ should involve the negative K -groups, compare Example 176.

Analogously to Conjecture 80 the following can be derived from the K -theoretic Farrell–Jones Conjecture 58, compare [208].

81 **Conjecture 81** The map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} \operatorname{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \operatorname{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always injective.

In general one does not expect this map to be an isomorphism. There should be additional contributions coming from negative K -groups. Conjecture 81 is true for groups satisfying a mild homological finiteness condition, see Theorem 145.

Remark 82: (*The Conjectures as Generalized Induction Theorems.*) The above discussion shows that one may think of the Farrell–Jones Conjectures 58 and the Baum–Connes Conjecture 59 as “generalized induction theorems”. The prototype of an induction theorem is Artin’s theorem about the complex representation ring $R_{\mathbb{C}}(G)$ of a finite group G . Let us recall Artin’s theorem.

For finite groups H the complex representation ring $R_{\mathbb{C}}(H)$ coincides with $K_0(\mathbb{C}H)$. Artin’s Theorem [283, theorem 17 in 9.2 on page 70] implies that the obvious induction homomorphism

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\text{cyc}}(G)} R_{\mathbb{C}}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Note that this is a very special case of Theorem 172 or 173, compare Remark 177.

Artin’s theorem says that rationally one can compute $R_{\mathbb{C}}(G)$ if one knows all the values $R_{\mathbb{C}}(C)$ (including all maps coming from induction with group homomorphisms induced by conjugation with elements in G) for all cyclic subgroups $C \subseteq G$. The idea behind the Farrell–Jones Conjectures 58 and the Baum–Connes Conjecture 59 is analogous. We want to compute the functors $K_n(RG)$, $L_n(RG)$ and $K_n(C_r^*(G))$ from their values (including their functorial properties under induction) on elements of the family $\mathcal{FIN}$ or $\mathcal{VCY}$.

The situation in the Farrell–Jones and Baum–Connes Conjectures is more complicated than in Artin’s Theorem, since we have already seen in Remarks 15, 26 and 35 that a computation of $K_n(RG)$, $L_n^{(-\infty)}(RG)$ and $K_n(C_r^*(G))$ does involve also the values $K_p(RH)$, $L_p^{(-\infty)}(RH)$ and $K_p(C_r^*(H))$ for $p \leq n$. A degree mixing occurs.

G-Theory

Instead of considering finitely generated projective modules one may apply the standard K -theory machinery to the category of finitely generated modules. This leads to the definition of the groups $G_n(R)$ for $n \geq 0$. For instance $G_0(R)$ is the abelian group whose generators are isomorphism classes $[M]$ of finitely generated R -modules and whose relations are given by $[M_0] - [M_1] + [M_2]$ for any exact sequence $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ of finitely generated modules. One may ask whether versions of the Farrell–Jones Conjectures for G -theory instead of K -theory might be true. The answer is negative as the following discussion explains.

For a finite group H the ring $\mathbb{C}H$ is semisimple. Hence any finitely generated $\mathbb{C}H$ -module is automatically projective and $K_0(\mathbb{C}H) = G_0(\mathbb{C}H)$. Recall that a group G is called *virtually poly-cyclic* if there exists a subgroup of finite index $H \subseteq G$ together with a filtration $\{1\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots \subseteq H_r = H$ such that

H_{i-1} is normal in H_i and the quotient H_i/H_{i-1} is cyclic. More generally for all $n \in \mathbb{Z}$ the forgetful map

$$f: K_n(\mathbb{C}G) \rightarrow G_n(\mathbb{C}G)$$

is an isomorphism if G is virtually poly-cyclic, since then $\mathbb{C}G$ is regular [273, theorem 8.2.2 and theorem 8.2.20] and the forgetful map f is an isomorphism for regular rings, compare [268, corollary 53.26 on page 293]. In particular this applies to virtually cyclic groups and so the left hand side of the Farrell–Jones assembly map does not see the difference between K - and G -theory if we work with complex coefficients. We obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_{H \in \operatorname{Sub} \mathcal{F}_{\mathcal{L}, \mathcal{N}}(G)} K_0(\mathbb{C}H) & \longrightarrow & K_0(\mathbb{C}G) \\ \cong \downarrow & & \downarrow f \\ \operatorname{colim}_{H \in \operatorname{Sub} \mathcal{F}_{\mathcal{L}, \mathcal{N}}(G)} G_0(\mathbb{C}H) & \longrightarrow & G_0(\mathbb{C}G) \end{array} \quad (2.11)$$

where, as indicated, the left hand vertical map is an isomorphism. Conjecture 79, which is implied by the Farrell–Jones Conjecture, says that the upper horizontal arrow is an isomorphism. A G -theoretic analogue of the Farrell–Jones Conjecture would say that the lower horizontal map is an isomorphism. There are however cases where the upper horizontal arrow is known to be an isomorphism, but the forgetful map f on the right is not injective or not surjective:

If G contains a non-abelian free subgroup, then the class $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ vanishes [202, theorem 9.66 on page 364] and hence the map $f: K_0(\mathbb{C}G) \rightarrow G_0(\mathbb{C}G)$ has an infinite kernel ($[\mathbb{C}G]$ generates an infinite cyclic subgroup in $K_0(\mathbb{C}G)$). The Farrell–Jones Conjecture for $K_0(\mathbb{C}G)$ is known for non-abelian free groups.

The Farrell–Jones Conjecture is also known for $A = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2$ and hence $K_0(\mathbb{C}A)$ is countable, whereas $G_0(\mathbb{C}A)$ is not countable [202, example 10.13 on page 375]. Hence the map f cannot be surjective.

At the time of writing we do not know a counterexample to the statement that for an amenable group G , for which there is an upper bound on the orders of its finite subgroups, the forgetful map $f: K_0(\mathbb{C}G) \rightarrow G_0(\mathbb{C}G)$ is an isomorphism. We do not know a counterexample to the statement that for a group G , which is not amenable, $G_0(\mathbb{C}G) = \{0\}$. We also do not know whether $G_0(\mathbb{C}G) = \{0\}$ is true for $G = \mathbb{Z} * \mathbb{Z}$.

For more information about $G_0(\mathbb{C}G)$ we refer for instance to [202, sect. 9.5.3].

Bass Conjectures

Complex representations of a finite group can be studied using characters. We now want to define the Hattori–Stallings rank of a finitely generated projective $\mathbb{C}G$ -module which should be seen as a generalization of characters to infinite groups.

Let $\text{con}(G)$ be the set of conjugacy classes (g) of elements $g \in G$. Denote by $\text{con}(G)_f$ the subset of $\text{con}(G)$ consisting of those conjugacy classes (g) for which each representative g has finite order. Let $\text{class}_0(G)$ and $\text{class}_0(G)_f$ be the $\mathbb{C}$ -vector space with the set $\text{con}(G)$ and $\text{con}(G)_f$ as basis. This is the same as the $\mathbb{C}$ -vector space of $\mathbb{C}$ -valued functions on $\text{con}(G)$ and $\text{con}(G)_f$ with finite support. Define the *universal $\mathbb{C}$ -trace* as

$$\text{tr}_{\mathbb{C}G}^u : \mathbb{C}G \rightarrow \text{class}_0(G), \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda_g \cdot (g). \quad (2.12)$$

It extends to a function $\text{tr}_{\mathbb{C}G}^u : M_n(\mathbb{C}G) \rightarrow \text{class}_0(G)$ on (n, n) -matrices over $\mathbb{C}G$ by taking the sum of the traces of the diagonal entries. Let P be a finitely generated projective $\mathbb{C}G$ -module. Choose a matrix $A \in M_n(\mathbb{C}G)$ such that $A^2 = A$ and the image of the $\mathbb{C}G$ -map $r_A : \mathbb{C}G^n \rightarrow \mathbb{C}G^n$ given by right multiplication with A is $\mathbb{C}G$ -isomorphic to P . Define the *Hattori–Stallings rank* of P as

$$\text{HS}_{\mathbb{C}G}(P) = \text{tr}_{\mathbb{C}G}^u(A) \in \text{class}_0(G). \quad (2.13)$$

The Hattori–Stallings rank depends only on the isomorphism class of the $\mathbb{C}G$ -module P and induces a homomorphism $\text{HS}_{\mathbb{C}G} : K_0(\mathbb{C}G) \rightarrow \text{class}_0(G)$.

Conjecture 83: (*Strong Bass Conjecture for $K_0(\mathbb{C}G)$.*)

83

The $\mathbb{C}$ -vector space spanned by the image of the map

$$\text{HS}_{\mathbb{C}G} : K_0(\mathbb{C}G) \rightarrow \text{class}_0(G)$$

is $\text{class}_0(G)_f$.

This conjecture is implied by the surjectivity of the map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C}, \quad (2.14)$$

(compare Conjecture 79) and hence by the K -theoretic Farrell–Jones Conjecture for $K_0(\mathbb{C}G)$. We will see below that the surjectivity of the map (2.14) also implies that the map $K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \text{class}_0(G)$, which is induced by the Hattori–Stallings rank, is injective. Hence we do expect that the Hattori–Stallings rank induces an isomorphism

$$K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \text{class}_0(G)_f.$$

There are also versions of the Bass conjecture for other coefficients than $\mathbb{C}$. It follows from results of Linnell [191, theorem 4.1 on page 96] that the following version is implied by the Strong Bass Conjecture for $K_0(\mathbb{C}G)$.

Conjecture 84: (*The Strong Bass Conjecture for $K_0(\mathbb{Z}G)$.*) The image of the composition

84

$$K_0(\mathbb{Z}G) \rightarrow K_0(\mathbb{C}G) \xrightarrow{\text{HS}_{\mathbb{C}G}} \text{class}_0(G)$$

is contained in the $\mathbb{C}$ -vector space of those functions $f: \text{con}(G) \rightarrow \mathbb{C}$ which vanish for $(g) \in \text{con}(g)$ with $g \neq 1$.

The conjecture says that for every finitely generated projective $\mathbb{Z}G$ -module P the Hattori–Stallings rank of $\mathbb{C}G \otimes_{\mathbb{Z}G} P$ looks like the Hattori–Stallings rank of a free $\mathbb{C}G$ -module. A natural explanation for this behaviour is the following conjecture which clearly implies the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$.

85 **Conjecture 85:** (*Rational $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ -Conjecture.*) For every group G the map

$$\tilde{K}_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \tilde{K}_0(\mathbb{Q}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced by the change of coefficients is trivial.

Finally we mention the following variant of the Bass Conjecture.

86 **Conjecture 86:** (*The Weak Bass Conjecture.*) Let P be a finitely generated projective $\mathbb{Z}G$ -module. The value of the Hattori–Stallings rank of $\mathbb{C}G \otimes_{\mathbb{Z}G} P$ at the conjugacy class of the identity element is given by

$$\text{HS}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{Z}G} P)((1)) = \dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P).$$

Here $\mathbb{Z}$ is considered as a $\mathbb{Z}G$ -module via the augmentation.

The K -theoretic Farrell–Jones Conjecture implies all four conjectures above. More precisely we have the following proposition.

87 **Proposition 87**
(i) The map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{F}, \mathcal{N}}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always injective. If the map is also surjective (compare Conjecture 79) then the Hattori–Stallings rank induces an isomorphism

$$K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \text{class}_0(G)_f$$

and in particular the Strong Bass Conjecture for $K_0(\mathbb{C}G)$ and hence also the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ hold.

(ii) The surjectivity of the map

$$A_{\text{Vcy}}: H_0^G(E_{\text{Vcy}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Rational $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ Conjecture and hence also the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$.

- (iii) The Strong Bass Conjecture for $K_0(\mathbb{C}G)$ implies the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$. The Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ implies the Weak Bass Conjecture.

Proof (i) follows from the following commutative diagram, compare [198, lemma 2.15 on page 220].

$$\begin{array}{ccccc}
 \operatorname{colim}_{H \in \operatorname{Sub} \mathcal{FIN}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\quad\quad\quad} & K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} & & \\
 \cong \downarrow & & \downarrow & & \\
 \operatorname{colim}_{H \in \operatorname{Sub} \mathcal{FIN}(G)} \operatorname{class}_0(H) & \xrightarrow{\quad\quad\quad} & \operatorname{class}_0(G)_f & \xrightarrow{\quad i \quad} & \operatorname{class}_0(G) .
 \end{array}$$

Here the vertical maps are induced by the Hattori–Stallings rank, the map i is the natural inclusion and in particular injective and we have the indicated isomorphisms.

(ii) According to Proposition 73 the surjectivity of the map $A_{\mathcal{V}\mathcal{C}\mathcal{Y}}$ appearing in 87 implies the surjectivity of the corresponding assembly map $A_{\mathcal{FIN}}$ (rationalized and with $\mathbb{Z}$ as coefficient ring) for the family of finite subgroups. The map $A_{\mathcal{FIN}}$ is natural with respect to the change of the coefficient ring from $\mathbb{Z}$ to $\mathbb{Q}$. By Theorem 173 we know that for every coefficient ring R there is an isomorphism from

$$\bigoplus_{p,q,p+q=0} \bigoplus_{(C) \in (\mathcal{FCY})} H_p(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C \cdot K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$$

to the 0-dimensional part of the left hand side of the rationalized $\mathcal{FIN}$ -assembly map $A_{\mathcal{FIN}}$. The isomorphism is natural with respect to a change of coefficient rings. To see that the Rational $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ Conjecture follows, it hence suffices to show that the summand corresponding to $C = \{1\}$ and $p = q = 0$ is the only one where the map induced from $\mathbb{Z} \rightarrow \mathbb{Q}$ is possibly non-trivial. But $K_q(\mathbb{Q}C) = 0$ if $q < 0$, because $\mathbb{Q}C$ is semisimple and hence regular, and for a finite cyclic group $C \neq \{1\}$ we have by [198, lemma 7.4]

$$\Theta_C \cdot K_0(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} = \operatorname{coker} \left(\bigoplus_{D \subsetneq C} K_0(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} \right) = 0 ,$$

since by a result of Swan $K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism for a finite group H , see [298, proposition 9.1].

(iii) As already mentioned the first statement follows from [191, theorem 4.1 on page 96]. The second statement follows from the formula

$$\sum_{(g) \in \operatorname{con}(G)} \operatorname{HS}_{\mathbb{C}G}(\mathbb{C} \otimes_{\mathbb{Z}} P)(g) = \dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P) .$$

The next result is due to Berrick, Chatterji and Mislin [36, theorem 5.2]. The Bost Conjecture is a variant of the Baum–Connes Conjecture and is explained in Sect. 2.5.1.

88 **Theorem 88** If the assembly map appearing in the Bost Conjecture 104 is rationally surjective, then the Strong Bass Conjecture for $K_0(\mathbb{C}G)$ (compare 83) is true.

We now discuss some further questions and facts that seem to be relevant in the context of the Bass Conjectures.

89 **Remark 89:** (*Integral $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ -Conjecture.*) We do not know a counterexample to the Integral $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ Conjecture, i.e. to the statement that the map

$$\tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathbb{Q}G)$$

itself is trivial. But we also do not know a proof which shows that the K -theoretic Farrell–Jones Conjecture implies this integral version. Note that the Integral $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ Conjecture would imply that the following diagram commutes.

$$\begin{array}{ccc} K_0(\mathbb{Z}G) & \longrightarrow & K_0(\mathbb{Q}G) \\ p_* \downarrow & \text{dim}_{\mathbb{Z}} & \uparrow i \\ K_0(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \end{array}$$

Here p_* is induced by the projection $G \rightarrow \{1\}$ and i sends $1 \in \mathbb{Z}$ to the class of $\mathbb{Q}G$.

90 **Remark 90:** (*The passage from $\tilde{K}_0(\mathbb{Z}G)$ to $\tilde{K}_0(\mathcal{N}(G))$.*) Let $\mathcal{N}(G)$ denote the group von Neumann algebra of G . It is known that for every group G the composition

$$\tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathbb{Q}G) \rightarrow \tilde{K}_0(\mathbb{C}G) \rightarrow \tilde{K}_0(C_r^*(G)) \rightarrow \tilde{K}_0(\mathcal{N}(G))$$

is the zero-map (see for instance [202, theorem 9.62 on page 362]). Since the group von Neumann algebra $\mathcal{N}(G)$ is not functorial under arbitrary group homomorphisms such as $G \rightarrow \{1\}$, this does *not* imply that the diagram

$$\begin{array}{ccc} K_0(\mathbb{Z}G) & \longrightarrow & K_0(\mathcal{N}(G)) \\ p_* \downarrow & \text{dim}_{\mathbb{Z}} & \uparrow i \\ K_0(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \end{array}$$

commutes. However, commutativity would follow from the Weak Bass Conjecture 86. For a discussion of these questions see [93].

More information and further references about the Bass Conjecture can be found for instance in [24], [36, sect. 7], [42, 92, 93, 118, 191] [202, sect. 9.5.2], and [225, pp. 66ff].

Applications VII

2.4.2

Novikov Conjectures

In Sect. 2.2.9 we discussed the Novikov Conjectures. Recall that one possible reformulation of the original Novikov Conjecture says that for every group G the rationalized classical assembly map in L -theory

$$A: H_n(BG; \mathbf{L}^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_n^p(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective. Since A can be identified with $A_{\{\emptyset\} \rightarrow \mathcal{ALN}}$ and we know from Sect. 2.3.2 that the relative assembly map

$$A_{\{\emptyset\} \rightarrow \mathcal{FLN}}: H_n^G(E_{\{\emptyset\}}(G); \mathbf{L}_{\mathbb{Z}}^p) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_n^G(E_{\mathcal{FLN}}(G); \mathbf{L}_{\mathbb{Z}}^p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective we obtain the following proposition.

Proposition 91 The rational injectivity of the assembly map appearing in L -theoretic Farrell–Jones Conjecture (Conjecture 58) implies the L -theoretic Novikov Conjecture (Conjecture 51) and hence the original Novikov Conjecture 52.

91

Similarly the Baum–Connes Conjecture 59 implies the injectivity of the rationalized classical assembly map

$$A: H_n(BG; \mathbf{K}^{\text{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In the next subsection we discuss how one can relate assembly maps for topological K -theory with L -theoretic assembly maps. The results imply in particular the following proposition.

Proposition 92 The rational injectivity of the assembly map appearing in the Baum–Connes Conjecture (Conjecture 59) implies the Novikov Conjecture (Conjecture 52).

92

Finally we would like to mention that by combining the results about the rationalization of $A_{\{\emptyset\} \rightarrow \mathcal{FLN}}$ from Sect. 2.3.2 with the splitting result about $A_{\mathcal{FLN} \rightarrow \mathcal{VCY}}$ from Proposition 72 we obtain the following result

93 **Proposition 93** The rational injectivity of the assembly map appearing in the Farrell–Jones Conjecture for algebraic K -theory (Conjecture 58) implies the K -theoretic Novikov Conjecture, i.e. the injectivity of

$$A: H_n(BG; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

94 **Remark 94:** (*Integral Injectivity Fails.*) In general the classical assembly maps $A = A_{\square}$ themselves, i.e. without rationalizing, are not injective. For example one can use the Atiyah–Hirzebruch spectral sequence to see that for $G = \mathbb{Z}/5$

$$H_1(BG; \mathbf{K}^{\text{top}}) \quad \text{and} \quad H_1(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z}))$$

contain 5-torsion, whereas for every finite group G the topological K -theory of $\mathbb{C}G$ is torsionfree and the torsion in the L -theory of $\mathbb{Z}G$ is always 2-torsion, compare Proposition 169 (i) and Proposition 171 (i).

Relating Topological K -Theory and L -Theory

For every real C^* -algebra A there is an isomorphism $L_n^p(A)[1/2] \xrightarrow{\cong} K_n(A)[1/2]$ [269]. This can be used to compare L -theory to topological K -theory and leads to the following result.

95 **Proposition 95** Let $\mathcal{F} \subseteq \mathcal{FIN}$ be a family of finite subgroups of G . If the topological K -theory assembly map

$$A_{\mathcal{F}}: H_n^G(E_{\mathcal{F}}(G); \mathbf{K}^{\text{top}}) \left[\frac{1}{2} \right] \rightarrow K_n(C_r^*(G)) \left[\frac{1}{2} \right]$$

is injective, then for an arbitrary decoration j also the map

$$A_{\mathcal{F}}: H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{\mathbb{Z}}^{(j)}) \left[\frac{1}{2} \right] \rightarrow L_n^{(j)}(\mathbb{Z}G) \left[\frac{1}{2} \right]$$

is injective.

Proof First recall from Remark 22 that after inverting 2 there is no difference between the different decorations and we can hence work with the p -decoration. One can construct for any subfamily $\mathcal{F} \subseteq \mathcal{FIN}$ the following commutative diagram [200, sect. 7.5]

$$\begin{array}{ccc}
 H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{\mathbb{Z}}^p[1/2]) & \xrightarrow{A_{\mathcal{F}}^1} & L_n^p(\mathbb{Z}G)[1/2] \\
 i_1 \downarrow \cong & & j_1 \downarrow \cong \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{\mathbb{Q}}^p[1/2]) & \xrightarrow{A_{\mathcal{F}}^2} & L_n^p(\mathbb{Q}G)[1/2] \\
 i_2 \downarrow \cong & & j_2 \downarrow \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{\mathbb{R}}^p[1/2]) & \xrightarrow{A_{\mathcal{F}}^3} & L_n^p(\mathbb{R}G)[1/2] \\
 i_3 \downarrow \cong & & j_3 \downarrow \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{C_r^*(?; \mathbb{R})}^p[1/2]) & \xrightarrow{A_{\mathcal{F}}^4} & L_n^p(C_r^*(G; \mathbb{R}))[1/2] \\
 i_4 \downarrow \cong & & j_4 \downarrow \cong \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{R}}^{\text{top}}[1/2]) & \xrightarrow{A_{\mathcal{F}}^5} & K_n(C_r^*(G; \mathbb{R}))[1/2] \\
 i_5 \downarrow & & j_5 \downarrow \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{C}}^{\text{top}}[1/2]) & \xrightarrow{A_{\mathcal{F}}^6} & K_n(C_r^*(G))[1/2]
 \end{array}$$

Here

$$\mathbf{L}_{\mathbb{Z}}^p[1/2], \quad \mathbf{L}_{\mathbb{Q}}^p[1/2], \quad \mathbf{L}_{\mathbb{R}}^p[1/2], \quad \mathbf{L}_{C_r^*(?; \mathbb{R})}^p[1/2], \\
 \mathbf{K}_{\mathbb{R}}^{\text{top}}[1/2] \quad \text{and} \quad \mathbf{K}_{\mathbb{C}}^{\text{top}}[1/2]$$

are covariant $\text{Or}(G)$ -spectra (compare Sect. 2.7.2 and in particular Proposition 156) such that the n -th homotopy group of their evaluations at G/H are given by

$$\begin{aligned}
 & L_n^p(\mathbb{Z}H)[1/2], \quad L_n^p(\mathbb{Q}H)[1/2], \quad L_n^p(\mathbb{R}H)[1/2], \quad L_n^p(C_r^*(H; \mathbb{R}))[1/2], \\
 & K_n(C_r^*(H; \mathbb{R}))[1/2] \quad \text{respectively} \quad K_n(C_r^*(H))[1/2].
 \end{aligned}$$

All horizontal maps are assembly maps induced by the projection $\text{pr}: E_{\mathcal{F}}(G) \rightarrow \text{pt}$. The maps i_k and j_k for $k = 1, 2, 3$ are induced from a change of rings. The isomorphisms i_4 and j_4 come from the general isomorphism for any real C^* -algebra A

$$L_n^p(A)[1/2] \xrightarrow{\cong} K_n(A)[1/2]$$

and its spectrum version [269, theorem 1.11 on page 350]. The maps i_1, j_1, i_2 are isomorphisms by [256, page 376] and [258, proposition 22.34 on page 252]. The map i_3 is bijective since for a finite group H we have $\mathbb{R}H = C_r^*(H; \mathbb{R})$. The maps i_5

and j_5 are given by extending the scalars from $\mathbb{R}$ to $\mathbb{C}$ by induction. For every real C^* -algebra A the composition

$$K_n(A)[1/2] \rightarrow K_n(A \otimes_{\mathbb{R}} \mathbb{C})[1/2] \rightarrow K_n(M_2(A))[1/2]$$

is an isomorphism and hence j_5 is split injective. An $\mathrm{Or}(G)$ -spectrum version of this argument yields that also i_5 is split injective.

96 Remark 96 One may conjecture that the right vertical maps j_2 and j_3 are isomorphisms and try to prove this directly. Then if we invert 2 everywhere the Baum–Connes Conjecture 59 for the real reduced group C^* -algebra, would be equivalent to the Farrell–Jones Isomorphism Conjecture for $L_*(\mathbb{Z}G)[1/2]$.

2.4.3 Applications VIII

The Modified Trace Conjecture

Denote by Λ^G the subring of $\mathbb{Q}$ which is obtained from $\mathbb{Z}$ by inverting all orders $|H|$ of finite subgroups H of G , i.e.

$$\Lambda^G = \mathbb{Z} \left[|H|^{-1} \mid H \subset G, |H| < \infty \right]. \quad (2.15)$$

The following conjecture generalizes Conjecture 37 to the case where the group need no longer be torsionfree. For the standard trace compare (2.7).

97 Conjecture 97: (*Modified Trace Conjecture for a group G .*) Let G be a group. Then the image of the homomorphism induced by the standard trace

$$\mathrm{tr}_{C_r^*(G)}: K_0(C_r^*(G)) \rightarrow \mathbb{R} \quad (2.16)$$

is contained in Λ^G .

The following result is proved in [203, theorem 0.3].

98 Theorem 98 Let G be a group. Then the image of the composition

$$K_0^G(E_{\mathcal{FIN}}(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{A_{\mathcal{FIN}} \otimes_{\mathbb{Z}} \mathrm{id}} K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\mathrm{tr}_{C_r^*(G)}} \mathbb{R}$$

is Λ^G . Here $A_{\mathcal{FIN}}$ is the map appearing in the Baum–Connes Conjecture 59. In particular the Baum–Connes Conjecture 59 implies the Modified Trace Conjecture.

The original version of the Trace Conjecture due to Baum and Connes [27, page 21] makes the stronger statement that the image of $\mathrm{tr}_{C_r^*(G)}: K_0(C_r^*(G)) \rightarrow \mathbb{R}$ is the additive subgroup of $\mathbb{Q}$ generated by all numbers $\frac{1}{|H|}$, where $H \subset G$ runs through

all finite subgroups of G . Roy has constructed a counterexample to this version in [274] based on her article [275]. The examples of Roy do *not* contradict the Modified Trace Conjecture 97 or the Baum–Connes Conjecture 59.

The Stable Gromov–Lawson–Rosenberg Conjecture

The Stable Gromov–Lawson–Rosenberg Conjecture is a typical conjecture relating Riemannian geometry to topology. It is concerned with the question when a given manifold admits a metric of positive scalar curvature. To discuss its relation with the Baum–Connes Conjecture we will need the real version of the Baum–Connes Conjecture, compare Sect. 2.5.1.

Let $\Omega_n^{\text{Spin}}(BG)$ be the bordism group of closed Spin-manifolds M of dimension n with a reference map to BG . Let $C_r^*(G; \mathbb{R})$ be the real reduced group C^* -algebra and let $KO_n(C_r^*(G; \mathbb{R})) = K_n(C_r^*(G; \mathbb{R}))$ be its topological K -theory. We use KO instead of K as a reminder that we here use the real reduced group C^* -algebra. Given an element $[u: M \rightarrow BG] \in \Omega_n^{\text{Spin}}(BG)$, we can take the $C_r^*(G; \mathbb{R})$ -valued index of the equivariant Dirac operator associated to the G -covering $\overline{M} \rightarrow M$ determined by u . Thus we get a homomorphism

$$\text{ind}_{C_r^*(G; \mathbb{R})}: \Omega_n^{\text{Spin}}(BG) \rightarrow KO_n(C_r^*(G; \mathbb{R})). \quad (2.17)$$

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$ -genus $\widehat{A}(B)$ is 8. We fix such a choice, the particular choice does not matter for the sequel. Notice that $\text{ind}_{C_r^*(\{1\}; \mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G; \mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G; \mathbb{R}))$. In particular

$$\text{ind}_{C_r^*(G; \mathbb{R})}(M) = \text{ind}_{C_r^*(G; \mathbb{R})}(M \times B), \quad (2.18)$$

if we identify $KO_n(C_r^*(G; \mathbb{R})) = KO_{n+8}(C_r^*(G; \mathbb{R}))$ via Bott periodicity.

Conjecture 99: (*Stable Gromov–Lawson–Rosenberg Conjecture.*) Let M be a closed connected Spin-manifold of dimension $n \geq 5$. Let $u_M: M \rightarrow B\pi_1(M)$ be the classifying map of its universal covering. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if

$$\text{ind}_{C_r^*(\pi_1(M); \mathbb{R})}([M, u_M]) = 0 \quad \in KO_n(C_r^*(\pi_1(M); \mathbb{R})).$$

If M carries a Riemannian metric with positive scalar curvature, then the index of the Dirac operator must vanish by the Bochner–Lichnerowicz formula [267]. The converse statement that the vanishing of the index implies the existence of a Riemannian metric with positive scalar curvature is the hard part of the conjecture. The following result is due to Stolz. A sketch of the proof can be found in [297, sect. 3], details are announced to appear in a different paper.

Theorem 100 If the assembly map for the real version of the Baum–Connes Conjecture (compare Sect. 2.5.1) is injective for the group G , then the Stable Gromov–Lawson–Rosenberg Conjecture 99 is true for all closed Spin-manifolds of dimension ≥ 5 with $\pi_1(M) \cong G$.

The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov–Lawson–Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg–Witten invariants, occur. The unstable version of this conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\text{ind}_{C_r^*(\pi_1(M); \mathbb{R})}([M, u_M]) = 0$. Schick [278] constructs counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau (see also [90]). It is not known at the time of writing whether the unstable version is true for finite fundamental groups. Since the Baum–Connes Conjecture 59 is true for finite groups (for the trivial reason that $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G) = \text{pt}$ for finite groups G), Theorem 100 implies that the Stable Gromov–Lawson Conjecture 99 holds for finite fundamental groups (see also [270]).

The index map appearing in (2.17) can be factorized as a composition

$$\text{ind}_{C_r^*(G; \mathbb{R})} : \Omega_n^{\text{Spin}}(BG) \xrightarrow{D} KO_n(BG) \xrightarrow{A} KO_n(C_r^*(G; \mathbb{R})) , \quad (2.19)$$

where D sends $[M, u]$ to the class of the G -equivariant Dirac operator of the G -manifold $\overline{M}$ given by u and $A = A_{|\cdot|}$ is the real version of the classical assembly map. The homological Chern character defines an isomorphism

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \in \mathbb{Z}} H_{n+4p}(BG; \mathbb{Q}) .$$

Recall that associated to M there is the $\widehat{A}$ -class

$$\widehat{\mathcal{A}}(M) \in \prod_{p \geq 0} H^p(M; \mathbb{Q}) \quad (2.20)$$

which is a certain polynomial in the Pontrjagin classes. The map D appearing in (2.19) sends the class of $u: M \rightarrow BG$ to $u_*(\widehat{\mathcal{A}}(M) \cap [M])$, i.e. the image of the Poincaré dual of $\widehat{\mathcal{A}}(M)$ under the map induced by u in rational homology. Hence $D([M, u]) = 0$ if and only if $u_*(\widehat{\mathcal{A}}(M) \cap [M])$ vanishes. For $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ define the *higher $\widehat{A}$ -genus of (M, u) associated to x* to be

$$\widehat{A}_x(M, u) = \langle \widehat{\mathcal{A}}(M) \cup u^*x, [M] \rangle = \langle x, u_*(\widehat{\mathcal{A}}(M) \cap [M]) \rangle \in \mathbb{Q} . \quad (2.21)$$

The vanishing of $\widehat{\mathcal{A}}(M)$ is equivalent to the vanishing of all higher $\widehat{A}$ -genera $\widehat{A}_x(M, u)$. The following conjecture is a weak version of the Stable Gromov–Lawson–Rosenberg Conjecture.

Conjecture 101: (*Homological Gromov–Lawson–Rosenberg Conjecture.*) Let G be a group. Then for any closed Spin-manifold M , which admits a Riemannian metric

with positive scalar curvature, the $\widehat{A}$ -genus $\widehat{A}_x(M, u)$ vanishes for all maps $u: M \rightarrow BG$ and elements $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$.

From the discussion above we obtain the following result.

Proposition 102 If the assembly map

102

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow KO_n(C_r^*(G; \mathbb{R})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective for all $n \in \mathbb{Z}$, then the Homological Gromov–Lawson–Rosenberg Conjecture holds for G .

Generalizations and Related Conjectures

2.5

Variants of the Baum–Connes Conjecture

2.5.1

The Real Version

There is an obvious real version of the Baum–Connes Conjecture, which predicts that for all $n \in \mathbb{Z}$ and groups G the assembly map

$$A_{\mathcal{F}, \mathcal{L}, \mathcal{N}}^{\mathbb{R}}: H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{R}}^{\text{top}}) \rightarrow KO_n(C_r^*(G; \mathbb{R}))$$

is an isomorphism. Here $H_n^G(-; \mathbf{K}_{\mathbb{R}}^{\text{top}})$ is an equivariant homology theory whose distinctive feature is that $H_n^G(G/H; \mathbf{K}_{\mathbb{R}}^{\text{top}}) \cong KO_n(C_r^*(H; \mathbb{R}))$. Recall that we write $KO_n(-)$ only to remind ourselves that the C^* -algebra we apply it to is a real C^* -algebra, like for example the real reduced group C^* -algebra $C_r^*(G; \mathbb{R})$. The following result appears in [31].

Proposition 103 The Baum–Connes Conjecture 59 implies the real version of the Baum–Connes Conjecture.

103

In the proof of Proposition 95 we have already seen that after inverting 2 the “real assembly map” is a retract of the complex assembly map. In particular with 2-inverted or after rationalizing also injectivity results or surjectivity results about the complex Baum–Connes assembly map yield the corresponding results for the real Baum–Connes assembly map.

The Version for Maximal Group C^* -Algebras

For a group G let $C_{\max}^*(G)$ be its *maximal group C^* -algebra*, compare [242, 7.1.5 on page 229]. The maximal group C^* -algebra has the advantage that every homomorphism of groups $\phi: G \rightarrow H$ induces a homomorphism $C_{\max}^*(G) \rightarrow C_{\max}^*(H)$ of C^* -algebras. This is not true for the reduced group C^* -algebra $C_r^*(G)$. Here

is a counterexample: since $C_r^*(F)$ is a simple algebra if F is a non-abelian free group [245], there is no unital algebra homomorphism $C_r^*(F) \rightarrow C_r^*(\{1\}) = \mathbb{C}$.

One can construct a version of the Baum–Connes assembly map using an equivariant homology theory $H_n^G(-; \mathbf{K}_{\max}^{\text{top}})$ which evaluated on G/H yields the K -theory of $C_{\max}^*(H)$ (use Proposition 156 and a suitable modification of $\mathbf{K}^{\text{top}}$, compare Sect. 2.7.3).

Since on the left hand side of a $\mathcal{FIN}$ -assembly map only the maximal group C^* -algebras for finite groups H matter, and clearly $C_{\max}^*(H) = \mathbb{C}H = C_r^*(H)$ for such H , this left hand side coincides with the left hand side of the usual Baum–Connes Conjecture. There is always a C^* -homomorphism $p: C_{\max}^*(G) \rightarrow C_r^*(G)$ (it is an isomorphism if and only if G is amenable [242, theorem 7.3.9 on page 243]) and hence we obtain the following factorization of the usual Baum–Connes assembly map

$$\begin{array}{ccc}
 & & K_n(C_{\max}^*(G)) \\
 & \nearrow A_{\mathcal{FIN}}^{\max} & \downarrow K_n(p) \\
 H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\text{top}}) & \xrightarrow{A_{\mathcal{FIN}}} & K_n(C_r^*(G)) .
 \end{array} \quad (2.22)$$

It is known that the map $A_{\mathcal{FIN}}^{\max}$ is in general not surjective. The Baum–Connes Conjecture would imply that the map is $A_{\mathcal{FIN}}^{\max}$ is always injective, and that it is surjective if and only if the vertical map $K_n(p)$ is injective.

A countable group G is called *K-amenable* if the map $p: C_{\max}^*(G) \rightarrow C_r^*(G)$ induces a KK -equivalence (compare [78]). This implies in particular that the vertical map $K_n(p)$ is an isomorphism for all $n \in \mathbb{Z}$. Note that for K -amenable groups the Baum–Connes Conjecture holds if and only if the “maximal” version of the assembly map $A_{\mathcal{FIN}}^{\max}$ is an isomorphism for all $n \in \mathbb{Z}$. A-T-menable groups are K -amenable, compare Theorem 120. But $K_0(p)$ is not injective for every infinite group which has property (T) such as for example $SL_n(\mathbb{Z})$ for $n \geq 3$, compare for instance the discussion in [163]. There are groups with property (T) for which the Baum–Connes Conjecture is known (compare Sect. 2.6.1) and hence there are counterexamples to the conjecture that $A_{\mathcal{FIN}}^{\max}$ is an isomorphism.

In Theorem 48 and Remark 49 we discussed applications of the maximal C^* -algebra version of the Baum–Connes Conjecture.

The Bost Conjecture

Some of the strongest results about the Baum–Connes Conjecture are proven using the so called Bost Conjecture (see [186]). The Bost Conjecture is the version of the Baum–Connes Conjecture, where one replaces the reduced group C^* -algebra $C_r^*(G)$ by the Banach algebra $l^1(G)$ of absolutely summable functions on G . Again one can use the spectra approach (compare Sects. 2.7.2 and 2.7.3 and in particular Proposition 156) to produce a variant of equivariant K -homology denoted

$H_n^G(-; \mathbf{K}_l^{\text{top}})$ which this time evaluated on G/H yields $K_n(l^1(H))$, the topological K -theory of the Banach algebra $l^1(H)$.

As explained in the beginning of Chapt. 2.3, we obtain an associated assembly map and we believe that it coincides with the one defined using a Banach-algebra version of KK -theory in [186].

Conjecture 104: (*Bost Conjecture.*) Let G be a countable group. Then the assembly map

104

$$A_{\mathcal{F}\mathcal{L}\mathcal{N}}^{l^1} : H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_l^{\text{top}}) \rightarrow K_n(l^1(G))$$

is an isomorphism.

Again the left hand side coincides with the left hand side of the Baum–Connes assembly map because for finite groups H we have $l^1(H) = \mathbb{C}H = C_r^*(H)$. There is always a homomorphism of Banach algebras $q: l^1(G) \rightarrow C_r^*(G)$ and one obtains a factorization of the usual Baum–Connes assembly map

$$\begin{array}{ccc} & & K_n(l^1(G)) \\ & \nearrow A_{\mathcal{F}\mathcal{L}\mathcal{N}}^{l^1} & \downarrow K_n(q) \\ H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}^{\text{top}}) & \xrightarrow{A_{\mathcal{F}\mathcal{L}\mathcal{N}}} & K_n(C_r^*(G)) . \end{array}$$

Every group homomorphism $G \rightarrow H$ induces a homomorphism of Banach algebras $l^1(G) \rightarrow l^1(H)$. So similar as in the maximal group C^* -algebra case this approach repairs the lack of functoriality for the reduced group C^* -algebra.

The disadvantage of $l^1(G)$ is however that indices of operators tend to take values in the topological K -theory of the group C^* -algebras, not in $K_n(l^1(G))$. Moreover the representation theory of G is closely related to the group C^* -algebra, whereas the relation to $l^1(G)$ is not well understood.

For more information about the Bost Conjecture 104 see [186, 288].

The Baum–Connes Conjecture with Coefficients

The Baum–Connes Conjecture 59 can be generalized to the Baum–Connes Conjecture with Coefficients. Let A be a separable C^* -algebra with an action of the countable group G . Then there is an assembly map

$$KK_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); A) \rightarrow K_n(A \rtimes G) \quad (2.23)$$

defined in terms of equivariant KK -theory, compare Sects. 2.8.3 and 2.8.4.

Conjecture 105: (*Baum–Connes Conjecture with Coefficients.*) For every separable C^* -algebra A with an action of a countable group G and every $n \in \mathbb{Z}$ the assembly map (2.23) is an isomorphism.

105

There are counterexamples to the Baum–Connes Conjecture with Coefficients, compare Remark 122. If we take $A = \mathbb{C}$ with the trivial action, the map (2.23) can be identified with the assembly map appearing in the ordinary Baum–Connes Conjecture 59.

106

Remark 106: (*A Spectrum Level Description.*) There is a formulation of the Baum–Connes Conjecture with Coefficients in the framework explained in Sect. 2.7.2. Namely, construct an appropriate covariant functor $\mathbf{K}^{\text{top}}(A \rtimes \mathcal{G}^G(-)) : \text{Or}(G) \rightarrow \text{SPECTRA}$ such that

$$\pi_n(\mathbf{K}^{\text{top}}(A \rtimes \mathcal{G}^G(G/H))) \cong K_n(A \rtimes H)$$

holds for all subgroups $H \subseteq G$ and all $n \in \mathbb{Z}$, and consider the associated G -homology theory $H_*^G(-; \mathbf{K}^{\text{top}}(A \rtimes \mathcal{G}^G(-)))$. Then the map (2.23) can be identified with the map which the projection $\text{pr} : E_{\mathcal{F}, \mathbb{1}, \mathcal{N}}(G) \rightarrow \text{pt}$ induces for this homology theory.

107

Remark 107: (*Farrell–Jones Conjectures with Coefficients.*) One can also formulate a “Farrell–Jones Conjecture with Coefficients”. (This should not be confused with the Fibered Farrell–Jones Conjecture discussed in Sect. 2.5.2.) Fix a ring S and an action of G on it by isomorphisms of rings. Construct an appropriate covariant functor $\mathbf{K}(S \rtimes \mathcal{G}^G(-)) : \text{Or}(G) \rightarrow \text{SPECTRA}$ such that

$$\pi_n(\mathbf{K}(S \rtimes \mathcal{G}^G(G/H))) \cong K_n(S \rtimes H)$$

holds for all subgroups $H \subseteq G$ and $n \in \mathbb{Z}$, where $S \rtimes H$ is the associated twisted group ring. Now consider the associated G -homology theory $H_*^G(-; \mathbf{K}(S \rtimes \mathcal{G}^G(-)))$. There is an analogous construction for L -theory. A *Farrell–Jones Conjecture with Coefficients* would say that the map induced on these homology theories by the projection $\text{pr} : E_{\text{VCY}}(G) \rightarrow \text{pt}$ is always an isomorphism. We do not know whether there are counterexamples to the Farrell–Jones Conjectures with Coefficients, compare Remark 122.

The Coarse Baum Connes Conjecture

We briefly explain the Coarse Baum–Connes Conjecture, a variant of the Baum–Connes Conjecture, which applies to metric spaces. Its importance lies in the fact that isomorphism results about the Coarse Baum–Connes Conjecture can be used to prove injectivity results about the classical assembly map for topological K -theory. Compare also Sect. 2.8.10.

Let X be a proper (closed balls are compact) metric space and H_X a separable Hilbert space with a faithful nondegenerate $*$ -representation of $C_0(X)$, the algebra of complex valued continuous functions which vanish at infinity. A bounded linear

operator T has a support $\text{supp } T \subset X \times X$, which is defined as the complement of the set of all pairs (x, x') , for which there exist functions ϕ and $\phi' \in C_0(X)$ such that $\phi(x) \neq 0$, $\phi'(x') \neq 0$ and $\phi'T\phi = 0$. The operator T is said to be a finite propagation operator if there exists a constant α such that $d(x, x') \leq \alpha$ for all pairs in the support of T . The operator is said to be *locally compact* if ϕT and $T\phi$ are compact for every $\phi \in C_0(X)$. An operator is called *pseudolocal* if $\phi T \psi$ is a compact operator for all pairs of continuous functions ϕ and ψ with compact and disjoint supports.

The Roe-algebra $C^*(X) = C(X, H_X)$ is the operator-norm closure of the $*$ -algebra of all locally compact finite propagation operators on H_X . The algebra $D^*(X) = D^*(X, H_X)$ is the operator-norm closure of the pseudolocal finite propagation operators. One can show that the topological K -theory of the quotient algebra $D^*(X)/C^*(X)$ coincides up to an index shift with the analytically defined (non-equivariant) K -homology $K_*(X)$, compare Sect. 2.8.1. For a uniformly contractible proper metric space the coarse assembly map $K_n(X) \rightarrow K_n(C^*(X))$ is the boundary map in the long exact sequence associated to the short exact sequence of C^* -algebras

$$0 \rightarrow C^*(X) \rightarrow D^*(X) \rightarrow D^*(X)/C^*(X) \rightarrow 0.$$

For general metric spaces one first approximates the metric space by spaces with nice local behaviour, compare [263]. For simplicity we only explain the case, where X is a discrete metric space. Let $P_d(X)$ the Rips complex for a fixed distance d , i.e. the simplicial complex with vertex set X , where a simplex is spanned by every collection of points in which every two points are a distance less than d apart. Equip $P_d(X)$ with the spherical metric, compare [335].

A discrete metric space has *bounded geometry* if for each $r > 0$ there exists a $N(r)$ such that for all x the ball of radius r centered at $x \in X$ contains at most $N(r)$ elements.

Conjecture 108: (*Coarse Baum–Connes Conjecture.*) Let X be a proper discrete metric space of bounded geometry. Then for $n = 0, 1$ the coarse assembly map

108

$$\text{colim}_d K_n(P_d(X)) \rightarrow \text{colim}_d K_n(C^*(P_d(X))) \cong K_n(C^*(X))$$

is an isomorphism.

The conjecture is false if one drops the bounded geometry hypothesis. A counterexample can be found in [336, sect. 8]. Our interest in the conjecture stems from the following fact, compare [263, chapt. 8].

Proposition 109 Suppose the finitely generated group G admits a classifying space BG of finite type. If G considered as a metric space via a word length metric satisfies the Coarse Baum–Connes Conjecture 108 then the classical assembly map $A: K_*(BG) \rightarrow K_*(C_r^*G)$ which appears in Conjecture 31 is injective.

109

The Coarse Baum–Connes Conjecture for a discrete group G (considered as a metric space) can be interpreted as a case of the Baum–Connes Conjecture with Coefficients 105 for the group G with a certain specific choice of coefficients, compare [339].

Further information about the coarse Baum–Connes Conjecture can be found for instance in [151, 152, 154, 263, 334, 335, 337, 340], and [338].

The Baum–Connes Conjecture for Non-Discrete Groups

Throughout this subsection let T be a locally compact second countable topological Hausdorff group. There is a notion of a classifying space for proper T -actions $\underline{ET}$ (see [28, sects. 1 and 2] [304, sect. I.6], [207, sect. 1]) and one can define its equivariant topological K -theory $K_n^T(\underline{ET})$. The definition of a reduced C^* -algebra $C_r^*(T)$ and its topological K -theory $K_n(C_r^*(T))$ makes sense also for T . There is an assembly map defined in terms of equivariant index theory

$$A_{\mathcal{K}}: K_n^T(\underline{ET}) \rightarrow K_n(C_r^*(T)). \quad (2.24)$$

The Baum–Connes Conjecture for T says that this map is bijective for all $n \in \mathbb{Z}$ [28, conjecture 3.15 on page 254].

Now consider the special case where T is a connected Lie group. Let $\mathcal{K}$ be the family of compact subgroups of T . There is a notion of a T -CW-complex and of a classifying space $E_{\mathcal{K}}(T)$ defined as in Sects. 2.3.1 and 2.3.1. The classifying space $E_{\mathcal{K}}(T)$ yields a model for $\underline{ET}$. Let $K \subset T$ be a maximal compact subgroup. It is unique up to conjugation. The space T/K is contractible and in fact a model for $\underline{ET}$ (see [1, appendix, theorem A.5], [2, corollary 4.14], [207, sect. 1]). One knows (see [28, proposition 4.22], [170])

$$K_n^T(\underline{ET}) = K_n^T(T/K) = \begin{cases} R_{\mathbb{C}}(K) & n = \dim(T/K) \pmod{2}, \\ 0 & n = 1 + \dim(T/K) \pmod{2}, \end{cases}$$

where $R_{\mathbb{C}}(K)$ is the complex representation ring of K .

Next we consider the special case where T is a totally disconnected group. Let $\mathcal{KO}$ be the family of compact-open subgroups of T . A T -CW-complex and a classifying space $E_{\mathcal{KO}}(T)$ for T and $\mathcal{KO}$ are defined similar as in Sect. 2.3.1 and 2.3.1. Then $E_{\mathcal{KO}}(T)$ is a model for $\underline{ET}$ since any compact subgroup is contained in a compact-open subgroup, and the Baum–Connes Conjecture says that the assembly map yields for $n \in \mathbb{Z}$ an isomorphism

$$A_{\mathcal{KO}}: K_n^T(E_{\mathcal{KO}}(T)) \rightarrow K_n(C_r^*(T)). \quad (2.25)$$

For more information see [30].

2.5.2 Variants of the Farrell–Jones Conjecture

Pseudoisotopy Theory

An important variant of the Farrell–Jones Conjecture deals with the pseudoisotopy spectrum functor $\mathbf{P}$, which we already discussed briefly in Sect. 2.2.4. In fact it is

this variant of the Farrell–Jones Conjecture (and its fibered version which will be explained in the next subsection) for which the strongest results are known at the time of writing.

In Proposition 157 we will explain that every functor $E: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$, which sends equivalences of groupoids to stable weak equivalences of spectra, yields a corresponding equivariant homology theory $H_n^G(-; E)$. Now whenever we have a functor $F: \text{SPACES} \rightarrow \text{SPECTRA}$, we can precompose it with the functor “classifying space” which sends a groupoid $\mathcal{G}$ to its classifying space $B\mathcal{G}$. (Here $B\mathcal{G}$ is simply the realization of the nerve of $\mathcal{G}$ considered as a category.) In particular this applies to the pseudoisotopy functor P . Thus we obtain a homology theory $H_n^G(-; P \circ B)$ whose essential feature is that

$$H_n^G(G/H; P \circ B) \cong \pi_n(P(BH)) ,$$

i.e. evaluated at G/H one obtains the homotopy groups of the pseudoisotopy spectrum of the classifying space BH of the group H . As the reader may guess there is the following conjecture.

Conjecture 110: (*Farrell–Jones Conjecture for Pseudoisotopies of Aspherical Spaces.*) For every group G and all $n \in \mathbb{Z}$ the assembly map

110

$$H_n^G(E_{\text{VCY}}(G); P \circ B) \rightarrow H_n^G(\text{pt}; P \circ B) \cong \pi_n(P(BG))$$

is an isomorphism. Similarly for P^{diff} , the pseudoisotopy functor which is defined using differentiable pseudoisotopies.

A formulation of a conjecture for spaces which are not necessarily aspherical will be given in the next subsection, see in particular Remark 115.

Remark 111: (*Relating K -theory and Pseudoisotopy Theory.*) We already outlined in Sect. 2.2.4 the relationship between K -theory and pseudoisotopies. The comparison in positive dimensions described there can be extended to all dimensions. Vogell constructs in [309] a version of A -theory using retractive spaces that are bounded over $\mathbb{R}^k$ (compare Sects. 2.2.2 and 2.2.4). This leads to a functor $A^{-\infty}$ from spaces to non-connective spectra. Compare also [56, 310, 311] and [326]. We define $\mathbf{Wh}_{PL}^{-\infty}$ via the fibration sequence

111

$$X_+ \wedge A^{-\infty}(\text{pt}) \rightarrow A^{-\infty}(X) \rightarrow \mathbf{Wh}_{PL}^{-\infty}(X) ,$$

where the first map is the assembly map. The natural equivalence

$$\Omega^2 \mathbf{Wh}_{PL}^{-\infty}(X) \simeq P(X)$$

seems to be hard to trace down in the literature but should be true. We will assume it in the following discussion.

Precompose the functors above with the classifying space functor B to obtain functors from groupoids to spectra. The pseudoisotopy assembly map which ap-

pears in Conjecture 110 is an isomorphism if and only if the A -theory assembly map

$$H_{n+2}^G(\mathrm{Ev}_{\mathcal{C}\mathcal{Y}}(G); \mathbf{A}^{-\infty} \circ B) \rightarrow H_{n+2}^G(\mathrm{pt}; \mathbf{A}^{-\infty} \circ B) \cong \pi_{n+2}(\mathbf{A}^{-\infty}(BG))$$

is an isomorphism. This uses a 5-lemma argument and the fact that for a fixed spectrum $\mathbf{E}$ the assembly map

$$H_n^G(E_{\mathcal{F}}(G); B\mathcal{G}^G(-)_+ \wedge \mathbf{E}) \rightarrow H_n^G(\mathrm{pt}; B\mathcal{G}^G(-)_+ \wedge \mathbf{E})$$

is always bijective. There is a linearization map $\mathbf{A}^{-\infty}(X) \rightarrow \mathbf{K}(\mathbb{Z}\Pi(X)_{\oplus})$ (see the next subsection for the notation) which is always 2-connected and a rational equivalence if X is aspherical (recall that $\mathbf{K}$ denotes the non-connective K -theory spectrum). For finer statements about the linearization map, compare also [230].

The above discussion yields in particular the following, compare [111, 1.6.7 on page 261].

112

Proposition 112 The rational version of the K -theoretic Farrell–Jones Conjecture 58 is equivalent to the rational version of the Farrell–Jones Conjecture for Pseudoisotopies of Aspherical Spaces 110. If the assembly map in the conjecture for pseudoisotopies is (integrally) an isomorphism for $n \leq -1$, then so is the assembly map in the K -theoretic Farrell–Jones Conjecture for $n \leq 1$.

Fibered Versions

Next we present the more general fibered versions of the Farrell–Jones Conjectures. These fibered versions have better inheritance properties, compare Sect. 2.6.4.

In the previous section we considered functors $\mathbf{F}: \mathrm{SPACES} \rightarrow \mathrm{SPECTRA}$, like $\mathbf{P}$, $\mathbf{P}^{\mathrm{diff}}$ and $\mathbf{A}^{-\infty}$, and the associated equivariant homology theories $H_n^G(-; \mathbf{F} \circ B)$ (compare Proposition 157). Here B denotes the classifying space functor, which sends a groupoid $\mathcal{G}$ to its classifying space $B\mathcal{G}$. In fact all equivariant homology theories we considered so far can be obtained in this fashion for special choices of $\mathbf{F}$. Namely, let $\mathbf{F}$ be one of the functors

$$\mathbf{K}(R\Pi(-)_{\oplus}), \quad \mathbf{L}^{(-\infty)}(R\Pi(-)_{\oplus}) \quad \text{or} \quad \mathbf{K}^{\mathrm{top}}(C_r^*\Pi(-)_{\oplus}),$$

where $\Pi(X)$ denotes the fundamental groupoid of a space, $R\mathcal{G}_{\oplus}$ respectively $C_r^*\mathcal{G}_{\oplus}$ is the R -linear respectively the C^* -category associated to a groupoid $\mathcal{G}$ and $\mathbf{K}$, $\mathbf{L}^{(-\infty)}$ and $\mathbf{K}^{\mathrm{top}}$ are suitable functors which send additive respectively C^* -categories to spectra, compare the proof of Theorem 158. There is a natural equivalence $\mathcal{G} \rightarrow \Pi B\mathcal{G}$. Hence, if we precompose the functors above with the classifying space functor B we obtain functors which are equivalent to the functors we have so far been calling

$$\mathbf{K}_R, \quad \mathbf{L}_R^{(-\infty)} \quad \text{and} \quad \mathbf{K}^{\mathrm{top}},$$

compare Theorem 158. Note that in contrast to these three cases the pseudoisotopy functor $\mathbf{P}$ depends on more than just the fundamental groupoid. However Conjecture 110 above only deals with aspherical spaces.

Given a G -CW-complex Z and a functor $\mathbf{F}$ from spaces to spectra we obtain a functor $X \mapsto \mathbf{F}(Z \times_G X)$ which digests G -CW-complexes. In particular we can restrict it to the orbit category to obtain a functor

$$\mathbf{F}(Z \times_G -): \text{Or}(G) \rightarrow \text{SPECTRA}.$$

According to Proposition 156 we obtain a corresponding G -homology theory

$$H_n^G(-; \mathbf{F}(Z \times_G -))$$

and associated assembly maps. Note that restricted to the orbit category the functor $EG \times_G -$ is equivalent to the classifying space functor B and so $H_n^G(-; \mathbf{F} \circ B)$ can be considered as a special case of this construction.

Conjecture 113: (*Fibered Farrell–Jones Conjectures.*) Let R be a ring (with involution). Let $\mathbf{F}: \text{SPACES} \rightarrow \text{SPECTRA}$ be one of the functors

113

$$\mathbf{K}(R\Pi(-)_{\oplus}), \quad \mathbf{L}^{(-\infty)}(R\Pi(-)_{\oplus}), \quad \mathbf{P}(-), \quad \mathbf{P}^{\text{diff}}(-) \quad \text{or} \quad \mathbf{A}^{-\infty}(-).$$

Then for every free G -CW-complex Z and all $n \in \mathbb{Z}$ the associated assembly map

$$H_n^G(E_{\mathcal{VCY}}(G); \mathbf{F}(Z \times_G -)) \rightarrow H_n^G(\text{pt}; \mathbf{F}(Z \times_G -)) \cong \pi_n(\mathbf{F}(Z/G))$$

is an isomorphism.

Remark 114: (*A Fibered Baum–Connes Conjecture.*) With the family $\mathcal{FIN}$ instead of $\mathcal{VCY}$ and the functor $\mathbf{F} = \mathbf{K}^{\text{top}}(C_r^*\Pi(-)_{\oplus})$ one obtains a *Fibered Baum–Connes Conjecture*.

114

Remark 115: (*The Special Case $Z = \tilde{X}$.*) Suppose $Z = \tilde{X}$ is the universal covering of a space X equipped with the action of its fundamental group $G = \pi_1(X)$. Then in the algebraic K - and L -theory case the conjecture above specializes to the “ordinary” Farrell–Jones Conjecture 58. In the pseudoisotopy and A -theory case one obtains a formulation of an (unfibered) conjecture about $\pi_n(\mathbf{P}(X))$ or $\pi_n(\mathbf{A}^{-\infty}(X))$ for spaces X which are not necessarily aspherical.

115

Remark 116: (*Relation to the Original Formulation.*) In [111] Farrell and Jones formulate a fibered version of their conjectures for every (Serre) fibration $Y \rightarrow X$ over a connected CW-complex X . In our set-up this corresponds to choosing Z to be the total space of the fibration obtained from $Y \rightarrow X$ by pulling back along the

116

universal covering projection $\tilde{X} \rightarrow X$. This space is a free G -space for $G = \pi_1(X)$. Note that an arbitrary free G -CW-complex Z can always be obtained in this fashion from a map $Z/G \rightarrow BG$, compare [111, corollary 2.2.1 on page 264].

117

Remark 117: (*Relating K -theory and Pseudoisotopy Theory in the Fibered Case.*) The linearization map $\pi_n(\mathbf{A}^{-\infty}(X)) \rightarrow K_n(\mathbb{Z}\Pi(X))$ is always 2-connected, but for spaces which are not aspherical it need not be a rational equivalence. Hence the comparison results discussed in Remark 111 apply for the fibered versions only in dimensions $n \leq 1$.

The Isomorphism Conjecture for NK -Groups

In Remark 15 we defined the groups $NK_n(R)$ for a ring R . They are the simplest kind of Nil-groups responsible for the infinite cyclic group. Since the functor $\mathbf{K}_R$ is natural with respect to ring homomorphism we can define $\mathbf{NK}_R$ as the (objectwise) homotopy cofiber of $\mathbf{K}_R \rightarrow \mathbf{K}_{R[t]}$. There is an associated assembly map.

118

Conjecture 118: (*Isomorphism Conjecture for NK -groups.*) The assembly map

$$H_n^G(E_{\text{Vec}}(G); \mathbf{NK}_R) \rightarrow H_n^G(\text{pt}; \mathbf{NK}_R) \cong NK_n(RG)$$

is always an isomorphism.

There is a weak equivalence $\mathbf{K}_{R[t]} \simeq \mathbf{K}_R \vee \mathbf{NK}_R$ of functors from GROUPOIDS to SPECTRA . This implies for a fixed family $\mathcal{F}$ of subgroups of G and $n \in \mathbb{Z}$ that whenever two of the three assembly maps

$$A_{\mathcal{F}} : H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{R[t]}) \rightarrow K_n(R[t][G]) ,$$

$$A_{\mathcal{F}} : H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R) \rightarrow K_n(R[G]) ,$$

$$A_{\mathcal{F}} : H_n^G(E_{\mathcal{F}}(G); \mathbf{NK}_R) \rightarrow NK_n(RG)$$

are bijective, then so is the third (compare [19, sect. 7]). Similarly one can define a functor $\mathbf{E}_R$ from the category GROUPOIDS to SPECTRA and weak equivalences

$$\mathbf{K}_{R[t, t^{-1}]} \rightarrow \mathbf{E}_R \leftarrow \mathbf{K}_R \vee \Sigma \mathbf{K}_R \vee \mathbf{NK}_R \vee \mathbf{NK}_R ,$$

which on homotopy groups corresponds to the Bass–Heller–Swan decomposition (see Remark 15). One obtains a two-out-of-three statement as above with the $\mathbf{K}_{R[t]}$ -assembly map replaced by the $\mathbf{K}_{R[t, t^{-1}]}$ -assembly map.

Algebraic K -Theory of the Hecke Algebra

In Sect. 2.5.1 we mentioned the classifying space $E_{\mathcal{K}\mathcal{O}}(G)$ for the family of compact-open subgroups and the Baum–Connes Conjecture for a totally disconnected

group T . There is an analogous conjecture dealing with the algebraic K -theory of the Hecke algebra.

Let $\mathcal{H}(T)$ denote the *Hecke algebra* of T which consists of locally constant functions $G \rightarrow \mathbb{C}$ with compact support and inherits its multiplicative structure from the convolution product. The Hecke algebra $\mathcal{H}(T)$ plays the same role for T as the complex group ring $\mathbb{C}G$ for a discrete group G and reduces to this notion if T happens to be discrete. There is a T -homology theory $\mathcal{H}_*^T$ with the property that for any open and closed subgroup $H \subseteq T$ and all $n \in \mathbb{Z}$ we have $\mathcal{H}_n^T(T/H) = K_n(\mathcal{H}(H))$, where $K_n(\mathcal{H}(H))$ is the algebraic K -group of the Hecke algebra $\mathcal{H}(H)$.

Conjecture 119: (*Isomorphism Conjecture for the Hecke-Algebra.*) For a totally disconnected group T the assembly map

119

$$A_{\mathcal{K}\mathcal{O}}: \mathcal{H}_n^T(E_{\mathcal{K}\mathcal{O}}(T)) \rightarrow \mathcal{H}^T(\text{pt}) = K_n(\mathcal{H}(T)) \quad (2.26)$$

induced by the projection $\text{pr}: E_{\mathcal{K}\mathcal{O}}(T) \rightarrow \text{pt}$ is an isomorphism for all $n \in \mathbb{Z}$.

In the case $n = 0$ this reduces to the statement that

$$\text{colim}_{T/H \in \text{Or}_{\mathcal{K}\mathcal{O}}(T)} K_0(\mathcal{H}(H)) \rightarrow K_0(\mathcal{H}(T)). \quad (2.27)$$

is an isomorphism. For $n \leq -1$ one obtains the statement that $K_n(\mathcal{H}(G)) = 0$. The group $K_0(\mathcal{H}(T))$ has an interpretation in terms of the smooth representations of T . The G -homology theory can be constructed using an appropriate functor $E: \text{Or}_{\mathcal{K}\mathcal{O}}(T) \rightarrow \text{SPECTRA}$ and the recipe explained in Sect. 2.7.2. The desired functor E is given in [276].

Status of the Conjectures

2.6

In this section, we give the status, at the time of writing, of some of the conjectures mentioned earlier. We begin with the Baum–Connes Conjecture.

Status of the Baum–Connes-Conjecture

2.6.1

The Baum–Connes Conjecture with Coefficients

We begin with the Baum–Connes Conjecture with Coefficients 105. It has better inheritance properties than the Baum–Connes Conjecture 59 itself and contains it as a special case.

Theorem 120: (*Baum–Connes Conjecture with Coefficients and a - T -menable Groups.*) The discrete group G satisfies the Baum–Connes Conjecture with Coefficients 105 and is K -amenable provided that G is a - T -menable.

120

This theorem is proved in Higson–Kasparov [149, theorem 1.1], where more generally second countable locally compact topological groups are treated (see also [164]).

A group G is *a-T-menable*, or, equivalently, has the *Haagerup property* if G admits a metrically proper isometric action on some affine Hilbert space. *Metrically proper* means that for any bounded subset B the set $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite. An extensive treatment of such groups is presented in [66]. Any a-T-menable group is countable. The class of a-T-menable groups is closed under taking subgroups, under extensions with finite quotients and under finite products. It is not closed under semi-direct products. Examples of a-T-menable groups are countable amenable groups, countable free groups, discrete subgroups of $SO(n, 1)$ and $SU(n, 1)$, Coxeter groups, countable groups acting properly on trees, products of trees, or simply connected $CAT(0)$ cubical complexes. A group G has Kazhdan's *property (T)* if, whenever it acts isometrically on some affine Hilbert space, it has a fixed point. An infinite a-T-menable group does not have property (T). Since $SL(n, \mathbb{Z})$ for $n \geq 3$ has property (T), it cannot be a-T-menable.

Using the Higson–Kasparov result Theorem 120 and known inheritance properties of the Baum–Connes Conjecture with Coefficients (compare Sect. 2.6.4 and [233, 234]) Mislin describes an even larger class of groups for which the conjecture is known [225, theorem 5.23].

121

Theorem 121: (*The Baum–Connes Conjecture with Coefficients and the Class of Groups $LH\mathcal{ETH}$.*) The discrete group G satisfies the Baum–Connes Conjecture with Coefficients 105 provided that G belongs to the class $LH\mathcal{ETH}$.

The class $LH\mathcal{ETH}$ is defined as follows. Let $H\mathcal{TH}$ be the smallest class of groups which contains all a-T-menable groups and contains a group G if there is a 1-dimensional contractible G -CW-complex whose stabilizers belong already to $H\mathcal{TH}$. Let $H\mathcal{ETH}$ be the smallest class of groups containing $H\mathcal{TH}$ and containing a group G if either G is countable and admits a surjective map $p: G \rightarrow Q$ with Q and $p^{-1}(F)$ in $H\mathcal{ETH}$ for every finite subgroup $F \subseteq Q$ or if G admits a 1-dimensional contractible G -CW-complex whose stabilizers belong already to $H\mathcal{ETH}$. Let $LH\mathcal{ETH}$ be the class of groups G whose finitely generated subgroups belong to $H\mathcal{ETH}$.

The class $LH\mathcal{ETH}$ is closed under passing to subgroups, under extensions with torsion free quotients and under finite products. It contains in particular one-relator groups and Haken 3-manifold groups (and hence all knot groups). All these facts of the class $LH\mathcal{ETH}$ and more information can be found in [225].

Vincent Lafforgue has an unpublished proof of the Baum–Connes Conjecture with Coefficients 105 for word-hyperbolic groups.

122

Remark 122 There are counterexamples to the Baum–Connes Conjecture with (commutative) Coefficients 105 as soon as the existence of finitely generated groups containing arbitrary large expanders in their Cayley graph is shown [150, sect. 7]. The existence of such groups has been claimed by Gromov [138, 139]. Details of

the construction are described by Ghys in [134]. At the time of writing no counterexample to the Baum–Connes Conjecture 59 (without coefficients) is known to the authors.

The Baum–Connes Conjecture

Next we deal with the Baum–Connes Conjecture 59 itself. Recall that all groups which satisfy the Baum–Connes Conjecture with Coefficients 105 do in particular satisfy the Baum–Connes Conjecture 59.

Theorem 123: (*Status of the Baum–Connes Conjecture.*) A group G satisfies the Baum–Connes Conjecture 59 if it satisfies one of the following conditions.

123

- (i) It is a discrete subgroup of a connected Lie groups L , whose Levi–Malcev decomposition $L = RS$ into the radical R and semisimple part S is such that S is locally of the form

$$S = K \times SO(n_1, 1) \times \dots \times SO(n_k, 1) \times SU(m_1, 1) \times \dots \times SU(m_l, 1)$$

for a compact group K .

- (ii) The group G has property (RD) and admits a proper isometric action on a strongly bolic weakly geodesic uniformly locally finite metric space.
- (iii) G is a subgroup of a word hyperbolic group.
- (vi) G is a discrete subgroup of $Sp(n, 1)$.

Proof The proof under condition 123 is due to Julg–Kasparov [166]. The proof under condition 123 is due to Lafforgue [183] (see also [288]). Word hyperbolic groups have property (RD) [84]. Any subgroup of a word hyperbolic group satisfies the conditions appearing in the result of Lafforgue and hence satisfies the Baum–Connes Conjecture 59 [222, theorem 20]. The proof under condition 123 is due to Julg [165].

Lafforgue’s result about groups satisfying condition 123 yielded the first examples of infinite groups which have Kazhdan’s property (T) and satisfy the Baum–Connes Conjecture 59. Here are some explanations about condition 123.

A *length function* on G is a function $L: G \rightarrow \mathbb{R}_{\geq 0}$ such that $L(1) = 0$, $L(g) = L(g^{-1})$ for $g \in G$ and $L(g_1 g_2) \leq L(g_1) + L(g_2)$ for $g_1, g_2 \in G$ holds. The word length metric L_S associated to a finite set S of generators is an example. A length function L on G has *property (RD)* (“rapid decay”) if there exist $C, s > 0$ such that for any $u = \sum_{g \in G} \lambda_g \cdot g \in \mathbb{C}G$ we have

$$\|\varphi_G(u)\|_{\infty} \leq C \cdot \left(\sum_{g \in G} |\lambda_g|^2 \cdot (1 + L(g))^{2s} \right)^{1/2},$$

where $\|\varrho_G(u)\|_\infty$ is the operator norm of the bounded G -equivariant operator $l^2(G) \rightarrow l^2(G)$ coming from right multiplication with u . A group G has *property (RD)* if there is a length function which has property (RD). More information about property (RD) can be found for instance in [63, 184] and [307, chapt. 8]. Bolicity generalizes Gromov's notion of hyperbolicity for metric spaces. We refer to [169] for a precise definition.

124 Remark 124 We do not know whether all groups appearing in Theorem 123 satisfy also the Baum–Connes Conjecture with Coefficients 105.

125 Remark 125 It is not known at the time of writing whether the Baum–Connes Conjecture is true for $SL(n, \mathbb{Z})$ for $n \geq 3$.

126 Remark 126: (*The Status for Topological Groups.*) We only dealt with the Baum–Connes Conjecture for discrete groups. We already mentioned that Higson–Kasparov [149] treat second countable locally compact topological groups. The Baum–Connes Conjecture for second countable almost connected groups G has been proven by Chabert–Echterhoff–Nest [60] based on the work of Higson–Kasparov [149] and Lafforgue [186]. The Baum–Connes Conjecture with Coefficients 105 has been proved for the connected Lie groups L appearing in Theorem 123 (i) by [166] and for $Sp(n, 1)$ by Julg [165].

The Injectivity Part of the Baum–Connes Conjecture

In this subsection we deal with injectivity results about the assembly map appearing in the Baum–Connes Conjecture 59. Recall that rational injectivity already implies the Novikov Conjecture 52 (see Proposition 92) and the Homological Stable Gromov–Lawson–Rosenberg Conjecture 101 (see Propositions 102 and 76).

127 Theorem 127: (*Rational Injectivity of the Baum–Connes Assembly Map.*) The assembly map appearing in the Baum–Connes Conjecture 59 is rationally injective if G belongs to one of the classes of groups below.

- (i) Groups acting properly isometrically on complete manifolds with non-positive sectional curvature.
- (ii) Discrete subgroups of Lie groups with finitely many path components.
- (iii) Discrete subgroups of p -adic groups.

Proof The proof of assertions (i) and (ii) is due to Kasparov [171], the one of assertion (iii) to Kasparov–Skandalis [172].

A metric space (X, d) admits a *uniform embedding into Hilbert space* if there exist a separable Hilbert space H , a map $f: X \rightarrow H$ and non-decreasing functions φ_1 and φ_2 from $[0, \infty) \rightarrow \mathbb{R}$ such that $\varphi_1(d(x, y)) \leq \|f(x) - f(y)\| \leq \varphi_2(d(x, y))$ for $x, y \in X$ and $\lim_{r \rightarrow \infty} \varphi_i(r) = \infty$ for $i = 1, 2$. A metric is *proper* if for each $r > 0$ and $x \in X$ the closed ball of radius r centered at x is compact. The question whether a discrete group G equipped with a proper left G -invariant metric d admits a uniform embedding into Hilbert space is independent of the choice of d , since the induced coarse structure does not depend on d [289, page 808]. For more information about groups admitting a uniform embedding into Hilbert space we refer to [87, 140].

The class of finitely generated groups, which embed uniformly into Hilbert space, contains a subclass A , which contains all word hyperbolic groups, finitely generated discrete subgroups of connected Lie groups and finitely generated amenable groups and is closed under semi-direct products [338, definition 2.1, theorem 2.2 and proposition 2.6]. Gromov [138, 139] has announced examples of finitely generated groups which do not admit a uniform embedding into Hilbert space. Details of the construction are described in Ghys [134].

The next theorem is proved by Skandalis–Tu–Yu [289, theorem 6.1] using ideas of Higson [148].

Theorem 128: (*Injectivity of the Baum–Connes Assembly Map.*) Let G be a countable group. Suppose that G admits a G -invariant metric for which G admits a uniform embedding into Hilbert space. Then the assembly map appearing in the Baum–Connes Conjecture with Coefficients 105 is injective.

128

We now discuss conditions which can be used to verify the assumption in Theorem 128.

Remark 129: (*Linear Groups.*) A group G is called *linear* if it is a subgroup of $GL_n(F)$ for some n and some field F . Guentner–Higson–Weinberger [140] show that every countable linear group admits a uniform embedding into Hilbert space and hence Theorem 128 applies.

129

Remark 130: (*Groups Acting Amenably on a Compact Space.*) A continuous action of a discrete group G on a compact space X is called *amenable* if there exists a sequence

130

$$p_n: X \rightarrow M^1(G) = \left\{ f: G \rightarrow [0, 1] \mid \sum_{g \in G} f(g) = 1 \right\}$$

of weak- $*$ -continuous maps such that for each $g \in G$ one has

$$\lim_{n \rightarrow \infty} \sup_{x \in X} \|g * (p_n(x) - p_n(g \cdot x))\|_1 = 0.$$

Note that a group G is amenable if and only if its action on the one-point-space is amenable. More information about this notion can be found for instance in [5, 6].

Higson–Roe [153, theorem 1.1 and proposition 2.3] show that a finitely generated group equipped with its word length metric admits an amenable action on a compact metric space, if and only if it belongs to the class A defined in [338, definition 2.1], and hence admits a uniform embedding into Hilbert space. Hence Theorem 128 implies the result of Higson [148, theorem 1.1] that the assembly map appearing in the Baum–Connes Conjecture with Coefficients 105 is injective if G admits an amenable action on some compact space.

Word hyperbolic groups and the class of groups mentioned in Theorem 127 (ii) fall under the class of groups admitting an amenable action on some compact space [153, sect. 4].

131 **Remark 131** Higson [148, theorem 5.2] shows that the assembly map appearing in the Baum–Connes Conjecture with Coefficients 105 is injective if $\underline{E}G$ admits an appropriate compactification. This is a C^* -version of the result for K - and L -theory due to Carlsson–Pedersen [55], compare Theorem 146.

132 **Remark 132** We do not know whether the groups appearing in Theorem 127 and 128 satisfy the Baum–Connes Conjecture 59.

Next we discuss injectivity results about the classical assembly map for topological K -theory.

The *asymptotic dimension* of a proper metric space X is the infimum over all integers n such that for any $R > 0$ there exists a cover $\mathcal{U}$ of X with the property that the diameter of the members of $\mathcal{U}$ is uniformly bounded and every ball of radius R intersects at most $(n + 1)$ elements of $\mathcal{U}$ (see [137, page 28]).

The next result is due to Yu [337].

133 **Theorem 133:** (*The C^* -Theoretic Novikov Conjecture and Groups of Finite Asymptotic Dimension.*) Let G be a group which possesses a finite model for BG and has finite asymptotic dimension. Then the assembly map in the Baum–Connes Conjecture 31

$$K_n(BG) \rightarrow K_n(C_r^*(G))$$

is injective for all $n \in \mathbb{Z}$.

The Coarse Baum–Connes Conjecture

The coarse Baum–Connes Conjecture was explained in Sect. 2.5.1. Recall the descent principle (Proposition 109): if a countable group can be equipped with a G -

invariant metric such that the resulting metric space satisfies the Coarse Baum–Connes Conjecture, then the classical assembly map for topological K -theory is injective.

Recall that a discrete metric space has bounded geometry if for each $r > 0$ there exists a $N(r)$ such that for all x the ball of radius $N(r)$ centered at $x \in X$ contains at most $N(r)$ elements.

The next result is due to Yu [338, theorem 2.2 and proposition 2.6].

Theorem 134: (*Status of the Coarse Baum–Connes Conjecture.*) The Coarse Baum–Connes Conjecture 108 is true for a discrete metric space X of bounded geometry if X admits a uniform embedding into Hilbert space. In particular a countable group G satisfies the Coarse Baum–Connes Conjecture 108 if G equipped with a proper left G -invariant metric admits a uniform embedding into Hilbert space.

134

Also Yu’s Theorem 133 is proven via a corresponding result about the Coarse Baum–Connes Conjecture.

Status of the Farrell–Jones Conjecture

2.6.2

Next we deal with the Farrell–Jones Conjecture.

The Fibered Farrell–Jones Conjecture

The Fibered Farrell–Jones Conjecture 113 was discussed in Sect. 2.5.2. Recall that it has better inheritance properties (compare Sect. 2.6.4) and contains the ordinary Farrell–Jones Conjecture 58 as a special case.

Theorem 135: (*Status of the Fibered Farrell–Jones Conjecture.*)

135

- (i) Let G be a discrete group which satisfies one of the following conditions.
- (a) There is a Lie group L with finitely many path components and G is a co-compact discrete subgroup of L .
 - (b) The group G is virtually torsionfree and acts properly discontinuously, co-compactly and via isometries on a simply connected complete nonpositively curved Riemannian manifold.

Then

- (1) The version of the Fibered Farrell–Jones Conjecture 113 for the topological and the differentiable pseudoisotopy functor is true for G .
- (2) The version of the Fibered Farrell–Jones Conjecture 113 for K -theory and $R = \mathbb{Z}$ is true for G in the range $n \leq 1$, i.e. the assembly map is bijective for $n \leq 1$.

Moreover we have the following statements.

- (ii) The version of the Fibered Farrell–Jones Conjecture 113 for K -theory and $R = \mathbb{Z}$ is true in the range $n \leq 1$ for braid groups.
- (iii) The L -theoretic version of the Fibered Farrell–Jones Conjecture 113 with $R = \mathbb{Z}$ holds after inverting 2 for elementary amenable groups.

Proof (i) For assertion (1) see [111, theorem 2.1 on page 263], [111, proposition 2.3] and [119, theorem A]. Assertion (2) follows from (1) by Remark 117.

(ii) See [119].

(iii) is proven in [117, theorem 5.2]. For crystallographic groups see also [333].

A surjectivity result about the Fibered Farrell–Jones Conjecture for Pseudoisotopies appears as the last statement in Theorem 139.

The rational comparison result between the K -theory and the pseudoisotopy version (see Proposition 112) does not work in the fibered case, compare Remark 117. However, in order to exploit the good inheritance properties one can first use the pseudoisotopy functor in the fibered set-up, then specialize to the unfibered situation and finally do the rational comparison to K -theory.

136

Remark 136 The version of the Fibered Farrell–Jones Conjecture 113 for L -theory and $R = \mathbb{Z}$ seems to be true if G satisfies the condition (a) appearing in Theorem 135. Farrell and Jones [111, remark 2.1.3 on page 263] say that they can also prove this version without giving the details.

137

Remark 137 Let G be a virtually poly-cyclic group. Then it contains a maximal normal finite subgroup N such that the quotient G/N is a discrete cocompact subgroup of a Lie group with finitely many path components [331, theorem 3, remark 4 on page 200]. Hence by Sect. 2.6.4 and Theorem 135 the version of the Fibered Farrell–Jones Conjecture 113 for the topological and the differentiable pseudoisotopy functor, and for K -theory and $R = \mathbb{Z}$ in the range $n \leq 1$, is true for G . Earlier results of this type were treated for example in [100, 105].

The Farrell–Jones Conjecture

Here is a sample of some results one can deduce from Theorem 135.

138

Theorem 138: (*The Farrell–Jones Conjecture and Subgroups of Lie groups.*) Suppose H is a subgroup of G , where G is a discrete cocompact subgroup of a Lie group L with finitely many path components. Then

- (i) The version of the Farrell–Jones Conjecture for K -theory and $R = \mathbb{Z}$ is true for H rationally, i.e. the assembly map appearing in Conjecture 58 is an isomorphism after applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$.
- (ii) The version of the Farrell–Jones Conjecture for K -theory and $R = \mathbb{Z}$ is true for H in the range $n \leq 1$, i.e. the assembly map appearing in Conjecture 58 is an isomorphism for $n \leq 1$.

Proof The results follow from Theorem 135, since the Fibered Farrell–Jones Conjecture 113 passes to subgroups [111, theorem A.8 on page 289] (compare Sect. 2.6.4) and implies the Farrell–Jones Conjecture 58.

We now discuss results for torsion free groups. Recall that for $R = \mathbb{Z}$ the K -theoretic Farrell–Jones Conjecture in dimensions ≤ 1 together with the L -theoretic version implies already the Borel Conjecture 27 in dimension ≥ 5 (see Theorem 28).

A complete Riemannian manifold M is called *A-regular* if there exists a sequence of positive real numbers $A_0, A_1, A_2, \dots$ such that $\|\nabla^n K\| \leq A_n$, where $\|\nabla^n K\|$ is the supremum-norm of the n -th covariant derivative of the curvature tensor K . Every locally symmetric space is A-regular since ∇K is identically zero. Obviously every closed Riemannian manifold is A-regular.

Theorem 139: (*Status of the Farrell–Jones Conjecture for Torsionfree Groups.*) Consider the following conditions for the group G .

139

- (i) $G = \pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature which is A-regular.
- (ii) $G = \pi_1(M)$ for a closed Riemannian manifold M with non-positive sectional curvature.
- (iii) $G = \pi_1(M)$ for a complete Riemannian manifold with negatively pinched sectional curvature.
- (iv) G is a torsion free discrete subgroup of $GL(n, \mathbb{R})$.
- (v) G is a torsion free solvable discrete subgroup of $GL(n, \mathbb{C})$.
- (vi) $G = \pi_1(X)$ for a non-positively curved finite simplicial complex X .
- (vii) G is a strongly poly-free group in the sense of Aravinda–Farrell–Roushon [10, definition 1.1]. The pure braid group satisfies this hypothesis.

Then

- (1) Suppose that G satisfies one of the conditions (i) to (vii). Then the K -theoretic Farrell–Jones Conjecture is true for $R = \mathbb{Z}$ in dimensions $n \leq 1$. In particular Conjecture 3 holds for G .
- (2) Suppose that G satisfies one of the conditions (i), (ii), (iii) or (iv). Then G satisfies the Farrell–Jones Conjecture for Torsion Free Groups and L -theory 19 for $R = \mathbb{Z}$.
- (3) Suppose that G satisfies (ii). Then the Farrell–Jones Conjecture for Pseudoisotopies of Aspherical Spaces 110 holds for G .
- (4) Suppose that G satisfies one of the conditions (i), (iii), or (iv). Then the assembly map appearing in the version of the Fibered Farrell–Jones Conjecture for Pseudoisotopies 113 is surjective, provided that the G -space Z appearing in Conjecture 113 is connected.

Proof Note that (ii) is a special case of (i) because every closed Riemannian manifold is A-regular. If M is a pinched negatively curved complete Riemannian

manifold, then there is another Riemannian metric for which M is negatively curved complete and A-regular. This fact is mentioned in [115, page 216] and attributed there to Abresch [3] and Shi [285]. Hence also (iii) can be considered as a special case of (i). The manifold $M = G \backslash GL(n, \mathbb{R})/O(n)$ is a non-positively curved complete locally symmetric space and hence in particular A-regular. So (iv) is a special case of (i).

Assertion (1) under the assumption (i) is proved by Farrell–Jones in [115, proposition 0.10 and lemma 0.12]. The earlier work [110] treated the case (ii). Under assumption (v) assertion (1) is proven by Farrell–Linnell [117, theorem 1.1]. The result under assumption (vi) is proved by Hu [156], under assumption (vii) it is proved by Aravinda–Farrell–Roushon [10, theorem 1.3].

Assertion (2) under assumption (i) is proven by Farrell–Jones in [115]. The case (ii) was treated earlier in [112].

Assertion (3) is proven by Farrell–Jones in [110] and assertion (4) by Jones in [160].

140

Remark 140 As soon as certain collapsing results (compare [114], [116]) are extended to orbifolds, the results under (4) above would also apply to groups with torsion and in particular to $SL_n(\mathbb{Z})$ for arbitrary n .

The Farrell–Jones Conjecture for Arbitrary Coefficients

The following result due to Bartels–Reich [21] deals with algebraic K -theory for arbitrary coefficient rings R . It extends Bartels–Farrell–Jones–Reich [19].

141

Theorem 141 Suppose that G is the fundamental group of a closed Riemannian manifold with negative sectional curvature. Then the K -theoretic part of the Farrell–Jones Conjecture 58 is true for any ring R , i.e. the assembly map

$$A_{\mathcal{VCY}}: H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \rightarrow K_n(RG)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Note that the assumption implies that G is torsion free and hence the family $\mathcal{VCY}$ reduces to the family $\mathcal{CYC}$ of cyclic subgroups. Recall that for a regular ring R the theorem above implies that the classical assembly

$$A: H_n(BG; \mathbf{K}(R)) \rightarrow K_n(RG)$$

is an isomorphism, compare Proposition 66 (i).

Injectivity Part of the Farrell–Jones Conjecture

The next result about the classical K -theoretic assembly map is due to Bökstedt–Hsiang–Madsen [38].

Theorem 142: (*Rational Injectivity of the Classical K -Theoretic Assembly Map.*) Let G be a group such that the integral homology $H_j(BG; \mathbb{Z})$ is finitely generated for each $j \in \mathbb{Z}$. Then the rationalized assembly map

$$A: H_n(BG; \mathbb{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n^G(E_{\{1\}}(G); \mathbb{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective for all $n \in \mathbb{Z}$.

Because of the homological Chern character (see Remark 12) we obtain for the groups treated in Theorem 142 an injection

$$\bigoplus_{s+t=n} H_s(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (K_t(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}. \quad (2.28)$$

Next we describe a generalization of Theorem 142 above from the trivial family $\{1\}$ to the family $\mathcal{FIN}$ of finite subgroups due to Lück–Reich–Rognes–Varisco [208]. Let $\mathbf{K}_{\mathbb{Z}}^{\text{con}}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$ be the connective version of the functor $\mathbf{K}_{\mathbb{Z}}$ of (2.37). In particular $H_n(G/H; \mathbf{K}_{\mathbb{Z}}^{\text{con}})$ is isomorphic to $K_n(\mathbb{Z}H)$ for $n \geq 0$ and vanishes in negative dimensions. For a prime p we denote by $\mathbb{Z}_p$ the p -adic integers. Let $K_n(R; \mathbb{Z}_p)$ denote the homotopy groups $\pi_n(\mathbf{K}^{\text{con}}(R)_{\widehat{p}})$ of the p -completion of the connective K -theory spectrum of the ring R .

Theorem 143: (*Rational Injectivity of the Farrell–Jones Assembly Map for Connective K -theory.*) Suppose that the group G satisfies the following two conditions:

- (H) For each finite cyclic subgroup $C \subseteq G$ and all $j \geq 0$ the integral homology group $H_j(BZ_G C; \mathbb{Z})$ of the centralizer $Z_G C$ of C in G is finitely generated.
- (K) There exists a prime p such that for each finite cyclic subgroup $C \subseteq G$ and each $j \geq 1$ the map induced by the change of coefficients homomorphism

$$K_j(\mathbb{Z}C; \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_j(\mathbb{Z}_p C; \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Then the rationalized assembly map

$$A_{\text{vcy}}: H_n^G(E_{\text{vcy}}(G); \mathbf{K}_{\mathbb{Z}}^{\text{con}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an injection for all $n \in \mathbb{Z}$.

Remark 144 The methods of Chapter 2.9 apply also to $\mathbf{K}_{\mathbb{Z}}^{\text{con}}$ and yield under assumption (H) and (K) an injection

$$\bigoplus_{s+t=n, t \geq 0} \bigoplus_{(C) \in (\mathcal{FCY})} H_s(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot K_t(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Notice that in the index set for the direct sum appearing in the source we require $t \geq 0$. This reflects the fact that the result deals only with the connective K -

theory spectrum. If one drops the restriction $t \geq 0$ the Farrell–Jones Conjecture 58 predicts that the map is an isomorphism, compare Sect. 2.3.2 and Theorem 173. If we restrict the injection to the direct sum given by $C = 1$, we rediscover the map (2.28) whose injectivity follows already from Theorem 142.

The condition (K) appearing in Theorem 143 is conjectured to be true for all primes p (compare [280, 290] and [291]) but no proof is known. The weaker version of condition (K), where C is the trivial group is also needed in Theorem 142. But that case is known to be true and hence does not appear in its formulation. The special case of condition (K), where $j = 1$ is implied by the Leopoldt Conjecture for abelian fields (compare [229, IX, § 3]), which is known to be true [229, theorem 10.3.16]. This leads to the following result.

145

Theorem 145: (*Rational Contribution of Finite Subgroups to $\text{Wh}(G)$.*) Let G be a group. Suppose that for each finite cyclic subgroup $C \subseteq G$ and each $j \leq 4$ the integral homology group $H_j(BZ_G C)$ of the centralizer $Z_G C$ of C in G is finitely generated. Then the map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{FIN}}(G)} \text{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \text{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

is injective, compare Conjecture 81.

The result above should be compared to the result which is proven using Fuglede–Kadison determinants in [209, sect. 5], [202, theorem 9.38 on page 354]: for every (discrete) group G and every finite normal subgroup $H \subseteq G$ the map $\text{Wh}(H) \otimes_{\mathbb{Z}G} \mathbb{Z} \rightarrow \text{Wh}(G)$ induced by the inclusion $H \rightarrow G$ is rationally injective.

The next result is taken from Rosenthal [271], where the techniques and results of Carlsson–Pedersen [55] are extended from the trivial family $\{1\}$ to the family of finite subgroups $\mathcal{FIN}$.

146

Theorem 146 Suppose there exists a model E for the classifying space $E_{\mathcal{FIN}}(G)$ which admits a metrizable compactification $\bar{E}$ to which the group action extends. Suppose $\bar{E}^H$ is contractible and E^H is dense in $\bar{E}^H$ for every finite subgroup $H \subset G$. Suppose compact subsets of E become small near $\bar{E} - E$. Then for every ring R the assembly map

$$A_{\mathcal{FIN}} : H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \rightarrow K_n(RG)$$

is split injective.

A compact subset $K \subset E$ is said to become *small near $\bar{E} - E$* if for every neighbourhood $U \subset \bar{E}$ of a point $x \in \bar{E} - E$ there exists a neighbourhood $V \subset \bar{E}$ such that $g \in G$ and $gK \cap V \neq \emptyset$ implies $gK \subset U$. Presumably there is an analogous result for $L^{(-\infty)}$ -theory under the assumption that $K_{-n}(RH)$ vanishes for finite subgroups H

of G and n large enough. This would extend the corresponding result for the family $\mathcal{F} = \{1\}$ which appears in Carlsson–Pedersen [55].

We finally discuss injectivity results about assembly maps for the trivial family. The following result is due to Ferry–Weinberger [129, corollary 2.3] extending earlier work of Farrell–Hsiang [99].

Theorem 147 Suppose $G = \pi_1(M)$ for a complete Riemannian manifold of non-positive sectional curvature. Then the L -theory assembly map

147

$$A: H_n(BG; \mathbf{L}_{\mathbb{Z}}^{\varepsilon}) \rightarrow L_n^{\varepsilon}(\mathbb{Z}G)$$

is injective for $\varepsilon = h, s$.

In fact Ferry–Weinberger also prove a corresponding splitting result for the classical A -theory assembly map. In [155] Hu shows that a finite complex of non-positive curvature is a retract of a non-positively curved PL -manifold and concludes split injectivity of the classical L -theoretic assembly map for $R = \mathbb{Z}$.

The next result due to Bartels [17] is the algebraic K - and L -theory analogue of Theorem 133.

Theorem 148: (*The K - and L -Theoretic Novikov Conjecture and Groups of Finite Asymptotic Dimension.*) Let G be a group which admits a finite model for BG . Suppose that G has finite asymptotic dimension. Then

148

(i) The assembly maps appearing in the Farrell–Jones Conjecture 11

$$A: H_n(BG; \mathbf{K}(R)) \rightarrow K_n(RG)$$

is injective for all $n \in \mathbb{Z}$.

(ii) If furthermore R carries an involution and $K_{-j}(R)$ vanishes for sufficiently large j , then the assembly maps appearing in the Farrell–Jones Conjecture 19

$$A: H_n(BG; \mathbf{L}^{(-\infty)}(R)) \rightarrow L_n^{(-\infty)}(RG)$$

is injective for all $n \in \mathbb{Z}$.

Further results related to the Farrell–Jones Conjecture 58 can be found for instance in [9, 33].

List of Groups Satisfying the Conjecture

2.6.3

In the following table we list prominent classes of groups and state whether they are known to satisfy the Baum–Connes Conjecture 59 (with coefficients 105) or the Farrell–Jones Conjecture 58 (fibered 113). Some of the classes are redundant. A question mark means that the authors do not know about a corresponding result. The reader should keep in mind that there may exist results of which the authors are not aware.

type of group	Baum–Connes Conjecture 59 (with coefficients 105)	Farrell–Jones Conjecture 58 for K -theory for $R = \mathbb{Z}$ (fibered 113)	Farrell–Jones Conjecture 58 for L -theory for $R = \mathbb{Z}$ (fibered 113)
a-T-menable groups	true with coefficients (see Theorem 120)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
amenable groups	true with coefficients (see Theorem 120)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
elementary amenable groups	true with coefficients (see Theorem 120)	?	true fibered after inverting 2 (see Theorem 135)
virtually poly-cyclic	true with coefficients (see Theorem 120)	true rationally, true fibered in the range $n \leq 1$ (compare Remark 137)	true fibered after inverting 2 (see Theorem 135)
torsion free virtually solvable subgroups of $GL(n, \mathbb{C})$	true with coefficients (see Theorem 120)	true in the range ≤ 1 [117, theorem 1.1]	true fibered after inverting 2 [117, corollary 5.3]
discrete subgroups of Lie groups with finitely many path components	injectivity true (see Theorem 128 and Remark 130)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
subgroups of groups which are discrete cocompact subgroups of Lie groups with finitely many path components	injectivity true (see Theorem 128 and Remark 130)	true rationally, true fibered in the range $n \leq 1$ (see Theorem 135)	probably true fibered (see Remark 136). Injectivity is true after inverting 2 (see Propositions 74 and 95)
linear groups	injectivity is true (see Theorem 128 and Remark 129)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
arithmetic groups	injectivity is true (see Theorem 128 and Remark 129)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
torsion free discrete subgroups of $GL(n, \mathbb{R})$	injectivity is true (see Theorem 128 and Remark 130)	true in the range $n \leq 1$ (see Theorem 139)	true (see Theorem 139)

type of group	Baum–Connes Conjecture 59 (with coefficients 105)	Farrell–Jones Conjecture 58 for K -theory for $R=\mathbb{Z}$ (fibered 113)	Farrell–Jones Conjecture 58 for L -theory for $R=\mathbb{Z}$ (fibered 113)
Groups with finite BG and finite asymptotic dimension	injectivity is true (see Theorem 133)	injectivity is true for arbitrary coefficients R (see Theorem 148)	injectivity is true for regular R as coefficients (see Theorem 148)
G acts properly and isometrically on a complete Riemannian manifold M with non-positive sectional curvature	rational injectivity is true (see Theorem 127)	?	rational injectivity is true (see Propositions 74 and 95)
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature	rationally injective (see Theorem 127)	?	injectivity true (see Theorem 147)
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature which is A-regular	rationally injective (see Theorem 127)	true in the range $n \leq 1$, rationally surjective (see Theorem 139)	true (see Theorem 139)
$\pi_1(M)$ for a complete Riemannian manifold M with pinched negative sectional curvature	rational injectivity is true (see Theorem 128)	true in the range $n \leq 1$, rationally surjective (see Theorem 139)	true (see Theorem 139)
$\pi_1(M)$ for a closed Riemannian manifold M with non-positive sectional curvature	rationally injective (see Theorem 127)	true fibered in the range $n \leq 1$, true rationally (see Theorem 139)	true (see Theorem 139)
$\pi_1(M)$ for a closed Riemannian manifold M with negative sectional curvature	true for all subgroups (see Theorem 123)	true for all coefficients R (see Theorem 141)	true (see Theorem 139)

type of group	Baum–Connes Con- jecture 59 (with co- efficients 105)	Farrell–Jones Con- jecture 58 for K - theory and $R = \mathbb{Z}$ (fibered 113)	Farrell–Jones Conjec- ture 58 for L -theory for $R = \mathbb{Z}$ (fibered 113)
word hyperbolic groups	true for all sub- groups (see Theo- rem 123). Unpub- lished proof with co- efficients by V. Laf- orgue	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
one-relator groups	true with coeffi- cients (see Theorem 121)	rational injectivity is true for the fibered version (see [20])	injectivity is true af- ter inverting 2 (see Propositions 74 and 95)
torsion free one- relator groups	true with coeffi- cients (see Theorem 121)	true for R regular [313, theorem 19.4 on page 249 and the- orem 19.5 on page 250]	true after inverting 2 [47, corollary 8]
Haken 3-manifold groups (in particu- lar knot groups)	true with coeffi- cients (see Theorem 121)	true in the range $n \leq 1$ for R regular [313, theorem 19.4 on page 249 and the- orem 19.5 on page 250]	true after inverting 2 [47, corollary 8]
$SL(n, \mathbb{Z}), n \geq 3$	injectivity is true	compare Remark 140	injectivity is true af- ter inverting 2 (see Propositions 74 and 95)
Artin’s braid group B_n	true with coeffi- cients [225, theorem 5.25], [277]	true fibered in the range $n \leq 1$, true ra- tionally [119]	injectivity is true af- ter inverting 2 (see Propositions 74 and 95)
pure braid group C_n	true with coeffi- cients	true in the range $n \leq$ 1 (see Theorem 139)	injectivity is true af- ter inverting 2 (see Propositions 74 and 95)
Thompson’s group F	true with coeffi- cients [94]	?	injectivity is true af- ter inverting 2 (see Propositions 74 and 95)

149

Remark 149 The authors have no information about the status of these conjectures for mapping class groups of higher genus or the group of outer automorphisms of free groups. Since all of these spaces have finite models for $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ Theorem 143 applies in these cases.

Inheritance Properties

In this Subsection we list some inheritance properties of the various conjectures.

Directed Colimits

Let $\{G_i \mid i \in I\}$ be a directed system of groups. Let $G = \operatorname{colim}_{i \in I} G_i$ be the colimit. We do not require that the structure maps are injective. If the Fibered Farrell–Jones Conjecture 113 is true for each G_i , then it is true for G [117, theorem 6.1].

Suppose that $\{G_i \mid i \in I\}$ is a system of subgroups of G directed by inclusion such that $G = \operatorname{colim}_{i \in I} G_i$. If each G_i satisfies the Farrell–Jones Conjecture 58, the Baum–Connes Conjecture 59 or the Baum–Connes Conjecture with Coefficients 105, then the same is true for G [32, theorem 1.1], [225, lemma 5.3]. We do not know a reference in Farrell–Jones case. An argument in that case uses Lemma 63, the fact that $K_n(RG) = \operatorname{colim}_{i \in I} K_n(RG_i)$ and that for suitable models we have $E_{\mathcal{F}}(G) = \bigcup_{i \in I} G \times_{G_i} E_{\mathcal{F} \cap G_i}(G_i)$.

Passing to Subgroups

The Baum–Connes Conjecture with Coefficients 105 and the Fibered Farrell–Jones Conjecture 113 pass to subgroups, i.e. if they hold for G , then also for any subgroup $H \subseteq G$. This claim for the Baum–Connes Conjecture with Coefficients 105 has been stated in [28], a proof can be found for instance in [59, theorem 2.5]. For the Fibered Farrell–Jones Conjecture this is proved in [111, theorem A.8 on page 289] for the special case $R = \mathbb{Z}$, but the proof also works for arbitrary rings R .

It is not known whether the Baum–Connes Conjecture 59 or the Farrell–Jones Conjecture 58 itself passes to subgroups.

Extensions of Groups

Let $p: G \rightarrow K$ be a surjective group homomorphism. Suppose that the Baum–Connes Conjecture with Coefficients 105 or the Fibered Farrell–Jones Conjecture 113 respectively holds for K and for $p^{-1}(H)$ for any subgroup $H \subset K$ which is finite or virtually cyclic respectively. Then the Baum–Connes Conjecture with Coefficients 105 or the Fibered Farrell–Jones Conjecture 113 respectively holds for G . This is proved in [233, theorem 3.1] for the Baum–Connes Conjecture with Coefficients 105, and in [111, proposition 2.2 on page 263] for the Fibered Farrell–Jones Conjecture 113 in the case $R = \mathbb{Z}$. The same proof works for arbitrary coefficient rings.

It is not known whether the corresponding statement holds for the Baum–Connes Conjecture 59 or the Farrell–Jones Conjecture 58 itself.

Let $H \subseteq G$ be a normal subgroup of G . Suppose that H is a-T-menable. Then G satisfies the Baum–Connes Conjecture with Coefficients 105 if and only if G/H does [59, corollary 3.14]. The corresponding statement is not known for the Baum–Connes Conjecture 59.

Products of Groups

The group $G_1 \times G_2$ satisfies the Baum–Connes Conjecture with Coefficients 105 if and only if both G_1 and G_2 do [59, theorem 3.17], [233, corollary 7.12]. The corresponding statement is not known for the Baum–Connes Conjecture 59.

Let $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$ denote the infinite dihedral group. Whenever a version of the Fibered Farrell–Jones Conjecture 113 is known for $G = \mathbb{Z} \times \mathbb{Z}$, $G = \mathbb{Z} \times D_\infty$ and $D_\infty \times D_\infty$, then that version of the Fibered Farrell–Jones Conjecture is true for $G_1 \times G_2$ if and only if it is true for G_1 and G_2 .

Subgroups of Finite Index

It is not known whether the Baum–Connes Conjecture 59, the Baum–Connes Conjecture with Coefficients 105, the Farrell–Jones Conjecture 58 or the Fibered Farrell–Jones Conjecture 113 is true for a group G if it is true for a subgroup $H \subseteq G$ of finite index.

Groups Acting on Trees

Let G be a countable discrete group acting without inversion on a tree T . Then the Baum–Connes Conjecture with Coefficients 105 is true for G if and only if it holds for all stabilizers of the vertices of T . This is proved by Oyono-Oyono [234, theorem 1.1]. This implies that Baum–Connes Conjecture with Coefficients 105 is stable under amalgamated products and HNN-extensions. Actions on trees in the context the Farrell–Jones Conjecture 58 will be treated in [20].

2.7

Equivariant Homology Theories

We already defined the notion of a G -homology theory in Sect. 2.3.1. If G -homology theories for different G are linked via a so called induction structure one obtains the notion of an equivariant homology theory. In this section we give a precise definition and we explain how a functor from the orbit category $\text{Or}(G)$ to the category of spectra leads to a G -homology theory (see Proposition 156) and how more generally a functor from the category of groupoids leads to an equivariant homology theory (see Proposition 157). We then describe the main examples of such spectra valued functors which were already used in order to formulate the Farrell–Jones and the Baum–Connes Conjectures in Chapt. 2.3.

2.7.1

The Definition of an Equivariant Homology Theory

The notion of a G -homology theory $\mathcal{H}_*^G$ with values in Λ -modules for a commutative ring Λ was defined in Sect. 2.3.1. We now recall the axioms of an equivariant homology theory from [201, sect. 1]. We will see in Sect. 2.7.3 that the G -homology theories we used in the formulation of the Baum–Connes and the Farrell–Jones Conjectures in Chapt. 2.3 are in fact the values at G of suitable equivariant homology theories.

Let $\alpha: H \rightarrow G$ be a group homomorphism. Given a H -space X , define the *induction of X with α* to be the G -space $\text{ind}_\alpha X$ which is the quotient of $G \times X$ by the right H -action $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g, x) \in G \times X$. If $\alpha: H \rightarrow G$ is an inclusion, we also write ind_H^G instead of ind_α .

An *equivariant homology theory $\mathcal{H}_*^?$ with values in Λ -modules* consists of a G -homology theory $\mathcal{H}_*^G$ with values in Λ -modules for each group G together with the following so called *induction structure*: given a group homomorphism $\alpha: H \rightarrow G$ and a H -CW-pair (X, A) such that $\ker(\alpha)$ acts freely on X , there are for each $n \in \mathbb{Z}$ natural isomorphisms

$$\text{ind}_\alpha: \mathcal{H}_n^H(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(\text{ind}_\alpha(X, A))$$

satisfying the following conditions.

- (i) Compatibility with the boundary homomorphisms

$$\partial_n^G \circ \text{ind}_\alpha = \text{ind}_\alpha \circ \partial_n^H.$$

- (ii) Functoriality

Let $\beta: G \rightarrow K$ be another group homomorphism such that $\ker(\beta \circ \alpha)$ acts freely on X . Then we have for $n \in \mathbb{Z}$

$$\text{ind}_{\beta \circ \alpha} = \mathcal{H}_n^K(f_1) \circ \text{ind}_\beta \circ \text{ind}_\alpha: \mathcal{H}_n^H(X, A) \rightarrow \mathcal{H}_n^K(\text{ind}_{\beta \circ \alpha}(X, A)),$$

where $f_1: \text{ind}_\beta \text{ind}_\alpha(X, A) \xrightarrow{\cong} \text{ind}_{\beta \circ \alpha}(X, A)$, $(k, g, x) \mapsto (k\beta(g), x)$ is the natural K -homeomorphism.

- (iii) Compatibility with conjugation

For $n \in \mathbb{Z}$, $g \in G$ and a G -CW-pair (X, A) the homomorphism

$$\text{ind}_{c(g): G \rightarrow G}: \mathcal{H}_n^G(X, A) \rightarrow \mathcal{H}_n^G(\text{ind}_{c(g): G \rightarrow G}(X, A))$$

agrees with $\mathcal{H}_n^G(f_2)$, where the G -homeomorphism

$$f_2: (X, A) \rightarrow \text{ind}_{c(g): G \rightarrow G}(X, A)$$

sends x to $(1, g^{-1}x)$ and $c(g): G \rightarrow G$ sends g' to $gg'g^{-1}$.

This induction structure links the various homology theories for different groups G .

If the G -homology theory $\mathcal{H}_*^G$ is defined or considered only for proper G -CW-pairs (X, A) , we call it a *proper G -homology theory $\mathcal{H}_*^G$ with values in Λ -modules*.

Example 150. Let $\mathcal{K}_*$ be a homology theory for (non-equivariant) CW-pairs with values in Λ -modules. Examples are singular homology, oriented bordism theory or topological K -homology. Then we obtain two equivariant homology theories with values in Λ -modules, whose underlying G -homology theories for a group G are given by the following constructions

$$\mathcal{H}_n^G(X, A) = \mathcal{K}_n(G \backslash X, G \backslash A);$$

$$\mathcal{H}_n^G(X, A) = \mathcal{K}_n(EG \times_G (X, A)).$$

Example 151. Given a proper G -CW-pair (X, A) , one can define the G -bordism group $\Omega_n^G(X, A)$ as the abelian group of G -bordism classes of maps $f: (M, \partial M) \rightarrow (X, A)$ whose sources are oriented smooth manifolds with cocompact orientation preserving proper smooth G -actions. The definition is analogous to the one in the non-equivariant case. This is also true for the proof that this defines a proper G -homology theory. There is an obvious induction structure coming from induction of equivariant spaces. Thus we obtain an equivariant proper homology theory $\Omega_*^?$.

Example 152. Let Λ be a commutative ring and let

$$M: \text{GROUPOIDS} \rightarrow \Lambda\text{-MODULES}$$

be a contravariant functor. For a group G we obtain a covariant functor

$$M^G: \text{Or}(G) \rightarrow \Lambda\text{-MODULES}$$

by its composition with the transport groupoid functor $\mathcal{G}^G$ defined in (2.30). Let $H_*^G(-; M)$ be the G -homology theory given by the Bredon homology with coefficients in M^G as defined in Example 64. There is an induction structure such that the collection of the $H_*^G(-; M)$ defines an equivariant homology theory $H_*^?(-; M)$. This can be interpreted as the special case of Proposition 157, where the covariant functor $\text{GROUPOIDS} \rightarrow \Omega\text{-SPECTRA}$ is the composition of M with the functor sending a Λ -module to the associated Eilenberg–MacLane spectrum. But there is also a purely algebraic construction.

The next lemma was used in the proof of the Transitivity Principle 65.

153

Lemma 153 Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in Λ -modules. Let G be a group and let $\mathcal{F}$ a family of subgroups of G . Let Z be a G -CW-complex. Consider $N \in \mathbb{Z} \cup \{\infty\}$. For $H \subseteq G$ let $\mathcal{F} \cap H$ be the family of subgroups of H given by $\{K \cap H \mid K \in \mathcal{F}\}$. Suppose for each $H \subset G$, which occurs as isotropy group in Z , that the map induced by the projection $\text{pr}: E_{\mathcal{F} \cap H}(H) \rightarrow \text{pt}$

$$\mathcal{H}_n^H(\text{pr}): \mathcal{H}_n^H(E_{\mathcal{F} \cap H}(H)) \rightarrow \mathcal{H}_n^H(\text{pt})$$

is bijective for all $n \in \mathbb{Z}, n \leq N$.

Then the map induced by the projection $\text{pr}_2: E_{\mathcal{F}}(G) \times Z \rightarrow Z$

$$\mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z) \rightarrow \mathcal{H}_n^G(Z)$$

is bijective for $n \in \mathbb{Z}, n \leq N$.

Proof We first prove the claim for finite-dimensional G -CW-complexes by induction over $d = \dim(Z)$. The induction beginning $\dim(Z) = -1$, i.e. $Z = \emptyset$, is trivial.

In the induction step from $(d-1)$ to d we choose a G -pushout

$$\begin{array}{ccc} \coprod_{i \in I_d} G/H_i \times S^{d-1} & \longrightarrow & Z_{d-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_d} G/H_i \times D^d & \longrightarrow & Z_d . \end{array}$$

If we cross it with $E_{\mathcal{F}}(G)$, we obtain another G -pushout of G -CW-complexes. The various projections induce a map from the Mayer–Vietoris sequence of the latter G -pushout to the Mayer–Vietoris sequence of the first G -pushout. By the Five-Lemma it suffices to prove that the following maps

$$\mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i \in I_d} G/H_i \times S^{d-1}\right) \rightarrow \mathcal{H}_n^G\left(\coprod_{i \in I_d} G/H_i \times S^{d-1}\right);$$

$$\mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z_{d-1}) \rightarrow \mathcal{H}_n^G(Z_{d-1});$$

$$\mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i \in I_d} G/H_i \times D^d\right) \rightarrow \mathcal{H}_n^G\left(\coprod_{i \in I_d} G/H_i \times D^d\right)$$

are bijective for $n \in \mathbb{Z}, n \leq N$. This follows from the induction hypothesis for the first two maps. Because of the disjoint union axiom and G -homotopy invariance of $\mathcal{H}_*^?$ the claim follows for the third map if we can show for any $H \subseteq G$ which occurs as isotropy group in Z that the map

$$\mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times G/H) \rightarrow \mathcal{H}_n^G(G/H) \quad (2.29)$$

is bijective for $n \in \mathbb{Z}, n \leq N$. The G -map

$$G \times_H \text{res}_G^H E_{\mathcal{F}}(G) \rightarrow G/H \times E_{\mathcal{F}}(G) \quad (g, x) \mapsto (gH, gx)$$

is a G -homeomorphism where res_G^H denotes the restriction of the G -action to an H -action. Obviously $\text{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F} \cap H}(H)$. We conclude from the induction structure that the map (2.29) can be identified with the map

$$\mathcal{H}_n^G(\text{pr}): \mathcal{H}_n^H(E_{\mathcal{F} \cap H}(H)) \rightarrow \mathcal{H}_n^H(\text{pt}),$$

which is bijective for all $n \in \mathbb{Z}, n \leq N$ by assumption. This finishes the proof in the case that Z is finite-dimensional. The general case follows by a colimit argument using Lemma 63.

Constructing Equivariant Homology Theories

2.7.2

Recall that a (non-equivariant) spectrum yields an associated (non-equivariant) homology theory. In this section we explain how a spectrum over the orbit category of a group G defines a G -homology theory. We would like to stress that our approach using spectra over the orbit category should be distinguished from

approaches to equivariant homology theories using spectra with G -action or the more complicated notion of G -equivariant spectra in the sense of [190], see for example [53] for a survey. The latter approach leads to a much richer structure but only works for compact Lie groups.

We briefly fix some conventions concerning spectra. We always work in the very convenient category *SPACES of compactly generated spaces* (see [295], [330, I.4]). In that category the adjunction homeomorphism $\text{map}(X \times Y, Z) \xrightarrow{\cong} \text{map}(X, \text{map}(Y, Z))$ holds without any further assumptions such as local compactness and the product of two CW -complexes is again a CW -complex. Let SPACES^+ be the category of pointed compactly generated spaces. Here the objects are (compactly generated) spaces X with base points for which the inclusion of the base point is a cofibration. Morphisms are pointed maps. If X is a space, denote by X_+ the pointed space obtained from X by adding a disjoint base point. For two pointed spaces $X = (X, x)$ and $Y = (Y, y)$ define their *smash product* as the pointed space

$$X \wedge Y = X \times Y / (\{x\} \times Y \cup X \times \{y\}) ,$$

and the *reduced cone* as

$$\text{cone}(X) = X \times [0, 1] / (X \times \{1\} \cup \{x\} \times [0, 1]) .$$

A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n): E(n) \wedge S^1 \rightarrow E(n+1)$. A *map* of spectra $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{E}'$ is a sequence of maps $f(n): E(n) \rightarrow E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e. we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \text{id}_{S^1})$ for all $n \in \mathbb{Z}$. Maps of spectra are sometimes called functions in the literature, they should not be confused with the notion of a map of spectra in the stable category (see [4, III.2.]). The category of spectra and maps will be denoted *SPECTRA*. Recall that the homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) = \text{colim}_{k \rightarrow \infty} \pi_{i+k}(E(k)) ,$$

where the system $\pi_{i+k}(E(k))$ is given by the composition

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism S and the homomorphism induced by the structure map. A *weak equivalence* of spectra is a map $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

Given a spectrum $\mathbf{E}$ and a pointed space X , we can define their smash product $X \wedge \mathbf{E}$ by $(X \wedge \mathbf{E})(n) := X \wedge E(n)$ with the obvious structure maps. It is a classical result that a spectrum $\mathbf{E}$ defines a homology theory by setting

$$H_n(X, A; \mathbf{E}) = \pi_n((X_+ \cup_{A_+} \text{cone}(A_+)) \wedge \mathbf{E}) .$$

We want to extend this to G -homology theories. This requires the consideration of spaces and spectra over the orbit category. Our presentation follows [82], where more details can be found.

In the sequel $\mathcal{C}$ is a small category. Our main example is the orbit category $\text{Or}(G)$, whose objects are homogeneous G -spaces G/H and whose morphisms are G -maps.

Definition 154 A *covariant (contravariant) $\mathcal{C}$ -space* X is a covariant (contravariant) functor

154

$$X: \mathcal{C} \rightarrow \text{SPACES}.$$

A map between $\mathcal{C}$ -spaces is a natural transformation of such functors. Analogously a *pointed $\mathcal{C}$ -space* is a functor from $\mathcal{C}$ to SPACES^+ and a $\mathcal{C}$ -spectrum a functor to SPECTRA .

Example 155. Let Y be a left G -space. Define the associated *contravariant $\text{Or}(G)$ -space* $\text{map}_G(-, Y)$ by

$$\text{map}_G(-, Y): \text{Or}(G) \rightarrow \text{SPACES}, \quad G/H \mapsto \text{map}_G(G/H, Y) = Y^H.$$

If Y is pointed then $\text{map}_G(-, Y)$ takes values in pointed spaces.

Let X be a contravariant and Y be a covariant $\mathcal{C}$ -space. Define their *balanced product* to be the space

$$X \times_{\mathcal{C}} Y := \coprod_{c \in \text{ob}(\mathcal{C})} X(c) \times Y(c) / \sim,$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for all morphisms $\phi: c \rightarrow d$ in $\mathcal{C}$ and points $x \in X(d)$ and $y \in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$. If X and Y are pointed, then one defines analogously their *balanced smash product* to be the pointed space

$$X \wedge_{\mathcal{C}} Y = \bigvee_{c \in \text{ob}(\mathcal{C})} X(c) \wedge Y(c) / \sim.$$

In [82] the notation $X \otimes_{\mathcal{C}} Y$ was used for this space. Doing the same construction level-wise one defines the *balanced smash product* $X \wedge_{\mathcal{C}} \mathbf{E}$ of a contravariant pointed $\mathcal{C}$ -space and a covariant $\mathcal{C}$ -spectrum $\mathbf{E}$.

The proof of the next result is analogous to the non-equivariant case. Details can be found in [82, lemma 4.4], where also cohomology theories are treated.

Proposition 156: (*Constructing G -Homology Theories.*) Let $\mathbf{E}$ be a covariant $\text{Or}(G)$ -spectrum. It defines a G -homology theory $H_*^G(-; \mathbf{E})$ by

156

$$H_n^G(X, A; \mathbf{E}) = \pi_n(\text{map}_G(-, (X_+ \cup_{A_+} \text{cone}(A_+))) \wedge_{\text{Or}(G)} \mathbf{E}).$$

In particular we have

$$H_n^G(G/H; \mathbf{E}) = \pi_n(\mathbf{E}(G/H)).$$

Recall that we seek an equivariant homology theory and not only a G -homology theory. If the $\text{Or}(G)$ -spectrum in Proposition 156 is obtained from a GROUPOIDS-spectrum in a way we will now describe, then automatically we obtain the desired induction structure.

Let GROUPOIDS be the category of small groupoids with covariant functors as morphisms. Recall that a groupoid is a category in which all morphisms are isomorphisms. A covariant functor $f: \mathcal{G}_0 \rightarrow \mathcal{G}_1$ of groupoids is called *injective*, if for any two objects x, y in $\mathcal{G}_0$ the induced map $\text{mor}_{\mathcal{G}_0}(x, y) \rightarrow \text{mor}_{\mathcal{G}_1}(f(x), f(y))$ is injective. Let $\text{GROUPOIDS}^{\text{inj}}$ be the subcategory of GROUPOIDS with the same objects and injective functors as morphisms. For a G -set S we denote by $\mathcal{G}^G(S)$ its associated *transport groupoid*. Its objects are the elements of S . The set of morphisms from s_0 to s_1 consists of those elements $g \in G$ which satisfy $gs_0 = s_1$. Composition in $\mathcal{G}^G(S)$ comes from the multiplication in G . Thus we obtain for a group G a covariant functor

$$\mathcal{G}^G: \text{Or}(G) \rightarrow \text{GROUPOIDS}^{\text{inj}}, \quad G/H \mapsto \mathcal{G}^G(G/H). \quad (2.30)$$

A functor of small categories $F: \mathcal{C} \rightarrow \mathcal{D}$ is called an *equivalence* if there exists a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that both $F \circ G$ and $G \circ F$ are naturally equivalent to the identity functor. This is equivalent to the condition that F induces a bijection on the set of isomorphisms classes of objects and for any objects $x, y \in \mathcal{C}$ the map $\text{mor}_{\mathcal{C}}(x, y) \rightarrow \text{mor}_{\mathcal{D}}(F(x), F(y))$ induced by F is bijective.

157

Proposition 157: (*Constructing Equivariant Homology Theories.*) Consider a covariant $\text{GROUPOIDS}^{\text{inj}}$ -spectrum

$$\mathbf{E}: \text{GROUPOIDS}^{\text{inj}} \rightarrow \text{SPECTRA}.$$

Suppose that $\mathbf{E}$ respects equivalences, i.e. it sends an equivalence of groupoids to a weak equivalence of spectra. Then $\mathbf{E}$ defines an equivariant homology theory $H_*^?(-; \mathbf{E})$, whose underlying G -homology theory for a group G is the G -homology theory associated to the covariant $\text{Or}(G)$ -spectrum $\mathbf{E} \circ \mathcal{G}^G: \text{Or}(G) \rightarrow \text{SPECTRA}$ in the previous Proposition 156, i.e.

$$H_*^G(X, A; \mathbf{E}) = H_*^G(X, A; \mathbf{E} \circ \mathcal{G}^G).$$

In particular we have

$$H_n^G(G/H; \mathbf{E}) \cong H_n^H(\text{pt}; \mathbf{E}) \cong \pi_n(\mathbf{E}(I(H))),$$

where $I(H)$ denotes H considered as a groupoid with one object. The whole construction is natural in $\mathbf{E}$.

Proof We have to specify the induction structure for a homomorphism $\alpha: H \rightarrow G$. We only sketch the construction in the special case where α is injective and $A = \emptyset$. The details of the full proof can be found in [276, theorem 2.10 on page 21].

The functor induced by α on the orbit categories is denoted in the same way

$$\alpha: \mathrm{Or}(H) \rightarrow \mathrm{Or}(G), \quad H/L \mapsto \mathrm{ind}_\alpha(H/L) = G/\alpha(L).$$

There is an obvious natural equivalence of functors $\mathrm{Or}(H) \rightarrow \mathrm{GROUPOIDS}^{\mathrm{inj}}$

$$T: \mathcal{G}^H \rightarrow \mathcal{G}^G \circ \alpha.$$

Its evaluation at H/L is the equivalence of groupoids $\mathcal{G}^H(H/L) \rightarrow \mathcal{G}^G(G/\alpha(L))$ which sends an object hL to the object $\alpha(h)\alpha(L)$ and a morphism given by $h \in H$ to the morphism $\alpha(h) \in G$. The desired isomorphism

$$\mathrm{ind}_\alpha: H_n^H(X; \mathbf{E} \circ \mathcal{G}^H) \rightarrow H_n^G(\mathrm{ind}_\alpha X; \mathbf{E} \circ \mathcal{G}^G)$$

is induced by the following map of spectra

$$\begin{aligned} \mathrm{map}_H(-, X_+) \wedge_{\mathrm{Or}(H)} \mathbf{E} \circ \mathcal{G}^H &\xrightarrow{\mathrm{id} \wedge \mathbf{E}(T)} \mathrm{map}_H(-, X_+) \wedge_{\mathrm{Or}(H)} \mathbf{E} \circ \mathcal{G}^G \circ \alpha \\ &\xleftarrow{\sim} (\alpha_* \mathrm{map}_H(-, X_+)) \wedge_{\mathrm{Or}(G)} \mathbf{E} \circ \mathcal{G}^G \xleftarrow{\sim} \mathrm{map}_G(-, \mathrm{ind}_\alpha X_+) \wedge_{\mathrm{Or}(G)} \mathbf{E} \circ \mathcal{G}^G. \end{aligned}$$

Here $\alpha_* \mathrm{map}_H(-, X_+)$ is the pointed $\mathrm{Or}(G)$ -space which is obtained from the pointed $\mathrm{Or}(H)$ -space $\mathrm{map}_H(-, X_+)$ by induction, i.e. by taking the balanced product over $\mathrm{Or}(H)$ with the $\mathrm{Or}(H)$ - $\mathrm{Or}(G)$ bimodule $\mathrm{mor}_{\mathrm{Or}(G)}(?, \alpha(?))$ [82, definition 1.8]. Notice that $\mathbf{E} \circ \mathcal{G}^G \circ \alpha$ is the same as the restriction of the $\mathrm{Or}(G)$ -spectrum $\mathbf{E} \circ \mathcal{G}^G$ along α which is often denoted by $\alpha^*(\mathbf{E} \circ \mathcal{G}^G)$ in the literature [82, definition 1.8]. The second map is given by the adjunction homeomorphism of induction α_* and restriction α^* (see [82, lemma 1.9]). The third map is the homeomorphism of $\mathrm{Or}(G)$ -spaces which is the adjoint of the obvious map of $\mathrm{Or}(H)$ -spaces $\mathrm{map}_H(-, X_+) \rightarrow \alpha^* \mathrm{map}_G(-, \mathrm{ind}_\alpha X_+)$ whose evaluation at H/L is given by ind_α .

K - and L -theory Spectra over Groupoids

2.7.3

Let RINGS be the category of associative rings with unit. An *involution* on a R is a map $R \rightarrow R$, $r \mapsto \bar{r}$ satisfying $\overline{\bar{r}} = r$, $\overline{x+y} = \bar{x} + \bar{y}$ and $\overline{x \cdot y} = \bar{y} \cdot \bar{x}$ for all $x, y \in R$. Let $\mathrm{RINGS}^{\mathrm{inv}}$ be the category of rings with involution. Let C^* -ALGEBRAS be the category of C^* -algebras. There are classical functors for $j \in -\infty \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$

$$\mathbf{K}: \mathrm{RINGS} \rightarrow \mathrm{SPECTRA}; \quad (2.31)$$

$$\mathbf{L}^{(j)}: \mathrm{RINGS}^{\mathrm{inv}} \rightarrow \mathrm{SPECTRA}; \quad (2.32)$$

$$\mathbf{K}^{\mathrm{top}}: C^*\text{-ALGEBRAS} \rightarrow \mathrm{SPECTRA}. \quad (2.33)$$

The construction of such a non-connective algebraic K -theory functor goes back to Gersten [133] and Wagoner [312]. The spectrum for quadratic algebraic L -theory is constructed by Ranicki in [258]. In a more geometric formulation it goes back to Quinn [250]. In the topological K -theory case a construction using Bott periodicity for C^* -algebras can easily be derived from the Kuiper–Mingo Theorem (see [281,

sect. 2.2]). The homotopy groups of these spectra give the algebraic K -groups of Quillen (in high dimensions) and of Bass (in negative dimensions), the decorated quadratic L -theory groups, and the topological K -groups of C^* -algebras.

We emphasize again that in all three cases we need the non-connective versions of the spectra, i.e. the homotopy groups in negative dimensions are non-trivial in general. For example the version of the Farrell–Jones Conjecture where one uses connective K -theory spectra is definitely false in general, compare Remark 15.

Now let us fix a coefficient ring R (with involution). Then sending a group G to the group ring RG yields functors $R(-): \text{GROUPS} \rightarrow \text{RINGS}$, respectively $R(-): \text{GROUPS} \rightarrow \text{RINGS}^{\text{inv}}$, where GROUPS denotes the category of groups. Let $\text{GROUPS}^{\text{inj}}$ be the category of groups with injective group homomorphisms as morphisms. Taking the reduced group C^* -algebra defines a functor $C_r^*: \text{GROUPS}^{\text{inj}} \rightarrow C^*\text{-ALGEBRAS}$. The composition of these functors with the functors (2.31), (2.32) and (2.33) above yields functors

$$\mathbf{KR}(-): \text{GROUPS} \rightarrow \text{SPECTRA}; \quad (2.34)$$

$$\mathbf{L}^{(j)}R(-): \text{GROUPS} \rightarrow \text{SPECTRA}; \quad (2.35)$$

$$\mathbf{K}^{\text{top}}C_r^*(-): \text{GROUPS}^{\text{inj}} \rightarrow \text{SPECTRA}. \quad (2.36)$$

They satisfy

$$\pi_n(\mathbf{KR}(G)) = K_n(RG);$$

$$\pi_n(\mathbf{L}^{(j)}R(G)) = L_n^{(j)}(RG);$$

$$\pi_n(\mathbf{K}^{\text{top}}C_r^*(G)) = K_n(C_r^*(G)),$$

for all groups G and $n \in \mathbb{Z}$. The next result essentially says that these functors can be extended to groupoids.

158

Theorem 158: (*K- and L-theory Spectra over Groupoids.*) Let R be a ring (with involution). There exist covariant functors

$$\mathbf{K}_R: \text{GROUPOIDS} \rightarrow \text{SPECTRA}; \quad (2.37)$$

$$\mathbf{L}_R^{(j)}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}; \quad (2.38)$$

$$\mathbf{K}^{\text{top}}: \text{GROUPOIDS}^{\text{inj}} \rightarrow \text{SPECTRA} \quad (2.39)$$

with the following properties:

- (i) If $F: \mathcal{G}_0 \rightarrow \mathcal{G}_1$ is an equivalence of (small) groupoids, then the induced maps $\mathbf{K}_R(F)$, $\mathbf{L}_R^{(j)}(F)$ and $\mathbf{K}^{\text{top}}(F)$ are weak equivalences of spectra.
- (ii) Let $I: \text{GROUPS} \rightarrow \text{GROUPOIDS}$ be the functor sending G to G considered as a groupoid, i.e. to $\mathcal{G}^G(G/G)$. This functor restricts to a functor $\text{GROUPS}^{\text{inj}} \rightarrow \text{GROUPOIDS}^{\text{inj}}$.

There are natural transformations from $\mathbf{KR}(-)$ to $\mathbf{K}_R \circ I$, from $\mathbf{L}^{(j)} R(-)$ to $\mathbf{L}_R^{(j)} \circ I$ and from $\mathbf{KC}_r^*(-)$ to $\mathbf{K}^{\text{top}} \circ I$ such that the evaluation of each of these natural transformations at a given group is an equivalence of spectra.

(iii) For every group G and all $n \in \mathbb{Z}$ we have

$$\pi_n(\mathbf{K}_R \circ I(G)) \cong K_n(RG) ;$$

$$\pi_n(\mathbf{L}_R^{(j)} \circ I^{\text{inv}}(G)) \cong L_n^{(j)}(RG) ;$$

$$\pi_n(\mathbf{K}^{\text{top}} \circ I(G)) \cong K_n(C_r^*(G)) .$$

Proof We only sketch the strategy of the proof. More details can be found in [82, sect. 2].

Let $\mathcal{G}$ be a groupoid. Similar to the group ring RG one can define an R -linear category $R\mathcal{G}$ by taking the free R -modules over the morphism sets of $\mathcal{G}$. Composition of morphisms is extended R -linearly. By formally adding finite direct sums one obtains an additive category $R\mathcal{G}_{\oplus}$. Pedersen–Weibel [237] (compare also [51]) define a non-connective algebraic K -theory functor which digests additive categories and can hence be applied to $R\mathcal{G}_{\oplus}$. For the comparison result one uses that for every ring R (in particular for RG) the Pedersen–Weibel functor applied to $R_{\oplus}$ (a small model for the category of finitely generated free R -modules) yields the non-connective K -theory of the ring R and that it sends equivalences of additive categories to equivalences of spectra. In the L -theory case $R\mathcal{G}_{\oplus}$ inherits an involution and one applies the construction of [258, example 13.6 on page 139] to obtain the $L^{(1)} = L^h$ -version. The versions for $j \leq 1$ can be obtained by a construction which is analogous to the Pedersen–Weibel construction for K -theory, compare [55, sect. 4]. In the C^* -case one obtains from $\mathcal{G}$ a C^* -category $C_r^*(\mathcal{G})$ and assigns to it its topological K -theory spectrum. There is a construction of the topological K -theory spectrum of a C^* -category in [82, sect. 2]. However, the construction given there depends on two statements, which appeared in [130, proposition 1 and proposition 3], and those statements are incorrect, as already pointed out by Thomason in [302]. The construction in [82, sect. 2] can easily be fixed but instead we recommend the reader to look at the more recent construction of Joachim [159].

Assembly Maps in Terms of Homotopy Colimits

2.7.4

In this section we describe a homotopy-theoretic formulation of the Baum–Connes and Farrell–Jones Conjectures. For the classical assembly maps which in our set-up correspond to the trivial family such formulations were described in [328].

For a group G and a family $\mathcal{F}$ of subgroups we denote by $\text{Or}_{\mathcal{F}}(G)$ the *restricted orbit category*. Its objects are homogeneous spaces G/H with $H \in \mathcal{F}$. Morphisms are G -maps. If $\mathcal{F} = \mathcal{ALL}$ we get back the (full) orbit category, i.e. $\text{Or}(G) = \text{Or}_{\mathcal{ALL}}(G)$.

159 **Meta-Conjecture 159:** (*Homotopy-Theoretic Isomorphism Conjecture.*) Let G be a group and $\mathcal{F}$ a family of subgroups. Let $\mathbf{E}: \text{Or}(G) \rightarrow \text{SPECTRA}$ be a covariant functor. Then

$$A_{\mathcal{F}}: \text{hocolim}_{\text{Or}_{\mathcal{F}}(G)} \mathbf{E}|_{\text{Or}_{\mathcal{F}}(G)} \rightarrow \text{hocolim}_{\text{Or}(G)} \mathbf{E} \simeq \mathbf{E}(G/G)$$

is a weak equivalence of spectra.

Here hocolim is the homotopy colimit of a covariant functor to spectra, which is itself a spectrum. The map $A_{\mathcal{F}}$ is induced by the obvious functor $\text{Or}_{\mathcal{F}}(G) \rightarrow \text{Or}(G)$. The equivalence $\text{hocolim}_{\text{Or}(G)} \mathbf{E} \simeq \mathbf{E}(G/G)$ comes from the fact that G/G is a final object in $\text{Or}(G)$. For information about homotopy-colimits we refer to [40], [82, sect. 3] and [88].

160 **Remark 160** If we consider the map on homotopy groups that is induced by the map $A_{\mathcal{F}}$ which appears in the Homotopy-Theoretic Isomorphism Conjecture above, then we obtain precisely the map with the same name in Meta-Conjecture 57 for the homology theory $H_*^G(-; \mathbf{E})$ associated with $\mathbf{E}$ in Proposition 156, compare [82, sect. 5]. In particular the Baum–Connes Conjecture 59 and the Farrell–Jones Conjecture 58 can be seen as special cases of Meta-Conjecture 159.

161 **Remark 161:** (*Universal Property of the Homotopy-Theoretic Assembly Map.*) The Homotopy-Theoretic Isomorphism Conjecture 159 is in some sense the most conceptual formulation of an Isomorphism Conjecture because it has a universal property as the universal approximation from the left by a (weakly) excisive $\mathcal{F}$ -homotopy invariant functor. This is explained in detail in [82, sect. 6]. This universal property is important if one wants to identify different models for the assembly map, compare e.g. [19, sect. 6] and [142].

2.7.5 Naturality under Induction

Consider a covariant functor $\mathbf{E}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$ which respects equivalences. Let $H_*^?(-; \mathbf{E})$ be the associated equivariant homology theory (see Proposition 157). Then for a group homomorphism $\alpha: H \rightarrow G$ and H -CW-pair (X, A) we obtain a homomorphism

$$\text{ind}_{\alpha}: H_n^H(X, A; \mathbf{E}) \rightarrow H_n^G(\text{ind}_{\alpha}(X, A); \mathbf{E})$$

which is natural in (X, A) . Note that we did not assume that $\ker(\alpha)$ acts freely on X . In fact the construction sketched in the proof of Proposition 157 still works even though ind_{α} may not be an isomorphism as it is the case if $\ker(\alpha)$ acts freely. We still have functoriality as described in 2.7.1 towards the beginning of Sect. 2.7.1.

Now suppose that $\mathcal{H}$ and $\mathcal{G}$ are families of subgroups for H and G such that $\alpha(K) \in \mathcal{G}$ holds for all $K \in \mathcal{H}$. Then we obtain a G -map $f: \text{ind}_\alpha E_{\mathcal{H}}(H) \rightarrow E_{\mathcal{G}}(G)$ from the universal property of $E_{\mathcal{G}}(G)$. Let $p: \text{ind}_\alpha \text{pt} = G/\alpha(H) \rightarrow \text{pt}$ be the projection. Let $I: \text{GROUPS} \rightarrow \text{GROUPOIDS}$ be the functor sending G to $\mathcal{G}^G(G/G)$. Then the following diagram, where the horizontal arrows are induced from the projections to the one point space, commutes for all $n \in \mathbb{Z}$.

$$\begin{array}{ccc} H_n^H(E_{\mathcal{H}}(H); \mathbf{E}) & \xrightarrow{A_{\mathcal{H}}} & H_n^H(\text{pt}; \mathbf{E}) = \pi_n(\mathbf{E}(I(H))) \\ H_n^G(f) \circ \text{ind}_\alpha \downarrow & & \downarrow H_n^G(p) \circ \text{ind}_\alpha = \pi_n(\mathbf{E}(I(\alpha))) \\ H_n^G(E_{\mathcal{G}}(G); \mathbf{E}) & \xrightarrow{A_{\mathcal{G}}} & H_n^G(\text{pt}; \mathbf{E}) = \pi_n(\mathbf{E}(I(G))) . \end{array}$$

If we take the special case $\mathbf{E} = \mathbf{K}_R$ and $\mathcal{H} = \mathcal{G} = \mathcal{VCY}$, we get the following commutative diagram, where the horizontal maps are the assembly maps appearing in the Farrell–Jones Conjecture 58 and α_* is the change of rings homomorphism (induction) associated to α .

$$\begin{array}{ccc} H_n^H(E_{\mathcal{VCY}}(H); \mathbf{K}_R) & \xrightarrow{A_{\mathcal{VCY}}} & K_n(RH) \\ H_n^G(f) \circ \text{ind}_\alpha \downarrow & & \downarrow \alpha_* \\ H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) & \xrightarrow{A_{\mathcal{VCY}}} & K_n(RG) . \end{array}$$

We see that we can define a kind of induction homomorphism on the source of the assembly maps which is compatible with the induction structure given on their target. We get analogous diagrams for the L -theoretic version of the Farrell–Jones–Isomorphism Conjecture 58, for the Bost Conjecture 104 and for the Baum–Connes Conjecture for maximal group C^* -algebras (see (2.22) in Sect. 2.5.1).

Remark 162 The situation for the Baum–Connes Conjecture 59 itself, where one has to work with reduced C^* -algebras, is more complicated. Recall that not every group homomorphism $\alpha: H \rightarrow G$ induces a homomorphism of C^* -algebras $C_r^*(H) \rightarrow C_r^*(G)$. (It does if $\ker(\alpha)$ is finite.) But it turns out that the source $H_n^H(E_{\mathcal{FIN}}(H); \mathbf{K}^{\text{top}})$ always admits such a homomorphism. The point is that the isotropy groups of $E_{\mathcal{FIN}}(H)$ are all finite and the spectra-valued functor $\mathbf{K}^{\text{top}}$ extends from $\text{GROUPOIDS}^{\text{inj}}$ to the category $\text{GROUPOIDS}^{\text{finker}}$, which has small groupoids as objects but as morphisms only those functors $f: \mathcal{G}_0 \rightarrow \mathcal{G}_1$ with finite kernels (in the sense that for each object $x \in \mathcal{G}_0$ the group homomorphism $\text{aut}_{\mathcal{G}_0}(x) \rightarrow \text{aut}_{\mathcal{G}_1}(f(x))$ has finite kernel). This is enough to get for any group homomorphism $\alpha: H \rightarrow G$ an induced map $\text{ind}_\alpha: H_n^H(X, A; \mathbf{K}^{\text{top}}) \rightarrow$

$H_n^G(\text{ind}_\alpha(X, A); \mathbf{K}^{\text{top}})$ provided that X is proper. Hence one can define an induction homomorphism for the source of the assembly map as above.

In particular the Baum–Connes Conjecture 59 predicts that for any group homomorphism $\alpha: H \rightarrow G$ there is an induced induction homomorphism $\alpha_*: K_n(C_r^*(H)) \rightarrow K_n(C_r^*(G))$ on the targets of the assembly maps although there is no induced homomorphism of C^* -algebras $C_r^*(H) \rightarrow C_r^*(G)$ in general.

2.8 Methods of Proof

In Chapt. 2.3, we formulated the Baum–Connes Conjecture 58 and the Farrell–Jones Conjecture 59 in abstract homological terms. We have seen that this formulation was very useful in order to understand formal properties of assembly maps. But in order to actually prove cases of the conjectures one needs to interpret the assembly maps in a way that is more directly related to geometry or analysis. In this chapter we wish to explain such approaches to the assembly maps. We briefly survey some of the methods of proof that are used to attack the Baum–Connes Conjecture 59 and the Farrell–Jones Conjecture 58.

2.8.1 Analytic Equivariant K -Homology

Recall that the covariant functor $\mathbf{K}^{\text{top}}: \text{GROUPOIDS}^{\text{inj}} \rightarrow \text{SPECTRA}$ introduced in (2.39) defines an equivariant homology theory $H_*^?(-; \mathbf{K}^{\text{top}})$ in the sense of Sect. 2.7.1 such that

$$H_n^G(G/H; \mathbf{K}^{\text{top}}) = H_n^H(\text{pt}; \mathbf{K}^{\text{top}}) = \begin{cases} R(H) & \text{for even } n; \\ 0 & \text{for odd } n, \end{cases}$$

holds for all groups G and subgroups $H \subseteq G$ (see Proposition 157). Next we want to give for a proper G -CW-complex X an analytic definition of $H_n^G(X; \mathbf{K}^{\text{top}})$.

Consider a locally compact proper G -space X . Recall that a G -space X is called *proper* if for each pair of points x and y in X there are open neighborhoods V_x of x and W_y of y in X such that the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of G is finite. A G -CW-complex X is proper if and only if all its isotropy groups are finite [197, theorem 1.23]. Let $C_0(X)$ be the C^* -algebra of continuous functions $f: X \rightarrow \mathbb{C}$ which vanish at infinity. The C^* -norm is the supremum norm. A *generalized elliptic G -operator* is a triple (U, φ, F) , which consists of a unitary representation $U: G \rightarrow \mathcal{B}(H)$ of G on a Hilbert space H , a $*$ -representation $\varphi: C_0(X) \rightarrow \mathcal{B}(H)$ such that $\varphi(f \circ l_{g^{-1}}) = U(g) \circ \varphi(f) \circ U(g)^{-1}$ holds for $g \in G$, and a bounded selfadjoint G -operator $F: H \rightarrow H$ such that the operators $\varphi(f)(F^2 - 1)$ and $[\varphi(f), F]$ are compact for all $f \in C_0(X)$. Here $\mathcal{B}(H)$ is the C^* -algebra of bounded operators $H \rightarrow H$, $l_{g^{-1}}: H \rightarrow H$ is given by multiplication with g^{-1} , and $[\varphi(f), F] = \varphi(f) \circ F - F \circ \varphi(f)$.

We also call such a triple (U, φ, F) an *odd cycle*. If we additionally assume that H comes with a $\mathbb{Z}/2$ -grading such that φ preserves the grading if we equip $C_0(X)$ with the trivial grading, and F reverses it, then we call (U, φ, F) an *even cycle*. This means that we have an orthogonal decomposition $H = H_0 \oplus H_1$ such that U, φ and F look like

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix} \quad \varphi = \begin{pmatrix} \varphi_0 & 0 \\ 0 & \varphi_1 \end{pmatrix} \quad F = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}.$$

An important example of an even cocycle is described in Sect. 2.8.5. A cycle (U, φ, f) is called *degenerate*, if for each $f \in C_0(X)$ we have $[\varphi(f), F] = \varphi(f)(F^2 - 1) = 0$. Two cycles (U_0, φ_0, F_0) and (U_1, φ_1, F_1) of the same parity are called *homotopic*, if $U_0 = U_1$, $\varphi_0 = \varphi_1$ and there exists a norm continuous path $F_t, t \in [0, 1]$ in $\mathcal{B}(H)$ such that for each $t \in [0, 1]$ the triple (U_0, φ_0, F_t) is again a cycle of the same parity. Two cycles (U_0, φ_0, F_0) and (U_1, φ_1, F_1) are called *equivalent*, if they become homotopic after taking the direct sum with degenerate cycles of the same parity. Let $K_n^G(C_0(X))$ for even n be the set of equivalence classes of even cycles and $K_n^G(C_0(X))$ for odd n be the set of equivalence classes of odd cycles. These become abelian groups by the direct sum. The neutral element is represented by any degenerate cycle. The inverse of an even cycle is represented by the cycle obtained by reversing the grading of H . The inverse of an odd cycle (U, φ, F) is represented by $(U, \varphi, -F)$.

A proper G -map $f: X \rightarrow Y$ induces a map of C^* -algebras $C_0(f): C_0(Y) \rightarrow C_0(X)$ by composition and thus in the obvious way a homomorphism of abelian groups $K_0^G(f): K_0^G(C_0(X)) \rightarrow K_0^G(C_0(Y))$. It depends only on the proper G -homotopy class of f . One can show that this construction defines a G -homology theory on the category of finite proper G -CW-complexes. It extends to a G -homology theory K_*^G for all proper G -CW-complexes by

$$K_n^G(X) = \operatorname{colim}_{Y \in I(X)} K_n^G(C_0(Y)), \quad (2.40)$$

where $I(X)$ is the set of proper finite G -CW-subcomplexes $Y \subseteq X$ directed by inclusion. This definition is forced upon us by Lemma 63. The groups $K_n^G(X)$ and $K_n^G(C_0(X))$ agree for finite proper G -CW-complexes, in general they are different.

The cycles were introduced by Atiyah [11]. The equivalence relation, the group structure and the homological properties of $K_n^G(X)$ were established by Kasparov [168]. More information about analytic K -homology can be found in Higson–Roe [154].

The Analytic Assembly Map

2.8.2

For every G -CW-complex X the projection $\operatorname{pr}: X \rightarrow \operatorname{pt}$ induces a map

$$H_n^G(X; \mathbf{K}^{\operatorname{top}}) \rightarrow H_n^G(\operatorname{pt}; \mathbf{K}^{\operatorname{top}}) = K_n(C_r^*(G)). \quad (2.41)$$

In the case where X is the proper G -space $E_{\mathcal{F}, \mathbf{1}, \mathcal{N}}(G)$ we obtain the assembly map appearing in the Baum–Connes Conjecture 59. We explain its analytic analogue

$$\mathrm{ind}_G: K_n^G(X) \rightarrow K_n(C_r^*(G)). \quad (2.42)$$

Note that we need to assume that X is a proper G -space since $K_n^G(X)$ was only defined for such spaces. It suffices to define the map for a finite proper G -CW-complex X . In this case it assigns to the class in $K_n^G(X) = K_n^G(C_0(X))$ represented by a cycle (U, φ, F) its G -index in $K_n(C_r^*(G))$ in the sense of Mishchenko–Fomenko [223]. At least in the simple case, where G is finite, we can give its precise definition. The odd K -groups vanish in this case and $K_0(C_r^*(G))$ reduces to the complex representation ring $R(G)$. If we write F in matrix form as in (2.8.1) then $P: H \rightarrow H$ is a G -equivariant Fredholm operator. Hence its kernel and cokernel are G -representations and the G -index of F is defined as $[\ker(P)] - [\mathrm{coker}(P)] \in R(G)$. In the case of an infinite group the kernel and cokernel are a priori not finitely generated projective modules over $C_r^*(G)$, but they are after a certain perturbation. Moreover the choice of the perturbation does not affect $[\ker(P)] - [\mathrm{coker}(P)] \in K_0(C_r^*(G))$.

The identification of the two assembly maps (2.41) and (2.42) has been carried out in Hambleton–Pedersen [142] using the universal characterization of the assembly map explained in [82, sect. 6]. In particular for a proper G -CW-complex X we have an identification $H_n^G(X; \mathbf{K}^{\mathrm{top}}) \cong K_n^G(X)$. Notice that $H_n^G(X; \mathbf{K}^{\mathrm{top}})$ is defined for all G -CW-complexes, whereas $K_n^G(X)$ has only been introduced for proper G -CW-complexes.

Thus the Baum–Connes Conjecture 59 gives an index-theoretic interpretations of elements in $K_0(C_r^*(G))$ as generalized elliptic operators or cycles (U, φ, F) . We have explained already in Sect. 2.2.8 an application of this interpretation to the Trace Conjecture for Torsionfree Groups 37 and in Sect. 2.4.3 to the Stable Gromov–Lawson–Rosenberg Conjecture 99.

2.8.3

Equivariant KK -Theory

Kasparov [170] developed *equivariant KK -theory*, which we will briefly explain next. It is one of the basic tools in the proofs of theorems about the Baum–Connes Conjecture 59.

A G - C^* -algebra A is a C^* -algebra with a G -action by $*$ -automorphisms. To any pair of separable G - C^* -algebras (A, B) Kasparov assigns abelian groups $KK_n^G(A, B)$. If G is trivial, we write briefly $KK_n(A, B)$. We do not give the rather complicated definition but state the main properties.

If we equip $\mathbb{C}$ with the trivial G -action, then $KK_n^G(C_0(X), \mathbb{C})$ reduces to the abelian group $K_n^G(C_0(X))$ introduced in Sect. 2.8.1. The topological K -theory $K_n(A)$ of a C^* -algebra coincides with $KK_n(\mathbb{C}, A)$. The equivariant KK -groups are covariant in the second and contravariant in the first variable under homomorphism of C^* -algebras. One of the main features is the bilinear Kasparov product

$$KK_i^G(A, B) \times KK_j^G(B, C) \rightarrow KK_{i+j}^G(A, C), \quad (\alpha, \beta) \mapsto \alpha \otimes_B \beta. \quad (2.43)$$

It is associative and natural. A homomorphism $\alpha: A \rightarrow B$ defines an element in $KK_0(A, B)$. There are natural *descent homomorphisms*

$$j_G: KK_n^G(A, B) \rightarrow KK_n(A \rtimes_r G, B \rtimes_r G), \quad (2.44)$$

where $A \rtimes_r G$ and $B \rtimes_r G$ denote the reduced crossed product C^* -algebras.

The Dirac-Dual Dirac Method

2.8.4

A G - C^* -algebra A is called *proper* if there exists a locally compact proper G -space X and a G -homomorphism $\sigma: C_0(X) \rightarrow \mathcal{B}(A)$, $f \mapsto \sigma_f$ satisfying $\sigma_f(ab) = a\sigma_f(b) = \sigma_f(a)b$ for $f \in C_0(X)$, $a, b \in A$ and for every net $\{f_i \mid i \in I\}$, which converges to 1 uniformly on compact subsets of X , we have $\lim_{i \in I} \|\sigma_{f_i}(a) - a\| = 0$ for all $a \in A$. A locally compact G -space X is proper if and only if $C_0(X)$ is proper as a G - C^* -algebra.

Given a proper G -CW-complex X and a G - C^* -algebra A , we put

$$KK_n^G(X; A) = \operatorname{colim}_{Y \in I(X)} KK_n^G(C_0(Y), A), \quad (2.45)$$

where $I(Y)$ is the set of proper finite G -CW-subcomplexes $Y \subseteq X$ directed by inclusion. We have $KK_n^G(X; \mathbb{C}) = K_n^G(X)$. There is an analytic index map

$$\operatorname{ind}_G^A: KK_n^G(X; A) \rightarrow K_n(A \rtimes_r G), \quad (2.46)$$

which can be identified with the assembly map appearing in the Baum–Connes Conjecture with Coefficients 105. The following result is proved in Tu [305] extending results of Kasparov–Skandalis [169, 172].

Theorem 163 The Baum–Connes Conjecture with coefficients 105 holds for a proper G - C^* -algebra A , i.e. $\operatorname{ind}_G^A: KK_n^G(E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G); A) \rightarrow K_n(A \rtimes G)$ is bijective.

163

Now we are ready to state the *Dirac-dual Dirac method* which is the key strategy in many of the proofs of the Baum–Connes Conjecture 59 or the Baum–Connes Conjecture with coefficients 105.

Theorem 164: (*Dirac-Dual Dirac Method.*) Let G be a countable (discrete) group. Suppose that there exist a proper G - C^* -algebra A , elements $\alpha \in KK_i^G(A, \mathbb{C})$, called the *Dirac element*, and $\beta \in KK_i^G(\mathbb{C}, A)$, called the *dual Dirac element*, satisfying

164

$$\beta \otimes_A \alpha = 1 \in KK_0^G(\mathbb{C}, \mathbb{C}).$$

Then the Baum–Connes Conjecture 59 is true, or, equivalently, the analytic index map

$$\operatorname{ind}_G: K_n^G(X) \rightarrow K_n(C_r^*(G))$$

of 2.42 is bijective.

Proof The index map ind_G is a retract of the bijective index map ind_G^A from Theorem 163. This follows from the following commutative diagram

$$\begin{array}{ccccc}
 K_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G)) & \xrightarrow{-\otimes_{\mathbb{C}}\beta} & KK_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); A) & \xrightarrow{-\otimes_A\alpha} & K_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G)) \\
 \downarrow \text{ind}_G & & \downarrow \text{ind}_G^A & & \downarrow \text{ind}_G \\
 K_n(C_r^*(G)) & \xrightarrow{-\otimes_{C_r^*(G)}j_G(\beta)} & K_n(A \rtimes_r G) & \xrightarrow{-\otimes_{A \rtimes_r G}j_G(\alpha)} & K_n(C_r^*(G))
 \end{array}$$

and the fact that the composition of both the top upper horizontal arrows and lower upper horizontal arrows are bijective.

2.8.5 An Example of a Dirac Element

In order to give a glimpse of the basic ideas from operator theory we briefly describe how to define the Dirac element α in the case where G acts by isometries on a complete Riemannian manifold M . Let $T_{\mathbb{C}}M$ be the complexified tangent bundle and let $\text{Cliff}(T_{\mathbb{C}}M)$ be the associated Clifford bundle. Let A be the proper G - C^* -algebra given by the sections of $\text{Cliff}(T_{\mathbb{C}}M)$ which vanish at infinity. Let H be the Hilbert space $L^2(\wedge T_{\mathbb{C}}^*M)$ of L^2 -integrable differential forms on $T_{\mathbb{C}}M$ with the obvious $\mathbb{Z}/2$ -grading coming from even and odd forms. Let U be the obvious G -representation on H coming from the G -action on M . For a 1-form ω on M and $u \in H$ define a $*$ -homomorphism $\varrho: A \rightarrow \mathcal{B}(H)$ by

$$\varrho_{\omega}(u) := \omega \wedge u + i_{\omega}(u) .$$

Now $D = (d + d^*)$ is a symmetric densely defined operator $H \rightarrow H$ and defines a bounded selfadjoint operator $F: H \rightarrow H$ by putting $F = \frac{D}{\sqrt{1+D^2}}$. Then (U, ϱ, F) is an even cocycle and defines an element $\alpha \in K_0^G(M) = KK_0^G(C_0(M), \mathbb{C})$. More details of this construction and the construction of the dual Dirac element β under the assumption that M has non-positive curvature and is simply connected, can be found for instance in [307, chapt. 9].

2.8.6 Banach KK -Theory

Skandalis showed that the Dirac-dual Dirac method cannot work for all groups [287] as long as one works with KK -theory in the unitary setting. The problem is that for a group with property (T) the trivial and the regular unitary representation cannot be connected by a continuous path in the space of unitary representations, compare also the discussion in [163]. This problem can be circumvented if one drops the condition unitary and works with a variant of KK -theory for Banach algebras as worked out by Lafforgue [183, 185, 186].

Controlled Topology and Algebra

2.8.7

To a topological problem one can often associate a notion of “size”. We describe a prototypical example. Let M be a Riemannian manifold. Recall that an h -cobordism W over $M = \partial^- W$ admits retractions $r^\pm: W \times I \rightarrow W$, $(x, t) \mapsto r_t^\pm(x, t)$ which retract W to $\partial^\pm W$, i.e. which satisfy $r_0^\pm = \text{id}_W$ and $r_1^\pm(W) \subset \partial^\pm W$. Given $\varepsilon > 0$ we say that W is ε -controlled if the retractions can be chosen in such a way that for every $x \in W$ the paths (called tracks of the h -cobordism) $p_x^\pm: I \rightarrow M$, $t \mapsto r_1^\pm \circ r_t^\pm(x)$ both lie within an ε -neighbourhood of their starting point. The usefulness of this concept is illustrated by the following theorem [124].

Theorem 165 Let M be a compact Riemannian manifold of dimension ≥ 5 . Then there exists an $\varepsilon = \varepsilon_M > 0$, such that every ε -controlled h -cobordism over M is trivial.

165

If one studies the s -Cobordism Theorem 5 and its proof one is naturally lead to algebraic analogues of the notions above. A (geometric) R -module over the space X is by definition a family $M = (M_x)_{x \in X}$ of free R -modules indexed by points of X with the property that for every compact subset $K \subset X$ the module $\bigoplus_{x \in K} M_x$ is a finitely generated R -module. A morphism ϕ from M to N is an R -linear map $\phi = (\phi_{y,x}): \bigoplus_{x \in X} M_x \rightarrow \bigoplus_{y \in X} N_y$. Instead of specifying fundamental group data by paths (analogues of the tracks of the h -cobordism) one can work with modules and morphisms over the universal covering $\tilde{X}$, which are invariant under the operation of the fundamental group $G = \pi_1(X)$ via deck transformations, i.e. we require that $M_{gx} = M_x$ and $\phi_{gy,gx} = \phi_{y,x}$. Such modules and morphisms form an additive category which we denote by $\mathcal{C}^G(\tilde{X}; R)$. In particular one can apply to it the non-connective K -theory functor $\mathbf{K}$ (compare [237]). In the case where X is compact the category is equivalent to the category of finitely generated free RG -modules and hence $\pi_* \mathbf{K} \mathcal{C}^G(\tilde{X}; R) \cong K_*(RG)$. Now suppose $\tilde{X}$ is equipped with a G -invariant metric, then we will say that a morphism $\phi = (\phi_{y,x})$ is ε -controlled if $\phi_{y,x} = 0$, whenever x and y are further than ε apart. (Note that ε -controlled morphisms do not form a category because the composition of two such morphisms will in general be 2ε -controlled.)

Theorem 165 has the following algebraic analogue [251] (see also sect. 4 in [240]).

Theorem 166 Let M be a compact Riemannian manifold with fundamental group G . There exists an $\varepsilon = \varepsilon_M > 0$ with the following property. The K_1 -class of every G -invariant automorphism of modules over $\tilde{M}$ which together with its inverse is ε -controlled lies in the image of the classical assembly map

166

$$H_1(BG; \mathbf{K}R) \rightarrow K_1(RG) \cong K_1(\mathcal{C}^G(\tilde{M}; R)) .$$

To understand the relation to Theorem 165 note that for $R = \mathbb{Z}$ such an ε -controlled automorphism represents the trivial element in the Whitehead group which is in bijection with the h -cobordisms over M , compare Theorem 5.

There are many variants to the simple concept of “metric ε -control” we used above. In particular it is very useful to not measure size directly in M but instead use a map $p: M \rightarrow X$ to measure size in some auxiliary space X . (For example we have seen in Sects. 2.2.2 and 2.2.4 that “bounded” control over $\mathbb{R}^k$ may be used in order to define or describe negative K -groups.)

Before we proceed we would like to mention that there are analogous control-notions for pseudoisotopies and homotopy equivalences. The tracks of a pseudoisotopy $f: M \times I \rightarrow M \times I$ are defined as the paths in M which are given by the composition

$$p_x: I = \{x\} \times I \subset M \times I \xrightarrow{f} M \times I \xrightarrow{p} M$$

for each $x \in M$, where the last map is the projection onto the M -factor. Suppose $f: N \rightarrow M$ is a homotopy equivalence, $g: M \rightarrow N$ its inverse and h_t and h'_t are homotopies from $f \circ g$ to id_M respectively from $g \circ f$ to id_N then the tracks are defined to be the paths in M that are given by $t \mapsto h_t(x)$ for $x \in M$ and $t \mapsto f \circ h'_t(y)$ for $y \in N$. In both cases, for pseudoisotopies and for homotopy equivalences, the tracks can be used to define ε -control.

2.8.8 Assembly as Forget Control

If instead of a single problem over M one defines a family of problems over $M \times [1, \infty)$ and requires the control to tend to zero for $t \rightarrow \infty$ in a suitable sense, then one obtains something which is a homology theory in M . Relaxing the control to just bounded families yields the classical assembly map. This idea appears in [252] in the context of pseudoisotopies and in a more categorical fashion suitable for higher algebraic K -theory in [55] and [241]. We spell out some details in the case of algebraic K -theory, i.e. for geometric modules.

Let M be a Riemannian manifold with fundamental group G and let $\mathcal{S}(1/t)$ be the space of all functions $[1, \infty) \rightarrow [0, \infty)$, $t \mapsto \delta_t$ such that $t \mapsto t \cdot \delta_t$ is bounded. Similarly let $\mathcal{S}(1)$ be the space of all functions $t \mapsto \delta_t$ which are bounded. Note that $\mathcal{S}(1/t) \subset \mathcal{S}(1)$. A G -invariant morphism ϕ over $\tilde{M} \times [1, \infty)$ is $\mathcal{S}$ -controlled for $\mathcal{S} = \mathcal{S}(1)$ or $\mathcal{S}(1/t)$ if there exists an $\alpha > 0$ and a $\delta_t \in \mathcal{S}$ (both depending on the morphism) such that $\phi_{(x,t),(x',t')} \neq 0$ implies that $|t - t'| \leq \alpha$ and $d_{\tilde{M}}(x, x') \leq \delta_{\min\{t, t'\}}$. We denote by $\mathcal{C}^G(\tilde{M} \times [1, \infty), \mathcal{S}; R)$ the category of all $\mathcal{S}$ -controlled morphisms. Furthermore $\mathcal{C}^G(\tilde{M} \times [1, \infty), \mathcal{S}; R)^\infty$ denotes the quotient category which has the same objects, but where two morphisms are identified, if their difference factorizes over an object which lives over $\tilde{M} \times [1, N]$ for some large but finite number N . This passage to the quotient category is called “taking germs at infinity”. It is a special case of a Karoubi quotient, compare [51].

Theorem 167: (*Classical Assembly as Forget Control.*) Suppose M is aspherical, i.e. M is a model for BG , then for all $n \in \mathbb{Z}$ the map

$$\pi_n(\mathbf{KC}^G(\tilde{M} \times [1, \infty), \mathcal{S}(1/t); R)^\infty) \rightarrow \pi_n(\mathbf{KC}^G(\tilde{M} \times [1, \infty), \mathcal{S}(1); R)^\infty)$$

can be identified up to an index shift with the classical assembly map that appears in Conjecture 11, i.e. with

$$H_{n-1}(BG; \mathbf{K}(R)) \rightarrow K_{n-1}(RG).$$

Note that the only difference between the left and the right hand side is that on the left morphism are required to become smaller in a $1/t$ -fashion, whereas on the right hand side they are only required to stay bounded in the $[1, \infty)$ -direction.

Using so called equivariant continuous control (see [7] and [19, sect. 2] for the equivariant version) one can define an equivariant homology theory which applies to arbitrary G -CW-complexes. This leads to a “forget-control description” for the generalized assembly maps that appear in the Farrell–Jones Conjecture 58. Alternatively one can use stratified spaces and stratified Riemannian manifolds in order to describe generalized assembly maps in terms of metric control. Compare [111, 3.6 on p. 270] and [252, appendix].

Methods to Improve Control

2.8.9

From the above description of assembly maps we learn that the problem of proving surjectivity results translates into the problem of improving control. A combination of many different techniques is used in order to achieve such control-improvements. We discuss some prototypical arguments which go back to [98] and [102] and again restrict attention to K -theory. Of course this can only give a very incomplete impression of the whole program which is mainly due to Farrell–Hsiang and Farrell–Jones. The reader should consult [120] and [162] for a more detailed survey.

We restrict to the basic case, where M is a compact Riemannian manifold with negative sectional curvature. In order to explain a contracting property of the geodesic flow $\Phi: \mathbb{R} \times \tilde{S}M \rightarrow \tilde{S}M$ on the unit sphere bundle $\tilde{S}M$, we introduce the notion of foliated control. We think of $\tilde{S}M$ as a manifold equipped with the one-dimensional foliation by the flow lines of Φ and equip it with its natural Riemannian metric. Two vectors v and w in $\tilde{S}M$ are called foliated (α, δ) -controlled if there exists a path of length α inside one flow line such that v lies within distance $\delta/2$ of the starting point of that path and w lies within distance $\delta/2$ of its endpoint.

Two vectors v and $w \in \tilde{S}M$ are called asymptotic if the distance between their associated geodesic rays is bounded. These rays will then determine the same point on the sphere at infinity which can be introduced to compactify $\tilde{M}$ to a disk. Recall that the universal covering of a negatively curved manifold is diffeomorphic to $\mathbb{R}^n$.

Suppose v and w are α -controlled asymptotic vectors, i.e. their distance is smaller than $\alpha > 0$. As a consequence of negative sectional curvature the vectors $\Phi_t(v)$ and $\Phi_t(w)$ are foliated $(C\alpha, \delta_t)$ -controlled, where $C > 1$ is a constant and $\delta_t > 0$ tends to zero when t tends to ∞ . So roughly speaking the flow contracts the directions transverse to the flow lines and leaves the flow direction as it is, at least if we only apply it to asymptotic vectors.

This property can be used in order to find foliated (α, δ) -controlled representatives of K -theory classes with arbitrary small δ if one is able to define a suitable transfer from M to $\tilde{S}M$, which yields representatives whose support is in an asymptotic starting position for the flow. Here one needs to take care of the additional problem that in general such a transfer may not induce an isomorphism in K -theory.

Finally one is left with the problem of improving foliated control to ordinary control. Corresponding statements are called “Foliated Control Theorems”. Compare [18, 101, 103, 104] and [108].

If such an improvement were possible without further hypothesis, we could prove that the classical assembly map, i.e. the assembly map with respect to the trivial family is surjective. We know however that this is not true in general. It fails for example in the case of topological pseudoisotopies or for algebraic K -theory with arbitrary coefficients. In fact the geometric arguments that are involved in a “Foliated Control Theorem” need to exclude flow lines in $\tilde{S}M$ which correspond to “short” closed geodesic loops in SM . But the techniques mentioned above can be used in order to achieve ε -control for arbitrary small $\varepsilon > 0$ outside of a suitably chosen neighbourhood of “short” closed geodesics. This is the right kind of control for the source of the assembly map which involves the family of cyclic subgroups. (Note that a closed loop in M determines the conjugacy class of a maximal infinite cyclic subgroup inside $G = \pi_1(M)$.) We see that even in the torsionfree case the class of cyclic subgroups of G naturally appears during the proof of a surjectivity result.

Another source for processes which improve control are expanding self-maps. Think for example of an n -torus $\mathbb{R}^n/\mathbb{Z}^n$ and the self-map f_s which is induced by $m_s: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \mapsto sx$ for a large positive integer s . If one pulls an automorphism back along such a map one can improve control, but unfortunately the new automorphism describes a different K -theory class. Additional algebraic arguments nevertheless make this technique very successful. Compare for example [98]. Sometimes a clever mixture between flows and expanding self-maps is needed in order to achieve the goal, compare [105]. Recent work of Farrell–Jones (see [114–116] and [161]) makes use of a variant of the Cheeger–Fukaya–Gromov collapsing theory.

Remark 168: (*Algebraicizing the Farrell–Jones Approach.*) In this Subsection we sketched some of the geometric ideas which are used in order to obtain control over an h -cobordism, a pseudoisotopy or an automorphism of a geometric module representing a single class in K_1 . In Sect. 2.8.8 we used families over the cone

$M \times [1, \infty)$ in order to describe the whole algebraic K -theory assembly map at once in categorical terms without ever referring to a single K -theory element. The recent work [21] shows that the geometric ideas can be adapted to this more categorical set-up, at least in the case where the group is the fundamental group of a Riemannian manifold with strictly negative curvature. However serious difficulties had to be overcome in order to achieve this. One needs to formulate and prove a Foliated Control Theorem in this context and also construct a transfer map to the sphere bundle for higher K -theory which is in a suitable sense compatible with the control structures.

The Descent Principle

2.8.10

In Theorem 167 we described the classical assembly map as a forget control map using G -invariant geometric modules over $\tilde{M} \times [1, \infty)$. If in that context one does not require the modules and morphisms to be invariant under the G -action one nevertheless obtains a forget control functor between additive categories for which we introduce the notation

$$\mathcal{D}(1/t) = \mathcal{C}(\tilde{M} \times [1, \infty), \mathcal{S}(1/t); R)^\infty \rightarrow \mathcal{D}(1) = \mathcal{C}(\tilde{M} \times [1, \infty), \mathcal{S}(1); R)^\infty.$$

Applying K -theory yields a version of a “coarse” assembly map which is the algebraic K -theory analogue of the map described in Sect. 2.5.1. A crucial feature of such a construction is that the left hand side can be interpreted as a locally finite homology theory evaluated on $\tilde{M}$. It is hence an invariant of the proper homotopy type of $\tilde{M}$. Compare [7] and [326]. It is usually a lot easier to prove that this coarse assembly map is an equivalence. Suppose for example that M has non-positive curvature, choose a point $x_0 \in M$ (this will destroy the G -invariance) and with increasing $t \in [1, \infty)$ move the modules along geodesics towards x_0 . In this way one can show that the coarse assembly map is an isomorphism. Such coarse assembly maps exist also in the context of algebraic L -theory and topological K -theory, compare [151, 263].

Results about these maps (compare e.g. [17, 55, 336, 338]) lead to injectivity results for the classical assembly map by the “descent principle” (compare [52, 55, 263]) which we will now briefly describe in the context of algebraic K -theory. (We already stated an analytic version in Sect. 2.5.1.) For a spectrum E with G -action we denote by E^{hG} the homotopy fixed points. Since there is a natural map from fixed points to homotopy fixed points we obtain a commutative diagram

$$\begin{array}{ccc} K(\mathcal{D}(1/t))^G & \longrightarrow & K(\mathcal{D}(1))^G \\ \downarrow & & \downarrow \\ K(\mathcal{D}(1/t))^{hG} & \longrightarrow & K(\mathcal{D}(1))^{hG}. \end{array}$$

If one uses a suitable model K -theory commutes with taking fixed points and hence the upper horizontal map can be identified with the classical assembly map by Theorem 167. Using that K -theory commutes with infinite products [54], one can show by an induction over equivariant cells, that the vertical map on the left is an equivalence. Since we assume that the map $\mathbf{K}(\mathcal{D}(1/t)) \rightarrow \mathbf{K}(\mathcal{D}(1))$ is an equivalence, a standard property of the homotopy fixed point construction implies that the lower horizontal map is an equivalence. It follows that the upper horizontal map and hence the classical assembly map is split injective. A version of this argument which involves the assembly map for the family of finite subgroups can be found in [271].

2.8.11 Comparing to Other Theories

Every natural transformation of G -homology theories leads to a comparison between the associated assembly maps. For example one can compare topological K -theory to periodic cyclic homology [72], i.e. for every Banach algebra completion $\mathcal{A}(G)$ of $\mathbb{C}G$ inside $C_r^*(G)$ there exists a commutative diagram

$$\begin{array}{ccc} K_*(BG) & \longrightarrow & K_*(\mathcal{A}(G)) \\ \downarrow & & \downarrow \\ H_*(BG; HP_*(\mathbb{C})) & \longrightarrow & HP_*(\mathcal{A}(G)) . \end{array}$$

This is used in [72] to prove injectivity results for word hyperbolic groups. Similar diagrams exist for other cyclic theories (compare for example [246]).

A suitable model for the cyclotomic trace $trc: K_n(RG) \rightarrow TC_n(RG)$ from (connective) algebraic K -theory to topological cyclic homology [38] leads for every family $\mathcal{F}$ to a commutative diagram

$$\begin{array}{ccc} H_n(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{Z}}^{con}) & \longrightarrow & K_n^{con}(\mathbb{Z}G) \\ \downarrow & & \downarrow \\ H_n(E_{\mathcal{F}}(G); \mathbf{TC}_{\mathbb{Z}}) & \longrightarrow & TC_n(\mathbb{Z}G) . \end{array}$$

Injectivity results about the left hand and the lower horizontal map lead to injectivity results about the upper horizontal map. This is the principle behind Theorem 142 and 143.

2.9 Computations

Our ultimate goal is to compute K - and L -groups such as $K_n(RG)$, $L_n^{(-\infty)}(RG)$ and $K_n(C_r^*(G))$. Assuming that the Baum–Connes Conjecture 59 or the Farrell–Jones Conjecture 58 is true, this reduces to the computation of the left hand side of the

corresponding assembly map, i.e. to $H_n^G(E_{\mathcal{FLN}}(G); \mathbf{K}^{\text{top}})$, $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$ and $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{(-\infty)})$. This is much easier since here we can use standard methods from algebraic topology such as spectral sequences, Mayer–Vietoris sequences and Chern characters. Nevertheless such computations can be pretty hard. Roughly speaking, one can obtain a general reasonable answer after rationalization, but integral computations have only been done case by case and no general pattern is known.

K - and L -Groups for Finite Groups

2.9.1

In all these computations the answer is given in terms of the values of $K_n(RG)$, $L_n^{(-\infty)}(RG)$ and $K_n(C_r^*(G))$ for finite groups G . Therefore we briefly recall some of the results known for finite groups focusing on the case $R = \mathbb{Z}$.

Topological K -Theory for Finite Groups

Let G be a finite group. By $r_F(G)$, we denote the number of isomorphism classes of irreducible representations of G over the field F . By $r_{\mathbb{R}}(G; \mathbb{R})$, $r_{\mathbb{C}}(G; \mathbb{C})$, respectively $r_{\mathbb{H}}(G; \mathbb{H})$ we denote the number of isomorphism classes of irreducible real G -representations V , which are of real, complex respectively of quaternionic type, i.e. $\text{aut}_{\mathbb{R}G}(V)$ is isomorphic to the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$. Let $RO(G)$ respectively $R(G)$ be the *real* respectively the *complex representation ring*.

Notice that $\mathbb{C}G = l^1(G) = C_r^*(G) = C_{\max}^*(G)$ holds for a finite group, and analogous for the real versions.

Proposition 169 Let G be a finite group.

169

(i) We have

$$K_n(C_r^*(G)) \cong \begin{cases} R(G) \cong \mathbb{Z}^{r_{\mathbb{C}}(G)} & \text{for } n \text{ even;} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

(ii) There is an isomorphism of topological K -groups

$$K_n(C_r^*(G; \mathbb{R})) \cong K_n(\mathbb{R})^{r_{\mathbb{R}}(G; \mathbb{R})} \times K_n(\mathbb{C})^{r_{\mathbb{C}}(G; \mathbb{C})} \times K_n(\mathbb{H})^{r_{\mathbb{H}}(G; \mathbb{H})}.$$

Moreover $K_n(\mathbb{C})$ is 2-periodic with values $\mathbb{Z}$, 0 for $n = 0, 1$, $K_n(\mathbb{R})$ is 8-periodic with values $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0 for $n = 0, 1, \dots, 7$ and $K_n(\mathbb{H}) = K_{n+4}(\mathbb{R})$ for $n \in \mathbb{Z}$.

Proof One gets isomorphisms of C^* -algebras

$$C_r^*(G) \cong \prod_{j=1}^{r_{\mathbb{C}}(G)} M_{n_j}(\mathbb{C})$$

and

$$C_r^*(G; \mathbb{R}) \cong \prod_{i=1}^{r_{\mathbb{R}}(G; \mathbb{R})} M_{m_i}(\mathbb{R}) \times \prod_{i=1}^{r_{\mathbb{R}}(G; \mathbb{C})} M_{n_i}(\mathbb{C}) \times \prod_{i=1}^{r_{\mathbb{R}}(G; \mathbb{H})} M_{p_i}(\mathbb{H})$$

from [283, theorem 7 on page 19, corollary 2 on page 96, page 102, page 106]. Now the claim follows from Morita invariance and the well-known values for $K_n(\mathbb{R})$, $K_n(\mathbb{C})$ and $K_n(\mathbb{H})$ (see for instance [301, page 216]).

To summarize, the values of $K_n(C_r^*(G))$ and $K_n(C_r^*(G; \mathbb{R}))$ are explicitly known for finite groups G and are in the complex case in contrast to the real case always torsion free.

Algebraic K -Theory for Finite Groups

Here are some facts about the algebraic K -theory of integral group rings of finite groups.

170

Proposition 170 Let G be a finite group.

- (i) $K_n(\mathbb{Z}G) = 0$ for $n \leq -2$.
- (ii) We have

$$K_{-1}(\mathbb{Z}G) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s,$$

where

$$r = 1 - r_{\mathbb{Q}}(G) + \sum_{p \mid |G|} r_{\mathbb{Q}_p}(G) - r_{\mathbb{F}_p}(G)$$

and the sum runs over all primes dividing the order of G . (Recall that $r_F(G)$ denotes the number of isomorphism classes of irreducible representations of G over the field F .) There is an explicit description of the integer s in terms of global and local Schur indices [58]. If G contains a normal abelian subgroup of odd index, then $s = 0$.

- (iii) The group $\tilde{K}_0(\mathbb{Z}G)$ is finite.
- (iv) The group $\text{Wh}(G)$ is a finitely generated abelian group and its rank is $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$.
- (v) The groups $K_n(\mathbb{Z}G)$ are finitely generated for all $n \in \mathbb{Z}$.
- (vi) We have $K_{-1}(\mathbb{Z}G) = 0$, $\tilde{K}_0(\mathbb{Z}G) = 0$ and $\text{Wh}(G) = 0$ for the following finite groups $G = \{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, D_6 , D_8 , where D_m is the dihedral group of order m .

If p is a prime, then $K_{-1}(\mathbb{Z}[\mathbb{Z}/p]) = K_{-1}(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}/p]) = 0$.

We have

$$K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}, \quad \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) = 0, \quad \text{Wh}(\mathbb{Z}/6) = 0$$

$$K_{-1}(\mathbb{Z}[D_{12}]) \cong \mathbb{Z}, \quad \tilde{K}_0(\mathbb{Z}[D_{12}]) = 0, \quad \text{Wh}(D_{12}) = 0.$$

(vii) Let $\text{Wh}_2(G)$ denote the cokernel of the assembly map

$$H_2(BG; \mathbf{K}(\mathbb{Z})) \rightarrow K_2(\mathbb{Z}G) .$$

We have $\text{Wh}_2(G) = 0$ for $G = \{1\}, \mathbb{Z}/2, \mathbb{Z}/3$ and $\mathbb{Z}/4$. Moreover $|\text{Wh}_2(\mathbb{Z}/6)| \leq 2$, $|\text{Wh}_2(\mathbb{Z}/2 \times \mathbb{Z}/2)| \geq 2$ and $\text{Wh}_2(D_6) = \mathbb{Z}/2$.

Proof (i) and (ii) are proved in [58].

(iii) is proved in [298, proposition 9.1 on page 573].

(iv) This is proved for instance in [232].

(v) See [181, 248].

(vi) and (vii) The computation $K_{-1}(\mathbb{Z}G) = 0$ for $G = \mathbb{Z}/p$ or $\mathbb{Z}/p \times \mathbb{Z}/p$ can be found in [22, theorem 10.6, p. 695] and is a special case of [58].

The vanishing of $\tilde{K}_0(\mathbb{Z}G)$ is proven for $G = D_6$ in [262, theorem 8.2] and for $G = D_8$ in [262, theorem 6.4]. The cases $G = \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6$, and $(\mathbb{Z}/2)^2$ are treated in [79, corollary 5.17]. Finally, $\tilde{K}_0(\mathbb{Z}D_{12}) = 0$ follows from [79, theorem 50.29 on page 266] and the fact that $\mathbb{Q}D_{12}$ as a $\mathbb{Q}$ -algebra splits into copies of $\mathbb{Q}$ and matrix algebras over $\mathbb{Q}$, so its maximal order has vanishing class group by Morita equivalence.

The claims about $\text{Wh}_2(\mathbb{Z}/n)$ for $n = 2, 3, 4, 6$ and for $\text{Wh}_2((\mathbb{Z}/2)^2)$ are taken from [85, proposition 5], [89, p. 482] and [296, pp. 218 and 221].

We get $K_2(\mathbb{Z}D_6) \cong (\mathbb{Z}/2)^3$ from [296, theorem 3.1]. The assembly map $H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) \rightarrow K_2(\mathbb{Z}[\mathbb{Z}/2])$ is an isomorphism by [89, theorem on p. 482]. Now construct a commutative diagram

$$\begin{array}{ccc} H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) & \xrightarrow{\cong} & H_2(BD_6; \mathbf{K}(\mathbb{Z})) \\ \cong \downarrow & & \downarrow \\ K_2(\mathbb{Z}[\mathbb{Z}/2]) & \longrightarrow & K_2(\mathbb{Z}D_6) \end{array}$$

whose lower horizontal arrow is split injective and whose upper horizontal arrow is an isomorphism by the Atiyah–Hirzebruch spectral sequence. Hence the right vertical arrow is split injective and $\text{Wh}_2(D_6) = \mathbb{Z}/2$.

Let us summarize. We already mentioned that a complete computation of $K_n(\mathbb{Z})$ is not known. Also a complete computation of $\tilde{K}_0(\mathbb{Z}[p])$ for arbitrary primes p is out of reach (see [221, pp. 29,30]). There is a complete formula for $K_{-1}(\mathbb{Z}G)$ and $K_n(\mathbb{Z}G) = 0$ for $n \leq -2$ and one has a good understanding of $\text{Wh}(G)$ (see [232]). We have already mentioned Borel’s formula for $K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $n \in \mathbb{Z}$ (see Remark 14). For more rational information see also 177.

Algebraic L -Theory for Finite Groups

Here are some facts about L -groups of finite groups.

171 Proposition 171 Let G be a finite group. Then

- (i) For each $j \leq 1$ the groups $L_n^{(j)}(\mathbb{Z}G)$ are finitely generated as abelian groups and contain no p -torsion for odd primes p . Moreover, they are finite for odd n .
- (ii) For every decoration $\langle j \rangle$ we have

$$L_n^{(j)}(\mathbb{Z}G)[1/2] \cong L_n^{(j)}(\mathbb{R}G)[1/2] \cong \begin{cases} \mathbb{Z}[1/2]^{r_{\mathbb{R}}(G)} & n \equiv 0 \pmod{4}; \\ \mathbb{Z}[1/2]^{r_{\mathbb{C}}(G)} & n \equiv 2 \pmod{4}; \\ 0 & n \equiv 1, 3 \pmod{4}. \end{cases}$$

- (iii) If G has odd order and n is odd, then $L_n^{\varepsilon}(\mathbb{Z}G) = 0$ for $\varepsilon = p, h, s$.

Proof (i) See for instance [143].

(ii) See [258, proposition 22.34 on page 253].

(iii) See [13] or [143].

The L -groups of $\mathbb{Z}G$ are pretty well understood for finite groups G . More information about them can be found in [143].

2.9.2 Rational Computations for Infinite Groups

Next we state what is known rationally about the K - and L -groups of an infinite (discrete) group, provided the Baum–Connes Conjecture 59 or the relevant version of the Farrell–Jones Conjecture 58 is known.

In the sequel let $(\mathcal{F}\mathcal{C}\mathcal{Y})$ be the set of conjugacy classes (C) for finite cyclic subgroups $C \subseteq G$. For $H \subseteq G$ let $N_G H = \{g \in G \mid gHg^{-1} = H\}$ be its *normalizer*, let $Z_G H = \{g \in G \mid ghg^{-1} = h \text{ for } h \in H\}$ be its *centralizer*, and put

$$W_G H := N_G H / (H \cdot Z_G H),$$

where $H \cdot Z_G H$ is the normal subgroup of $N_G H$ consisting of elements of the form hu for $h \in H$ and $u \in Z_G H$. Notice that $W_G H$ is finite if H is finite.

Recall that the *Burnside ring* $A(G)$ of a finite group is the Grothendieck group associated to the abelian monoid of isomorphism classes of finite G -sets with respect to the disjoint union. The ring multiplication comes from the cartesian product. The zero element is represented by the empty set, the unit is represented by $G/G = \text{pt}$. For a finite group G the abelian groups $K_q(C_r^*(G))$, $K_q(RG)$ and $L^{(-\infty)}(RG)$ become modules over $A(G)$ because these functors come with a Mackey structure and $[G/H]$ acts by $\text{ind}_H^G \circ \text{res}_G^H$.

We obtain a ring homomorphism

$$\chi^G: A(G) \rightarrow \prod_{(H) \in \mathcal{FIN}} \mathbb{Z}, \quad [S] \mapsto (|S^H|)_{(H) \in \mathcal{FIN}},$$

which sends the class of a finite G -set S to the element given by the cardinalities of the H -fixed point sets. This is an injection with finite cokernel. This leads to an isomorphism of $\mathbb{Q}$ -algebras

$$\chi_{\mathbb{Q}}^G := \chi^G \otimes_{\mathbb{Z}} \text{id}_{\mathbb{Q}}: A(G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{(H) \in (\mathcal{FIN})} \mathbb{Q}. \quad (2.47)$$

For a finite cyclic group C let

$$\theta_C \in A(C) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|C|] \quad (2.48)$$

be the element which is sent under the isomorphism $\chi_{\mathbb{Q}}^G: A(C) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{(H) \in \mathcal{FIN}} \mathbb{Q}$ of (2.47) to the element, whose entry is one if $(H) = (C)$ and is zero if $(H) \neq (C)$. Notice that θ_C is an idempotent. In particular we get a direct summand $\theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$ in $K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and analogously for $K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L^{(-\infty)}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Rationalized Topological K -Theory for Infinite Groups

The next result is taken from [203, theorem 0.4 and page 127]. Recall that Λ^G is the ring $\mathbb{Z} \subseteq \Lambda^G \subseteq \mathbb{Q}$ which is obtained from $\mathbb{Z}$ by inverting the orders of the finite subgroups of G .

Theorem 172: (*Rational Computation of Topological K -theory for Infinite Groups.*) Suppose that the group G satisfies the Baum–Connes Conjecture 59. Then there is an isomorphism

$$\begin{aligned} \bigoplus_{p+q=n} \bigoplus_{(C) \in (\mathcal{FCY})} K_p(BZ_G C) \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \Lambda^G \\ \xrightarrow{\cong} K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G. \end{aligned}$$

If we tensor with $\mathbb{Q}$, we get an isomorphism

$$\begin{aligned} \bigoplus_{p+q=n} \bigoplus_{(C) \in (\mathcal{FCY})} H_p(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \xrightarrow{\cong} K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned}$$

Rationalized Algebraic K -Theory for Infinite Groups

Recall that for algebraic K -theory of the integral group ring we know because of Proposition 73 that in the Farrell–Jones Conjecture we can reduce to the family of finite subgroups. A reduction to the family of finite subgroups also works if the coefficient ring is a regular $\mathbb{Q}$ -algebra, compare 70. The next result is a variation of [201, theorem 0.4].

173

Theorem 173: (*Rational Computation of Algebraic K -theory.*) Suppose that the group G satisfies the Farrell–Jones Conjecture 58 for the algebraic K -theory of RG , where either $R = \mathbb{Z}$ or R is a regular ring with $\mathbb{Q} \subset R$. Then we get an isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C) \in (\mathcal{F} \mathcal{C} \mathcal{Y})} H_p(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

174

Remark 174 If in Theorem 173 we assume the Farrell–Jones Conjecture for the algebraic K -theory of RG but make no assumption on the coefficient ring R , then we still obtain that the map appearing there is split injective.

Example 175. (*The Comparison Map from Algebraic to Topological K -theory.*) If we consider $R = \mathbb{C}$ as coefficient ring and apply $- \otimes_{\mathbb{Z}} \mathbb{C}$ instead of $- \otimes_{\mathbb{Z}} \mathbb{Q}$, the formulas simplify. Suppose that G satisfies the Baum–Connes Conjecture 59 and the Farrell–Jones Conjecture 58 for algebraic K -theory with $\mathbb{C}$ as coefficient ring. Recall that $\text{con}(G)_f$ is the set of conjugacy classes (g) of elements $g \in G$ of finite order. We denote for $g \in G$ by $\langle g \rangle$ the cyclic subgroup generated by g .

Then we get the following commutative square, whose horizontal maps are isomorphisms and whose vertical maps are induced by the obvious change of theory homomorphisms (see [201, theorem 0.5])

$$\begin{array}{ccc} \bigoplus_{p+q=n} \bigoplus_{(g) \in \text{con}(G)_f} H_p(Z_G \langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \xrightarrow{\cong} & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\ \downarrow & & \downarrow \\ \bigoplus_{p+q=n} \bigoplus_{(g) \in \text{con}(G)_f} H_p(Z_G \langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\text{top}}(\mathbb{C}) & \xrightarrow[\cong]{} & K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

The Chern character appearing in the lower row of the commutative square above has already been constructed by different methods in [26]. The construction in [201] works also for $\mathbb{Q}$ (and even smaller rings) and other theories like algebraic K - and L -theory. This is important for the proof of Theorem 98 and to get the commutative square above.

Example 176. (*A Formula for $K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$.*) Suppose that the Farrell–Jones Conjecture is true rationally for $K_0(\mathbb{Z}G)$, i.e. the assembly map

$$A_{\mathcal{VCY}}: H_0^G(\text{Evcy}(G); \mathbb{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Then we obtain

$$K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \bigoplus_{(C) \in (\mathcal{FCY})} H_1(BZ_GC; \mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_{-1}(RC) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Notice that $\tilde{K}_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains only contributions from $K_{-1}(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$ for finite cyclic subgroups $C \subseteq G$.

Remark 177 Note that these statements are interesting already for finite groups. For instance Theorem 172 yields for a finite group G and $R = \mathbb{C}$ an isomorphism

177

$$\bigoplus_{(C) \in (\mathcal{FCY})} \Lambda_G \otimes_{\Lambda_G[W_GC]} \theta_C \cdot R(C) \otimes_{\mathbb{Z}} \Lambda_G \cong R(G) \otimes_{\mathbb{Z}} \Lambda_G.$$

which in turn implies Artin’s Theorem discussed in Remark 82.

Rationalized Algebraic L -Theory for Infinite Groups

Here is the L -theory analogue of the results above. Compare [201, theorem 0.4].

Theorem 178: (*Rational Computation of Algebraic L -theory for Infinite Groups.*) Suppose that the group G satisfies the Farrell–Jones Conjecture 58 for L -theory. Then we get for all $j \in \mathbb{Z}, j \leq 1$ an isomorphism

178

$$\bigoplus_{p+q=n} \bigoplus_{(C) \in (\mathcal{FCY})} H_p(BZ_GC; \mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot L_q^{(j)}(RC) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} L_n^{(j)}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 179: (*Separation of Variables.*) Notice that in Theorem 172, 173 and 178 we see again the principle we called *separation of variables* in Remark 13. There is a group homology part which is independent of the coefficient ring R and the K - or L -theory under consideration and a part depending only on the values of the theory under consideration on RC or $C_r^*(C)$ for all finite cyclic subgroups $C \subseteq G$.

179

Integral Computations for Infinite Groups

2.9.3

As mentioned above, no general pattern for integral calculations is known or expected. We mention at least one situation where a certain class of groups can

be treated simultaneously. Let $\mathcal{MFI}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ which are maximal in $\mathcal{FIN}$. Consider the following assertions on the group G .

- (M) $M_1, M_2 \in \mathcal{MFI}, M_1 \cap M_2 \neq 1 \Rightarrow M_1 = M_2$;
- (NM) $M \in \mathcal{MFI} \Rightarrow N_G M = M$;
- (VCL_I) If V is an infinite virtually cyclic subgroup of G , then V is of type I (see Lemma 71);
- (FJK_N) The Isomorphism Conjecture of Farrell–Jones for algebraic K -theory 58 is true for $\mathbb{Z}G$ in the range $n \leq N$ for a fixed element $N \in \mathbb{Z} \amalg \{\infty\}$, i.e. the assembly map $A: \mathcal{H}_n^G(E_{\text{Vcy}}(G); \mathbf{K}_R) \xrightarrow{\cong} K_n(RG)$ is bijective for $n \in \mathbb{Z}$ with $n \leq N$.

Let $\tilde{K}_n(C_r^*(H))$ be the cokernel of the map $K_n(C_r^*(\{1\})) \rightarrow K_n(C_r^*(H))$ and $\tilde{L}_n^{(j)}(RG)$ be the cokernel of the map $L_n^{(j)}(R) \rightarrow L_n^{(j)}(RG)$. This coincides with $\tilde{L}_n^{(j)}(R)$, which is the cokernel of the map $L_n^{(j)}(\mathbb{Z}) \rightarrow L_n^{(j)}(R)$ if $R = \mathbb{Z}$ but not in general. Denote by $\text{Wh}_n^R(G)$ the n -th Whitehead group of RG which is the $(n-1)$ -th homotopy group of the homotopy fiber of the assembly map $BG_+ \wedge \mathbf{K}(R) \rightarrow \mathbf{K}(RG)$. It agrees with the previous defined notions if $R = \mathbb{Z}$. The next result is taken from [83, theorem 4.1].

180

Theorem 180 Let $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ be a ring such that the order of any finite subgroup of G is invertible in Λ . Let $(\mathcal{MFI})$ be the set of conjugacy classes (H) of subgroups of G such that H belongs to $\mathcal{MFI}$. Then:

- (i) If G satisfies (M), (NM) and the Baum–Connes Conjecture 59, then for $n \in \mathbb{Z}$ there is an exact sequence of topological K -groups

$$0 \rightarrow \bigoplus_{(H) \in (\mathcal{MFI})} \tilde{K}_n(C_r^*(H)) \rightarrow K_n(C_r^*(G)) \rightarrow K_n(G \backslash E_{\mathcal{FIN}}(G)) \rightarrow 0,$$

which splits after applying $- \otimes_{\mathbb{Z}} \Lambda$.

- (ii) If G satisfies (M), (NM), (VCL_I) and the L -theory part of the Farrell–Jones Conjecture 58, then for all $n \in \mathbb{Z}$ there is an exact sequence

$$\begin{aligned} \dots \rightarrow H_{n+1}(G \backslash E_{\mathcal{FIN}}(G); \mathbf{L}^{(-\infty)}(R)) &\rightarrow \bigoplus_{(H) \in (\mathcal{MFI})} \tilde{L}_n^{(-\infty)}(RH) \\ &\rightarrow L_n^{(-\infty)}(RG) \rightarrow H_n(G \backslash E_{\mathcal{FIN}}(G); \mathbf{L}^{(-\infty)}(R)) \rightarrow \dots \end{aligned}$$

It splits after applying $- \otimes_{\mathbb{Z}} \Lambda$, more precisely

$$L_n^{(-\infty)}(RG) \otimes_{\mathbb{Z}} \Lambda \rightarrow H_n(G \backslash E_{\mathcal{FIN}}(G); \mathbf{L}^{(-\infty)}(R)) \otimes_{\mathbb{Z}} \Lambda$$

is a split-surjective map of Λ -modules.

- (iii) If G satisfies the assertions (M), (NM) and the Farrell–Jones Conjecture 58 for $L_n(RG)[1/2]$, then the conclusion of assertion (ii) still holds if we invert

2 everywhere. Moreover, in the case $R = \mathbb{Z}$ the sequence reduces to a short exact sequence

$$\begin{aligned} 0 \rightarrow \bigoplus_{(H) \in (\mathcal{MF})} \tilde{L}_n^{(j)}(\mathbb{Z}H) \left[\frac{1}{2} \right] &\rightarrow L_n^{(j)}(\mathbb{Z}G) \left[\frac{1}{2} \right] \\ &\rightarrow H_n(G \backslash E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G); \mathbb{L}(\mathbb{Z})) \left[\frac{1}{2} \right] \rightarrow 0, \end{aligned}$$

which splits after applying $-\otimes_{\mathbb{Z}[\frac{1}{2}]} \Lambda \left[\frac{1}{2} \right]$.

(iv) If G satisfies (M), (NM), and (FJ) K_N , then there is for $n \in \mathbb{Z}, n \leq N$ an isomorphism

$$H_n(E_{\mathcal{V} \subset \mathcal{Y}}(G), E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G); \mathbf{K}_R) \oplus \bigoplus_{(H) \in (\mathcal{MF})} \mathrm{Wh}_n^R(H) \xrightarrow{\cong} \mathrm{Wh}_n^R(G),$$

where $\mathrm{Wh}_n^R(H) \rightarrow \mathrm{Wh}_n^R(G)$ is induced by the inclusion $H \rightarrow G$.

Remark 181: (*Role of $G \backslash E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$.*) Theorem 180 illustrates that for such computations a good understanding of the geometry of the orbit space $G \backslash E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ is necessary.

181

Remark 182 In [83] it is explained that the following classes of groups do satisfy the assumption appearing in Theorem 180 and what the conclusions are in the case $R = \mathbb{Z}$. Some of these cases have been treated earlier in [34, 212].

182

- Extensions $1 \rightarrow \mathbb{Z}^n \rightarrow G \rightarrow F \rightarrow 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
- Fuchsian groups F ;
- One-relator groups G .

Theorem 180 is generalized in [204] in order to treat for instance the semi-direct product of the discrete three-dimensional Heisenberg group by $\mathbb{Z}/4$. For this group $G \backslash E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ is S^3 .

A calculation for 2-dimensional crystallographic groups and more general cocompact NEC-groups is presented in [212] (see also [236]). For these groups the orbit spaces $G \backslash E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ are compact surfaces possibly with boundary.

Example 183. Let F be a cocompact Fuchsian group with presentation

$$F = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_t \mid c_1^{\gamma_1} = \dots = c_t^{\gamma_t} = c_1^{-1} \cdots c_t^{-1} [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

for integers $g, t \geq 0$ and $\gamma_i > 1$. Then $G \backslash E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ is a closed orientable surface of genus g . The following is a consequence of Theorem 180 (see [212] for more details).

- There are isomorphisms

$$K_n(C_r^*(F)) \cong \begin{cases} (2 + \sum_{i=1}^t (\gamma_i - 1)) \cdot \mathbb{Z} & n = 0 ; \\ (2g) \cdot \mathbb{Z} & n = 1 . \end{cases}$$

- The inclusions of the maximal subgroups $\mathbb{Z}/\gamma_i = \langle c_i \rangle$ induce an isomorphism

$$\bigoplus_{i=1}^t \mathrm{Wh}_n(\mathbb{Z}/\gamma_i) \xrightarrow{\cong} \mathrm{Wh}_n(F)$$

for $n \leq 1$.

- There are isomorphisms

$$L_n(\mathbb{Z}F)[1/2] \cong \begin{cases} (1 + \sum_{i=1}^t [\frac{\gamma_i}{2}]) \cdot \mathbb{Z}[1/2] & n \equiv 0 \quad (4) ; \\ (2g) \cdot \mathbb{Z}[1/2] & n \equiv 1 \quad (4) ; \\ (1 + \sum_{i=1}^t [\frac{\gamma_i-1}{2}]) \cdot \mathbb{Z}[1/2] & n \equiv 2 \quad (4) ; \\ 0 & n \equiv 3 \quad (4) , \end{cases}$$

where $[r]$ for $r \in \mathbb{R}$ denotes the largest integer less than or equal to r .

From now on suppose that each γ_i is odd. Then the number m above is odd and we get for $\varepsilon = p$ and s

$$L_n^\varepsilon(\mathbb{Z}F) \cong \begin{cases} \mathbb{Z}/2 \oplus (1 + \sum_{i=1}^t \frac{\gamma_i-1}{2}) \cdot \mathbb{Z} & n \equiv 0 \quad (4) ; \\ (2g) \cdot \mathbb{Z} & n \equiv 1 \quad (4) ; \\ \mathbb{Z}/2 \oplus (1 + \sum_{i=1}^t \frac{\gamma_i-1}{2}) \cdot \mathbb{Z} & q \equiv 2 \quad (4) ; \\ (2g) \cdot \mathbb{Z}/2 & n \equiv 3 \quad (4) . \end{cases}$$

For $\varepsilon = h$ we do not know an explicit formula. The problem is that no general formula is known for the 2-torsion contained in $\tilde{L}_{2q}^h(\mathbb{Z}[\mathbb{Z}/m])$, for m odd, since it is given by the term $\hat{H}^2(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/m]))$, see [14, theorem 2].

Information about the left hand side of the Farrell–Jones assembly map for algebraic K -theory in the case where G is $SL_3(\mathbb{Z})$ can be found in [306].

2.9.4 Techniques for Computations

We briefly outline some methods that are fundamental for computations and for the proofs of some of the theorems above.

Equivariant Atiyah–Hirzebruch Spectral Sequence

Let $\mathcal{H}_*^G$ be a G -homology theory with values in Λ -modules. Then there are two spectral sequences which can be used to compute it. The first one is the rather

obvious equivariant version of the Atiyah–Hirzebruch spectral sequence. It converges to $\mathcal{H}_n^G(X)$ and its E^2 -term is given in terms of Bredon homology

$$E_{p,q}^2 = H_p^G(X; \mathcal{H}_q^G(G/H))$$

of X with respect to the coefficient system, which is given by the covariant functor $\text{Or}(G) \rightarrow \Lambda\text{-MODULES}$, $G/H \mapsto \mathcal{H}_q^G(G/H)$. More details can be found for instance in [82, theorem 4.7].

p -Chain Spectral Sequence

There is another spectral sequence, the p -chain spectral sequence [83]. Consider a covariant functor $E: \text{Or}(G) \rightarrow \text{SPECTRA}$. It defines a G -homology theory $\mathcal{H}_*^G(-; E)$ (see Proposition 156). The p -chain spectral sequence converges to $\mathcal{H}_n^G(X)$ but has a different setup and in particular a different E^2 -term than the equivariant Atiyah–Hirzebruch spectral sequence. We describe the E^1 -term for simplicity only for a proper G -CW-complex.

A p -chain is a sequence of conjugacy classes of finite subgroups

$$(H_0) < \dots < (H_p)$$

where $(H_{i-1}) < (H_i)$ means that H_{i-1} is subconjugate, but not conjugate to (H_i) . Notice for the sequel that the group of automorphism of G/H in $\text{Or}(G)$ is isomorphic to NH/H . To such a p -chain there is associated the NH_p/H_p - NH_0/H_0 -set

$$\begin{aligned} S((H_0) < \dots < (H_p)) \\ = \text{map}(G/H_{p-1}, G/H_p)^G \times_{NH_{p-1}/H_{p-1}} \dots \times_{NH_1/H_1} \text{map}(G/H_0, G/H_1)^G. \end{aligned}$$

The E^1 -term $E_{p,q}^1$ of the p -chain spectral sequence is

$$\bigoplus_{(H_0) < \dots < (H_p)} \pi_q \left((X^{H_p} \times_{NH_p/H_p} S((H_0) < \dots < (H_p)))_+ \wedge_{NH_0/H_0} E(G/H_0) \right),$$

where Y_+ means the pointed space obtained from Y by adjoining an extra base point. There are many situations where the p -chain spectral sequence is much more useful than the equivariant Atiyah–Hirzebruch spectral sequence. Sometimes a combination of both is necessary to carry through the desired calculation.

Equivariant Chern Characters

Equivariant Chern characters have been studied in [201] and [203] and allow to compute equivariant homology theories for proper G -CW-complexes. The existence of the equivariant Chern character says that under certain conditions the Atiyah–Hirzebruch spectral sequence collapses and, indeed, the source of the equivariant Chern character is canonically isomorphic to $\bigoplus_{p+q} E_{p,q}^2$, where $E_{p,q}^2$ is the E^2 -term of the equivariant Atiyah–Hirzebruch spectral sequence.

The results of Sect. 2.9.2 are essentially proved by applying the equivariant Chern character to the source of the assembly map for the family of finite subgroups.

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IV.3

Comparison Between Algebraic and Topological K -Theory for Banach Algebras and C^* -Algebras*

Jonathan Rosenberg

3.1	<i>Some Problems in Operator Theory</i>	844
	Toeplitz Operators and K -Theory	844
	K -Theory of Banach Algebras	846
	Essentially Normal Operators	847
	Smooth Extensions and K_2	848
	Multiplicative Commutators	849
	AF Algebras and Dimension Groups	849
3.2	<i>"Lie Groups Made Discrete" and Early Explorations</i>	850
	Basic Concepts and Notations	850
	Direct Calculation in the Abelian Case	853
	"Lie Groups Made Discrete" and Suslin's Theorems on $K_*(\mathbb{R})$, $K_*(\mathbb{C})$	854
	Karoubi's Early Work on Algebraic K -Theory of Operator Algebras	857
3.3	<i>Recent Progress on Algebraic K-Theory of Operator Algebras</i>	860
	Algebraic K -Theory Invariants for Operator Algebras	860
	The Work of Suslin–Wodzicki on Excision	861
	Resolution of the Karoubi Conjecture	862
	Other Miscellaneous Results	865
	<i>References</i>	870

* Partially supported by NSF Grant DMS-0103647.

Summary. For a Banach algebra, one can define two kinds of K -theory: topological K -theory, which satisfies Bott periodicity, and algebraic K -theory, which usually does not. It was discovered, starting in the early 80's, that the "comparison map" from algebraic to topological K -theory is a surprisingly rich object. About the same time, it was also found that the *algebraic* (as opposed to topological) K -theory of operator algebras does have some direct applications in operator theory. This article will summarize what is known about these applications and the comparison map.

3.1

Some Problems in Operator Theory

3.1.1

Toeplitz Operators and K -Theory

The connection between operator theory and K -theory has very old roots, although it took a long time for the connection to be understood. We begin with an example. Think of S^1 as the unit circle in the complex plane and let $\mathcal{H} \subset L^2(S^1)$ be the Hilbert space H^2 of functions all of whose negative Fourier coefficients vanish. In other words, if we identify functions with their formal Fourier expansions,

$$\mathcal{H} = \left\{ \sum_{n=0}^{\infty} c_n z^n \quad \text{with} \quad \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\}.$$

Now let $f \in C(S^1)$ and let M_f be the operator of multiplication by f on $L^2(S^1)$. This operator does not necessarily map $\mathcal{H}$ into itself, so let $P: L^2(S^1) \rightarrow \mathcal{H}$ be the orthogonal projection and let $T_f = PM_f$, viewed as an operator from $\mathcal{H}$ to itself. This is called the *Toeplitz operator* with continuous symbol f . In terms of the orthonormal basis $e_0(z) = 1, e_1(z) = z, e_2(z) = z^2, \dots$ of $\mathcal{H}$, T_f is given by the (one-sided) infinite matrix with entries $\langle T_f e_i, e_j \rangle = c_{j-i}$, where $f(z) = \sum c_n z^n$ is the formal Fourier expansion of f . This is precisely a *Toeplitz matrix*, i.e., a matrix with constant entries along any diagonal. The operator T_f may also be viewed as a singular integral operator, since by the Cauchy integral formula, one has

$$T_f \varphi(z) = \frac{1}{2\pi i} \oint_{S^1} \frac{f(\zeta) \varphi(\zeta)}{\zeta - z} d\zeta$$

for $|z| < 1$, and the same formula is "formally" valid for $|z| = 1$.

A natural question now arises: when is T_f invertible? And when this is the case, can one give a formula for the inverse? In other words, how does one solve the singular integral equation $T_f \varphi(z) = g(z)$? The following result is "classical" and was first proved by Krein back in the 1950's, though his formulation looked quite different.

1

Theorem 1 Let T_f be the Toeplitz operator on H^2 defined as above, for $f \in C(S^1)$. Then T_f is invertible if and only if f is everywhere non-vanishing (so that f can be viewed as a map $S^1 \rightarrow \mathbb{C}^\times$) and if the winding number of f , i.e., the degree of the map $\frac{f}{|f|}: S^1 \rightarrow S^1$, is zero.

Sketch of a modern proof

(For more details, see [18, chapt. 7, especially theorem 7.23 and proposition 7.24].) Let $\mathcal{T}$ be the C^* -algebra¹ generated by all the operators $T_f, f \in C(S^1)$, i.e., the norm closure of the algebra generated by these operators and their adjoints. $\mathcal{T}$ is called the *Toeplitz algebra*. The first thing to observe is that there is a surjective $*$ -homomorphism $\sigma: \mathcal{T} \rightarrow C(S^1)$, the “symbol map,” induced by $T_f \mapsto f$, fitting into a short exact sequence of C^* -algebras

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \rightarrow 0, \quad (3.1)$$

where $\mathcal{K}$ is the algebra of compact operators on H^2 . In particular, $\mathcal{T}$ is commutative modulo compact operators.

To begin with, it is obvious that

$$T_f^* = (PM_f P)^*|_{H^2} = (PM_f^* P)|_{H^2} = (PM_{\bar{f}} P)|_{H^2} = T_{\bar{f}}$$

and that the map $f \mapsto T_f$ is linear, and

$$\|T_f\| = \|PM_f\| \leq \|P\| \|M_f\| = \|f\|_\infty.$$

So since polynomials in z are dense in $C(S^1)$, for proving commutativity of $\mathcal{T}$ modulo compacts and multiplicativity of σ it is enough to check that $T_{z^j} T_{z^k} \equiv T_{z^{j+k}} \pmod{\mathcal{K}}$. This is immediate since

$$T_{z^j} T_{z^k} e_m = T_{z^{j+k}} e_m = e_{m+j+k}$$

for m sufficiently large ($m \geq |j| + |k|$). Thus $\mathcal{T}/(\mathcal{T} \cap \mathcal{K})$ is commutative, and σ by construction is surjective. Next, we show that $\mathcal{K} \subset \mathcal{T}$. For this it suffices to show that the action of $\mathcal{T} \cap \mathcal{K}$ on $\mathcal{H}$ is irreducible, and since T_z is the unilateral shift (sending $e_j \mapsto e_{j+1}$), which is known to be irreducible, the result follows. (In fact, the rank-one operators $\xi \mapsto \langle \xi, e_j \rangle e_k$, which generate a dense subalgebra of $\mathcal{K}$, can all be written as polynomials in T_z and its adjoint $T_{z^{-1}}$. For example, $T_{z^{-1}} T_z - T_z T_{z^{-1}}$ is orthogonal projection onto the span of e_0 .) Finally, we need to show that the kernel of σ is precisely $\mathcal{K}$; this can be checked by showing that the map $f \mapsto T_f \pmod{\mathcal{K}}$ is an isometry – a detailed proof is in [18, proof of Theorem 7.11].

Now we get to the more interesting part of the proof, the part that involves K -theory. The idea is to use the long exact K -theory sequences

$$\begin{array}{ccccc} K_1(\mathcal{T}) & \xrightarrow{\sigma_*} & K_1(C(S^1)) & \xrightarrow{\partial} & K_0(\mathcal{K}) = \mathbb{Z} \\ \downarrow & & \downarrow & & \parallel \\ 0 = K_1(\mathcal{L}) & \longrightarrow & K_1(\mathcal{Q}) & \xrightarrow{\partial} & K_0(\mathcal{K}) = \mathbb{Z} \end{array} \quad (3.2)$$

¹ By definition, a C^* -algebra is a Banach algebra with involution $*$, isometrically $*$ -isomorphic to a norm-closed self-adjoint algebra of operators on a Hilbert space.

associated to (3.1) and to the algebra $\mathcal{L}$ of all bounded linear operators on H^2 and its quotient $\mathcal{Q} = \mathcal{L}/\mathcal{K}$, the so-called *Calkin algebra*. The downward-pointing arrows here are induced by the inclusion $\mathcal{T} \hookrightarrow \mathcal{L}$. Note that we are using excision for K_0 to identify the relative groups $K_0(\mathcal{T}, \mathcal{K})$ and $K_0(\mathcal{L}, \mathcal{K})$ with $K_0(\mathcal{K}) = \mathbb{Z}$. Now one can show that $\partial([f])$ is (up to a sign depending on orientation conventions) the winding number of f . (To prove this, one can first show that $\partial([f])$ only depends on the homotopy class of f as a map $S^1 \rightarrow \mathbb{C}^\times$, and then compute for $f(z) = z$, which generates $\pi_1(S^1)$.) If T_f is invertible, then from (3.1), $\sigma(T_f) = f$ is invertible. And by exactness of (3.2), $\partial([f]) = 0$, so the winding number condition in the theorem is satisfied. In the other direction, suppose f is invertible in $C(S^1)$. Then f defines a class in $K_1(C(S^1))$ and $\partial([f])$ is an obstruction to lifting f to an invertible element of $\mathcal{T}$. So if the winding number condition in the theorem is satisfied, the obstruction vanishes. From the bottom part of the commuting diagram (3.2), together with the interpretation of the inverse image of $\mathcal{Q}^\times$ in $\mathcal{L}$ as the set of Fredholm operators and $\partial: K_1(\mathcal{Q}) \rightarrow K_0(\mathcal{K})$ as the Fredholm index, T_f is a Fredholm operator of index 0. Thus $\dim \ker T_f = \dim \ker T_f^* = \dim \ker T_{\bar{f}}$. But one can show that $\ker T_f$ and $\ker T_{\bar{f}}$ can't both be non-trivial [18, proposition 7.24], so T_f is invertible.

3.1.2 K -Theory of Banach Algebras

The connection between Fredholm operators and K -theory, which appeared to some extent in the above proof, first appeared in [33]. This marked the beginning of formal connections between operator theory and K -theory. About the same time, Wood [69] noticed that topological K -theory can be defined for Banach algebras, in such a way that Bott periodicity holds, just as it does for topological K -theory of spaces. However, it took a while for specialists in Banach algebras to notice the possibilities that K -theory afforded for solving certain kinds of problems. Direct applications of K -theory to operator algebras did not surface until the early 70's, with publication of works like [58] and [11]. In the rest of this section, we will discuss a few of the other early connections between K -theory and problems in operator algebras, and in Sect. 3.2 which follows, we will discuss some of the motivation for studying the comparison map between algebraic and topological K -theory for Banach algebras.

In [58] and [59], Taylor began to consider direct applications of K -theory of Banach algebras to problems in harmonic analysis. Part of the motivation was to give new proofs of results like the Cohen idempotent theorem (which says that the idempotent finite measures on a locally compact abelian group are generated by those of the form $\chi(h) dh$, with H a compact subgroup, dh its Haar measure, and χ a character on H). One of the things he found was:

2

Theorem 2: *Taylor* If A is a unital commutative Banach algebra and if X is its maximal ideal space, then the Gelfand transform $A \rightarrow C(X)$ induces an isomorphism on topological K -theory.

An immediate corollary is that topological K -theory vanishes for the radical of A (the intersection of all the maximal ideals), and thus for purposes of studying topological K -theory, it is no loss of generality to assume that A is semisimple, or even that A is a C^* -algebra. The corresponding result for algebraic K_1 is easily seen to be false, however. (Just consider the algebra of dual numbers, $\mathbb{C}[x]/(x^2)$.)

Essentially Normal Operators

3.1.3

At about the same time, interest in K -theory for C^* -algebras began to explode, thanks to the work of Brown, Douglas, and Fillmore (“BDF” [7, 8]) on extensions of C^* -algebras, followed quickly by the work of Kasparov on “operator K -homology” ([40, 41]). The BDF work grew out of the study of a rather concrete problem in operator theory: classification of *essentially normal operators*, bounded operators T on an infinite-dimensional separable Hilbert space $\mathcal{H}$ for which $T^*T - TT^*$ is compact. Given such an operator, 1 , T , T^* , and $\mathcal{K}$ (the algebra of compact operators) generate a C^* -algebra $E \subset \mathcal{L}$ containing $\mathcal{K}$ as an ideal and with $E/\mathcal{K} = A$ a unital commutative C^* -algebra, hence with $A \cong C(X)$, where X is the “essential spectrum” of T . Thus T defines an extension of C^* -algebras

$$0 \rightarrow \mathcal{K} \rightarrow E \xrightarrow{q} C(X) \rightarrow 0. \quad (3.3)$$

The similarity with (3.1) is not an accident; in fact, the Toeplitz extension is the special case where $\mathcal{H} = H^2$ and T is the Toeplitz operator T_z . The original problem was to determine when T can be written in the form $N + K$ with N normal (i.e., $N^*N = NN^*$) and K compact. (Clearly any operator T of the form $N + K$ satisfies the original condition $T^*T - TT^* \in \mathcal{K}$.) If we can write $T = N + K$ in this fashion, then the map $q(T) \mapsto N$ defines a splitting of the exact sequence (3.3) (assuming we choose N so that its spectrum is no larger than the essential spectrum of T). So classification of essentially normal operators comes down to classification of C^* -algebra extensions by $\mathcal{K}$, modulo split extensions. This was the motivation for the BDF project.

The important discovery in the BDF work was that extensions of the form (3.3) (modulo split extensions, in some sense) can be made into an abelian group $\text{Ext}(X)$, and that Ext is part of a homology theory which is dual to (topological) K -theory. The addition operation on extensions makes use of the fact that $M_2(\mathcal{K}) \cong \mathcal{K}$. Given two such extensions E_1 and E_2 , then

$$E_1 \oplus_A E_2 =_{\text{def}} \{(e_1, e_2) \in E_1 \oplus E_2 : e_1 \equiv e_2 \pmod{\mathcal{K}}\}$$

is an extension of A by $\mathcal{K} \oplus \mathcal{K}$, and if we add to $E_1 \oplus_A E_2 \subset \mathcal{L} \oplus \mathcal{L} \subset M_2(\mathcal{L}) \cong \mathcal{L}$ the ideal $M_2(\mathcal{K}) \cong \mathcal{K}$, we get an extension of A by $\mathcal{K}$. In fact, Ext extends to a contravariant functor on the category of separable nuclear C^* -algebras (where we replace $A = C(X)$ by more general C^* -algebras) – the duality with K -theory comes from the fact that the long exact K -theory sequence of (3.3) gives a homomorphism $\partial: K_1(A) \rightarrow K_0(\mathcal{K}) = \mathbb{Z}$ just as in the above proof of

Theorem 1. And should this “primary obstruction” to splitting of (3.3) vanish, there is a secondary obstruction that comes from the exact sequence

$$0 \rightarrow K_0(\mathcal{K}) = \mathbb{Z} \rightarrow K_0(E) \xrightarrow{q^*} K_0(A) \rightarrow K_{-1}(\mathcal{K}) = 0 ,$$

which defines an element of $\text{Ext}_{\mathbb{Z}}^1(K_0(A), \mathbb{Z})$. In fact, Brown showed [11] that these invariants give rise to a “universal coefficient theorem” (UCT) exact sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(K^0(X), \mathbb{Z}) \rightarrow \text{Ext}(X) \rightarrow \text{Hom}_{\mathbb{Z}}(K^{-1}(X), \mathbb{Z}) \rightarrow 0 .$$

3.1.4 Smooth Extensions and K_2

A bounded operator T on a Hilbert space $\mathcal{H}$ is said to be of *determinant class* if $T - 1$ belongs to the ideal $\mathcal{L}^1 \subseteq \mathcal{L}(\mathcal{H})$ of trace-class operators. There is a well-defined notion of determinant for operators of determinant class. As expected, it is defined to be 0 if T is not invertible. If T is invertible, then one can show that $T = \exp(S)$ for some trace-class operator S , and we define $\det T = \det(\exp(S))$ to be $e^{\text{Tr}(S)}$, according to the usual relationship between the trace and the determinant. (One needs to check that this is independent of the choice of S .) The determinant defined this way is multiplicative (on operators of determinant class); in fact it defines a homomorphism $\det: K_1(\mathcal{L}, \mathcal{L}^1) \rightarrow \mathbb{C}^\times$. Using this notion of determinant, Helton and Howe [28] defined an interesting invariant for a special subclass of the essentially normal operators. It was then shown by Brown ([10, 11]) that this invariant can be viewed as having something to do with algebraic K_2 . The idea is this. Suppose one has an extension of the form (3.3), and suppose X is a smooth manifold (possibly with boundary). Inside E , which is an extension of $C(X)$ by $\mathcal{K}$, suppose one has a subalgebra $\mathfrak{A}$ which is an extension

$$0 \rightarrow \mathcal{L}^1 \rightarrow \mathfrak{A} \xrightarrow{q} C^\infty(X) \rightarrow 0 . \quad (3.4)$$

of $C^\infty(X)$ by $\mathcal{L}^1$, the trace-class operators. Thus operators T in $\mathfrak{A}$ are not only essentially normal; they have trace-class self-commutators (i.e., $T^*T - TT^* \in \mathcal{L}^1$). Suppose T and S are two invertible operators in $\mathfrak{A}$. Then the images modulo $\mathcal{L}^1$ of T , T^* , S , and S^* commute, and so the multiplicative commutator $TST^{-1}S^{-1}$ is 1 modulo $\mathcal{L}^1$, and so is of determinant class. In particular, $\det(TST^{-1}S^{-1})$ is defined. Brown noticed that

$$\det(TST^{-1}S^{-1}) = \det \circ \partial(\{q(T), q(S)\}) ,$$

where $\partial: K_2(C^\infty(X)) \rightarrow K_1(\mathfrak{A}, \mathcal{L}^1)$ is the connecting map in the long exact K -theory sequence of (3.4), we view $\det$ as a function on $K_1(\mathfrak{A}, \mathcal{L}^1)$ via the natural map $K_1(\mathfrak{A}, \mathcal{L}^1) \rightarrow K_1(\mathcal{L}, \mathcal{L}^1)$, and $\{q(T), q(S)\} \in K_2(C^\infty(X))$ is the Steinberg symbol of the functions $q(T)$ and $q(S)$. In particular, one obtains the relation $\det(TST^{-1}S^{-1}) = 1$ when the symbols satisfy $q(T) + q(S) = 1$, which is not at all obvious from the operator-theoretic point of view.

Multiplicative Commutators

3.1.5

Algebraic K_1 and K_2 are also related to a number of other problems about multiplicative commutators in various operator algebras. For example, one has:

Theorem 3: *Brown and Schochet [9] $K_1(\mathcal{L}, \mathcal{K}) = 0$.*

3

This is proved by showing explicitly that every invertible operator $\equiv 1 \pmod{\mathcal{K}}$ is a product of a finite number of (multiplicative) commutators of such operators. Thus there is a huge difference between the algebraic K -theory of $\mathcal{K}$ and that of $\mathcal{L}^1$. (Recall that we have the determinant map $\det: K_1(\mathcal{A}, \mathcal{L}^1) \rightarrow \mathbb{C}^\times$, which is surjective.) Brown and Schochet also remark [9, remark 3] that their methods also show that $K_1(\tilde{\mathcal{K}}, \mathcal{K}) = 0$, with $\tilde{\mathcal{K}} = \mathcal{K} + \mathbb{C} \cdot 1$ the algebra obtained by adjoining a unit to $\mathcal{K}$. (The two statements are not the same since K_1 does not in general satisfy the excision property.) A subsequent paper [12], using refinements of the same techniques, showed that the group of invertible operators in $\mathcal{L}$ which are $\equiv 1 \pmod{\mathcal{K}}$ is perfect, with all even cohomology groups nontrivial. These groups are of course related by the Hurewicz homomorphism to the higher algebraic K -theory $K_*(\mathcal{L}, \mathcal{K})$ (about which we will say more later). A related later paper by de la Harpe and Skandalis [17] showed that if A is a *stable* C^* -algebra, i.e., if $A \cong A \otimes \mathcal{K}$,² then the connected component of the identity in the group of invertible operators of the form $1 + a$, $a \in \mathcal{K}$, is always perfect.

AF Algebras and Dimension Groups

3.1.6

One other important source for interest in K -theory of operator algebras comes from the study of so-called AF algebras, or C^* -algebra inductive limits of finite-dimensional semisimple algebras over $\mathbb{C}$. (The abbreviation AF stands for “approximately finite-dimensional.”) Such algebras were first introduced by Bratteli [5], who showed how to classify them by means of equivalence classes of certain combinatorial constructs now called “Bratteli diagrams.” However, this method of classification was almost uncomputable. A major breakthrough came a few years

² Here $A \otimes \mathcal{K}$ is the C^* -algebra completion of the algebraic tensor product $A \odot \mathcal{K}$. For general C^* -algebras A and B , there can be more than one C^* -algebra completion of $A \odot B$, but there is always a maximal one $A \otimes_{\max} B$, defined by completing $A \odot B$ in the norm

$$\left\| \sum_{i=1}^n a_i \otimes b_i \right\|_{\max} = \sup \left\{ \left\| \sum_{i=1}^n \varphi_1(a_i) \varphi_2(b_i) \right\| : \varphi_1 \text{ and } \varphi_2 \right.$$

commuting representations of A and B $\left. \right\}$,

as well as a minimal one $A \otimes_{\min} B$, the completion of $A \odot B \subset \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ when A is represented on a Hilbert space $\mathcal{H}_1$ and B is represented on a Hilbert space $\mathcal{H}_2$. (One can show this is independent of the choices of faithful representations of A and B .) But if one of the two algebras is *nuclear*, and in particular if B is commutative or $B = \mathcal{K}$, all completions coincide.

later when Elliott [20] showed that AF algebras are classified by their K_0 groups, together with the natural ordering on K_0 induced by the monoid of finitely generated projective modules, and in the unital case, the “order unit” corresponding to the rank-one free module. (The invariant consisting of K_0 and this extra order structure is often called the *dimension group*.) This classification theorem was made even more satisfying by a subsequent paper of Effros, Handelman, and Shen [19], which gave an abstract characterization of the possible dimension groups of AF algebras – they are exactly the unperforated ordered abelian groups satisfying the Riesz interpolation property. There has been much subsequent literature on classification of various classes of C^* -algebras via topological K -theory and the order structure on it, but we do not go into this here.

“Lie Groups Made Discrete” and Early Explorations

3.2

Topological K -theory, first introduced for compact spaces by Atiyah and Hirzebruch, was extended to Banach algebras as early as the work of Wood [69] in the mid-60’s. As higher algebraic K -theory began to be developed in the 1970’s, the question arose of trying to understand the similarities and differences between the two theories in the cases where both of them made sense. These explorations eventually went off in two different directions, with a certain overlap between them. The first of the directions had to do with relating purely algebraic and topological or “quasi-topological” K -theories for algebraic varieties, especially over $\mathbb{C}$. This subject is intimately connected with the Riemann–Roch problem (see [4] and [60], for example) and led to the development of semi-topological K -theory (see [23]). This line of development will *not* be the primary theme of this article, but the interested reader should consult the chapter by Friedlander and Walker for a treatment of at least some of this topic. Instead we will discuss another thread in the subject, of relating algebraic and topological K -theory for Banach algebras in general and for C^* -algebras in particular. This subject is also related to the use of algebraic K -theory as a language for discussing certain problems in operator theory.

3.2.1

Basic Concepts and Notations

In order to make it possible to give precise statements for all results, we begin by establishing some definitions and notation. The definitions here do not always coincide with those in use when the results were first established, but we have translated everything into terms consistent with these “modern” definitions.

First we need to make precise exactly what we mean by algebraic and topological K -theory for Banach algebras. Let A be a Banach algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. (The Banach norm $\|\cdot\|$ on A is implicit.) For the moment we assume A is unital, though

it will be necessary from time to time to talk about non-unital Banach algebras as well. (Just as an example, stable C^* -algebras, which already appeared in Sect. 3.1.5 above, are necessarily non-unital.) By $K_n(A)$ we will mean the usual (Quillen) algebraic K -groups of A for $n \geq 0$. However, since the topological K -groups $K_n^{\text{top}}(A)$ are periodic in n (with period 2 if $\mathbb{F} = \mathbb{C}$, period 8 if $\mathbb{F} = \mathbb{R}$), and since we want to compare $K_n(A)$ with $K_n^{\text{top}}(A)$, it is also necessary to have a good definition of $K_n(A)$ for $n < 0$. Accordingly, we let $\mathbb{K}(A)$ be the non-connective delooping of the algebraic K -theory spectrum of A , as defined in [25] and [64], and let $K_n(A)$ denote the n -th homotopy group of $\mathbb{K}(A)$, whether or not n is positive. The groups $K_n(A)$ for $n < 0$ then agree with the “Bass negative K -groups” defined in [37] or [3], and in fact all the standard constructions of deloopings of the algebraic K -theory spectrum are known to be naturally equivalent [44, §§5–6].

By the same token, we let $\mathbb{K}^{\text{top}}(A)$ be the topological K -theory spectrum of A . This is an Ω -spectrum in which every second (or eighth, depending on whether $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) space is $GL(A)$, the infinite general linear group of A , with the Hausdorff group topology defined by the norm on A (not the discrete topology on $GL(A)$, which we’ll denote by $GL(A)^\delta$, used to define $\mathbb{K}(A)$). More specifically, when $\mathbb{F} = \mathbb{C}$, $\mathbb{K}^{\text{top}}(A)$ is given by the homotopy equivalences

$$\begin{aligned} K_0(A) \times BGL(A) &\xrightarrow{\cong} \Omega GL(A), \\ GL(A) &\xrightarrow{\cong} \Omega BGL(A) = \Omega(K_0(A) \times BGL(A)) \end{aligned}$$

of the Bott Periodicity Theorem [69], and by similar maps when $\mathbb{F} = \mathbb{R}$. The homotopy groups $K_n^{\text{top}}(A)$ of $\mathbb{K}^{\text{top}}(A)$ are thus periodic in n (with period 2 if $\mathbb{F} = \mathbb{C}$, period 8 if $\mathbb{F} = \mathbb{R}$).

Basic to what follows is [49, theorem 1.1]:

Theorem 4 Let A be a Banach algebra (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). There is a functorial “comparison map” of spectra $c: \mathbb{K}(A) \rightarrow \mathbb{K}^{\text{top}}(A)$ induced by the “change of topology” map $GL(A)^\delta \rightarrow GL(A)$. The induced map $c_*: K_0(A) \rightarrow K_0^{\text{top}}(A)$ is the identity, and the induced map $c_*: K_1(A) \rightarrow K_1^{\text{top}}(A)$ is the quotient map $GL(A)/E(A) \rightarrow GL(A)/GL(A)_0$. (Here $E(A)$ is the group generated by the elementary matrices, and $GL(A)_0 \supseteq E(A)$ is the identity component of $GL(A)$.)

Recall also that $\mathbb{K}(A)$ is a $\mathbb{K}(\mathbb{F})$ -module spectrum and that $\mathbb{K}^{\text{top}}(A)$ is a $\mathbb{K}^{\text{top}}(\mathbb{F})$ -module spectrum. The map c is compatible with the product structures, in that the diagram

$$\begin{array}{ccc} \mathbb{K}(\mathbb{F}) \times \mathbb{K}(A) & \xrightarrow{\mu} & \mathbb{K}(A) \\ \downarrow (c_{\mathbb{F}}, c_A) & & \downarrow c \\ \mathbb{K}^{\text{top}}(\mathbb{F}) \times \mathbb{K}^{\text{top}}(A) & \xrightarrow{\mu_{\text{top}}} & \mathbb{K}^{\text{top}}(A), \end{array}$$

μ denoting the multiplication maps, is homotopy commutative.

Proof (Sketch)

The “change of topology” map of topological groups

$$GL(A)^\delta \rightarrow GL(A)$$

induces a map of classifying spaces $BGL(A)^\delta \rightarrow BGL(A)$. Apply the Quillen +-construction. Since $BGL(A)$ is already an H-space, this does nothing to $BGL(A)$, and we get a map $(BGL(A)^\delta)^+ \rightarrow BGL(A)$ and thus a map $K_0(A) \times (BGL(A)^\delta)^+ \rightarrow K_0(A) \times BGL(A)$. This is an infinite loop space map, and induces a map c of connective K -theory spectra $\mathbb{K}(A)\langle 0 \rangle \rightarrow \mathbb{K}^{\text{top}}(A)\langle 0 \rangle$ with the desired properties. So it's only necessary to deloop it. This could be done using the Pedersen–Weibel construction in [44], or we can do it inductively, one step at a time, as follows. The single delooping of $K_0(A) \times (BGL(A)^\delta)^+$, which on the spectrum level we'll denote by $\Sigma(\mathbb{K}(A)\langle -1 \rangle)$, is a direct summand in the K -theory space of the Laurent polynomial ring $A[t, t^{-1}]$, i.e., in $K_0(A[t, t^{-1}]) \times (BGL(A[t, t^{-1}])^\delta)^+$. Now by Stone–Weierstraß, $A[t, t^{-1}]$ is a dense subalgebra of the Banach algebra $C(S^1, A)$ (in the complex case), or of

$$\{f \in C(S^1, A_{\mathbb{C}}) : f(z^{-1}) = \overline{f(z)}\}$$

in the real case. (Note this is *not* the same as the algebra of real-valued continuous functions $S^1 \rightarrow A$, since the Laurent polynomial variable t should be identified with the complex variable z on the unit circle in the complex plane, and $z^{-1} = \bar{z}$.) Let us denote the completion of $A[t, t^{-1}]$ in both cases by ΣA , and call it the “suspension” of A . As before we have a map of spectra $\mathbb{K}(\Sigma A)\langle 0 \rangle \rightarrow \mathbb{K}^{\text{top}}(\Sigma A)\langle 0 \rangle$. However, by the Fundamental Theorem of K -theory,

$$\mathbb{K}(A[t, t^{-1}]) \simeq \mathbb{K}(A) \oplus \Sigma \mathbb{K}(A) \oplus \text{Nil terms} ,$$

and similarly $\mathbb{K}^{\text{top}}(\Sigma A) \simeq \mathbb{K}^{\text{top}}(A) \oplus \Sigma \mathbb{K}^{\text{top}}(A)$ by Bott periodicity (for KR in the real case). We thus obtain a commutative diagram of spectra

$$\begin{array}{ccccc} \Sigma(\mathbb{K}(A)\langle -1 \rangle) & \xrightarrow{\Sigma(c \text{ delooped})} & & \Sigma(\mathbb{K}^{\text{top}}(A)\langle -1 \rangle) & \\ \downarrow & & & \downarrow & \\ \mathbb{K}(A[t, t^{-1}])\langle 0 \rangle & \longrightarrow & \mathbb{K}(\Sigma A)\langle 0 \rangle & \xrightarrow{c_{\Sigma A}} & \mathbb{K}^{\text{top}}(\Sigma A)\langle 0 \rangle , \end{array}$$

with the vertical dotted arrows split inclusions, which gives the inductive step.

The compatibility of the map c with products follows from the way the products are defined. The product in topological K -theory comes from a group homomorphism $\mu_{\text{top}}: GL(\mathbb{F}) \times GL(A) \rightarrow GL(A)$ (see for example [48, theorem 5.3.1, pp. 280–281], and the product in algebraic K -theory comes from a map

$\mu: GL(\mathbb{F}) \times GL(A) \rightarrow GL(A)$ defined by exactly the same formula, so clearly the diagram

$$\begin{array}{ccc} GL(\mathbb{F})^\delta \times GL(A)^\delta & \xrightarrow{\mu} & GL(A)^\delta \\ \downarrow & & \downarrow \\ GL(\mathbb{F}) \times GL(A) & \xrightarrow{\mu_{\text{top}}} & GL(A) , \end{array}$$

commutes. So apply the classifying space functor, the plus construction, etc.

Now we can formulate the basic problems to be studied in this article:

Problem 5

1. How close is the map $c: \mathbb{K}(A) \rightarrow \mathbb{K}^{\text{top}}(A)$ to being an equivalence?
2. When c is far from being an equivalence, can we still say anything intelligent about $\mathbb{K}(A)$?

We will sometimes consider K -theory with coefficients. With A as before, $\mathbb{K}(A; \mathbb{Z}/n)$, the algebraic K -theory spectrum with coefficients in $\mathbb{Z}/n$, is obtained by smashing $\mathbb{K}(A)$ with the mod n Moore spectrum (the cofiber of the map $\mathbb{S} \xrightarrow{n} \mathbb{S}$ of degree n , where $\mathbb{S}$ is the sphere spectrum). This definition agrees in positive degrees with, but is not precisely identical to, the (older) definition of mod n K -theory in [6].

Direct Calculation in the Abelian Case

3.2.2

In considering Problems 5(1–2), one must certainly begin with the case of the simplest Banach algebras, namely the archimedean local fields $\mathbb{R}$ and $\mathbb{C}$, and after that with commutative Banach algebras. Taylor's Theorem 2 shows that the study of the commutative case reduces to the study of the algebras of continuous functions, $C^{\mathbb{R}}(X)$ and $C^{\mathbb{C}}(X)$. Already in [43, §7], Milnor did a direct analysis of these cases in low dimensions, and found:

Theorem 6 Let X be a compact Hausdorff space, let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $A = C^{\mathbb{F}}(X)$. Then the map $c_*: K_j(A) \rightarrow K_j^{\text{top}}(A)$ is surjective for $j = 1$, with kernel $C(X, \mathbb{F}^\times)_0$, the identity component of the continuous functions from X to $\mathbb{F}^\times$. If $\mathbb{F} = \mathbb{R}$, since $\mathbb{R}_0^\times = \mathbb{R}_+^\times$ is contractible,

$$\exp: C^{\mathbb{R}}(X) \xrightarrow{\cong} C(X, \mathbb{R}^\times)_0 ,$$

while if $\mathbb{F} = \mathbb{C}$, since $\mathbb{C}^\times$ has the homotopy type of a circle,

$$\exp: C^{\mathbb{C}}(X) \twoheadrightarrow C(X, \mathbb{C}^\times)_0$$

with kernel $C(X, \mathbb{Z})$ and with $C(X, \mathbb{C}^\times)/C(X, \mathbb{C}^\times)_0 \cong H^1(X, \mathbb{Z})$ (Čech cohomology). Furthermore, c_* is surjective also for $j = 2$ and $\mathbb{F} = \mathbb{R}$.

5

6

This shows in particular that c_* can have a huge kernel when $j = 1$, since $C^{\mathbb{R}}(X)$ is always a $\mathbb{Q}$ -vector space of uncountable dimension. It is also true that c_* can have a huge kernel when $j = 2$, since for example by [43, theorem 11.10], $K_2(\mathbb{R})$ and $K_2(\mathbb{C})$ must be uncountable, while on the other hand $K_2^{\text{top}}(\mathbb{R}) = \mathbb{Z}/2$ and $K_2^{\text{top}}(\mathbb{C}) = \mathbb{Z}$. So in general we cannot expect c_* to be close to an isomorphism, and we can already see that the presence of large uniquely divisible groups is part of the explanation. This suggests that examining c_* with finite coefficients might be more valuable.

“Lie Groups Made Discrete” and Suslin’s Theorems on $K_*(\mathbb{R}), K_*(\mathbb{C})$

3.2.3

The algebraic K -theory of $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is more accessible than that of general Banach algebras, since it can be obtained from applying the Quillen $+$ -construction to $BGL(\mathbb{F})^{\delta}$, and $GL(\mathbb{F})$ is an inductive limit of Lie groups. Thus understanding $\mathbb{K}(\mathbb{F}; \mathbb{Z}/n)$ is related to understanding the group homology with finite coefficients of “Lie groups made discrete.” This was studied by Friedlander (as early as the mid-1970’s) and Friedlander–Mislin (see, e.g., [22]), using the machinery of étale homotopy theory, and by Milnor [42].

The most optimistic possible conjecture is that for any Lie group G , the natural map $BG^{\delta} \rightarrow BG$ is a homology isomorphism with finite coefficients. As Milnor shows in [42], this is indeed the case for solvable Lie groups. Milnor also proves that for G any Lie group with finitely many components, the map $H^*(BG; \mathbb{Z}/n) \rightarrow H^*(BG^{\delta}; \mathbb{Z}/n)$ is split injective.³

Around the same time as Milnor’s work, Suslin began to investigate $\mathbb{K}(\mathbb{F}; \mathbb{Z}/n)$ (for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, as well as for more general local or algebraically closed fields) by using completely different techniques coming from algebraic geometry. We quickly summarize his remarkable results.

7

Theorem 7: *Suslin [52]* If $F \hookrightarrow L$ is an extension of algebraically closed fields, then for any positive integer n , the induced map $\mathbb{K}(F; \mathbb{Z}/n) \rightarrow \mathbb{K}(L; \mathbb{Z}/n)$ is an equivalence.

Comments on the proof

Suslin begins by observing that $L = \varinjlim A$, where A runs over the finitely generated F -subalgebras of L . Since F is algebraically closed, the Nullstellensatz implies that for any such A , the map $F \hookrightarrow A$ has an F -linear algebra splitting, and in particular, $K_*(F; \mathbb{Z}/n) \rightarrow K_*(A; \mathbb{Z}/n)$ is split injective. Thus $K_*(F; \mathbb{Z}/n) \rightarrow K_*(L; \mathbb{Z}/n)$

³ One might even hope that injectivity would be true for more general locally compact groups, but this cannot even be the case for general profinite groups, as demonstrated in [50].

is injective. However, this is the “trivial” part of the proof, as it would have applied just as well to the integral K -groups.

The finite coefficients are used (though the divisibility of $L^\times$ and of $\text{Pic}^0(C)$, C a smooth curve over L) in the course of proving the rigidity theorem 8 below. This is then applied with A a smooth finitely generated F -subalgebra of L , $h_0: A \rightarrow L$ the inclusion, and $h_1: A \rightarrow L$ factoring through an F -algebra homomorphism $A \rightarrow F$. Passage to the limit over all such A 's gives the surjectivity of $K_*(F; \mathbb{Z}/n) \rightarrow K_*(A; \mathbb{Z}/n)$.

The proof is completed with:

Theorem 8: *Suslin rigidity theorem [52]* If $F \hookrightarrow L$ is an extension of algebraically closed fields, if A is a smooth affine F -algebra without zero-divisors, and if $h_0, h_1: A \rightarrow L$ are two F -homomorphisms, then for any positive integer n , $(h_0)_* \simeq (h_1)_*$ as maps $\mathbb{K}_*(A; \mathbb{Z}/n) \rightarrow \mathbb{K}_*(L; \mathbb{Z}/n)$.

8

Theorem 7 implies:

Corollary 9 If F is an algebraically closed field of characteristic 0, then $\mathbb{K}(F; \mathbb{Z}/n) \simeq \mathbb{K}(\mathbb{C}; \mathbb{Z}/n)$. And if F is an algebraically closed field of characteristic $p > 0$, then for $(n, p) = 1$, $K_i(F; \mathbb{Z}/n) \cong K_i^{\text{top}}(\mathbb{C}; \mathbb{Z}/n)$, while $K_i(F; \mathbb{Z}/p) = 0$ for $i > 0$.

9

Proof Theorem 7 implies that the homotopy type of $\mathbb{K}(F; \mathbb{Z}/n)$ is the same as for $F = \overline{\mathbb{Q}}$ (in the characteristic 0 case) or for $F = \overline{\mathbb{F}_p}$ (in the characteristic p case). The first statement follows from Theorem 7 applied to $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$; the second follows from Quillen's calculation [46] of the homotopy type of $\mathbb{K}(\mathbb{F}_q)$.

More relevant for our purposes is:

Theorem 10: *Suslin [54]* Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then the comparison map c of Theorem 4 induces isomorphisms $c_*: K_j(\mathbb{F}; \mathbb{Z}/n) \xrightarrow{\cong} K_j^{\text{top}}(\mathbb{F}; \mathbb{Z}/n)$ for all positive integers n and for all $j \geq 0$. We can rephrase this by saying that c induces an equivalence of spectra $\mathbb{K}(\mathbb{F}; \mathbb{Z}/n) \xrightarrow{\cong} \mathbb{K}^{\text{top}}(\mathbb{F}; \mathbb{Z}/n)\langle 0 \rangle$, where the spectrum on the right is the connective topological K -theory spectrum, often denoted $\text{bu}(\mathbb{Z}/n)$ or $\text{bo}(\mathbb{Z}/n)$.

10

Comparison of this result with Corollary 9 yields the remarkable conclusion that for algebraically closed fields F , the homotopy type of $\mathbb{K}(F; \mathbb{Z}/n)$ is almost independent of F . (The only variations show up when n is a multiple of the characteristic.)

However, this is taking us somewhat far afield, as our interest here is in Banach algebras. The proof of Theorem 10 follows a surprising detour; it depends on:

11

Theorem 11: Gabber [24], Gillet–Thomason [26] Let A be a commutative ring in which the integer $n > 0$ is invertible, and let I be an ideal of A contained in the radical of A , such that the pair (A, I) is Henselian. (This means that the conclusion of Hensel’s Lemma holds for the map $A \twoheadrightarrow A/I$, i.e., that if $f \in A[t]$ and if the reduction $\bar{f} \in (A/I)[t]$ of $f \bmod I$ has a root $\bar{\alpha} \in A/I$ such that $\bar{f}'(\bar{\alpha})$ is a unit in A/I , then $\bar{\alpha}$ can be lifted to a root α of f in A .) Then $K_*(A, A/I; \mathbb{Z}/n) = 0$.

Comments on the proof of Theorem 10

Theorem 11 has a fairly obvious application to the computation of $K_*(\mathbb{Q}_p; \mathbb{Z}/n)$ or of mod n K -theory of other non-archimedean local fields F , since if $\mathcal{O}$ is the ring of integers in F and $\mathfrak{p}$ is its maximal ideal, then $(\mathcal{O}, \mathfrak{p})$ is Henselian, but the most ingenious part of [54] is the development of a trick for handling the case of the archimedean fields $\mathbb{R}$ and $\mathbb{C}$.

First there is a relatively straightforward reduction of the problem to proving that the identity map $BSL_k(\mathbb{F})^\delta \rightarrow BSL_k(\mathbb{F})$ induces an isomorphism on mod n homology in a range of dimensions (depending on k but increasing to infinity as $k \rightarrow \infty$). But since $G_k = SL_k(\mathbb{F})$ is a Lie group, it turns out that there is a good model for the fiber of the map $BG_k^\delta \rightarrow BG_k$, which Suslin denotes $(BG_k)_\varepsilon$, obtained by fixing a left-invariant Riemannian metric on G_k and choosing ε small enough so that if U_ε denotes the open ε -ball around the identity e of G_k , then there is a unique geodesic arc joining any two points in U_ε . This guarantees that any intersection of left translates of U_ε , if non-empty, is contractible. One then takes $(BG_k)_\varepsilon$ to be the geometric realization of the simplicial set whose m -simplices are m -tuples $[g_1, \dots, g_m]$ such that $U_\varepsilon \cap g_1 U_\varepsilon \cap \dots \cap g_m U_\varepsilon \neq \emptyset$.

Now because of the Serre spectral sequence of the fibration

$$(BG_k)_\varepsilon \rightarrow BG_k^\delta \rightarrow BG_k$$

as well as Milnor’s results, it turns out it suffices to prove that the natural map $(BG_k)_\varepsilon \rightarrow BG_k$ induces the zero map on mod n homology. To prove this, one similarly translates Theorem 11 into a statement about mod n homology, namely that the map $BGL_k(R, I) \rightarrow BGL(R, I)$ induces the zero map on mod n homology in the limit as $k \rightarrow \infty$. This is then used in a strange way – we take R to be the local

ring of germs of $\mathbb{F}$ -valued continuous functions on $\overbrace{G_k \times \dots \times G_k}^j$ near $(e, \dots, e)$, and I to be its maximal ideal of functions vanishing at $(e, \dots, e)$. Disentangling everything turns out to give the result one needs in degree j , since j -chains on $(BG_k)_\varepsilon$ (where one can pass to the limit as $\varepsilon \rightarrow 0$) are basically elements of R .

One can also find an exposition of the proof in [51].

Karoubi's Early Work on Algebraic K -Theory of Operator Algebras

3.2.4

The first substantial work on Problems 5 for infinite-dimensional Banach algebras, aside from the few special results already mentioned, was undertaken by Karoubi. In this subsection we summarize some of the results in two important papers of Karoubi, [38] and [39]. In all of this section, all Banach and C^* -algebras will be over $\mathbb{C}$, not $\mathbb{R}$.

In the category of C^* -algebras, it is rather artificial to restrict attention to unital algebras, so at this point it's necessary to say something about algebraic K -theory for *non-unital* algebras (over a field of characteristic zero). The problem is that algebraic K -theory does not in general satisfy excision, so that the algebraic K -theory of a non-unital algebra A should be interpreted as the *relative* K -theory of a pair (B, A) , where B is an algebra containing A as an ideal. When A is a nonunital C^* -algebra, there are two canonical choices for B , both of which are C^* -algebras: $\tilde{A} = A + 1 \cdot \mathbb{C}$, the algebra obtained by adjoining a unit to A , and $\mathcal{M}(A)$, the *multiplier algebra* of A . The latter, first introduced in [34] and [13], is the *largest* unital C^* -algebra containing A as an *essential* ideal, just as $\tilde{A}$ is the smallest such C^* -algebra. For example, if X is a locally compact Hausdorff space and if $A = C_0(X)$, $\tilde{A} = C(X_+)$ and $\mathcal{M}(A) = C(\beta X)$, where X_+ is the one-point compactification of X and βX is the Stone-Ćech compactification of X . It turns out that $\mathcal{M}(\mathcal{K}) = \mathcal{L}$, the algebra of bounded operators on the same Hilbert space where $\mathcal{K}$ is the algebra of compact operators. Below, when we talk about the algebraic K -theory of $\mathcal{K}$, we will implicitly mean the K -theory of $(\mathcal{L}, \mathcal{K})$. (Later on, in Sect. 3.3.2, it will turn out it doesn't matter, and the pair $(\tilde{\mathcal{K}}, \mathcal{K})$ would give the same results.)

Karoubi noticed that the periodicity of $K^{\text{top}}(\mathbb{C})$ can be attributed to two special elements, the Bott element $\beta \in K_2^{\text{top}}(\mathbb{C})$ and the inverse Bott element $\beta^{-1} \in K_{-2}^{\text{top}}(\mathbb{C})$. The class β , once we use finite coefficients, does lie in the image of the comparison map $K_2(\mathbb{C}; \mathbb{Z}/n) \rightarrow K_2^{\text{top}}(\mathbb{C}; \mathbb{Z}/n)$ of Theorem 4. (This follows immediately from Theorem 10, but it can also be proved directly – see [38, proposition 5.5].) However, β^{-1} *cannot* lie in the image of the comparison map, even with finite coefficients, since $\mathbb{C}$ is a regular ring and thus its negative K -groups vanish. However, Karoubi noticed that topological K -theory is the same for $\mathbb{C}$ and for the algebra $\mathcal{K}$ of compact operators. (More precisely, the *non-unital* homomorphism $\mathbb{C} \hookrightarrow \mathcal{K}$ sending 1 to a rank-one projection induces an isomorphism on topological K -theory. The excision property of topological K -theory implies functoriality for *non-unital* homomorphisms.) And there *is* an algebraic inverse Bott element in $K_{-2}(\mathcal{K})$ which maps to $\beta^{-1} \in K_{-2}^{\text{top}}(\mathbb{C})$ under the composite

$$K_{-2}(\mathcal{K}) \xrightarrow{c_{\mathcal{K}}} K_{-2}^{\text{top}}(\mathcal{K}) \xrightarrow{\cong} K_{-2}^{\text{top}}(\mathbb{C}) \cong \mathbb{Z}.$$

Karoubi proves this using two simple observations. The first is:

Theorem 12: [38, *théorème 3.6*] If A is a C^* -algebra (with or without unit), the map $c: K_{-1}(A) \rightarrow K_{-1}^{\text{top}}(A)$ is surjective.

12

Sketch of proof [38, §III]

It suffices to consider the case where A has a unit (since if A is non-unital, $K_{-1}(A) \cong K_{-1}(\tilde{A})$, where $\tilde{A}$ is the C^* -algebra obtained by adjoining a unit to A). Recall that the Bass definition of $K_{-1}(A)$ is in terms of a direct summand in $K_0(A[t, t^{-1}])$, and that the Laurent polynomial ring $A[t, t^{-1}]$ embeds densely in $C(S^1, A)$. But $K_0(C(S^1, A)) \cong K_0(A) \oplus K_1^{\text{top}}(A)$, and $K_1^{\text{top}}(A) \cong K_{-1}^{\text{top}}(A)$ by Bott periodicity. So we just need to show that the summand $K_{-1}(A)$ in $K_0(A[t, t^{-1}])$ surjects onto $K_1^{\text{top}}(A)$ under the map induced by the inclusion $A[t, t^{-1}] \hookrightarrow C(S^1, A)$. Since elements of $K_1^{\text{top}}(A)$ are represented by unitary matrices over A and we can always replace A by $M_r(A)$ for some r , it suffices to show that if $u \in A$ is unitary (i.e., u is invertible and $u^{-1} = u^*$), the corresponding class in $K_1^{\text{top}}(A)$ lies in the image of $K_0(A[t, t^{-1}])$. Since the C^* -algebra generated by u is a quotient of $C(S^1)$ (since u is normal and has spectrum in the unit circle), under a $*$ -homomorphism sending the standard generator z of $C(S^1)$ (the identity map $S^1 \rightarrow S^1 \subset \mathbb{C}$, when we think of S^1 as the unit circle in the complex plane) to u , it suffices to deal with the case where $A = C(S^1)$ and we are considering the class $[z]$. Then we just need to show that the Bott element in $K_1^{\text{top}}(C(S^1)) \cong K_0(C(T^2))$ lies in the image of $K_0(C(S^1)[t, t^{-1}])$. However, one can write the Bott element out in terms of a very explicit 2×2 matrix with entries that are functions of z and t that are Laurent polynomials in the t -variable (see [38, pp. 269–270]), so that does it.

Now we obtain the desired result on the inverse Bott element as follows:

13

Theorem 13: *Karoubi* The comparison map $c: K_{-2}(\mathcal{K}) \rightarrow K_{-2}^{\text{top}}(\mathcal{K})$ is surjective.

Proof Consider the exact sequence of C^* -algebras

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{Q} = \mathcal{L}/\mathcal{K} \rightarrow 0,$$

where $\mathcal{Q}$ is the Calkin algebra. Since $\mathcal{L}$, the algebra of all bounded operators on a separable Hilbert space, is “flasque” by the “Eilenberg swindle” (all finitely generated projective $\mathcal{L}$ -modules are stably isomorphic to 0), all its K -groups, whether topological or algebraic, vanish. So now consider the commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 = K_{-1}(\mathcal{L}) & \longrightarrow & K_{-1}(\mathcal{Q}) & \longrightarrow & K_{-2}(\mathcal{K}) & \longrightarrow & 0 = K_{-2}(\mathcal{L}) \\ & & \parallel & & \downarrow & & \parallel \\ 0 = K_{-1}^{\text{top}}(\mathcal{L}) & \longrightarrow & K_{-1}^{\text{top}}(\mathcal{Q}) & \longrightarrow & K_{-2}^{\text{top}}(\mathcal{K}) & \longrightarrow & 0 = K_{-2}^{\text{top}}(\mathcal{L}), \end{array}$$

where the surjectivity of the arrow $K_{-1}(\mathcal{Q}) \rightarrow K_{-1}^{\text{top}}(\mathcal{Q})$ follows from Theorem 12. The result follows by diagram chasing.

In fact because of the multiplicative structure on K -theory one can do much better than this, and Karoubi managed to prove:

Theorem 14: *Karoubi* The comparison map $c: K_*(\mathcal{K}; \mathbb{Z}/n) \rightarrow K_*^{\text{top}}(\mathcal{K}; \mathbb{Z}/n)$ is an isomorphism (in all degrees), and the map $c: K_j(\mathcal{K}) \rightarrow K_j^{\text{top}}(\mathcal{K})$ is surjective for all j and an isomorphism for $j \leq 0$.

14

Proof The first step is to prove the statement about K -theory with finite coefficients. Choose $\gamma \in K_{-2}(\mathcal{K})$ mapping to $\beta^{-1} \in K_{-2}^{\text{top}}(\mathcal{K})$; this is possible by Theorem 13. Let β_n be the mod n Bott element in $K_2(\mathbb{C}; \mathbb{Z}/n)$. (Recall Suslin's Theorem 10.) Then the cup-product $\beta_n \cdot \gamma \in K_0(\mathcal{K}; \mathbb{Z}/n) \cong \mathbb{Z}/n$ maps to $\beta \cdot \beta^{-1} = 1 \in K_0^{\text{top}}(\mathcal{K}; \mathbb{Z}/n) \cong \mathbb{Z}/n$ (by the last part of Theorem 4, the compatibility with products), and so is 1. So the product with γ is inverse to the product with β_n on $K_*(\mathcal{K}; \mathbb{Z}/n)$, and so $K_*(\mathcal{K}; \mathbb{Z}/n)$ is Bott-periodic and canonically isomorphic to $K_*^{\text{top}}(\mathcal{K}; \mathbb{Z}/n) = \mathbb{Z}/n[\beta, \beta^{-1}]$.

Now we lift the mod n result to an integral result for K_2 . Recall that by Theorem 3, $K_1(\mathcal{K}) = 0$. Because of this fact and the above result on mod n K -theory, we have the commuting diagram of long exact sequences

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & K_2(\mathcal{K}) & \xrightarrow{n} & K_2(\mathcal{K}) & \longrightarrow & K_2(\mathcal{K}; \mathbb{Z}/n) \cong \mathbb{Z}/n \longrightarrow K_1(\mathcal{K}) = 0 \\
 & & \downarrow c & & \downarrow c & & \cong \downarrow c \quad \parallel \\
 0 & \longrightarrow & K_2^{\text{top}}(\mathcal{K}) \cong \mathbb{Z} & \xrightarrow{n} & K_2^{\text{top}}(\mathcal{K}) \cong \mathbb{Z} & \longrightarrow & K^{\text{top}}(\mathcal{K}; \mathbb{Z}/n) \cong \mathbb{Z}/n \longrightarrow K_1^{\text{top}}(\mathcal{K}) = 0.
 \end{array}$$

From this it follows that the comparison map $c: K_2(\mathcal{K}) \rightarrow K_2^{\text{top}}(\mathcal{K}) \cong \mathbb{Z}$ hits a generator mod n for each n , and thus this map is integrally surjective.

Hence we can choose an algebraic Bott element $\delta \in K_2(\mathcal{K})$ mapping to $\beta \in K_2^{\text{top}}(\mathcal{K})$. We could then deduce that multiplication by γ is inverse to multiplication by δ , and thus that the algebraic K -theory of $\mathcal{K}$ is Bott-periodic and canonically isomorphic to the topological K -theory, provided we had a good cup-product structure on K -theory for non-commutative rings. Unfortunately there is a problem with this that comes from failure of excision in algebraic K -theory in positive degrees. This is exactly why Karoubi can only conclude that $c: K_j(\mathcal{K}) \rightarrow K_j^{\text{top}}(\mathcal{K})$ is surjective for all j and an isomorphism for $j \leq 0$.

The above result on the K -theory of $\mathcal{K}$ (or rather, Karoubi's first partial results in this direction, since the paper [38] predated Theorem 14) motivated a rather audacious conjecture in [38] about the K -theory of stable C^* -algebras, which came to be known as the *Karoubi Conjecture*.

Conjecture 15: *Karoubi Conjecture* [38] For any stable C^* -algebra A , the comparison map $c: \mathbb{K}(A) \rightarrow \mathbb{K}^{\text{top}}(A)$ is an equivalence.

15

The original formulation of this conjecture in Karoubi's paper seems a bit vague about what definition of algebraic K -theory should be used here for non-unital algebras. Fortunately we shall see later (Sect. 3.3.2) that all possible definitions coincide. In fact it would appear that Karoubi wants to work with $K_*(A \otimes \mathcal{L}, A \otimes \mathcal{K})$, which presents a problem since the minimal C^* -algebra tensor product is not an exact functor in general. Fortunately all the difficulties resolve themselves *a posteriori*.

It is also worth mentioning that Karoubi's paper [39] deals not only with C^* -algebras, but also with Banach algebras, especially the Schatten ideals $\mathcal{L}^p(\mathcal{H})$ in $\mathcal{L}(\mathcal{H})$. (The ideal $\mathcal{L}^p(\mathcal{H})$, $1 \leq p < \infty$ is contained in $\mathcal{K}(\mathcal{H})$; a compact operator T lies in $\mathcal{L}^p(\mathcal{H})$ when the eigenvalues (counted with multiplicities) of the self-adjoint compact operator $(T^*T)^{\frac{1}{2}}$ form an l^p sequence. Thus $\mathcal{L}^1$ is the ideal of trace-class operators discussed previously.) All the ideals $\mathcal{L}^p$ have the same topological K -theory, but roughly speaking, the algebraic K -theory of $\mathcal{L}^p$ becomes more and more "stable" (resembling the K -theory of $\mathcal{K}$) as $p \rightarrow \infty$. This is reflected in:

16

Theorem 16: *Karoubi, [39, propositions 3.5 and 3.9, corollaire 4.2, and théorème 4.13]* For all $p \geq 1$, $K_{-1}(\mathcal{L}^p) = 0$ and $c: K_{-2}(\mathcal{L}^p) \rightarrow K_{-2}^{\text{top}}(\mathcal{L}^p) \cong \mathbb{Z}$ is surjective. However, for integers $n \geq 1$, $c: K_{2n}(\mathcal{L}^p) \rightarrow K_{2n}^{\text{top}}(\mathcal{L}^p) \cong \mathbb{Z}$ is the 0-map for $p \leq 2n - 1$ and is surjective for $n = 1, p > 1$.

The result for K_2 suggests that by using products one should obtain surjectivity of $c: K_{2n}(\mathcal{L}^p) \rightarrow K_{2n}^{\text{top}}(\mathcal{L}^p) \cong \mathbb{Z}$ for p large enough compared with n , but failure of excision gets in the way of proving this in an elementary fashion. This issue is discussed in more detail in [68, §2], where additional results along these lines are obtained.

Recent Progress on Algebraic K -Theory of Operator Algebras

3.3

3.3.1

Algebraic K -Theory Invariants for Operator Algebras

For some purposes, it is useful to study the homotopy fiber $\mathbb{K}^{\text{rel}}(A)$ of the comparison map $c: \mathbb{K}(A) \rightarrow \mathbb{K}^{\text{top}}(A)$ of Theorem 4. We call this spectrum (or the set of its homotopy groups) the *relative K -theory*; it measures the difference between the algebraic and topological theories. Obviously we get a long exact sequence of K -groups

$$\cdots \rightarrow K_{j+1}^{\text{top}}(A) \rightarrow K_j^{\text{rel}}(A) \rightarrow K_j(A) \xrightarrow{c} K_j^{\text{top}}(A) \rightarrow K_{j-1}^{\text{rel}}(A) \rightarrow \cdots \quad (3.5)$$

Since (for any unital Banach algebra A) $K_1(A)$ surjects onto $K_1^{\text{top}}(A)$ and $K_0(A) \rightarrow K_0^{\text{top}}(A)$ is an isomorphism, $K_0^{\text{rel}}(A) = 0$. We have $K_j^{\text{rel}}(\mathbb{C}) = \mathbb{Z}$ for $j = -3, -5, \dots$ and

$K_j^{\text{rel}}(\mathbb{C}) = 0$ for other negative values of j . The Karoubi Conjecture (Conjecture 15) amounts to the assertion that $\mathbb{K}^{\text{rel}}(A)$ is trivial for stable C^* -algebras.

A number of papers in the literature, such as [14, 15, 35], and [36], attempt to detect classes in relative K -theory through secondary index invariants or regulators. (“Primary” index invariants detect classes in topological K -theory.) For example, suppose τ is a p -summable Fredholm module over A . This consists of a representation of A on a Hilbert space $\mathcal{H}$, together with an operator $F \in \mathcal{L}(\mathcal{H})$ that satisfies $F^2 = 1$ and that commutes with A modulo the Schatten class $\mathcal{L}^p(\mathcal{H})$. When p is even, one additionally requires that $\mathcal{H}$ is $\mathbb{Z}/2$ -graded, that the action of A on $\mathcal{H}$ preserves the grading, and that T is odd with respect to the grading. (The prototype for this situation is the case where $A = C^\infty(M)$, M a compact $(p-1)$ -dimensional smooth manifold, and T is obtained by functional calculus from a first-order elliptic differential operator, such as the Dirac operator or signature operator.) In [14] and [15], Connes and Karoubi set up, for each $(p+1)$ -summable Fredholm module τ , a commutative diagram with exact rows, where the top row comes from (3.5):

$$\begin{array}{ccccccccc}
 K_{p+2}(A) & \xrightarrow{c} & K_{p+2}^{\text{top}}(A) & \longrightarrow & K_{p+1}^{\text{rel}}(A) & \longrightarrow & K_{p+1}(A) & \xrightarrow{c} & K_{p+1}^{\text{top}}(A) \\
 & & \text{Ind}_\tau \downarrow & & \text{Ind}_\tau^{\text{sec}} \downarrow & & \text{Ind}_\tau^{\text{sec}} \downarrow & & \\
 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\pi i} & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times & \longrightarrow & 0.
 \end{array}$$

The downward arrow Ind_τ is the usual index and the downward arrows $\text{Ind}_\tau^{\text{sec}}$ are the secondary index invariants. When $A = C^\infty(S^1)$ (this is only a Fréchet algebra, but standard properties of topological K -theory for Banach algebras apply to it as well) and τ corresponds to the smooth Toeplitz extension (3.4), $\text{Ind}_\tau^{\text{sec}}$ recovers the determinant invariant discussed above in Sect. 3.1.4. Other papers such as [35] and [36] relate other secondary invariants defined analytically (for example, via the eta invariant) to the Connes–Karoubi construction.

The Work of Suslin–Wodzicki on Excision

3.3.2

As we saw in Sect. 3.2.4, the Karoubi Conjecture (Conjecture 15) and related conjectures about the K -theory of operator algebras are dependent on understanding to what extent the K -theory of nonunital Banach algebras satisfies excision. Work on this topic was begun by Wodzicki ([66, 67]) and completed in collaboration with Suslin [55]. Wodzicki started by studying excision in cyclic homology, then moved on to the study of rational K -theory, and finally Suslin and Wodzicki clarified the status of excision in integral algebraic K -theory. As the papers [67] and [55] are massive and deep, there is no room to discuss them here in detail, so we will be content with a short synopsis. For simplicity we specialize the results to algebras over a field F of characteristic 0, the only case of interest to us. Then (in [66]) Wodzicki calls an F -algebra A *homologically unital*, or *H-unital* for short, if the standard bar complex $B_\bullet(A)$ is acyclic, i.e., if $\text{Tor}_\bullet^{\tilde{A}}(F, F) = 0$, where $\tilde{A} = A + F \cdot 1$ is A with

unit adjoined. In [66] and [67], Wodzicki shows that C^* -algebras, Banach algebras with bounded approximate unit [66, proposition 5], and many familiar Fréchet algebras such as $\mathcal{A}(\mathbb{R}^n)$ [67, corollary 6.3], are H-unital. Furthermore, any tensor product (over F) of an H-unital algebra with a unital F -algebra is H-unital [67, corollary 9.7]. The main result of [66] is that an F -algebra satisfies excision in cyclic homology if and only if it is H-unital. It is also pointed out, as a consequence of Goodwillie's Theorem [27], that if an F -algebra satisfies excision in rational algebraic K -theory, then it must satisfy excision in cyclic homology and thus be H-unital.

In [55], Suslin and Wodzicki managed to prove the converse, that if A is an H-unital F -algebra, then A satisfies excision in rational algebraic K -theory, i.e., $K_*(B, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is independent of B , for B an F -algebra containing A as an ideal. Since Weibel had already shown [65] that K -theory with $\mathbb{Z}/p$ -coefficients satisfies excision for $\mathbb{Q}$ -algebras, this implies:

17

Theorem 17: *Suslin–Wodzicki [55]* Let A be an algebra over a field F of characteristic 0. Then A satisfies excision for algebraic K -theory if and only if A is H-unital. In particular, C^* -algebras satisfy excision for algebraic K -theory.

The proof of the Suslin–Wodzicki Theorem is rather complicated, but ultimately, via the use of the Volodin approach to K -theory, it comes down to showing that the inclusion

$$A \hookrightarrow A_1 = \begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}$$

induces an isomorphism on Lie algebra homology $H_{\bullet}^{\text{Lie}}(\mathfrak{gl}(A)) \cong H_{\bullet}^{\text{Lie}}(\mathfrak{gl}(A_1))$. This in turn follows from showing that $HC_{\bullet}(A) \cong HC_{\bullet}(A_1)$, which can be deduced from the H-unitality of A . (By the way, if one is only interested in C^* -algebras A , then since they satisfy $A^2 = A$, the proof in [55] can be shortened somewhat, as explained on page 89.)

3.3.3 Resolution of the Karoubi Conjecture

The Karoubi Conjecture is now known to be true, thanks to a combination of the work of Higson [29] and the Suslin–Wodzicki Theorem discussed above in Sect. 3.3.2. The method of Higson is somewhat indirect, and is based on the following intermediate result of independent interest:

18

Theorem 18: [29, theorem 3.2.2] Let k be a functor from the category of C^* -algebras and $*$ -homomorphisms (or a suitable full subcategory, such as the category of separable C^* -algebras) to the category of abelian groups. Assume that k is **stable**, i.e., that the morphism $A \rightarrow A \otimes \mathcal{K}$ (C^* -algebra tensor product) given by $a \mapsto a \otimes e$, where e is a rank-one projection in $\mathcal{K}$, always induces an isomorphism

$k(A) \rightarrow k(A \otimes \mathcal{K})$. Also assume that k is **split exact**, i.e., that it sends split short exact sequences of C^* -algebras to split short exact sequences of abelian groups. Then k is homotopy-invariant.

A few ideas from the proof

The idea is to use the hypotheses to construct a pairing of k with Fredholm modules. More precisely, suppose $\varphi = (\varphi_+, \varphi_-)$ is a Fredholm pair; i.e., φ_+ and φ_- are $*$ -representations of a C^* -algebra B on a Hilbert space $\mathcal{H}$, such that $\varphi_+(a) - \varphi_-(a) \in \mathcal{K}(\mathcal{H})$ for all $a \in B$. From this data, by a construction originally due to Cuntz, one gets a split short exact sequence (for any C^* -algebra A)

$$0 \longrightarrow A \otimes \mathcal{K} \longrightarrow A \otimes B_\varphi \xrightleftharpoons[p]{1 \otimes \varphi} A \otimes B \longrightarrow 0,$$

where $B_\varphi = \{(b, x) \in B \oplus \mathcal{K}(\mathcal{H}) \mid \varphi(b) - x \in \mathcal{K}(\mathcal{H})\}$. (Note that this is independent of whether one uses φ_+ or φ_- .) Since k was assumed stable and split exact, we get a map

$$\varphi_*: k(A \otimes B) \rightarrow \ker(p_*) \xrightarrow{\cong} k(A \otimes \mathcal{K}) \xrightarrow{\cong} k(A)$$

with certain good functorial properties. The next step (which is not so difficult) is to show that this pairing can be expressed a pairing with Fredholm modules of the more conventional sort (where one has a $*$ -representation φ of B on a Hilbert space $\mathcal{H}$ and a unitary operator F that commutes with the representation modulo compacts). One simply lets $\varphi_+ = \varphi$, $\varphi_- = \text{Ad}(F) \circ \varphi$. Then one shows that this pairing is invariant under *operatorial homotopy*, i.e., norm-continuous deformation of the F , keeping φ fixed and with the “commutation modulo compacts” condition satisfied at all times. The final, and hardest, step is to construct an operatorial homotopy $(\varphi, \{F_t\}_{t \in [0,1]})$ of Fredholm modules over $C([0,1])$, such that the pairing of k with (φ, F_0) , $k(A \otimes C([0,1])) \rightarrow k(A)$, corresponds to evaluation of functions at 0, and the pairing of k with (φ, F_1) corresponds to evaluation of functions at 1. This step of the proof is highly reminiscent of the proof [41, §6, theorem 1] that operatorial homotopy invariance of Kasparov’s KK -functor implies homotopy invariance in the most general sense, and establishes the theorem.

From this and the Suslin–Wodzicki Theorem we immediately deduce

Theorem 19 The Karoubi Conjecture is true. In other words, if $A \cong A \otimes \mathcal{K}$ is a stable C^* -algebra, then the comparison map $c: \mathbb{K}(A) \rightarrow \mathbb{K}^{\text{top}}(A)$ of Theorem 4 is an equivalence.

Proof For each integer j , let $k_j(A) = K_j(A \otimes \mathcal{K})$. Then k_j is a functor from C^* -algebras to abelian groups – note that since A is H -unital, we do not need to specify which unital algebra contains $A \otimes \mathcal{K}$ as an ideal, by Theorem 17. We claim this functor is split exact. Indeed, if

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} C \longrightarrow 0$$

is split exact, then so is

$$0 \longrightarrow A \otimes \mathcal{K} \longrightarrow B \otimes \mathcal{K} \xrightarrow{\varphi} C \otimes \mathcal{K} \longrightarrow 0$$

(because the C^* -algebra tensor product with $\mathcal{K}$ is an exact functor, since $\mathcal{K}$ is nuclear), and we can apply the long exact sequence in K -theory. Furthermore, k_j is stable, since if e is a rank-one projection in $\mathcal{K}$ and $\varphi: A \rightarrow A \otimes \mathcal{K}$ is given by $a \mapsto a \otimes e$, then $k_j(\varphi): K_j(A \otimes \mathcal{K}) \rightarrow K_j(A \otimes \mathcal{K} \otimes \mathcal{K})$ is the morphism on K -theory induced by $a \otimes e \mapsto a \otimes e \otimes e$, and there is an isomorphism $\mathcal{K} \otimes \mathcal{K} \xrightarrow{\cong} \mathcal{K}$ sending $e \otimes e \mapsto e$. Hence by Theorem 18, k_j is homotopy-invariant.

Now we conclude the proof by showing by induction that $c_*: k_j(A) \rightarrow K_j^{\text{top}}(A)$ is an isomorphism for all C^* -algebras A and all j . Clearly this is true for $j = 0$. Next, we prove it for j positive. Assume by induction that $c_*: k_j(A) \rightarrow K_j^{\text{top}}(A)$ is an isomorphism for all C^* -algebras A . We have a short exact sequence of C^* -algebras:

$$0 \rightarrow C_0([0, 1)) \otimes A \rightarrow C_0([0, 1)) \otimes A \rightarrow A \rightarrow 0.$$

The middle algebra is contractible, so by the homotopy invariance result just proved, $k_{j+1}(C_0([0, 1)) \otimes A) = 0$ and $k_j(C_0([0, 1)) \otimes A) = 0$. A similar result holds for topological K -theory. Thus the long exact sequences in K -theory give a commuting diagram

$$\begin{array}{ccc} k_{j+1}(A) & \xrightarrow{\partial} & K_j(C_0([0, 1)) \otimes A) \\ c_* \downarrow & \cong & \downarrow c_* \\ K_{j+1}^{\text{top}}(A) & \xrightarrow{\partial} & K_j^{\text{top}}(C_0([0, 1)) \otimes A) . \end{array}$$

and thus $c_*: k_{j+1}(A) \rightarrow K_{j+1}^{\text{top}}(A)$ is an isomorphism. This completes the inductive step.

The result for $j \leq 0$ is already contained in [38, théorème 5.18] and is essentially identical to the proof of Theorem 14, using the product structure on $K_*(\mathcal{K})$.

Unfortunately this proof does not necessarily explain “why” the Karoubi Conjecture is true, since, unlike the proof of the Brown–Schochet Theorem (Theorem 3), it is not constructive.

A number of modifications or variants on Theorem 19 are now known. For example, one has the “unstable Karoubi Conjecture” in [53]: if A is a stable C^* -algebra, then the natural map $B(GL_n(A)^\delta) \rightarrow BGL_n(A)$ is an isomorphism on integral homology for all n . Here $GL_n(A)$ is to be interpreted as $GL_n(\tilde{A}, A)$, i.e., the group of matrices in $GL_n(\tilde{A})$ which are congruent to 1 modulo A . There is a Fréchet analogue of the Karoubi Conjecture in [57], with $\mathcal{K}$ replaced by the algebra of smoothing operators, or in other words by infinite matrices with rapidly decreasing entries, a version of the theorem for certain generalized stable algebras in [31], and a pro- C^* -algebra analogue in [32].

Other Miscellaneous Results

3.3.4

In this final section, we mention a number of other results and open problems related to algebraic K -theory of operator algebras. These involve K -regularity, negative K -theory, and K -theory with finite coefficients.

K -Regularity

We begin with a few results about K -regularity, or in other words, results that say that C^* -algebras behave somewhat like regular rings with respect to algebraic K -theory. As motivation for this subject, note that in [56], Swan defined a commutative ring R with unit, and with no nilpotent elements, to be *seminormal* if for any $b, c \in R$ with $b^3 = c^2$, there is an element $a \in R$ with $a^2 = b$ and $a^3 = c$. This condition guarantees that $\text{Pic } R[X_1, \dots, X_n] \cong \text{Pic } R$ for all n , which we can call Pic-regularity. Swan’s condition is clearly satisfied for commutative C^* -algebras, since if $R = C(X)$ for some compact Hausdorff space X , and if b and c are as indicated, one can take

$$a(x) = \begin{cases} c(x)/b(x), & b(x) \neq 0, \\ 0, & b(x) = c(x) = 0, \end{cases}$$

and check that a is continuous and thus lies in R . Hence commutative C^* -algebras are Pic-regular. This suggests that they might be K -regular as well, since Pic and K_0 are closely related.

In [29, §6], Higson proved the K -regularity of stable C^* -algebras as part of his work on the Karoubi Conjecture. In other words, we have

Theorem 20: *Higson; see also [31, theorem 18]* If A is a stable C^* -algebra, then for any n , the natural map $\mathbb{K}(A) \rightarrow \mathbb{K}(A[t_1, \dots, t_n])$ (which is obviously split by the map induced by sending $t_j \mapsto 0$) is an equivalence. In other words, stable C^* -algebras are K -regular.

20

Proof For any j , the functor $k_j^n = A \mapsto K_j(A[t_1, \dots, t_n])$ satisfies the conditions of Theorem 18. (Here we are using the fact that H-unitality of A implies H-unitality of

the polynomial ring $A[t_1, \dots, t_n]$.) Hence k_j^n is a homotopy functor. So we have an isomorphism $k_j^n(A \otimes C([0, 1])) \cong k_j^n(A)$ induced in one direction by the inclusion of A in $A \otimes C([0, 1]) \cong C([0, 1], A)$ and in the other direction by evaluation at either 0 or 1. Now consider the homomorphism φ from $C([0, 1], A)[t_1, \dots, t_n]$ to itself defined by

$$\varphi(f)(s, t_1, \dots, t_n) = f(s, st_1, \dots, st_n), \quad s \text{ the coordinate on } [0, 1].$$

Then φ followed by evaluation at $s = 1$ is the identity on $A[t_1, \dots, t_n]$, so it induces the identity on $K_j(A[t_1, \dots, t_n])$, but on the other hand, φ followed by evaluation at $s = 0$ sends $A[t_1, \dots, t_n]$ to A . Hence $K_j(A[t_1, \dots, t_n])$ factors through $K_j(A)$.

Other results on K -regularity of C^* -algebras may be found in [49]. For example, there is some evidence there that all C^* -algebras should be K_0 -regular (i.e., that one should have isomorphisms $K_0(A[t_1, \dots, t_n]) \cong K_0(A)$ for all n , when A is a C^* -algebra). There are simple counterexamples there to show this cannot be true for Banach algebras. Commutative C^* -algebras are in some sense at the opposite extreme from stable C^* -algebras, and for these one has basically the same K -regularity result, though the method of proof is totally different.

21 Theorem 21: *Rosenberg [49, theorem 3.1]* If A is a commutative C^* -algebra, then for any n , the natural map $\mathbb{K}(A) \rightarrow \mathbb{K}(A[t_1, \dots, t_n])$ (which is obviously split by the map induced by sending $t_j \mapsto 0$) is an equivalence. In other words, commutative C^* -algebras are K -regular.

As observed in [49], to prove the general case, one may by excision (Sect. 3.3.2) reduce to the case where A is unital, and one may by a transfer argument reduce to the case $\mathbb{F} = \mathbb{C}$. So we may take $A = C(Y)$. It was also observed in [49] that any finitely generated subalgebra $\mathbb{C}[f_1, \dots, f_n]$ of A is reduced (contains no nilpotent elements), hence by the Nullstellensatz is isomorphic to the algebra $\mathbb{C}[X]$ of regular functions on some affine algebraic set $X \subseteq \mathbb{C}^N$, $N \leq n$, not necessarily irreducible. Then the inclusion $\mathbb{C}[f_1, \dots, f_n] \hookrightarrow A$ is dual to a continuous map $Y \rightarrow X$. Thus it suffices to show:

22 Theorem 22 Let $A = C(Y)$, where Y is a compact Hausdorff space, be a (complex) commutative C^* -algebra, and let $X \subseteq \mathbb{C}^N$ be an affine algebraic set. Suppose one is given a continuous map $\varphi: Y \rightarrow X$, and let $\varphi^*: \mathbb{C}[X] \rightarrow C(Y)$ be the dual map on functions. Then $(\varphi^*)_*$ vanishes identically on $N^j K_m(A)$ for any $j \geq 0$ and $m \geq 0$.

Proof The proof of this given in [49] was based on the (basically correct) idea of chopping up Y and factoring φ through smooth varieties, but the technical details were incorrect.⁴ Indeed, as pointed out to me by Mark Walker, it was

⁴ I thank Mark Walker for pointing this out to me.

claimed in [49] that one can find a closed covering of X such that a resolution of singularities $p: \widehat{X} \rightarrow X$ of X (in the sense of [30]) splits topologically over each member of the closed cover, and this simply isn't true. (It would be OK with a *locally* closed cover, however.) Walker [personal communication] has found another proof of Theorem 20; see also [23, theorem 5.3]; to set the record straight, we give still another proof here.

Let $p: \widehat{X} \rightarrow X$ be a resolution of singularities of X (in the sense of [30]).⁵ This has the following properties of interest to us:

1. $\widehat{X}$ is a smooth quasiprojective variety (not necessarily irreducible, since we aren't assuming this of X), and p is a proper surjective algebraic morphism.
2. There is a Zariski-closed subset X_1 of X , such that $X \setminus X_1$ is a smooth quasiprojective variety Zariski-dense in X , and such that if $\widehat{X}_1 = p^{-1}(X_1)$, then p gives an isomorphism from $\widehat{X} \setminus \widehat{X}_1$ to $X \setminus X_1$, and a proper surjective morphism from $\widehat{X}_1$ to X_1 .

We now prove the theorem by induction on the dimension of X . To start the induction, if $\dim X = 0$, then X is necessarily smooth and the theorem is trivial. So assume we know the result when X has smaller dimension, and observe that the inductive hypothesis applies to the singular set X_1 . Also note, as observed in [49], that there is no loss of generality in assuming $Y \subseteq X$. Let $Y_1 = Y \cap X_1$. From the diagram

$$\begin{array}{ccccc} Y_1 & \hookrightarrow & Y & \longrightarrow & (Y, Y_1) \\ \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\ X_1 & \hookrightarrow & X & \longrightarrow & (X, X_1), \end{array}$$

we get a commuting diagram of exact sequences of K -groups

$$\begin{array}{ccccccc} N^j K_{m+1}(X_1) & \xrightarrow{\partial} & N^j K_m(X, X_1) & \longrightarrow & N^j K_m(X) & \longrightarrow & \cdots \\ \downarrow (\varphi^*)_* & & \downarrow (\varphi^*)_* & & \downarrow (\varphi^*)_* & & \\ N^j K_{m+1}(C(Y_1)) & \xrightarrow{\partial} & N^j K_m(C_0(Y \setminus Y_1)) & \longrightarrow & N^j K_m(C(Y)) & \longrightarrow & \cdots \\ & & & & & & \\ \cdots & \longrightarrow & N^j K_m(X) & \longrightarrow & N^j K_m(X_1) & \xrightarrow{\partial} & N^j K_{m-1}(X, X_1) \\ & & \downarrow (\varphi^*)_* & & \downarrow (\varphi^*)_* & & \downarrow (\varphi^*)_* \\ \cdots & \longrightarrow & N^j K_m(C(Y)) & \longrightarrow & N^j K_m(C(Y_1)) & \xrightarrow{\partial} & N^j K_{m-1}(C_0(Y \setminus Y_1)). \end{array}$$

⁵ We don't need the full force of the existence of a such a resolution, but it makes the argument a little easier. The interested reader can think of how to formulate everything without using $\widehat{X}$.

Here we have used excision (Sect. 3.3.2) on the bottom rows and have identified K -theory of the coordinate ring of an affine variety with the K -theory of its category of vector bundles. The K -groups of (X, X_1) denote *relative* K -theory of vector bundles in the sense of [16], and NK -theory for varieties is defined by setting $NK_m(X) = \ker(K_m(X \times \mathbb{A}^1) \rightarrow K_m(X))$, etc. By inductive hypothesis, the maps $N^j K_m(X_1) \xrightarrow{(\varphi^*)^*} N^j K_{m+1}(C(Y_1))$ vanish, so by diagram chasing, it's enough to show that the maps $N^j K_m(X, X_1) \rightarrow N^j K_m(C_0(Y \setminus Y_1))$ vanish.

Since $X \setminus X_1$ is smooth, one might think this should be automatic, but that's not the case since algebraic K -theory doesn't satisfy excision. However, we are saved by the fact that we have excision in the *target* algebra. The map $p: \widehat{X} \rightarrow X$ is an isomorphism from $\widehat{X} \setminus \widehat{X}_1$ to $X \setminus X_1$, and induces maps $p^*: N^j K_m(X, X_1) \rightarrow N^j K_m(\widehat{X}, \widehat{X}_1)$. Since φ lifts over $Y \setminus Y_1$, the map $N^j K_m(X, X_1) \rightarrow N^j K_m(C_0(Y \setminus Y_1))$ factors through $N^j K_m(\widehat{X}, \widehat{X}_1)$. (Here the approach of [16] is essential since $\widehat{X}$ may not be affine, and so we can't work just with K -theory of rings.) But $N^j K_m(\widehat{X}, \widehat{X}_1)$ vanishes since $\widehat{X}$ and $\widehat{X}_1$ are smooth.

Negative K -Theory

In [47] and [49], the author began a study of the *negative* algebraic K -theory of C^* -algebras. The most manageable case to study should be commutative C^* -algebras. By Theorem 21, such algebras are K -regular, so they satisfy the Fundamental Theorem in the simple form $K_j(A[t, t^{-1}]) \cong K_j(A) \oplus K_{j-1}(A)$. A conjecture from [47] and [49], complementary to the results of Higson in [29], is:

23 **Conjecture 23:** *Rosenberg* Negative K -theory is a homotopy functor on the category of commutative C^* -algebras. Thus $X \mapsto K_j(C_0(X))$ is a homotopy functor on the category of locally compact Hausdorff spaces and proper maps when $j \leq 0$.

24 **Corollary 24** On the category of (second countable) locally compact Hausdorff spaces, $X \mapsto K_j(C_0(X))$ coincides with **connective** K -theory $bu^{-j}(X)$, for $j \leq 0$.

Proof (from [49]) that the Corollary follows from the Conjecture

Let

$$k^{-j}(X) = \begin{cases} K_j^{\text{top}}(C_0(X)), & j > 0 \\ K_j(C_0(X)), & j \leq 0. \end{cases}$$

Then Conjecture 24 implies that k^* is a homotopy functor, and it satisfies the excision and long exact sequence axioms, by Theorem 17 and the long exact sequences in algebraic and topological K -theory, pasted together at $j = 0$, where they coincide. It is also clear that k^* is additive on infinite disjoint unions, i.e., that

$k^*(\coprod_i X_i) = \bigoplus_i k^*(X_i)$. Thus it is an additive cohomology theory (with compact supports). There is an obvious natural transformation of cohomology theories $k^* \rightarrow K^*$ (ordinary topological K -theory with compact supports), induced by $c_*: K_j(C_0(X)) \rightarrow K_j^{\text{top}}(C_0(X))$, which is an isomorphism on k^{-j} , $j \leq 0$. And k^* is a connective theory, since $\mathbb{C}$ is a regular ring and thus $k^{-j}(\text{pt}) = K_j(\mathbb{C}) = 0$ for $j < 0$. Thus by the universal property of the connective cover of a spectrum [2, p. 145], $k^* \rightarrow K^*$ factors through bu^* . Since $k^*(X) \rightarrow K^*(X)$ is an isomorphism for X a point, it is an isomorphism for any X with X_+ a finite CW-complex, and then by additivity, for X_+ any compact metric space (since any compact metric space is a countable inverse limit of finite complexes).

While a proof of Conjecture 24 is outlined in [47], Mark Walker has kindly pointed out that the proof is faulty. The author still believes that the same method should work, and indeed it does in certain special cases, but it seems to be hard to get the technical details to work. In fact, it is even conceivable that negative K -theory is a homotopy functor for arbitrary C^* -algebras, but a proof of this would require a totally new technique.

K -Theory with Finite Coefficients

In this last section, we discuss results on K -theory with finite coefficients that generalize Theorem 10. These results can be viewed as analytic counterparts to the work of Friedlander–Mislin and Milnor discussed above in Sect. 3.2, and to the results of Thomason ([60–63]) for algebraic varieties.

Theorem 25: *Fischer [21], Prasolov ([1, 45])* Let A be a commutative C^* -algebra. Then the comparison map for A with finite coefficients, 25

$$c: K_i(A; \mathbb{Z}/n) \rightarrow K_i^{\text{top}}(A; \mathbb{Z}/n)$$

is an isomorphism for $i \geq 0$.

The method of proof of this theorem is copied closely from the proof of Suslin's theorem, Theorem 10. Thus it relies on Theorem 11 on Henselian rings, and is quite special to the commutative case. However, it is conceivable that one has:

Conjecture 26: *Rosenberg [47, conjecture 4.1]* Let A be a C^* -algebra. Then the comparison map for A with finite coefficients, 26

$$c: K_i(A; \mathbb{Z}/n) \rightarrow K_i^{\text{top}}(A; \mathbb{Z}/n)$$

is an isomorphism for $i \geq 0$.

In support of this, we have:

27

Theorem 27: *Rosenberg [47, theorem 4.2]* Let A be a type I C^* -algebra which has a finite composition series, each of whose composition factors has the form $A \otimes M_n(\mathbb{F})$ ($n \geq 0$) or $A \otimes \mathcal{K}$, where A is commutative. Then the comparison map for A with finite coefficients,

$$c: K_i(A; \mathbb{Z}/n) \rightarrow K_i^{\text{top}}(A; \mathbb{Z}/n)$$

is an isomorphism for $i \geq 0$.

This is proved by piecing together Theorems 25 and 19, using excision (Theorem 17). The main obstruction to extending the proof to more general classes of C^* -algebras is the lack of a good result on (topological) inductive limits of C^* -algebras. Such a result would necessarily be delicate, because we know that *algebraic* K -theory behaves differently under *algebraic* inductive limits and *topological* inductive limits. For example, the *algebraic* inductive limit $\varinjlim M_n(\mathbb{C})$ has the same K -theory as $\mathbb{C}$, and thus its negative K -theory vanishes, whereas the C^* -algebra inductive limit $\varinjlim M_n(\mathbb{C})$ is $\mathcal{K}$, which has infinitely many non-zero negative K -groups.

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Part V

Other Forms of *K*-Theory

Semi-topological K -Theory

V.1

Eric M. Friedlander and Mark E. Walker *

1.1	<i>Introduction</i>	878
1.2	<i>Definition of Semi-topological K-Theory</i>	881
	Semi-topological K -Theory of Projective Varieties: $\mathcal{K}^{semi}$	883
	Semi-topological K -Theory of Quasi-projective Varieties: $\mathcal{K}^{qst}$	886
	The Recognition Principle	889
	Semi-topological K -Theory for Real Varieties	891
1.3	<i>Algebraic, Semi-topological, and Topological Theories</i>	894
	Motivic, Morphic, and Singular Cohomology	895
	The Chern Class Maps	898
	Finite Coefficients and the Bott Element	903
	Real Analogues	906
1.4	<i>Spectral Sequences and Computations</i>	906
	The Spectral Sequence	907
	Generalized Cycle Map and Weights	909
	Computations	912
1.5	<i>Conjectures</i>	914
	Integral Versions of the "Classical" Conjectures	915
	K -Theoretic Analogue of the Hodge Conjecture	918
	<i>References</i>	921

* Both authors were partially supported by the N.S.F and the N.S.A.

Introduction

The *semi-topological K-theory* of a complex variety X , written $K_*^{sst}(X)$, interpolates between the algebraic K-theory, $K_*^{alg}(X)$, of X and the topological K-theory, $K_{top}^*(X^{an})$, of the analytic space X^{an} associated to X . (The superscript “sst” stands for “singular semi-topological”.) In a similar vein, the *real semi-topological K-theory*, written $K\mathbb{R}_*^{sst}(Y)$, of a real variety Y interpolates between the algebraic K-theory of Y and the Atiyah Real K-theory of the associated space with involution $Y_{\mathbb{R}}(\mathbb{C})$. We intend this survey to provide both motivation and coherence to the field of semi-topological K-theory. We explain the many foundational results contained in the series of papers by the authors [27, 31, 32], as well as in the recent paper by the authors and Christian Haesemeyer [21]. We shall also mention various conjectures that involve challenging problems concerning both algebraic cycles and algebraic K-theory.

Our expectation is that the functor $K_*^{sst}(-)$ is better suited for the study of complex algebraic varieties than either algebraic K-theory or topological K-theory. For example, applied to the point $X = \text{Spec } \mathbb{C}$, $K_i^{alg}(-)$ yields uncountable abelian groups for $i > 0$, whereas $K_i^{sst}(\text{Spec } \mathbb{C})$ is 0 for i odd and $\mathbb{Z}$ for i even (i.e., it coincides with the topological K-theory of a point). On the other hand, topological K-theory is a functor on homotopy types and ignores finer algebro-geometric structure of varieties, whereas semi-topological and algebraic K-theory agree on finite coefficients

$$K_*^{alg}(-, \mathbb{Z}/n) \cong K_*^{sst}(-, \mathbb{Z}/n) \quad (1.1)$$

and the rational semi-topological K-groups $K_*^{sst}(X, \mathbb{Q})$ contain information about the cycles on X and, conjecturally, the rational Hodge filtration on singular cohomology $H^*(X^{an}, \mathbb{Q})$.

To give the reader some sense of the definition of semi-topological K-theory, we mention that $K_0^{sst}(X)$ is the Grothendieck group of algebraic vector bundles modulo algebraic equivalence: two bundles on X are *algebraically equivalent* if each is given as the specialization to a closed point on a connected curve C of a common vector bundle on $C \times X$. In particular, the ring $K_0^{sst}(X)$ (with product given by tensor product of vector bundles) is rationally isomorphic to the ring $A^*(X)$ of algebraic cycles modulo algebraic equivalence (with the product given by intersection of cycles) under the Chern character map

$$ch : K_0^{sst}(X, \mathbb{Q}) \xrightarrow{\cong} A^*(X, \mathbb{Q}) .$$

This should be compared with the similar relationship between $K_0^{alg}(X)$ and the Chow ring $CH^*(X)$ of algebraic cycles modulo rational equivalence. Taking into consideration also the associated topological theories, we obtain the following heuristic diagram, describing six cohomology theories of interest.

As we discuss below, the authors have constructed a precise counter-part of this heuristic diagram by establishing a homotopy commutative diagram of spectra –

Table 1.1. Six cohomology theories together with their “base values” for a smooth variety X

K-theory	Cohomology (i.e., cycle theory)
<i>Algebraic K-theory</i> , $K_*^{alg}(-)$ $K_0^{alg}(X)$ = algebraic vector bundles modulo rational equivalence	<i>Motivic Cohomology</i> , $H_{\mathcal{M}}^*(-, \mathbb{Z}(*))$ $H_{\mathcal{M}}^{2*}(X, \mathbb{Z}(*)) = CH^*(X)$ = cycles modulo rational equivalence
<i>Semi-topological K-theory</i> , $K_*^{sst}(-)$ $K_0^{sst}(X)$ = algebraic vector bundles modulo algebraic equivalence	<i>Morphic Cohomology</i> , $L^*H^*(-)$ $L^*H^{2*}(X) = A^*(X)$ = cycles modulo algebraic equivalence
<i>Topological K-theory</i> , $K_{top}^*(-)$ $K_{top}^0(X^{an})$ = topological vector bundles modulo topological equivalence	<i>Singular Cohomology</i> , $H_{sing}^*(-)$ $H_{sing}^{2*}(X^{an})$ = integral, rectifiable cycles modulo topological equivalence

see (1.9) in Sect. 1.3 below. For example, there are Chern character maps joining the theories in the left column and the theories (with rational coefficients) in the right column, and if X is smooth, these Chern character maps are rational isomorphisms in all degrees. In particular, such isomorphisms extend the rational isomorphism $K_0^{sst}(-)_{\mathbb{Q}} \cong A^*(-)_{\mathbb{Q}}$ mentioned above.

For certain special varieties X (e.g., projective smooth toric varieties), the natural map

$$K_*^{sst}(X) \rightarrow K_{top}^{-*}(X^{an})$$

is an isomorphism whenever $* \geq 0$ (see [21]). Such an isomorphism can be interpreted (as we now interpret the Lawson Suspension Theorem for $X = \mathbb{P}^n$) as asserting that some construction involving algebraic morphisms is a “small” homotopy-theoretic model for an analogous construction involving continuous maps between analytic spaces. In general, however, $K_*^{sst}(X)$ differs considerably from topological K -theory; for example, $K_0^{sst}(X)$ need not be finitely generated even for a smooth projective variety X . Nonetheless, there are natural transformations

$$\mathcal{K}^{alg}(-) \rightarrow \mathcal{K}^{sst}(-) \rightarrow \mathcal{K}_{top}(-)$$

from $(Sch/\mathbb{C})$ to $Spectra$ with many good properties, perhaps the most striking of which is the isomorphism for finite coefficients mentioned above (1.1). Understanding multiplication in $K_*^{sst}(X)$ by the Bott element

$$\beta \in K_2^{sst}(\text{Spec } \mathbb{C}) \cong K_{top}^{-2}(pt)$$

is also interesting: for X is smooth, the natural map $K_*^{sst}(X) \rightarrow K_{top}^{-*}(X^{an})$ induces an isomorphism

$$K_*^{sst}(X)[\beta^{-1}] \cong K_{top}^{-*}(X^{an})[\beta^{-1}] = KU^{-*}(X^{an})$$

upon inverting the Bott element. On the other hand, the kernel of $K_0^{sst}(X) \rightarrow K_{top}^0(X^{an})$ is rationally isomorphic to the Griffiths group (of algebraic cycles on X homologically equivalent to 0 modulo those algebraically equivalent to 0). Moreover, a filtration on $K_{top}^*(X) \otimes \mathbb{Q}$ associated to $\mathcal{K}^{sst}(-) \rightarrow \mathcal{K}_{top}(-)$ and multiplication by $\beta \in K_2^{sst}(pt)$ is conjecturally equivalent to the rational Hodge filtration.

To formulate K -theories, we require some process of “homotopy theoretic group completion” as first became evident in Quillen’s formulation of algebraic K -theory of rings using the Quillen plus construction. This can be contrasted with the simpler constructions of cohomology theories: these derive from structures (e.g., the monoid of effective cycles) which are commutative, whereas the direct sum of vector bundles is only commutative up to coherent isomorphism. For this reason, the constructions we present involve use of the machinery of operads and the utilization of certain other homotopy-theoretic techniques.

As the reader will see, the analytic topology on a real or complex variety is used in the construction of semi-topological K -theory. Thus far, there is no reasonable definition of semi-topological K -theory for varieties over other base fields, although one might anticipate that p -adic fields and real closed fields might lend themselves to such a theory.

We conclude this introduction with a few brief comments to guide the reader toward more details about the topics we discuss. First, the original paper of H.B. Lawson [39] initiated the study of the analytic spaces of Chow varieties, varieties that parametrize effective algebraic cycles on a complex projective variety. Lawson’s remarkable theorem enables one to compute the homotopy groups of the topological abelian group of algebraic cycles of a given dimension on a projective space $\mathbb{P}^n$ (i.e., his theorem enables one to compute what is now known as the Lawson homology of $\mathbb{P}^n$). In [14] the first author pointed out that the group of connected components of the topological abelian group of r -cycles on a complex projective variety X is naturally isomorphic to the group of algebraic r -cycles on X modulo algebraic equivalence. A key insight was provided by Lawson and M.-L. Michelson [41] who proved that the universal total Chern class can be interpreted as a map induced by the inclusion of linear cycles on projective spaces into all algebraic cycles. Important formal properties of Lawson homology were developed by the first author in collaboration with O. Gabber [20] and by P. Lima-Filho [46]. B. Mazur and the first author investigated filtrations on homology associated with Lawson homology in [24], and the first author studied complementary filtrations on cycles in [15, 19]. H.B. Lawson and the first author introduced the concept of a cocycle leading to morphic cohomology theory [23] and established a duality relationship between morphic cohomology and Lawson homology in [22]. Consideration of quasi-projective varieties using similar methods has led to awkward questions of point-set topology, so that plausible definitions are difficult to handle

when the varieties are not smooth (cf. [28]). When contemplating the formulation of semi-topological K -theory for quasi-projective varieties, the authors introduce singular semi-topological complexes, which appear to give a good formulation of morphic cohomology for any quasi-projective algebraic variety [32].

As Lawson homology and morphic cohomology developed, it became natural to seek a companion K -theory. In [23], H.B. Lawson and the first author showed how to obtain characteristic classes in morphic cohomology for algebraic vector bundles. “Holomorphic K -theory” was briefly introduced by Lawson, Lima-Filho, and Michelsohn in [40]. Following an outline of the first author [16], the authors established the foundations and general properties of semi-topological K -theory in a series of papers [27, 31, 32] and extended this theory to real quasi-projective varieties in [30]. Many of the results sketched in this survey were first formulated and proved in these papers. A surprisingly difficult result proved by the authors is the assertion that there is a natural rational isomorphism (given by the Chern character) relating semi-topological K -theory and morphic cohomology of smooth varieties [32, 4.7].

Most recently, the authors together with C. Haesemeyer established a spectral sequence relating morphic cohomology and semi-topological K -theory compatible with the motivic and Atiyah–Hirzebruch spectral sequences [21]. Moreover, that paper uses the notion of integral weight filtrations on Borel–Moore homology (due to Deligne [11] and Gillet–Soulé [33]), which, in conjunction with the spectral sequence, enable them to establish that $K_*^{sst}(X) \rightarrow K_{top}^*(X^{an})$ is an isomorphism for many of the special varieties for which one might hope this to be true.

The subject now is ready for the computation of $K_*^{sst}(-)$ for more complicated varieties and for applications of this theory to the study of geometry. Since many of the most difficult and long-standing conjectures about complex algebraic varieties are related to such computations, one suspects that general results will be difficult to achieve. We anticipate that the focus on algebraic equivalence given by morphic cohomology and semi-topological K -theory might lead to insights into vector bundles and algebraic cycles on real and complex varieties.

Definition of Semi-topological K -Theory

1.2

Originally [31] semi-topological K -theory was defined only for projective, weakly normal complex algebraic varieties and these original constructions involved consideration of topological spaces of algebraic morphisms from such a variety X to the family of Grassmann varieties $\text{Grass}_n(\mathbb{C}^N)$. The assumption that X is projective implies that the set of algebraic morphisms $\text{Hom}(X, \text{Grass}_n(\mathbb{C}^N))$ coincides with the set of closed points of an ind-variety, and thus we may topologize $\text{Hom}(X, \text{Grass}_n(\mathbb{C}^N))$ by giving it the associated analytic topology. If, in addition, X is weakly normal, then $\text{Hom}(X, \text{Grass}_n(\mathbb{C}^N))$ maps injectively to $\text{Maps}(X^{an}, \text{Grass}_n(\mathbb{C}^N)^{an})$, the set of all continuous maps, and we may also endow $\text{Hom}(X, \text{Grass}_n(\mathbb{C}^N))$ with the subspace topology of the space $\mathcal{M}aps(X^{an}, \text{Grass}_n(\mathbb{C}^N)^{an})$, the

set $\text{Maps}(X^{an}, \text{Grass}_n(\mathbb{C}^N)^{an})$ endowed with the usual compact-open topology. In fact, these topologies coincide, and $\mathcal{M}or(X, \text{Grass}_n(\mathbb{C}^N))$ denotes this topological space. The collection of spaces $\mathcal{M}or(X, \text{Grass}_n(\mathbb{C}^N))$ for varying n and N leads to the construction of a spectrum $\mathcal{K}^{semi}(X)$ whose homotopy groups are the semi-topological K -groups of X .

The authors [31] subsequently extended the theory so constructed to all quasi-projective complex varieties U by providing the set of algebraic morphisms $\text{Hom}(U^w, \text{Grass}_n(\mathbb{C}^N))$ (where $U^w \rightarrow U$ is the weak normalization of U) with a natural topology (again using $\mathcal{M}or(U, \text{Grass}_n(\mathbb{C}^N))^{an}$ to denote the resulting space). We were, however, unable to verify many of the desired formal properties of this construction $\mathcal{K}_*^{semi}(-)$ when applied to non-projective varieties.

Inspired by a suggestion of V. Voevodsky, the authors reformulated semi-topological K -theory in [27]. The resulting functor from quasi-projective complex varieties to spectra, $\mathcal{K}^{sst}(-)$, when applied to a weakly normal projective variety X gives a spectrum weakly homotopy equivalent to the spectrum $\mathcal{K}^{semi}(X)$. We have shown that the functor $U \mapsto \mathcal{K}_*^{sst}(U)$ satisfies many desirable properties, and thus now view the groups $\mathcal{K}_*^{sst}$ as the semi-topological K -groups of a variety.

In this section, we begin with the definition of $\mathcal{K}^{semi}(X)$ (restricted to weakly normal, projective complex varieties). Although supplanted by the more general construction $\mathcal{K}^{sst}$ discussed below, the motivation underlying the construction of $\mathcal{K}^{semi}$ is more geometric and transparent.

We shall see that the definition is formulated so that there are natural homotopy classes of maps of spectra

$$\mathcal{K}^{alg}(X) \rightarrow \mathcal{K}^{semi}(X) \rightarrow \mathcal{K}_{top}(X^{an}) . \quad (1.2)$$

(Here, $\mathcal{K}_{top}(X^{an})$ denotes the (-1) -connected cover of $ku(X^{an})$, the mapping spectrum from X^{an} to $\mathbf{bu}$.) These maps are induced by the natural maps of simplicial sets given in degree d by

$$\begin{aligned} \text{Hom}(\Delta^d \times X, \text{Grass}_n(\mathbb{C}^N)) &\rightarrow \text{Maps}(\Delta_{top}^d, \mathcal{M}or(X, \text{Grass}_n(\mathbb{C}^N))^{an}) \\ &\rightarrow \text{Maps}(\Delta_{top}^d, \mathcal{M}aps(X^{an}, \text{Grass}_n(\mathbb{C}^N))^{an}) . \end{aligned}$$

In formulating $\mathcal{K}^{semi}(-)$ (and later $\mathcal{K}^{sst}(-)$), we are motivated by the property that if one applies the connected component functor $\pi_0(-)$ to the maps of (1.2), then one obtains the natural maps

$$K_0^{alg}(X) \rightarrow K_0^{alg}(X)/\text{algebraic equivalence} \rightarrow K_{top}^0(X^{an})$$

from the Grothendieck group of algebraic vector bundles to the Grothendieck group of algebraic vector bundles modulo algebraic equivalence to the Grothendieck group of topological vector bundles.

The reader will find that in order to define the spectra appearing in (1.2), we use operads to stabilize and group complete the associated mapping spaces. This use of operads makes the following discussion somewhat technical.

Semi-topological K -Theory of Projective Varieties:

 $\mathcal{K}^{semi}$

1.2.1

Let X be a projective, weakly normal complex variety and define $\text{Grass}(\mathbb{C}^N) = \coprod_n \text{Grass}_n(\mathbb{C}^N)$ where $\text{Grass}_n(\mathbb{C}^N)$ is the projective variety parameterizing rank n quotient complex vector spaces of $\mathbb{C}^N$. Since X and $\text{Grass}(\mathbb{C}^N)$ are projective varieties, the set $\text{Hom}(X, \text{Grass}(\mathbb{C}^N))$ is the set of closed points of an infinite disjoint union (indexed by degree) of quasi-projective complex varieties. We write this ind-variety as $\mathcal{M}or(X, \text{Grass}(\mathbb{C}^N))$ and we let $\mathcal{M}or(X, \text{Grass}(\mathbb{C}^N))^{an}$ denote the associated topological space endowed with the analytic topology. Since X is weakly normal, one can verify that $\mathcal{M}or(X, \text{Grass}(\mathbb{C}^N))^{an}$ is naturally a subspace of $\mathcal{M}aps(X^{an}, \text{Grass}(\mathbb{C}^N)^{an})$, the space of all continuous maps endowed with the compact-open topology. Considering the system of ind-varieties $\mathcal{M}or(X, \text{Grass}(\mathbb{C}^N))$ for $N \geq 0$, where the map $\text{Grass}(\mathbb{C}^N) \rightarrow \text{Grass}(\mathbb{C}^{N+1})$ is given by composing with the projection map $\mathbb{C}^{N+1} \rightarrow \mathbb{C}^N$ onto the first N coordinates, gives the ind-variety $\mathcal{M}or(X, \text{Grass})$ and the associated space

$$\mathcal{M}or(X, \text{Grass})^{an} = \lim_{\substack{\longrightarrow \\ N}} \mathcal{M}or(X, \text{Grass}(\mathbb{C}^N))^{an} ,$$

which we may identify with a subspace of $\mathcal{M}aps(X^{an}, \text{Grass}^{an})$.

The following proposition indicates that the space $\mathcal{M}or(X, \text{Grass})^{an}$ possesses interesting K -theoretic information. We shall say that two algebraic bundles $V_1 \rightarrow X, V_2 \rightarrow X$ are *algebraically equivalent* if there exists a connected smooth curve C and an algebraic vector bundle $\mathcal{V} \rightarrow C \times X$ such that each of $V_1 \rightarrow X, V_2 \rightarrow X$ is given by restriction of $\mathcal{V}$ to some $\mathbb{C}$ -point of C . We define *algebraic equivalence* on either the monoid of isomorphism classes of algebraic vector bundles or the associate Grothendieck group $K_0^{alg}(X)$ to be equivalence relation generated by algebraic equivalence of bundles.

Proposition 1 (cf. [31, 2.10, 2.12]) There is a natural isomorphism of abelian monoids

$$\pi_0(\mathcal{M}or(X, \text{Grass})^{an}) \cong \frac{\{\text{algebraic vector bundles on } X \text{ generated by global sections}\}}{\text{algebraic equivalence}} ,$$

(where the abelian monoid law for the left-hand side is described below). Moreover, the group completion of the above map can be identified with the following natural isomorphism of abelian groups

$$K_0^{semi}(X) = \pi_0(\mathcal{M}or(X, \text{Grass})^{an})^+ \cong \frac{K_0(X)}{\text{algebraic equivalence}} .$$

The proof of Proposition 1 is straight-forward, perhaps disguising several interesting and important features. First, the condition that two points in $\mathcal{M}or(X, \text{Grass})^{an}$ lie in the same topological component is equivalent to the condition

that they lie in the same Zariski component of the ind-variety $\mathcal{M}or(X, \text{Grass})$. Second, upon group completion, one obtains all (virtual) vector bundles so that one may drop the condition that the vector bundles be generated by their global sections.

To define the higher semi-topological K -groups, we introduce the structure of an H -space on $\mathcal{M}or(X, \text{Grass})$. Direct sum of bundles determines algebraic pairings

$$\mathcal{M}or(X, \text{Grass}_n(\mathbb{C}^N)) \times \mathcal{M}or(X, \text{Grass}_m(\mathbb{C}^M)) \rightarrow \mathcal{M}or(X, \text{Grass}_{n+m}(\mathbb{C}^{N+M})) ,$$

for all M, N . Once one chooses a linear injection $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \hookrightarrow \mathbb{C}^\infty$, these pairings may be stabilized in a suitable fashion by letting $M, N \mapsto \infty$ to endow $\mathcal{M}or(X, \text{Grass})$ with an operation. Under this operation, the associated space $\mathcal{M}or(X, \text{Grass})^{an}$ is a homotopy-commutative H -space. Using one of several techniques of infinite loop spaces, one shows that this H -space admits a *homotopy-theoretic group completion*

$$\mathcal{M}or(X, \text{Grass})^{an} \rightarrow (\mathcal{M}or(X, \text{Grass})^{an})^{h+} ;$$

by definition, this is a map of H -spaces which induces group completion on π_0 and whose map on (integral, singular) homology can be identified with the map

$$H_* (\mathcal{M}or(X, \text{Grass})^{an}) \rightarrow \mathbb{Z}[\pi_0^+] \otimes_{\mathbb{Z}[\pi_0]} H_* (\mathcal{M}or(X, \text{Grass})^{an}) .$$

2

Definition 2 Let X be a weakly normal, projective complex variety. We define

$$\mathcal{M}or(X, \text{Grass})^{an} \rightarrow \mathcal{K}^{semi}(X)$$

to be a homotopy-theoretic group completion of the homotopy commutative H -space $\mathcal{M}or(X, \text{Grass})^{an}$. We call $\mathcal{K}^{semi}(X)$ the *semi-topological K -theory space* of X , and we define the *semi-topological K -groups* of X by the formula

$$K_n^{semi}(X) = \pi_n(\mathcal{K}^{semi}(X)) , \quad n \geq 0 .$$

In particular, $K_0^{semi}(X)$ is naturally isomorphic to $K_0^{alg}(X)/(\text{algebraic equivalence})$. For any finitely generated abelian group A , we define the semi-topological K -groups of X with coefficients in A by the formula

$$K_n^{sst}(X, A) = \pi_n(\mathcal{K}^{semi}(X; A)) , \quad n \geq 0 .$$

3

Remark 3 An equivalent construction of $\mathcal{K}^{semi}(X)$ is given by Lawson, Lima-Filho, and Michelsohn in [40]. See also [49]. In these papers, the term “holomorphic K -theory” is used instead of “semi-topological K -theory”.

The construction of the group-like H -space $\mathcal{K}^{semi}(X)$ from the H -space $\mathcal{M}or(X, \text{Grass})^{an}$ can be enriched to yield an Ω -spectrum in several ways. For example, one can let $\mathcal{I} = \{\mathcal{I}(n), n \geq 0\}$, denote the E_∞ -operad with $\mathcal{I}(n)$ defined to be the

contractible space of all linear injections from $(\mathbb{C}^\infty)^{\oplus n}$ into $\mathbb{C}^\infty$. (Thus $\mathcal{I}$ is closely related to the linear isometries operad.) Then $\mathcal{I}$ “acts” on $\mathcal{M}or(X, \text{Grass})^{an}$ via a family of pairings

$$\mathcal{I}(n) \times (\mathcal{M}or(X, \text{Grass})^{an})^n \rightarrow \mathcal{M}or(X, \text{Grass})^{an}, \quad n \geq 0.$$

Intuitively, this action can be describe as follows: given a point in $\mathcal{I}(n)$ and n quotients of the form $\mathcal{O}_X^\infty \twoheadrightarrow \mathcal{E}_i$, the point of $\mathcal{I}(n)$ allows one to move the n quotient objects into general position so that the sum of these maps is surjective. In particular, the case $n = 2$ together with a specific choice of a point in $\mathcal{I}(2)$ defines the H -space operation for $\mathcal{M}or(X, \text{Grass})^{an}$ given above. Such an action of the operad $\mathcal{I}$ determines an Ω -spectrum $\Omega^\infty \mathcal{M}or(X, \text{Grass})^{an}$ using the machinery of May [50, §14], and the 0-th space of this spectrum provides a model for the homotopy-theoretic group completion $\mathcal{K}^{semi}(X)$.

It is useful to know that the algebraic K -theory space defined for a variety Y over an arbitrary field F admits a parallel construction. Recall that the standard algebraic k -simplex Δ^k over $\text{Spec } F$ is the affine variety $\text{Spec } F[x_0, \dots, x_k]/(\sum_i x_i - 1)$ and that these standard simplices determine a cosimplicial variety $\Delta_F^\bullet$. The simplicial set $d \mapsto \text{Hom}(X \times_F \Delta_F^d, \text{Grass}(F^\infty))$ admits the structure of a homotopy-commutative H -space (if “space” is interpreted to mean “simplicial set”). In fact, $\text{Hom}(X \times_F \Delta_F^\bullet, \text{Grass}(F^\infty))$ is an $\mathcal{I}(\Delta_F^\bullet)$ -space where $\mathcal{I}(\Delta_F^\bullet)$ is a suitable simplicial analogue of the E_∞ -operad $\mathcal{I}$ introduced above. The following result is due to the second author and D. Grayson.

Theorem 4 (cf. [31, 6.8] [35, 3.3]) Given a smooth, algebraic variety X over a field F , the homotopy-theoretic group completion of the homotopy-commutative H -space

$$\text{Hom}(X \times_F \Delta_F^\bullet, \text{Grass}(F^\infty))$$

is weakly homotopy equivalent to $\mathcal{K}^{alg}(X)$, the algebraic K -theory space of X . In fact, the spectrum associated to the $\mathcal{I}(\Delta_F^\bullet)$ -space $\text{Hom}(X \times_F \Delta_F^\bullet, \text{Grass}(F^\infty))$ is weakly equivalent to the algebraic K -theory spectrum of X .

Theorem 4 leads easily to the existence of a natural map

$$\mathcal{K}^{alg}(X) \rightarrow \mathcal{K}^{semi}(X)$$

of spectra (more precisely, $\mathcal{K}^{alg}(X) \rightarrow \text{Sing.}(\mathcal{K}^{semi}(X))$) representing the algebraic K -theory and semi-topological K -theory of smooth, projective complex varieties. Furthermore, if we replace $\mathcal{M}or(X, \text{Grass})^{an}$ by $\mathcal{M}aps(X^{an}, \text{Grass}^{an})$, then we can proceed with the same construction as above to form a space (in fact, an Ω -spectrum) $\mathcal{K}_{top}(X^{an})$ that receives a map from $\mathcal{K}^{semi}(X)$. It follows from [51, I.1] that we have

$$K_{top}^{-n}(X^{an}) := \pi_n \mathcal{K}_{top}(X^{an}) \cong ku^{-n}(X^{an}), \quad n \geq 0,$$

where ku^* denotes the connective topological K -theory of a space (i.e., the generalized cohomology theory represented by the connective spectrum bu). In

other words, the spectrum $\mathcal{K}_{top}(X^{an})$ is the (-1) -connected cover of the mapping spectrum from X^{an} to **bu**. Moreover, the subspace inclusion $\mathcal{M}or(X, \text{Grass})^{an} \subset \mathcal{M}aps(X^{an}, \text{Grass}^{an})$ induces a natural map $\mathcal{K}^{semi}(X) \rightarrow \mathcal{K}_{top}(X^{an})$ such that the composition

$$\mathcal{K}^{alg}(X) \rightarrow \mathcal{K}^{semi}(X) \rightarrow \mathcal{K}_{top}(X) \quad (1.3)$$

induces the usual map from algebraic to topological K -theory.

When $X = \text{Spec } \mathbb{C}$, we clearly have $\mathcal{K}^{semi}(\text{Spec } \mathbb{C}) = \mathcal{K}_{top}(pt)$. A more interesting computation is the following integral analogue of the Quillen–Lichtenbaum conjecture for smooth projective complex curves.

5

Theorem 5 (cf. [31, 7.5]) If C is a smooth, projective complex curve, then the natural map

$$\mathcal{K}^{semi}(C) \rightarrow \mathcal{K}_{top}(C^{an})$$

is a weak homotopy equivalence, inducing isomorphisms

$$K_n^{semi}(C) \cong K_{top}^{-n}(C^{an}), \quad n \geq 0.$$

The proof of Theorem 5 uses a result of F. Kirwan [38, 1.1] on the moduli space of vector bundles on curves. Specifically, Kirwan shows that the composition of

$$A_d(n)^{an} \rightarrow \mathcal{M}or(C, \text{Grass}_n(\mathbb{C}^\infty))_d^{an} \rightarrow \mathcal{M}aps(C^{an}, \text{Grass}_n(\mathbb{C}^\infty)^{an})_d$$

induces an isomorphism in cohomology up to dimension k provided

$$d \geq 2n(2g + k + 1) + n \max(k + 1 + n(2g + k + 1), \frac{1}{4}n^2g). \quad (1.4)$$

Here, g is the genus of C , the subscripts d refer to taking the open and closed subspaces consisting of degree d maps, and $A_d(n)$ refers to the subvariety of $\mathcal{M}or_d(C, \text{Grass}_n(\mathbb{C}^\infty))$ parameterizing quotients $\mathcal{O}_X^\infty \twoheadrightarrow V$ satisfying the extra condition that $H^1(C, V) = 0$. Theorem 5 is deduced from this result of Kirwan by showing that the homotopy-theoretic group completions of each of the spaces in the chain

$$\coprod_{d,n} A_d(n)^{an} \rightarrow \coprod_{d,n} \mathcal{M}or_d(C, \text{Grass}_n(\mathbb{C}^\infty))_d^{an} \rightarrow \coprod_{d,n} \mathcal{M}aps(C^{an}, \text{Grass}_n(\mathbb{C}^\infty)^{an})_d$$

can be obtained by taking suitable limits (technically, mapping telescopes) of self-maps of each of the spaces. The point is that the first map here becomes an equivalence upon taking such limits since the condition defining $A_d(n)$ as a subvariety of $\mathcal{M}or_d(C, \text{Grass}_n(\mathbb{C}^\infty))^{an}$ becomes trivial, and the second map becomes an equivalence since the inequality (1.4) is met in all degrees in the limit.

For higher dimensional varieties X , the map $\mathcal{K}^{semi}(X) \rightarrow \mathcal{K}_{top}(X^{an})$ is rarely a weak homotopy equivalence. For example, if S is a smooth, projective complex surface, the map $\mathcal{K}^{semi}(S) \rightarrow \mathcal{K}_{top}(S^{an})$ usually fails to induce an isomorphism at π_0 (although it does induce an isomorphism on all higher homotopy groups.) In

fact, $K_0^{semi}(S, \mathbb{Q}) \cong K_{top}^0(S^{an}, \mathbb{Q})$ if and only if $H^2(S^{an}, \mathbb{Q})$ consists only of algebraic cohomology classes [21, 6.17].

Semi-topological K -Theory of Quasi-projective Varieties: $\mathcal{K}^{sst}$

1.2.2

The extension of the definition of semi-topological K -theory from projective complex varieties to quasi-projective complex varieties has a somewhat confusing history. Initially, the authors (see especially [31]) carried out this extension in seemingly the most natural way possible: one imposes a suitable topology on the set $\text{Hom}(X^w, \text{Grass})$ to form a space $\mathcal{M}or(X, \text{Grass})^{an}$ and then repeats the constructions of the previous section to yield a group-like H -space (in fact, a spectrum) $\mathcal{K}^{semi}(X)$. We will not go into the details of the topology on $\mathcal{M}or(X, \text{Grass})^{an}$ – we refer the interested reader to [28] for a careful description.

It gradually became apparent that annoying point-set topology considerations prevents one from establishing the desired formal properties of the theory $\mathcal{K}^{semi}(-)$ for arbitrary quasi-projective varieties. On the other hand, the authors have developed a closely related and conjecturally equivalent theory that allows for such properties to be proven. This newer theory, $\mathcal{K}^{sst}$, is now viewed by the authors as *the* semi-topological K -theory.

To motivate the definition of $\mathcal{K}^{sst}(-)$, we return to the case of weakly normal, projective complex varieties and consider what happens if we replace spaces with singular simplicial sets in the construction of $\mathcal{K}^{semi}(-)$. That is, for such a variety X we replace the space $\mathcal{M}or(X, \text{Grass})^{an}$ with the simplicial set $d \mapsto \text{Maps}(\Delta_{top}^d, \mathcal{M}or(X, \text{Grass})^{an})$ and we replace the E_∞ topological operad $\mathcal{I}$ with the associated simplicial one, $\mathcal{I}(\Delta_{top}^\bullet)$, defined by $\mathcal{I}(\Delta_{top}^\bullet)(n) = (d \mapsto \text{Maps}(\Delta_{top}^d, \mathcal{I}(n)))$. An important observation is that since Δ_{top}^d is compact and since $\mathcal{M}or(X, \text{Grass})^{an}$ is an inductive limit of analytic spaces associated to quasi-projective varieties, we have the natural isomorphism

$$\text{Maps}(\Delta_{top}^d, \mathcal{M}or(X, \text{Grass})^{an}) \cong \lim_{\Delta_{top}^d \rightarrow U^{an}} \text{Hom}(U \times X, \text{Grass}),$$

where the limit ranges over the filtered category whose objects are continuous maps $\Delta_{top}^d \rightarrow U^{an}$, with U a quasi-projective complex variety, and in which a morphism is given by a morphism of varieties $U \rightarrow V$ causing the evident triangle to commute. In other words, if we define

$$\text{Hom}(\Delta_{top}^d \times X, \text{Grass}) = \lim_{\Delta_{top}^d \rightarrow U^{an}} \text{Hom}(U \times X, \text{Grass})$$

then we have $\text{Maps}(\Delta_{top}^d, \mathcal{M}or(X, \text{Grass})) \cong \text{Hom}(\Delta_{top}^d \times X, \text{Grass})$.

For readers inclined to categorical constructions it might be helpful to observe that $\text{Hom}(\Delta_{top}^d \times X, \text{Grass})$ is the result of applying to the topological space Δ_{top}^d

the Kan extension of the presheaf $\text{Hom}(- \times X, \text{Grass})$ on $\text{Sch}/\mathbb{C}$ along the functor $\text{Sch}/\mathbb{C} \rightarrow \text{Top}$ given by $U \mapsto U^{an}$.

Just as in the construction of $\mathcal{K}^{semi}$, it's easy to show that we have the action of the simplicial E_∞ operad $\mathcal{I}(\Delta_{top}^\bullet)$ on the simplicial set $\text{Hom}(\Delta_{top}^\bullet \times X, \text{Grass})$, and hence we obtain an associated Ω -spectrum

$$\Omega^\infty | \text{Hom}(\Delta_{top}^\bullet \times X, \text{Grass}) | .$$

Finally, this Ω -spectrum is readily seen to be equivalent to the spectrum $\mathcal{K}^{semi}(X)$ constructed above (assuming X is projective and weakly normal).

The idea in defining $\mathcal{K}^{sst}$, then, is to just take the simplicial set $\text{Hom}(X \times \Delta_{top}^\bullet, \text{Grass})$ itself for the starting point of the construction. For observe that the definition of this simplicial set does not depend on X being either projective or weakly normal, and so we may use it for arbitrary varieties. Theorem 4 suggests another alternative – one could simply take the algebraic K -theory functor taking values in spectra, $\mathcal{K}^{alg}$, and “semi-topologize” it by applying it degree-wise to $\Delta_{top}^\bullet \times X$ via the Kan extension formula. The following proposition shows that the two constructions result in equivalent theories.

6

Proposition 6 (cf. [27, 1.3]) For any quasi-projective complex variety X , there are natural weak homotopy equivalences of spectra

$$\begin{aligned} |d \mapsto \mathcal{K}^{alg}(\Delta_{top}^d \times X)| &\rightarrow \Omega^\infty | \text{Hom}(\Delta_{top}^\bullet \times \Delta^\bullet \times X, \text{Grass}) | \\ &\leftarrow \Omega^\infty | \text{Hom}(\Delta_{top}^\bullet \times X, \text{Grass}) | \end{aligned}$$

where $\mathcal{K}^{alg}(-) : (\text{Sch}/\mathbb{C}) \rightarrow \underline{\text{Spectra}}$ is a fixed choice of a functorial model of the algebraic K -theory spectrum of quasi-projective complex varieties and $\mathcal{K}^{alg}(\Delta_{top}^d \times X)$ is the value of the Kan extension of $\mathcal{K}^{alg}(- \times X)$ applied to Δ_{top}^d .

The choice of $|d \mapsto \mathcal{K}^{alg}(\Delta_{top}^d \times X)|$ as the primary definition of semi-topological K -theory is justified by the “Recognition Principle”, which appears below as Theorem 12. As we shall see, this definition is but one of an interesting collection of “singular semi-topological constructions”.

7

Definition 7 For any quasi-projective complex variety X , the (singular) semi-topological K -theory spectrum of X is the Ω -spectrum

$$\mathcal{K}^{sst}(X) = \mathcal{K}(\Delta_{top}^\bullet \times X) = |d \mapsto \mathcal{K}^{alg}(\Delta_{top}^d \times X)| .$$

The semi-topological K -groups of X with coefficients in the abelian group A are given by

$$K_n^{sst}(X, A) = \pi_n \mathcal{K}^{sst}(X, A) , \quad n \geq 0 .$$

We find that we may easily construct maps as in (1.3) for any quasi-projective complex variety.

Proposition 8 (cf. [27, 1.4]) There are natural maps of spectra (in the stable homotopy category)

8

$$\mathcal{K}^{alg}(X) \rightarrow \mathcal{K}^{sst}(X) \rightarrow \mathcal{K}_{top}(X^{an}) . \quad (1.5)$$

Furthermore, if X is projective and weakly normal, there is a weak equivalence of spectra

$$\mathcal{K}^{sst}(X) \simeq \mathcal{K}^{semi}(X) .$$

The map $\mathcal{K}^{alg}(X) \rightarrow \mathcal{K}^{sst}(X)$ is the canonical map $\mathcal{K}^{alg}(\Delta_{top}^0 \times X) \rightarrow |d \mapsto \mathcal{K}(\Delta_{top}^d \times X)|$. The map $\mathcal{K}^{sst}(X) \rightarrow \mathcal{K}_{top}(X^{an})$ is the map in the stable homotopy category (using Proposition 6) associated to the map

$$\Omega^\infty | \text{Hom}(\Delta_{top}^\bullet \times X, \text{Grass}) | \rightarrow \mathcal{K}_{top}(X^{an})$$

that is given by the adjoint of the map $| \text{Sing}_\bullet(-) | \rightarrow id$ together with the natural inclusion $\text{Hom}(-, \text{Grass}) \subset \text{Maps}((-)^{an}, \text{Grass}^{an})$.

Upon taking homotopy groups with coefficients in an abelian group A , we thus have the chain of maps

$$K_*^{alg}(X, A) \rightarrow K_*^{sst}(X, A) \rightarrow K_{top}^{-*}(X^{an}, A) \quad (1.6)$$

whose composition is the usual map from algebraic to topological K -theory with coefficients in A .

The following property of $\mathcal{K}^{sst}(-)$ is one indication that Definition 7 is a suitable generalization of $\mathcal{K}^{semi}(-)$ to all quasi-projective complex varieties.

Proposition 9 [30, 2.5] For any quasi-projective complex variety X , there is a natural isomorphism

9

$$\mathcal{K}_0^{sst}(X) \cong \frac{K_0^{alg}(X)}{\text{algebraic equivalence}} .$$

The Recognition Principle

1.2.3

The formulation of $\mathcal{K}^{sst}(-)$ in Definition 7 is a special case of the following *singular topological construction*.

Definition 10 Let $\mathcal{F}$ be a contravariant functor from $Sch/\mathbb{C}$ to a suitable category $\mathcal{C}$ (such as chain complexes of abelian groups, spaces, or spectra). For any compact Hausdorff space T and variety $X \in Sch/\mathbb{C}$, define

10

$$\mathcal{F}(T \times X) = \lim_{\substack{\rightarrow \\ T \rightarrow U^{an}}} \mathcal{F}(U \times X) .$$

Define $\mathcal{F}^{sst}$ to be the functor from $Sch/\mathbb{C}$ to $\mathcal{C}$ by

$$\mathcal{F}^{sst}(X) = Tot \left(d \mapsto \mathcal{F}(\Delta_{top}^d \times X) \right),$$

where Tot refers to a suitable notion of “total object” (such as total complex of a bicomplex or geometric realization of a bisimplicial space or spectrum). We call $\mathcal{F}^{sst}$ the *singular semi-topological functor associated to $\mathcal{F}$* .

The usefulness of this singular semi-topological construction arises in large part from the validity of the following *Recognition Principle*. This theorem should be compared with an analogous theorem of V. Voevodsky [58, 5.9].

11 Theorem 11 [32, 2.7] Suppose $\mathcal{F} \rightarrow \mathcal{G}$ is a natural transformation of contravariant functors from $Sch/\mathbb{C}$ to chain complexes of abelian groups, group-like H -spaces, or spectra. Suppose this map is a weak equivalence locally in the h -topology (for example, suppose it is a weak equivalence on all smooth varieties). Then the associated map

$$Tot \left(d \mapsto \mathcal{F}(\Delta_{top}^d) \right) \rightarrow Tot \left(d \mapsto \mathcal{G}(\Delta_{top}^d) \right)$$

is a weak homotopy equivalence.

As a sample application (many more will be discussed below) of the Recognition Principle, we have following theorem relating algebraic and semi-topological K -theory. The authors had originally established the validity of Theorem 12 using a much more involved argument (see [27, 3.8]), an argument which pointed the way toward the formulation and applications of Theorem 11.

12 Theorem 12 (cf. [27, 3.7]) For a quasi-projective complex variety X and positive integer n , we have an isomorphism

$$K_q^{alg}(X; \mathbb{Z}/n) \xrightarrow{\cong} K_q^{sst}(X; \mathbb{Z}/n), \quad q \geq 0.$$

Sketch of Proof

Via an evident spectral sequence argument, it suffices to prove $K_q^{alg}(X; \mathbb{Z}/n) \rightarrow (d \mapsto K_q^{alg}(\Delta_{top}^d \times X; \mathbb{Z}/n))$ is a homotopy equivalence of simplicial abelian groups (the source being constant). By the Recognition Principle, it suffices to prove the map of presheaves $K_q^{alg}(X; \mathbb{Z}/n) \rightarrow K_q^{alg}(- \times X; \mathbb{Z}/n)$ is locally an isomorphism in the h topology (where again the source is constant). This holds already for the étale topology by Suslin rigidity [55].

Semi-topological K -Theory for Real Varieties

1.2.4

In this section, we summarize results of [30] which show that the semi-topological K -theory for real varieties satisfies analogues of the pleasing properties of $\mathcal{K}^{sst}(-)$ for complex varieties. We take these properties as confirmation that the definition of $\mathcal{K}^{\mathbb{R}^{sst}}(-)$ given here is the “correct” analogue of $\mathcal{K}^{sst}(-)$, but are frustrated by the fact that this extension does not suggest a generalization to other fields.

The reader should observe that the definition of $\mathcal{K}^{\mathbb{R}^{sst}}(-)$ below is so formulated that if Y is a quasi-projective complex variety then (see Proposition 18)

$$\mathcal{K}^{\mathbb{R}^{sst}}(Y|_{\mathbb{R}}) = \mathcal{K}^{sst}(Y) ,$$

where $Y|_{\mathbb{R}}$ denotes the complex variety Y regarded as a real variety via restriction of scalars. Thus, any result concerning $\mathcal{K}^{\mathbb{R}^{sst}}(-)$ that applies to all quasi-projective real varieties incorporates the analogous statement for $\mathcal{K}^{sst}(-)$ applied to quasi-projective complex varieties.

As in the complex case, the motivation for the definition of real semi-topological K -theory is most easily seen in the projective case first, and in this case we first define $\mathcal{K}^{\mathbb{R}^{semi}}$, an equivalent but more geometric version of $\mathcal{K}^{\mathbb{R}^{sst}}$ defined below.

Definition 13 Let Y be a projective real variety. We define

13

$$\mathcal{M}or_{\mathbb{R}}(Y, \text{Grass})^{an} = \lim_{\substack{\rightarrow \\ N}} \mathcal{M}or_{\mathbb{R}}(Y, \text{Grass}(\mathbb{R}^N))(\mathbb{R}) ,$$

where $\mathcal{M}or_{\mathbb{R}}(Y, \text{Grass}(\mathbb{R}^N))(\mathbb{R})$ denotes the space of real points of the real ind-variety $\mathcal{M}or_{\mathbb{R}}(Y, \text{Grass}(\mathbb{R}^N))$ parameterizing morphisms over $\mathbb{R}$ from Y to $\text{Grass}(\mathbb{R}^N)$. As in the complex setting, $\mathcal{M}or_{\mathbb{R}}(Y, \text{Grass})^{an}$ admits the structure of a homotopy-commutative H -space and we let

$$\mathcal{M}or_{\mathbb{R}}(Y, \text{Grass})^{an} \rightarrow \mathcal{K}^{\mathbb{R}^{semi}}(Y)$$

denote the homotopy-theoretic group completion. We call $\mathcal{K}^{\mathbb{R}^{semi}}(Y)$ the *real semi-topological K -theory space*.

As in the complex case, for Y a projective, weakly normal real variety, we have that

$$\text{Maps}\left(\Delta_{top}^d, \mathcal{M}or_{\mathbb{R}}(Y, \text{Grass})\right) = \lim_{\substack{\rightarrow \\ \Delta_{top}^d \rightarrow U(\mathbb{R})}} \text{Hom}(U \times_{\mathbb{R}} Y, \text{Grass}) ,$$

where the limit ranges over pairs $(U, \Delta_{top}^d \rightarrow U(\mathbb{R}))$ consisting of a real variety U and a continuous maps from Δ_{top}^d to the its space of real points $U(\mathbb{R})$. As before, this leads naturally to the following definition:

14 **Definition 14** For a quasi-projective real variety Y , the *real (singular) semi-topological K -theory space* of Y is defined by

$$\mathcal{KR}^{sst}(Y) \equiv |d \mapsto \mathcal{K}^{alg}(\Delta_{top}^d \times_{\mathbb{R}} Y)|$$

where

$$\mathcal{K}^{alg}(\Delta_{top}^d \times_{\mathbb{R}} Y) = \lim_{\substack{\rightarrow \\ \Delta_{top}^d \rightarrow U(\mathbb{R})}} \mathcal{K}^{alg}(U \times_{\mathbb{R}} Y) .$$

The *real (singular) semi-topological K -groups* of Y are defined by

$$K\mathbb{R}_n^{sst}(Y) = \pi_n \mathcal{KR}^{sst}(Y) .$$

In other words, the theory $\mathcal{KR}^{sst}$ is induced from algebraic K -theory using Kan extension along the functor $Sch/\mathbb{R} \rightarrow Top$ sending U to $U(\mathbb{R})$.

15 **Proposition 15** (cf. [30, 2.5]) If Y is a weakly normal, projective real variety, then there is a natural weak equivalence of spectra

$$\mathcal{KR}^{semi}(Y) \simeq \mathcal{KR}^{sst}(Y) .$$

The explicit description of $K\mathbb{R}_0^{sst}(Y)$ is perhaps a bit unexpected. If $V_1 \rightarrow X, V_2 \rightarrow Y$ are algebraic vector bundles on the real variety Y , then we say that V_1, V_2 are *real algebraically equivalent* if there exists a smooth, connected real curve C , an algebraic vector bundle $\mathcal{V} \rightarrow Y \times C$, and real points $c_1, c_2 \in C(\mathbb{R})$ lying in the same real analytic component of $C(\mathbb{R})$ such that $V_i \rightarrow Y$ is the fibre of $\mathcal{V} \rightarrow Y \times C$ over $Y \times \{c_i\}$. We refer to the equivalence relation on $K_0^{alg}(Y)$ generated by real algebraic equivalence as *real algebraic equivalence*.

The condition that two bundles be joined via real algebraic equivalence is significantly stronger than what might be termed “algebraic equivalence for real varieties” (i.e., requiring only that c_1, c_2 belong to the same algebraic component of C). Nevertheless, the next proposition and the subsequent theorem indicated that this stronger condition is the appropriate one.

16 **Proposition 16** (cf. [30, 1.6]) For any quasi-projective real variety Y ,

$$K\mathbb{R}_0^{sst}(Y) \cong \frac{K_0^{alg}(Y)}{\text{real algebraic equivalence}} .$$

If Y is a real variety, we write $Y_{\mathbb{R}}(\mathbb{C})$ for the topological space $Y(\mathbb{C})^{an}$ equipped with the involution $y \mapsto \bar{y}$ induced by complex conjugation – in Atiyah’s terminology [4] $Y_{\mathbb{R}}(\mathbb{C})$ is a *Real space*. As with any Real space, we may associated to $Y_{\mathbb{R}}(\mathbb{C})$ its *Atiyah’s Real K -theory space* $\mathcal{KR}_{top}(Y_{\mathbb{R}}(\mathbb{C}))$. We remind the reader

that $\mathcal{KR}_{top}(Y_{\mathbb{R}}(\mathbb{C}))$ is constructed using the category of *Real vector bundles*. Such a bundle is a complex topological vector bundle $V \rightarrow Y_{\mathbb{R}}(\mathbb{C})$ equipped with an involution $\tau : V \rightarrow V$ covering the involution of $Y_{\mathbb{R}}(\mathbb{C})$ such that for each $y \in Y_{\mathbb{R}}(\mathbb{C})$, the map $\mathbb{C}^r \cong V_y \xrightarrow{\tau} V_{\bar{y}} \cong \mathbb{C}^r$ is given by complex conjugation.

Proposition 17 (cf. [30, 2.5, 4.3]) For a quasi-projective real variety Y , there is a natural (up to weak equivalence) triple of spectra 17

$$\mathcal{K}^{alg}(Y) \rightarrow \mathcal{KR}^{sst}(Y) \rightarrow \mathcal{KR}_{top}(Y_{\mathbb{R}}(\mathbb{C})) . \quad (1.7)$$

We may of course view any quasi-projective complex variety Y as a quasi-projective real variety $Y|_{\mathbb{R}}$ via restriction of scalars, and, conversely, any real variety U admits a base change $U_{\mathbb{C}} = U \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C}$ to a complex variety. If Y is a quasi-projective complex variety, then $(Y|_{\mathbb{R}})_{\mathbb{C}} = Y \coprod Y$ and the non-trivial element of the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $(Y|_{\mathbb{R}})_{\mathbb{C}}$ by interchanging the copies of Y . It follows that $\text{Hom}(U \times_{\mathbb{R}} Y|_{\mathbb{R}}, \text{Grass}) = \text{Hom}(U_{\mathbb{C}} \times_{\mathbb{C}} Y, \text{Grass})$ for any real variety U , from which the following result may be deduced.

Proposition 18 [30, 2.4, 4.3] If Y is a complex, quasi-projective variety and $X = Y|_{\mathbb{R}}$, then 18

$$\mathcal{KR}^{sst}(X) = \mathcal{K}^{sst}(Y) \quad \text{and} \quad \mathcal{KR}_{top}(X_{\mathbb{R}}(\mathbb{C})) = \mathcal{K}_{top}(Y)$$

and, moreover, in this case the maps of (1.7) coincide with the maps of (1.5).

The following theorem, generalizing Theorems 5 and 12, provides further evidence of the “correctness” of our definition of $\mathcal{KR}^{sst}(-)$.

Theorem 19 (cf. [30, 3.9, 6.9]) Let Y be a quasi-projective real variety. Then 19

$$K_*^{alg}(Y, \mathbb{Z}/n) \cong K_*^{sst}(Y, \mathbb{Z}/n)$$

for any positive integer n .

Furthermore, if C is a smooth real curve, then

$$K_q^{sst}(C) \cong K_{top}^{-q}(C_{\mathbb{R}}(\mathbb{C})) , \quad q \geq 0 .$$

For example, if C is a smooth, projective real curve of genus g such that $C(\mathbb{R}) \neq \emptyset$, then we have

$$K_0^{semi}(C) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/2)^{c-1} ,$$

where c is the number of connected components of the space $C(\mathbb{R})^{an}$ (cf. [30, 1.7]). This example shows that real algebraic equivalence differs from algebraic equivalence for real varieties, since modding out $K_0(C)$ by the latter equivalence relation yields the group $\mathbb{Z} \oplus \mathbb{Z}$ (cf. [30, 1.8]).

We interpret the next theorem as asserting that the triple (1.7) for the real variety Y is a retract of the triple (1.5) for its base change to $\mathbb{C}$, $Y_{\mathbb{C}}$, once one inverts the prime 2. In establishing this theorem, the authors first constructed a good transfer map $\pi_* : \mathcal{K}^{sst}(Y_{\mathbb{C}}) \rightarrow \mathcal{K}^{sst}(Y)$ (see [30, §5]).

20

Theorem 20 (cf. [30, 5.4, 5.6]) Let Y be a quasi-projective real variety and let

$$\pi : Y_{\mathbb{C}} = Y \times_{\mathrm{Spec} \mathbb{R}} \mathrm{Spec} \mathbb{C} \rightarrow Y$$

denote the natural map of $\mathbb{R}$ -varieties. Then we have a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathcal{K}(Y) & \rightarrow & \mathcal{K}\mathbb{R}^{sst}(Y) & \rightarrow & \mathcal{K}\mathbb{R}_{top}(Y_{\mathbb{R}}(\mathbb{C})) \\ \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* \\ \mathcal{K}(Y_{\mathbb{C}}) & \rightarrow & \mathcal{K}^{sst}(Y_{\mathbb{C}}) & \rightarrow & \mathcal{K}_{top}(Y(\mathbb{C})^{an}) \\ \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* \\ \mathcal{K}(Y) & \rightarrow & \mathcal{K}\mathbb{R}^{sst}(Y) & \rightarrow & \mathcal{K}\mathbb{R}_{top}(Y_{\mathbb{R}}(\mathbb{C})) \end{array} \quad (1.8)$$

with the property that the vertical compositions are weakly equivalent to multiplication by 2 with respect to the H-space structures.

The reader seeking to extend the construction of $\mathcal{K}^{sst}(-)$ and $\mathcal{K}\mathbb{R}^{sst}(-)$ to varieties over some other ground field F would likely have to address the following two questions:

- What is the correct notion of “ F -algebraic equivalence” if F is not algebraically closed and not equal to $\mathbb{R}$? Specifically, what condition on a pair of F -points of a variety is analogous to the condition that two points $c_1, c_2 \in C(\mathbb{R})$ lie in the same real analytic component of $C(\mathbb{R})^{an}$?
- What should play the role of $\mathcal{K}\mathbb{R}_{top}(-)$ or $\mathcal{K}_{top}$ if F is not equal to $\mathbb{R}$ or $\mathbb{C}$?

Algebraic, Semi-topological, and Topological Theories

1.3

In this section we state the major results relating semi-topological K -theory to algebraic K -theory, topological K -theory, and morphic cohomology and we provide some indications of proofs.

The connections between semi-topological K -theory and these other cohomology theories are well summarized by the existence of and properties enjoyed by the commutative diagram

$$\begin{array}{ccccc} K_*^{alg}(X; A) & \rightarrow & K_*^{sst}(X; A) & \rightarrow & K_{top}^{-*}(X^{an}; A) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_q H_{\mathcal{M}}^{2q-*}(X, A(q)) & \rightarrow & \bigoplus_q L^q H^{2q-*}(X; A) & \rightarrow & \bigoplus_q H_{sing}^{2q-*}(X^{an}; A) \end{array} \quad (1.9)$$

where X is a smooth, quasi-projective complex variety and A is an arbitrary abelian group. In this diagram, the top row is the chain of maps (1.6) defined in the previous section, and the maps in the bottom row are defined in a similar manner below. The vertical arrows are dashed to indicate that one must interpret them non-literally in one of three ways: (1) One may interpret them as homomorphisms from K -theories to cycle theories (heading downward in the diagram) whose targets land, to put it a bit imprecisely, in the groups of units of the ring cohomology theories along the bottom row – that is, one may take these arrows to be total Chern class maps; (2) one may interpret them as natural transformations of ring theories from K -theories to cycles theories (heading downward in the diagram) provided one takes $A = \mathbb{Q}$ – that is, one may take these arrows to be Chern character maps; or (3) one may interpret these dashed lines as indicating the existence of three compatible spectral sequences whose E_2 -terms are cycles theories and whose abutments are K -theories. We discuss the first two interpretations of these vertical arrows in this section and leave the spectral sequence interpretation for the next.

Motivic, Morphic, and Singular Cohomology

1.3.1

Before describing the many nice properties enjoyed by diagram (1.9), we first remind the reader of the definition of morphic cohomology and define the maps along the bottom row of this diagram.

Once again, the definition of morphic cohomology is more intuitive in the case of projective varieties, and, in fact, we first describe Lawson homology, the homology theory dual to morphic cohomology for smooth varieties, in this case. The definition of Lawson homology can be motivated by the Dold–Thom Theorem [12] that gives the isomorphism

$$\pi_n \left((\mathbb{L}_d S^d(Y))^+ \right) \cong H_n^{\text{sing}}(Y),$$

where Y is a compact space, $S^d(-)$ denotes taking the d -th symmetric power of a space, and $(-)^+$ denotes forming the topological abelian group associated to a topological abelian monoid via (naive) group completion. Observe that if we take Y to be the analytic realization of a projective variety X , then $S^d(X^{\text{an}})$ is the space of effective 0-cycles of degree d on X and this space coincides with the analytic realization of the Chow variety $C_{0,d}(X)$. Thus, in this context, the Dold–Thom theorem becomes

$$\pi_n \mathcal{Z}_0(X)^{\text{an}} \cong H_n^{\text{sing}}(X^{\text{an}}), \quad \text{where } \mathcal{Z}_0(X)^{\text{an}} = \left(\coprod_d C_{0,d}(X)^{\text{an}} \right)^+.$$

Observe that $\mathcal{Z}_0(X)^{\text{an}}$ coincides the space of all 0-cycles on X . A fascinating theorem of F. Almgren [2] generalizes the Dold–Thom Theorem by asserting that for sufficiently well-behaved spaces Y (i.e., for Lipschitz neighborhood retracts) and for any $r \geq 0$ the topological abelian group $\mathcal{Z}_r^{\text{curr}}(Y)$ of “integral r -cycles” (i.e., closed rectifiable currents on Y) has the property that

$$\pi_n \mathcal{Z}_r^{\text{curr}}(Y) = H_{n+r}^{\text{sing}}(Y).$$

This result motivated Blaine Lawson to investigate spaces of *algebraic* r -cycles on a complex projective variety X as a “small model” for the space of integral $2r$ -cycles on X^{an} . Namely, the collection of effective r -cycles for any $r \geq 0$ of a fixed degree d on a projective variety X is given as the set of closed points of the Chow variety $\mathcal{C}_{r,d}(X)$. Letting $\mathcal{C}_r(X) = \coprod_d \mathcal{C}_{r,d}(X)$, we see that $\mathcal{C}_r(X)^{an}$ is a topological abelian monoid under addition of cycles. We define

$$\mathcal{Z}_r(X)^{an} = (\mathcal{C}_r(X)^{an})^+ ,$$

the associated topological abelian group given by (naive) group completion. (Up to homotopy, one may equivalently use a homotopy-theoretic group completion, defined by the bar construction, in place of naive group completion [47].) Then $\mathcal{Z}_r(X)^{an}$ is the topological space of all r -cycles on X , and the Lawson homology groups are defined to be the homotopy groups of this space:

21

Definition 21 For a projective, complex variety X , we define the *Lawson homology groups* of X to be

$$L_r H_n(X) = \pi_{n-2r} \mathcal{Z}_r(X)^{an} ,$$

where

$$\mathcal{Z}_r(X)^{an} = (\mathcal{C}_r(X)^{an})^+ \quad \text{and} \quad \mathcal{C}_r(X) = \coprod_d \mathcal{C}_{r,d}(X) .$$

In fact, this definition generalizes in a straightforward fashion to quasi-projective varieties. Namely, for U quasi-projective, one chooses a compactification $U \subset X$ (i.e., an open, dense embedding with X projective) and defines

$$L_r H_n(U) = \pi_{n-2r} \mathcal{Z}_r(X)^{an} / \mathcal{Z}_r(X - U)^{an}$$

where $\mathcal{Z}_r(X)^{an} / \mathcal{Z}_r(X - U)^{an}$ denotes the quotient topological abelian group. By the Dold–Thom Theorem [12], we have

$$L_0 H_n(U) = H_n^{BM}(U^{an}) , \quad \text{for all } n ,$$

where H^{BM} denotes Borel–Moore homology, and it is easy to prove (cf. [20]) that

$$L_r H_{2r}(X) = \pi_0 \mathcal{Z}_r(X)^{an} \cong A_r(X)$$

where $A_r(X)$ denotes the group of cycles of dimension r on X modulo algebraic equivalence.

The definition of the morphic cohomology groups is also quite natural. The key motivational observation is that the quotient topological group

$$\mathcal{Z}_0(\mathbb{P}^q)^{an} / \mathcal{Z}_0(\mathbb{P}^{q-1})^{an}$$

is a model for the Eilenberg–MacLane space $K(\mathbb{Z}, 2q)$ by the Dold–Thom theorem, and thus represents the functor $H_{sing}^{2q}(-, \mathbb{Z})$. That is, the homotopy groups of

$\mathcal{M}aps(Y, \mathcal{Z}_0(\mathbb{P}^q)^{an}/\mathcal{Z}_0(\mathbb{P}^{q-1})^{an})$ give the singular homology groups of a space Y . Replacing $\mathcal{M}aps(-, -)$ by $\mathcal{M}or(-, -)^{an}$ as in the definition of $\mathcal{K}^{semi}$, we arrive at the definition of morphic cohomology for projective varieties.

Definition 22 For a smooth, projective complex variety X , we define the *morphic cohomology groups of X* to be

22

$$L^q H^n(X) = \pi_{2q-n} \mathcal{M}or(X, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1}))^{an}$$

where we define

$$\mathcal{M}or(X, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1}))^{an} = [\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^q))]^{an+} / [\mathcal{M}or(X, \mathcal{C}_0(\mathbb{P}^{q-1}))^{an}]^{+}.$$

As before, the definition extends naturally to all quasi-projective varieties, but we omit the details.

The connection between morphic cohomology and singular cohomology can be seen from the definition of the former: since $\mathcal{M}or(-, -)^{an}$ is a subspace of $\mathcal{M}aps((-)^{an}, (-)^{an})$, we obtain a natural map

$$\mathcal{M}or(X, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1}))^{an} \rightarrow \mathcal{M}aps(X^{an}, \mathcal{Z}_0(\mathbb{P}^q)^{an}/\mathcal{Z}_0(\mathbb{P}^{q-1})^{an})$$

which induces the map

$$L^q H^n(X) \rightarrow H_{sing}^n(X^{an}).$$

The connection between morphic cohomology and motivic cohomology is suggested by the following fact:

Proposition 23 (cf. [26, 4.4, 8.1], [56, 2.1]) For a smooth, quasi-projective complex variety X , we have

23

$$\pi_n \operatorname{Hom}(X \times \Delta^\bullet, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1})) \cong H_{\mathcal{M}}^{2q-n}(X, \mathbb{Z}(q))$$

where $\operatorname{Hom}(X \times \Delta^\bullet, \mathcal{Z}_0(\mathbb{P}^q)/\mathcal{Z}_0(\mathbb{P}^{q-1}))$ denotes the quotient simplicial abelian group

$$\operatorname{Hom}(X \times \Delta^\bullet, \mathcal{C}_0(\mathbb{P}^q))^+ / \operatorname{Hom}(X \times \Delta^\bullet, \mathcal{C}_0(\mathbb{P}^{q-1}))^+.$$

(Here, the superscript $+$ signifies taking degree-wise group completion of a simplicial abelian monoid.)

For a projective variety X , it's not hard to establish the isomorphism

$$L^q H^n(X) \cong \pi_{2q-n} \operatorname{Hom}(X \times \Delta_{top}^\bullet, \mathcal{C}_0(\mathbb{P}^q))^+ / \operatorname{Hom}(X \times \Delta_{top}^\bullet, \mathcal{C}_0(\mathbb{P}^{q-1}))^+$$

in much the same way that the equivalence $\mathcal{K}^{semi} \simeq \mathcal{K}^{sst}$ is proven for such varieties. Indeed, we may thus use this isomorphism to *define* morphic cohomology

for non-projective varieties. Although less “geometric”, the construction given in the following definition of morphic cohomology is more amenable.

24 **Definition 24:** *Revised Definition of Morphic Cohomology* For a smooth, quasi-projective complex variety X , the *morphic cohomology groups* of X are defined to be

$$L^q H^n(X) = \pi_{2q-n} \operatorname{Hom} \left(X \times \Delta_{\text{top}}^\bullet, \mathcal{Z}_0(\mathbb{P}^q) / \mathcal{Z}_0(\mathbb{P}^{q-1}) \right),$$

where

$$\begin{aligned} \operatorname{Hom} \left(X \times \Delta_{\text{top}}^\bullet, \mathcal{Z}_0(\mathbb{P}^q) / \mathcal{Z}_0(\mathbb{P}^{q-1}) \right) = \\ \operatorname{Hom} \left(X \times \Delta_{\text{top}}^\bullet, \mathcal{C}_0(\mathbb{P}^q) \right)^+ / \operatorname{Hom} \left(X \times \Delta_{\text{top}}^\bullet, \mathcal{C}_0(\mathbb{P}^{q-1}) \right)^+. \end{aligned}$$

(The superscripts $+$ denote taking degree-wise group completion of a simplicial abelian monoid.)

In other words, we simply define morphic cohomology to be the “semi-topologized” theory associated to motivic cohomology.

25 **Definition 25** The maps along the bottom row of (1.9) are given by applying π_* to the sequence of natural maps

$$\begin{aligned} \operatorname{Hom} \left(X \times \Delta^\bullet, \mathcal{Z}_0(\mathbb{P}^q) / \mathcal{Z}_0(\mathbb{P}^{q-1}) \right) &\rightarrow \operatorname{Hom} \left(X \times \Delta_{\text{top}}^\bullet, \mathcal{Z}_0(\mathbb{P}^q) / \mathcal{Z}_0(\mathbb{P}^{q-1}) \right) \\ &\rightarrow \operatorname{Maps} \left(X^{\text{an}} \times \Delta_{\text{top}}^\bullet, \mathcal{Z}_0(\mathbb{P}^q)^{\text{an}} / \mathcal{Z}_0(\mathbb{P}^{q-1})^{\text{an}} \right). \end{aligned}$$

In summary, we have the following heuristic overview: Motivic, morphic, and singular cohomology are defined as the homotopy groups of, respectively, the “algebraic space” of algebraic morphisms, the topological space of algebraic morphisms, and the topological space of topological morphisms from a given variety to the object $\mathcal{Z}_0(\mathbb{P}^q) / \mathcal{Z}_0(\mathbb{P}^{q-1})$. Moreover, the maps joining these three theories are given by the canonical maps from the algebraic space of algebraic morphisms to the topological space of algebraic morphisms to the topological space of topological morphisms.

1.3.2 The Chern Class Maps

The relation between semi-topological K -theory and morphic cohomology is given, as one would expect, by the total Chern class map and the closely related Chern character. The former has the advantage that it is defined integrally, whereas the latter has the advantage that it determines a natural transformation of ring-valued cohomology theories. Each map induces a rational isomorphism

from the rational semi-topological K -groups to the rational morphic cohomology groups of a smooth, quasi-projective complex variety. These isomorphisms generalize the isomorphisms on π_0 groups

$$c : K_0^{sst}(X)_{\mathbb{Q}} \xrightarrow{\cong} A^0(X)_{\mathbb{Q}} \times \left(\{1\} \times \bigoplus_{q \geq 1} A^q(X)_{\mathbb{Q}} \right)^{\times}$$

and

$$ch : K_0^{sst}(X)_{\mathbb{Q}} \xrightarrow{\cong} A^*(X)_{\mathbb{Q}} = L^* H^{2*}(X; \mathbb{Q}) .$$

The first of these isomorphisms, the total Chern class map

$$c(\alpha) = (\text{rank}(\alpha), 1 + c_1(\alpha) + \cdots) ,$$

is an isomorphism of abelian groups, where the group law for the second component of the target is given by cup product (i.e., intersection of cycles). The second of these isomorphisms, the Chern character, is an isomorphism of rings and is defined via the usual universal polynomials (with $\mathbb{Q}$ coefficients) in the individual Chern classes $c_i, i \geq 1$. Each of these isomorphisms may be deduced easily from the corresponding and well-known isomorphisms relating $K_0^{alg}(X)$ and $CH^*(X)$ by simply modding out by algebraic equivalence.

Lawson and Michelsohn recognized that sending an arbitrary subspace $W \subset \mathbb{C}^{N+1}$ to the linear cycle $\mathbb{P}(W^*) \subset \mathbb{P}((\mathbb{C}^{N+1})^*) \cong \mathbb{P}^N$ stabilizes (by letting N approach infinity) to give the universal total Chern class map [41]. (Here, $\mathbb{P}(W^*) = \text{Proj}(S^*(W^*))$ is the projective variety parameterizing one dimensional subspaces of W^* , the linear dual of W .) Indeed, in [9], Boyer, Lawson, Lima-Filho, Mann, and Michelsohn show that this model of the total Chern class is a map of infinite loop spaces, thereby answering a question of G. Segal. This result is extended in Theorem 27 below. We find it more convenient when stabilizing with respect to N and when considering the pairing determined by external direct sum of vector spaces to send a *quotient* space of the form $\mathbb{C}^{N+1} \twoheadrightarrow V$ to the linear cycle $\mathbb{P}(V) \subset \mathbb{P}^N$. This becomes a model for the total Segre class. The total Segre and Chern class maps differ only slightly: We define $\text{Seg}(\alpha) = (\text{rank}(\alpha), 1 - s_1(\alpha) + s_2(\alpha) - \cdots)$ where s_q are the Segre class maps, defined by the formula

$$1 + s_1(x) + s_2(x) + \cdots = (1 + c_1(x) + c_2(x) + \cdots)^{-1} .$$

It follows from [32, 1.4] that there is a natural isomorphism of the form

$$\begin{aligned} \text{Hom} \left(X \times \Delta_{top}^{\bullet}, \mathcal{C}_r(\mathbb{P}^N) \right)^+ &\cong \\ \bigoplus_{q=0}^{N-r} \text{Hom} \left(X \times \Delta_{top}^{\bullet}, \mathcal{C}_0(\mathbb{P}^q) \right)^+ &/ \text{Hom} \left(X \times \Delta_{top}^{\bullet}, \mathcal{C}_0(\mathbb{P}^{q-1}) \right)^+ \end{aligned}$$

from which one deduces the isomorphism

$$\pi_n \left(\text{Hom} \left(X \times \Delta_{top}^\bullet, \mathcal{C}_r(\mathbb{P}^\infty) \right)^+ \right) \cong \bigoplus_{q \geq 0} L^q H^{2q-n}(X),$$

for any $r \geq 0$ and any smooth, quasi-projective complex variety X . Thus, morphic cohomology is “represented” by the ind-variety $\mathcal{C}_r(\mathbb{P}^\infty)$ for any $r \geq 0$ just as semi-topological K -theory is represented by $\text{Grass}(\mathbb{C}^\infty)$. Now, a point in the latter ind-variety is given by a quotient $\mathbb{C}^\infty \twoheadrightarrow V$ (that factors through $\mathbb{C}^N$ for $N \gg 0$), which in turn determines an effective cycle $\mathbb{P}(V) \subset \mathbb{P}^\infty$ of degree 1 and dimension $\dim(V) - 1$ by taking associated projective spaces. Thus we have a map

$$\text{Grass}(\mathbb{C}^\infty) \rightarrow \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)_1^+, \quad (1.10)$$

where $\mathcal{C}_{r-1}(\mathbb{P}^\infty)_1^+$ denotes the subset of the abelian group $\mathcal{C}_{r-1}(\mathbb{P}^\infty)^+$ consisting of (not necessarily effective) cycles of degree 1. (As a technical point, when $r = 0$ one sets $\mathcal{C}_{-1}(\mathbb{P}^\infty)^+ = \mathbb{Z}$, the free abelian group generated by the “empty cycle” which has degree 1 by convention.) In essence, the map (1.10) induces the total Segre class map by taking homotopy-theoretic group completions, although some further details are needed to make this precise.

The ind-variety $\coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)$ admits a natural product given by linear join of cycles. Namely, we first specify a linear embedding $\mathbb{P}^\infty \coprod \mathbb{P}^\infty \hookrightarrow \mathbb{P}^\infty$ by choosing a surjection $\mathbb{C}^\infty \twoheadrightarrow \mathbb{C}^\infty \oplus \mathbb{C}^\infty$ of vector spaces. Then given a pair of effective cycles α and β in $\mathbb{P}^\infty$, we may embed them as cycles in general position in $\mathbb{P}^\infty$ by use of this embedding (regarding α as a cycle on the first copy of $\mathbb{P}^\infty$ and β as a cycle on the second), so that their linear join (i.e., the cycle formed by the union of all lines intersecting both α and β) is well-behaved. This pairing extends to all cycles by linearity and restricts to a pairing

$$\coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)_1^+ \times \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)_1^+ \rightarrow \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)_1^+$$

since the linear join of cycles having degrees d and e has degree de . For a complex variety X , this product endows

$$\left(\text{Hom} \left(X \times \Delta_{top}^\bullet, \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty) \right)^{an} \right)_1^+$$

with the structure of a homotopy-commutative H -space, whose associated homotopy-theoretic group completion is written $\mathcal{H}_{mult}^{sst}(X)$. It is apparent from its definition that the space $\mathcal{H}_{mult}^{sst}(X)$ should be closely related to the morphic cohomology of X and, since cup product in morphic cohomology can be defined by linear

join, that the H -space structure of this space should be related to cup product. The precise connection is given by the isomorphism of groups

$$\pi_n \mathcal{H}_{mult}^{sst}(X) \cong L^0 H^{-n}(X) \times \left(\{1\} \times \bigoplus_{q \geq 1} L^q H^{2q-n}(X) \right)^\times,$$

where $\left(\{1\} \times \bigoplus_{q \geq 1} L^q H^{2q-n}(X) \right)^\times$ is a subgroup of the multiplicative group of units of the ring $L^* H^*(X)$.

A key observation is that the map (1.10) is additive in that it takes direct sum to linear join – that is, given any quasi-projective variety X , the induced map

$$\mathrm{Hom}(X \times \Delta_{top}^\bullet, \mathrm{Grass}(\mathbb{C}^\infty)) \rightarrow \left(\mathrm{Hom} \left(X \times \Delta_{top}^\bullet, \coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty) \right)^{an} \right)_1^+,$$

is a map of H -spaces. In fact, it can easily be enriched to be a map of $\mathcal{I}$ -spaces, where $\mathcal{I}$ is the E_∞ operad discussed above. Upon taking the homotopy-theoretic group completion of this map, we obtain the *total Segre class* map

$$Seg^{sst} : \mathcal{K}^{sst}(X) \rightarrow \mathcal{H}_{mult}^{sst}(X),$$

which is a map of group-like H -spaces (in fact, of spectra). Upon taking homotopy groups, we get

$$Seg^{sst} : K_n^{sst}(X) \rightarrow L^0 H^{-n}(X) \times \left(\{1\} \times \bigoplus_{q \geq 1} L^q H^{2q-n}(X) \right)^\times.$$

Remark 26 Lima-Filho [49, 4.1] has constructed a similar map of spectra, for just projective varieties, resulting in a total Chern class map.

26

The above construction of the (semi-topological) total Segre class involves, in a suitable sense, only constructions on the ind-varieties $\mathrm{Grass}(\mathbb{C}^\infty)$ and $\coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)$ representing semi-topological K -theory and morphic cohomology. Given that $\mathrm{Grass}(\mathbb{C}^\infty)$ and $\coprod_r \mathcal{C}_{r-1}(\mathbb{P}^\infty)$ (resp., the corresponding analytic spaces) can also be used to define algebraic K -theory and motivic cohomology (resp., topological K -theory and singular cohomology), it is unsurprising that one also obtains total Segre class maps

$$Seg^{alg} : K_n^{alg}(X) \rightarrow H_{\mathcal{M}}^{-n}(X, \mathbb{Z}(0)) \times \left(\{1\} \times \bigoplus_{q \geq 1} H_{\mathcal{M}}^{2q-n}(X, \mathbb{Z}(q)) \right)^\times$$

and

$$Seg^{top} : K_{top}^{-n} \rightarrow H_{sing}^{-n}(X, \mathbb{Z}) \times \left(\{1\} \times \bigoplus_{q \geq 1} H_{sing}^{2q-n}(X, \mathbb{Z}) \right)^\times$$

in the algebraic and topological setting via highly analogous constructions. We obtain the following theorem.

27

Theorem 27 (cf. [30, 8.6]) For any quasi-projective variety X , the total Segre class maps

$$\begin{aligned} \text{Seg}^{\text{alg}}: K_n^{\text{alg}}(X) &\rightarrow L^0 H^{-n}(X) \times \left(\{1\} \times \bigoplus_{q \geq 1} L^q H^{2q-n}(X) \right)^{\times}, \\ \text{Seg}^{\text{sst}}: K_n^{\text{sst}}(X) &\rightarrow H_{\mathcal{M}}^{-n}(X, \mathbb{Z}(0)) \times \left(\{1\} \times \bigoplus_{q \geq 1} H_{\mathcal{M}}^{2q-n}(X, \mathbb{Z}(q)) \right)^{\times}, \quad \text{and} \\ \text{Seg}^{\text{top}}: K_{\text{top}}^{-n} &\rightarrow H_{\text{sing}}^{-n}(X, \mathbb{Z}) \times \left(\{1\} \times \bigoplus_{q \geq 1} H_{\text{sing}}^{2q-n}(X, \mathbb{Z}) \right)^{\times} \end{aligned}$$

are induced by natural transformations of H -spaces (in fact, of spectra). Moreover, they form the vertical arrows of the commutative diagram (1.9), provided one interprets the entries along the bottom row as groups in a suitable fashion.

The topological version of this theorem was first proven by Boyer et al in [9], settling in the affirmative a conjecture of Segal that the total Chern class map is a natural transformation of generalized cohomology theories. In addition, Lima-Filho [49, §4] has established an equivalent version (in the special case where X is projective) of the right half of diagram (1.9) in which the verticle arrows are the total Chern class maps.

By applying the familiar universal polynomials (which have coefficients in $\mathbb{Q}$) that define the Chern character from the individual Chern classes, we obtain the Chern character maps

$$\begin{aligned} ch^{\text{alg}}: K_*^{\text{alg}}(X) &\rightarrow H_{\mathcal{M}}^*(X; \mathbb{Q}(*)), \\ ch^{\text{sst}}: K_*^{\text{sst}}(X) &\rightarrow L^* H^*(X; \mathbb{Q}), \quad \text{and} \\ ch^{\text{top}}: K_{\text{top}}^*(X^{\text{an}}) &\rightarrow H_{\text{sing}}^*(X; \mathbb{Q}). \end{aligned}$$

28

Theorem 28 (cf. [32, 4.7]) For a smooth, quasi-projective complex variety X , the Chern character maps are ring maps and they induce rational isomorphisms:

$$\begin{aligned} ch^{\text{alg}}: K_*^{\text{alg}}(X)_{\mathbb{Q}} &\xrightarrow{\cong} H_{\mathcal{M}}^*(X; \mathbb{Q}(*)), \\ ch^{\text{sst}}: K_*^{\text{sst}}(X)_{\mathbb{Q}} &\xrightarrow{\cong} L^* H^*(X; \mathbb{Q}), \quad \text{and} \\ ch^{\text{top}}: K_{\text{top}}^*(X^{\text{an}})_{\mathbb{Q}} &\xrightarrow{\cong} H_{\text{sing}}^*(X; \mathbb{Q}). \end{aligned}$$

Sketch of Proof

The result is well-known in the algebraic [7, 42] and topological [5] settings. For the semi-topological context, the proof is easy to describe at a heuristic level (although the rigorous details turn out to be more complicated than one might guess): One first shows, without much difficulty, that it suffices to prove that the semi-topological total Segre class map induces an isomorphism on rational homotopy groups. Since this is a map of H -spaces and since the result is known in the algebraic context for all smooth varieties, the Recognition Principle (Theorem 11) implies the desired result. (One difficulty in making this argument rigorous is proving that the usual Chern character isomorphism in algebraic K -theory coincides with the map given by universal polynomials from the map Seg^{alg} .)

Remark 29 Cohen and Lima-Filho [10] have claimed a proof of the second isomorphism of Theorem 28, but their proof is invalid.

29

Finite Coefficients and the Bott Element

1.3.3

In this section, we describe two important properties of the horizontal maps in the diagram (1.9) – that is, we describe results about the comparison of algebraic and semi-topological theories and about the comparison of semi-topological and topological theories.

The first property is given by the following result, the first half of which has already been stated above as Theorem 12.

Theorem 30 (cf. [27, 3.7], [54]) The left-hand horizontal maps of (1.9) are isomorphisms if X is smooth and $A = \mathbb{Z}/n$ for $n > 0$. That is, for $n > 0$ we have isomorphisms

30

$$K_m^{\text{alg}}(X; \mathbb{Z}/n) \xrightarrow{\cong} K_m^{\text{sst}}(X; \mathbb{Z}/n)$$

and

$$H_{\mathcal{M}}^p(X, \mathbb{Z}/n(q)) \xrightarrow{\cong} L^q H^p(X, \mathbb{Z}/n)$$

for all integers p, q, m and all quasi-projective complex varieties X .

Remark 31 In fact, this theorem is valid even for X singular. For the second isomorphism, one must define the morphic cohomology so that cdh descent holds.

31

As mentioned in the sketch of proof of Theorem 12, the first isomorphism follows from the Recognition Principle and rigidity for algebraic K -theory with finite coefficients. The second isomorphism was proven originally by Suslin–Voevodsky [54] but the Recognition Principle can also be used to give another proof (but one which mimics much of Suslin–Voevodsky’s original proof): The map in question is induced by the natural transformation of functors from $Sch/\mathbb{C}$ to chain complexes

$$\begin{aligned} z_0^{equi}(X \times \Delta^\bullet, \mathbb{P}^q) / z_0^{equi}(X \times \Delta^\bullet, \mathbb{P}^{q-1}) \otimes \mathbb{Z}/n \\ \rightarrow z_0^{equi}(- \times X \times \Delta^\bullet, \mathbb{P}^q) / z_0^{equi}(- \times X \times \Delta^\bullet, \mathbb{P}^{q-1}) \otimes \mathbb{Z}/n \end{aligned}$$

(with the first one being constant). By rigidity [54], this map is locally a quasi-isomorphism for the étale topology, and hence the induced map

$$\begin{aligned} z_0^{equi}(X \times \Delta^\bullet, \mathbb{P}^q) / z_0^{equi}(X \times \Delta^\bullet, \mathbb{P}^{q-1}) \otimes \mathbb{Z}/n \\ \rightarrow z_0^{equi}(\Delta_{top}^\bullet \times X \times \Delta^\bullet, \mathbb{P}^q) / z_0^{equi}(\Delta_{top}^\bullet \times X \times \Delta^\bullet, \mathbb{P}^{q-1}) \otimes \mathbb{Z}/n \end{aligned}$$

is a quasi-isomorphism by the Recognition Principle. These complexes define the motivic cohomology and morphic cohomology of smooth varieties, respectively.

Since using finite coefficients makes the maps from algebraic theories to semi-topological theories into equivalences, one might ask what modification of the maps from semi-topological theories to topological theories converts them to equivalences. The answer is that one needs only invert the Bott element (for K -theory) and the s element (for cycle theories). In other words, the *integral* analogue of Thomason’s theorem [57, 4.11] relating Bott inverted algebraic K -theory and topological K -theory with finite coefficients holds in the context of semi-topological K -theory.

Note that since we have the isomorphism $K_*^{sst}(\mathrm{Spec} \mathbb{C}) \cong K_{top}^{-*}(pt)$ we have in particular that $K_2^{sst}(\mathrm{Spec} \mathbb{C}) \cong \mathbb{Z}$. Let β be the generator of $K_2^{sst}(\mathrm{Spec} \mathbb{C})$ associated to the canonical map $S^2 = \mathbb{P}^1(\mathbb{C}) \rightarrow \mathrm{Grass}$ induced by the surjection $\mathbb{C}^\infty \rightarrow \mathbb{C}^2$ (defined by projection onto the first 2 coordinate) and call β the “Bott element”. Obviously, under the map from semi-topological K -theory to topological K -theory, β maps to the usual Bott element in topology. Moreover, it’s evident that that under the composition

$$K_2^{sst}(\mathrm{Spec} \mathbb{C}) \rightarrow K_2^{sst}(\mathrm{Spec} \mathbb{C}; \mathbb{Z}/n) \cong K_2^{alg}(\mathrm{Spec} \mathbb{C}; \mathbb{Z}/n) \cong \mu_n(\mathbb{C}), \quad n > 0,$$

the element β maps to a generator of $\mu_n(\mathbb{C})$ (i.e., a primitive n -th root of unity), so that β maps to the Bott element in algebraic K -theory with finite coefficients.

Since $K_*^{sst}(X)$ is a (graded) module over the (graded) ring $K_*^{sst}(\mathrm{Spec} \mathbb{C}) \cong \mathbb{Z}[\beta]$, we may formally invert the action of β on $K_*^{sst}(X)$. Doing the same to $K_{top}^*(X^{an})$ results in the 2-periodic (non-connective) K -theory ring $KU^*(X^{an})$. Clearly, $K_*^{sst}(X)[\beta^{-1}]$ maps to $KU^*(X^{an})$, and the theorem is simply that this map is an isomorphism in all degrees:

Theorem 32 (cf. [32], [60]) For a smooth, quasi-projective complex variety X , the upper right-hand horizontal map of (1.9) become isomorphisms upon inverting the Bott element:

32

$$K_*^{sst}(X) [\beta^{-1}] \xrightarrow{\cong} K_{top}^{-*}(X^{an}) [\beta^{-1}] = KU^{-*}(X^{an}) .$$

We know of three separate proofs of this theorem, two of which use the analogous result for morphic cohomology. This result involves inverting the so-called “ s operation” in morphic cohomology, defined originally by the first author and B. Mazur [24] in the context of Lawson homology. The original definition involved a map defined on the level of cycle spaces; the definition given here is equivalent, under duality, to the induced map on homotopy groups.

Definition 33 For a quasi-projective complex variety X , the s operation

33

$$s : L^t H^n(X) \rightarrow L^{t+1} H^n(X)$$

is defined as multiplication by the s element $s \in L^1 H^0(\text{Spec } \mathbb{C})$, which is given by $s = c_{2,1}(\beta)$ where $c_{2,1} : K_2^{sst}(\text{Spec } \mathbb{C}) \rightarrow L^1 H^0(\text{Spec } \mathbb{C})$ is the Chern class map.

The element s is a generator of $L^1 H^0(\text{Spec } \mathbb{C}) \cong \mathbb{Z}$ and it clearly maps to a unit of the graded ring $H_{sing}^*(X^{an})$, for all X , but is never a unit in the bigraded ring $L^* H^*(X)$. Let $L^t H^n(X)[s^{-1}]$ denote the degree (t, n) piece of the result of inverting s in the bi-graded ring $L^* H^*(X)$.

Proposition 34 For a smooth, quasi-projective complex variety X , the canonical map

34

$$L^t H^n(X) [s^{-1}] \rightarrow H_{sing}^n(X^{an})$$

is an isomorphism for all t, n .

The proposition follows directly from the facts that morphic cohomology is isomorphic to Lawson homology, that under this isomorphism multiplication by s corresponds to cap product by s (which is a map of the form $L_t H_n(X) \rightarrow L_{t-1} H_n(X)$), and that $L_t H_n(X) \cong H_n^{BM}(X^{an})$ for $t \leq 0$.

One proof of Theorem 32 (cf. [32, 5.8]) is given by establishing the desired isomorphism in the case of $\mathbb{Z}/n$ coefficients and $\mathbb{Q}$ coefficients separately. For $\mathbb{Z}/n$ coefficients, by using Theorem 30 it suffices to establish the analogous result comparing Bott inverted algebraic K -theory with $\mathbb{Z}/n$ coefficients to topological K -theory with $\mathbb{Z}/n$ coefficients – that this map is an isomorphism is (a special case of) Thomason’s theorem [57, 4.11]. For $\mathbb{Q}$ coefficients, the result follows directly from the rational isomorphisms of Theorem 28, using Proposition 34 and the fact that $ch^{sst}(\beta) = s$.

A second proof is quite similar to the proof of the $\mathbb{Q}$ coefficients case above, except that one uses the integral “Atiyah–Hirzebruch-like” spectral sequence relating morphic cohomology and semi-topological K -theory that has been established by the authors and Christian Haesemeyer [21, 2.10]. This spectral sequence is described in the next section. Here again the point is that inverting the Bott element corresponds under this spectral sequence to inverting the s element in morphic cohomology, and thus Proposition 34 applies.

The third proof of Theorem 32 does not use morphic cohomology in any fashion, but it applies only to smooth, projective varieties. This proof is given by the second author in [60].

Note that the whereas the first proof uses Thomason’s theorem, the latter two proofs do not. In light of Theorem 12, these proofs therefore represent, in particular, new proofs of Thomason’s theorem for the special case of smooth complex varieties.

We close this section by presenting diagram (1.9) again, this time with arrows suitable decorated to indicate their properties:

$$\begin{array}{ccccc}
 K_*^{alg}(X) & \xrightarrow{\mathbb{Z}/n\text{-equiv.}} & K_*^{sst}(X) & \xrightarrow{\frac{1}{\beta}\text{-equiv.}} & K_{top}^{-*}(X^{an}) \\
 \downarrow \mathbb{Q}\text{-equiv.} & & \downarrow \mathbb{Q}\text{-equiv.} & & \downarrow \mathbb{Q}\text{-equiv.} \\
 \bigoplus_q H_{\mathcal{M}}^{2q-*}(X, \mathbb{Z}(q)) & \xrightarrow{\mathbb{Z}/n\text{-equiv.}} & \bigoplus_q L^q H^{2q-*}(X) & \xrightarrow{\frac{1}{s}\text{-equiv.}} & \bigoplus_q H_{sing}^{2q-*}(X^{an}) \dots
 \end{array} \tag{1.11}$$

1.3.4

Real Analogues

The real analogues of the results of Sect. 1.3.3 are developed by the authors in [30]. In particular, the real morphic cohomology of the variety X defined over $\mathbb{R}$ is formulated in terms of morphisms defined over $\mathbb{R}$ from X to Chow varieties, and semi-topological real Chern and Segre classes are defined. Moreover, the real analogue of Theorem 12 is proved. In [32], the semi-topological real total Segre class is shown to be a rational isomorphism for smooth, quasi-projective varieties defined over $\mathbb{R}$. Indeed, once one inverts the prime 2, the semi-topological total Segre class is a retract of the semi-topological total Segre class of the complexified variety $X_{\mathbb{C}}$, thanks to an argument using transfers.

1.4

Spectral Sequences and Computations

In this section we describe the construction of the “semi-topological Atiyah–Hirzebruch spectral sequence” relating morphic cohomology to semi-topological K -theory. We also provide computations of semi-topological K -groups for certain special varieties. These computations essentially all boil down to proving that

for certain varieties, the map from semi-topological to topological K -theory is an isomorphism, at least in a certain range. These two topics are related, since the primary technique exploited in this section for such computations is the fact that the map from Lawson homology to Borel–Moore singular homology is an isomorphism in certain degrees for a special class of varieties. Such isomorphisms, in the case of smooth varieties, imply isomorphisms from morphic cohomology to singular cohomology, which, by using the spectral sequence, imply isomorphisms relating semi-topological to topological K -theory. Nearly all of the results in this section are found in the recent paper [21] of the two authors and C. Haesemeyer.

The Spectral Sequence

1.4.1

The “classical” Atiyah–Hirzebruch spectral sequence relates the singular cohomology groups of a finite dimensional CW complex Y with its topological K -groups, and is given by

$$E_2^{p,q}(top) = H_{sing}^{p-q}(Y, \mathbb{Z}) \Rightarrow ku^{p+q}(Y).$$

Recall that ku^* denotes the generalized cohomology theory associated to the (-1) -connected spectrum $\mathbf{bu}$. (In non-positive degrees, ku^* coincides with K_{top}^* .) One method of constructing this spectral sequence is to observe that the homotopy groups of the spectrum $\mathbf{bu}$ are $\pi_{2n}\mathbf{bu} = \mathbb{Z}$, $\pi_{2n+1}\mathbf{bu} = 0$, for $n \geq 0$. Thus, the Postnikov tower of the spectrum $\mathbf{bu}$ is the tower of spectra

$$\cdots \rightarrow \mathbf{bu}[4] \rightarrow \mathbf{bu}[2] \rightarrow \mathbf{bu}$$

and there are fibration sequences

$$\mathbf{bu}[2q+2] \rightarrow \mathbf{bu}[2q] \rightarrow K(\mathbb{Z}, 2q), \quad q \geq 0,$$

where $K(\mathbb{Z}, 2q)$ denotes the Eilenberg–MacLane spectrum whose only non-vanishing homotopy group is $\mathbb{Z}$ in degree $2q$. By applying $\mathcal{M}aps(Y, -)$, one obtains the tower of spectra

$$\cdots \rightarrow \mathcal{M}aps(Y, \mathbf{bu}[4]) \rightarrow \mathcal{M}aps(Y, \mathbf{bu}[2]) \rightarrow \mathcal{M}aps(Y, \mathbf{bu}) \quad (1.12)$$

and fibration sequences of spectra

$$\mathcal{M}aps(Y, \mathbf{bu}[2q+2]) \rightarrow \mathcal{M}aps(Y, \mathbf{bu}[2q]) \rightarrow \mathcal{M}aps(Y, K(\mathbb{Z}, 2q)), \quad q \geq 0.$$

These data determine a collection of long exact sequences that form an exact couple, and the isomorphisms

$$\pi_n \mathcal{M}aps(Y, K(\mathbb{Z}, 2q)) \cong H_{sing}^{2q-n}(Y, \mathbb{Z}) \quad \text{and} \quad \pi_n \mathcal{M}aps(Y, \mathbf{bu}) = ku^{-n}(Y^{an}), \quad n \in \mathbb{Z},$$

show that the associated spectral sequence has the form indicated above.

One of the more significant developments in algebraic K -theory in recent years is the construction of a purely algebraic analogue of the Atiyah–Hirzebruch spectral

sequence, one which relates the motivic cohomology groups of a smooth variety to its algebraic K -groups. The construction of this spectral sequence is given (in various forms) in the papers [8, 25, 34, 43, 53]. To construct the spectral sequence for arbitrary smooth varieties (as is done in [25, 34, 43, 53]), the essential point is to reproduce the tower (1.12) at the algebraic level. Namely, for a smooth, quasi-projective variety over an arbitrary ground field F , one constructs a natural tower of spectra

$$\cdots \rightarrow \mathcal{K}^{(q+1)}(X) \rightarrow \mathcal{K}^{(q)}(X) \rightarrow \cdots \rightarrow \mathcal{K}^{(1)}(X) \rightarrow \mathcal{K}^{(0)}(X) = \mathcal{K}^{alg}(X) \quad (1.13)$$

together with natural fibration sequences of the form

$$\mathcal{K}^{(q+1)}(X) \rightarrow \mathcal{K}^{(q)}(X) \rightarrow \mathcal{H}_{\mathcal{M}}(X, \mathbb{Z}(q)) , \quad (1.14)$$

where $\mathcal{H}_{\mathcal{M}}(X, \mathbb{Z}(q))$ is a suitable spectrum (in fact, a spectrum associated to a chain complex of abelian groups) whose homotopy groups give the motivic cohomology groups of X :

$$\pi_n \mathcal{H}_{\mathcal{M}}(X, \mathbb{Z}(q)) = H_{\mathcal{M}}^{2q-n}(X, \mathbb{Z}(q)) .$$

Such a tower and collection of fibration sequences leads immediately to the *motivic spectral sequence*:

$$E_2^{p,q}(alg) = H_{\mathcal{M}}^{p-q}(X, \mathbb{Z}(q)) \Rightarrow K_{-p-q}^{alg}(X) .$$

The *semi-topological spectral sequence* is defined by simply “semi-topologizing” the motivic version. That is, once one observes that the spectra appearing in (1.13) and (1.14) are defined for all $X \in Sch/\mathbb{C}$ (not just smooth varieties) and that they represent functors from $Sch/\mathbb{C}$ to spectra (see [25] and [21, §2]), then one may form the tower

$$\begin{aligned} \cdots \rightarrow \mathcal{K}^{(q+1)}(X \times \Delta_{top}^\bullet) &\rightarrow \mathcal{K}^{(q)}(X \times \Delta_{top}^\bullet) \rightarrow \cdots \\ \cdots \rightarrow \mathcal{K}^{(1)}(X \times \Delta_{top}^\bullet) &\rightarrow \mathcal{K}^{(0)}(X \times \Delta_{top}^\bullet) = \mathcal{K}^{alg}(X \times \Delta_{top}^\bullet) \end{aligned} \quad (1.15)$$

and the collection of fibration sequences

$$\mathcal{K}^{(q+1)}(X \times \Delta_{top}^\bullet) \rightarrow \mathcal{K}^{(q)}(X \times \Delta_{top}^\bullet) \rightarrow \mathcal{H}_{\mathcal{M}}(X \times \Delta_{top}^\bullet, \mathbb{Z}(q)) \quad (1.16)$$

in the usual manner. Since we have $\mathcal{K}^{sst}(X) = \mathcal{K}^{alg}(X \times \Delta_{top}^\bullet)$ and we also have (essentially by definition – at least, using the definition given in this paper)

$$\pi_n \mathcal{H}_{\mathcal{M}}(X \times \Delta_{top}^\bullet, \mathbb{Z}(q)) = L^q H^{2q-n}(X) ,$$

the collection of long exact sequences associated to (1.16) determines the semi-topological spectral sequence

$$E_2^{p,q}(sst) = L^q H^{p-q}(X) \Rightarrow K_{-p-q}^{sst}(X) . \quad (1.17)$$

Moreover, just as there is a natural map $\mathcal{K}^{sst}(X) \rightarrow \mathcal{K}_{top}(X^{an})$, one can define natural maps $\mathcal{K}^{(q)}(X \times \Delta_{top}^\bullet) \rightarrow \mathcal{M}aps(X^{an}, \mathbf{bu}[2q])$ for all $q \geq 0$ [21, 3.4], and thus one obtains a map from the semi-topological version of the Atiyah–Hirzebruch spectral sequence to the classical one. There is an obvious map from the motivic spectral sequence to the semi-topological spectral sequence, and thus we have the following theorem.

Theorem 35 [21, 3.6] For a smooth, quasi-projective complex variety X , we have natural maps of convergent spectral sequences of “Atiyah–Hirzebruch type”

35

$$\begin{array}{ccc} E_2^{p,q}(alg) = H_{\mathcal{M}}^{p-q}(X, \mathbb{Z}(q)) & \Rightarrow & K_{-p-q}^{alg}(X) \\ \downarrow & & \\ E_2^{p,q}(sst) = L^q H^{p-q}(X) & \Rightarrow & K_{-p-q}^{sst}(X) \\ \downarrow & & \\ E_2^{p,q}(top) = H_{sing}^{p-q}(Y, \mathbb{Z}) & \Rightarrow & ku^{p+q}(Y) \end{array}$$

given by the usual maps from motivic to morphic to singular cohomology and from algebraic to semi-topological to topological K -theory.

Generalized Cycle Map and Weights

1.4.2

The concept of a *weight filtration* on the singular cohomology of a complex variety was introduced by Deligne [11] (for rational coefficients). This notion was extended to arbitrary coefficients in a paper of Gillet–Soulé [33]. For our purposes, the analogous notion of a weight filtration for Borel–Moore singular cohomology, H_*^{BM} , turns out to be of more use.

For any U , the weight filtration on $H_n^{BM}(U^{an})$ has the form

$$\cdots \subset W_t H_n^{BM}(U^{an}) \subset W_{t+1} H_n^{BM}(U^{an}) \subset \cdots \subset H_n^{BM}(U^{an})$$

and is “concentrated” in the range $-n \leq t \leq d - n$, where $d = \dim(U)$, in the sense that $W_t H_n^{BM}(U^{an}) = 0$ for $t < -n$ and $W_t H_n^{BM}(U^{an}) = H_n^{BM}(U^{an})$ for $t \geq d - n$. (That the filtration is concentrated in this range is not so obvious from the definition below, but see [33, §2].) We have found it useful to consider a slight variation on the groups $W_t H_n^{BM}(U^{an})$, which are written $\tilde{W}_t H_n^{BM}(U)$. The groups $\tilde{W}_t H_n^{BM}(U)$ do not form a filtration on $H_n^{BM}(U^{an})$, but rather map surjectively to $W_t H_n^{BM}(U^{an})$ (with torsion kernel). The groups $\tilde{W}_t H_n^{BM}(U)$, however, enjoy better formal properties than do the groups comprising the weight filtration.

The essential idea underlying the definition of the weight filtration is that the n -th homology group of a smooth, projective complex variety X is of pure weight $-n$, by which we mean

$$W_t H_n^{BM}(X^{an}) = \begin{cases} 0, & \text{if } t < -n \quad \text{and} \\ H_n^{BM}(X^{an}), & \text{if } t \geq -n, \end{cases}$$

and, more generally, elements in $H_*^{BM}(U^{an})$ have weight t if they “come from” elements of weight t in $H_*^{BM}(X^{an})$ for X smooth and projective under a suitable construction.

In detail, given a quasi-projective complex variety U , one chooses a projective compactification $\overline{U}$ (so that $U \subset \overline{U}$ is open and dense) and lets $Y = \overline{U} - U$ be the reduced closed complement. One then constructs a pair of “smooth hyperenvelopes” $\overline{U}_\bullet \rightarrow \overline{U}$ and $Y_\bullet \rightarrow Y$ together with a map $Y_\bullet \rightarrow \overline{U}_\bullet$ of such extending the map $Y \hookrightarrow \overline{U}$. In general, a “smooth hyperenvelope” $X_\bullet \rightarrow X$ is an augmented simplicial variety such that each X_n is smooth and the induced map $X_n \rightarrow (\text{cosk}_{n-1}(X_\bullet))_n$ is a proper map that is surjective on F -points for any field F . Loosely speaking, such a smooth hyper-envelope over X is formed by first choosing a resolution of singularities $X_0 \rightarrow X$ of X (more specifically, a projective map with X_0 smooth that induces a surjection on F -points for any field F), then by choosing a resolution of singularities $X_1 \rightarrow X_0 \times_X X_0 = (\text{cosk}_0(X_\bullet))_1$, then by choosing a resolution of singularities $X_2 \rightarrow (\text{cosk}_2(X_\bullet))_1$, and so on.

Let $\mathbb{Z}\text{Sing}_\bullet(-)$ denote the functor taking a space to the chain complex that computes its singular homology (i.e., $\mathbb{Z}\text{Sing}_\bullet(-)$ is the chain complex associated to the simplicial abelian group $d \mapsto \mathbb{Z}\text{Maps}(\Delta_{\text{top}}^d, -)$). Then the total complex associated to the map of bicomplexes (i.e., the tri-complex)

$$\mathbb{Z}\text{Sing}_\bullet(Y_\bullet^{an}) \rightarrow \mathbb{Z}\text{Sing}_\bullet(\overline{U}_\bullet^{an})$$

gives the Borel–Moore homology of U^{an} . That is, letting $U_i = \overline{U}_i \coprod Y_{i-1}$ (with $Y_{-1} = \emptyset$), we have

$$H_n^{BM}(U^{an}) \cong h_n(\text{Tot}(\cdots \rightarrow \mathbb{Z}\text{Sing}_\bullet(U_1) \rightarrow \mathbb{Z}\text{Sing}_\bullet(U_0))) .$$

Observe that the definition of $W_t H_n$ for a smooth, projective variety X given below amounts to setting

$$W_t H_n(X) = h_n(\text{tr}_{\geq -t} \mathbb{Z}\text{Sing}_\bullet(X^{an})) ,$$

where $\text{tr}_{\geq -t}$ denotes the good truncation of chain complexes at homological degree $-t$. In heuristic terms, the weight filtration on $H_*^{BM}(U^{an})$ is defined from the “left-derived functor” of $\text{tr}_{\geq -t}$, if we interpret the smooth, projective varieties U_i as forming a resolution of U .

This idea is formalized in the following definition, which also includes a definition of related functors $\widetilde{W}_t H_n^{BM}$.

Definition 36 Given a quasi-projective complex variety U , define

$$\widetilde{W}_t H_n^{BM}(U) = h_n(\cdots \rightarrow \text{tr}_{\geq -t} \mathbb{Z}\text{Sing}_\bullet(U_1) \rightarrow \text{tr}_{\geq -t} \mathbb{Z}\text{Sing}_\bullet(U_0)) ,$$

where $\text{tr}_{\geq -t}$ denotes the good truncation of a chain complex at homological degree $-t$ and the U_i 's are constructed as above.

Define $W_t H_n^{BM}(U^{an})$ to be the image of $\tilde{W}_t H_n^{BM}(U)$ in $H_n^{BM}(U^{an})$ under the canonical map:

$$W_t H_n^{BM}(U^{an}) = \text{image}(\tilde{W}_t H_n^{BM}(U) \rightarrow H_n^{BM}(U^{an})) .$$

For an alternative formulation of the weight filtration, observe that associated to the bicomplex $\cdots \rightarrow \mathbb{Z} \text{Sing}_\bullet(U_1) \rightarrow \mathbb{Z} \text{Sing}_\bullet(U_0)$, we have the spectral sequence

$$E_{p,q}^2 = h_p(\cdots \rightarrow H_q(U_1^{an}) \rightarrow H_q(U_0^{an})) \Rightarrow H_{p+q}^{BM}(U^{an}) . \quad (1.18)$$

The weight filtration on $H_*^{BM}(U^{an})$ may equivalently be defined to be the filtration induced by this spectral sequence [33, §3]:

$$W_t H_n^{BM}(U^{an}) = \text{image}(h_n(\mathbb{Z} \text{Sing}_\bullet(U_{n+t}) \rightarrow \cdots \rightarrow \mathbb{Z} \text{Sing}_\bullet(U_0)) \rightarrow H_n^{BM}(U^{an})) .$$

In other words, the groups $W_t H_n^{BM}(U^{an})$ are the D_∞ terms of the spectral sequence (1.18). What's more, the groups $\tilde{W}_t H_n^{BM}(U)$ are equal to the D_2 terms of this spectral sequence. This implies that for a situation in which the spectral sequence (1.18) degenerates at the E_2 terms, the map $\tilde{W}_t H_n^{BM}(U) \rightarrow W_t H_n^{BM}(U^{an})$ is an isomorphism for all t and n . In particular, since (1.18) degenerates rationally by Deligne's result [11], we have $\tilde{W}_t H_n^{BM}(U)_\mathbb{Q} \cong W_t H_n^{BM}(U^{an})_\mathbb{Q}$.

For a simple example, suppose U happens to be smooth and admits a smooth compactification X such that $Y = X - U$ is again smooth. Then the spectral sequence (1.18) degenerates (integrally) and it really just amounts to a single long exact sequence

$$\cdots \rightarrow H_n^{sing}(Y^{an}) \rightarrow H_n^{sing}(X^{an}) \rightarrow H_n^{BM}(U^{an}) \rightarrow H_{n-1}^{sing}(Y^{an}) \rightarrow \cdots .$$

It follows that $\tilde{W}_t H_n^{BM}(U) = W_t H_n^{BM}(U^{an})$ and

$$W_t H_n^{BM}(U^{an}) = \begin{cases} 0 , & \text{if } t < -n , \\ \text{image}(H_n^{sing}(X^{an}) \rightarrow H_n^{BM}(U^{an})) , & \text{if } t = -n , \text{ and} \\ H_n^{BM}(U^{an}) , & \text{if } t > -n . \end{cases}$$

Thus, in this situation the information encoded by the weight filtration on $H_*^{BM}(U^{an})$ concerns which classes in $H_*^{BM}(U^{an})$ can be lifted to the homology of a smooth compactification $H_*(X^{an})$.

It is a non-trivial theorem, due to Deligne [11] for $\mathbb{Q}$ -coefficients and Gillet-Soulé [33, §2] in general, that the weight $W_* H_*^{BM}$ filtration is independent of the choices made in its construction. Using their techniques, the authors and Christian Haesemeyer have also proven that $\tilde{W}_t H^{BM}$ is independent of the choices made [21, 5.9].

1.4.3 Computations

The *generalized cycle map* (see [24] and [48]) is the map from the Lawson homology of a complex variety X to its Borel–Moore homology

$$L_t H_n(X) \rightarrow H_n^{\text{sing}}(X^{\text{an}}).$$

For smooth varieties, the generalized cycle map corresponds under duality to the map from morphic cohomology to singular cohomology

$$L^{d-t} H^{2d-n}(X) \rightarrow H_{\text{sing}}^{2d-n}(X^{\text{an}})$$

described in Sect. 1.3.1. For projective (but possibly singular) varieties, the generalized cycle map is defined by applying π_{n-2t} to the diagram of spaces

$$\mathcal{Z}_t(X) \xrightarrow{s} \Omega^{2t} \mathcal{Z}_t(X \times \mathbb{A}^t) \xleftarrow{\sim} \Omega^{2t} \mathcal{Z}_0(X),$$

where the first map is the “ s map” defined by Friedlander and Mazur [24] (see also Definition 33) and the second map is the homotopy equivalence induced by flat pullback of cycles along the projection $X \times \mathbb{A}^t \rightarrow X$. This definition is extended to quasi-projective varieties in [20].

Observe that the singular chain complex associated to the space $\Omega^{2t} \mathcal{Z}_0(X)$ is quasi-isomorphic to $tr_{\geq -2t} \mathbb{Z} \text{Sing}_{\bullet}(X^{\text{an}})[2t]$, the complex used to define $\tilde{W}_{2t} H_*^{BM}$. This observation leads to a proof that the generalized cycle class map from $L_t H_n$ lands in the weight $-2t$ part of Borel–Moore homology. This fact, as well as other properties relating the weight filtration on Borel–Moore homology and the generalized cycle map, is formalized by the following result.

37 Proposition 37 (cf. [21, 5.11, 5.12])

1. For any quasi-projective variety U , the generalized cycle map factors as

$$L_t H_n(U) \rightarrow \tilde{W}_{-2t} H_n^{BM}(U) \twoheadrightarrow W_{-2t} H_n^{BM}(U) \subset H_n^{BM}(U),$$

and each of these maps is covariantly functorial for proper morphisms and contravariantly functorial for open immersions. The map $L_t H_n(U) \rightarrow \tilde{W}_{-2t} H_n^{BM}(U)$ is called the *refined cycle map*.

2. Each of the theories $L_t H_*(-)$, $\tilde{W}_t H_*^{BM}(-)$, $H_*^{BM}((-)^{\text{an}})$ has a long exact localization sequence associated to an open immersion $U \subset X$ with closed complement $Y = X - U$, and the maps

$$L_t H_*(-) \rightarrow \tilde{W}_t H_*^{BM}(-) \rightarrow H_*^{BM}((-)^{\text{an}})$$

are compatible with these long exact sequences.

38

Remark 38 The weight filtration itself, $W_t H_*^{BM}(-)$, is *not* always compatible with localization sequences, and the construction $\tilde{W}_t H_*^{BM}$ was introduced to rectify this defect.

The first part of Proposition 37 clearly provides an obstruction for the generalized cycle map to be an isomorphism in certain degrees for certain kinds of varieties.

Definition 39 Define $\mathcal{C}$ to be the collection of smooth, quasi-projective complex varieties U such that the refined cycle map $L_t H_n(U) \rightarrow W_t H_n(U)$ is an isomorphism for all t and n .

39

Theorem 40 (cf. [21, 6.3]) Assume X is a quasi-projective complex variety of dimension d that belongs to the class $\mathcal{C}$ and let A be any abelian group.

40

1. The generalized cycle map

$$L_t H_n(X, A) \rightarrow H_n^{BM}(X^{an}, A)$$

is an isomorphism for $n \geq d + t$ and a monomorphism for $n = d + t - 1$. If X is smooth and projective, this map is an isomorphism for $n \geq 2t$.

2. If X is smooth, the canonical map

$$K_q^{sst}(X, A) \rightarrow K_{top}^{-q}(X^{an}, A)$$

is an isomorphism for $q \geq d - 1$ and a monomorphism for $q = d - 2$. If X is smooth and projective, this map is an isomorphism for $q \geq 0$.

The proof of the first part of Theorem 40 is achieved via a careful analysis of the spectral sequence (1.18), and the proof of the second part follows from a careful analysis of the semi-topological spectral sequence and its comparison with the classical Atiyah–Hirzebruch spectral sequence (Theorem 35).

The conclusion of the second part of Theorem 40 (in the not-necessarily-projective case) is what we term the *Semi-topological Quillen–Lichtenbaum Conjecture*, discussed in more detail in Sect. 1.5 below.

The validity of the following assertions for the class $\mathcal{C}$ is the primary reason the groups $\widetilde{W}_* H_*^{BM}(-)$ were introduced. The corresponding statement for the class of varieties for which $L_t H_n(-) \rightarrow W_{-2t} H_n^{BM}((-)^{an})$ is an isomorphism in all degrees is false.

Proposition 41 (cf. [21, 6.9]) The class $\mathcal{C}$ is closed under the following constructions:

41

1. Closure under localization: Let $Y \subset X$ be a closed immersion with Zariski open complement U . If two of X , Z , and U belong to $\mathcal{C}$, so does the third.
2. Closure for bundles: For a vector bundle $E \rightarrow X$, the variety X belongs to $\mathcal{C}$ if and only if $\mathbb{P}(E)$ does. In this case, E belongs to $\mathcal{C}$ as well.
3. Closure under blow-ups: Let $Z \subset X$ be a regular closed immersion and such that Z belongs to $\mathcal{C}$. Then X is in $\mathcal{C}$ if and only if the blow-up X_Z of X along Z is in $\mathcal{C}$.

Recall that the class of *linear varieties* is the smallest collection $\mathcal{L}$ of complex varieties such that (1) $\mathbb{A}^n$ belongs to $\mathcal{L}$, for all $n \geq 0$, and (2) if X is a quasi-projective complex variety, $Z \subset X$ is a closed subscheme, $U = X - Z$ is the open complement, and Z and either X or U belongs to $\mathcal{L}$, then so does the remaining member of the triple (X, Z, U) . Examples of linear varieties include toric and cellular varieties.

42

Theorem 42 (cf. [21, §6]) The following complex varieties belong to $\mathcal{C}$:

1. A quasi-projective curve.
2. A smooth, quasi-projective surface having a smooth compactification with all of H_{sing}^2 algebraic.
3. A smooth projective rational three-fold.
4. A smooth quasi-projective linear variety (e.g., a smooth quasi-projective toric variety).
5. A toric fibration (e.g., an affine or projective bundle) over one of the above varieties.

In particular, if X is smooth and one of the above types of varieties, then for any abelian group A the natural map

$$K_n^{\text{sst}}(X, A) \rightarrow K_{\text{top}}^{-n}(X^{\text{an}}, A)$$

is an isomorphism for $n \geq \dim(X) - 1$ and a monomorphism for $n = \dim(X) - 2$. If X is in addition projective, this map is an isomorphism for all $n \geq 0$.

As mentioned in the introduction, when X is a smooth, projective complex variety belonging to $\mathcal{C}$, Theorem 42 implies that the subspace inclusion

$$\mathcal{M}or(X, \text{Grass}) \subset \mathcal{M}aps(X^{\text{an}}, \text{Grass}^{\text{an}})$$

becomes a homotopy equivalence upon taking homotopy-theoretic group completions. In fact, both homotopy-theoretic group completions can be described precisely by taking mapping telescopes of self-maps (essentially defined as “addition by a fixed ample line bundle”) of the spaces above [31, 3.5]. This result therefore gives examples when the stabilized space of all continuous maps between two complex (ind-)varieties can be represented up-to-homotopy equivalence by the stabilized space of all algebraic morphisms between them.

1.5

Conjectures

In this section, we discuss various conjectures relating semi-topological K -theory to topological K -theory and relating morphic cohomology to singular cohomology.

Integral Versions of the “Classical” Conjectures

One important feature of semi-topological K -theory and morphic cohomology is that they allow for the formulation of plausible analogues for arbitrary coefficients of the classical conjectures in algebraic K -theory and motivic cohomology for finite coefficients. For example, the Quillen–Lichtenbaum and Beilinson–Lichtenbaum Conjectures, each of which concerns theories with $\mathbb{Z}/n$ -coefficients, admit *integral* analogues in the semi-topological world. Moreover, in light of Theorem 30, these semi-topological conjectures imply their classical counter-parts (for complex varieties).

Perhaps the most fundamental of these conjectures, formulated originally by A. Suslin, concerns a conjectural description of morphic cohomology in terms of singular cohomology. To understand *Suslin’s Conjecture*, as we have termed it, recall that if we define $\mathbb{Z}^{sst}$ to be the complex of abelian sheaves

$$\mathbb{Z}^{sst}(t) = \mathrm{Hom} \left(- \times \Delta_{top}^\bullet, \mathcal{C}_0(\mathbb{P}^t) \right)^+ / \mathrm{Hom} \left(- \times \Delta_{top}^\bullet, \mathcal{C}_0(\mathbb{P}^{t-1}) \right)^+ [-2t],$$

then the morphic cohomology groups of a smooth variety X are given by

$$L^t H^n(X) = h^{n-2t} \Gamma(X, \mathbb{Z}^{sst}(t)).$$

In fact, Zariski descent for morphic cohomology [18] implies that

$$L^t H^n(X) \cong \mathbb{H}_{Zar}^{n-2t}(X, \mathbb{Z}^{sst}(t))$$

for X smooth, where $\mathbb{H}_{Zar}$ denotes taking the hypercohomology in the Zariski topology. The comparison of $\mathbb{Z}^{sst}(t)$ with singular cohomology uses the morphism of sites $\varepsilon : CW_{open} \rightarrow (Sch/C_{Zar})$, where CW denotes the category of topological spaces homeomorphic to finite dimensional CW -complexes, associated to the functor $U \mapsto U^{an}$ taking a complex variety to its associated analytic space. If $\mathbb{Z}$ denotes the sheaf associated to the constant presheaf $T \mapsto \mathbb{Z}$ defined on CW , then we have

$$H_{sing}^n(X^{an}, \mathbb{Z}) \cong H_{sheaf}^n(X^{an}, \mathbb{Z}) \cong \mathbb{H}_{Zar}^n(X^{an}, \mathbb{R}\varepsilon_* \mathbb{Z}),$$

for any $X \in Sch/C$. It’s not hard to see that the map from morphic cohomology to singular cohomology is induced by a map of chain complexes of sheaves

$$\mathbb{Z}^{sst}(q) \rightarrow \mathbb{R}\varepsilon_* \mathbb{Z}.$$

More generally, for any abelian group A , if we define $A^{sst}(q) = \mathbb{Z}^{sst}(q) \otimes A$, then there is a natural map

$$A^{sst}(q) \rightarrow \mathbb{R}\varepsilon_* A$$

of complexes of sheaves that induces the map from morphic cohomology with A -coefficients to singular cohomology with A -coefficients.

To formulate Suslin’s Conjecture, we need the following result:

43

Theorem 43 [21, 7.3] For any abelian group A , the map $A^{sst}(q) \rightarrow \mathbb{R}\mathcal{E}_*A$ factors (in the derived category of sheaves) as

$$A^{sst}(q) \rightarrow tr^{\leq q} \mathbb{R}\mathcal{E}_*A \rightarrow \mathbb{R}\mathcal{E}_*A,$$

where $tr^{\leq q}$ represents the “good truncation” at degree q of a cochain complex.

The proof of Theorem 43 may be of independent interest, and so we sketch it here. (This is proved formally in [21], building on ideas from [13].) It suffices to prove $L^t H^n(-, A)$ vanishes locally on a smooth variety whenever $n > t$. Using duality relating morphic cohomology to Lawson homology [17, 22], we see that it suffices to prove that $L_t H_m(-, A)$ vanishes at the generic point of X for $m < t + \dim(X)$. Localization for Lawson homology and the rational injectivity of the Hurewicz map for a topological abelian group shows that it suffices to verify that the canonical map

$$\lim_{\substack{\rightarrow \\ Y \subset X, \text{codim}(Y) \geq 1}} H_m^{sing}(Z_t(Y), A) \rightarrow H_m^{sing}(Z_t(X), A)$$

is an isomorphism for $n < d - t - 1$ and a surjection for $n = d - t - 1$. For a given X and a given $t \geq 0$, the proof of this statement can be reduced to proving the analogous statements for

$$\lim_{\substack{\rightarrow \\ Y \subset X, \text{codim}(Y) \geq 1}} H_m^{sing}(C_{t,e}(Y), A) \rightarrow H_m^{sing}(C_{t,e}(X), A), \quad e > 0.$$

Finally, these statements concerning the (singular) algebraic varieties $C_{t,e}(X)$ are then proved using the Lefschetz theorem as proved by Andreotti and Frankel [3].

44

Conjecture 44: *Suslin’s Conjecture* For any abelian group A , the map of complexes

$$A^{sst}(q) \rightarrow tr^{\leq q} \mathbb{R}\mathcal{E}_*A$$

is a quasi-isomorphism on the category of smooth, quasi-projective complex varieties.

Equivalently (see [21, 7.9]), for all smooth, quasi-projective complex varieties X , the map

$$L^t H^n(X, A) \rightarrow H_{sing}^n(X, A)$$

is an isomorphism for $n \leq t$ and a monomorphism for $n = t + 1$.

Suslin’s Conjecture is clearly analogous to the Beilinson–Lichtenbaum Conjecture, which can be stated as follows.

45

Conjecture 45: *Beilinson–Lichtenbaum Conjecture* (See [6] and [45].) Let F be an arbitrary field, let $\pi : (Sch/F)_{\acute{e}t} \rightarrow (Sch/F)_{Zar}$ be the evident morphism of sites,

and let m be a positive integer not divisible by the characteristic of F . Then the canonical map of complexes of sheaves on $(Sm/F)_{Zar}$

$$\mathbb{Z}/m(q) \rightarrow tr^{\leq q} \mathbb{R}\pi_* \mu_m^{\otimes q}$$

is a quasi-isomorphism.

Equivalently, for all smooth, quasi-projective F -varieties X , the map

$$H_{\mathcal{M}}^n(X, \mathbb{Z}/n(q)) \rightarrow H_{\acute{e}t}^n(X, \mu_m^{\otimes q})$$

is an isomorphism for $n \leq t$ and a monomorphism for $n = t + 1$.

In light of Theorem 30 and the fact that étale and singular cohomology with finite coefficients of complex varieties coincide, the following result is evident.

Proposition 46 Suslin's Conjecture implies the Beilinson–Lichtenbaum Conjecture for complex varieties. 46

In a parallel fashion, the Quillen–Lichtenbaum Conjecture, which asserts an isomorphism between algebraic and topological K -theory with finite coefficients in a certain range, admits an integral, semi-topological analogue:

Conjecture 47: *Semi-topological Quillen–Lichtenbaum Conjecture* For a smooth, quasi-projective complex variety X and abelian group A , the canonical map 47

$$K_n^{sst}(X, A) \rightarrow K_{top}^{-n}(X, A)$$

is an isomorphism for $n \geq \dim(X) - 1$ and a monomorphism for $n = \dim(X) - 2$.

Using the isomorphism of Theorem 30, we see in the case $A = \mathbb{Z}/m$ that this conjecture is equivalent to the assertion that

$$K_n^{alg}(X, \mathbb{Z}/m) \rightarrow K_{top}^{-n}(X, \mathbb{Z}/m)$$

is an isomorphism for $n \geq \dim(X) - 1$ and a monomorphism for $n = \dim(X) - 2$. This special case is the “classical” *Quillen–Lichtenbaum Conjecture* (see [52] and [44]) for complex varieties.

Evidence for the semi-topological Quillen–Lichtenbaum Conjecture is supplied by Theorem 32, which may be interpreted as saying the map $K_*^{sst}(X) \rightarrow K_{top}^{-*}(X^{an})$ is an isomorphism “stably”. In addition, we have the following result establishing split surjectivity of this map in a range.

Theorem 48 For a smooth, quasi-projective complex variety X , the map 48

$$K_n^{sst}(X) \rightarrow K_{top}^{-n}(X^{an})$$

is a split surjection for $n \geq 2 \dim(X)$.

When X is projective, this is proven by the second author in [60] using the theory of “semi-topological K -homology”. Thanks to the recently established motivic spectral sequence (1.17), a proof for the general case is obtained by mimicking the argument of [29, 1.4].

Using the semi-topological Atiyah–Hirzebruch spectral sequence (1.17), one may readily deduce that Suslin’s Conjecture implies the semi-topological Quillen–Lichtenbaum Conjecture.

49 **Theorem 49** [21, 6.1] For a smooth, quasi-projective complex variety X and an abelian group A , if

$$L^q H^n(X, A) \rightarrow H^n(X^{an}, A)$$

is an isomorphism for $n \leq q$ and a monomorphism for $n = q + 1$, then the map

$$K_i^{sst}(X, A) \rightarrow ku^{-i}(X^{an}, A)$$

is an isomorphism for $i \geq \dim(X) - 1$ and a monomorphism for $i = \dim(X) - 2$.

In other words, Suslin’s Conjecture implies the semi-topological Quillen–Lichtenbaum Conjecture.

The results described in Sect. 1.4.3 lead to the following theorem:

50 **Theorem 50** (cf. [21, 7.14]) Suslin’s Conjecture and the semi-topological Quillen–Lichtenbaum Conjectures hold for the following complex varieties:

1. smooth quasi-projective curves,
2. smooth quasi-projective surfaces,
3. smooth projective rational three-folds,
4. smooth quasi-projective linear varieties (for example, smooth quasi-projective toric and cellular varieties), and
5. smooth toric fibrations (e.g., affine and projective bundles) over one of the above varieties.

Consequently, the “classical” Quillen–Lichtenbaum Conjecture and the Beilinson–Lichtenbaum Conjecture hold for these varieties.

We remind the reader that Voevodsky (using results of M. Rost) has recently proven the Beilinson–Lichtenbaum and “classical” Quillen–Lichtenbaum Conjectures [59].

1.5.2 K -Theoretic Analogue of the Hodge Conjecture

The Hodge conjecture [37] concerns which rational singular cohomology classes of a smooth, complex variety arise from cycles – more precisely, it asserts that

for a smooth, projective complex variety X , every class in $H_{\text{sing}}^{p,p}(X^{\text{an}}, \mathbb{Q})$ lies in the image of the rational cycle class map $A^p(X)_{\mathbb{Q}} \rightarrow H_{\text{sing}}^{2p}(X^{\text{an}}, \mathbb{Q})$, where

$$H^{p,p}(X^{\text{an}}, \mathbb{Q}) = H_{\text{sing}}^{2p}(X^{\text{an}}, \mathbb{Q}) \cap H^{p,p}(X^{\text{an}}, \mathbb{C})$$

and $H^{p,p}(X^{\text{an}}, \mathbb{C})$ refers to the Hodge decomposition of a complex Kahler manifold. It is easy to show the image of the rational cycle class map is contained in $H_{\text{sing}}^{p,p}(X^{\text{an}}, \mathbb{Q})$, and thus the Hodge Conjecture becomes (in the language of morphic cohomology) that the rational cycle class map

$$L^p H^{2p}(X, \mathbb{Q}) \rightarrow H^{p,p}(X^{\text{an}}, \mathbb{Q})$$

is a surjection.

The Generalized Hodge Conjecture, as corrected by Grothendieck [36], asserts that the *Hodge filtration* and the *coniveau filtration* on the rational homology of a smooth, projective complex variety coincide. The rational Hodge filtration is given as

$$H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) = F_h^0 H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) \supset F_h^1 H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) \supset F_h^2 H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) \supset \cdots,$$

where $F_h^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q})$ is defined by Grothendieck [36] to be the maximal sub-mixed Hodge structure of $H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q})$ contained in $H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) \cap \bigoplus_{p \geq j} H_{\text{sing}}^{p, m-p}(X^{\text{an}}, \mathbb{C})$. The coniveau filtration, $N^* H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q})$, is given by

$$N^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) = \bigcup_{\substack{Y \subset X \\ \text{codim}(Y) = j}} \ker \left(H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) \rightarrow H_{\text{sing}}^m((X - Y)^{\text{an}}, \mathbb{Q}) \right)$$

The containment $N^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) \subset F_h^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q})$ always holds, and so the Generalized Hodge Conjecture amounts to the assertion that the opposite containment also holds – i.e., every class in $F_h^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q})$ vanishes on the complement of a closed subscheme of codimension j .

For any smooth, quasi-projective complex variety X , the *topological filtration* of $H_{\text{sing}}^m(X^{\text{an}})$ is given by considering the images of the powers of the s map:

$$T^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Z}) = \text{image} \left(L^{m-j} H^m(X) \rightarrow H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Z}) \right).$$

(We set $T^j = T^0$ if $j < 0$.) Recall that Suslin's Conjecture predicts that the map $L^m H^m(X) \rightarrow H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Z})$ is an isomorphism, so that, conjecturally, $T^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Z})$ may be identified with the image of $s^j : L^{m-j} H_{\text{sing}}^m(X) \rightarrow L^m H^m(X)$.

The following result of the first author and B. Mazur (originally stated in the context of Lawson homology) relates the three filtrations above. Note that we have modified the indexing conventions here from those of the original.

51 Proposition 51 (cf. [24]) For a smooth, projective complex variety X , we have

$$T^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) \subset N^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) \subset F_h^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}),$$

for all j and m .

The following conjecture thus represents a (possibly stronger) version of the Generalized Hodge Conjecture:

52 Conjecture 52: *Friedlander–Mazur Conjecture* (cf. [24]) For a smooth, projective complex variety, we have

$$T^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}) = F_h^j H_{\text{sing}}^m(X^{\text{an}}, \mathbb{Q}),$$

for all m and j .

Proposition 51 shows that the Friedlander–Mazur Conjecture implies the Generalized Hodge Conjecture. In the case of abelian varieties, S. Abdulali has established the following converse.

53 Theorem 53 (cf. [1]) For abelian varieties for which the Generalized Hodge Conjecture is known the Friedlander–Mazur Conjecture also holds.

Semi-topological K -theory provides another perspective on the Hodge Conjecture, one which could prove to be of some use. Since the topological filtration on singular cohomology is defined in terms of the s map, it is natural to define the *topological filtration* on topological K -theory in terms of the Bott map (i.e., multiplication by the Bott element) in semi-topological K -theory. That is, an element of $K_{\text{top}}^0(X^{\text{an}})$ lies in the j -th filtered piece of the topological filtration if it comes from semi-topological K -theory after applying the j -th power of the Bott map:

$$T^j K_{\text{top}}^0(X^{\text{an}}) = \tag{1.19}$$

$$\text{image} \left(K_{2d-2j}^{\text{sst}}(X) \xrightarrow{\beta^j} K_{2d}^{\text{sst}}(X) \rightarrow K_{\text{top}}^{-2d}(X^{\text{an}}) \xrightarrow{\cong} K_{\text{top}}^0(X^{\text{an}}) \right).$$

(As before, $T^j = T^0$ for $j < 0$.) The form of this definition appears more sensible once one recalls that the map

$$K_{2d}^{\text{sst}}(X) \rightarrow K_{\text{top}}^{-2d}(X^{\text{an}})$$

is known to be a split surjection (see Theorem 48). We thus have a filtration of the form

$$K_{\text{top}}^0(X^{\text{an}}) = T^0 K_{\text{top}}^0(X^{\text{an}}) \supset T^1 K_{\text{top}}^0(X^{\text{an}}) \supset \cdots \supset T^d K_{\text{top}}^0(X^{\text{an}}) \supset T^{d+1} K_{\text{top}}^0(X^{\text{an}}) = 0,$$

for a smooth, quasi-projective complex variety X . In fact, for $j \leq \frac{d+1}{2}$, the semi-topological Quillen–Lichtenbaum Conjecture predicts that each map in (1.19) is an isomorphism, so that conjecturally we have $T^0 K_{top}^0(X^{an}) = T^1 K_{top}^0(X^{an}) = \dots = T^{\frac{d+1}{2}} K_{top}^0(X^{an})$.

Rationally, under the Chern character isomorphism, the Bott element in K -theory corresponds to the s element in cohomology, and thus the topological filtrations for K -theory and cohomology defined above are closely related. The precise statement is the following.

Theorem 54 (cf. [32, 5.9]) For any smooth, quasi-projective complex variety X of dimension d , the Chern character restricts to an isomorphism

54

$$ch^{top} : T^j K_{top}^0(X^{an})_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{q \geq 0} T^{q+j-d} H_{sing}^{2q}(X^{an}, \mathbb{Q}),$$

for all j . In other words, the weight q piece of $T^j K_{top}^0(X^{an})_{\mathbb{Q}}$ is mapped isomorphically via the Chern character to $T^{q+j-d} H_{sing}^{2q}(X^{an}, \mathbb{Q})$.

Using the Chern character isomorphism, one can transport the rational Hodge Filtration in singular cohomology to a filtration of $K_{top}^0(X)_{\mathbb{Q}}$. In this manner, the Friedlander–Mazur Conjecture, which implies the Generalized Hodge Conjecture, can be stated in purely K -theoretic terms. It would be interesting to find an intrinsic description of the “Hodge filtration” of $K_{top}^0(X)_{\mathbb{Q}}$.

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Equivariant K -Theory *

V.2

Alexander S. Merkurjev

2.1	<i>Introduction</i>	927
2.2	<i>Basic Results in the Equivariant K-Theory</i>	928
	Definitions	928
	Torsors	930
	Basic Results in Equivariant K -Theory	932
2.3	<i>Category $\mathcal{C}(G)$ of G-Equivariant K-Correspondences</i>	934
2.4	<i>Equivariant K-Theory of Projective Homogeneous Varieties</i>	936
	Split Case	936
	Quasi-split Case	937
	General Case	937
2.5	<i>K-Theory of Toric Varieties</i>	938
	K -Theory of Toric Models	938
	K -Theory of Toric Varieties	939
2.6	<i>Equivariant K-Theory of Solvable Algebraic Groups</i>	940
	Split Unipotent Groups	940
	Split Algebraic Tori	941
	Quasi-trivial Algebraic Tori	944
	Coflasque Algebraic Tori	944
2.7	<i>Equivariant K-Theory of some Reductive Groups</i>	945
	Spectral Sequence	945
	K -Theory of Simply Connected Group	946
2.8	<i>Equivariant K-Theory of Factorial Groups</i>	948

* This work has been supported by the NSF grant DMS 0098111.

2.9 *Applications* 950

K-Theory of Classifying Varieties..... 950

 Equivariant Chow Groups..... 951

 Group Actions on the *K'*-Groups..... 952

References 953

Introduction

The equivariant K -theory was developed by R. Thomason in [21]. Let an algebraic group G act on a variety X over a field F . We consider G -modules, i.e., $\mathcal{O}_X$ -modules over X that are equipped with an G -action compatible with one on X . As in the non-equivariant case there are two categories: the abelian category $\mathcal{M}(G; X)$ of coherent G -modules and the full subcategory $\mathcal{P}(G; X)$ consisting of locally free $\mathcal{O}_X$ -modules. The groups $K'_n(G; X)$ and $K_n(G; X)$ are defined as the K -groups of these two categories respectively.

In the second section we present definitions and formulate basic theorems in the equivariant K -theory such as the localization theorem, projective bundle theorem, strong homotopy invariance property and duality theorem for regular varieties.

In the following section we define an additive category $\mathcal{C}(G)$ of G -equivariant K -correspondences that was introduced by I. Panin in [15]. This category is analogous to the category of Chow correspondences presented in [9]. Many interesting functors in the equivariant K -theory of algebraic varieties factor through $\mathcal{C}(G)$. The category $\mathcal{C}(G)$ has more objects (for example, separable F -algebras are also the objects of $\mathcal{C}(G)$) and has much more morphisms than the category of G -varieties. For instance, every projective homogeneous variety is isomorphic to a separable algebra (Theorem 16).

In Sect. 2.4, we consider the equivariant K -theory of projective homogeneous varieties developed by I. Panin in [15]. The following section is devoted to the computation of the K -groups of toric models and toric varieties (see [12]).

In Sects. 2.6 and 2.7, we construct a spectral sequence

$$E_{p,q}^2 = \mathrm{Tor}_p^{R(G)} \left(\mathbb{Z}, K'_q(G; X) \right) \Rightarrow K'_{p+q}(X),$$

where G is a split reductive group with the simply connected commutator subgroup and X is a G -variety.

The rest of the paper addresses the following question. Let G be an algebraic group. Under what condition on G the G -action on a G -variety X can be extended to a linear action on every vector bundle $E \rightarrow X$ making it a G -vector bundle on X ? If $X = G$ and E is a line bundle, then the existence of a G -structure on E implies that E is trivial. Thus, if the answer is positive, the Picard group $\mathrm{Pic}(G)$ must be trivial. It turns out that the triviality of $\mathrm{Pic}(G)$ implies positive solution at least stably, on the level of coherent G -modules. We prove that for a factorial group G the restriction homomorphism $K'_0(G; X) \rightarrow K'_0(X)$ is surjective (Theorem 39). Our exposition is different from the one presented in [11].

In the last section we consider some applications.

We use the word *variety* for a separated scheme of finite type over a field. If X is a variety over a field F and L/F is a field extension, then we write X_L for the variety $X \otimes_F L$ over L . By X_{sep} we denote $X_{F_{\mathrm{sep}}}$, where F_{sep} is a separable closure of F . If R is a commutative F -algebra, we write $X(R)$ for the set $\mathrm{Mor}_F(\mathrm{Spec} R, X)$ of R -points of X .

An *algebraic group* is a smooth affine group variety of finite type over a field.

Basic Results in the Equivariant K -Theory

In this section we review the equivariant K -theory developed by R. Thomason in [21].

Definitions

Let G be an algebraic group over a field F . A variety X over F is called a G -variety if an action morphism $\theta : G \times X \rightarrow X$ of the group G on X is given, which satisfies the usual associative and unital identities for an action. In other words, to give a structure of a G -variety on a variety X is to give, for every commutative F -algebra R , a natural in R action of the group of R -points $G(R)$ on the set $X(R)$.

A G -module M over X is a quasi-coherent $\mathcal{O}_X$ -module M together with an isomorphism of $\mathcal{O}_{G \times X}$ -modules

$$\varphi = \varphi_M : \theta^*(M) \xrightarrow{\sim} p_2^*(M) ,$$

(where $p_2 : G \times X \rightarrow X$ is the projection), satisfying the cocycle condition

$$p_{23}^*(\varphi) \circ (\mathrm{id}_G \times \theta)^*(\varphi) = (m \times \mathrm{id}_X)^*(\varphi) ,$$

where $p_{23} : G \times G \times X \rightarrow G \times X$ is the projection and $m : G \times G \rightarrow G$ is the product morphism (see [14, Ch. 1, §3] or [21]).

A morphism $\alpha : M \rightarrow N$ of G -modules is called a G -morphism if

$$\varphi_N \circ \theta^*(\alpha) = p_2^*(\alpha) \circ \varphi_M .$$

Let M be a quasi-coherent $\mathcal{O}_X$ -module. For a point $x : \mathrm{Spec} R \rightarrow X$ of X over a commutative F -algebra R , write $M(x)$ for the R -module of global sections of the sheaf $x^*(M)$ over $\mathrm{Spec} R$. Thus, M defines the functor sending R to the family $\{M(x)\}$ of R -modules indexed by the R -valued point $x \in X(R)$. To give a G -module structure on M is to give natural in R isomorphisms of R -modules

$$\varphi_{g,x} : M(x) \rightarrow M(gx)$$

for all $g \in G(R)$ and $x \in X(R)$ such that $\varphi_{gg',x} = \varphi_{g,g'x} \circ \varphi_{g',x}$.

Example 1. Let X be a G -variety. A G -vector bundle on X is a vector bundle $E \rightarrow X$ together with a linear G -action $G \times E \rightarrow E$ compatible with the one on X . The sheaf of sections P of a G -vector bundle E has a natural structure of a G -module. Conversely, a G -module structure on the sheaf P of sections of a vector bundle $E \rightarrow X$ yields structure of a G -vector bundle on E . Indeed, for a commutative F -algebra R and a point $x \in X(R)$, the fiber of the map $E(R) \rightarrow X(R)$ over x is canonically isomorphic to $P(x)$.

We write $\mathcal{M}(G; X)$ for the abelian category of coherent G -modules over a G -variety X and G -morphisms. We set for every $n \geq 0$:

$$K'_n(G; X) = K_n(\mathcal{M}(G; X)) .$$

A flat morphism $f : X \rightarrow Y$ of varieties over F induces an exact functor

$$\mathcal{M}(G; Y) \rightarrow \mathcal{M}(G; X), \quad M \mapsto f^*(M)$$

and therefore defines the *pull-back* homomorphism

$$f^* : K'_n(G; Y) \rightarrow K'_n(G; X) .$$

A *G -projective morphism* $f : X \rightarrow Y$ is a morphism that factors equivariantly as a closed embedding into the projective bundle variety $\mathbb{P}(E)$, where E is a G -vector bundle on Y . Such a morphism f yields the *push-forward* homomorphisms [21, 1.5]

$$f_* : K'_n(G; X) \rightarrow K'_n(G; Y) .$$

If G is the trivial group, then $\mathcal{M}(G; X) = \mathcal{M}(X)$ is the category of coherent $\mathcal{O}_X$ -modules over X and therefore, $K'_n(G; X) = K'_n(X)$.

Consider the full subcategory $\mathcal{P}(G; X)$ of $\mathcal{M}(G; X)$ consisting of locally free $\mathcal{O}_X$ -modules. This category is naturally equivalent to the category of vector G -vector bundles on X (Example 1). The category $\mathcal{P}(G; X)$ has a natural structure of an exact category. We set

$$K_n(G; X) = K_n(\mathcal{P}(G; X)) .$$

The functor $K_n(G; *)$ is contravariant with respect to arbitrary G -morphisms of G -varieties. If G is a trivial group, we have $K_n(G; X) = K_n(X)$.

The tensor product of G -modules induces a ring structure on $K_0(G; X)$ and a module structure on $K_n(G; X)$ and $K'_n(G; X)$ over $K_0(G; X)$.

The inclusion of categories $\mathcal{P}(G; X) \hookrightarrow \mathcal{M}(G; X)$ induces an homomorphism

$$K_n(G; X) \rightarrow K'_n(G; X) .$$

Example 2. Let $\mu : G \rightarrow \mathrm{GL}(V)$ be a finite dimensional representation of an algebraic group G over a field F . One can view the G -module V as a G -vector bundle on $\mathrm{Spec} F$. Clearly, we obtain an equivalence of the abelian category $\mathrm{Rep}(G)$ of finite dimensional representations of G and the categories $\mathcal{P}(G; \mathrm{Spec} F) = \mathcal{M}(G; \mathrm{Spec} F)$. Hence there are natural isomorphisms

$$R(G) \xrightarrow{\sim} K_0(G; \mathrm{Spec} F) \xrightarrow{\sim} K'_0(G; \mathrm{Spec} F) ,$$

where $R(G) = K_0(\mathrm{Rep}(G))$ is the *representation ring* of G . For every G -variety X over F , the pull-back map

$$R(G) \simeq K_0(G; \mathrm{Spec} F) \rightarrow K_0(G; X)$$

with respect to the structure morphism $X \rightarrow \operatorname{Spec} F$ is a ring homomorphism, making $K_0(G; X)$ (and similarly $K'_0(G; X)$) a module over $R(G)$. Note that as a group, $R(G)$ is free abelian with basis given by the classes of all irreducible representations of G over F .

Let $\pi : H \rightarrow G$ be an homomorphism of algebraic groups over F and let X be a G -variety over F . The composition

$$H \times X \xrightarrow{\pi \times \operatorname{id}_X} G \times X \xrightarrow{\theta} X$$

makes X an H -variety. Given a G -module M with the G -module structure defined by an isomorphism φ , we can introduce an H -module structure on M via $(\pi \times \operatorname{id}_X)^*(\varphi)$. Thus, we obtain exact functors

$$\operatorname{Res}_\pi : \mathcal{M}(G; X) \rightarrow \mathcal{M}(H; X) , \quad \operatorname{Res}_\pi : \mathcal{P}(G; X) \rightarrow \mathcal{P}(H; X)$$

inducing the *restriction* homomorphisms

$$\operatorname{res}_\pi : K'_n(G; X) \rightarrow K'_n(H; X) , \quad \operatorname{res}_\pi : K_n(G; X) \rightarrow K_n(H; X) .$$

If H is a subgroup of G , we write $\operatorname{res}_{G/H}$ for the restriction homomorphism res_π , where $\pi : H \hookrightarrow G$ is the inclusion.

2.2.2 Torsors

Let G and H be algebraic groups over F and let $f : X \rightarrow Y$ be a $G \times H$ -morphism of $G \times H$ -varieties. Assume that f is a G -torsor (in particular, G acts trivially on Y). Let M be a coherent H -module over Y . Then $f^*(M)$ has a structure of a coherent $G \times H$ -module over X given by $p^*(\varphi_M)$, where p is the composition of the projection $G \times H \times X \rightarrow H \times X$ and the morphism $\operatorname{id}_H \times f : H \times X \rightarrow H \times Y$.

Thus, there are exact functors

$$f^0 : \mathcal{M}(H; Y) \rightarrow \mathcal{M}(G \times H; X) , \quad M \mapsto p^*(M) ,$$

$$f^0 : \mathcal{P}(H; Y) \rightarrow \mathcal{P}(G \times H; X) , \quad P \mapsto p^*(P) .$$

3

Proposition 3 (Cf. [21, Prop. 6.2]) The functors f^0 are equivalences of categories. In particular, the homomorphisms

$$K'_n(H; Y) \rightarrow K'_n(G \times H; X) ,$$

$$K_n(H; Y) \rightarrow K_n(G \times H; X) ,$$

induced by f^0 , are isomorphisms.

Proof Under the isomorphisms

$$G \times X \xrightarrow{\sim} X \times_Y X, \quad (g, x) \mapsto (gx, x),$$

$$G \times G \times X \xrightarrow{\sim} X \times_Y X \times_Y X, \quad (g, g', x) \mapsto (gg'x, g'x, x)$$

the action morphism θ is identified with the first projection $p_1 : X \times_Y X \rightarrow X$ and the morphisms $m \times \text{id}, \text{id} \times \theta$ are identified with the projections $p_{13}, p_{12} : X \times_Y X \times_Y X \rightarrow X \times_Y X$. Hence, the isomorphism φ giving a G -module structure on a $\mathcal{O}_X$ -module M can be identified with the *descent data*, i.e. with an isomorphism

$$\varphi : p_1^*(M) \xrightarrow{\sim} p_2^*(M)$$

of $\mathcal{O}_{X \times_Y X}$ -modules satisfying the usual cocycle condition

$$p_{23}^*(\varphi) \circ p_{12}^*(\varphi) = p_{13}^*(\varphi).$$

More generally, a $G \times H$ -module structure on M is the descent data commuting with an H -module structure on M . The statement follows now from the theory of faithfully flat descent [13, Prop.2.22].

Example 4. Let $f : X \rightarrow Y$ be a G -torsor and let $\varphi : G \rightarrow \mathbf{GL}(V)$ be a finite dimensional representation. The group G acts linearly on the affine space $\mathbb{A}(V)$ of V , so that the product $X \times \mathbb{A}(V)$ is a G -vector bundle on X . We write E_φ for the vector bundle on Y such that $f^*(E_\varphi) \simeq X \times \mathbb{A}(V)$, i.e., $E_\varphi = G \backslash (X \times \mathbb{A}(V))$. The assignment $\varphi \mapsto E_\varphi$ gives rise to a group homomorphism

$$r : R(G) \rightarrow K_0(Y).$$

Note that the homomorphism r coincides with the composition

$$R(G) \xrightarrow{\sim} K_0(G; \text{Spec } F) \xrightarrow{p^*} K_0(G; X) \xrightarrow{\sim} K_0(Y),$$

where $p : X \rightarrow \text{Spec } F$ is the structure morphism.

Let G be an algebraic group over F and let H be a subgroup of G .

Corollary 5 For every G -variety X , there are natural isomorphisms

$$K_n(G; X \times (G/H)) \simeq K_n(H; X), \quad K'_n(G; X \times (G/H)) \simeq K'_n(H; X).$$

Proof Consider $X \times G$ as a $G \times H$ -variety with the action morphism given by the rule $(g, h) \cdot (x, g') = (hx, gg'h^{-1})$. The statement follows from Proposition 3 applied to the G -torsor $p_2 : X \times G \rightarrow X$ and to the H -torsor $X \times G \rightarrow X \times (G/H)$ given by $(x, g) \mapsto (gx, gH)$.

Let $\varphi : H \rightarrow \mathbf{GL}(V)$ be a finite dimensional representation. Consider G as an H -torsor over G/H with respect to the H -action given by $h * g = gh^{-1}$. The vector bundle $E_\varphi = H \backslash (G \times \mathbb{A}(V))$ constructed in Example 4 has a natural structure of a G -vector bundle. Corollary 5 with $X = \operatorname{Spec} F$ implies:

6 **Corollary 6** The assignment $\varphi \mapsto E_\varphi$ gives rise to an isomorphism $R(H) \xrightarrow{\sim} K_0(G; G/H)$.

7 **Corollary 7** There is a natural isomorphism $K_n(G/H) \xrightarrow{\sim} K_n(H; G)$.

Proof Apply Proposition 3 to the H -torsor $G \rightarrow G/H$.

2.2.3 Basic Results in Equivariant K -Theory

We formulate basic statements in the equivariant algebraic K -theory developed by R. Thomason in [21]. In all of them G is an algebraic group over a field F and X is a G -variety.

Let $Z \subset X$ be a closed G -subvariety and let $U = X \setminus Z$. Since every coherent G -module over U extends to a coherent G -module over X [21, Cor. 2.4], the category $\mathcal{M}(G; U)$ is equivalent to the factor category of $\mathcal{M}(G; X)$ by the subcategory $\mathcal{M}'$ of coherent G -modules supported on Z . By Quillen's devissage theorem [17, §5, Th. 4], the inclusion of categories $\mathcal{M}(G; Z) \subset \mathcal{M}'$ induces an isomorphism $K'_n(G; Z) \xrightarrow{\sim} K'_n(\mathcal{M}')$. The localization in algebraic K -theory [17, §5, Th. 5] yields *connecting homomorphisms*

$$K'_{n+1}(G; U) \xrightarrow{\delta} K'_n(\mathcal{M}') \simeq K'_n(G; Z)$$

and the following:

8 **Theorem 8** [21, Th. 2.7] (Localization) The sequence

$$\dots \rightarrow K'_{n+1}(G; U) \xrightarrow{\delta} K'_n(G; Z) \xrightarrow{i_*} K'_n(G; X) \xrightarrow{j^*} K'_n(G; U) \xrightarrow{\delta} \dots,$$

where $i : Z \rightarrow X$ and $j : U \rightarrow X$ are the embeddings, is exact.

9 **Corollary 9** Let X be a G -variety. Then the natural closed G -embedding $f : X_{\text{red}} \rightarrow X$ induces the isomorphism $f_* : K_n(G; X_{\text{red}}) \rightarrow K_n(G; X)$.

Let X be a G -variety and let E be a G -vector bundle of rank $r + 1$ on X . The projective bundle variety $\mathbb{P}(E)$ has natural structure of a G -variety so that the

natural morphism $p : \mathbb{P}(E) \rightarrow X$ is G -equivariant. We write $\mathcal{L}$ for the G -module of sections of the tautological line bundle on $\mathbb{P}(E)$.

A modification of the Quillen's proof [17, §8] of the standard projective bundle theorem yields:

Theorem 10 [21, Th. 3.1] (Projective bundle theorem) The correspondence

10

$$(a_0, a_1, \dots, a_r) \mapsto \sum_{i=0}^r [\mathcal{L}^{\otimes i}] \otimes p^* a_i$$

induces isomorphisms

$$K_n(G; X)^{r+1} \rightarrow K_n(G; \mathbb{P}(E)), \quad K'_n(G; X)^{r+1} \rightarrow K'_n(G; \mathbb{P}(E)).$$

Let X be a G -variety and let $E \rightarrow X$ be a G -vector bundle on X . Let $f : Y \rightarrow X$ be a torsor under the vector bundle variety E (considered as a group scheme over X) and G acts on Y so that f and the action morphism $E \times_X Y \rightarrow Y$ are G -equivariant. For example, one can take the trivial torsor $Y = E$.

Theorem 11 [21, Th. 4.1] (Strong homotopy invariance property) The pull-back homomorphism

11

$$f^* : K'_n(G; X) \rightarrow K'_n(G; Y)$$

is an isomorphism.

The idea of the proof is construct an exact sequence of G -vector bundles on X :

$$0 \rightarrow E \rightarrow W \xrightarrow{\varphi} \mathbb{A}_X^1 \rightarrow 0,$$

where $\mathbb{A}_X^1$ is the trivial line bundle, such that $\varphi^{-1}(1) \simeq Y$. Thus, Y is isomorphic to the open complement of the projective bundle variety $\mathbb{P}(E)$ in $\mathbb{P}(V)$. Then one uses the projective bundle theorem and the localization to compute the equivariant K' -groups of Y .

Corollary 12 Let $G \rightarrow \mathrm{GL}(V)$ be a finite dimensional representation. Then the projection $p : X \times \mathbb{A}(V) \rightarrow X$ induces the pull-back isomorphism

12

$$p^* : K'_n(G; X) \xrightarrow{\sim} K'_n(G; X \times \mathbb{A}(V)).$$

Let X be a regular G -variety. By [21, Lemma 5.6], every coherent G -module over X is a factor module of a locally free coherent G -module. Therefore, every coherent G -module has a finite resolution by locally free coherent G -modules. The resolution theorem [17, §4, Th. 3] then yields:

13

Theorem 13 [21, Th. 5.7] (Duality for regular varieties) Let X be a regular G -variety over F . Then the canonical homomorphism $K_n(G; X) \rightarrow K'_n(G; X)$ is an isomorphism.

Category $\mathcal{C}(G)$ of G -Equivariant K -Correspondences

2.3

Let G be an algebraic group over a field F and let A be a separable F -algebra, i.e. A is isomorphic to a product of simple algebras with centers separable field extensions of F . An G - A -module over a G -variety X is a G -module M over X which is endowed with the structure of a left $A \otimes_F \mathcal{O}_X$ -module such that the G -action on M is A -linear. An G - A -morphism of G - A -modules is a G -morphism that is also a morphism of $A \otimes_F \mathcal{O}_X$ -modules.

We consider the abelian category $\mathcal{M}(G; X, A)$ of G - A -modules and G - A -morphisms and set

$$K'_n(G; X, A) = K_n(\mathcal{M}(G; X, A)) .$$

The functor $K'_n(G; *, A)$ is contravariant with respect to flat G - A -morphisms and is covariant with respect to projective G - A -morphisms of G -varieties. The category $\mathcal{M}(G; X, F)$ is isomorphic to $\mathcal{M}(G; X)$, and thus it follows that $K'_n(G; X, F) = K'_n(G; X)$.

Consider also the full subcategory $\mathcal{P}(G; X, A)$ of $\mathcal{M}(G; X, A)$ consisting of all G - A -modules which are locally free $\mathcal{O}_X$ -modules. The K -groups of the category $\mathcal{P}(G; X, A)$ are denoted by $K_n(G; X, A)$. The group $K_n(G; X, F)$ coincides with $K_n(G; X)$.

In [15], I. Panin has defined the *category of G -equivariant K -correspondences* $\mathcal{C}(G)$ whose objects are the pairs (X, A) , where X is a smooth projective G -variety over F and A is a separable F -algebra. Morphisms in $\mathcal{C}(G)$ are defined as follows:

$$\mathrm{Mor}_{\mathcal{C}(G)}((X, A), (Y, B)) = K_0(G; X \times Y, A^{\mathrm{op}} \otimes_F B) ,$$

where A^{op} stands for the algebra opposite to A . If $u : (X, A) \rightarrow (Y, B)$ and $v : (Y, B) \rightarrow (Z, C)$ are two morphisms in $\mathcal{C}(G)$, then their composition is defined by the formula

$$v \circ u = p_{13*}(p_{23}^*(v) \otimes_B p_{12}^*(u)) ,$$

where p_{12}, p_{13} and p_{23} are the projections from $X \times Y \times Z$ to $X \times Y, X \times Z$ and $Y \times Z$ respectively. The identity endomorphism of (X, A) in $\mathcal{C}(G)$ is the class $[A \otimes_F \mathcal{O}_\Delta]$, where $\Delta \subset X \times X$ is the diagonal, in the group

$$K'_0(G; X \times X, A^{\mathrm{op}} \otimes_F A) \simeq K_0(G; X \times X, A^{\mathrm{op}} \otimes_F A) = \mathrm{End}_{\mathcal{C}(G)}(X, A) .$$

We will simply write X for (X, F) and A for $(\text{Spec } F, A)$ in $\mathcal{C}(G)$.

The category $\mathcal{C}(G)$ for the trivial group G is simply denoted by $\mathcal{C}$. There is the forgetful functor $\mathcal{C}(G) \rightarrow \mathcal{C}$.

Note that an element $u \in K_0(G; X \times Y, A^{\text{op}} \otimes_F B)$, i.e. a morphism $u : (X, A) \rightarrow (Y, B)$ can be considered also as a morphism $u^{\text{op}} : (Y, B^{\text{op}}) \rightarrow (X, A^{\text{op}})$. Thus, the category $\mathcal{C}(G)$ has the *involution functor* taking (X, A) to (X, A^{op}) .

For every variety Z over F and every $n \in \mathbb{Z}$ we have the *realization functor*

$$\mathcal{K}_n^Z : \mathcal{C}(G) \rightarrow \text{Abelian Groups},$$

taking a pair (X, A) to $K'_n(G; Z \times X, A)$ and a morphism

$$v \in \text{Hom}_{\mathcal{C}(G)}((X, A), (Y, B)) = K_0(G; X \times Y, A^{\text{op}} \otimes_F B)$$

to

$$\mathcal{K}_n^Z(v) : K'_n(G; Z \times X, A) \rightarrow K'_n(G; Z \times Y, B)$$

given by the formula

$$\mathcal{K}_n^Z(v)(u) = v \circ u.$$

Note that we don't need to assume Z neither smooth nor projective to define $\mathcal{K}_n^Z$. We simply write $\mathcal{K}_n$ for $\mathcal{K}_n^{\text{Spec } F}$.

Example 14. Let X be a smooth projective variety over F . The identity $[\mathcal{O}_X] \in K_0(X)$ defines two morphisms $u : X \rightarrow \text{Spec } F$ and $v : \text{Spec } F \rightarrow X$ in $\mathcal{C}$. If $p_*[\mathcal{O}_X] = 1 \in K_0(F)$, where $p : X \rightarrow \text{Spec } F$ is the structure morphism (for example, if X is a projective homogeneous variety), then the composition $u \circ v$ in $\mathcal{C}$ is the identity. In other words, the morphism p splits canonically in $\mathcal{C}$, i.e., the point $\text{Spec } F$ is a canonical "direct summand" of X in $\mathcal{C}$, although X may have no rational points. The application of the resolution functor $\mathcal{K}_n^Z$ for a variety Z over F shows that the group $K'_n(Z)$ is a canonical direct summand of $K'_n(X \times Z)$.

Let G be a split reductive group over a field F with simply connected commutator subgroup and let $B \subset G$ be a Borel subgroup. By [20, Th.1.3], $R(B)$ is a free $R(G)$ -module.

The following statement is a slight generalization of [15, Th. 6.6].

Proposition 15 Let $Y = G/B$ and let $u_1, u_2, \dots, u_m$ be a basis of $R(B) = K_0(G; Y)$ over $R(G)$. Then the element

$$u = (u_i) \in R(B)^m = K_0(G; Y)^m = K_0(G; Y, F^m)$$

defines an isomorphism $F^m \xrightarrow{\sim} Y$ in the category $\mathcal{C}(G)$.

Proof Denote by $p : G/B \rightarrow \operatorname{Spec} F$ the structure morphism. Since G/B is a projective variety, the push-forward homomorphism

$$p_* : R(B) = K_0(G; G/B) \rightarrow K_0(G; \operatorname{Spec} F) = R(G)$$

is well defined. The $R(G)$ -bilinear form on $R(B)$ defined by the formula

$$\langle u, v \rangle_G = p_*(u \cdot v)$$

is unimodular ([6], [15, Th. 8.1.], [11, Prop. 2.17]).

Let $v_1, v_2, \dots, v_m$ be the dual $R(G)$ -basis of $R(B)$ with respect to the unimodular bilinear form. The element $v = (v_i) \in K_0(G; Y, F^m)$ can be considered as a morphism $Y \rightarrow F^m$ in $\mathcal{C}(G)$. The fact that u and v are dual bases is equivalent to the equality $v \circ u = \operatorname{id}$. In order to prove that $u \circ v = \operatorname{id}$ it suffices to show that the $R(G)$ -module $K_0(G, Y \times Y)$ is generated by m^2 elements (see [15, Cor. 7.3]). It is proved in [15, Prop. 8.4] for a simply connected group G , but the proof goes through for a reductive group G with simply connected commutator subgroup.

Equivariant K -Theory of Projective Homogeneous Varieties

2.4

Let G be a semisimple group over a field F . A G -variety X is called *homogeneous* (resp. *projective homogeneous*) if X_{sep} is isomorphic (as a G_{sep} -variety) to G_{sep}/H for a closed (resp. a (reduced) parabolic) subgroup $H \subset G_{\operatorname{sep}}$.

2.4.1 Split Case

Let G be a simply connected split algebraic group over F , let $P \subset G$ be a parabolic subgroup and set $X = G/P$. The center C of G is a finite diagonalizable group scheme and $C \subset P$; we write C^* for the character group of C . For a character $\chi \in C^*$, we say that a representation $\varphi : P \rightarrow \operatorname{GL}(V)$ is χ -homogeneous if the restriction of φ on C is given by multiplication by χ . Let $R(P)^{(\chi)}$ be the subgroup of $R(P)$ generated by the classes of χ -homogeneous representations of P .

By [20, Th.1.3], there is a basis $u_1, u_2, \dots, u_k$ of $R(P)$ over $R(G)$ such that each $u_i \in R(P)^{(\chi_i)}$ for some $\chi_i \in C^*$. As in the proof of Proposition 15, the elements u_i define an isomorphism $u : E \rightarrow X$ in the category $\mathcal{C}(G)$, where $E = F^k$.

For every $i = 1, 2, \dots, k$, choose a representation $\varphi_i : G \rightarrow \operatorname{GL}(V_i)$ such that $[\varphi_i] \in R(G)^{(\chi_i)}$. Consider the vector spaces V_i as G -vector bundles on $\operatorname{Spec} F$ with trivial G -action. The classes of the dual vector spaces

$$v_i = [V_i^*] \in K_0(G; \operatorname{Spec} F, \operatorname{End}(V_i^*))$$

define isomorphisms $v_i : \text{End}(V_i) \rightarrow F$ in $\mathcal{C}(G)$. Let V be the E -module $V_1 \times V_2 \times \cdots \times V_k$. Taking the product of all the v_i we get an isomorphism $v : \text{End}_E(V) \rightarrow E$ in $\mathcal{C}(G)$. The composition $w = u \circ v$ is then an isomorphism $w : \text{End}_E(V) \rightarrow X$.

Now we let the group G act on itself by conjugation, on X by left translations, on w via the representations φ_i . Let $\overline{G} = G/C$ be the adjoint group associated with G . We claim that all the G -actions factor through $\overline{G}$. This is obvious for the actions on G and X . Since the elements u_i are χ_i -homogeneous and the center C acts on V_i^* via φ_i by the character χ_i^{-1} , the class w also admits a $\overline{G}$ -structure.

Quasi-split Case

2.4.2

Let G be a simply connected quasi-split algebraic group over F , let $P \subset G$ be a parabolic subgroup and set $X = G/P$. The absolute Galois group $\Gamma = \text{Gal}(F_{\text{sep}}/F)$ acts naturally on the representation ring $R(P_{\text{sep}})$. By [20, Th.1.3], the basis $u_1, u_2, \dots, u_k \in R(P_{\text{sep}})$ over $R(G_{\text{sep}})$ considered in 2.4.1, can be chosen Γ -invariant. Let E be the étale F -algebra corresponding to the Γ -set of the u_i 's. As in the proof of Proposition 15, the element $u \in K_0(G; X, E)$ defines an isomorphism $u : E \rightarrow X$ in the category $\mathcal{C}(G)$.

Since the group Γ permutes the χ_i defined in 2.4.1, one can choose the representations φ_i whose classes in the representation ring $R(G_{\text{sep}})$ are also permuted by Γ . Hence as in 2.4.1, there is an E -module V and an isomorphism $w : \text{End}_E(V) \rightarrow X$ which admits a $\overline{G}$ -structure.

General Case

2.4.3

Let G be a simply connected algebraic group over F , let X be a projective homogeneous variety of G . Choose a quasi-split inner twisted form G^q of G . The group G is obtained from G^q by twisting with respect to a cocycle γ with coefficients in the quasi-split adjoint group $\overline{G}^q$. Let X^q be the projective homogeneous G^q -variety which is a twisted form of X . As in 2.4.2, find an isomorphism $w^q : \text{End}_E(V) \rightarrow X^q$ in $\mathcal{C}(G^q)$ for a certain étale F -algebra E and an E -module V . Note that all the structures admit $\overline{G}^q$ -operators. Twisting by the cocycle γ we get an isomorphism $w : A \rightarrow X$ in $\mathcal{C}(G)$ for a separable F -algebra A with center E . We have proved

Theorem 16 (Cf. [15, Th. 12.4]) Let G be a simply connected group over a field F and let X be a projective homogeneous G -variety. Then there exist a separable F -algebra A and an isomorphism $A \simeq X$ in the category $\mathcal{C}(G)$. In particular, $K_*(G; X) \simeq K_*(G; A)$ and $K_*(X) \simeq K_*(A)$.

16

Corollary 17 The restriction homomorphism $K_0(G; X) \rightarrow K_0(X)$ is surjective.

17

Proof The statement follows from the surjectivity of the restriction homomorphism $K_0(G; A) \rightarrow K_0(A)$.

We will generalize Corollary 17 in Theorem 39.

2.5

***K*-Theory of Toric Varieties**

Let a torus T act on a normal geometrically irreducible variety X defined over a field F . The variety X is called a *toric T -variety* if there is an open orbit which is a principal homogeneous space of T . A toric T -variety is called a *toric model of T* if the open orbit has a rational point. A choice of a rational point x in the open orbit gives an open T -equivariant embedding $T \hookrightarrow X, t \mapsto tx$.

2.5.1

***K*-Theory of Toric Models**

We will need the following:

18

Proposition 18 [12, Proposition 5.6] Let X be a smooth toric T -model defined over a field F . Then there is a torus S over F , an S -torsor $\pi : U \rightarrow X$ and an S -equivariant open embedding of U into an affine space $\mathbb{A}$ over F on which S acts linearly.

19

Remark 19 It turns out that the canonical homomorphism $S_{\text{sep}}^* \rightarrow \text{Pic}(X_{\text{sep}})$ is an isomorphism, so that $\pi : U \rightarrow X$ is the *universal torsor* in the sense of [2, 2.4.4]. Thus, the Proposition 18 asserts that the universal torsor of X can be equivariantly imbedded into an affine space as an open subvariety.

Let $\varphi : S \rightarrow \text{GL}(V)$ be a representation over F . Suppose that there is an action of an étale F -algebra A on V commuting with the S -action. Then A acts on the vector bundle E_φ (see Example 4), therefore, E_φ defines an element $u_\varphi \in K_0(X, A)$, i.e., a morphism $u_\varphi : A \rightarrow X$ in $\mathcal{C}$. The composition

$$K_0(A) \xrightarrow{\alpha_\varphi} R(S) \xrightarrow{r} K_0(X),$$

where r is defined in Example 4 and α_φ is induced by the exact functor $M \mapsto M \otimes_A V$, is given by the rule $x \mapsto u_\varphi \circ x$.

Let φ be an irreducible representation. Since S is a torus, φ is the corestriction in a finite separable field extension L_φ/F of a 1-dimensional representation of S . Thus, there is an action of L_φ on V that commutes with the S -action. Note that the element u_φ defined above is represented by an element of the Picard group $\text{Pic}(X \otimes_F L_\varphi)$.

Now we consider two irreducible representations ϱ and μ of the torus S over F , and apply the construction described above to the torus $S \times S$ and its representation

$$\varrho \otimes \mu : S \times S \rightarrow \mathbf{GL}(V_\varrho \otimes_F V_\mu) .$$

The composition

$$K_0(L_\varrho \otimes_F L_\mu) \xrightarrow{\alpha_{\varrho,\mu}} R(S \times S) \xrightarrow{r} K_0(X \times X)$$

coincides with the map

$$x \mapsto u_\varrho^{\text{op}} \circ x \circ u_\mu ,$$

where the composition is taken in $\mathcal{C}$ and $u_\mu : X \rightarrow L_\mu$, $u_\varrho^{\text{op}} : L_\varrho \rightarrow X$, $x : L_\mu \rightarrow L_\varrho$ are considered as the morphisms in $\mathcal{C}$.

Now let Φ be a finite set of irreducible representations of S . Set

$$A = \prod_{\varrho \in \Phi} L_\varrho, \quad u = \sum_{\varrho \in \Phi} u_\varrho, \quad \alpha = \sum_{\varrho, \mu \in \Phi} \alpha_{\varrho, \mu} .$$

The element u_ϱ is represented by an element of the Picard group $\text{Pic}(X \otimes_F A)$.

Then the composition

$$K_0(A \otimes_F A) \xrightarrow{\alpha} R(S \times S) \xrightarrow{r} K_0(X \times X)$$

is given by the rule $x \mapsto u^{\text{op}} \circ x \circ u$, where u is considered as a morphism $X \rightarrow A$.

The homomorphism r coincides with the composition

$$R(S \times S) = K_0(S \times S; \text{Spec } F) \xrightarrow{\sim} K_0(S \times S; \mathbb{A} \times \mathbb{A}) \rightarrow$$

$$K_0(S \times S; U \times U) = K_0(X \times X)$$

and hence r is surjective. By the representation theory of algebraic tori, the sum of all the $\alpha_{\varrho,\mu}$ is an isomorphism. It follows that for sufficiently large (but finite!) set Φ of irreducible representations of S the identity $\text{id}_X \in K_0(X \times X)$ belongs to the image of $r \circ \alpha$. In other words, there exists $x \in K_0(A \otimes_F A)$ such that $u^{\text{op}} \circ x \circ u = \text{id}_X$ in $\mathcal{C}$, i.e. $v = u^{\text{op}} \circ x$ is a left inverse to $u : X \rightarrow A$ in $\mathcal{C}$. We have proved the following:

Theorem 20 [12, Th. 5.7] Let X be a smooth projective toric model of an algebraic torus defined over a field F . Then there exist an étale F -algebra A and elements $u, v \in K_0(X, A)$ such that the composition $X \xrightarrow{u} A \xrightarrow{v} X$ in $\mathcal{C}$ is the identity and u is represented by a class in $\text{Pic}(X \otimes_F A)$.

20

K -Theory of Toric Varieties

2.5.2

Let T be a torus over F . The natural G -equivariant bilinear map

$$T(F_{\text{sep}}) \otimes T_{\text{sep}}^* \rightarrow F_{\text{sep}}^\times, \quad x \otimes \chi \mapsto \chi(x)$$

induces a pairing of the Galois cohomology groups

$$H^1(F, T(F_{\text{sep}})) \otimes H^1(F, T_{\text{sep}}^*) \rightarrow H^2(F, F_{\text{sep}}^\times) = \text{Br}(F),$$

where $\text{Br}(F)$ is the Brauer group of F . There is a natural isomorphism $\text{Pic}(T) \simeq H^1(F, T_{\text{sep}}^*)$ (see [23]). A principal homogeneous T -space U defines an element $[U] \in H^1(F, T(F_{\text{sep}}))$. Therefore, the pairing induces the homomorphism

$$\lambda^U : \text{Pic}(T) \rightarrow \text{Br}(F), \quad [Q] \mapsto [U] \cup [Q].$$

Let X be a toric variety of the torus T with the open orbit U which is a principal homogeneous space over T .

21 Theorem 21 [12, Th. 7.6] Let Y be a smooth projective toric variety over a field F . Then there exist an étale F -algebra A , a separable F -algebra B of rank n^2 over its center A and morphisms $u : Y \rightarrow B$, $v : B \rightarrow Y$ in $\mathcal{C}$ such that $v \circ u = \text{id}$. The morphism u is represented by a locally free $\mathcal{O}_Y$ -module in $\mathcal{P}(Y, B)$ of rank n . The class of the algebra B in $\text{Br}(A)$ belongs to the image of $\lambda^{U_A} : \text{Pic}(T_A) \rightarrow \text{Br}(A)$.

22 Corollary 22 The homomorphism $\mathcal{K}_n(u) : K_n(X) \rightarrow K_n(A)$ identifies $K_n(X)$ with the direct summand of $K_n(A)$ which is equal to the image of the projector $\mathcal{K}_n(u \circ v) : K_n(A) \rightarrow K_n(A)$. In particular, $K_0(X)$ is a free abelian group of finite rank.

Equivariant K -Theory of Solvable Algebraic Groups

We consider separately the equivariant K -theory of unipotent groups and algebraic tori.

2.6.1 Split Unipotent Groups

A unipotent group U is called *split* if there is a chain of subgroups of U with the subsequent factor groups isomorphic to the additive group G_a . For example, the unipotent radical of a Borel subgroup of a (quasi-split) reductive group is split.

23 Theorem 23 Let U be a split unipotent group and let X be a U -variety. Then the restriction homomorphism $K'_n(U; X) \rightarrow K'_n(X)$ is an isomorphism.

Proof Since U is split, it is sufficient to prove that for a subgroup $U' \subset U$ with $U/U' \simeq G_a$, the restriction homomorphism $K'_n(U; X) \rightarrow K'_n(U'; X)$ is an isomorphism. By Corollary 5, this homomorphism coincides with the pull-back $K'_n(U; X) \rightarrow K'_n(U; X \times G_a)$ with respect to the projection $X \times G_a \rightarrow X$, that is an isomorphism by the homotopy invariance property (Corollary 12).

Split Algebraic Tori

2.6.2

Let T be a split torus over a field F . Choose a basis $\chi_1, \chi_2, \dots, \chi_r$ of the character group T^* . We define an action of T on the affine space $\mathbb{A}^r$ by the rule $t \cdot x = y$ where $y_i = \chi_i(t)x_i$. Write H_i ($i = 1, 2, \dots, r$) for the coordinate hyperplane in $\mathbb{A}^r$ defined by the equation $x_i = 0$. Clearly, H_i is a closed T -subvariety in $\mathbb{A}^r$ and $T = \mathbb{A}^r - \bigcup_{i=1}^r H_i$. For every subset $I \subset \{1, 2, \dots, r\}$ set $H_I = \bigcap_{i \in I} H_i$.

In [8], M. Levine has constructed a spectral sequence associated to a family of closed subvarieties of a given variety. This sequence generalizes the localization exact sequence. We adapt this sequence to the equivariant algebraic K -theory and also change the indices making this spectral sequence of homological type.

Let X be a T -variety over F . The family of closed subsets $Z_i = X \times H_i$ in $X \times \mathbb{A}^r$ gives then a spectral sequence

$$E_{p,q}^1 = \coprod_{|I|=p} K'_q(T; X \times H_I) \Rightarrow K'_{p+q}(T; X \times T).$$

By Corollary 5, the group $K'_{p+q}(T; X \times T)$ is isomorphic to $K'_{p+q}(X)$.

In order to compute $E_{p,q}^1$, note that H_I is an affine space over F , hence the pull-back $K'_q(T; X) \rightarrow K'_q(T; X \times H_I)$ is an isomorphism by the homotopy invariance property (Corollary 12). Thus,

$$E_{p,q}^1 = \coprod_{|I|=p} K'_q(T; X) \cdot e_I$$

and by [8, p.419], the differential map $d : E_{p+1,q}^1 \rightarrow E_{p,q}^1$ is given by the formula

$$d(x \cdot e_I) = \sum_{k=0}^p (-1)^k (1 - \chi_{i_k}^{-1}) x \cdot e_{I - \{i_k\}}, \quad (2.1)$$

where $I = \{i_0 < i_1 < \dots < i_p\}$.

Consider the Koszul complex C_* built upon the free $R(T)$ -module $R(T)^r$ and the system of the elements $1 - \chi_i^{-1} \in R(T)$. More precisely,

$$C_p = \coprod_{|I|=p} R(T) \cdot e_I$$

and the differential $d : C_{p+1} \rightarrow C_p$ is given by the rule formally coinciding with (2.1), where $x \in R(T)$.

The representation ring $R(T)$ is the group ring over the character group T^* . The Koszul complex gives the resolution $C_* \rightarrow \mathbb{Z} \rightarrow 0$ of $\mathbb{Z}$ by free $R(T)$ -modules, where we view $\mathbb{Z}$ as a $R(T)$ -module via the *rank* homomorphism $R(T) \rightarrow \mathbb{Z}$ taking the class of a representation to its dimension. It follows from (2.1) that the complex $E_{*,q}^1$ coincides with

$$C_* \otimes_{R(T)} K'_q(T; X) .$$

Hence, being the homology group of $E_{*,q}^1$, the term $E_{p,q}^2$ is equal to

$$\mathrm{Tor}_p^{R(T)}(\mathbb{Z}, K'_q(T; X)) .$$

We have proved:

24 Proposition 24 Let T be a split torus over a field F and let X be a T -variety. Then there is a spectral sequence

$$E_{p,q}^2 = \mathrm{Tor}_p^{R(T)}(\mathbb{Z}, K'_q(T; X)) \Rightarrow K'_{p+q}(X) .$$

We are going to prove that if X is smooth projective, the spectral sequence degenerates.

Let G be an algebraic group and let $H \subset G$ be a subgroup. Suppose that there exists a group homomorphism $\pi : G \rightarrow H$ such that $\pi|_H = \mathrm{id}_H$. For a smooth projective G -variety X we write $\check{X}$ for the variety X together with the new G -action $g * x = \pi(g)x$.

25 Lemma 25 If the restriction homomorphism $K_0(G; \check{X} \times X) \rightarrow K_0(H; X \times X)$ is surjective, then the restriction homomorphism $K'_n(G; X) \rightarrow K'_n(H; X)$ is a split surjection.

Proof Since the restriction map

$$\mathrm{res}_{G/H} : \mathrm{Hom}_{\mathcal{C}(G)}(\check{X}, X) = K_0(G; \check{X} \times X) \rightarrow$$

$$K_0(H; X \times X) = \mathrm{Hom}_{\mathcal{C}(H)}(X, X)$$

is surjective, there is $v \in \mathrm{Hom}_{\mathcal{C}(G)}(\check{X}, X)$ such that $\mathrm{res}_{G/H}(v) = \mathrm{id}_X$ in $\mathcal{C}(G)$.

Consider the diagram

$$\begin{array}{ccccc} K'_n(H; X) & \xrightarrow{\mathrm{res}_\pi} & K'_n(G; \check{X}) & \xrightarrow{\mathcal{K}_n(v)} & K'_n(G; X) \\ & & \mathrm{res}_{G/H} \downarrow & & \mathrm{res}_{G/H} \downarrow \\ & & K'_n(H; X) & = & K'_n(H; X) \end{array} ,$$

where the square is commutative since $\mathrm{res}_{G/H}(v) = \mathrm{id}_X$. The equality $\mathrm{res}_{G/H} \circ \mathrm{res}_\pi = \mathrm{id}$ implies that the composition in the top row splits the restriction homomorphism $K'_n(G; X) \rightarrow K'_n(H; X)$.

Let T be a split torus over F , and let $\chi \in T^*$ be a character such that $T^*/(\mathbb{Z} \cdot \chi)$ is a torsion-free group. Then $T' = \ker(\chi)$ is a subtorus in T . Denote by $\pi : T \rightarrow T'$ a splitting of the embedding $T' \hookrightarrow T$.

Proposition 26 Let X be a smooth projective T -variety. Then the restriction homomorphism $K'_n(T; X) \rightarrow K'_n(T'; X)$ is a split surjection. 26

Proof We use the notation $\dot{X}$ as above. Since $T/T' \simeq \mathbf{G}_m$, by Corollary 12, Corollary 5 and the localization (Theorem 8), we have the surjection

$$K'_0(T; \dot{X} \times X) \xrightarrow{\sim} K'_0(T; \dot{X} \times X \times \mathbb{A}_F^1) \twoheadrightarrow K'_0(T; \dot{X} \times X \times \mathbf{G}_m) \simeq K'_0(T'; X \times X)$$

which is nothing but the restriction homomorphism. The statement follows from Lemma 25.

Corollary 27 The sequence 27

$$0 \rightarrow K'_n(T; X) \xrightarrow{1-\chi} K'_n(T; X) \xrightarrow{\text{res}} K'_n(T'; X) \rightarrow 0$$

is split exact.

Proof We consider $X \times \mathbb{A}_F^1$ as a T -variety with respect to the T -action on $\mathbb{A}_F^1$ given by the character χ . In the localization exact sequence

$$\dots \rightarrow K'_n(T; X) \xrightarrow{i_*} K'_n(T; X \times \mathbb{A}_F^1) \xrightarrow{j^*} K'_n(T; X \times \mathbf{G}_m) \xrightarrow{\delta} \dots,$$

where $i : X = X \times \{0\} \hookrightarrow X \times \mathbb{A}_F^1$ and $j : X \times \mathbf{G}_m \hookrightarrow X \times \mathbb{A}_F^1$ are the embeddings, the second term is identified with $K'_n(T; X)$ by Corollary 12 and the third one with $K'_n(T'; X)$ since $T/T' \simeq \mathbf{G}_m$ as T -varieties (Corollary 5). With these identifications, j^* is the restriction homomorphism which is a split surjection by Proposition 26. By the projection formula, i_* is the multiplication by $i_*(1)$. Let t be the coordinate of $\mathbb{A}^1$. It follows from the exactness of the sequence of T -modules over $X \times \mathbb{A}_F^1$:

$$0 \rightarrow \mathcal{O}_{X \times \mathbb{A}^1}[\chi^{-1}] \xrightarrow{t} \mathcal{O}_{X \times \mathbb{A}^1} \rightarrow i_*(\mathcal{O}_X) \rightarrow 0$$

that $i_*(1) = 1 - \chi^{-1}$.

Proposition 28 Let T be a split torus and let X be a smooth projective T -variety. Then the spectral sequence in Proposition 24 degenerates, i.e., 28

$$\text{Tor}_p^{R(T)}(\mathbb{Z}, K'_n(T; X)) = \begin{cases} K'_n(X), & \text{if } p = 0, \\ 0, & \text{if } p > 0. \end{cases}$$

Proof Let $\chi_1, \chi_2, \dots, \chi_r$ be a $\mathbb{Z}$ -basis of the character group T^* . Since $R(T)$ is a Laurent polynomial ring in the variables χ_i , and by Corollary 27, the elements $1 - \chi_i \in R(T)$ form a $R(T)$ -regular sequence, the result follows from [19, IV-7].

2.6.3 Quasi-trivial Algebraic Tori

An algebraic torus T over a field F is called *quasi-trivial* if the character Galois module T_{sep}^* is permutation. In other words, T is isomorphic to the torus $\text{GL}_1(C)$ of invertible elements of an étale F -algebra C . The torus $T = \text{GL}_1(C)$ is embedded as an open subvariety of the affine space $\mathbb{A}(C)$. By the classical homotopy invariance and localization, the pull-back homomorphism

$$\mathbb{Z} \cdot 1 = K_0(\mathbb{A}(C)) \rightarrow K_0(T)$$

is surjective. We have proved

29 Proposition 29 For a quasi-trivial torus T , one has $K_0(T) = \mathbb{Z} \cdot 1$.

We generalize this statement in Theorem 30.

2.6.4 Coflasque Algebraic Tori

An algebraic torus T over F is called *coflasque* if for every field extension L/F the Galois cohomology group $H^1(L, T_{\text{sep}}^*)$ is trivial, or equivalently, if $\text{Pic}(T_L) = 0$. For example, quasi-trivial tori are coflasque.

30 Theorem 30 Let T be a coflasque torus and let U be a principal homogeneous space of T . Then $K_0(U) = \mathbb{Z} \cdot 1$.

Proof Let X be a smooth projective toric model of T (for the existence of X see [1]). The variety $Y = T \backslash (X \times U)$ is then a toric variety of T that has an open orbit isomorphic to U .

By Theorem 21, there exist an étale F -algebra A , a separable F -algebra B of rank n^2 over its center A and morphisms $u : Y \rightarrow B$, $v : B \rightarrow Y$ in $\mathcal{C}$ such that $v \circ u = \text{id}$. The morphism u is represented by a locally free $\mathcal{O}_Y$ -module in $\mathcal{P}(Y, B)$ of rank n . The class of the algebra B in $\text{Br}(A)$ belongs to the image of $\lambda^{U_A} : \text{Pic}(T_A) \rightarrow \text{Br}(A)$. The torus T is coflasque, hence the group $\text{Pic}(T_A)$ is trivial and therefore, the algebra B splits, $B \simeq M_n(A)$, so that $K_0(B^{\text{op}})$ is isomorphic canonically to $K_0(A)$.

Applying the realization functor to the morphism $u^{\text{op}} : B^{\text{op}} \rightarrow Y$ we get a (split) surjection

$$\mathcal{K}_0(u^{\text{op}}) : K_0(B^{\text{op}}) \rightarrow K_0(Y) .$$

Under the identification of $K_0(B^{\text{op}})$ with $K_0(A)$ we get a (split) surjection

$$\mathcal{K}_0(w^{\text{op}}) : K_0(A) \rightarrow K_0(Y) ,$$

where w is a certain element in $K_0(Y, A)$ represented by a locally free $\mathcal{O}_Y$ -module of rank one, i.e., by an element of $\text{Pic}(Y \otimes_F A)$.

It follows that $K_0(Y)$ is generated by the push-forwards of the classes of $\mathcal{O}_Y$ -modules from $\text{Pic}(Y_E)$ for all finite separable field extensions E/F . Since the pull-back homomorphism $K_0(Y) \rightarrow K_0(U)$ is surjective, the analogous statement holds for the open subset $U \subset Y$. But by [18, Prop. 6.10], there is an injection $\text{Pic}(U_E) \hookrightarrow \text{Pic}(T_E) = 0$, hence $\text{Pic}(U_E) = 0$ and therefore $K_0(U) = \mathbb{Z} \cdot 1$.

Equivariant K -Theory of some Reductive Groups

2.7

Spectral Sequence

2.7.1

Let G be a split reductive group over a field F . Choose a maximal split torus $T \subset G$.

Let X be a G -variety. The group $K'_n(G; X)$ (resp. $K'_n(T; X)$) is a module over the representation ring $R(G)$ (resp. $R(T)$). The restriction map $K'_n(G; X) \rightarrow K'_n(T; X)$ is an homomorphism of modules with respect to the restriction ring homomorphism $R(G) \rightarrow R(T)$ and hence it induces an $R(T)$ -module homomorphism

$$\eta : R(T) \otimes_{R(G)} K'_n(G; X) \rightarrow K'_n(T; X) .$$

Proposition 31 Assume that the commutator subgroup of G is simply connected. Then the homomorphism η is an isomorphism.

31

Proof Let $B \subset G$ be a Borel subgroup containing T . Set $Y = G/B$. By Proposition 15, there is an isomorphism $u : F^m \xrightarrow{\sim} Y$ in the category $\mathcal{C}(G)$ defined by some elements $u_1, u_2, \dots, u_m \in K_0(G; Y) = R(B)$ that form a basis of $R(B)$ over $R(G)$. Applying the realization functor (see Sect. 2.3)

$$\mathcal{K}_n^X : \mathcal{C}(G) \rightarrow \text{Abelian Groups} ,$$

to the isomorphism u , we obtain an isomorphism

$$K'_n(G; X)^m \xrightarrow{\sim} K'_n(G; X \times Y) .$$

Identifying $K'_n(G; X)^m$ with $R(B) \otimes_{R(G)} K'_n(G; X)$ using the same elements u_i we get a canonical isomorphism

$$R(B) \otimes_{R(G)} K'_n(G; X) \xrightarrow{\sim} K'_n(G; X \times Y) .$$

Composing this isomorphism with the canonical isomorphism (Corollary 5)

$$K'_n(G; X \times Y) \xrightarrow{\sim} K'_n(B; X) ,$$

and identifying $K'_n(B; X)$ with $K'_n(T; X)$ via the restriction homomorphism (Theorem 23) we get the isomorphism η .

Since $R(T)$ is free $R(G)$ -module by [20, Th.1.3], in the assumptions of Proposition 31 one has

$$\mathrm{Tor}_p^{R(G)}(\mathbb{Z}, K'_n(G; X)) \simeq \mathrm{Tor}_p^{R(T)}(\mathbb{Z}, K'_n(T; X)) , \quad (2.2)$$

where we view $\mathbb{Z}$ as a $R(G)$ -module via the *rank* homomorphism $R(G) \rightarrow \mathbb{Z}$.

Proposition 24 then yields:

32 Theorem 32 [11, Th. 4.3] Let G be a split reductive group defined over F with the simply connected commutator subgroup and let X be a G -variety. Then there is a spectral sequence

$$E_{p,q}^2 = \mathrm{Tor}_p^{R(G)}(\mathbb{Z}, K'_q(G; X)) \Rightarrow K'_{p+q}(X) .$$

33 Corollary 33 The restriction homomorphism $K'_0(G; X) \rightarrow K'_0(X)$ induces an isomorphism $\mathbb{Z} \otimes_{R(G)} K'_0(G; X) \simeq K'_0(X)$.

In the smooth projective case, Proposition 28 and (2.2) give the following generalization of Corollary 33:

34 Corollary 34 If X is a smooth projective G -variety, then the spectral sequence in Theorem 32 degenerates. i.e.,

$$\mathrm{Tor}_p^{R(G)}(\mathbb{Z}, K'_n(G; X)) = \begin{cases} K'_n(X) , & \text{if } p = 0 , \\ 0 , & \text{if } p > 0 . \end{cases}$$

2.7.2 K -Theory of Simply Connected Group

The following technical statement is very useful.

35 Proposition 35 Let G be an algebraic group over F and let $f : X \rightarrow Y$ be a G -torsor over F . For every point $y \in Y$ let X_y be the fiber $f^{-1}(y)$ of f over y (so that X_y is a principal homogeneous space of G over the residue field $F(y)$). Assume that $K_0(X_y) = \mathbb{Z} \cdot 1$ for every point $y \in Y$. Then the restriction homomorphism $K'_0(Y) \simeq K'_0(G; X) \rightarrow K'_0(X)$ is surjective.

Proof We prove that the restriction homomorphism $\text{res}^X : K'_0(G; X) \rightarrow K'_0(X)$ is surjective by induction on the dimension of X . Assume that we have proved the statement for all varieties of dimension less than the dimension of X . We would like to prove that res^X is surjective.

We prove this statement by induction on the number of irreducible components of Y . Suppose first that Y is irreducible. By Corollary 9, we may assume that Y is reduced.

Let $y \in Y$ be the generic point and let $v \in K'_0(X)$. Since $K'_0(X_y) = K_0(X_y) = \mathbb{Z} \cdot 1$, the restriction homomorphism $K'_0(G; X_y) \rightarrow K'_0(X_y)$ is surjective. It follows that there exists a non-empty open subset $U' \subset Y$ such that the pull-back of v in $K'_0(U)$, where $U = f^{-1}(U')$, belongs to the image of the restriction homomorphism $K'_0(G; U) \rightarrow K'_0(U)$. Set $Z = X \setminus U$ (considered as a reduced closed subvariety of X). Since $\dim(Z) < \dim(X)$ and $Z \rightarrow Y \setminus U'$ is a G -torsor, by the induction hypothesis, the left vertical homomorphism in the commutative diagram with the exact rows

$$\begin{array}{ccccccc} K'_0(G; Z) & \xrightarrow{i_*} & K'_0(G; X) & \xrightarrow{j^*} & K'_0(G; U) & \rightarrow & 0 \\ \text{res}^Z \downarrow & & \text{res}^X \downarrow & & \text{res}^U \downarrow & & \\ K'_0(Z) & \xrightarrow{i_*} & K'_0(X) & \xrightarrow{j^*} & K'_0(U) & \rightarrow & 0 \end{array}$$

is surjective. Hence, by diagram chase, $v \in \text{im}(\text{res}^X)$.

Now let Y be an arbitrary variety. Choose an irreducible component Z' of Y and set $Z = f^{-1}(Z')$, $U = X \setminus Z$. The number of irreducible components of U is less than one of X . By the first part of the proof and the induction hypothesis, the homomorphisms res_Z and res_U in the commutative diagram above are surjective. It follows that res^X is also surjective.

I. Panin has proved in [16] that for a principal homogeneous space X of a simply connected group of inner type, $K_0(X) = \mathbb{Z} \cdot 1$. In the next statement we extend this result to arbitrary simply connected groups (and later in Theorem 38 - to factorial groups).

Proposition 36 Let G be a simply connected group and let X be a principal homogeneous space of G . Then $K_0(X) = \mathbb{Z} \cdot 1$.

36

Proof Suppose first that G is a quasi-split group. Choose a maximal torus T of a Borel subgroup B of G . A fiber of the projection $f : T \backslash X \rightarrow B \backslash X$ is isomorphic to the unipotent radical of B and hence is isomorphic to an affine space. By [17, §7, Prop. 4.1], the pull-back homomorphism

$$f^* : K_0(B \backslash X) \rightarrow K_0(T \backslash X)$$

is an isomorphism.

The character group T^* is generated by the fundamental characters and therefore, T^* is a permutation Galois module, so that T is a quasi-trivial torus. Every principal homogeneous space of T is trivial, hence by Propositions 29 and 35, the restriction homomorphism

$$K_0(T \backslash X) = K_0(T; X) \rightarrow K_0(X)$$

is surjective. Thus, the pull-back homomorphism $g^* : K_0(B \backslash X) \rightarrow K_0(X)$ with respect to the projection $g : X \rightarrow B \backslash X$ is surjective.

Let G_1 be the algebraic group of all G -automorphisms of X . Over F_{sep} , the groups G and G_1 are isomorphic, so that G_1 is a simply connected group. The variety X can be viewed as a G_1 -torsor [10, Prop. 1.2]. In particular, $B \backslash X$ is a projective homogeneous variety of G_1 .

In the commutative diagram

$$\begin{array}{ccc} K_0(G_1; B \backslash X) & \rightarrow & K_0(G_1; X) \\ \text{res} \downarrow & & \text{res} \downarrow \\ K_0(B \backslash X) & \xrightarrow{g^*} & K_0(X) \end{array}$$

the left vertical homomorphism is surjective by Corollary 17. Since g^* is also surjective, so is the right vertical restriction. It follows from Proposition 3 that

$$K_0(G_1; X) = K_0(\text{Spec } F) = \mathbb{Z} \cdot 1,$$

hence, $K_0(X) = \mathbb{Z} \cdot 1$.

Now let G be an arbitrary simply connected group. Consider the projective homogeneous variety Y of all Borel subgroups of G . For every point $y \in Y$, the group $G_{F(y)}$ is quasi-split. The fiber of the projection $X \times Y \rightarrow Y$ over y is the principal homogeneous space $X_{F(y)}$ of $G_{F(y)}$. By the first part of the proof, $K_0(X_{F(y)}) = \mathbb{Z} \cdot 1$. Hence by Proposition 35, the pull-back homomorphism

$$K_0(Y) \rightarrow K_0(X \times Y)$$

is surjective. It follows from Example 14 that the natural homomorphism $\mathbb{Z} \cdot 1 = K_0(F) \rightarrow K_0(X)$ is a direct summand of this surjection and therefore, is surjective. Therefore, $K_0(X) = \mathbb{Z} \cdot 1$.

2.8

Equivariant K -Theory of Factorial Groups

An algebraic group G over a field F is called *factorial* if for any finite field extension E/F the Picard group $\text{Pic}(G_E)$ is trivial.

37

Proposition 37 [11, Prop. 1.10] A reductive group G is factorial if and only if the commutator subgroup G' of G is simply connected and the torus G/G' is coflasque.

In particular, simply connected groups and coflasque tori are factorial.

Theorem 38 Let G be a factorial group and let X be a principal homogeneous space of G . Then $K_0(X) = \mathbb{Z} \cdot 1$.

38

Proof Let G' be the commutator subgroup of G and let $T = G/G'$. The group G' is simply connected and the torus T is coflasque. The variety X is a G' -torsor over $Y = G' \backslash X$. By Propositions 35 and 36, the restriction homomorphism

$$K_0(Y) = K_0(G'; X) \rightarrow K_0(X)$$

is surjective. The variety Y is a principal homogeneous space of T and by Theorem 30, $K_0(Y) = \mathbb{Z} \cdot 1$, whence the result.

Theorem 39 [11, Th. 6.4] Let G be a reductive group defined over a field F . Then the following condition are equivalent:

39

1. G is factorial.
2. For every G -variety X , the restriction homomorphism

$$K'_0(G; X) \rightarrow K'_0(X)$$

is surjective.

Proof (1) $\Rightarrow$ (2). Consider first the case when there is a G -torsor $X \rightarrow Y$. Then the restriction homomorphism $K_0(G; X) \rightarrow K_0(X)$ is surjective by Proposition 35 and Theorem 38.

In the general case, choose a faithful representation $G \hookrightarrow S = \mathrm{GL}(V)$. Let $\mathbb{A}$ be the affine space of the vector space $\mathrm{End}(V)$ so that S is an open subvariety in $\mathbb{A}$. Consider the commutative diagram

$$\begin{array}{ccccc} K'_0(G; X) & \xrightarrow{\sim} & K'_0(G; \mathbb{A} \times X) & \rightarrow & K'_0(G; S \times X) \\ \text{res} \downarrow & & \text{res} \downarrow & & \text{res} \downarrow \\ K'_0(X) & \xrightarrow{\sim} & K'_0(\mathbb{A} \times X) & \rightarrow & K'_0(S \times X) \end{array} .$$

The group G acts freely on $S \times X$ so that we have a G -torsor $S \times X \rightarrow Y$. In fact, Y exists in the category of algebraic spaces and may not be a variety. One should use the equivariant K' -groups of algebraic spaces as defined in [21]. By the first part of the proof, the right vertical map is surjective. By localization, the right horizontal arrows are the surjections. Finally, the composition in the bottom row is an isomorphism since it has splitting $K'_0(S \times X) \rightarrow K'_0(X)$ by the pull-back with respect to the closed embedding $X = \{1\} \times X \hookrightarrow S \times X$ of finite Tor-dimension (see [17, §7, 2.5]). Thus, the left vertical restriction homomorphism is surjective.

(2) $\Rightarrow$ (1). Taking $X = G_E$ for a finite field extension E/F , we have a surjective homomorphism

$$\mathbb{Z} \cdot 1 = K_0(E) = K_0(G; G_E) \rightarrow K_0(G_E) ,$$

i.e. $K_0(G_E) = \mathbb{Z} \cdot 1$. Hence, the first term of the topological filtration $K_0(G_E)^{(1)}$ of $K_0(G_E)$ (see [17, §7.5]), that is the kernel of the rank homomorphism $K_0(G_E) \rightarrow \mathbb{Z}$, is trivial. The Picard group $\text{Pic}(G_E)$ is a factor group of $K_0(G_E)^{(1)}$ and hence is also trivial, i.e., G is a factorial group.

In the end of the section we consider the smooth projective case.

40

Theorem 40 [11, Th. 6.7] Let G be a factorial reductive group and let X be a smooth projective G -variety over F . Then the restriction homomorphism

$$K'_n(G; X) \rightarrow K'_n(X)$$

is split surjective.

Proof Consider the smooth variety $X \times X$ with the action of G given by $g(x, x') = (x, gx')$. By Theorem 39, the restriction homomorphism $K'_0(G; X \times X) \rightarrow K'_0(X \times X)$ is surjective. Hence by Lemma 25, applied to the trivial subgroup of G , the restriction homomorphism $K'_n(G; X) \rightarrow K'_n(X)$ is a split surjection.

2.9

Applications

2.9.1

K -Theory of Classifying Varieties

Let G be an algebraic group over a field F . Choose a faithful representation $\mu : G \hookrightarrow \text{GL}_n$ and consider the factor variety $X = \text{GL}_n / \mu(G)$. For every field extension E/F , the set $H^1(E, G)$ of isomorphism classes of principal homogeneous spaces of G over E can be identified with the orbit space of the action of $\text{GL}_n(E)$ on $X(E)$ [7, Cor. 28.4]:

$$H^1(E, G) = \text{GL}_n(E) \backslash X(E) .$$

The variety X is called a *classifying variety* of G . The $\text{GL}_n(E)$ -orbits in the set $X(E)$ classify principal homogeneous spaces of G over E .

We can compute the Grothendieck ring of a classifying variety X of G . M. Rost used this result for the computation of orders of the Rost's invariants (see [5]). As shown in Example 4, the G -torsor $\text{GL}_n \rightarrow X$ induces the homomorphism $r : R(G) \rightarrow K_0(X)$ taking the class of a finite dimensional representation $\varphi : G \rightarrow \text{GL}(V)$ to the class the vector bundle E_φ .

41

Theorem 41 Let X be a classifying variety of an algebraic group G . The homomorphism r gives rise to an isomorphism

$$\mathbb{Z} \otimes_{R(\text{GL}_n)} R(G) \simeq K_0(X) .$$

In particular, the group $K_0(X)$ is generated by the classes of the vector bundles E_ϱ for all finite dimensional representations ϱ of G over F .

Proof The Corollary 33 applied to the smooth $\mathbf{GL}_n$ -variety X yields an isomorphism

$$\mathbb{Z} \otimes_{R(\mathbf{GL}_n)} K_0(\mathbf{GL}_n; X) \simeq K_0(X) .$$

On the other hand,

$$K_0(\mathbf{GL}_n; X) \simeq R(G)$$

by Corollary 6.

Note that the structure of the representation ring of an algebraic group is fairly well understood in terms of the associated root system and indices of the Tits algebras of G (see [22], [5, Part 2, Th. 10.11]).

Equivariant Chow Groups

2.9.2

For a variety X over a field F we write $\mathrm{CH}_i(X)$ for the *Chow group* of equivalence classes of dimension i cycles on X [4, I.1.3]. Let G be an algebraic group over F . For X a G -variety, D. Edidin and W. Graham have defined in [3] the *equivariant Chow groups* $\mathrm{CH}_i^G(X)$. There is an obvious *restriction* homomorphism

$$\mathrm{res} : \mathrm{CH}_i^G(X) \rightarrow \mathrm{CH}_i(X) .$$

Theorem 42 Let X be a G -variety of dimension d , where G is a factorial group. Then the restriction homomorphism

42

$$\mathrm{res} : \mathrm{CH}_{d-1}^G(X) \rightarrow \mathrm{CH}_{d-1}(X)$$

is surjective.

Proof The proof is essentially the same as the one of Theorem 39. We use the homotopy invariance property and localization for the equivariant Chow groups. In the case of a torsor the proof goes the same lines as in Proposition 35. The only statement to check is the triviality of $\mathrm{CH}^1(Y) = \mathrm{Pic}(Y)$ for a principal homogeneous space Y of G . By [18, Prop. 6.10], the group $\mathrm{Pic}(Y)$ is isomorphic to a subgroup of $\mathrm{Pic}(G)$, which is trivial since G is a factorial group.

Let $\mathrm{Pic}^G(X)$ denote the group of line G -bundles on X . If X is smooth irreducible, the natural homomorphism $\mathrm{Pic}^G(X) \rightarrow \mathrm{CH}_{d-1}^G(X)$ is an isomorphism [3, Th. 1].

43 **Corollary 43** (Cf. [14, Cor. 1.6]) Let X be a smooth G -variety, where G is a factorial group. Then the restriction homomorphism

$$\mathrm{Pic}^G(X) \rightarrow \mathrm{Pic}(X)$$

is surjective. In other words, every line bundle on X has a structure of a G -vector bundle.

2.9.3 Group Actions on the K' -Groups

Let G be an algebraic group and let X be a G -variety over F . For every element $g \in G(F)$ write λ_g for the automorphism $x \mapsto gx$ of X . The group $G(F)$ acts naturally on $K'_n(X)$ by the pull-back homomorphisms λ_g^* .

44 **Theorem 44** [11, Prop.7.20] Let G be a reductive group and let X be a G -variety. Then

1. The group $G(F)$ acts trivially on $K'_0(X)$.
2. If X is smooth and projective, the group $G(F)$ acts trivially on $K'_n(X)$ for every $n \geq 0$.

Proof By [11, Lemma 7.6], there exists an exact sequence

$$1 \rightarrow P \rightarrow \tilde{G} \xrightarrow{\pi} G \rightarrow 1$$

with a factorial reductive group $\tilde{G}$ and a quasi-trivial torus P . It follows from the exactness of the sequence

$$\tilde{G}(F) \xrightarrow{\pi(F)} G(F) \rightarrow H^1(F, P(F_{\mathrm{sep}}))$$

and triviality of $H^1(F, P(F_{\mathrm{sep}}))$ (Hilbert Theorem 90) that the homomorphism $\pi(F) : \tilde{G}(F) \rightarrow G(F)$ is surjective. Hence, we can replace G by $\tilde{G}$ and assume that G is factorial.

By definition of a G -module M , the isomorphism

$$\varphi : \theta^*(M) \xrightarrow{\sim} p_2^*(M),$$

where $\theta : G \times X \rightarrow X$ is the action morphism, induces an isomorphism of two compositions $\theta^* \circ \mathrm{res}$ and $p_2^* \circ \mathrm{res}$ in the diagram

$$\mathcal{M}(G; X) \xrightarrow{\mathrm{res}} \mathcal{M}(X) \xrightarrow[\substack{\theta^* \\ p_2^*}]{\theta^*} \mathcal{M}(G \times X).$$

Hence the compositions

$$K'_n(G; X) \xrightarrow{\mathrm{res}} K'_n(X) \xrightarrow[\substack{\theta^* \\ p_2^*}]{\theta^*} K'_n(G \times X)$$

are equal.

For any $g \in G(F)$ write ε_g for the morphism $X \rightarrow G \times X$, $x \mapsto (g, x)$. Then clearly $p_2 \circ \varepsilon_g = \text{id}_X$ and $\theta \circ \varepsilon_g = \lambda_g$. The pull-back homomorphism ε_g^* is defined since ε_g is of finite Tor-dimension [17, §7, 2.5]. Thus, we have $\varepsilon_g^* \circ p_2^* = \text{id}$ and $\varepsilon_g^* \circ \theta^* = \lambda_g^*$ on $K'_n(X)$, hence

$$\text{res} = \varepsilon_g^* \circ p_2^* \circ \text{res} = \varepsilon_g^* \circ \theta^* \circ \text{res} = \lambda_g^* \circ \text{res} : K'_n(G; X) \rightarrow K'_n(X) .$$

By Theorem 39, the restriction homomorphism res is surjective for $n = 0$, hence $\lambda_g^* = \text{id}$. In the case of smooth projective X the restriction is surjective for every $n \geq 0$ (Theorem 40), hence again $\lambda_g^* = \text{id}$.

Corollary 45 Let G be a reductive group and let X be a smooth G -variety. Then the group $G(F)$ acts trivially on $\text{Pic}(X)$.

45

Proof The Picard group $\text{Pic}(X)$ is isomorphic to a subfactor of $K_0(X)$ and $G(F)$ acts trivially on $K_0(X)$ by Theorem 44.

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K(1)-Local Homotopy Theory, Iwasawa Theory and Algebraic K-Theory

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3.1	<i>Introduction</i>	957
3.2	<i>The Iwasawa Algebra</i>	960
	Definition of Λ and Λ'	960
	Modules over the Iwasawa Algebra	961
	Tate Twisting	963
	Modules over Λ'	963
3.3	<i>K(1)-local Homotopy Theory</i>	965
	ℓ -Adic Completion of Spectra	966
	ℓ -Adic Topological K-Theory	968
	K(1)-Localization	968
	The $\widehat{K}$ -Based Adams Spectral Sequence	969
	The Structure of $L_{K(1)}\mathcal{S}$: General Results	972
	Iwasawa Theory for $\widehat{K}$ -Finite Spectra	974
3.4	<i>Iwasawa Theory</i>	976
	Notation	976
	Iwasawa Theory for the Cyclotomic Extension	979
3.5	<i>Algebraic K-Theory Spectra</i>	986
3.6	<i>K-Theoretic Interpretation of some Conjectures in Iwasawa Theory</i>	988
	Conjectures Concerning the Semi-discrete Primes	988

* Supported by a grant from the National Science Foundation.

	Iwasawa's μ -Invariant Conjecture	993
	Totally Real Fields	993
3.7	<i>The K-Theory of $\mathbb{Z}$</i>	998
3.8	<i>Homotopy-type of KR at the Prime 2</i>	1003
	The Construction JKR	1003
	The 2-Adic Lichtenbaum–Quillen Conjecture for $\mathbb{Z}[1/2]$	1004
	The 2-Adic Lichtenbaum–Quillen Conjecture for General $\mathbb{R}$	1006
	<i>References</i>	1008

Introduction

The *Iwasawa algebra* Λ is a power series ring $Z_\ell[[T]]$, ℓ a fixed prime. It arises in number theory as the pro-group ring of a certain Galois group, and in homotopy theory as a ring of operations in ℓ -adic complex K -theory. Furthermore, these two incarnations of Λ are connected in an interesting way by algebraic K -theory. The main goal of this paper is to explore this connection, concentrating on the ideas and omitting most proofs.

Let F be a number field. Fix a prime ℓ and let F_∞ denote the ℓ -adic cyclotomic tower – that is, the extension field formed by adjoining the ℓ^n -th roots of unity for all $n \geq 1$. The central strategy of Iwasawa theory is to study number-theoretic invariants associated to F by analyzing how these invariants change as one moves up the cyclotomic tower. Number theorists would in fact consider more general towers, but we will be concerned exclusively with the cyclotomic case. This case can be viewed as analogous to the following geometric picture: Let X be a curve over a finite field $\mathbb{F}$, and form a tower of curves over X by extending scalars to the algebraic closure of $\mathbb{F}$, or perhaps just to the ℓ -adic cyclotomic tower of $\mathbb{F}$. This analogy was first considered by Iwasawa, and has been the source of many fruitful conjectures.

As an example of a number-theoretic invariant, consider the norm inverse limit A_∞ of the ℓ -torsion part of the class groups in the tower. Then A_∞ is a profinite module over the pro-group ring $\Lambda'_F = Z_\ell[[G(F_\infty/F)]]$. Furthermore, $\Lambda'_F = \Lambda_F[\Delta_F]$, where $\Lambda_F \cong \Lambda$, and Δ_F is a cyclic group of order dividing $\ell - 1$ (if ℓ is odd) or dividing 2 (if $\ell = 2$). The beautiful fact about the Iwasawa algebra is that finitely-generated modules over it satisfy a classification theorem analogous to the classification theorem for modules over a principal ideal domain. The difference is that isomorphisms must be replaced by pseudo-isomorphisms; these are the homomorphisms with finite kernel and cokernel. Then the game is to see how modules such as A_∞ fit into this classification scheme. For a survey of Iwasawa theory, see [13].

The Iwasawa algebra also arises in homotopy theory. Let $\mathcal{K}$ denote the periodic complex K -theory spectrum, and $\widehat{\mathcal{K}}$ denote its ℓ -adic completion in the sense of Bousfield. Then the ring of degree zero operations $[\widehat{\mathcal{K}}, \widehat{\mathcal{K}}]$ is isomorphic to $\Lambda' = \Lambda[\Delta]$, where Λ is again a power series ring over Z_ℓ , and Δ is cyclic of order $\ell - 1$ (if ℓ is odd) or of order 2 (if $\ell = 2$). The isomorphism comes about by regarding Λ' as the pro-group ring of the group of ℓ -adic Adams operations. Hence the classification theory for Λ -modules can be applied to $\widehat{\mathcal{K}} X = \widehat{\mathcal{K}}^0 X \oplus \widehat{\mathcal{K}}^1 X$, at least when X is a spectrum with $\widehat{\mathcal{K}} X$ finitely-generated over Λ .

One can go further by passing to $L_{K(1)}\mathcal{S}$, the Bousfield $K(1)$ -localization of the stable homotopy category $\mathcal{S}$. This is a localized world in which all spectra with vanishing $\widehat{\mathcal{K}}$ have been erased. The category $L_{K(1)}\mathcal{S}$ is highly algebraic; for example, for ℓ odd its objects are determined up to a manageable ambiguity by $\widehat{\mathcal{K}} X$ as Λ' -module. This suggests studying $L_{K(1)}\mathcal{S}$ from the perspective of Iwasawa theory. Call the objects X with $\widehat{\mathcal{K}} X$ finitely-generated as Λ -module $\widehat{\mathcal{K}}$ -finite. The $\widehat{\mathcal{K}}$ -finite

objects are the ones to which Iwasawa theory directly applies; they can be characterized as the objects whose homotopy groups are finitely-generated Z_ℓ -modules, and as the objects that are weakly dualizable in the sense of axiomatic stable homotopy theory. Within this smaller category there is a notion of pseudo-equivalence in $L_{K(1)}\mathcal{S}$ analogous to pseudo-isomorphism for Λ -modules, an analogous classification theorem for objects, and an Iwasawa-theoretic classification of the thick subcategories [15].

Algebraic K-theory provides a link from the number theory to the homotopy theory. Let $R = \mathcal{O}_F[\frac{1}{\ell}]$, and let KR denote the algebraic K-theory spectrum of R . By deep work of Thomason [44], the famous Lichtenbaum–Quillen conjectures can be viewed as asserting that the ℓ -adic completion $KR^\wedge$ is essentially $K(1)$ -local, meaning that for some $d \geq 0$ the natural map $KR^\wedge \rightarrow L_{K(1)}KR$ induces an isomorphism on π_n for $n \geq d$; here $d = 1$ is the expected value for number rings. Since the Lichtenbaum–Quillen conjectures are now known to be true for $\ell = 2$, by Voevodsky’s work on the Milnor conjecture, Rognes–Weibel [41] and Østvær [39], it seems very likely that they are true in general. In any case, it is natural to ask how $L_{K(1)}KR$ fits into the classification scheme alluded to above.

The first step is to compute $\widehat{\mathcal{K}} KR$. Let M_∞ denote the Galois group of the maximal abelian ℓ -extension of F_∞ that is unramified away from ℓ . We call M_∞ the *basic Iwasawa module*.

Theorem 1 [9,33] Let ℓ be any prime. Then there are isomorphisms of Λ' -modules

$$\widehat{\mathcal{K}}^n KR \cong \begin{cases} \Lambda' \otimes_{\Lambda'_F} Z_\ell & \text{if } n = 0 \\ \Lambda' \otimes_{\Lambda'_F} M_\infty & \text{if } n = -1. \end{cases}$$

For ℓ odd this theorem depends on Thomason [44]. For $\ell = 2$ it depends on the work of Rognes–Weibel and Østvær cited above. Theorem 1 leads to a complete description of the homotopy-type $L_{K(1)}KR$, and hence also $KR^\wedge$ in cases where the Lichtenbaum–Quillen conjecture is known.

It is known that M_∞ is a finitely-generated Λ_F -module. Many famous conjectures in number theory can be formulated in terms of its Iwasawa invariants, including the Leopoldt conjecture, the Gross conjecture, and Iwasawa’s μ -invariant conjecture. Consequently, all of these conjectures can be translated into topological terms, as conjectures about the structure of $\widehat{\mathcal{K}} KR$ or the homotopy-type of $L_{K(1)}KR$.

As motivation for making such a translation, we recall a theorem of Soulé. One of many equivalent forms of Soulé’s theorem says that A_∞ contains no negative Tate twists of Z_ℓ . This is a purely number-theoretic assertion. The only known proof, however, depends in an essential way on higher K-theory and hence on homotopy theory. It reduces to the fact that the homotopy groups $\pi_{2n}KR$ are finite for $n > 0$. There are at least two different ways of proving this last assertion – one can work with either the plus construction for $BGL(R)$, or Quillen’s Q -construction – but they both ultimately reduce to finiteness theorems for general linear group homology

due to Borel, Borel–Serre, and Raghunathan. These are in essence analytic results. One can therefore view Soule’s theorem as a prototype of the strategy:

$$\begin{aligned} (\text{analytic input}) &\Rightarrow (\text{estimates on the homotopy-type } KR) \\ &\Rightarrow (\text{number-theoretic output}) \end{aligned}$$

Notice, however, that only the bare homotopy groups of KR have been exploited to prove Soule’s theorem. The homotopy-type of KR contains far more information, since it knows everything about the basic Iwasawa module M_∞ . For example, the homotopy groups alone cannot decide the fate of the algebraic Gross conjecture; as explained in Sect. 3.6.1, this is a borderline case one step beyond Soule’s theorem, for which one additional bit of structure would be needed. For Iwasawa’s μ -invariant conjecture, on the other hand, knowledge of π_*KR alone is of little use. But certain crude estimates on the homotopy-type of KR would suffice, and the conjecture is equivalent to the assertion that $\widehat{\mathcal{K}}^{-1}KR$ is ℓ -torsionfree.

There is a curious phenomenon that arises here. By its definition, the spectrum KR has no homotopy groups in negative degrees. But $K(1)$ -localizations are never connective, and much of the number theory is tied up in the *negative* homotopy groups of $L_{K(1)}KR$. For example, the Leopoldt conjecture is equivalent to the finiteness of $\pi_{-2}L_{K(1)}KR$. Even the Gross conjecture mentioned above involves the part of $\pi_0L_{K(1)}KR$ that doesn’t come from KR , and indirectly involves $\pi_{-1}L_{K(1)}S^0 \cong Z_\ell$. It is tempting to think of KR as a sort of homotopical L -function, with $L_{K(1)}KR$ as its analytic continuation and with functional equation given by some kind of Artin–Verdier–Brown–Comenetz duality. (Although in terms of the generalized Lichtenbaum conjecture on values of ℓ -adic L -functions at integer points – see Sect. 3.6 – the values at negative integers are related to *positive* homotopy groups of $L_{K(1)}KR$, while the values at positive integers are related to the negative homotopy groups!) Speculation aside, Theorem 1.1 shows that all of these conjectures are contained in the structure of $\widehat{\mathcal{K}}^{-1}KR$.

Another goal of this paper is to explain some examples of the actual or conjectural homotopy-type KR in cases where it can be determined more or less explicitly. For example, if one assumes not only the Lichtenbaum–Quillen conjecture, but also the Kummer–Vandiver conjecture, then one can give a fairly explicit description of the ℓ -adic homotopy-type of $K\mathbb{Z}$. Most interesting of all at present is the case $\ell = 2$, since in that case the Lichtenbaum–Quillen conjecture itself is now known. Then $KR^\wedge$ can be described completely in Iwasawa-theoretic terms; and if the Iwasawa theory is known one obtains an explicit description of the homotopy-type $KR^\wedge$.

Organization of the paper: Sect. 3.2 introduces the theory of modules over the Iwasawa algebra. Sect. 3.3 is an overview of $K(1)$ -local homotopy theory, including the $\widehat{\mathcal{K}}$ -based Adams spectral sequence and the structure of the category $L_{K(1)}\mathcal{S}$. Here we take the viewpoint of axiomatic stable homotopy theory, following [17, 18]. But we also refine this point of view by exploiting the Iwasawa algebra [15]. In Sect. 3.4 we study the Iwasawa theory of the ℓ -adic cyclotomic tower of a number

field, using a novel étale homotopy-theoretic approach due to Bill Dwyer. Sect. 3.5 is a brief discussion of algebraic K-theory spectra and the Lichtenbaum–Quillen conjectures for more general schemes. Although our focus in this paper is on $\mathcal{O}_F[\frac{1}{\ell}]$, it will be clear from the discussion that many of the ideas apply in a much more general setting. In Sect. 3.6 we explain some standard conjectures in number theory, and show how they can be reinterpreted homotopically in terms of the spectrum KR . We also discuss the analytic side of the picture; that is, the connection with L-functions. In particular, we state a generalized Lichtenbaum conjecture on special values of L-functions, and prove one version of it. Sect. 3.7 is a study of the example $K\mathbb{Z}$. Finally, Sect. 3.8 is devoted to KR at the prime 2. Here we start from scratch with $K\mathbb{Z}[\frac{1}{2}]$ and Bökstedt’s $JK\mathbb{Z}$ construction, and then study KR in general.

Acknowledgements: I would like to thank Ethan Devinatz and John Palmieri for helpful conversations. I would also like to thank Ralph Greenberg for many helpful conversations and tutorials on number theory; any errors or misconceptions that remain are, of course, the responsibility of the author alone.

3.2 The Iwasawa Algebra

In this section we introduce the Iwasawa algebra Λ and some basic properties of its modules, as well as the related algebra Λ' . General references for this material include [38] and [46]. We also define Tate twists, and discuss some properties of modules over Λ' in the case $\ell = 2$ that are not so well known.

3.2.1 Definition of Λ and Λ'

Let Γ' denote the automorphism group of $\mathbb{Z}/\ell^\infty$. Thus Γ' is canonically isomorphic to the ℓ -adic units $Z_\ell^\times$, with the isomorphism $c : \Gamma' \rightarrow Z_\ell^\times$ given by $\gamma(x) = c(\gamma)x$ for $\gamma \in \Gamma'$, $x \in \mathbb{Z}/\ell^\infty$. In fact, if A is any abelian group isomorphic to $\mathbb{Z}/\ell^\infty$, we again have a canonical isomorphism $\Gamma' \cong \text{Aut } A$, defined in the same way. In particular, Γ' is canonically identified with the automorphism group of the group of ℓ -power roots of unity in any algebraically closed field of characteristic different from ℓ . Note also that there is a unique product decomposition $\Gamma' = \Gamma \times \Delta$, where Γ corresponds under c to the units congruent to 1 mod ℓ (resp. 1 mod 4) if ℓ is odd (resp. $\ell = 2$), and Δ corresponds to the $\ell - 1$ -st roots of unity (resp. ± 1) if ℓ is odd (resp. $\ell = 2$). The restriction of c to Δ is denoted ω and called the *Teichmüller character*.

We write Λ for $Z_\ell[[\Gamma]]$ and Λ' for $Z_\ell[[\Gamma']]$. Let γ_0 be a topological generator of Γ . To be specific, we take $c(\gamma_0) = 1 + \ell$ if ℓ is odd, and $c(\gamma_0) = 5$ if $\ell = 2$. Then it was observed by Serre that there is an isomorphism of profinite rings

$$Z_\ell[[\Gamma]] \cong Z_\ell[[T]] ,$$

such that $\gamma_0 \mapsto T + 1$. Note that $\Lambda' = \Lambda[\Delta]$.

The ring Λ is a regular noetherian local domain of Krull dimension two. In particular, it has global dimension two, and every module over it admits a projective resolution of length at most two. It is also complete with respect to its maximal ideal $\mathcal{M} = (\ell, T)$, with residue field $\mathbb{Z}/\ell$, and therefore is a profinite topological ring.

The height one prime ideals are all principal, and are of two types. First there is the prime ℓ , which plays a special role. Second, there are the prime ideals generated by the irreducible distinguished polynomials $f(T)$. Here a polynomial $f \in \mathbb{Z}_\ell[T]$ is *distinguished* if it is monic and $f(T) = T^n \bmod \ell$, $n = \deg f$. Note that each height one prime is now equipped with a canonical generator; we will occasionally not bother to distinguish between the ideal and its generator.

Modules over the Iwasawa Algebra

3.2.2

An *elementary cyclic module* is a Λ -module that is either free of rank one or of the form Λ/q^i , where q is either ℓ or an irreducible distinguished polynomial. A finitely-generated Λ -module E is *elementary* if it is a direct sum of elementary cyclic modules. The primes q and exponents i that appear are uniquely determined by E , up to ordering.

A *pseudo-isomorphism* is a homomorphism of Λ -modules with finite kernel and cokernel. The main classification theorem then reads:

Theorem 2 Let M be a finitely-generated Λ -module. Then there is an elementary module E and a pseudo-isomorphism $\phi : M \rightarrow E$. Up to isomorphism, E is uniquely determined by M .

2

Remark 3 If M is a Λ -torsion module, then one can also find a pseudo-isomorphism $\psi : E \rightarrow M$. In general, however, this isn't true; if M is Λ -torsionfree but not free, there is no pseudo-isomorphism from a free module to M .

3

Recall that the *support* of a module M , denoted $\text{Supp } M$, is the set of primes q such that $M_q \neq 0$. Note that $\text{Supp } M$ is *closed under specialization*, i.e., if $q \in \text{Supp } M$ and $q \subset q'$, then $q' \in \text{Supp } M$. For example, the nonzero finite modules are the modules with support $\{\mathcal{M}\}$, while the Λ -torsion modules are the modules with $(0) \notin \text{Supp } M$. Now let N be a finitely-generated torsion module and let $q_1, \dots, q_m$ be the height one primes in $\text{Supp } M$. These are, of course, precisely the q 's that appear in the associated elementary module E . Thus we can write

$$E = E_1 \oplus \dots \oplus E_m \qquad E_i = \bigoplus_{j=1}^{r_i} \Lambda/q_i^{s_{ij}}.$$

The data $(q_1; s_{11}, \dots, s_{1r_1}), \dots, (q_m; s_{m1}, \dots, s_{mr_m})$ constitute the *torsion invariants* of N . The *multiplicity* of q_i in N is $n_i = \sum_j s_{ij}$, the *frequency* is r_i , and the *lengths* are the exponents s_{ij} . The *divisor* of N is the formal sum $\mathcal{D}(N) = \sum n_i q_i$. Sometimes we

write $\langle q, N \rangle$ for the multiplicity of q in N . In the case of the prime ℓ , the multiplicity is also denoted $\mu(N)$. If M is an arbitrary finitely-generated Λ -module, the above terms apply to its Λ -torsion submodule tM .

Thus one can study finitely-generated Λ -modules M up to several increasingly coarse equivalence relations:

- Up to isomorphism. This can be difficult, but see for example [20].
- Up to pseudo-isomorphism. At this level M is determined by its Λ -rank and torsion invariants.
- Up to divisors. If we don't know the torsion invariants of M , it may still be possible to determine the divisor $\sum n_i q_i$ of tM . This information can be conveniently packaged in a *characteristic series* for tM ; that is, an element $g \in \Lambda$ such that $g = u \prod q_i^{n_i}$ for some unit $u \in \Lambda$.
- Up to support. Here we ask only for the Λ -rank of M and the support of tM .

We mention a few more interesting properties of Λ -modules; again, see [38] for details and further information.

A finitely-generated Λ -module M has a unique maximal finite submodule, denoted M^0 .

4 **Proposition 4** M has projective dimension at most one if and only if $M^0 = 0$.

Now let M^* denote the Λ -dual $\text{Hom}_\Lambda(M, \Lambda)$.

5 **Proposition 5** If M is a finitely-generated Λ -module, then M^* is a finitely-generated free module. Furthermore, the natural map $M \rightarrow M^{**}$ has kernel tM and finite cokernel.

We call the cokernel of the map $M \rightarrow M^{**}$ the *freeness defect* of M ; it is zero if and only if M/tM is free. In fact:

6 **Proposition 6** Suppose N is finitely-generated and Λ -torsionfree, and let $N \rightarrow F$ be any pseudo-isomorphism to a free module (necessarily injective). Then F/N is Pontrjagin dual to $\text{Ext}_\Lambda^1(N, \Lambda)$, and is isomorphic to the freeness defect of N .

In particular, then, F/N is independent of F and the choice of pseudo-isomorphism.

Call a finitely-generated Λ -module L *semi-discrete* if Γ acts discretely on L . Thus L is finitely-generated as Z_ℓ -module with Γ_n , the unique subgroup of index ℓ^n of Γ , acting trivially on L for some n . The semi-discrete modules fit into the pseudo-isomorphism theory as follows: Let $\omega_n = (1 + T)^{\ell^n} - 1$. Then $\omega_n = v_0 v_1 \dots v_n$ for certain irreducible distinguished polynomials v_i ; in fact, the v_i 's are just the cyclotomic polynomials. We call these the *semi-discrete* primes because they are pulled back from $Z_\ell[\mathbb{Z}/\ell^n]$ under the map induced by the natural epimorphism

from Γ to the discrete group $\mathbb{Z}/\ell^n$. Clearly L is semi-discrete if and only if it has support in the semi-discrete primes and each v_i occurs with length at most one. Finally, note that for any finitely-generated Λ -module M , the ascending chain

$$M^\Gamma \subset M^{\Gamma_1} \subset M^{\Gamma_2} \subset \dots$$

terminates. Hence M has a maximal semi-discrete submodule M^δ , with $M^\delta = M^{\Gamma_n}$ for $n \gg 0$.

Tate Twisting

3.2.3

Let M be a Λ -module. The n -th Tate twist $M(n)$ has the same underlying Z_ℓ -module, but with the Γ -action twisted by the rule

$$\gamma \cdot x = c(\gamma)^n \gamma x.$$

If Z_ℓ has trivial Γ -action, then clearly

$$M(n) = M \otimes_{Z_\ell} (Z_\ell(n))$$

as Λ -modules. Thinking of Λ as the power series ring $Z_\ell[[T]]$, we can interpret this twisting in another way. Given any automorphism ϕ of Λ – where we mean automorphism as topological ring – any module M can be twisted to yield a new module M_ϕ in which $\lambda \cdot x = \phi(\lambda)x$. In particular, any linear substitution $T \mapsto cT + d$ with $c, d \in Z_\ell$, c a unit and $d = 0 \bmod \ell$ defines such an automorphism. Among these we single out the case $c = c_0$, $d = c_0 - 1$. This automorphism will be denoted τ and called the *Tate automorphism*, the evident point being that $M(1) = M_\tau$.

Note that τ permutes the height one primes. In particular, we write $\tau_n = \tau^n(T)$. Written as an irreducible distinguished polynomial, $\tau_n = (T - (c_0^n - 1))$. These are the *Tate primes*, which will play an important role in the sequel.

Modules over Λ'

3.2.4

Suppose first that ℓ is odd. Then the entire theory of finitely-generated Λ -modules extends in a straightforward way to Λ' -modules. The point is that $\Lambda' = \Lambda[\Delta]$, and Δ is finite of order prime to ℓ . Hence Λ' splits as a direct product of topological rings into $\ell - 1$ copies of Λ , and similarly for its category of modules. Explicitly, there are idempotents $e_i \in Z_\ell[\Delta] \subset \Lambda'$, $0 \leq i \leq \ell - 2$, such that for any module M we have $M = \bigoplus e_i M$ as Λ' -modules, with Δ acting on $e_i M$ as ω^i . Hence we can apply the structure theory to the summands $e_i M$ independently.

We also have $\text{Spec } \Lambda' = \coprod_{i=0}^{\ell-2} \text{Spec } \Lambda$. If $q \in \text{Spec } \Lambda$ and $n \in \mathbb{Z}$, we write (q, n) for the prime q in the n -th summand, n of course being interpreted modulo $\ell - 1$. Tate-twisting is defined as before, interpreting Λ' as $Z_\ell[[\Gamma']]$. In terms of the product decomposition above, this means that τ' permutes the factors by $(q, n) \mapsto (\tau(q), n+1)$. The *extended Tate primes* are defined by $\tau'_n = (\tau_n, n)$. Sometimes we will want to twist the Δ -action while leaving the Γ -action alone; we call this Δ -twisting.

Semi-discrete primes are defined just as for Λ . Note that the semi-discrete primes are invariant under Δ -twisting.

If $\ell = 2$, the situation is considerably more complicated, and indeed does not seem to be documented in the literature. The trouble is that now Δ has order 2. Thus we still have that Λ' is a noetherian, profinite, local ring, but it does not split as a product of Λ 's, and modules over it can have infinite projective dimension. Letting σ denote the generator of Δ , we will sometimes use the notation e_n as above for the elements $1 + \sigma$ (n even) or $1 - \sigma$ (n odd), even though these elements are not idempotent. The maximal ideal is $(2, T, 1 - \sigma)$, and there are two minimal primes $(1 \pm \sigma)$. The prime ideal spectrum can be written as a pushout

$$\begin{array}{ccc} \mathrm{Spec} \Lambda/2 & \rightarrow & \mathrm{Spec} \Lambda \\ \downarrow & & \downarrow \\ \mathrm{Spec} \Lambda & \rightarrow & \mathrm{Spec} \Lambda' \end{array}.$$

The extended Tate twist τ' is still simple enough; it merely acts by τ combined with an exchange of the two $\mathrm{Spec} \Lambda$ factors.

The pseudo-isomorphism theory is more complicated. For modules with vanishing μ -invariant, we have the following:

7

Theorem 7 Let M be a finitely-generated Λ' -module with $\mu(M) = 0$. Then M is pseudo-isomorphic to a module of the form

$$E = E^+ \oplus E^- \oplus E^f,$$

where E^+ (resp. E^-) is an elementary Λ -module supported on the distinguished polynomials, with σ acting trivially (resp. acting as -1), and E^f is Λ -free.

This is far from a complete classification, however, since we need to analyze E^f . Let Λ^l denote Λ with σ acting as -1 , and let Λ unadorned denote Λ with the trivial σ action. Next, let L_n denote $\Lambda \oplus \Lambda$ with σ acting by the matrix

$$\begin{pmatrix} 1 & T^n \\ 0 & -1 \end{pmatrix}$$

and let L_n^* denote its Λ -dual. Note that L_0 is free over Λ' of rank one, and hence $L_0 \cong L_0^*$, but with that exception no two of these modules are isomorphic.

We pause to remark that these modules L_n, L_n^* arise in nature, in topology as well as in number theory. In topology the module L_1^* occurs as the 2-adic topological K-theory of the cofibre of the unit map $S^0 \rightarrow \widehat{\mathcal{K}}$; that is, the first stage of an Adams resolution for the sphere (see Sect. 3.3.4 below). The modules L_1, L_2 occur in the Iwasawa theory of 2-adic local fields; for example, the torsion-free quotient of the basic Iwasawa module M_∞ associated to $\mathbb{Q}_2$ is isomorphic to L_2 (see [36]).

Theorem 8 Let N be a finitely-generated Λ -free Λ' -module. Then N is isomorphic as Λ' -module to a direct sum of modules of the form Λ , Λ^I , L_n ($n \geq 0$), and L_n^* ($n \geq 1$), with the number of summands of each type uniquely determined by N . Furthermore

- (a) Such modules N of fixed Λ -rank are classified by their Tate homology groups $H_*(\sigma; N)$, regarded as modules over the principal ideal domain $\Lambda/2$;
- (b) Any pseudo-isomorphism between two such modules is an isomorphism.

This gives a complete classification up to pseudo-isomorphism in the case $\mu = 0$. The proof of Theorem 8 given in [36] is *ad hoc*; it would be nice to have a more conceptual proof.

$K(1)$ -local Homotopy Theory

K -theoretic localization has been studied extensively by Bousfield, who indeed invented the subject. Many of the results in this section either come directly from Bousfield's work or are inspired by it. The most important references for us are [3, 4] and [5]; we will not attempt to give citations for every result below.

We will also take up the viewpoint of axiomatic stable homotopy theory, following Hovey, Palmieri and Strickland [18]. Further motivation comes from the elegant Hovey–Strickland memoir [17], in which $K(n)$ -localization for arbitrary n is studied. In the latter memoir the case $n = 1$ would be regarded as the trivial case; even in the trivial case, however, many interesting and nontrivial things can be said. In particular we will use Iwasawa theory to study the finer structure of $L_{K(1)}\mathcal{S}$. Although the connection with Iwasawa theory has been known since the beginning [4] [40], it seems not to have been exploited until now.

We begin in Sect. 3.3.1 with a discussion of ℓ -adic completion. All of our algebraic functors will take values in the category of $\text{Ext-}\ell$ -complete abelian groups, so we have included a brief introduction to this category. The short Sect. 3.3.2 introduces $\widehat{\mathcal{K}}$ and connects its ring of operations with the Iwasawa algebra. Some important properties of $K(1)$ -localization are summarized in Sect. 3.3.3.

Section 3.3.4 discusses the $\widehat{\mathcal{K}}$ -based Adams spectral sequence. In fact the spectral sequence we use is the so-called modified Adams spectral sequence, in a version for which the homological algebra takes place in the category of compact Λ' -modules.

Section 3.3.5 studies the structure of the category $L_{K(1)}\mathcal{S}$, beginning with some remarks concerning the small, dualizable, and weakly dualizable objects of $L_{K(1)}\mathcal{S}$. This last subcategory is the category in which we will usually find ourselves; it coincides with the thick subcategory generated by $\widehat{\mathcal{K}}$ itself, and with the subcategory of objects X such that $\widehat{\mathcal{K}}X$ is a finitely-generated Λ' -module. Most of the results here are special cases of results from [17], and ultimately depend on the work of Bousfield cited above. We also briefly discuss the Picard group of

$L_{K(1)}\mathcal{S}$ [16]. In Sect. 3.3.6, we summarize some new results from [15], with ℓ odd, including a spectrum analogue of the classification of Iwasawa modules, and an Iwasawa-theoretic classification of thick subcategories of the subcategory of weakly dualizable objects.

We remark that according to an unpublished paper of Franke [12], for ℓ odd $L_{K(1)}\mathcal{S}$ is equivalent to a certain category of chain complexes. We will not make any use of this, however. Nor will we make any use of Bousfield's united K -theory [5], but the reader should be aware of this homologically efficient approach to 2-primary K -theory.

3.3.1 ℓ -Adic Completion of Spectra

Fix a spectrum E . A spectrum Y is said to be E -acyclic if $E \wedge Y \cong *$, and E -local if $[W, Y] = 0$ for all E -acyclic W . There is a Bousfield localization functor $L_E : \mathcal{S} \rightarrow \mathcal{S}$ and a natural transformation $Id \rightarrow L_E$ such that $L_E X$ is E -local and the fibre of $X \rightarrow L_E X$ is E -acyclic [3].

The ℓ -adic completion of a spectrum is its Bousfield localization with respect to the mod ℓ Moore spectrum $M\mathbb{Z}/\ell$. We usually write $X^\wedge$ for $L_{M\mathbb{Z}/\ell} X$. Note that the $M\mathbb{Z}/\ell$ -acyclic spectra are the spectra with uniquely ℓ -divisible homotopy groups. The ℓ -adic completion can be constructed explicitly as the homotopy inverse limit $\text{holim}_n X \wedge M\mathbb{Z}/\ell^n$, or equivalently as the function spectrum $\mathcal{F}(\mathcal{N}, X)$, where $\mathcal{N} = \Sigma^{-1}M\mathbb{Z}/\ell^\infty$. Resolving $\mathcal{N}$ by free Moore spectra, it is easy to see that there is a universal coefficient sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}/\ell^\infty, [W, X]) \rightarrow [W, X^\wedge] \rightarrow \text{Hom}(\mathbb{Z}/\ell^\infty, [\Sigma^{-1}W, X]) \rightarrow 0.$$

To see what this has to do with ℓ -adic completion in the algebraic sense, we need a short digression.

An abelian group A is *Ext- ℓ complete* if

$$\text{Hom}(\mathbb{Z}[1/\ell], A) = 0 = \text{Ext}(\mathbb{Z}[1/\ell], A).$$

Thus the full subcategory $\mathcal{E}$ of all Ext- ℓ complete abelian groups is an abelian subcategory closed under extensions. In fact it is the smallest abelian subcategory containing the ℓ -complete abelian groups. The category $\mathcal{E}$ enjoys various pleasant properties, among which we mention the following:

- An Ext- ℓ complete abelian group has no divisible subgroups.
- The objects of $\mathcal{E}$ have a natural $\mathbb{Z}_\ell$ -module structure.
- $\mathcal{E}$ is closed under arbitrary inverse limits.
- If A, B are in $\mathcal{E}$, then so are $\text{Hom}(A, B)$ and $\text{Ext}(A, B)$.
- For any abelian group A , $\text{Hom}(\mathbb{Z}/\ell^\infty, A)$ and $\text{Ext}(\mathbb{Z}/\ell^\infty, A)$ are Ext- ℓ complete.
- The functor $eA = \text{Ext}(\mathbb{Z}/\ell^\infty, A)$ is idempotent with image $\mathcal{E}$. The kernel of the evident natural transformation $A \rightarrow eA$ is $\text{Div } A$, the maximal divisible subgroup of A .
- If A is finitely-generated, eA is just the usual ℓ -adic completion $A^\wedge$.

Here it is important to note that although an $\text{Ext-}\ell$ complete abelian group has no divisible subgroups, it may very well have divisible *elements*. A very interesting example of this phenomenon arises in algebraic K-theory, in connection with the so-called “wild kernel”; see [1].

We remark that although $\mathcal{E}$ is not closed under infinite direct sums, it nevertheless has an intrinsic coproduct for arbitrary collections of objects: $\coprod A_\alpha = e(\oplus A_\alpha)$. Hence $\mathcal{E}$ has arbitrary colimits, and these are constructed applying e to the ordinary colimit.

Finally, we note that any profinite abelian ℓ -group is $\text{Ext-}\ell$ complete, and furthermore if A, B are profinite then $\text{Hom}_{\text{cont}}(A, B)$ is $\text{Ext-}\ell$ complete. To see this, let $A = \varprojlim A_\alpha, B = \varprojlim B_\beta$ with A_α, B_β finite, and recall that the continuous homomorphisms are given by

$$\varprojlim_\beta \text{colim}_\alpha \text{Hom}(A_\alpha, B_\beta).$$

Any group with finite ℓ -exponent is $\text{Ext-}\ell$ complete, whence the claim. Thus the category of profinite abelian ℓ -groups and continuous homomorphisms embeds as a non-full subcategory of $\mathcal{E}$, compatible with the intrinsic Hom and Ext . This ends our digression.

Returning to the topology, the universal coefficient sequence shows that if X, Y are ℓ -complete, $[X, Y]$ is $\text{Ext-}\ell$ complete. Now let $\mathcal{S}_{\text{tor}}$ denote the full subcategory of ℓ -torsion spectra; that is, the spectra whose homotopy groups are ℓ -torsion groups. Then the functors

$$F(\mathcal{N}, -) : \mathcal{S}_{\text{tor}} \rightarrow \mathcal{S}^\wedge \qquad (-) \wedge \mathcal{N} : \mathcal{S}^\wedge \rightarrow \mathcal{S}_{\text{tor}}$$

are easily seen to be mutually inverse equivalences of categories. In fact, these functors are equivalences of abstract stable homotopy theories in the sense of [18]. To make sense of this, it is crucial to distinguish between various constructions performed in the ambient category $\mathcal{S}$ and the *intrinsic* analogues obtained by reflecting back into $\mathcal{S}_{\text{tor}}, \mathcal{S}^\wedge$. For example, $\mathcal{S}^\wedge$ is already closed under products and function spectra, so the ambient and intrinsic versions coincide, whereas intrinsic coproducts and smash products must be defined by completing the ambient versions. In $\mathcal{S}_{\text{tor}}$ it is the reverse: For smash product and coproducts the ambient and intrinsic versions coincide, while intrinsic products and function spectra must be defined by $(\coprod X_\alpha) \wedge \mathcal{N}$ and $\mathcal{F}(X, Y) \wedge \mathcal{N}$, respectively. Note also that $\mathcal{N}$ is the unit in $\mathcal{S}_{\text{tor}}$.

One also has to be careful about “small” objects. In the terminology of [18], an object W of $\mathcal{S}^\wedge$ is *small* if the natural map

$$\oplus [W, X_\alpha] \rightarrow \left[W, \coprod X_\alpha \right]$$

is an isomorphism. The symbol $\coprod$ on the right refers to the intrinsic coproduct – that is, the completed wedge. With this definition, the completed sphere $(S^0)^\wedge$ is not small. The problem is that an infinite direct sum of $\mathbb{Z}_\ell$ ’s is not $\text{Ext-}\ell$ complete. As we have just seen, however, the category $\mathcal{E}$ has its own intrinsic coproduct, and

it is easy to see that the functor $[(S^0)^\wedge, -]$ *does* commute with intrinsic coproducts in this sense. Thus there are two variants of “smallness” in the ℓ -complete world. To avoid confusion, however, we will keep the definition given above, and call objects that commute with intrinsic coproducts *quasi-small*. It follows immediately that an object is quasi-small if and only if it is $\mathcal{F}$ -small in the sense of [18]. We also find that the small objects in $\mathcal{S}^\wedge$ are the finite ℓ -torsion spectra.

3.3.2 ℓ -Adic Topological K -Theory

Let $\widehat{\mathcal{K}}$ denote the ℓ -completion of the periodic complex K -theory spectrum. We will use the notation $\widehat{\mathcal{K}}^* X = \widehat{\mathcal{K}}^0 X \oplus \widehat{\mathcal{K}}^1 X$. Then the ring of operations $\widehat{\mathcal{K}}^* \widehat{\mathcal{K}}$ is completely determined by $\widehat{\mathcal{K}}^* \widehat{\mathcal{K}}$, and has an elegant description in terms of the Iwasawa algebra ([27]; see also [30]).

9

Proposition 9 $\widehat{\mathcal{K}}^0 \widehat{\mathcal{K}}$ is isomorphic to Λ' . The isomorphism is uniquely determined by the correspondence $\psi^k \leftrightarrow \gamma$ with $c(\gamma) = k$ ($k \in \mathbb{Z}$, ℓ prime to k). Furthermore, $\widehat{\mathcal{K}}^1 \widehat{\mathcal{K}} = 0$.

Using the idempotents $e_i \in Z_\ell[\Delta]$, we obtain a splitting

$$\widehat{\mathcal{K}} \cong \bigvee_{i=0}^{\ell-2} e_i \widehat{\mathcal{K}}.$$

The zero-th summand is itself a $(2\ell - 2)$ -periodic ring spectrum, called the *Adams summand* and customarily denoted $E(1)^\wedge$; furthermore, $e_i \widehat{\mathcal{K}} \cong \Sigma^{2i} E(1)^\wedge$. Smashing with the Moore spectrum yields a similar decomposition, whose Adams summand $e_0 \mathcal{K} \wedge M\mathbb{Z}/\ell$ is usually denoted $K(1)$ – the first Morava K -theory.

Note also that $\widehat{\mathcal{K}}^0 S^{2n} = Z_\ell(n)$. More generally, $\widehat{\mathcal{K}}^0 S^{2n} \wedge X = \widehat{\mathcal{K}}^0 X(n)$. Thus Tate twisting corresponds precisely to double suspension.

One striking consequence of Proposition 9 is that the theory of Iwasawa modules can now be applied to K -theory. This will be one of the major themes of our paper.

3.3.3 $K(1)$ -Localization

We will be working almost exclusively in the $K(1)$ -local world. We cannot give a thorough introduction to this world here, but we will at least mention a few salient facts.

1. $L_{K(1)} X = (L_{\mathcal{K}} X)^\wedge$. The functor $L_{\mathcal{K}}$ is *smashing* in the sense that $L_{\mathcal{K}} X = X \wedge L_{\mathcal{K}} S^0$, but completion is not smashing and neither is $L_{K(1)}$.
2. A spectrum X is $K(1)$ -acyclic if and only if $\widehat{\mathcal{K}}^* X = 0$.
3. $K(1)$ -local spectra X manifest a kind of crypto-periodicity (an evocative term due to Bob Thomason). Although X itself is rarely periodic, each reduction $X \wedge M\mathbb{Z}/\ell^n$ is periodic, with the period increasing as n gets larger. A similar periodicity is familiar in number theory, for example in the Kummer congruences.

4. The crypto-periodicity has two striking consequences: (i) If X is both $K(1)$ -local and connective, then X is trivial; and (ii) The functor $L_{K(1)}$ is invariant under connective covers, in the sense that if $Y \rightarrow X$ has fibre bounded above, then $L_{K(1)}Y \rightarrow L_{K(1)}X$ is an equivalence. Hence the $K(1)$ -localization of a spectrum depends only on its “germ at infinity” (another evocative term, due to Bill Dwyer this time).
5. The $K(1)$ -local sphere fits into a fibre sequence of the form

$$L_{K(1)}S^0 \rightarrow \widehat{\mathcal{K}} \xrightarrow{\psi^q - 1} \widehat{\mathcal{K}}$$

for ℓ odd, or

$$L_{K(1)}S^0 \rightarrow KO^\wedge \xrightarrow{\psi^q - 1} KO^\wedge$$

for $\ell = 2$. Here q is chosen as follows: If ℓ is odd, q can be taken to be any topological generator of $\mathbb{Z}_\ell^\times$. In terms of the Iwasawa algebra, we could replace $\psi^q - 1$ by e_0T . If $\ell = 2$, any $q = \pm 3 \bmod 8$ will do; the point is that $q, -1$ should generate $\mathbb{Z}_2^\times$. In the literature $q = 3$ is the most popular choice, but $q = 5$ fits better with our conventions on Γ . One can identify $\widehat{KO}^0 \widehat{KO}$ canonically with Λ , and then $\psi^q - 1$ corresponds to T . In any event, the homotopy groups of $L_{K(1)}S^0$ can be read off directly from these fibre sequences.

6. There is an equivalence of stable homotopy categories

$$L_{\mathcal{K}}\mathcal{S}_{\text{tor}} \cong L_{K(1)}\mathcal{S}$$

given by the same functors discussed earlier in the context of plain ℓ -adic completion. The expression on the left is unambiguous; $L_{\mathcal{K}}(\mathcal{S}_{\text{tor}})$ and $(L_{\mathcal{K}}\mathcal{S})_{\text{tor}}$ are the same.

The $\widehat{\mathcal{K}}$ -Based Adams Spectral Sequence

3.3.4

The functors $\widehat{\mathcal{K}}^n$ take values in the category of compact Λ' -modules and continuous homomorphisms. This puts us in the general setting of “compact modules over complete group rings”, a beautiful exposition of which can be found in [38], Chapter V, Sect. 3.2. In particular, there is a contravariant equivalence of categories

$$(\text{discrete torsion } \Lambda'\text{-modules}) \cong (\text{compact } \Lambda'\text{-modules})$$

given by (continuous) Pontrjagin duality. This fits perfectly with the equivalence

$$L_{\mathcal{K}}\mathcal{S}_{\text{tor}} \cong L_{K(1)}\mathcal{S}$$

mentioned earlier, since there are universal coefficient isomorphisms

$$\widehat{\mathcal{K}}^n X \cong (\mathcal{K}_{n-1} X \wedge \mathcal{N})^\# ,$$

where $\#$ denotes Pontrjagin duality.

Furthermore, homological algebra in either of the equivalent categories above is straightforward and pleasant; again, see [38] for details. The main point to bear in mind is that Hom and Ext will always refer to the continuous versions; that is, to Hom and Ext in the category of compact Λ' -modules (or occasionally in the category of discrete torsion Λ' -modules). We note also that if M is finitely-generated over Λ and N is arbitrary, the continuous $Ext^p_{\Lambda'}(M, N)$ is the same as the ordinary Ext ([38], 5.2.22).

The Adams spectral sequence we will use is the so-called “modified” Adams spectral sequence, as discussed for example in [6] and [17], except that we prefer to work with compact Λ' -modules rather than with discrete torsion modules or with comodules. The “modified” spectral sequence works so beautifully here that we have no need for its unmodified antecedent, and consequently we will drop “modified” from the terminology.

Call $W \in L_{K(1)}\mathcal{S}$ *projective* if (i) $\widehat{\mathcal{K}}W$ is projective as compact Λ' -module; and (ii) for any $X \in L_{K(1)}\mathcal{S}$, the natural map

$$[X, W] \rightarrow Hom_{\Lambda'}(\widehat{\mathcal{K}}W, \widehat{\mathcal{K}}X)$$

is an isomorphism. There is the obvious analogous notion of injective object in $L_{\mathcal{K}\mathcal{S}_{tors}}$, and clearly W is projective if and only if $W \wedge \mathcal{N}$ is injective. It is not hard to show:

10

Proposition 10

- (a) If $M = M^0 \oplus M^1$ is any $\mathbb{Z}/2$ -graded projective compact Λ' -module, there is a projective spectrum W with $\widehat{\mathcal{K}}W \cong M$.
- (b) $L_{K(1)}\mathcal{S}$ has enough projectives, in the sense that for every X there exists a projective W and a map $X \rightarrow W$ inducing a surjection on $\widehat{\mathcal{K}}$.

Hence for any X we can iterate the construction of (b) to obtain an *Adams resolution*

$$\begin{array}{ccccccc} X & \leftarrow & X_1 & \leftarrow & X_2 & \leftarrow & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ W_0 & & W_1 & & W_2 & & \end{array},$$

where the triangles are cofibre sequences and the horizontal arrows shift dimensions and induce zero on $\widehat{\mathcal{K}}^*$. If Y is another object, applying the functor $[Y, -]$ yields an exact couple and a spectral sequence with

$$E_2^{s,t} = Ext_{\Lambda'}^s(\widehat{\mathcal{K}}X, \widehat{\mathcal{K}}\Sigma^t Y)$$

and abutment $[\Sigma^{t-s}Y, X]$. This is the (modified) Adams spectral sequence. It could be displayed as a right half-plane cohomology spectral sequence, but the custom in homotopy theory is to put $t - s$ on the horizontal axis and s on the vertical axis. This yields a display occupying the upper half-plane, with the differential d^r going up r and to the left 1, and with the i -th column of E_∞ corresponding to the associated graded module of $[\Sigma^i Y, X]$. Here the filtration on $[Y, X]$ is the obvious one obtained from the tower: Filtration n consists of the maps that lift to X_n .

It is easy to check that from E_2 on, the spectral sequence does not depend on the choice of Adams resolution. The filtration is also independent of this choice, and in fact has an alternate, elegant description: Let $\mathcal{A}^n[Y, X]$ denote the subgroup of maps that factor as a composite of n maps each of which induces the zero homomorphism on $\widehat{\mathcal{K}}$. This is the *Adams filtration*, and it coincides with the filtration obtained from any Adams resolution.

It remains to discuss convergence. In fact the Adams spectral sequence converges *uniformly* to the associated graded object of the Adams filtration. By “uniformly” we mean that there is a fixed d such that $\mathcal{A}^d[Y, X] = 0$, and the spectral sequence collapses at E_d . The simplest way for this to happen is to have every d -fold composite of maps in the tower $X_{n+d} \rightarrow \dots \rightarrow X_n$ be null; then we say the tower is *uniformly d -convergent*. This would be useful even if d depended on X, Y , but in the present situation we will have d depending only on whether $\ell = 2$ or ℓ odd.

If ℓ is odd, every compact Λ' -module has projective dimension ≤ 2 , and hence every $X \in L_{K(1)}\mathcal{S}$ admits an Adams resolution of length ≤ 2 . So we can trivially take $d = 3$, and furthermore the spectral sequence is confined to the lines $s = 0, 1, 2$, with d_2 the only possible differential. In general, this d_2 is definitely nonzero.

If $\ell = 2$, then Λ' has infinite global dimension, and most objects will not admit a finite Adams resolution. Nevertheless, it is easy to see that the Adams tower is uniformly convergent with $d \leq 6$. For if $\widehat{\mathcal{K}}X$ is Λ -projective, then one can use $W_0 = X \wedge C_\eta$ as the first term of an Adams resolution, where C_η is the mapping cone of $\eta \in \pi_1 S^0$. Then the cofibre X_1 is again Λ -projective, so one can iterate the process to obtain an Adams resolution in which the maps $X_n \rightarrow X_{n-1}$ are all multiplications by η . Since $\eta^4 = 0$, we can take $d = 4$ in this case. If $\widehat{\mathcal{K}}X$ is only $\mathbb{Z}_2$ -projective, then any choice of X_1 is Λ -projective, and we can take $d = 5$. Finally, if X is arbitrary then any choice of X_1 is $\mathbb{Z}_2$ -projective, and one can take $d = 6$. (Compare [17], Proposition 6.5.)

As an illustration of the Adams spectral sequence, we compute the Z_ℓ -ranks of $\pi_* X$ in terms of $\widehat{\mathcal{K}}^* X$. The following proposition is valid at any prime ℓ .

Proposition 11 Suppose $\widehat{\mathcal{K}}X$ is finitely-generated over Λ' . Then

11

$$\begin{aligned} \text{rank}_{Z_\ell} \pi_{2m} X &= \text{rank}_\Lambda e_m \widehat{\mathcal{K}}^0 X + \langle \tau'_m, \widehat{\mathcal{K}}^0 X \rangle + \langle \tau'_m, \widehat{\mathcal{K}}^1 X \rangle \\ \text{rank}_{Z_\ell} \pi_{2m+1} X &= \text{rank}_\Lambda e_m \widehat{\mathcal{K}}^1 X + \langle \tau'_m, \widehat{\mathcal{K}}^1 X \rangle + \langle \tau'_{m+1}, \widehat{\mathcal{K}}^0 X \rangle. \end{aligned}$$

The proof is straightforward, part of the point being that all the groups above the 1-line in the Adams E_2 -term are finite, and even when $\ell = 2$ only finitely many of these survive in each topological degree.

When the terms involving, say, $\widehat{\mathcal{K}}^0$ vanish, we can turn these formulae around to get

$$\begin{aligned} \langle \tau'_m, \widehat{\mathcal{K}}^1 X \rangle &= \text{rank}_{Z_\ell} \pi_{2m} X \\ \text{rank}_\Lambda e_m \widehat{\mathcal{K}}^1 X &= \text{rank}_{Z_\ell} \pi_{2m+1} X - \text{rank}_{Z_\ell} \pi_{2m} X. \end{aligned}$$

3.3.5 The Structure of $L_{K(1)}\mathcal{S}$: General Results

To begin, we have:

12 Theorem 12 $L_{K(1)}\mathcal{S}$ has no proper nontrivial localizing subcategories.

This means that a natural transformation of cohomology theories on $L_{K(1)}\mathcal{S}$ is an isomorphism provided that it is an isomorphism on a single nontrivial object. Next, recall that a full subcategory $\mathcal{C}$ of a stable homotopy category is *thick* if it is closed under retracts and under cofibrations (meaning that if $X \rightarrow Y \rightarrow Z$ is a cofibre sequence and any two of X, Y, Z lie in $\mathcal{C}$, so does the third). We write $Th(X)$ for the thick subcategory generated by an object X ; objects of this category are said to be X -finite. Recall also that $W \in L_{K(1)}\mathcal{S}$ is *small* if for any collection of objects X_α , the natural map

$$\oplus [W, X_\alpha] \rightarrow \left[W, \coprod X_\alpha \right]$$

is an isomorphism. Here the coproduct on the right is the intrinsic coproduct; that is, the $K(1)$ -localization of the wedge.

13 Theorem 13 The following are equivalent:

- (a) X is small.
- (b) $\mathcal{K}X$ is finite.
- (c) X is $M\mathbb{Z}/\ell$ -finite.
- (d) $X = L_{K(1)}F$ for some finite ℓ -torsion spectrum F .

Next recall (again from [18]) that an object X is *dualizable* if the natural map

$$\mathcal{F}(X, S) \wedge Y \rightarrow \mathcal{F}(X, Y)$$

is an equivalence for all Y . Here the smash product on the left is the intrinsic smash product; that is, the $K(1)$ -localization of the ordinary smash product. This property has the alternate name “strongly dualizable”, but following [18] we will say simply that X is dualizable. If we were working in the ordinary stable homotopy category $\mathcal{S}$, the small and dualizable objects would be the same, and would coincide with the finite spectra. In $L_{K(1)}\mathcal{S}$, however, the dualizable objects properly contain the small objects:

14 Theorem 14 The following are equivalent:

- (a) X is dualizable.
- (b) $\mathcal{K}X$ is a finitely-generated Z_ℓ -module.
- (c) X is quasi-small.

Recall that quasi-small means that the map occurring in the definition of small object becomes an isomorphism after Ext - ℓ -completion of the direct sum in its

source; this is equivalent to being $\mathcal{F}$ -small in the sense of [18]. Of course the localization of any finite spectrum is dualizable, but these are only a small subclass of all dualizable objects. We will return to this point in the next section.

Let $DX = \mathcal{F}(X, S)$. An object X is *weakly dualizable* if the natural map $X \rightarrow D^2X$ is an equivalence. Now there is another kind of duality in stable homotopy theory: Brown–Comenetz duality, an analogue of Pontrjagin duality. In $L_{K(1)}\mathcal{S}$ we define it as follows [17]: Fix an object $A \in L_{K(1)}\mathcal{S}$ and consider the functor $X \mapsto (\pi_0 A \wedge X \wedge \mathcal{N})^\#$, where $(-)^\#$ denotes the ℓ -primary Pontrjagin dual $\text{Hom}(-, \mathbb{Z}/\ell^\infty)$. This functor is cohomological, and therefore by Brown representability there is a unique spectrum dA representing it; we call dA the *Brown–Comenetz dual* of A . It is easy to see that $dA = \mathcal{F}(A, dS)$, and so understanding Brown–Comenetz duality amounts to understanding dS . It turns out that in $L_{K(1)}\mathcal{S}$ life is very simple, at least for ℓ odd: dS is just $L_{K(1)}S^2$. Hence Brown–Comenetz duality is just a Tate-twisted form of functional duality, with the appearance of the twist strongly reminiscent of number theory. If $\ell = 2$ the situation is, as usual, somewhat more complicated. Let V denote the cofibre of $\varepsilon : \Sigma M\mathbb{Z}/2 \rightarrow S^0$, where ε is one of the two maps of order 4. Then the Brown–Comenetz dual of $L_{K(1)}S^0$ is $\Sigma^2 L_{K(1)}V$.

As another example, note that the universal coefficient isomorphism given at the beginning of Sect. 3.3.4 says precisely that $d\widehat{\mathcal{K}} \cong \Sigma\widehat{\mathcal{K}}$.

Some parts of the following theorem are special cases of the results of [17]; the rest can be found in [15].

Theorem 15 The following are equivalent:

- (a) X is weakly dualizable.
- (b) X is $\widehat{\mathcal{K}}$ -finite.
- (c) $\widehat{\mathcal{K}}X$ is finitely-generated over Λ .
- (d) $\pi_n X$ is finitely-generated over $\mathbb{Z}_\ell$ for all n .
- (e) The natural map $X \rightarrow d^2X$ is an equivalence.

15

A $K(1)$ -local spectrum is *invertible* if $X \wedge Y \cong S$ for some Y . Here the smash product is of course the intrinsic smash product, and S is the $K(1)$ -local sphere. The set (and it is a set) of weak equivalence classes of such X forms a group under smash product, called the Picard group (see [16]). It is not hard to show that X is invertible if and only if $\widehat{\mathcal{K}}X$ is free of rank one as $\mathbb{Z}_\ell$ -module. For simplicity we restrict our attention to the subgroup of index two of $\text{Pic } L_{K(1)}\mathcal{S}$ consisting of the X with $\widehat{\mathcal{K}}^1 X = 0$, denoted $\text{Pic}^0 L_{K(1)}\mathcal{S}$. There is a natural homomorphism

$$\phi : \text{Pic}^0 L_{K(1)}\mathcal{S} \rightarrow \text{Pic } \Lambda',$$

where $\text{Pic } \Lambda' \cong \mathbb{Z}_\ell^\times$ is the analogous Picard group for the category of Λ' -modules. It is not hard to show that ϕ is onto at all primes ℓ , and is an isomorphism if ℓ is odd. The subtle point is that ϕ has a nontrivial kernel when $\ell = 2$. In fact, $\text{Ker } \phi$ has order two and is generated by the object $L_{K(1)}V$ defined above.

As an example for ℓ odd we mention the Δ -twists of the sphere. Let χ be a power of the Teichmüller character; in other words, χ is pulled back along the projection

$\Gamma' \rightarrow \Delta$. Let S_χ^0 denote the corresponding invertible spectrum. Thus Γ acts trivially on $\widehat{\mathcal{K}}^0 S_\chi^0$, but Δ acts via χ . We call these spectra Δ -twists of the sphere. It turns out that they arise in nature in an interesting way. Since $\Delta \cong \text{Aut } \mathbb{Z}/\ell$, Δ acts on the suspension spectrum of the classifying space $B\mathbb{Z}/\ell$. Hence the idempotents e_i defined above can be used to split $B\mathbb{Z}/\ell$, in exactly the same way that we used them to split $\widehat{\mathcal{K}}$. Then for $0 \leq i \leq \ell - 2$, $e_i L_{K(1)} B\mathbb{Z}/\ell \cong S_{\omega^i}^0$. For $i = 0$ one concludes that $L_{K(1)} B\Sigma_\ell \cong L_{K(1)} S^0$. For $i \not\equiv 0 \pmod{\ell - 1}$, however, $S_{\omega^i}^0$ cannot be the localization of a finite spectrum.

Note that maps out of invertible spectra can be viewed as ℓ -adic interpolations of ordinary homotopy groups.

3.3.6 Iwasawa Theory for $\widehat{\mathcal{K}}$ -Finite Spectra

If $X \in \text{Th}(\widehat{\mathcal{K}})$ then we can apply the classification theory for Λ -modules. In this section we assume ℓ is odd. Most of the results below are from [15], where details and further results can be found.

Let $\mathcal{C}$ be a thick subcategory of $L_{K(1)} \mathcal{S}$. A map $X \rightarrow Y$ is a $\mathcal{C}$ -equivalence if its fibre lies in $\mathcal{C}$. In the case when $\mathcal{C}$ is the category of small objects, we call a $\mathcal{C}$ -equivalence a *pseudo-equivalence*. An object X of $L_{K(1)} \mathcal{S}$ is *elementary* if (i) $X \cong X_0 \vee X_1$, where $\widehat{\mathcal{K}}^1 X_0 = 0 = \widehat{\mathcal{K}}^0 X_1$, and (ii) $\widehat{\mathcal{K}}^1 X$ is elementary as Λ -module. An elementary object X is determined up to equivalence by $\widehat{\mathcal{K}}^1 X$ as Λ' -module. Hence if E is a $\mathbb{Z}/2$ -graded elementary Λ' -module, it makes sense to write $\mathcal{M}_E$ for the corresponding elementary object.

16 Theorem 16 Any $\widehat{\mathcal{K}}$ -finite X is pseudo-equivalent to an elementary spectrum $\mathcal{M}_E$.

There need not be any pseudo-equivalence $X \rightarrow \mathcal{M}_E$ or $\mathcal{M}_E \rightarrow X$. In general, one can only find a third object Y and pseudo-equivalences

$$X \rightarrow Y \leftarrow \mathcal{M}_E.$$

The next result provides a simple but interesting illustration.

17 Theorem 17 X is the $K(1)$ -localization of a finite spectrum if and only if X is pseudo-equivalent to a finite wedge of spheres.

Notice that the collection of localizations of finite spectra does not form a thick subcategory. It is easy to see why this fails: The point is that $L_{K(1)} S^0$ has nontrivial negative homotopy groups, and no nontrivial element of such a group can be in the image of the localization functor. In particular, $\pi_{-1} L_{K(1)} S^0 = \mathbb{Z}_\ell$, and the cofibre of a generator is not the localization of any finite spectrum. Oddly enough, this cofibre comes up in connection with the Gross conjecture from algebraic number theory (see below).

It is possible to classify thick subcategories of the $\widehat{\mathcal{K}}$ -finite spectra. Define the *support* of X , denoted $\text{Supp } X$, to be the union of the supports of the Λ' -modules

$\widehat{\mathcal{K}}^n X$, $n \in \mathbb{Z}$. Note that $\text{Supp } X$ is invariant under Tate twisting and hence will typically contain infinitely many height one primes. On the other hand, it is clear that $\text{Supp } X$ is generated by $\text{Supp } \widehat{\mathcal{K}}^0 X \amalg \text{Supp } \widehat{\mathcal{K}}^1 X$ under Tate twisting.

A subset A of $\text{Spec } A'$ will be called *fit* if it is closed under specialization and under Tate twisting. For any subset A , we let $\mathcal{C}_A$ denote the full subcategory of $L_{K(1)}\mathcal{S}$ consisting of objects whose support lies in the fit subset generated by A . If A is given as a subset of $\text{Spec } A$ only, we interpret this to mean taking closure under Δ -twisting as well. In other words, we take the fit subset generated by all (q, i) , $q \in A$. It is easy to see that $\mathcal{C}_A$ is a thick subcategory. Note for example:

- If $A = \{\mathcal{M}\}$, $\mathcal{C}_A$ consists of the small objects;
- If A is the collection of all irreducible distinguished polynomials, $\mathcal{C}_A$ consists of the dualizable objects.

Theorem 18 Let $\mathcal{C}$ be a thick subcategory of $\text{Th}(\widehat{\mathcal{K}})$. Then there is a unique fit set of primes A such that $\mathcal{C} = \mathcal{C}_A$.

18

Some further examples:

- Let $\mathcal{C}$ be the collection of objects X with finite homotopy groups. Then A is generated by the complement of the set of extended Tate primes (in the set of height one primes).
- Let $\mathcal{C}$ be the collection of objects X with almost all homotopy groups finite. Then A is generated by the set of all height one primes.
- Let $\mathcal{C} = \text{Th}(L_{K(1)}S^0)$. Then A is generated by the set of extended Tate primes.
- Let $\mathcal{C}$ be the thick subcategory generated by the invertible spectra. Then A is generated by the set of linear distinguished polynomials.

The semi-discrete primes are also of interest. Note, for example, that if G is a finite group then $\widehat{\mathcal{K}} BG$ is semi-discrete; this follows from the famous theorem of Atiyah computing $\widehat{\mathcal{K}} BG$ in terms of the representation ring of G . In the next two propositions, the spectra occurring are implicitly localized with respect to $K(1)$.

Proposition 19 Let A be the set of semi-discrete primes.

19

- a) $\mathcal{C}_A$ is generated by the suspension spectra BC , C ranging over finite cyclic ℓ -groups.
- b) Fix a prime $p \neq \ell$. Then $\mathcal{C}_A$ is generated by the spectra $K\mathbb{F}_q$, q ranging over the powers of p .

Proposition 20 For any S -arithmetic group G , the classifying space BG is in $\mathcal{C}_A$, where A is the set of semi-discrete primes.

20

This last proposition is an easy consequence of well-known theorems of Borel-Serre; the point is that BG has a finite filtration whose layers are finite wedges of suspensions of classifying spaces of finite groups.

3.4 Iwasawa Theory

We assume throughout this section that ℓ is odd.

We begin by introducing our notation for various objects and Iwasawa modules associated to the ℓ -adic cyclotomic tower (Sect. 3.4.1). In Sect. 3.4.2 we use Dwyer's étale homotopy theory approach to prove many of the classical theorems of Iwasawa theory.

3.4.1 Notation

We regret having to subject the reader to a barrage of notation at this point, but we might as well get it over with, and at least have all the notation in one place for easy reference. Fix the number field F and odd prime ℓ . Recall that r_1 and r_2 denote respectively the number of real and complex places of F .

Warning: Some of our notation conflicts with standard usage in number theory. The main example is that for us, A_∞ and A'_∞ refer to norm inverse of ℓ -class groups, not the direct limits. Thus our A_∞, A'_∞ would usually be denoted X_∞, X'_∞ in the number theory literature.

The Cyclotomic Tower

By the *cyclotomic tower* we mean the ℓ -adic cyclotomic tower defined as follows: F_0 is the extension obtained by adjoining the ℓ -th roots of unity to F . We let $d = d_F$ denote the degree of F_0 over F ; note that d divides $\ell - 1$. The Galois group $G(F_0/F)$ will be denoted Δ_F ; it is a cyclic group of order d .

F_∞ is the extension obtained by adjoining all the ℓ -power roots of unity to F . The Galois group $G(F_\infty/F_0)$ will be denoted Γ_F ; it is isomorphic as profinite group to $\mathbb{Z}_\ell$. More precisely, the natural map $\Gamma_F \rightarrow \Gamma_{\mathbb{Q}}$ is an isomorphism onto a closed subgroup of index ℓ^m for some $m = m_F$. Note that when ℓ -th roots of unity are adjoined, we may have accidentally adjoined ℓ^j -th roots for some finite j as well. The Galois group $G(F_\infty/F)$ will be denoted Γ'_F ; it splits uniquely as $\Gamma' \times \Delta_F$.

It follows that there is a unique sequence of subextensions F_n/F_0 whose union is F_∞ , and with F_n of degree ℓ over F_{n-1} for all $n \geq 1$. Our *cyclotomic tower* over F is this tower $F \subset F_0 \subset F_1 \subset \dots$

For $0 \leq n \leq \infty$, we write $\mathcal{O}_n$ for $\mathcal{O}_{F_n}$ and R_n for $\mathcal{O}_{F_n}[1/\ell]$. Further variations of this obvious notational scheme will be used without comment.

Let Λ_F and Λ'_F denote respectively the pro-group rings of Γ_F and Γ'_F . Thus $\Lambda_F \subset \Lambda$, and at the same time Λ_F is abstractly isomorphic to Λ as profinite ring; in particular, Λ_F is a power series ring over $\mathbb{Z}_\ell$. We need a separate notation, however, for its power series generator. Set

$$T_F = (1 + T)^{\ell^m} - 1,$$

where $m = m_F$ is as above. Then Λ_F is identified with $\mathbb{Z}_\ell[[T_F]] \subset \mathbb{Z}_\ell[[T]] = \Lambda$.

Some Important Modules Over the Iwasawa Algebra of F

Starting from basic algebraic objects attached to number rings – class groups, unit groups and so on – we can construct associated Λ'_F -modules in two ways: (i) By taking ℓ -adic completions at each level of the cyclotomic tower and passing to the inverse limit over the appropriate norm maps, thereby obtaining a profinite Λ'_F -module; or (ii) passing to the direct limit over the appropriate inclusion-induced maps in the cyclotomic tower – in some cases, after first tensoring with $\mathbb{Z}/\ell^\infty$ – thereby obtaining, typically, a discrete torsion Λ'_F -module.

The basic examples:

Primes over ℓ : Let S denote the set of primes dividing ℓ in $\mathcal{O}_F$, and let s denote the cardinality of S . Each $\beta \in S$ is ramified in the cyclotomic tower, and furthermore there is some finite j such that all $\beta \in S_j$ are totally ramified in $\mathcal{O}_\infty/\mathcal{O}_j$. Thus S_∞ , the set of all primes over ℓ in $\mathcal{O}_\infty$, is finite, and the permutation representation of Γ'_F given by $Z_\ell S_\infty$ is the same thing as the norm inverse limit of the representations $Z_\ell S_n$. As with any permutation representation, there is a canonical epimorphism to the trivial module. Let B_∞ denote the kernel. Thus there is a short exact sequence of semidiscrete Λ'_F -modules

$$0 \rightarrow B_\infty \rightarrow Z_\ell S_\infty \rightarrow Z_\ell \rightarrow 0.$$

The letter B is chosen to suggest the Brauer group, since it follows from class field theory that B is naturally isomorphic to $\text{Hom}(\mathbb{Z}/\ell^\infty, \text{Br } R)$, the Tate module of the Brauer group of R .

Class groups: Let A (resp. A') denote the ℓ -torsion subgroup of the class group of $\mathcal{O}_F$ (resp. of R). Passing to the norm inverse limit with the A 's yields profinite Λ'_F -modules

$$A_\infty = \lim_n A_n \qquad A'_\infty = \lim_n A'_n.$$

Passing to the direct limit yields discrete torsion Λ'_F -modules

$$\mathcal{A}_\infty = \text{colim}_n A_n \qquad \mathcal{A}'_\infty = \text{colim}_n A'_n.$$

Note that there is a short exact sequence of the form $0 \rightarrow J_\infty \rightarrow A_\infty \rightarrow A'_\infty \rightarrow 0$, where J_∞ is a certain quotient of $Z_\ell S_\infty$ and in particular is semidiscrete.

Unit groups: Let E (resp. E') denote the ℓ -adic completion of the unit group $\mathcal{O}_F^\times$ (resp. $R^\times$). Do not confuse this construction with taking units of the associated local rings.

Passing to the norm inverse limit yields profinite Λ'_F -modules

$$E_\infty = \lim_n (\mathcal{O}_n^\times)^\wedge \qquad E'_\infty = \lim_n (R_n^\times)^\wedge.$$

Let $\mathcal{E}$ (resp. $\mathcal{E}'$) denote $\mathcal{O}_F^\times \otimes \mathbb{Z}/\ell^\infty$ (resp. $R^\times \otimes \mathbb{Z}/\ell^\infty$). Passing to the direct limit in the cyclotomic tower yields discrete torsion Λ'_F -modules

$$\mathcal{E}_\infty = \text{colim}_n \mathcal{O}_n^\times \otimes \mathbb{Z}/\ell^\infty \qquad \mathcal{E}'_\infty = \text{colim}_n R_n^\times \otimes \mathbb{Z}/\ell^\infty.$$

Local unit groups: For each prime $\beta \in S$, let $F_{\mathcal{P}}$ denote the local field obtained by $\mathcal{P}$ -adic completion. Let

$$U = \prod_{\beta \in S} (F_{\beta}^{\times})^{\wedge},$$

where as usual $(-)^{\wedge}$ denotes ℓ -adic completion. Let U_{∞} denote the norm inverse limit of the U_n 's. In view of the above remarks on primes over ℓ , the number of factors in the product defining U_n stabilizes to s_{∞} . This leads easily to the following description of the norm inverse limit U_{∞} : For each fixed prime β over ℓ in $\mathcal{O}_F$, let $U_{\infty, \beta}$ denote the norm inverse limit of the completed unit groups for the ℓ -adic cyclotomic tower over the completion F_{β} . Let $\Lambda'_{F_{\beta}}$ denote the pro-group ring analogous to Λ'_F . Then as Λ'_F -modules we have

$$U_{\infty} \cong \oplus_{\beta \in S} \Lambda'_F \otimes_{\Lambda'_{F_{\beta}}} U_{\infty, \beta}.$$

ℓ -extensions unramified away from ℓ : Let M_n denote the Galois group of the maximal abelian ℓ -extension of F_n that is unramified away from ℓ . Passing to the inverse limit over n yields M_{∞} , the maximal abelian ℓ -extension of F_{∞} that is unramified away from ℓ . Note that M_{∞} is a profinite Γ'_F -module by conjugation, and hence a profinite Λ'_F -module.

Algebraic and étale K -groups: One can play the same game using the algebraic K -groups $K_* R_n$ in the cyclotomic tower. If the Lichtenbaum–Quillen conjecture are true, however, these do not yield much new. Taking norm inverse limits of ℓ -completed groups to illustrate, the reason is that these groups become more and more periodic (conjecturally) as $n \rightarrow \infty$, while in low degrees $K_* R$ is essentially determined by the class group, unit group, and Brauer group – all of which we have already taken into account above.

In étale K -theory, however, there is one small but very useful exception. Equivalently, we can take $K(1)$ -localized K -theory $\pi_* L_{K(1)} KR$, and in any case we are in effect just looking at étale cohomology $H^*(R; Z_{\ell}(n))$ for $* = 1, 2$. In particular there is a short exact sequence

$$0 \rightarrow H^2(R; Z_{\ell}(1)) \rightarrow K_0^{\text{ét}} R \rightarrow Z_{\ell} \rightarrow 0,$$

where the first term in turn fits into a short exact sequence

$$0 \rightarrow A' \rightarrow H^2(R; Z_{\ell}(1)) \rightarrow B \rightarrow 0.$$

Writing $L_n = H^2(R_n; Z_{\ell}(1))$, we may again pass to an inverse limit in the cyclotomic tower, yielding a short exact sequence of Λ'_F -modules

$$0 \rightarrow A'_{\infty} \rightarrow L_{\infty} \rightarrow B_{\infty} \rightarrow 0.$$

As we will see below, L_{∞} is in many ways better behaved than A'_{∞} .

Iwasawa Theory for the Cyclotomic Extension

3.4.2

The beautiful fact is that virtually all of the profinite modules considered in the previous section are finitely-generated Λ_F -modules. (In the case of the algebraic K -groups, this would follow from the Lichtenbaum–Quillen conjecture; can it be shown directly?) Then one can ask: What are the Λ_F -ranks? What are the torsion invariants?

Finding the torsion invariants explicitly is an extremely difficult problem, but one can ask for qualitative information of a more general nature. In this section we will sketch an approach based on Poincaré–Artin–Verdier duality and étale homotopy theory. Some of this material comes from unpublished joint work of Bill Dwyer and the author, but the key ideas below are due to Dwyer, and the author is grateful for his permission to include this work here. While the method comes with a certain cost in terms of prerequisites, it yields many of the classical results (compare e.g. [19, 38, 46]) in an efficient, conceptual fashion.

We point out that number theorists usually study much more general Z_ℓ -extensions, not just the cyclotomic one. It would be interesting to apply this approach to the more general setting. Also, for us the ℓ -adic cyclotomic extension never means just the Z_ℓ -extension it contains; we invariably adjoin all the ℓ -power roots of unity and work with modules over Λ'_F . On the other hand, since it is usually easy to descend from F_0 to F , we will sometimes assume for simplicity that F contains the ℓ -th roots of unity.

Duality and the Seven-term Exact Sequence

We begin by considering the natural map

$$i : \coprod_{\beta|\ell} \operatorname{Spec} F_\beta \rightarrow \operatorname{Spec} R ,$$

in the étale topology. Let us abbreviate the target of i as Y , and the source as ∂Y . One can define homology groups $H_*(\partial Y; Z_\ell(n))$ by considering the s components of ∂Y separately and simply taking Galois homology as in [38]. Then local class field theory says that ∂Y behaves like a nonorientable 2-manifold with s components, in that there are natural local duality isomorphisms

$$H_k(\partial Y; Z_\ell(n)) \cong H^{2-k}(\partial Y; Z_\ell(1-n)) .$$

One could define homology groups for Y in a similar *ad hoc* way. Let Ω_F denote the maximal ℓ -extension of F_∞ that is unramified away from ℓ . Then the étale cohomology for Y is the same as the profinite group cohomology of the Galois group $G(\Omega_F/F)$, so we could define homology by the same device. But we also want to define relative homology for the pair $Y, \partial Y$, and here the *ad hoc* approach begins to break down. Instead, we will use pro-space homology as in [7]. It is then

possible to regard $H_*(Y, \partial Y; Z_\ell(n))$ as the homology of the “cofibre” of i . Then $(Y, \partial Y)$ behaves like a nonorientable 3-manifold with boundary, in that Artin–Verdier duality (or rather a special case of it usually called Poincaré duality) yields duality isomorphisms

$$H_k(Y; Z_\ell(n)) \cong H^{3-k}(Y, \partial Y; Z_\ell(1-n))$$

and similarly with the roles of H_* , H^* reversed. A general discussion and proof of Artin–Verdier duality can be found in [28]. The proof makes use of local duality and a relatively small dose of global class field theory, including the Hilbert classfield.

Bearing in mind that $H_2(\partial Y; Z_\ell) = 0$ by local duality, we get a seven-term exact sequence (with trivial Z_ℓ coefficients understood)

$$0 \rightarrow H_2 Y \rightarrow H_2(Y, \partial Y) \rightarrow H_1(\partial Y) \rightarrow H_1 Y \rightarrow H_1(Y, \partial Y) \rightarrow Z_\ell S \rightarrow Z_\ell \rightarrow 0.$$

Taking into account both local and Artin–Verdier duality, and using notation from the previous section, we have at each stage of the cyclotomic tower a seven-term exact sequence

$$0 \rightarrow D_n \rightarrow E'_n \rightarrow U'_n \rightarrow M_n \rightarrow L_n \rightarrow Z_\ell S_n \rightarrow Z_\ell \rightarrow 0.$$

Here we have abbreviated $H_2(Y_n)$ as D_n ; this group is the *Leopoldt defect* as will be explained below. Recall that E'_n , U'_n are ℓ -completed global and local unit groups, M_n is the Galois group of the maximal abelian ℓ -extension unramified away from ℓ , L_n is the interesting part of the zero-th étale K-theory, and S_n is the set of primes dividing ℓ .

Taking norm inverse limits we obtain a seven-term exact sequence of profinite Λ'_F -modules

$$0 \rightarrow D_\infty \rightarrow E'_\infty \rightarrow U'_\infty \rightarrow M_\infty \rightarrow L_\infty \rightarrow Z_\ell S_\infty \rightarrow Z_\ell \rightarrow 0.$$

Note that we deduce at once a standard result of class field theory, namely a short exact sequence

$$0 \rightarrow U'_\infty / E'_\infty \rightarrow M_\infty \rightarrow A'_\infty \rightarrow 0,$$

We will see later that $D_\infty = 0$, justifying the notation. If the Leopoldt conjecture holds (see Sect. 3.6), then all the D_n ’s vanish as well. Thus an optimistic title for this section would be “Duality and the six-term exact sequence”.

The Local Case

We digress to consider the local case, as both warm-up and input for the global case.

Let K be a finite extension of $\mathbb{Q}_\ell$, ℓ odd. Then in a notation that should be self-explanatory, we have K_∞ , Λ'_K , $M_{K,\infty}$, $U'_{K,\infty}$, etc. For example, $M_{K,\infty}$ is the Galois group of the maximal abelian ℓ -extension of K_∞ . Local class field theory gives isomorphisms

$$M_{K,m} \cong U'_{K,m},$$

$m \leq \infty$. Our goal is to determine explicitly the structure of $M_{K,\infty}$ as Λ'_K -module. For simplicity we will assume $\mu_\ell \subset K$, and to avoid notational clutter we will drop the subscript K from Λ_K , etc. First, there is a Serre spectral sequence

$$H_p(\Gamma; H_q K_\infty) \Rightarrow H_{p+q} K .$$

From this we obtain at once a short exact sequence

$$0 \rightarrow (M_\infty)_\Gamma \rightarrow H_1 K \rightarrow Z_\ell \rightarrow 0 .$$

Since $H_1 K \cong H^1(K; Z_\ell(1)) \cong U'$ by local duality, this shows that $(M_\infty)_\Gamma$ has Z_ℓ -rank d , where $d = [K : \mathbb{Q}_\ell]$.

Second, there is a universal coefficient spectral sequence

$$Ext_\Lambda^p(H_q K_\infty, \Lambda(1)) \Rightarrow H_{2-p-q} K_\infty .$$

Notice here that the Tate twist $\Lambda(1)$ is isomorphic to Λ as Λ -modules, but the twist is necessary in order to come out with the right Λ -module structure on the E_∞ -term. In fact the natural abutment of the spectral sequence is $H^{p+q}(K; \Lambda(1))$, but this group is isomorphic to the homology group above by local duality plus a form of Shapiro's lemma. This spectral sequence yields at once a short exact sequence

$$0 \rightarrow Ext_\Lambda^1(Z_\ell, \Lambda(1)) \rightarrow M_\infty \rightarrow Hom_\Lambda(M_\infty, \Lambda(1)) \rightarrow 0 .$$

Since Λ -duals are always free, combining these results yields:

Proposition 21 There are isomorphisms of Λ -modules

21

$$U'_\infty \cong M_\infty \cong \Lambda^d \oplus Z_\ell(1) .$$

Now recall that for a number field F we defined a norm inverse limit of local units U'_∞ , using all the primes over ℓ . Again assuming for simplicity that F contains the ℓ -th roots of unity, we have:

Corollary 22 If F is a number field,

22

$$U'_\infty \cong \Lambda_F^{2r_2} \oplus \left(\bigoplus_{\beta \in S} \Lambda_F \otimes_{\Lambda_{F_\beta}} Z_\ell(1) \right) .$$

The Global Case

We consider four miniature spectral sequences associated to the cyclotomic tower. They are miniature in the sense that they are first quadrant spectral sequences occupying a small rectangle near the origin. The first three collapse automatically. Since we are concerned mainly with the Λ_F -structure here, we will ignore the Δ_F module structure for the time being. To avoid notational clutter, in this section we will drop the subscript F from Γ_F , Λ_F , etc.

First spectral sequence: There is a Serre spectral sequence

$$H_p(\Gamma; H_q Y_\infty) \Rightarrow H_{p+q} Y_0 .$$

From it we obtain immediately:

23 Proposition 23 There are short exact sequences

a)

$$0 \rightarrow (D_\infty)_\Gamma \rightarrow D_0 \rightarrow (M_\infty)^\Gamma \rightarrow 0$$

and

b)

$$0 \rightarrow (M_\infty)_\Gamma \rightarrow M_0 \rightarrow Z_\ell \rightarrow 0 .$$

In particular, as a corollary of (b) we get:

24 Corollary 24 M_∞ is a finitely-generated Λ -module.

Proof By a version of Nakayama's lemma, it suffices to show $(M_\infty)_\Gamma$ is finitely-generated as Z_ℓ -module. Hence by (b), it suffices to show $M_0 = H_1(F_0; Z_\ell)$ is finitely-generated as Z_ℓ -module. This in turn reduces to showing $H^1(F_0; \mu_\ell)$ is finite. But there is a Kummer exact sequence

$$0 \rightarrow R_0^\times / \ell \rightarrow H^1(F_0; \mu_\ell) \rightarrow A_0[\ell] \rightarrow 0 ,$$

where $A_0[\ell]$ denotes the elements annihilated by ℓ , so this follows from the finite-generation of the unit group and finiteness of the class group.

Second spectral sequence: There is a relative Serre spectral sequence

$$H^p(\Gamma; H_q(Y_\infty, \partial Y_\infty)) \Rightarrow H_{p+q}(Y_0, \partial Y_0) .$$

From it we obtain immediately:

25 Proposition 25

a) $(L_\infty)_\Gamma = L_0$.

b) There is a short exact sequence

$$0 \rightarrow (E'_\infty)_\Gamma \rightarrow E'_0 \rightarrow (L_\infty)^\Gamma \rightarrow 0 .$$

In particular, as a corollary of (a) we get:

26 Corollary 26 L_∞, A'_∞ , and hence A_∞ are finitely-generated Λ -modules.

This follows because L_0 is a finitely-generated Z_ℓ -module – thanks to the fact that both the class group and S_∞ are finite.

The analogue of Proposition 25a for A'_∞ is false; this is one reason that L_∞ is easier to work with. In fact Proposition 25a yields an interesting corollary. For a fixed $\beta \in S_0$, let s_β denote the number of primes over β in S_∞ .

Corollary 27 Let ℓ^v denote the minimum value of s_β , $\beta \in S$. Then the cokernel of the natural map $\phi : (A'_\infty)_\Gamma \rightarrow A'_0$ is cyclic of order ℓ^v . 27

In particular, ϕ is onto if and only if at least one $\beta \in S_0$ is nonsplit (i.e., inert or ramified) in F_1/F_0 .

Proof There are isomorphisms

$$\text{Coker } \phi \cong \text{Ker } ((B_\infty)_\Gamma \rightarrow B_0) \cong \text{Coker } ((Z_\ell S_\infty)^\Gamma \rightarrow Z_\ell) \cong \mathbb{Z}/\ell^v,$$

where the first follows from Proposition 25a and the snake lemma, the second is elementary, and the third is obvious.

Remark 28 The kernel of ϕ is isomorphic to the image of the boundary map $B_\infty^\Gamma \rightarrow (A'_\infty)_\Gamma$, or equivalently the cokernel of $L_\infty^\Gamma \rightarrow B_\infty^\Gamma$. This seems harder to analyze, although it follows for example that if $s_0 = 1$ then ϕ is injective. If $s_\infty = 1$ then $B_\infty = 0$ and ϕ is an isomorphism. 28

Next we illustrate the significance of part (b). Note that $(E'_\infty)_\Gamma$ is the subgroup of *universal norms* in E'_0 ; that is, the units which are in the image of the norm map from every level of the tower. (Clearly $(E'_\infty)_\Gamma$ is contained in the universal norms, and a *lim*¹ argument shows every universal norm comes from E'_∞ .) Hence if $s = 1$ and $(A'_\infty)^\Gamma = 0$, every element of E'_0 is a universal norm. We also get a theorem of Kuz'min on the Gross–Sinnott kernel (see [24], Theorem 3.3): Let $\mathcal{N}_F$ denote the subgroup of E'_0 consisting of elements that are local universal norms at every prime over ℓ .

Corollary 29 There is a canonical short exact sequence 29

$$0 \rightarrow (E'_\infty)_\Gamma \rightarrow \mathcal{N}_F \rightarrow (A'_\infty)^\Gamma \rightarrow 0.$$

Proof This is immediate from the commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & (E'_\infty)_\Gamma & \rightarrow & E'_0 & \rightarrow & L_\infty^\Gamma \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & (U'_\infty)_\Gamma & \rightarrow & U'_0 & \rightarrow & (Z_\ell S_\infty)^\Gamma \rightarrow 0. \end{array}$$

The second spectral sequence also shows that $(E'_\infty)^\Gamma = 0$, but this was obvious *a priori*.

Third spectral sequence: There is a universal coefficient spectral sequence

$$\mathrm{Ext}_{\Lambda}^p(H_q Y_{\infty}, \Lambda(1)) \Rightarrow H_{3-p-q}(Y_{\infty}, \partial Y_{\infty}).$$

Notice here that the Tate twist $\Lambda(1)$ is isomorphic to Λ as Λ -modules, but the twist is necessary in order to come out with the right Λ -module structure on the E_{∞} -term. In fact the natural abutment of the spectral sequence is $H^{p+q}(Y_0; \Lambda(1))$, but this group is isomorphic to the homology group above by Artin–Verdier duality plus a form of Shapiro’s lemma. Recalling that Λ has global dimension two, we obtain at once:

30 Proposition 30

a) There is a short exact sequence

$$0 \rightarrow Z_{\ell}(1) \rightarrow E'_{\infty} \rightarrow \mathrm{Hom}_{\Lambda}(M_{\infty}, \Lambda)(1) \rightarrow 0$$

b) There is an isomorphism $L_{\infty} \cong \mathrm{Ext}_{\Lambda}^1(M_{\infty}, \Lambda)(1)$

c) $\mathrm{Ext}_{\Lambda}^2(M_{\infty}, \Lambda) = 0$. Hence M_{∞} has no nonzero finite submodules, and has projective dimension at most one.

Since any module of the form $\mathrm{Ext}_{\Lambda}^1(N, \Lambda)$ is a Λ -torsion module, part (b) shows at once:

31 Corollary 31 L_{∞} is a Λ -torsion module. Hence A_{∞} and A'_{∞} are also Λ -torsion modules.

Next we have:

32 Corollary 32 $E'_{\infty} \cong \Lambda^{r_2} \oplus Z_{\ell}(1)$. Moreover, M_{∞} has Λ -rank r_2 .

Proof Any module of the form $\mathrm{Hom}_{\Lambda}(N, \Lambda)$ with N finitely-generated is a free module. By Proposition 30a we conclude that $E'_{\infty} \cong Z_{\ell}(1) \oplus \Lambda^t$ for some t . Furthermore t is the Z_{ℓ} -rank of $(E'_{\infty})_{\Gamma}$. Now since L_{∞} is a finitely-generated Λ -torsion module, $(L_{\infty})^{\Gamma}$ and $(L_{\infty})_{\Gamma}$ have the same Z_{ℓ} -rank; hence $(L_{\infty})^{\Gamma}$ has rank $s_0 - 1$. Then Proposition 25b shows that $(E'_{\infty})_{\Gamma}$ has rank r_2 . This completes the proof of the first statement. The second is then immediate from Proposition 30a.

33 Corollary 33 The freeness defect of M_{∞}/tM_{∞} is Pontrjagin dual to the maximal finite submodule of A'_{∞} .

Proof By Proposition 2.4, the freeness defect is Pontrjagin dual to $\mathrm{Ext}_{\Lambda}^1(M_{\infty}/tM_{\infty}, \Lambda)$, and this latter group is easily seen to be isomorphic to the maximal finite submodule of $\mathrm{Ext}_{\Lambda}^1(M_{\infty}, \Lambda)$. Now use Proposition 30b.

We also get an important corollary on the Leopoldt defect groups D_n .

Corollary 34

34

- a) The norm inverse limit $D_\infty = H_2(Y_\infty; Z_\ell)$ vanishes.
- b) $D_0 = M^T$.
- c) The direct limit $\mathcal{D}_\infty$ is just D_n for large enough n , and is in fact isomorphic to the maximal semidiscrete submodule M_∞^δ .

Proof

- a) Note that we have now computed the Λ -rank of every term in the seven-term exact sequence except D_∞ . Counting ranks then forces $\text{rank}_\Lambda D_\infty = 0$; in other words, D_∞ is a Λ -torsion module. But the seven-term exact sequence also shows that it embeds in $E'_\infty/Z_\ell(1)$, and so is Λ -torsionfree. Hence it is zero.
- b) This is now immediate from Proposition 23a.
- c) This follows from the validity of (b) at each level of the cyclotomic tower.

Fourth spectral sequence:

Reversing the roles of Y and $(Y, \partial Y)$ in the third spectral sequence yields our last spectral sequence:

$$\text{Ext}_\Lambda^p (H_q(Y_\infty, \partial Y_\infty), \Lambda(1)) \Rightarrow H_{3-p-q} Y_\infty .$$

Some of the information from this spectral sequence is already known: We find that $\text{Hom}_\Lambda(L_\infty, \Lambda) = 0$, confirming that L_∞ is a Λ -torsion module; we find that $\text{Ext}_\Lambda^2(E'_\infty, \Lambda) = 0$, confirming that E'_∞ has no finite submodules. The remaining parts of the spectral sequence yield an exact sequence

$$0 \rightarrow \text{Ext}_\Lambda^1(L_\infty, \Lambda(1)) \rightarrow M_\infty \rightarrow \text{Hom}_\Lambda(E'_\infty, \Lambda(1)) \xrightarrow{d_2} \text{Ext}_\Lambda^2(L_\infty, \Lambda(1)) \rightarrow 0 .$$

and an isomorphism $Z_\ell \cong \text{Ext}_\Lambda^1(E'_\infty, \Lambda(1))$.

The isomorphism yields no new information, but we get something interesting from the exact sequence. First we should justify the surjectivity of the indicated d_2 . This can be seen by direct inspection, or as follows: We have seen in Corollary 33 that the freeness defect of M_∞/tM_∞ is Pontrjagin dual to the maximal finite submodule of L_∞ , which in turn is dual to the indicated Ext^2 . Hence d_2 is surjective. Thus we have:

Corollary 35

35

- a) The torsion submodule tM_∞ of M_∞ is isomorphic to $\text{Ext}_\Lambda^1(L_\infty, \Lambda(1))$. Hence there is a short exact sequence

$$0 \rightarrow \text{Ext}_\Lambda^1(B_\infty, \Lambda(1)) \rightarrow tM_\infty \rightarrow \text{Ext}_\Lambda^1(A'_\infty, \Lambda(1)) \rightarrow 0 .$$

- b) The torsion-free quotient of M_∞ embeds with finite index in the free module $\text{Hom}_\Lambda(E'_\infty, \Lambda(1))$.

One can also get the following theorem of Iwasawa, in effect by the ordinary universal coefficient theorem:

36 **Theorem 36** There is a short exact sequence of Λ' -modules

$$0 \rightarrow (\mathcal{A}'_\infty)^\#(1) \rightarrow M_\infty \rightarrow (\mathcal{E}'_\infty)^\#(1) \rightarrow 0.$$

3.5 Algebraic K-Theory Spectra

We continue to fix a prime ℓ .

Let X be a sufficiently nice scheme. By this we mean that X is a separated noetherian regular scheme of finite Krull dimension and with all residue fields of characteristic different from ℓ , and that we reserve the right to impose additional hypotheses as needed. In the present context, it is common to assume also that X has finite étale cohomological dimension for ℓ -torsion sheaves, written $cd_{\text{ét}}^\ell X < \infty$. We will, however, avoid this last hypothesis as much as possible.

Now let $\mathbb{H}_{\text{ét}}^\bullet(X, \underline{K})$ denote the Thomason–Jardine hypercohomology spectrum associated to the algebraic K -theory presheaf on the étale site of X . Up to connective covers, the ℓ -adic completion of this spectrum is equivalent to the Dwyer–Friedlander étale K -theory spectrum of X . See [22, 44] or [34] for details. The key fact about $\mathbb{H}_{\text{ét}}^\bullet(X, \underline{K})$ is that it admits a conditionally convergent right half-plane cohomology spectral sequence

$$E_2^{p, -q} = H_{\text{ét}}^p\left(X; Z_\ell\left(\frac{q}{2}\right)\right) \Rightarrow \pi_{q-p} \mathbb{H}_{\text{ét}}^\bullet(X, \underline{K})^\wedge.$$

Here $Z_\ell(\frac{q}{2})$ is to be interpreted as zero if q is odd, and as always our étale cohomology groups are continuous étale cohomology groups in the sense of Jannsen [21]. The condition $cd_{\text{ét}}^\ell X < \infty$ is often invoked to ensure actual convergence of the spectral sequence, but it is not a necessary condition.

There is a natural augmentation map $\eta : KX \rightarrow \mathbb{H}_{\text{ét}}^\bullet(X, \underline{K})$. The Dwyer–Friedlander spectrum-level version of the Lichtenbaum–Quillen conjecture can then be stated as follows:

37 **Conjecture 37** Let X be a sufficiently nice scheme. Then for some $d \geq 0$, the completed augmentation

$$\eta^\wedge : (KX)^\wedge \rightarrow \mathbb{H}_{\text{ét}}^\bullet(X, \underline{K})^\wedge$$

induces a weak equivalence on $d - 1$ -connected covers.

In a monumental paper [44], Thomason proved the $K(1)$ -local Lichtenbaum–Quillen conjecture.

Theorem 38 (Thomason) Let X be a sufficiently nice scheme. Then $L_{K(1)}\eta$ is a weak equivalence .

38

For Thomason, “sufficiently nice” includes several further technical hypotheses, including $cd_{\acute{e}t}^{\ell}X < \infty$. He also assumes $\sqrt{-1} \in X$ in the case $\ell = 2$. But for $\ell = 2$ the Lichtenbaum–Quillen conjecture itself has now been proved in many cases [41] [39], and in those cases the assumption $\sqrt{-1} \in X$ can be dropped.

Now $(\mathbb{H}_{\acute{e}t}(X, \underline{K}))^{\wedge}$ is *essentially $K(1)$ -local*, meaning that the map to its $K(1)$ -localization induces a weak equivalence on some connected cover. Hence the Lichtenbaum–Quillen conjecture can be re-interpreted as follows:

Conjecture 39 Let X be a sufficiently nice scheme. Then $(KX)^{\wedge}$ is essentially $K(1)$ -local.

39

Thomason’s theorem (or the actual Lichtenbaum–Quillen conjecture, when known) also has the corollary:

Corollary 40 $\widehat{\mathcal{K}}^* KX \cong \widehat{\mathcal{K}}^* \mathbb{H}_{\acute{e}t}(X, \underline{K})$.

40

Since $\widehat{\mathcal{K}}^* \mathbb{H}_{\acute{e}t}(X, \underline{K})$ is computable, this leads to explicit computations of $\widehat{\mathcal{K}}^* KX$ ([9, 10, 33]).

It is natural to ask how $L_{K(1)}KX$ fits into the $K(1)$ -local world described in Sect. 3.3. We will assume that X satisfies the $K(1)$ -local Lichtenbaum–Quillen conjecture. Thus X could be any scheme satisfying the hypotheses of Thomason’s theorem, or any scheme for which the actual Lichtenbaum–Quillen conjecture is known. In many interesting cases, $L_{K(1)}KX$ belongs to the thick subcategory of $\widehat{\mathcal{K}}$ -finite spectra:

Proposition 41 Suppose that (a) $H_{\acute{e}t}^i(X; \mathbb{Z}/\ell(m))$ is finite for every i, m ; and (b) the descent filtration on $\pi_n(\mathbb{H}_{\acute{e}t}(X, \underline{K}))^{\wedge}$ terminates for each n . Then $L_{K(1)}KX$ is $\widehat{\mathcal{K}}$ -finite.

41

Proof By (a), $H_{\acute{e}t}^i(X; \mathbb{Z}_{\ell}(m))$ is a finitely-generated $\mathbb{Z}_{\ell}$ -module for all i, m . Hence by (b) and the descent spectral sequence, the same is true of $\pi_n(\mathbb{H}_{\acute{e}t}(X, \underline{K}))^{\wedge}$. Using Theorem 15 and the fact that $(\mathbb{H}_{\acute{e}t}(X, \underline{K}))^{\wedge}$ is essentially $K(1)$ -local, it follows that $L_{K(1)}\mathbb{H}_{\acute{e}t}(X, \underline{K})$ is $\widehat{\mathcal{K}}$ -finite. Since $L_{K(1)}KX \cong L_{K(1)}\mathbb{H}_{\acute{e}t}(X, \underline{K})$ by assumption, this completes the proof.

There are many examples of schemes X satisfying conditions (a) and (b). The most important for our purposes is $X = \text{Spec } R$, where $R = \mathcal{O}_F[\frac{1}{\ell}]$ as in Sect. 3.4.1. Condition (b) holds even when $\ell = 2$ and F has a real embedding; in this case descent filtration 5 vanishes. See for example [35], Proposition 2.10.

Given such an X , we can then analyze $L_{K(1)}KX$ up to pseudo-equivalence, or up to some weaker mod $\mathcal{C}$ equivalence in the sense of Sect. 3.3.6. The case $X = \text{Spec } R$ is the topic of the next section.

K-Theoretic Interpretation of some Conjectures in Iwasawa Theory

3.6

A fundamental problem of Iwasawa theory is to determine the pseudo-isomorphism-type of M_∞ . In Corollary 32 we determined the rank; the much more difficult question that remains is to determine the torsion invariants. Several classical conjectures from number theory – the Leopoldt conjecture and Iwasawa’s μ -invariant conjecture, for example – have interpretations in terms of the basic Iwasawa module M_∞ . Using a theorem of the author and Bill Dwyer, we show how to reinterpret these conjectures in terms of the homotopy-type of KR (Sects. 3.6.1, 3.6.2). In the case of totally real fields, these conjectures have an analytic interpretation also, in terms of ℓ -adic L-functions. In Sect. 3.6.3 we indicate how to make the connection between the algebraic and analytic points of view, and discuss the generalized Lichtenbaum conjecture.

3.6.1

Conjectures Concerning the Semi-discrete Primes

As a first step we consider the multiplicity of the extended Tate primes $\tau'_n = (\tau_n, n)$ in M .

42

Conjecture 42: (*Basic Conjecture I*)

$$\langle \tau'_n, M_\infty \rangle = \begin{cases} 0 & \text{if } n \neq 1 \\ s-1 & \text{if } n = 1 \end{cases}$$

Note that since M_∞ has no nonzero finite submodules, for $n \neq 1$ the conjecture has the more transparent reformulation

$$\text{Hom}_{\Lambda'_F}(Z_\ell(n), M_\infty) = 0 .$$

In the case $n = 1$ it is known (see below) that

$$\text{rank}_{Z_\ell} \text{Hom}_{\Lambda'_F}(Z_\ell(n), M_\infty) = s-1 .$$

Hence the content of the conjecture is that $e_1 M_\infty$ does not involve any elementary factors of the form Λ_F / τ_1^k for $k > 1$.

I don’t know where this conjecture was first formulated, although it seems to be standard; see [24]. For the case $n \neq 1$ it is equivalent to a conjecture in Galois (or étale) cohomology of Schneider [42]. For $n = 1$ it is an algebraic version of a conjecture of Gross [14]. In all cases there are analytic analogues of the conjecture for totally real fields, expressed in terms of special values of ℓ -adic L-functions. The link between the algebraic and analytic versions is provided by Iwasawa’s Main Conjecture as proved by Wiles [47], as will be discussed briefly below.

If we consider the Basic Conjecture for all the F_m 's at once, it can be formulated in another way. If S is a set of height one primes of Λ , and M, N are finitely-generated Λ -modules, write

$$M \sim_S N$$

if M and N have the same elementary summands at all primes in S . Equivalently, the torsion submodules are isomorphic after localizing at any prime in S . Let S_δ denote the set of semi-discrete primes in Λ , and let $\langle S_\delta \rangle$ denote the fit subset it generates.

Conjecture 43: (*Basic Conjecture II*)

43

$$M_\infty \sim_{\langle S_\delta \rangle} B_\infty(1) .$$

Equivalently, $(M_\infty(-n))^\delta = 0$ for $n \neq 1$, and $M_\infty(-1) \sim_{S_\delta} B_\infty$.

It is easy to see that Conjecture II is equivalent to Conjecture I for all the F_n 's. It can also be interpreted as giving the torsion invariants of M_∞ at the Tate twists of the semi-discrete primes; it says that v_i only occurs 1-twisted with length one, and with frequency determined in a simple way by the splitting behaviour of the primes over ℓ in the cyclotomic tower.

The conjecture has an equivalent formulation in terms of class groups:

Conjecture 44: (*Basic Conjecture III*)

44

For all $n \in \mathbb{Z}$, $\langle t'_n, A'_\infty \rangle = 0$. Equivalently (if we consider all F_m at once), for all $n \in \mathbb{Z}$ we have

$$(A'_\infty(-n))^\delta = 0 .$$

The equivalence of I–II with III is immediate from the formula for tM_∞ given in Corollary 35.

We will see that these conjectures incorporate versions of the Leopoldt and Gross conjectures, and that the case $n > 1$ has been proved by Soulé. First, however, we bring topological K -theory back into the picture.

The following theorem was proved by Bill Dwyer and the author in [9]; see also [33].

Theorem 45

45

$$\begin{aligned} \widehat{\mathcal{K}}^0 KR &\cong \Lambda' \otimes_{\Lambda'_F} Z_\ell \\ \widehat{\mathcal{K}}^{-1} KR &\cong \Lambda' \otimes_{\Lambda'_F} M_\infty \end{aligned}$$

This theorem yields a description of the homotopy-type $L_{K(1)}KR$, and hence conjecturally of KR itself. First of all – since we are working at an odd prime ℓ –

one can always find a certain residue field $\mathbb{F}$ of R such that $KR \rightarrow K\mathbb{F}$ is a retraction at ℓ . More explicitly, one chooses $\mathbb{F}$ so that if $\mathbb{F}_\infty$ is the ℓ -adic cyclotomic extension of $\mathbb{F}$, then $G(\mathbb{F}_\infty/\mathbb{F}) = \Gamma'_F$. The existence of such residue fields follows from the Tchebotarev density theorem. Let $K^{\text{red}}R$ denote the fibre of this map – the *reduced K -theory spectrum* of R . Then $K^{\text{red}}R$ is the fibre of a map between wedges of copies of $\Sigma^{-1}\widehat{\mathcal{K}}$, where the map is given by a length one resolution of M_∞ as Λ'_F -module.

Theorem 45 also shows that the Basic Conjecture may be translated into a conjecture about the action of the Adams operations on $\widehat{\mathcal{K}}^{-1}KR$. It can also be formulated in terms of the homotopy groups of $L_{K(1)}KR$. For the case $n \neq 1$ we have:

46

Proposition 46 For $n \neq 1$, the following are equivalent:

- a) The Basic Conjecture for n
- b) $\widehat{\mathcal{K}}^{2n-1}KR$ has no nonzero fixed points for the Adams operations.
- c) $\pi_{2n-2}L_{K(1)}KR$ is finite.

The equivalence of (a) and (b) is immediate from Theorem 45; note that (b) is just another way of saying that $\langle \tau'_n, \widehat{\mathcal{K}}^{-1}KR \rangle = 0$. The equivalence of (b) and (c) is immediate from the Adams spectral sequence; cf. Proposition 11. The case $n = 1$ will be discussed further below.

In terms of the category of $\widehat{\mathcal{K}}$ -finite spectra discussed in Sect. 3.3.5, we can reformulate the Basic Conjecture in its second version as follows: Fix a residue field $\mathbb{F}$ of R as above. Now let $\beta_1, \dots, \beta_s$ denote the primes over ℓ in R_0 , and let d_i denote the number of primes over β_i in R_∞ . Finally, let $\mathbb{F}_i$ denote the extension of $\mathbb{F}$ of degree d_i .

47

Proposition 47 Assume for simplicity that $\mu_\ell \subset F$. Let A denote the complement of the set of Tate-twisted semi-discrete primes (in the set of height one primes of Λ'), and let $\mathcal{C} = \mathcal{C}_A$. Then Basic Conjecture II is equivalent to the existence of a mod $\mathcal{C}$ equivalence

$$L_{K(1)}KR \sim_{\mathcal{C}} L_{K(1)} \left(K\mathbb{F} \vee \Sigma \overline{K\mathbb{F}_1} \vee \Sigma K\mathbb{F}_2 \vee \dots \vee \Sigma K\mathbb{F}_s \vee \left(\bigvee^{\substack{r_2 \\ \Sigma^{-1}\widehat{\mathcal{K}}}} \right) \right),$$

where $\overline{K\mathbb{F}_1}$ denotes the cofibre of the natural map $K\mathbb{F} \rightarrow K\mathbb{F}_1$. (The ordering of the β_i 's is immaterial.)

We next consider various special cases of the Basic Conjecture.

Soulé's Theorem: This is the case $n \geq 2$ of the Basic Conjecture, which was proved in [43]. A proof can be given as follows: It suffices to show that $\pi_{2n}L_{K(1)}KR$ is finite for $n \geq 1$. By theorems of Borel and Quillen the groups $\pi_{2n}KR$ are finite for $n \geq 1$, so if the Lichtenbaum–Quillen conjecture holds for R we are done. But in fact all

we need is that $\pi_{2n}KR$ maps onto $\pi_{2n}L_{K(1)}KR$, and then the surjectivity theorem of Dwyer and Friedlander [9] completes the proof.

The Leopoldt Conjecture. The Leopoldt conjecture asserts that the map $\phi : E' \rightarrow U'$ from global to local ℓ -adically completed units is injective. The kernel D of ϕ is the Leopoldt defect defined earlier. There are several interesting equivalent versions (see [38]):

Theorem 48 The following are equivalent:

48

- 1) The Leopoldt conjecture for F
- 2) $M_{\infty}^{\Gamma_F} = 0$
- 3) F has exactly $r_2 + 1$ independent Z_{ℓ} -extensions

Note this is the case $n = 0$ of Basic Conjecture I.

In topological terms, we then have:

Proposition 49 The following are equivalent:

49

- 1) The Leopoldt conjecture for F
- 2) There are no nonzero fixed points for the action of the Adams operations on $\widehat{\mathcal{K}}^{-1}KR$.
- 3) $\pi_{-2}L_{K(1)}KR$ is finite.

Here we encounter a recurring and tantalizing paradox: On the one hand, the algebraic K -groups of R vanish by definition in negative degrees; on the face of it, then, they are useless for analyzing condition (3) above in the spirit of Soulé's theorem. On the other hand, we know that $L_{K(1)}KR$ is determined by any of its connective covers, and hence if Lichtenbaum–Quillen conjecture holds it is determined, in principle, by KR . The problem lies in making this determination explicit. Smashing with a Moore spectrum $M\mathbb{Z}/\ell^v$ makes the homotopy groups periodic, thereby relating negative homotopy to positive homotopy, but it is difficult to get much mileage out of this.

The Gross Conjecture. The case $n = 1$ of Basic Conjecture I is an algebraic version of the Gross conjecture [14]. For simplicity we will assume $\mu_{\ell} \subset R$, so that $\Lambda'_F = \Lambda_F$. Since the original Gross conjecture concerns totally real fields, this might seem like a strange assumption, but in the version to be discussed here the assumption can be eliminated by a simple descent. The algebraic Gross conjecture says that τ_1 occurs with multiplicity $s - 1$ in M_{∞} . The multiplicity is at least $s - 1$ by Corollary 35a. More precisely, we have:

Proposition 50 τ_1 occurs in M_{∞} with frequency $s - 1$. In other words,

50

$$\text{rank}_{Z_{\ell}} \text{Hom}_{\Lambda_F}(Z_{\ell}(1), M_{\infty}) = s - 1.$$

Proof Let ν denote the frequency. Using Theorem 45, Proposition 11 and Corollary 32, we have

$$\text{rank}_{Z_\ell} \pi_1 L_{K(1)} KR = r_2 + \nu.$$

But $\pi_1 L_{K(1)} KR \cong \pi_1 KR^\wedge \cong (R^\times)^\wedge$, and $(R^\times)^\wedge$ has rank $r_2 + s - 1$ as desired. (This is just Soulé's étale cohomology argument translated into topological terms.)

Now the homotopy groups by themselves can only detect the frequencies of the τ_n 's, not the multiplicities. Nevertheless, there is a curious homotopical interpretation of the Gross conjecture that we now explain.

Let ξ denote a generator of $\pi_{-1} L_{K(1)} S^0 \cong Z_\ell$, and recall that ξ has Adams filtration one.

51

Theorem 51 The Gross conjecture holds for R if and only if multiplication by ξ :

$$\pi_1 L_{K(1)} KR \rightarrow \pi_0 L_{K(1)} KR$$

has rank $s - 1$.

Proof The algebraic Gross conjecture holds for R if and only if

$$\text{rank}_{Z_\ell} \text{Hom}_{\Lambda_F}(M_\infty, \Lambda_F / \tau_1^2) = 2r_2 + s - 1.$$

Now let X denote the cofibre of $\xi : L_{K(1)} S^{-1} \rightarrow L_{K(1)} S^0$. Then X can be described as the unique object of $L_{K(1)} \mathcal{S}$ with $\widehat{\mathcal{K}}^1 X = 0$ and $\widehat{\mathcal{K}}^0 X = \Lambda / T^2$ (with trivial Δ action). The usual Adams spectral sequence argument shows that the Z_ℓ -rank of $[\Sigma X, L_{K(1)} KR]$ is the same as the rank of the Hom term appearing in the lemma above. Now consider the exact sequence

$$\pi_2 L_{K(1)} KR \xrightarrow{\xi} \pi_1 L_{K(1)} KR \rightarrow [\Sigma X, L_{K(1)} KR] \rightarrow \pi_1 L_{K(1)} KR \xrightarrow{\xi} \pi_0 L_{K(1)} KR.$$

Since $\pi_2 L_{K(1)} KR \cong (K_2 R)^\wedge$ is finite, we see that $[\Sigma X, L_{K(1)} KR]$ has the desired rank $2r_2 + s - 1$ if and only if $\xi : \pi_1 L_{K(1)} KR \rightarrow \pi_0 L_{K(1)} KR$ has rank $s - 1$.

52

Remark 52 Recall here that $\pi_0 L_{K(1)} KR \cong K_0^{\text{ét}} R$, and that there is a short exact sequence

$$0 \rightarrow L \rightarrow \pi_0 L_{K(1)} KR \rightarrow Z_\ell \rightarrow 0$$

with $L \cong H_{\text{ét}}^2(R; Z_\ell(1))$ and $\text{rank}_{Z_\ell} L = s - 1$. Since ξ has Adams filtration one, the image of multiplication by ξ lies in L .

Iwasawa's μ -Invariant Conjecture

3.6.2

Recall that if M is a finitely-generated Λ -module, $\mu(M)$ denotes the multiplicity of the prime ℓ in the associated elementary module E ; in other words, it measures the number of Λ/ℓ^a 's occurring in E , weighted by the exponents a .

Conjecture 53: (*Iwasawa's μ -invariant conjecture*) For any number field F , $\mu(M_\infty) = 0$. Equivalently, $\mu(A_\infty) = 0$.

53

The A_∞ version of the conjecture was motivated by the analogy with curves over a finite field; see [38] or [13]. Note that since $M_\infty^0 = 0$, for M_∞ the conjecture is equivalent to the statement that M_∞ is ℓ -torsion-free.

Proposition 54 The following are equivalent:

54

- 1) Iwasawa's μ -invariant conjecture
- 2) $\widehat{\mathcal{K}}^{-1}KR$ is ℓ -torsion-free
- 3) (Here we assume $\mu_\ell \subset R$ for simplicity.) If $\mathcal{C}$ denotes the thick subcategory of dualizable $K(1)$ -local spectra, $L_{K(1)}KR$ is equivalent mod $\mathcal{C}$ to a wedge of r_2 copies of $\Sigma^{-1}\widehat{\mathcal{K}}$.

The equivalence of (1) and (2) is immediate from Theorem 45, together with the fact that M_∞ has no finite submodules. The equivalence of (2) and (3) is an easy consequence of Theorem 45, the fact that M_∞ has Λ -rank r_2 , and the characterization of dualizable objects in Sect. 3.3.5.

Totally Real Fields

3.6.3

When the number field F is totally real, the Basic Conjectures can be formulated in terms of special values of ℓ -adic L -functions. Since the connection is not always easy to extract from the literature, we give a brief discussion here. See [38] or [13] for further information.

Algebraic Aspects

On the algebraic side, the Iwasawa theory of totally real fields simplifies somewhat for the following reason. Let F be such a field, let F_0 denote as usual the extension obtained by adjoining the ℓ -th roots of unity, and let F_0^+ denote the fixed field of complex conjugation – that is, the unique element σ of order 2 in Δ_F – acting on F_0 . Then σ acts on the various Iwasawa modules M_∞ , A'_∞ , etc. associated to F as above, and these split into ± 1 -eigenspaces: $M_\infty = M_\infty^+ \oplus M_\infty^-$, and so on. In terms of the idempotents e_i of $Z_\ell\Delta_F$, we are merely sorting the summands $e_i M$ according to the parity of i . Now it is not hard to show that every unit of $\mathcal{O}_0$ is the product of a unit of $\mathcal{O}_0^+$ and root of unity. The next result then follows easily from Corollary 32.

55 Proposition 55 $E_{\infty}^{-} = (E'_{\infty})^{-} = Z_{\ell}(1)$, and therefore M_{∞}^{+} is a Λ_F -torsion module.

Here the second statement follows from Corollary 35; note the Tate twist there that reverses the ± 1 -eigenspaces of σ .

As an illustration we compare the Iwasawa modules tM_{∞}^{+} and A_{∞}^{-} . Define J_{∞} by the short exact sequence

$$0 \rightarrow J_{\infty} \rightarrow A_{\infty} \rightarrow A'_{\infty} \rightarrow 0,$$

and note that there is an exact sequence of Λ'_F -modules

$$0 \rightarrow E_{\infty} \rightarrow E'_{\infty} \rightarrow Z_{\ell} S_{\infty} \rightarrow J_{\infty} \rightarrow 0.$$

Taking (-1) -eigenspaces, the proposition yields:

56 Corollary 56 $(Z_{\ell} S_{\infty})^{-} \cong J_{\infty}^{-}$.

Now recall from Sect. 3.2.3 the divisor $\mathcal{D}$ of a module, and note that $\mathcal{D}$ is additive on short exact sequences. Hence we have the divisor equation

$$\mathcal{D}(A_{\infty}^{-}) = \mathcal{D}((A'_{\infty})^{-}) + \mathcal{D}(B_{\infty}^{-}).$$

Finally, for a Λ'_F -module N let $\dot{N}$ denote N with the twisted Γ'_F -action $\gamma \cdot x = c(\gamma)\gamma^{-1}x$; note this twist takes $Z_{\ell}(n)$ to $Z_{\ell}(1-n)$. Then Corollary 35 yields the important fact:

57 Corollary 57 $\mathcal{D}(A_{\infty}^{-}) = \mathcal{D}(tM_{\infty}^{+})$.

A conjecture of Greenberg (see [13]) asserts that A_{∞}^{+} is finite. Assuming this conjecture, the contributions of the units and the class group to M_{∞} can be neatly separated into the $+$ and $-$ summands. In fact the torsion-free and torsion parts would then also separate into the $+$ and $-$ summands, except for the part coming from B_{∞} . Even without the Greenberg conjecture, this splitting into $+$ and $-$ summands is very useful; see for example the reflection principle as discussed in [46], Sect. 10, or [38], XI, Sect. 4.

L-functions

On the analytic side, the totally real fields are distinguished by their interesting ℓ -adic L -functions. Now an L -function typically involves a choice of Dirichlet character, or more generally a representation of the Galois group, so we emphasize from the outset that we are only going to consider a very special case: characters of Δ_F . These are the characters ω^i , where ω is the Teichmüller character and $0 \leq i \leq d$ ($d = [F_0 : F]$).

The following theorem was first proved for abelian fields F by Leopoldt and Kubota. It was proved in general, for arbitrary abelian L -functions, by Deligne and Ribet.

Theorem 58 Let F be a totally real field, and assume ℓ is odd. For each character $\chi = \omega^i$ of Δ_F with i even, there is a unique continuous function

$$L_\ell(s, \chi) : Z_\ell - \{1\} \rightarrow \mathbb{Q}_\ell$$

such that for all $n \geq 1$

$$L_\ell(1 - n, \chi) = L(1 - n, \chi \omega^{-n}) \prod_{\beta | \ell} (1 - \chi \omega^{-n}(\beta) N(\beta)^{n-1}).$$

Moreover, there are unique power series $g_i(T) \in \Lambda_F$ such that

$$L_\ell(1 - s, \chi) = \begin{cases} g_i(c_0^s - 1) & \text{if } i \neq 0 \\ g_i(c_0^s - 1)/(c_0^s - 1) & \text{if } i = 0. \end{cases}$$

Several remarks are in order. The L -function appearing on the right of the first equality is a classical complex L -function. The indicated values, however, are known to lie in $\mathbb{Q}(\mu_{\ell-1})$ and hence may be regarded as lying in $\mathbb{Q}_\ell$. One could try to define L_ℓ for odd χ or arbitrary F by the same interpolation property, but then the classical L -function values on the righthand side would be zero; indeed the functional equation for such L -functions (see [37], pp. 126–7 for a short and clear overview of this equation) shows that for $n \geq 1$ $L(1 - n, \omega^j)$ is nonzero if and only if F is totally real and $j = n \bmod 2$. Thus L_ℓ would be identically zero.

Turning to the Euler factors, recall that $N(\beta)$ is the cardinality of the associated residue field and that for any character χ , $\chi(\beta)$ is defined as follows: Let F^χ denote the fixed field of the kernel of χ , so that χ is pulled back from a faithful character of $G(F^\chi/F)$. If β is unramified in F^χ/F , we set $\chi(\beta) = \chi(\sigma_\beta)$, where σ_β is the associated Frobenius element. In practical terms, this means that $\chi(\beta) = 1$ if and only if β splits completely in F^χ . If β is ramified, we set $\chi(\beta) = 0$.

Iwasawa's Main Conjecture – motivated by the analogy with curves over a finite field – predicted that the power series g_i were twisted versions of characteristic series for the appropriate eigenspaces of A_∞ . The conjecture was proved by Wiles [47], and can also be formulated in terms of M_∞ ; the two versions are related by Corollary 57.

Theorem 59 With the notation of the preceeding theorem, the power series $g_i(T)$ is a characteristic series for the Λ_F -module $e_i M_\infty$.

Analytic Versions of the Basic Conjecture

Combining these two results, we can now translate Basic Conjecture I for F_0 into a statement about zeros of $L_\ell(s, \chi)$. The translation is not perfect; for example, as it stands we can only hope to get information about the even eigenspaces $e_i M_\infty$. If we assume Greenberg's conjecture, however, the torsion in the odd eigenspaces all comes from $B_\infty(1)$. Let us consider case by case:

The case $n > 1$: Note that τ_n is in the support of $e_i M$ – or in other words, $Z_\ell(n)$ occurs in $e_i M$ – if and only if the characteristic series for $e_i M$ vanishes at $c_0^n - 1$. By Wiles theorem and the second part of the Deligne–Ribet theorem, this in turn is equivalent to the vanishing of $L_\ell(1 - n, \omega^i)$. Assuming Greenberg’s conjecture, we get the clean statement that Basic Conjecture I for $n > 1$ is equivalent to the nonvanishing of the $L_\ell(1 - n, \omega^i)$. Now observe that the Euler factors appearing in the definition of $L_\ell(s, \chi)$ are units in Z_ℓ when $n > 1$. Hence $L_\ell(1 - n, \omega^i) = 0$ if and only if $L(s, \chi \omega^{-n}) = 0$. But as noted above, for $n \geq 1$, $L(1 - n, \omega^j)$ is nonzero if and only if F is totally real and $j = n \bmod 2$. Here we have $j = i - n$ with i even. This yields Basic Conjecture I for $n > 1$; that is, Soulé’s theorem – at least for the even eigenspaces $e_i M$.

The case $n = 1$: In this case a typical Euler factor has the form $1 - \chi \omega^{-1}(\mathcal{P})$, and hence will vanish precisely when β splits completely in $F^{\chi \omega^{-1}}$. Let m_χ denote the total number of such primes β . Note that the classical L -function factor does not vanish. Then it is natural to conjecture that $L_\ell(s, \chi)$ has a zero of order m_χ at $s = 0$. I will call this the analytic Gross conjecture, even though it is only a part of a special case of the general conjecture made by Gross in [14]; the general version not only considers much more general characters but also predicts the exact value of the leading coefficient. In view of Wiles’ theorem, we see at once that the analytic Gross conjecture is equivalent to the algebraic Gross conjecture given earlier, but restricted to the even eigenspaces $e_i M$. Once again, if we assume Greenberg’s conjecture, the algebraic Gross conjecture for the odd eigenspaces is automatic.

The case $n = 0$: Here we are just outside the range where $L_\ell(1 - n, \chi)$ is specified by Theorem 58. On the other hand, Wiles’ theorem tells us that if χ is nontrivial then $L_\ell(1, \chi)$ is defined, and is nonzero if and only if the algebraic Leopoldt conjecture holds for $e_i M$ ($\chi = \omega^i$). If $\chi = \omega^0$ is the trivial character, so that $L_\ell(s, \chi)$ is the ℓ -adic zeta function $\zeta_\ell(s)$, then a theorem of Colmez says that the algebraic Leopoldt conjecture for $e_0 M$ is equivalent to the assertion that $\zeta_\ell(s)$ has a simple pole at $s = 1$. Indeed this last equivalence would follow immediately from Wiles’ theorem, but Wiles’ proof uses Colmez’ theorem, so such an argument would be circular. In any case, we can now formulate an analytic Leopoldt conjecture – nonvanishing of $L_\ell(1, \chi)$ when χ is nontrivial, and the simple pole when χ is trivial – and the analytic form is equivalent to the algebraic Leopoldt conjecture for the even eigenspaces $e_i M_\infty$.

The case $n < 0$: Again we are outside the range where $L_\ell(1 - n, \chi)$ is specified. But in this case the classical values $L(1 - n, \chi)$ are obviously nonzero, by the Euler product formula. So (as far as I know) it is a reasonable conjecture that $L_\ell(1 - n, \chi)$ is also nonzero. Again, this corresponds to Basic Conjecture I for $n < 0$ and even eigenspaces.

The Generalized Lichtenbaum Conjecture

Finally, this is a good place to mention the generalized Lichtenbaum conjecture – or theorem, depending on which version of it is considered. Here is a version that

is known: For $x, y \in \mathbb{Q}_\ell$, write $x \sim_\ell y$ if $v_\ell x = v_\ell y$. Recall that $d = [F_0 : F]$, and let e_i , $0 \leq i < d$, denote the idempotents in $Z_\ell[\Delta_F]$ corresponding to the characters ω^i .

Theorem 60 Let F be a totally real field, and assume ℓ is odd. Then for $n > 1$ and $i = n \bmod 2$,

60

$$L(1-n, \omega^i) \sim_\ell \frac{|e_{-i} \pi_{2n-2} L_{K(1)} KR_0|}{|e_{-i} \pi_{2n-1} L_{K(1)} KR_0|}.$$

Remark 61 If the Lichtenbaum–Quillen conjecture is true for R_0 , then the homotopy groups in the fraction can be replaced by the corresponding ℓ -completed K -groups of R_0 . These homotopy groups coincide with the étale cohomology groups $H_{\text{ét}}^2(R_0; Z_\ell(n))$ (the numerator) and $H_{\text{ét}}^1(R_0; Z_\ell(n))$ (the denominator). In this form, a more general version of Theorem 60 is given in [24].

61

We sketch the proof, leaving the details to the reader. We use the abbreviation $\pi_m = \pi_m L_{K(1)} KR$. Of course we should first show that the groups appearing in the fraction are finite, so that the theorem makes sense. For the numerator this is clear from Soulé’s theorem, even before applying the idempotent e_{-i} . For the denominator we have:

Lemma 62

62

$$e_{-i} \pi_{2n-1} \cong \begin{cases} Z_\ell / (c_0^n - 1) & \text{if } i + n = 0 \bmod d \\ 0 & \text{otherwise.} \end{cases}$$

The proof is similar to some of the arguments below, but easier. Now using Wiles’ theorem we have

$$L(1-n, \omega^i) \sim_\ell L_\ell(1-n, \omega^{i+n}) \sim_\ell \begin{cases} g_{i+n}(c_0^n - 1) & \text{if } i + n \neq 0 \bmod d \\ g_{i+n}(c_0^n - 1) / (c_0^n - 1) & \text{if } i + n = 0 \bmod d. \end{cases}$$

Furthermore, for any power series g prime to τ_n we have $g(c_0^n - 1) \sim_\ell |Z_\ell(n)g|$. The next lemma is an interesting exercise in Λ -modules:

Lemma 63 Suppose M, N are finitely-generated Λ -torsion modules with disjoint support, and $M^0 = 0 = N^0$. Then $\text{Ext}_\Lambda^1(M, N)$ is finite, and $|\text{Ext}_\Lambda^1(M, N)|$ depends only on the divisor of M . In fact, if f is a characteristic series for M ,

63

$$|\text{Ext}_\Lambda^1(M, N)| = |N/fN|.$$

We conclude that

$$\begin{aligned} g_{i+n}(c_0^n - 1) &\sim_\ell \left| \text{Ext}_{\Lambda_F}^1(e_{i+n}M_\infty, Z_\ell(n)) \right| = \left| e_{-i} \text{Ext}_{\Lambda_F}^1(M_\infty, Z_\ell(n)) \right| \\ &= \left| e_{-i} \text{Ext}_{\Lambda'}^1(\widehat{\mathcal{K}} \cdot KR_0, \widehat{\mathcal{K}} \cdot S^{2n-2}) \right| = \left| e_{-i} \pi_{2n-2} \right|. \end{aligned}$$

Taking into account the first lemma, Theorem 60 follows.

Note that taking $i = 0$ yields the more familiar formula for the Dedekind zeta function:

$$\zeta_F(1-n) \sim_\ell \frac{|\pi_{2n-2} L_{K(1)} KR|}{|\pi_{2n-1} L_{K(1)} KR|}.$$

for $n > 1$ even. If we formulate the theorem in terms of the ℓ -adic L -function, we can conjecturally get values at positive integers also:

64

Corollary 64 Let F be a totally real field, and assume ℓ is odd. Suppose also that Basic Conjecture I holds for F_0 . Then for all $n \neq 0, 1$ and $i = n \bmod 2$,

$$L_\ell(1-n, \omega^{i+n}) \sim_\ell \frac{|e_{-i} \pi_{2n-2} L_{K(1)} KR_0|}{|e_{-i} \pi_{2n-1} L_{K(1)} KR_0|}.$$

The proof is the same as before. Note that we cannot take $n = 0$ because $e_0 \pi_{-1} L_{K(1)} KR_0 \cong \pi_{-1} L_{K(1)} KR$ always contains a copy of Z_ℓ in Adams filtration one coming from $\pi_{-1} L_{K(1)} S^0$ under the unit map $S^0 \rightarrow KR$; this Z_ℓ is also detected by mapping to a suitable residue field. We cannot take $n = 1$ because the primes over ℓ contribute to the ranks of π_0 and π_1 .

3.7 The K-Theory of $\mathbb{Z}$

Since $\mathbb{Q}$ is totally real, the material of the previous section applies to it. Furthermore, for any prime ℓ there is a unique prime over ℓ in $\mathbb{Q}_\infty$; in other words, $s_\infty = 1$. Hence $B_\infty = 0$, and $A_\infty = A'_\infty = L_\infty$, simplifying the analysis further. Note also that $\Gamma'_\mathbb{Q} = \Gamma'$, and hence $\Lambda'_\mathbb{Q} = \Lambda'$.

Many of the conjectures mentioned above are known for abelian fields, and in particular $\mathbb{Q}$, where the proofs are often easier. For example, the Leopoldt and Gross conjectures, and Iwasawa's μ -invariant conjecture, are known for abelian fields. Iwasawa's Main Conjecture, which was proved by Mazur–Wiles in the abelian case and by Wiles in general, has a much easier proof for the case $\mathbb{Q}$; see [46]. Furthermore the L -function values appearing in the generalized Lichtenbaum conjecture can be computed *a priori* in a more elementary form (see [46], Sect. 3.4):

65

Proposition 65 $L(1-n, \chi) = \frac{-B_{n,\chi}}{n}$, where $B_{n,\chi}$ is the n -th generalized Bernoulli number.

We recall that $B_{n,\omega^0} = B_n$; here B_n is the usual Bernoulli number, in the notation for which $B_n = 0$ if $n > 1$ is odd.

In the case $F = \mathbb{Q}$ there is an older and stronger version of the Greenberg conjecture, namely the Kummer–Vandiver conjecture. This conjecture asserts that for any prime ℓ , ℓ does not divide the order of the class group of $\mathbb{Q}_0^+$. Recall that for ℓ odd, $\mathbb{Q}_0$ means $\mathbb{Q}$ with the ℓ -th roots of unity adjoined, while the “+” means take the maximal real subfield. It is known to be true for all primes ℓ up to eight million or so. In our notation, the conjecture says that $A_0^+ = 0$.

Proposition 66 Suppose the Kummer–Vandiver conjecture holds for the prime ℓ . Then $A_n^+ = 0$ for all n , and hence $A_\infty^+ = 0$.

66

The proof is immediate, since $B_\infty = 0$ and $A_n = A'_n$ for all n ; hence $(A_\infty)_{\Gamma_n} = A_n$ for all n by Proposition 25a.

Theorem 67 Suppose the Kummer–Vandiver conjecture holds for ℓ . Then if M_∞ is the basic Iwasawa module associated to $R = \mathbb{Z}[1/\ell]$, we have

67

$$e_i M_\infty \cong \begin{cases} \Lambda & \text{if } i \text{ odd} \\ \Lambda/g_i & \text{if } i \text{ even} \end{cases}$$

where g_i is the power series associated to the ℓ -adic L -function $L_\ell(s, \omega^i)$ as in Theorem 58. Furthermore, $e_0 M_\infty = 0$.

Proof If i is odd, then the Kummer–Vandiver conjecture and the formula for tM given in Corollary 35 imply that M_∞^- is Λ -torsionfree of rank $r_2 \mathbb{Q}_0 = \frac{\ell-1}{2}$. Moreover, by Corollary 35 the freeness defect vanishes and hence M_∞^- is actually free of rank r_2 , and indeed M_∞^- is the twisted Λ -dual of E'_∞ . A standard argument with Dirichlet’s unit theorem then shows that all the odd characters occur; that is, $e_i M$ is free of rank one for each odd i .

If i is even, then $e_i M_\infty$ is Λ -torsion. The Main Conjecture then shows that $e_i M_\infty$ has characteristic series g_i , but in this case a more elementary argument proves a much stronger statement, namely that $e_i M_\infty \cong \Lambda/g_i$ ([46], Theorem 10.16; this step does not depend on the Kummer–Vandiver conjecture). When $i = 0$, g_0 is the numerator of the ℓ -adic zeta function, which has a simple pole at $s = 1$. Hence $g_0(0)$ is a unit in $\mathbb{Z}_\ell$, g_0 is a unit in Λ , and $\Lambda/g_0 = 0$. Again, see [46] for further details.

Now write $B_m/m = c_m/d_m$, where c_m and d_m are relatively prime.

Theorem 68 Assume the Lichtenbaum–Quillen conjecture and the Kummer–Vandiver conjecture at all primes. Then for $n \geq 2$ the K -groups of $\mathbb{Z}$ are given as follows, where $n = 2m - 2$ or $n = 2m - 1$ as appropriate:

$n \bmod 8$	π_n
0	0
1	$\mathbb{Z} \oplus \mathbb{Z}/2$
2	$\mathbb{Z}/c_m \oplus \mathbb{Z}/2$
3	$\mathbb{Z}/4d_m$
4	0
5	$\mathbb{Z}$
6	$\mathbb{Z}/c_m$
7	$\mathbb{Z}/2d_m$

Proof The 2-primary part of this result will be discussed in the next section. So we assume ℓ is odd, and show that the theorem holds at ℓ .

Given the structure of M_∞ , and assuming the Lichtenbaum–Quillen conjecture, there are two ways to proceed. First, one can easily compute the étale cohomology groups occurring in the descent spectral sequence by using the universal coefficient spectral sequence

$$\mathrm{Ext}_{\Lambda'}^p(H_q^{\mathrm{\acute{e}t}} R, Z_\ell(m)) \Rightarrow H_{\mathrm{\acute{e}t}}^{p+q}(R; Z_\ell(m)) ;$$

the descent spectral sequence collapses and one can read off the result from the Kummer–Vandiver conjecture. Alternatively, in the spirit of this paper, one can use the $\widehat{\mathcal{K}}$ -based Adams spectral sequence. This is what we will do; in any case the calculations involved are almost identical because in this simple situation the Adams and descent spectral sequences are practically the same thing.

Now we know that as Λ' -modules

$$\widehat{\mathcal{K}}^n K\mathbb{Z} \cong \begin{cases} M_\infty & \text{if } n = -1 \\ Z_\ell & \text{if } n = 0 . \end{cases}$$

Assuming the Lichtenbaum–Quillen conjecture, $K\mathbb{Z}^\wedge$ is essentially $K(1)$ -local, so we can use the $\widehat{\mathcal{K}}$ -based Adams spectral sequence to compute its homotopy. Displayed in its customary upper half-plane format, the Adams spectral sequence will have only two nonzero rows, namely filtrations zero and one. Hence the spectral sequence collapses, and there are no extensions because the bottom row is always Z_ℓ -torsionfree. First of all we have

$$E_2^{2m,0} = \mathrm{Hom}_{\Lambda'}(Z_\ell, Z_\ell(m)) \cong \begin{cases} Z_\ell & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E_2^{2m+1,0} = \text{Hom}_{\Lambda'}(M_\infty(-1), Z_\ell(m)) \cong \begin{cases} Z_\ell & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} . \end{cases}$$

The (-1) -twist occurs because we set up our Adams spectral sequence using $\widehat{\mathcal{K}}^\cdot$ where $\cdot = 0, 1$; since $\widehat{\mathcal{K}}^{-1}K\mathbb{Z}[\frac{1}{\ell}] = M_\infty$ we then have $\widehat{\mathcal{K}}^1K\mathbb{Z}[\frac{1}{\ell}] = M_\infty(-1)$. This accounts for all the non-torsion in the theorem. Next we have

$$E_2^{2m-1,1} = \text{Ext}_{\Lambda'}(Z_\ell, Z_\ell(m)) \cong \begin{cases} Z_\ell/(c_0^m - 1) & \text{if } m = 0 \bmod \ell - 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall our convention that $c_0 = 1 + \ell$, although the choice doesn't really matter. The Clausen–von Staudt theorem then implies that the denominator of the indicated Bernoulli number and $((1 + \ell)^m - 1)$ have the same powers of ℓ . This accounts for everything in degrees $n = 3 \bmod 4$, as well as the absence of ℓ -torsion in degrees $n = 1 \bmod 4$. Now let m be even. Then

$$E_2^{2m,1} = \text{Ext}_{\Lambda'}^1(M_\infty(-1), Z_\ell(m)) = 0$$

because $e_i(M_\infty(-1))$ is Λ -free for i even. This proves the theorem in degrees $n = 0 \bmod 4$. Finally,

$$E_2^{2m-2,1} = \text{Ext}_{\Lambda'}^1(\Lambda/g_m, Z_\ell(m)) .$$

If $m = 0 \bmod \ell - 1$, then $\Lambda/g_m = 0$. Since then ℓ does not divide c_m , by the Clausen–von Staudt theorem, the theorem is proved for this case. If $m \neq 0 \bmod \ell - 1$, the Ext group above is isomorphic to $Z_\ell/g_m(c_0^m - 1)$, and hence is cyclic of order ℓ^v , where $v = v_\ell(g_m(c_0^m - 1))$. But

$$g_m(c_0^m - 1) \sim_\ell L_\ell(1 - m, \omega^m) \sim_\ell L(1 - m, \omega^0) = -\frac{B_m}{m} .$$

This completes the proof.

Theorem 68 has the following converse (cf. [25]):

Theorem 69 Suppose $(K_{4n}\mathbb{Z})_\ell = 0$ for $2n \leq \ell - 3$. Then the Kummer–Vandiver conjecture holds at ℓ .

69

Proof sketch: Suppose the Kummer–Vandiver conjecture fails at ℓ . Then for some even i , $0 \leq i \leq \ell - 3$, $e_i(M_\infty(-1))$ is not free (either because of a torsion submodule, or because the torsion-free part is not free). It follows that

$$E_2^{2m,1} = \text{Ext}_{\Lambda'}^1(e_i(M_\infty(-1)), Z_\ell(m)) \neq 0 \text{ for all } m = i \bmod \ell - 1 .$$

Taking $i = 2n$, we conclude that $(K_{4n}\mathbb{Z})_\ell \neq 0$. (Here we do not need the Lichtenbaum–Quillen conjecture, because of the Dwyer–Friedlander surjectivity theorem [9].)

We conclude by discussing the conjectural homotopy-type of $K\mathbb{Z}[\frac{1}{\ell}]$. Consider first $L_{K(1)}K\mathbb{Z}[\frac{1}{\ell}]$. For a Λ -module M , write $M[i]$ for the Λ' -module obtained by letting Δ act on M as ω^i . Then we have seen above that

$$\widehat{\mathcal{K}}^0 K\mathbb{Z}[\frac{1}{\ell}] \cong Z_\ell[0]$$

and

$$\widehat{\mathcal{K}}^{-1} K\mathbb{Z}[\frac{1}{\ell}] \cong \Lambda[1] \oplus \Lambda[3] \oplus \dots \oplus \Lambda[\ell-2] \oplus (\Lambda/g_2)[2] \oplus \dots \oplus (\Lambda/g_{\ell-3})[\ell-3] .$$

Since these Λ' -modules have projective dimension one, $L_{K(1)}K\mathbb{Z}[\frac{1}{\ell}]$ splits into wedge summands corresponding to the indicated module summands. The $\widehat{\mathcal{K}}^0$ term contributes a copy of $L_{K(1)}S^0$. Each $\Lambda[i]$ contributes a desuspended Adams summand of $\widehat{\mathcal{K}}$; taken together, these free summands contribute a copy of $\Sigma KO^\wedge$. Finally, let X_i denote the fibre of $g_i : \Sigma^{-1}e_i\widehat{\mathcal{K}} \rightarrow \Sigma^{-1}e_i\widehat{\mathcal{K}}$. Then we have

$$L_{K(1)}K\mathbb{Z}[\frac{1}{\ell}] \cong L_{K(1)}S^0 \bigvee \Sigma KO^\wedge \bigvee X_2 \bigvee \dots \bigvee X_{\ell-3} .$$

Of course this description is only “explicit” to the extent that one knows the g_i ’s explicitly, at least up to units. But even our assumption of the Kummer–Vandiver conjecture does not yield this information. There is an auxiliary conjecture – see [46], Corollary 10.17 and its proof – that implies g_i is a unit times a certain linear distinguished polynomial. In any event, if the Lichtenbaum–Quillen conjecture holds at ℓ we conclude:

$$K\mathbb{Z}[\frac{1}{\ell}]^\wedge \cong j^\wedge \bigvee \Sigma bo^\wedge \bigvee Y_2 \bigvee \dots \bigvee Y_{\ell-3} ,$$

where $j^\wedge$ is the completed connective J-spectrum, bo is the (-1) -connected cover of KO , and the Y ’s are the (-1) -connected covers of the X ’s.

The spectrum $j^\wedge$ has the following algebraic model: Let p be any prime that generates the ℓ -adic units; such primes exist by Dirichlet’s theorem on arithmetic progressions. Then $j^\wedge \cong K\mathbb{F}_p^\wedge$, and the retraction map $K\mathbb{Z}[\frac{1}{\ell}]^\wedge \rightarrow j^\wedge$ in the conjectural equivalence above would correspond to the mod p reduction map in algebraic K -theory.

We remark also that when ℓ is a regular prime, all the X ’s and Y ’s are contractible. Even when ℓ is irregular, the proportion of nontrivial Y ’s tends to be low. For example, when $\ell = 37$ – the smallest irregular prime – there will be just one nontrivial summand Y_{32} , while for $\ell = 691$ – the first prime to appear in a Bernoulli numerator, if the Bernoulli numbers are ordered as usual – there are two: Y_{12}, Y_{200} . These assertions follow from the tables in [46], p. 350.

Homotopy-type of KR at the Prime 2

3.8

In this section we work at the prime 2 exclusively. If the number field F has at least one real embedding – and to avoid trivial exceptions, we will usually assume that it does – then R has infinite étale cohomological dimension for 2-torsion sheaves. This makes life harder. It is also known, however, that the higher cohomology all comes from the Galois cohomology of $\mathbb{R}$, and after isolating the contribution of the reals, one finds that life is not so hard after all.

On the topological side, the element of order 2 in Γ' causes trouble in a similar way. In particular, the ring of $\mathcal{K}$ -operations Λ' has infinite global dimension. Perhaps the best way around this problem would be to work with Bousfield's united K-theory [5], which combines complex, real and self-conjugate K-theory so as to obtain, loosely speaking, a ring of operations with global dimension two. We hope to pursue this approach in a future paper, but we will not use united K-theory here.

The Construction JKR

3.8.1

We begin by turning back the clock two or three decades. After Quillen's landmark work on the K-theory of finite fields, it was natural to speculate on the K-theory of $\mathbb{Z}$. The ranks of the groups were known, as well as various torsion subgroups at 2: (i) a cyclic subgroup in degrees $n = 7 \bmod 8$ coming from the image of the classical J-homomorphism in $\pi_* S^0$; (ii) a similar cyclic subgroup in degrees $n = 3 \bmod 8$, of order 16 and containing the image of J with index 2; and (iii) subgroups of order 2 in degrees $n = 1, 2 \bmod 8$, again coming from $\pi_* S^0$ and detected by the natural map $K\mathbb{Z} \rightarrow bo$.

The simplest guess compatible with this data is the following (I first heard this, or something like it, from Mark Mahowald): Define $JK\mathbb{Z}[\frac{1}{2}]$ by the following homotopy fibre square:

$$\begin{array}{ccc} JK\mathbb{Z}[\frac{1}{2}] & \rightarrow & bo^\wedge \\ \downarrow & & \downarrow \\ K\mathbb{Z}_3^\wedge & \xrightarrow[\theta]{} & bu^\wedge \end{array},$$

where θ is Quillen's Brauer lift, as extended to spectra by May. Then the algebraic data was consistent with the conjecture that $K\mathbb{Z}[\frac{1}{2}]^\wedge \cong JK\mathbb{Z}[\frac{1}{2}]$. But on the face of it, there is not even an obvious map from $K\mathbb{Z}[\frac{1}{2}]^\wedge$ to $JK\mathbb{Z}[\frac{1}{2}]$. There are natural maps from $K\mathbb{Z}[\frac{1}{2}]$ to $K\mathbb{F}_3$ and bo , but no apparent reason why these maps should be homotopic when pushed into bu .

Nevertheless, Bökstedt showed in [2] that there is a natural map from $K\mathbb{Z}[\frac{1}{2}]^\wedge$ to $JK\mathbb{Z}[\frac{1}{2}]$. The clearest way to construct such a map is to appeal to the work of Suslin.

Choose an embedding of the 3-adic integers $\mathbb{Z}_3$ into $\mathbb{C}$, and form the commutative diagram of rings

$$\begin{array}{ccc} \mathbb{Z}[\frac{1}{2}] & \rightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{Z}_3 & \rightarrow & \mathbb{C} \end{array}.$$

Applying the completed K -theory functor yields a strictly commutative diagram of spectra

$$\begin{array}{ccc} K\mathbb{Z}[\frac{1}{2}]^\wedge & \rightarrow & K\mathbb{R}^\wedge \\ \downarrow & & \downarrow \\ K\mathbb{Z}_3^\wedge & \rightarrow & K\mathbb{C}^\wedge \end{array}.$$

But by the well-known work of Suslin, $K\mathbb{R}^\wedge$ is $bo^\wedge$, $K\mathbb{C}^\wedge$ is $bu^\wedge$, and there is a commutative diagram

$$\begin{array}{ccc} K\mathbb{Z}_3^\wedge & \rightarrow & K\mathbb{C}^\wedge \\ \cong \downarrow & \nearrow \theta & \\ K\mathbb{F}_3^\wedge & & \end{array},$$

in which the vertical map is an equivalence. Here the Brauer lift θ should be chosen to be compatible with the embedding of $\mathbb{Z}_3$; alternatively, one could even use the above diagram to define the Brauer lift. In any case, this yields the desired map $\phi : K\mathbb{Z}[\frac{1}{2}]^\wedge \rightarrow JK\mathbb{Z}[\frac{1}{2}]$.

Of course this map is not canonical, as it stands. *A priori*, it depends on the choice of embedding of the 3-adic integers and on the choice of Brauer lift θ . Furthermore, the choice of the prime 3 was arbitrary to begin with; in fact, one could replace 3 by any prime $p = \pm 3 \bmod 8$. The alternatives 3 and $-3 \bmod 8$ definitely yield nonequivalent spectra $K\mathbb{F}_p^\wedge$, but the homotopy-type of $JK\mathbb{Z}[\frac{1}{2}]$ is the same.

3.8.2 The 2-Adic Lichtenbaum–Quillen Conjecture for $\mathbb{Z}[1/2]$

Building on work of Voevodsky, Rognes and Weibel [41] proved the algebraic form of the 2-adic Lichtenbaum–Quillen conjecture for number rings; that is, they computed $\pi_* KR^\wedge$ in terms of the étale cohomology groups $H_{\text{ét}}^*(R; \mathbb{Z}_2(n))$. In the case $R = \mathbb{Z}[\frac{1}{2}]$, the computation shows that $\pi_* K\mathbb{Z}[\frac{1}{2}]^\wedge \cong \pi_* JK\mathbb{Z}[\frac{1}{2}]$, but not that the isomorphism is induced by the map ϕ defined above. On the other hand, Bökstedt also showed that ϕ is surjective on homotopy groups. Since the groups in question are finitely-generated $\mathbb{Z}_2$ -modules, it follows that the naive guess is in fact true:

Theorem 70 There is a weak equivalence $K\mathbb{Z}[\frac{1}{2}]^\wedge \xrightarrow{\cong} JK\mathbb{Z}[\frac{1}{2}]$.

This result has many interesting consequences. For example, it leads immediately to a computation of the mod 2 homology of $GL\mathbb{Z}[\frac{1}{2}]$, since one can easily compute $H_*\Omega_0^\infty JK\mathbb{Z}[\frac{1}{2}]$; cf. [32]. Note this also gives the 2-primary part of Theorem 68. We also have:

Corollary 71 $K\mathbb{Z}[\frac{1}{2}]^\wedge$ is essentially $K(1)$ -local; in fact, $K\mathbb{Z}[\frac{1}{2}]^\wedge \rightarrow L_{K(1)}K\mathbb{Z}[\frac{1}{2}]$ induces an equivalence on (-1) -connected covers. 71

This is immediate since $JK\mathbb{Z}[\frac{1}{2}]$ has the stated properties.

Before stating the next corollary, we need to discuss the connective J -spectrum at 2. Perhaps the best definition is to simply take the (-1) -connected cover of $L_{\mathcal{K}}S^0$; or, for present purposes, the (-1) -connected cover of $L_{K(1)}S^0$. As noted earlier, for any $q = \pm 3 \bmod 8$ there is a noncanonical equivalence $L_{K(1)}S^0 \cong JO(q)^\wedge$, where $JO(q)$ is the fibre of $\psi^q - 1 : KO \rightarrow KO$. The (-1) -connected cover $jo(q)^\wedge$ has homotopy groups corresponding to the image of the classical J -homomorphism and the Adams μ -family in degrees $n = 1, 2 \bmod 8$, with one small discrepancy: There are extra $\mathbb{Z}/2$ summands in π_0 and π_1 . These elements can be eliminated via suitable fibrations, although there is no real need to do so. Nevertheless, to be consistent with received notation we will let $\tilde{j}$ denote the (-1) -connected cover of $L_{K(1)}S^0$, while $j^\wedge$ will denote the 2-adic completion of the traditional connective j -spectrum, in which the two spurious $\mathbb{Z}/2$'s have been eliminated. These are both ring spectra, and the natural map $j^\wedge \rightarrow \tilde{j}$ is a map of ring spectra.

We remark that these spectra have interesting descriptions in terms of algebraic K -theory: We have $\tilde{j} \cong KO\mathbb{F}_q^\wedge$, where $KO\mathbb{F}_q$ is the K -theory spectrum associated to the category of nondegenerate quadratic spaces over $\mathbb{F}_q$ ([11], p. 84ff and p. 176ff), while $j^\wedge \cong KND\mathbb{F}_q^\wedge$, where ND refers to the property “determinant times spinor norm equals 1” ([11], p. 68).

Corollary 72 Let X be any module spectrum over $K\mathbb{Z}[\frac{1}{2}]^\wedge$. For example, X could be the completed K -theory spectrum of any $\mathbb{Z}[\frac{1}{2}]$ -algebra (not necessarily commutative), or scheme over $Spec\mathbb{Z}[\frac{1}{2}]$, or category of coherent sheaves over such a scheme. 72

Then X has a natural module structure over the 2-adic connective J -spectrum $j^\wedge$; in fact, it has a natural module structure over $\tilde{j}$.

This result follows from the previous corollary. By taking connective covers in the diagram

$$\begin{array}{ccc} S^0 & \rightarrow & L_{K(1)}K\mathbb{Z}[\frac{1}{2}] \\ \downarrow & \nearrow & \\ L_{K(1)}S^0 & & \end{array}$$

we get unique maps of ring spectra $j^\wedge \rightarrow K\mathbb{Z}[\frac{1}{2}]^\wedge$ or $\tilde{j} \rightarrow K\mathbb{Z}[\frac{1}{2}]^\wedge$ factoring the unit map.

73 **Corollary 73** After localization at 2, the induced homomorphism $\pi_* S^0 \rightarrow K_* \mathbb{Z}$ factors through $\pi_* j^\wedge$.

This corollary was proved for all ℓ in [29], by a completely different method independent of the Lichtenbaum–Quillen conjecture. It is also known that the homotopy of $j^\wedge$ injects into $K_* \mathbb{Z}$, with only the following exception:

74 **Corollary 74** Let $Im J_n \subset \pi_n S^0$ denote the image of the classical J -homomorphism. Then for $n > 1$ and $n = 0, 1 \bmod 8$, $Im J_n$ maps to zero in $K_n \mathbb{Z}$.

This is immediate since $\pi_{8k} JK\mathbb{Z} = 0$ for $k > 0$, and $Im J_{8k+1} = \eta Im J_{8k}$. As far as I know, the only other proof of this fact is the original proof of Waldhausen [45].

The 2-Adic Lichtenbaum–Quillen Conjecture for General $\mathbb{R}$

3.8.3

In the general case a more sophisticated, systematic construction is required. One approach is the étale K -theory spectrum of Dwyer and Friedlander [7]. Their spectrum $K_{\text{ét}} R$ can be thought of as a kind of twisted $bu^\wedge$ -valued function spectrum on $\text{Spec } R$, where the latter is thought of as a space in its étale topology. In fact this “space” is in essence the classifying space of the Galois group $G(\Omega/F)$, where Ω is the maximal algebraic extension of F unramified away from 2; the only subtlety is that one must take into account the profinite topology on this Galois group.

Consider first the case $R = \mathbb{Z}[\frac{1}{2}]$. Then the 2-adic cohomological type of $\text{Spec } \mathbb{Z}[\frac{1}{2}]$ turns out to be very simple [8]. Define a homomorphism ξ from the free product $(\mathbb{Z}/2) * \mathbb{Z}$ to $G(\Omega/F)$ by sending the involution to complex conjugation and sending the free generator to any lift of $\gamma_q \in \Gamma_{\mathbb{Q}}$, where $q = \pm 3 \bmod 8$ as discussed above. This yields a map of classifying spaces

$$B\xi : \mathbb{R}P^\infty \vee S^1 \rightarrow \left(\text{Spec } \mathbb{Z}[\tfrac{1}{2}] \right)_{\text{ét}}$$

inducing an isomorphism on cohomology with locally constant 2-torsion coefficients. Hence we can replace $(\text{Spec } \mathbb{Z}[\frac{1}{2}])_{\text{ét}}$ with $\mathbb{R}P^\infty \vee S^1$ as the domain of our twisted function spectrum. The upshot is that up to connective covers there is a homotopy fibre square (suppressing 2-adic completions)

$$\begin{array}{ccc} K_{\text{ét}} \mathbb{Z}[\tfrac{1}{2}] & \rightarrow & bu^{h\mathbb{Z}/2} \\ \downarrow & & \downarrow \\ bu^{h\mathbb{Z}} & \rightarrow & bu \end{array}$$

corresponding to the pushout diagram of spaces

$$\begin{array}{ccc} * & \rightarrow & \mathbb{R}P^\infty \\ \downarrow & & \downarrow \\ S^1 & \rightarrow & \mathbb{R}P^\infty \vee S^1. \end{array}$$

Here $(-)^{hG}$ denotes the homotopy-fixed point construction $F^G(EG, -)$. Since – up to connective covers – we have $bu^{h\mathbb{Z}/2} = bo$ and $bu^{h\mathbb{Z}} = K\mathbb{F}_q$, we recover the spectrum $JK\mathbb{Z}[\frac{1}{2}]$.

For general R the cohomological type of $(Spec R)_{\acute{e}t}$ is not so simple; we have to just take $K_{\acute{e}t}R$ as it comes.

Another approach, due in different versions to Thomason and Jardine, starts from algebraic K -theory as a presheaf of spectra on the Grothendieck site $(Spec R)_{\acute{e}t}$. This leads to the étale hypercohomology spectrum $\mathbb{H}_{\acute{e}t}(Spec R; \underline{K})$ discussed in Sect. 3.5.

The strong form of the Lichtenbaum–Quillen conjecture was proved by Rognes–Weibel for purely imaginary fields, and by Østvær [39] for fields with a real embedding:

Theorem 75 The natural map $KR^{\wedge} \rightarrow \mathbb{H}_{\acute{e}t}(Spec R; \underline{K})^{\wedge}$ induces a weak equivalence on 0-connected covers. **75**

As before, we get the corollary:

Corollary 76 $KR^{\wedge}$ is essentially $K(1)$ -local. In fact, the natural map $KR^{\wedge} \rightarrow L_{K(1)}KR$ induces an equivalence on 0-connected covers. **76**

The map on π_0 in the corollary is injective, but gives an isomorphism if and only if there is a unique prime dividing 2 in $\mathcal{O}_F$.

To analyze the homotopy-type of $KR^{\wedge}$, then, it suffices to analyze the homotopy-type of $L_{K(1)}KR = L_{K(1)}\mathbb{H}_{\acute{e}t}(Spec R; \underline{K})^{\wedge}$. Now the topological K -theory of KR is given by the same formula as before:

$$\widehat{\mathcal{K}}^i KR \cong \begin{cases} \Lambda' \otimes_{\Lambda'_F} M_{\infty} & \text{if } i = -1 \\ \Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_2 & \text{if } i = -0. \end{cases}$$

The difference is that when F has at least one real embedding, both $\widehat{\mathcal{K}}^0$ and $\widehat{\mathcal{K}}^{-1}$ have infinite projective dimension as Λ' -modules, which complicates the analysis. To get around this we isolate the real embeddings, since these are the source of the homological difficulties. Define $K^{rel}R$ by the fibre sequence

$$K^{rel}R \rightarrow KR \rightarrow \prod_{i=1}^{r_1} bo.$$

We can then compute the topological K -theory of the relative term. Although the basic Iwasawa module M_{∞} will have infinite projective dimension, it fits into a canonical short exact sequence of Λ'_F -modules

$$0 \rightarrow M_{\infty} \rightarrow N_{\infty} \rightarrow \Lambda_F^{r_1} \rightarrow 0,$$

where N_{∞} has projective dimension one as Λ'_F -module. We then have:

$$\widehat{\mathcal{K}}^i K^{rel} R \cong \begin{cases} \Lambda' \otimes \Lambda'_F N_\infty & \text{if } i = -1 \\ 0 & \text{if } i = -0 \end{cases}$$

Then the homotopy-type of $K^{rel} R$ is completely determined by the Λ'_F -module N_∞ , in exactly the same way that the homotopy-type of $K^{red} R$ is completely determined by M_∞ in the odd-primary case. Finally, one can explicitly compute the connecting map $\prod^{r_1} bo \rightarrow \Sigma K^{rel} R$, yielding a complete description of the homotopy-type of KR .

Of course, Theorem 77 cannot be proved from the fibre sequence defining $K^{rel} R$ alone; this only gives $\widehat{\mathcal{K}}^* K^{rel} R$ up to an extension, and it is essential to determine this extension explicitly. The proof makes use of an auxiliary Grothendieck site associated to $Spec R$, defined by Zink [48]. In effect, one partially compactifies $Spec R$ by adjoining the real places as points at infinity. Up to connective covers $K^{rel} R$ is the hypercohomology of a relative K -theory presheaf on the Zink site, and this description leads to the computation above.

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The K -Theory of Triangulated Categories

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4.1	<i>Historical Survey</i>	1012
4.2	<i>Introduction</i>	1014
4.3	<i>Waldhausen Model Categories and Triangulated Categories</i>	1016
4.4	<i>Virtual Triangles</i>	1019
4.5	<i>Categories with Squares</i>	1020
4.6	<i>Regions</i>	1023
4.7	<i>The Simplicial Set</i>	1025
4.8	<i>What It All Means</i>	1029
4.9	<i>Waldhausen Models and the Existence of Large Simplices</i>	1042
4.10	<i>The Main Theorems</i>	1051
4.11	<i>Computational Problems</i>	1054
4.12	<i>Functoriality Problems</i>	1055
4.13	<i>Localisation</i>	1057
4.14	<i>Bounded δ-Functors</i>	1059
4.15	<i>Devissage</i>	1065
4.16	<i>About the Proofs</i>	1068
4.17	<i>Appendix: Examples of $D(\mathcal{A}) = D(\mathcal{B})$</i>	1069
	<i>References</i>	1073

* This manuscript contains the expanded notes of a talk I gave in Paris, at the working seminar run by Georges Maltsiniotis and Bernhard Keller. I would like to thank Keller and Maltsiniotis for inviting me to speak, and Jussieu for its hospitality during my visit.

Summary. The purpose of this survey is to explain the open problems in the K -theory of triangulated categories. The survey is intended to be very easy for non-experts to read; I gave it to a couple of fourth-year undergraduates, who had little trouble with it. Perhaps the hardest part is the first section, which discusses the history of the subject. It is hard to give a brief historical account without assuming prior knowledge. The students are advised to skip directly to Sect. 4.2.

4.1 Historical Survey

The fact that the groups K_0 and K_1 are related to derived categories is so obvious that it was observed right at the beginnings of the subject. We remind the reader.

Let $\mathcal{A}$ be a small abelian (or exact) category. Let $D^b(\mathcal{A})$ be its bounded derived category.¹ The category $D^b(\mathcal{A})$ is a triangulated category. What we will now do is define, for every triangulated category $\mathcal{T}$, an abelian group $K_0(\mathcal{T})$. This definition has the virtue that there is a natural isomorphism $K_0(\mathcal{A}) = K_0(D^b(\mathcal{A}))$. By $K_0(\mathcal{A})$ we understand the usual Grothendieck group of the exact category $\mathcal{A}$, while $K_0(D^b(\mathcal{A}))$ is as follows:

Definition 1 Let $\mathcal{T}$ be a small triangulated category. Consider the abelian group freely generated by the isomorphism classes $[X]$ of objects $X \in \mathcal{T}$. The group $K_0(\mathcal{T})$ is obtained by dividing by the relations generated by all expressions $[X] - [Y] + [Z]$, where there exists a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

The relation between $K_1(\mathcal{A})$ and $D^b(\mathcal{A})$ is not so simple. For example it was not known, until quite recently, how to give a definition of $K_1(\mathcal{A})$ which builds on $D^b(\mathcal{A})$. But the fact that K_1 is related (more loosely) to derived categories was known. This goes back to Whitehead's work on determinants of automorphisms of chain complexes and simple homotopy type.

Practically as soon as higher K -theory was defined, its relation with derived category was implicit. One of the first theorems in Quillen's foundational paper on the subject is the resolution theorem [77, Theorem 3 and Corollary 1 of §4]. The theorem says approximately the following:

Theorem 2 (Modified version of Quillen's theorem) Let $i : \mathcal{A} \rightarrow \mathcal{B}$ be a fully faithful, exact embedding of the exact category $\mathcal{A}$ into the exact category $\mathcal{B}$. Assume that the induced map of bounded derived categories $D^b(i) : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$ is an equivalence. Then the induced map in Quillen's K -theory $K(i) : K(\mathcal{A}) \rightarrow K(\mathcal{B})$ is a homotopy equivalence.

¹ For an abelian category $\mathcal{A}$, the definition of $D^b(\mathcal{A})$ is classical. See Verdier [94], or Hartshorne [41, Chapter I]. When $\mathcal{A}$ is only an exact category there was some confusion about how to define $D^b(\mathcal{A})$; see [64].

The reader is referred to Quillen's original paper, or to Theorem 73 of this article, for Quillen's precise formulation (which does not explicitly mention derived categories).

To make K -theory into a useful tool, it is important to understand how $K(\mathcal{A})$ changes with $\mathcal{A}$. Let $f : \mathcal{B} \rightarrow \mathcal{C}$ be an exact functor of exact categories. It induces a continuous map $K(f) : K(\mathcal{B}) \rightarrow K(\mathcal{C})$. The homotopy fiber of this map is a spectrum, and it turns out to be very useful to describe it in some computable way, for example as $K(\mathcal{A})$ for some $\mathcal{A}$. The first theorem of this sort was Quillen's localisation theorem [77, Theorem 5 of §5]. Quillen's theorem was very powerful, with many important consequences, for example in algebraic geometry. But, while on the subject of the algebro-geometric applications, it should be noted that to apply the theorem effectively one had to restrict to smooth varieties, or varieties with very mild singularities. Important work followed, trying to generalise this to singular varieties. The reader is referred to Levine [57, 58] and Weibel [97, 98]. The definitive treatment did not come until Thomason [89], and for his work Thomason needed a more powerful foundational basis. It turns out that the homotopy fiber of the map $K(f)$ above can be expressed as $K(\mathcal{A})$, but only if we are willing to understand by this the Waldhausen K -theory of a suitable Waldhausen category $\mathcal{A}$. In other words, to obtain a sufficiently powerful general theorem one needed the domain of the K -theory functor to be expanded. Progress depended on K -theory being defined in greater generality.

Waldhausen's work [96] provided a far more general setting for studying K -theory. To every Waldhausen model category $\mathcal{C}$ one attaches a K -theory spectrum $K(\mathcal{C})$. There is a brief discussion of Waldhausen model categories, and of their relation with triangulated categories, in Sect. 4.3. For our purposes the important observation is that, once again, there is a clear relation with triangulated categories. To each Waldhausen category $\mathcal{C}$ one can associate a triangulated category $\mathrm{ho}(\mathcal{C})$. Waldhausen's approximation theorem says that, under some technical hypotheses,

Theorem 3 (Waldhausen Approximation Theorem, without the technical hypotheses) Suppose $i : \mathcal{C} \rightarrow \mathcal{D}$ is an exact functor of Waldhausen model categories. Suppose the induced map of triangulated categories

$$\mathrm{ho}(i) : \mathrm{ho}(\mathcal{C}) \rightarrow \mathrm{ho}(\mathcal{D})$$

is an equivalence. Then the K -theory map $K(i) : K(\mathcal{C}) \rightarrow K(\mathcal{D})$ is a homotopy equivalence.

All of this suggests very naturally that K -theory and triangulated categories ought to be related. We still do not understand the relation, and this survey is mainly about the many open problems in the field.

But while we are still on the history of the problem, let me discuss the work that has been done. In the light of Waldhausen's approximation theorem, it is natural to ask whether Waldhausen's K -theory depends only on triangulated categories.

Given a Waldhausen category $\mathcal{C}$, Waldhausen defined a spectrum $K(\mathcal{C})$. Does this spectrum only depend on $\mathrm{ho}(\mathcal{C})$? If so, is the dependence functorial? I believe the question was first asked in Thomason [89].

The answer turns out to be No. In a paper by myself [65] I produce an example of a pair of Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, and a triangulated functor $f : \mathrm{ho}(\mathcal{C}) \rightarrow \mathrm{ho}(\mathcal{D})$ which cannot possibly induce a map in Waldhausen K -theory. More recently Schlichting [85] produces a pair of Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, with $\mathrm{ho}(\mathcal{C}) \simeq \mathrm{ho}(\mathcal{D})$ but $K(\mathcal{C}) \not\simeq K(\mathcal{D})$.

This establishes that Waldhausen's K -theory $K(\mathcal{C})$ depends on more than just $\mathrm{ho}(\mathcal{C})$. But it still leaves unresolved the question of whether we can recover Quillen's K -theory of an abelian (or exact?) category $\mathcal{A}$ from the triangulated category $D^b(\mathcal{A})$. This question has interested people since the 1980's. Kapranov tells me that they held a seminar about it in Moscow at the time. There were several counterexamples produced. The reader can see some of them in Hinich and Schechtman [42, 43] and Vaknin [91, 92]. By the mid 1980's, the consensus was that it could not be done.

Then in the late 1980's and early 1990's I proved a theorem, establishing the unexpected. For abelian categories, all of Quillen's higher K -theory may be recovered directly from the derived category. In the first half of this survey I state carefully the results I proved, and in the second half I explain the many open problems that remain.

Still in the historical survey, I should mention that Matthias Künzer also worked on this. He produced a construction and several very interesting conjectures. Unfortunately none of this ever appeared in print. His constructions were actually quite similar to mine. The key difference was that his constructions did not come with coherent differentials (these will be described in detail in Definition 21). For what it may be worth, let me quote Thomason who said that the key input in my work was the coherent differentials.

Also deserving mention is the fascinating work of Maltsiniotis, Cisinski and (somewhat later) Garkusha. Their work begins with something intermediate between the Waldhausen category $\mathcal{C}$ and its triangulated category $\mathrm{ho}(\mathcal{C})$. Starting from the derivator associated to $\mathcal{C}$, one can define a K -theory by modifying Waldhausen's construction in a straight-forward way. It is interesting to study this, and the reader can find excellent accounts in

<http://www.math.jussieu.fr/~maltsin/Gtder.html>

<http://www.ictp.trieste.it/~garkusha/papers/sdc.ps>

For a recent result, about the limitations on what one can expect to achieve using derivators, see Toën [90, Proposition 2.17 and Corollary 2.18].

4.2

Introduction

The aim of this manuscript is to explain just how little we know about the K -theory of triangulated categories. There are many fascinating open problems in the subject. I am going to try to make the point that a bright young math-

ematician, with plenty of imagination, could make impressive progress in the field. What we now know is enough to establish that the field is interesting. But the most basic, immediate questions that beg to be answered are completely open.

The best way to explain how little we know is to tell you all of it. Therefore we begin with a fairly careful account of all the existing theorems in the field.

Unfortunately, this requires us to be a little technical. It forces us to introduce five simplicial sets and four maps connecting them. Let $\mathcal{T}$ be a triangulated category with a bounded t -structure. Let $\mathcal{A}$ be the heart. Suppose $\mathcal{T}$ has at least one Waldhausen model. The first half of the manuscript produces five simplicial sets and four maps

$$S_*(\mathcal{A}) \xrightarrow{\alpha} {}^wS_*(\mathcal{T}) \xrightarrow{\beta} S_*(d\mathcal{T}) \xrightarrow{\gamma} S_*({}^v\mathcal{T}) \xrightarrow{\delta} S_*(\mathrm{Gr}^b\mathcal{A}).$$

The only simplicial set the reader might already be familiar with is $S_*(\mathcal{A})$, the Waldhausen S_* -construction applied to the abelian category $\mathcal{A}$.

The main theorem is Theorem 50. It tells us

- (i) The composite $\delta\gamma\beta\alpha$ induces a homotopy equivalence.
- (ii) The map α induces a homotopy equivalence.
- (iii) The simplicial set $S_*({}^v\mathcal{T})$ has a homotopy type which depends only on $\mathcal{A}$.
That is, $S_*({}^v\mathcal{T}) \cong S_*({}^vD^b(\mathcal{A}))$.

Perhaps part (i) of this is the most striking. Each of the simplicial sets ${}^wS_*(\mathcal{T})$, $S_*(d\mathcal{T})$ and $S_*({}^v\mathcal{T})$ defines a K -theory for our triangulated category $\mathcal{T}$. We have three candidates for what the right definition might be. By Theorem 50(i), all of them contain the Quillen K -theory of $\mathcal{A}$ as a retract. Any half-way sensible definition of the K -theory of derived categories contains Quillen's K -theory. Passing to the derived category most certainly does not lose K -theoretic information.

I have tried to organise the material so that the introductory part, the part where we define the four simplicial maps α , β , γ and δ , is short. I tried to condense this part of the manuscript without sacrificing the accuracy. It is helpful to have the exact statements of the theorems we now know. It helps delineate the extent of our ignorance.

After setting up the simplicial machinery and stating Theorem 50, we very briefly explain how it can be used to draw very strong conclusions about K -theory. This part is very brief. As I have already said, we focus mostly on the shortcomings of the theory, as it now stands. This allows us to highlight the many open problems.

In this entire document we will consider only small categories. The abelian categories, triangulated categories and Waldhausen model categories will all be small categories.

Waldhausen Model Categories and Triangulated Categories

4.3

In this survey we assume some familiarity with triangulated categories. It also helps to know a little bit about their models. This modest introductory section will attempt to provide the very minimum, bare essentials. Instead of developing the axiomatic formalism, we will give the key examples of interest.

Example 4. Let $\mathcal{A}$ be an abelian category. The category $C(\mathcal{A})$ is the category of chain complexes in $\mathcal{A}$. The objects are the chain complexes

$$\cdots \xrightarrow{\partial} x_{i-1} \xrightarrow{\partial} x_i \xrightarrow{\partial} x_{i+1} \xrightarrow{\partial} \cdots,$$

where $\partial\partial = 0$. The morphisms are the chain maps; that is the commutative diagrams

$$\begin{array}{ccccccc} \cdots & \longrightarrow & x_{i-1} & \longrightarrow & x_i & \longrightarrow & x_{i+1} & \longrightarrow & \cdots \\ & & \downarrow f_{i-1} & & \downarrow f_i & & \downarrow f_{i+1} & & \\ \cdots & \longrightarrow & y_{i-1} & \longrightarrow & y_i & \longrightarrow & y_{i+1} & \longrightarrow & \cdots \end{array}.$$

So far, we have defined a category.

It is customary to consider $C(\mathcal{A})$ as a Waldhausen category. This means endowing it with a great deal of extra structure. First of all, we consider three subcategories $cC(\mathcal{A})$, $fC(\mathcal{A})$ and $wC(\mathcal{A})$. These subcategories all have the same objects, namely all the objects of $C(\mathcal{A})$. It is the morphisms that are restricted. The restrictions are

- (i) A morphism in $cC(\mathcal{A})$, also called a *cofibration in $C(\mathcal{A})$* , is a chain map of chain complexes so that, for every $i \in \mathbb{Z}$, the map $f_i : x_i \rightarrow y_i$ is a split monomorphism. (The splittings are not assumed to be chain maps).
- (ii) A morphism in $fC(\mathcal{A})$, also called a *fibration in $C(\mathcal{A})$* , is a chain map of chain complexes so that, for every $i \in \mathbb{Z}$, the map $f_i : x_i \rightarrow y_i$ is a split epimorphism. (Once again, the splittings are not assumed to be chain maps).
- (iii) A morphism in $wC(\mathcal{A})$, also called a *weak equivalence in $C(\mathcal{A})$* , is a chain map of chain complexes inducing an isomorphism in homology.

One also assumes that there is a functor, called the *cylinder functor*, taking a morphism in $C(\mathcal{A})$ to an object, called the *mapping cylinder*. Let me not remind the reader of the detail of this construction. In Example 6 we will see the related construction of the *mapping cone*, which is more relevant for us. An important consequence of the existence of mapping cylinders (or mapping cones) is that the

category $C(\mathcal{A})$ has an automorphism, called the suspension functor, and denoted $\Sigma : C(\mathcal{A}) \rightarrow C(\mathcal{A})$. It takes the complex

$$\cdots \xrightarrow{\partial} x_{i-1} \xrightarrow{\partial} x_i \xrightarrow{\partial} x_{i+1} \xrightarrow{\partial} \cdots$$

to the complex

$$\cdots \xrightarrow{-\partial} x_i \xrightarrow{-\partial} x_{i+1} \xrightarrow{-\partial} x_{i+2} \xrightarrow{-\partial} \cdots .$$

In other words, Σ shifts the degrees by one, and changes the sign of the differential ∂ .

Remark 5 The data above, that is the three subcategories $cC(\mathcal{A})$, $fC(\mathcal{A})$ and $wC(\mathcal{A})$ and the cylinder functor, satisfy a long list of compatibility conditions. We omit all of them. The interested reader can find a much more thorough treatment in Chapter 1 of Thomason's [89]. Thomason calls the categories satisfying this long list of properties *biWaldhausen complicial categories*. In this paper we will call them *Waldhausen model categories*, or just *Waldhausen categories* for brevity. The experts, please note: what we call Waldhausen model categories is exactly the same as Thomason's biWaldhausen complicial categories. This allows us to freely quote results from [89].

5

Example 6. Suppose we start with a Waldhausen model category, like $C(\mathcal{A})$. We can form a category, often denoted $\text{ho}C(\mathcal{A})$. It is called the *homotopy category of $C(\mathcal{A})$* , and is obtained from $C(\mathcal{A})$ by formally inverting the weak equivalences. In the case of the Waldhausen category $C(\mathcal{A})$, the category $\text{ho}C(\mathcal{A})$ is usually called the *derived category of $\mathcal{A}$* , and denoted $D(\mathcal{A})$. The suspension functor descends to an automorphism of $\text{ho}C(\mathcal{A}) = D(\mathcal{A})$. The category $D(\mathcal{A})$ is a *triangulated category*; it satisfies a very short list of axioms. Basically, the only construction one has is the mapping cone. Suppose we are given two chain complexes X and Y , and a map of chain complexes $f : X \rightarrow Y$. That is, we are given a commutative diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial} & x_{i-1} & \xrightarrow{\partial} & x_i & \xrightarrow{\partial} & x_{i+1} \xrightarrow{\partial} \cdots \\ & & \downarrow f_{i-1} & & \downarrow f_i & & \downarrow f_{i+1} \\ \cdots & \xrightarrow{\partial} & y_{i-1} & \xrightarrow{\partial} & y_i & \xrightarrow{\partial} & y_{i+1} \xrightarrow{\partial} \cdots \end{array}$$

We can form the mapping cone, which is a chain complex

$$\cdots \longrightarrow x_i \oplus y_{i-1} \xrightarrow{\begin{pmatrix} -\partial & 0 \\ f_i & \partial \end{pmatrix}} x_{i+1} \oplus y_i \xrightarrow{\begin{pmatrix} -\partial & 0 \\ f_{i+1} & \partial \end{pmatrix}} x_{i+2} \oplus y_{i+1} \longrightarrow \cdots .$$

It turns out that this mapping cone, which we will denote $\text{Cone}(f)$, is well-defined in the category $\text{hoC}(\mathcal{A}) = D(\mathcal{A})$. One can look at the maps

$$X \xrightarrow{f} Y \xrightarrow{g} \text{Cone}(f) .$$

Of course, there is nothing to stop us from iterating this process. We can continue to

$$X \xrightarrow{f} Y \xrightarrow{g} \text{Cone}(f) \xrightarrow{h} \text{Cone}(g) \xrightarrow{i} \text{Cone}(h) \xrightarrow{j} \cdots .$$

Contrary to what we might expect, this process soon begins to iterate. There is a natural commutative square in $D(\mathcal{A})$, where the vertical maps are isomorphisms

$$\begin{array}{ccc} \text{Cone}(g) & \xrightarrow{i} & \text{Cone}(h) \\ \downarrow \wr & & \downarrow \wr \\ \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \end{array} .$$

That is, up to suspension and sign, the diagram is periodic with period 3. We call any diagram

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

isomorphic to

$$X \xrightarrow{f} Y \xrightarrow{g} \text{Cone}(f) \xrightarrow{h} \text{Cone}(g)$$

a *distinguished triangle* in $D(\mathcal{A})$. There is a very short list of axioms which distinguished triangles satisfy, and that is all the structure there is in $D(\mathcal{A})$. The axiomatic treatment may be found, for example, in Verdier's thesis [94], in Hartshorne [41, Chapter 1], or in the recent book [73].

Remark 7 It is quite possible for a single triangulated category $\mathcal{T}$ to have many different Waldhausen models. For instance, there are many known examples of abelian categories $\mathcal{A}$ and $\mathcal{B}$, with $D(\mathcal{A}) = D(\mathcal{B})$.² The models $C(\mathcal{A})$ and $C(\mathcal{B})$ are quite different, non-isomorphic Waldhausen categories. The passage from $C(\mathcal{A})$ to $\mathrm{ho}C(\mathcal{A}) = D(\mathcal{A})$ loses a great deal of information. What we will try to explain is that higher K-theory is not among the information which is lost.

Virtual Triangles

We need to remind the reader briefly of some of the results in Vaknin's [93]. In any triangulated category $\mathcal{T}$, Vaknin defined a hierarchy of triangles. When we use the word *triangle* without an adjective, we mean a diagram

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

so that $vu = wv = \{\Sigma u\}w = 0$. Vaknin defines classes of triangles

$$\text{splitting} \subset \text{distinguished} \subset \text{exact} \subset \text{virtual}.$$

The definitions are as follows.

(i) A *splitting triangle* is a direct sum of three triangles

$$\begin{array}{ccccccc} A & \xrightarrow{1} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ & & & & & & \\ 0 & \longrightarrow & B & \xrightarrow{1} & B & \longrightarrow & 0 \\ & & & & & & \\ \Sigma^{-1}C & \longrightarrow & 0 & \longrightarrow & C & \xrightarrow{1} & C \end{array}.$$

(ii) A *distinguished triangle* is part of the structure that comes for free, just because $\mathcal{T}$ is a triangulated category.

(iii) A *triangle*

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

² The first example may have been the one in Beilinson's 1978 article [11]. By now, a quarter of a century later, we know a wealth of other examples. A very brief discussion is included in an appendix; see Sect. 4.17.

is *exact* if there exist maps u' , v' and w' so that the following three triangles

$$\begin{array}{ccccc} A & \xrightarrow{u'} & B & \xrightarrow{v} & C & \xrightarrow{w} & \Sigma A \\ A & \xrightarrow{u} & B & \xrightarrow{v'} & C & \xrightarrow{w} & \Sigma A \\ A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w'} & \Sigma A \end{array}$$

are all distinguished.

- (iv) A triangle T is *virtual* if there exists a splitting triangle S so that $S \oplus T$ is exact.

The important facts for us to observe here are

8 **Lemma 8** All distinguished triangles are virtual.

9 **Lemma 9** Homological functors take virtual triangles to long exact sequences.

Proof Lemma 8 may be found in Vaknin's [93, Remark 1.4]. For Lemma 9, see [93, Definition 1.6 and Theorem 1.11].

4.5

Categories with Squares

The input we will need to define K -theory is a *category with squares*. In Sect. 4.7 we will see how, starting with a category with squares, one can define a K -theory. This section prepares the background. We will see the definition of a category with squares, and also the key examples of interest.

10 **Definition 10** An additive category $\mathcal{T}$ will be called a *category with squares* provided

- (i) $\mathcal{T}$ has an automorphism $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$.
(ii) $\mathcal{T}$ comes with a collection of *special squares*

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array} \quad \begin{array}{c} \curvearrowright \\ (1) \end{array}$$

This means that the square

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

is commutative in $\mathcal{T}$, and there is a map $D \rightarrow \Sigma A$, which we denote by the curly arrow

$$\begin{array}{c} D \\ \searrow \\ A \end{array} \quad (1)$$

The (1) in the label of the arrow is to remind us that the map is of degree 1, that is a map $D \rightarrow \Sigma A$.

Definition 11 Given two categories with squares, a *special functor*

11

$$F : \mathcal{S} \rightarrow \mathcal{T}$$

is an additive functor such that

- (i) There is a natural isomorphism $\Sigma F \cong F\Sigma$.
- (ii) The functor F takes special squares in $\mathcal{S}$ to special squares in $\mathcal{T}$.

The next definition is a convenient tool in the discussion of the examples.

Definition 12 Let $\mathcal{T}$ be an additive category with an automorphism $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$. Suppose we are given a square

12

$$\begin{array}{ccc} C & \xrightarrow{\delta} & D \\ \beta \uparrow & & \uparrow \gamma \\ A & \xrightarrow{\alpha} & B \end{array} \quad \begin{array}{c} \searrow \\ \mu \end{array}$$

The *fold of this square* will be the the sequence

$$A \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} D \xrightarrow{\mu} \Sigma A.$$

Example 13. Let $\mathcal{T}$ be a triangulated category. Then $\mathcal{T}$ is an additive category, and it comes with an automorphism $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$. A square is defined to be special if and only if its fold is a distinguished triangle in $\mathcal{T}$. When we think of the triangulated category $\mathcal{T}$ as being the category with squares defined above, then we will denote it as ${}^d\mathcal{T}$.

Example 14. Given a triangulated category $\mathcal{T}$, we wish to consider yet another possible structure one can give it, as a category with squares. The suspension functor $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$ is the same as in ${}^d\mathcal{T}$. But there are more special squares. In the category which we will call ${}^v\mathcal{T}$, a square will be special if and only if its fold is a *virtual* triangle, in the sense of Vaknin [93] (see also Sect. 4.4).

Example 15. Let $\mathcal{A}$ be an abelian category. Let $\text{Gr}^b\mathcal{A}$ be the category of bounded, graded objects in $\mathcal{A}$. We remind the reader. A *graded object* of $\mathcal{A}$ is a sequence of objects $\{a_i \mid i \in \mathbb{Z}, a_i \in \mathcal{A}\}$. The sequence $\{a_i\}$ is *bounded* if $a_i = 0$ except for finitely many $i \in \mathbb{Z}$.

We define the functor $\Sigma : \text{Gr}^b\mathcal{A} \rightarrow \text{Gr}^b\mathcal{A}$ to be the shift. That is,

$$\Sigma\{a_i\} = \{b_i\}$$

with $b_i = a_{i+1}$. A square in $\text{Gr}^b\mathcal{A}$ is defined to be *special* if the fold

$$A \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} D \xrightarrow{\mu} \Sigma A$$

gives a long exact sequence in $\mathcal{A}$. That is, the fold gives us a sequence

$$\cdots \longrightarrow D_{i-1} \longrightarrow A_i \longrightarrow B_i \oplus C_i \longrightarrow D_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

and we require that this sequence be exact everywhere.

Example 16. In Definition 11, a *special functor* $\mathcal{S} \rightarrow \mathcal{T}$ was defined to be an additive functor taking special squares in $\mathcal{S}$ to special squares in $\mathcal{T}$. Let $\mathcal{T}$ be a triangulated category. Lemma 8 tells us that the identity functor $1 : \mathcal{T} \rightarrow \mathcal{T}$ gives a special functor $\gamma : {}^d\mathcal{T} \rightarrow {}^v\mathcal{T}$. Any special square in ${}^d\mathcal{T}$ is automatically a special square in ${}^v\mathcal{T}$.

Let $H : \mathcal{T} \rightarrow \mathcal{A}$ be a homological functor from the triangulated category $\mathcal{T}$ to the abelian category $\mathcal{A}$. Suppose H is bounded. That is, for each $t \in \mathcal{T}$ there exists $N \in \mathbb{N}$ with $H(\Sigma^i t) = 0$ unless $-N < i < N$. By Lemma 9, H takes virtual triangles in $\mathcal{T}$ to long exact sequences in $\mathcal{A}$. The functor taking $t \in \mathcal{T}$ to the graded object $\{H(\Sigma^i t) \mid i \in \mathbb{Z}\}$ is a special functor

$$\delta : {}^v\mathcal{T} \rightarrow \text{Gr}^b \mathcal{A}$$

of categories with squares. Summarising, we have produced special functors

$${}^d\mathcal{T} \xrightarrow{\gamma} {}^v\mathcal{T} \xrightarrow{\delta} \text{Gr}^b \mathcal{A}.$$

In some very simple cases, for example if $\mathcal{T} = D^b(k)$ is the derived category of a field k and H is ordinary homology, the maps γ and δ are equivalences of categories with squares.

Regions

4.6

In Sect. 4.5 we learned what is meant by a category with squares. We learned the definition, and the three examples we will refer to in this article. In the current section we will study regions $\mathcal{R} \subset \mathbb{Z} \times \mathbb{Z}$, and then in Sect. 4.7 we put it all together. The K -theory of a category with squares $\mathcal{T}$ is defined from the simplicial set of certain functors from regions $\mathcal{R} \subset \mathbb{Z} \times \mathbb{Z}$ to the category with squares $\mathcal{T}$.

Let us agree first that, from this point on, $\mathbb{Z}$ will be understood to be a category. The objects are the integers, and

$$\text{Hom}(i, j) = \begin{cases} \emptyset & \text{if } i > j \\ 1 & \text{if } i \leq j. \end{cases}$$

That is, $\text{Hom}(i, j)$ is either empty or has one element. It is non-empty exactly when $i \leq j$. There is only one possible composition law.

Definition 17 A *region* will mean a full subcategory $\mathcal{R} \subset \mathbb{Z} \times \mathbb{Z}$.

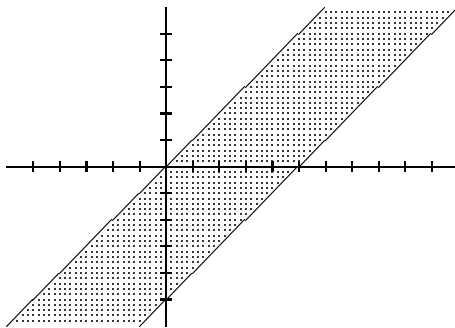
18 **Definition 18** Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two regions. A *morphism of regions* $\mathcal{R}_1 \rightarrow \mathcal{R}_2$ is a functor $F : \mathcal{R}_1 \rightarrow \mathcal{R}_2$, so that there exist two functors $f_1 : \mathbb{Z} \rightarrow \mathbb{Z}$, $f_2 : \mathbb{Z} \rightarrow \mathbb{Z}$ and a commutative square

$$\begin{array}{ccc} \mathcal{R}_1 & \xrightarrow{F} & \mathcal{R}_2 \\ \downarrow & & \downarrow \\ \mathbb{Z} \times \mathbb{Z} & \xrightarrow{f_1 \times f_2} & \mathbb{Z} \times \mathbb{Z} \end{array} .$$

19 **Remark 19** In this article, the regions we most care about are

$$\mathcal{R}_n = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq x - y \leq n + 1\} .$$

We consider them when $n \geq 0$. The picture is



Part of the reason we care about the $\mathcal{R}_n$'s is the following.

20 **Remark 20** Recall the category Δ of finite ordered sets. The objects are $\mathbf{n} = \{0, 1, \dots, n\}$. The morphisms are the order preserving maps. I assert that there is a functor θ from Δ to the category of regions in $\mathbb{Z} \times \mathbb{Z}$. We define the functor θ as follows.

- (i) On objects: For an object $\mathbf{n} \in \Delta$, put $\theta(\mathbf{n}) = \mathcal{R}_n$, as in Remark 19.
- (ii) On morphisms: Suppose we are given a morphism $\varphi : \mathbf{m} \rightarrow \mathbf{n}$ in Δ . We define $f : \mathbb{Z} \rightarrow \mathbb{Z}$ as follows. Any integer in $\mathbb{Z}$ can be expressed, uniquely, as $a(m+1) + b$, with $0 \leq b \leq m$. Put

$$f(a(m+1) + b) = a(n+1) + \varphi(b) .$$

Then f is an order-preserving map $\mathbb{Z} \rightarrow \mathbb{Z}$ (that is, a functor when we view $\mathbb{Z}$ as a category). The reader can show that

$$f \times f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$$

takes $\mathcal{R}_m \subset \mathbb{Z} \times \mathbb{Z}$ into $\mathcal{R}_n \subset \mathbb{Z} \times \mathbb{Z}$. We define $\theta(\varphi)$ to be the map $\mathcal{R}_m \rightarrow \mathcal{R}_n$ induced by $f \times f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$.

It is useful to note that $\theta(\varphi)$ takes the boundary of the region $\theta(\mathbf{m}) = \mathcal{R}_m$ to the boundary of the region $\theta(\mathbf{n}) = \mathcal{R}_n$. More explicitly, the boundary point $(y, y) \in \mathcal{R}_m$ gets mapped to the boundary point $(f(y), f(y)) \in \mathcal{R}_n$. The boundary point $(y + m + 1, y) \in \mathcal{R}_m$ gets mapped to the boundary point $(f(y) + n + 1, f(y)) \in \mathcal{R}_n$.

The Simplicial Set

4.7

Now we know what we mean by

- (i) Regions in $\mathbb{Z} \times \mathbb{Z}$.
- (ii) Categories with squares.

It is time to put it together and define K-theory. The key ingredient is

Definition 21 Let $\mathcal{T}$ be a category with squares. Let $\mathcal{R}$ be a region in $\mathbb{Z} \times \mathbb{Z}$. An *augmented diagram for the pair* $(\mathcal{R}, \mathcal{T})$ is defined to be

21

- (i) A functor $F : \mathcal{R} \rightarrow \mathcal{T}$.
- (ii) Suppose we are given four integers $i \leq i'$ and $j \leq j'$. These four integers define a commutative square in $\mathbb{Z} \times \mathbb{Z}$, namely

$$\begin{array}{ccc} (i, j') & \longrightarrow & (i', j') \\ \uparrow & & \uparrow \\ (i, j) & \longrightarrow & (i', j) \end{array} .$$

If this square happens to be contained in the region $\mathcal{R}$, then the functor F , of part (i) above, takes it to a commutative square in $\mathcal{T}$

$$\begin{array}{ccc} F(i, j') & \longrightarrow & F(i', j') \\ \uparrow & & \uparrow \\ F(i, j) & \longrightarrow & F(i', j) \end{array} .$$

We require that all such squares extend to special squares in $\mathcal{T}$. That is, we must be given a map

$$\delta_{i,j}^{i',j'} : F(i', j') \longrightarrow \Sigma F(i, j)$$

yielding a special square.

- (iii) The maps $\delta_{i,j}^{i',j'}$ should be compatible, in the following sense. Suppose we are given two squares in $\mathbb{Z} \times \mathbb{Z}$, one inside the other. That is, we have integers $I \leq i \leq i' \leq I'$ and $J \leq j \leq j' \leq J'$, giving in $\mathbb{Z} \times \mathbb{Z}$ the commutative diagram

$$\begin{array}{ccccccc}
 (I, J') & \longrightarrow & (i, J') & \longrightarrow & (i', J') & \longrightarrow & (I', J') \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 (I, j') & \longrightarrow & (i, j') & \longrightarrow & (i', j') & \longrightarrow & (I', j') \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 (I, j) & \longrightarrow & (i, j) & \longrightarrow & (i', j) & \longrightarrow & (I', j) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 (I, J) & \longrightarrow & (i, J) & \longrightarrow & (i', J) & \longrightarrow & (I', J) \quad .
 \end{array}$$

Suppose the small, middle square and the outside, large square both lie entirely in $\mathcal{R}$. That is, we have two squares in $\mathcal{R}$, one contained in the other

$$\begin{array}{ccc}
 (i, j') & \longrightarrow & (i', j') \\
 \uparrow & & \uparrow \\
 (i, j) & \longrightarrow & (i', j)
 \end{array}
 \qquad
 \begin{array}{ccc}
 (I, J') & \longrightarrow & (I', J') \\
 \uparrow & & \uparrow \\
 (I, J) & \longrightarrow & (I', J) \quad .
 \end{array}$$

Part (ii) above gives us two maps

$$\delta_{i,j}^{i',j'} : F(i', j') \longrightarrow \Sigma F(i, j)$$

$$\delta_{I,J}^{I',J'} : F(I', J') \longrightarrow \Sigma F(I, J) \quad .$$

The compatibility requirement is that $\delta_{i,j}^{i',j'}$ should be the composite

$$F(i', j') \xrightarrow{F(\alpha)} F(I', J') \xrightarrow{\delta_{I,J}^{I',J'}} \Sigma F(I, J) \xrightarrow{\Sigma F(\beta)} \Sigma F(i, j) \quad ,$$

where $\beta : (I, J) \rightarrow (i, j)$ and $\alpha : (i', j') \rightarrow (I', J')$ are the unique maps in $\mathbb{Z} \times \mathbb{Z}$.

Remark 22 The definition of augmented diagrams is clearly functorial in the pairs $\mathcal{R}, \mathcal{T}$. Given a morphism of regions $f : \mathcal{R} \rightarrow \mathcal{R}'$ and a special functor of categories with squares $g : \mathcal{T} \rightarrow \mathcal{T}'$, then composition induces a natural map

$$\left\{ \begin{array}{c} \text{Augmented diagrams} \\ \text{for the pair } (\mathcal{R}', \mathcal{T}) \end{array} \right\} \xrightarrow{(f,g)} \left\{ \begin{array}{c} \text{Augmented diagrams} \\ \text{for the pair } (\mathcal{R}, \mathcal{T}') \end{array} \right\} .$$

This says that there is a functor

$$\left\{ \begin{array}{c} \text{Regions} \\ \mathcal{R} \subset \mathbb{Z} \times \mathbb{Z} \end{array} \right\}^{\text{op}} \times \left\{ \begin{array}{c} \text{Categories} \\ \text{with squares} \end{array} \right\} \xrightarrow{\Phi} \{\text{Sets}\} ,$$

which takes the pair $(\mathcal{R}, \mathcal{T}) \in \{\text{Regions}\} \times \{\text{Categories with squares}\}$ to

$$\Phi(\mathcal{R}, \mathcal{T}) = \left\{ \begin{array}{c} \text{Augmented diagrams} \\ \text{for the pair } (\mathcal{R}, \mathcal{T}) \end{array} \right\} .$$

This functor is contravariant in the region $\mathcal{R}$, covariant in $\mathcal{T}$ (the category with squares).

Now, finally, we come to our simplicial set.

Definition 23 Remark 20 provides us with a functor

$$\theta : \Delta \rightarrow \{\text{regions in } \mathbb{Z} \times \mathbb{Z}\} .$$

Remark 22 gives a functor

$$\Phi : \left\{ \begin{array}{c} \text{Regions} \\ \mathcal{R} \subset \mathbb{Z} \times \mathbb{Z} \end{array} \right\}^{\text{op}} \times \left\{ \begin{array}{c} \text{Categories} \\ \text{with squares} \end{array} \right\} \longrightarrow \{\text{Sets}\} .$$

Let $\mathcal{T}$ be a category with squares. Then the functor taking $(-) \in \Delta$ to

$$\Phi(\theta(-), \mathcal{T})$$

is a functor $\Delta^{\text{op}} \rightarrow \{\text{Sets}\}$. We wish to consider a simplicial subset

$$S_*(\mathcal{T}) \subset \Phi(\theta(-), \mathcal{T}) .$$

The elements of $S_n(\mathcal{T})$ form a subset of

$$\Phi(\theta(\mathbf{n}), \mathcal{T}) = \Phi(\mathcal{R}_n, \mathcal{T}) .$$

The set $\Phi(\mathcal{R}_n, \mathcal{T})$ consists of all augmented diagrams for the pair $(\mathcal{R}_n, \mathcal{T})$. The subset $S_n(\mathcal{T})$ are the augmented diagrams which vanish on the boundary. Recall: An augmented diagram gives, among other things, a functor $F : \mathcal{R}_n \rightarrow \mathcal{T}$. The augmented diagram belongs to $S_n(\mathcal{T})$ if

$$F(y, y) = 0 = F(y + n + 1, y) .$$

24 Remark 24 At the end of Remark 20 we noted that, if $\varphi : \mathbf{m} \rightarrow \mathbf{n}$ is any morphism in Δ , then $\theta(\varphi)$ takes points on the boundary of the region $\mathcal{R}_m = \theta(\mathbf{m})$ to boundary points of $\mathcal{R}_n = \theta(\mathbf{n})$. Augmented diagrams which vanish on the boundary of $\mathcal{R}_n$ therefore go to augmented diagrams vanishing on the boundary of $\mathcal{R}_m$, and hence $S_*(\mathcal{T})$ really is a simplicial subset of $\Phi(\theta(-), \mathcal{T})$.

25 Remark 25 It is clear that $S_*(\mathcal{T})$ is functorial in $\mathcal{T}$. Given a special functor of categories with squares $\mathcal{S} \rightarrow \mathcal{T}$, then composition induces a map

$$S_*(\mathcal{S}) \rightarrow S_*(\mathcal{T}) .$$

In Example 16 we saw that, given a triangulated category $\mathcal{T}$, an abelian category $\mathcal{A}$ and a bounded homological functor $H : \mathcal{T} \rightarrow \mathcal{A}$, there are special functors of categories with squares

$${}^d\mathcal{T} \xrightarrow{\gamma} {}^v\mathcal{T} \xrightarrow{\delta} \mathrm{Gr}^b\mathcal{A} .$$

We conclude that there are simplicial maps of simplicial sets

$$S_*({}^d\mathcal{T}) \xrightarrow{\gamma} S_*({}^v\mathcal{T}) \xrightarrow{\delta} S_*(\mathrm{Gr}^b\mathcal{A}) .$$

Note that, in an abuse of notation, the letter γ stands for both the map ${}^d\mathcal{T} \rightarrow {}^v\mathcal{T}$ and for the map it induces on the simplicial sets, and similarly for the letter δ .

26 Definition 26 For a category with squares $\mathcal{T}$, its K -theory $K(\mathcal{T})$ is defined to be the loop space of the geometric realisation of the simplicial set $S_*(\mathcal{T})$. In symbols:

$$K(\mathcal{T}) = \Omega |S_*(\mathcal{T})| .$$

27 Remark 27 Taking loop spaces of the geometric realisation of the maps in Remark 25, we deduce continuous maps of spaces

$$K({}^d\mathcal{T}) \xrightarrow{\gamma} K({}^v\mathcal{T}) \xrightarrow{\delta} K(\mathrm{Gr}^b\mathcal{A}) .$$

What It All Means

4.8

Until now our treatment has been very abstract. We have constructed certain simplicial sets and simplicial maps. It might be helpful to work out explicitly what are the low-dimensional simplices. The definition says

$$S_n(\mathcal{T}) = \left\{ \begin{array}{l} \text{Augmented diagrams} \\ \text{for the pair } (\mathcal{R}_n, \mathcal{T}) \end{array} \left| \begin{array}{l} \text{The functor } F : \mathcal{R}_n \rightarrow \mathcal{T}, \text{ given} \\ \text{as part of the data of the} \\ \text{augmented diagram, satisfies} \\ F(y, y) = 0 = F(y + n + 1, y) \end{array} \right. \right\}.$$

Let us now work this out, in low dimensions, for the category with squares ${}^d\mathcal{T}$.

Case 28 $S_0({}^d\mathcal{T})$ is easy to compute. The region $\mathcal{R}_0$ is the region $0 \leq x - y \leq 1$, and all the points are boundary points. That is, for every $(x, y) \in \mathcal{R}_0$ we have that $x - y$ is either 0 or 1. There is only one element in $S_0(\mathcal{T})$. It is the diagram

28

$$\begin{array}{ccc} & & 0 \rightarrow \\ & & \uparrow \\ & 0 \rightarrow & 0 \\ & \uparrow & \\ \rightarrow & 0 & \end{array}$$

Case 29 Slightly more interesting is $S_1({}^d\mathcal{T})$. The region $\mathcal{R}_1$ is $0 \leq x - y \leq 2$, and the boundary consists of the points where $x - y$ is 0 or 2. A simplex is therefore a diagram

29

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \uparrow \\ & & & & & 0 \rightarrow & x_{n+1} \rightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & 0 \rightarrow & x_n \rightarrow & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 \rightarrow & x_{n-1} \rightarrow & 0 & & & & \\ & \uparrow & & & & & \\ & 0 & & & & & \end{array}$$

In this diagram, each square

$$\begin{array}{ccc} 0 & \longrightarrow & x_{n+1} \\ \uparrow & & \uparrow \\ x_n & \longrightarrow & 0 \end{array}$$

is a special square. It comes with a map $\delta_n : x_{n+1} \rightarrow \Sigma x_n$. In the case of the category with squares ${}^d\mathcal{T}$, the fact that the square is special means that we have a distinguished triangle

$$x_n \longrightarrow 0 \longrightarrow x_{n+1} \xrightarrow{\delta_n} \Sigma x_n .$$

In other words, the map $\delta_n : x_{n+1} \rightarrow \Sigma x_n$ must be an isomorphism. The diagram defining the simplex is canonically isomorphic to

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \uparrow \\ & & & & & 0 & \longrightarrow \Sigma x_0 \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & 0 & \longrightarrow & x_0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Sigma^{-1} x_0 & \longrightarrow & 0 & & \\ & & \uparrow & & & & \\ & & 0 & & & & \end{array}$$

Up to canonical isomorphism, the simplices in $S_1({}^d\mathcal{T})$ are just the objects of $\mathcal{T}$.

Case 30 Next we consider $S_2({}^d\mathcal{T})$. The region $\mathcal{R}_2$ is $0 \leq x - y \leq 3$, and the boundary consists of the points where $x - y$ is 0 or 3. A 2-simplex is a diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & & & 0 & \longrightarrow & y' & \longrightarrow z' \longrightarrow 0 \\
 & & & \uparrow & & \uparrow & \uparrow \\
 & & 0 & \longrightarrow & z & \longrightarrow & x' \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & x & \longrightarrow & y & \longrightarrow & 0 \\
 & & \uparrow & & & & \\
 & & 0 & & & & .
 \end{array}$$

The special squares

$$\begin{array}{ccc}
 0 \longrightarrow x' & 0 \longrightarrow y' & 0 \longrightarrow z' \\
 \uparrow & \uparrow & \uparrow \\
 x \longrightarrow 0 & y \longrightarrow 0 & z \longrightarrow 0
 \end{array}$$

have differentials

$$\delta_x : x' \rightarrow \Sigma x, \quad \delta_y : y' \rightarrow \Sigma y, \quad \delta_z : z' \rightarrow \Sigma z.$$

As in Case 29 above, these differentials must be isomorphisms. The diagram as a whole is therefore canonically isomorphic to

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & & & 0 & \longrightarrow & \Sigma y & \xrightarrow{v'} \Sigma z \longrightarrow 0 \\
 & & & \uparrow & & \uparrow u' & \uparrow \\
 & & 0 & \longrightarrow & z & \xrightarrow{w} \Sigma x & \longrightarrow 0 \\
 & & \uparrow & & \uparrow v & \uparrow & \\
 0 & \longrightarrow & x & \xrightarrow{u} & y & \longrightarrow & 0 \\
 & & \uparrow & & & & \\
 & & 0 & & & & .
 \end{array}$$

The isomorphism is such that, in the special squares

$$\begin{array}{ccc}
 0 \longrightarrow \Sigma x & 0 \longrightarrow \Sigma y & 0 \longrightarrow \Sigma z \\
 \uparrow & \uparrow & \uparrow \\
 x \longrightarrow 0 & y \longrightarrow 0 & z \longrightarrow 0
 \end{array}$$

the differentials are all identity maps. Next we will use the fact that the differentials are coherent, to compute the maps in the diagram.

Consider the following little bit of the larger diagram above

$$\begin{array}{ccccc}
 & & 0 & \longrightarrow & \Sigma y \\
 & & \uparrow & & \uparrow u' \\
 & & & w & \\
 0 & \longrightarrow & z & \longrightarrow & \Sigma x \\
 \uparrow & & \uparrow v & & \uparrow \\
 x & \xrightarrow{u} & y & \longrightarrow & 0
 \end{array} .$$

There are three squares in this bit, namely

$$\begin{array}{ccc}
 0 \longrightarrow \Sigma x & 0 \longrightarrow \Sigma y & z \xrightarrow{w} \Sigma x \\
 \uparrow & \uparrow & \uparrow v \\
 x \longrightarrow 0 & y \longrightarrow 0 & y \longrightarrow 0
 \end{array} .$$

These are three special squares, with compatible differentials. The differentials of the first two squares are

$$\delta_x = 1 : \Sigma x \rightarrow \Sigma x, \quad \delta_y = 1 : \Sigma y \rightarrow \Sigma y.$$

The compatibility says that the differential of the third square

$$\begin{array}{ccc}
 z & \xrightarrow{w} & \Sigma x \\
 \uparrow v & & \uparrow \\
 y & \longrightarrow & 0
 \end{array}$$

can be computed as either of the composites

$$\begin{array}{ccccc} \Sigma x & \xrightarrow{\delta_x} & \Sigma x & \xrightarrow{\Sigma u} & \Sigma y \\ \Sigma x & \xrightarrow{u'} & \Sigma y & \xrightarrow{\delta_y} & \Sigma y \end{array} .$$

We conclude that $u' = \Sigma u$. The diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \Sigma y & \xrightarrow{v'} & \Sigma z \\ \uparrow & & \uparrow & u' & \uparrow \\ z & \xrightarrow{w} & \Sigma x & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ y & \longrightarrow & 0 & & \end{array}$$

permits us to compute that $v' = \Sigma v$, and so on. The simplex becomes

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ & & 0 & \longrightarrow & \Sigma y & \xrightarrow{\Sigma v} & \Sigma z \longrightarrow 0 \\ & & \uparrow & & \uparrow & \Sigma u & \uparrow \\ 0 & \longrightarrow & z & \xrightarrow{w} & \Sigma x & \longrightarrow & 0 \\ & \uparrow & \uparrow & & \uparrow & & \\ & \uparrow & \uparrow & v & \uparrow & & \\ 0 & \longrightarrow & x & \xrightarrow{u} & y & \longrightarrow & 0 \\ & & \uparrow & & & & \\ & & 0 & & & & \end{array} .$$

In this diagram there are many special squares. So far, we have focused mainly on the special squares of the form

$$\begin{array}{ccc} 0 & \longrightarrow & \Sigma A \\ \uparrow & & \uparrow \\ A & \longrightarrow & 0 \end{array},$$

where the differential $\Sigma A \rightarrow \Sigma A$ is the identity. But there are other special squares. For example

$$\begin{array}{ccc} 0 & \longrightarrow & z \\ \uparrow & & \uparrow v \\ x & \xrightarrow{u} & y \end{array}.$$

The differential of this special square may be computed from the fact that, in the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & z & \xrightarrow{w} & \Sigma x \\ \uparrow & & \uparrow v & & \uparrow \\ x & \xrightarrow{u} & y & \longrightarrow & 0 \end{array}$$

the larger special square

$$\begin{array}{ccc} 0 & \longrightarrow & \Sigma x \\ \uparrow & & \uparrow \\ x & \longrightarrow & 0 \end{array}$$

has for its differential the map $1 : \Sigma x \rightarrow \Sigma x$. Compatibility tells us that the differential of

$$\begin{array}{ccc} 0 & \longrightarrow & z \\ \uparrow & & \uparrow v \\ x & \xrightarrow{u} & y \end{array}$$

must be $w : z \rightarrow \Sigma x$. But in Example 13 we defined special squares in ${}^d\mathcal{T}$ to be squares

$$\begin{array}{ccc}
 C & \xrightarrow{\delta} & D \\
 \beta \uparrow & & \uparrow \gamma \\
 A & \xrightarrow{\alpha} & B
 \end{array}
 \quad \begin{array}{c} \curvearrowright \\ \mu \end{array}$$

for which the sequence

$$A \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} D \xrightarrow{\mu} \Sigma A$$

is a distinguished triangle. In our case, this becomes a distinguished triangle

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x.$$

One of the miracles here is that the signs take care of themselves. The special square

$$\begin{array}{ccc}
 z & \xrightarrow{w} & \Sigma x \\
 \uparrow v & & \uparrow \\
 y & \longrightarrow & 0
 \end{array}$$

has a differential, which is easily computed to be $\Sigma u : \Sigma x \rightarrow \Sigma y$. This gives a distinguished triangle

$$y \xrightarrow{-v} z \xrightarrow{w} \Sigma x \xrightarrow{\Sigma u} \Sigma y.$$

The fact that the morphism $v : y \rightarrow z$ in the square is vertical automatically takes care of the sign.

We conclude that the only real restriction on the diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & & & 0 & \longrightarrow & \Sigma y & \xrightarrow{\Sigma v} & \Sigma z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & \Sigma u & & \uparrow & & \\
 0 & \longrightarrow & z & \xrightarrow{w} & \Sigma x & \longrightarrow & 0 \\
 & \uparrow & \uparrow v & & \uparrow & & \\
 0 & \longrightarrow & x & \xrightarrow{u} & y & \longrightarrow & 0 \\
 & & \uparrow & & & & \\
 & & 0 & & & &
 \end{array}$$

is the fact that

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x$$

is a distinguished triangle. The other special squares give distinguished triangles which are just rotations of the above. In conclusion: Any element in $S_2({}^d\mathcal{T})$ is canonically isomorphic to a diagram which arises as above from a distinguished triangle

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x.$$

There are three face maps

$$S_2({}^d\mathcal{T}) \rightrightarrows S_1({}^d\mathcal{T}).$$

In the above, we identified the elements of $S_2({}^d\mathcal{T})$ with distinguished triangles in $\mathcal{T}$. In Case 29, we identified the elements of $S_1({}^d\mathcal{T})$ with the objects of $\mathcal{T}$. The face maps above take the distinguished triangle

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x$$

to z , y and x , respectively.

Remark 31 We now have an explicit identification of the elements of $S_2({}^d\mathcal{T})$ and $S_1({}^d\mathcal{T})$, and of the three face maps

$$S_2({}^d\mathcal{T}) \rightrightarrows S_1({}^d\mathcal{T}).$$

Using this, one can compute the first homology group of the space $|S_*(^d\mathcal{T})|$. Since $|S_*(^d\mathcal{T})|$ is an H -space, we have

$$H_1|S_*(^d\mathcal{T})| = \pi_1|S_*(^d\mathcal{T})| = \pi_0K(^d\mathcal{T}) .$$

An explicit computation easily shows this to be the usual Grothendieck group of Definition 1.

Remark 32 In Case 29 we saw that the diagram for a 1-simplex has objects which repeat (up to suspension). In Case 30 we saw that the morphisms in a 2-simplex also repeat, again up to suspension. Take an element $x \in S_n(^d\mathcal{T})$, with $n \geq 2$. Then x is a diagram in $\mathcal{T}$. The objects of this diagram are all objects of 1-dimensional faces of x , and the morphisms are all composites of morphisms in 2-dimensional faces of x . From Cases 29 and 30 we conclude that the entire diagram is periodic.

More explicitly, a fundamental region for the diagram $x \in S_n(^d\mathcal{T})$ is given by

$$\mathcal{D}_n = \left\{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \left| \begin{array}{l} 0 \leq a \leq n \\ 0 \leq b \leq n \\ 0 \leq a - b \end{array} \right. \right\} .$$

Thus, a 1-simplex is completely determined by the diagram

$$\begin{array}{ccc} & & 0 \\ & & \uparrow \\ 0 & \longrightarrow & x \end{array}$$

and a 2-simplex is determined by

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \uparrow \\ & & & & | \\ 0 & \longrightarrow & 0 & \longrightarrow & z \\ & & \uparrow & & \uparrow \\ & & x & \xrightarrow{u} & y \\ 0 & \longrightarrow & x & \longrightarrow & y \end{array} .$$

$$\begin{array}{ccc} 0 & \longrightarrow & Z \\ \uparrow & & \uparrow v \\ x & \longrightarrow & y \end{array} \quad u$$

What is being asserted is the following. The region $\mathcal{R}_n$ contains the region $\mathcal{D}_n$. An element $x \in S_n({}^d\mathcal{T})$ is an augmented diagram for the pair $(\mathcal{R}_n, {}^d\mathcal{T})$. It restricts to an augmented diagram for the pair $(\mathcal{D}_n, {}^d\mathcal{T})$. The assertion is that the smaller diagram determines, up to canonical isomorphism, the larger one.

Case 33 Next we wish to study the elements of $S_3(\mathcal{T})$. By Remark 32, the simplex is determined by its restriction to $\mathcal{D}_3 \subset \mathcal{R}_3$. We have a diagram

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \uparrow \\
 & & & & 0 & \rightarrow & z \\
 & & & \uparrow & & & \uparrow \\
 & & 0 & \rightarrow & x & \rightarrow & y \\
 & \uparrow & & \uparrow & & & \uparrow \\
 0 & \rightarrow & u & \rightarrow & v & \rightarrow & w .
 \end{array}$$

A simplex in $S_3(\mathcal{T})$ is obtained from this by periodicity, up to suspension. The simplex will look like

$$\begin{array}{ccccccccccc}
& & & & & & & & & 0 & \\
& & & & & & & & \uparrow & & \\
& & & & & & & 0 & \longrightarrow & \Sigma w & \longrightarrow & \Sigma y & \longrightarrow & \Sigma z & \longrightarrow & 0 \\
& & & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\
& & & & & 0 & \longrightarrow & z & \longrightarrow & \Sigma v & \longrightarrow & \Sigma x & \longrightarrow & 0 & & \\
& & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & & & \\
& & & 0 & \longrightarrow & x & \longrightarrow & y & \longrightarrow & \Sigma u & \longrightarrow & 0 & & & & \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & & & & & \\
0 & \longrightarrow & u & \longrightarrow & v & \longrightarrow & w & \longrightarrow & 0 & & & & & & & \\
& & & & & & \uparrow & & & & & & & & & \\
& & & & & & 0 & . & & & & & & & &
\end{array}$$

What does it all mean?

We have two composable morphisms

$$u \rightarrow v \rightarrow w .$$

The special squares

$$\begin{array}{ccc} 0 & \rightarrow & x \\ \uparrow & & \uparrow \\ u & \rightarrow & v \end{array} \quad \begin{array}{ccc} 0 & \rightarrow & y \\ \uparrow & & \uparrow \\ u & \rightarrow & w \end{array} \quad \begin{array}{ccc} 0 & \rightarrow & z \\ \uparrow & & \uparrow \\ v & \rightarrow & w \end{array}$$

give three distinguished triangles

$$u \longrightarrow v \longrightarrow x \longrightarrow \Sigma u$$

$$u \longrightarrow w \longrightarrow y \longrightarrow \Sigma u$$

$$v \longrightarrow w \longrightarrow z \longrightarrow \Sigma v$$

and the special square

$$\begin{array}{ccc} 0 & \rightarrow & z \\ \uparrow & & \uparrow \\ x & \rightarrow & y \end{array}$$

tells us that the mapping cones x, y and z of the maps $u \rightarrow v, u \rightarrow w$ and $v \rightarrow w$ fit in a distinguished triangle

$$x \rightarrow y \rightarrow z \rightarrow \Sigma x .$$

This should hopefully look familiar. What we have here is an octahedron, with its four distinguished triangles and four commutative triangles.

Remark 34 Our octahedron is somewhat special. We have special squares

34

$$\begin{array}{ccc} x & \rightarrow & y \\ \uparrow & & \uparrow \\ v & \rightarrow & w \end{array} \quad \begin{array}{ccc} z & \rightarrow & \Sigma v \\ \uparrow & & \uparrow \\ y & \rightarrow & \Sigma u \end{array} .$$

These come with differentials, and fold to give distinguished triangles. Thus a 3-simplex in the simplicial set $S_*(^d\mathcal{T})$ is more than just an octahedron. It is an octahedron where the two commutative squares are special.

I observed the existence of such octahedra in [66, Remark 5.5]. This existence may be viewed as a refinement of the old octahedral lemma.

35

Remark 35 It is perhaps worth explaining this point even further. In Remark 32 we observed that a simplex in $S_n({}^d\mathcal{T})$ is determined by its restriction to the region $\mathcal{D}_n \subset \mathcal{R}_n$. But it is only right to warn the reader that not every augmented diagram for the pair $(\mathcal{D}_n, {}^d\mathcal{T})$, vanishing on the top diagonal, extends to a simplex in $S_n({}^d\mathcal{T})$. If the extension exists then it is unique up to canonical isomorphism; but there is no guarantee of existence. For clarity, let us illustrate this when $n = 3$.

An augmented diagram for the pair $(\mathcal{D}_3, {}^d\mathcal{T})$, vanishing on the top diagonal, is a diagram

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \uparrow \\
 & & & & 0 & \longrightarrow & z \\
 & & & & \uparrow & & \uparrow \\
 & & 0 & \longrightarrow & x & \longrightarrow & y \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & u & \longrightarrow & v & \longrightarrow & w
 \end{array}$$

together with compatible differentials, and where we have five special squares. By the periodicity of Remark 32, we can extend this to a diagram

$$\begin{array}{cccccccc}
 & & & & 0 & \longrightarrow & \Sigma w \\
 & & & & \uparrow & & \uparrow \\
 & & & 0 & \longrightarrow & z & \longrightarrow & \Sigma v \\
 & & & \uparrow & & \uparrow & & \uparrow \\
 & 0 & \longrightarrow & x & \longrightarrow & y & \longrightarrow & \Sigma u \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & u & \longrightarrow & v & \longrightarrow & w & \longrightarrow & 0 \quad .
 \end{array}$$

The periodicity provides us all the maps and differentials we might care to have. The problem is that nothing guarantees that

$$\begin{array}{ccc}
 z & \longrightarrow & \Sigma v \\
 \uparrow & & \uparrow \\
 y & \longrightarrow & \Sigma u
 \end{array}$$

should be a special square. In general it will not be.

It turns out that, for the categories with squares ${}^v\mathcal{T}$ and $\mathrm{Gr}^b\mathcal{A}$, this problem disappears; every augmented diagram for the pair $(\mathcal{D}_n, {}^v\mathcal{T})$ (resp. $(\mathcal{D}_n, \mathrm{Gr}^b\mathcal{A})$), vanishing on the top diagonal, extends to a simplex in $S_n({}^v\mathcal{T})$ (resp. $S_n(\mathrm{Gr}^b\mathcal{A})$). The point is that in the diagram above we have

$$\begin{array}{ccc} z & \longrightarrow & \Sigma v \\ \uparrow & & \uparrow \\ y & \longrightarrow & \Sigma u \\ \uparrow & & \uparrow \\ w & \longrightarrow & 0 \end{array}$$

The squares

$$\begin{array}{ccc} z & \longrightarrow & \Sigma v \\ \uparrow & & \uparrow \\ w & \longrightarrow & 0 \end{array} \quad \begin{array}{ccc} y & \longrightarrow & \Sigma u \\ \uparrow & & \uparrow \\ w & \longrightarrow & 0 \end{array}$$

are both special, being the rotations of given virtual triangles (resp. long exact sequences). The 2-out-of-3 property holds, implying that the square

$$\begin{array}{ccc} z & \longrightarrow & \Sigma v \\ \uparrow & & \uparrow \\ y & \longrightarrow & \Sigma u \end{array}$$

is also special. For ${}^v\mathcal{T}$ the 2-out-of-3 property is proved in [93, sect. 2.4]. For $\mathrm{Gr}^b\mathcal{A}$ the proof may be found in [70, lemma 4.3].

Remark 36 The elements of $S_n(\mathcal{T})$ can be thought of as refinements of the higher octahedra. Let $x \in S_n(\mathcal{T})$ be a simplex. It is an augmented diagram for the pair $(\mathcal{R}_n, {}^d\mathcal{T})$. In $\mathcal{R}_n \subset \mathbb{Z} \times \mathbb{Z}$, consider the intersection with $\mathbb{Z} \times \{0\}$. It is the set $\{(i, 0) \mid 0 \leq i \leq n+1\}$. On the region $\mathcal{R}_n \cap \{\mathbb{Z} \times \{0\}\}$, the restriction of $x \in S_n(\mathcal{T})$ is just

$$0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x_n \rightarrow 0.$$

This gives us $(n-1)$ composable morphisms. The restriction of x to the fundamental region $\mathcal{D}_n \subset \mathcal{R}_n$ of Remark 32 is a diagram which contains all the mapping cones on the maps $x_i \rightarrow x_j$. And the simplex keeps track of the relations among these.

Note that the simplex remembers more data than the higher octahedra of [8, Remarque 1.1.14]. We already observed this in the case of 3-simplices. Somehow the coherent differentials and all the special squares tell us of the existence of many distinguished triangles.

Waldhausen Models and the Existence of Large Simplices

4.9

Let $\mathcal{T}$ be a category with squares. In Sect. 4.7 we defined a simplicial set $S_*(\mathcal{T})$. In Sect. 4.8 we analysed the low-dimensional simplices of $S_*(^d\mathcal{T})$, where $^d\mathcal{T}$ is the category with squares obtained from a triangulated category $\mathcal{T}$ as in Example 13. The analysis of Sect. 4.8 tells us that the 1-simplices correspond to objects, the 2-simplices correspond to distinguished triangles, and the 3-simplices correspond to special octahedra. The refined octahedral axiom guarantees the existence of a great many 3-simplices. For $n \geq 4$, the n -simplices are complicated diagrams, and it is not clear if any non-degenerate examples exist. It is therefore of some interest to see how a Waldhausen model can be used to construct simplices.

Let $\mathcal{A}$ be an abelian category, $C(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. As in Sect. 4.3, our Waldhausen categories will all be assumed to be full subcategories of $C(\mathcal{A})$. We begin with a definition

Definition 37 A commutative square in $C(\mathcal{A})$

$$\begin{array}{ccc} b' & \xrightarrow{\delta} & c \\ \beta \uparrow & & \uparrow \gamma \\ a & \xrightarrow{\alpha} & b \end{array}$$

is called *bicartesian* if the sequence

$$0 \longrightarrow a \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} b \oplus b' \xrightarrow{(\gamma \quad \delta)} c \longrightarrow 0$$

is a short exact sequence of chain complexes.

Remark 38 Suppose we have a bicartesian square in $C(\mathcal{A})$ as in Definition 37. The fact that the composite

38

$$a \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} b \oplus b' \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} c$$

vanishes gives us a natural map from the mapping cone of $\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$ to c . This map must be a homology isomorphism. It therefore becomes invertible in $\text{ho}C(\mathcal{A}) = D(\mathcal{A})$. Unless confusion is likely to arise (that is, if there are several possibilities for α, β, γ and δ), we will omit them entirely in the notation. The map will be written

$$\text{Cone}(a \rightarrow b \oplus b') \longrightarrow c.$$

The key lemma is

Lemma 39 Let $C(\mathcal{A})$ be a Waldhausen category. Let

39

$$\begin{array}{ccc} b' & \twoheadrightarrow & c \\ \uparrow & & \uparrow \\ a & \twoheadrightarrow & b \end{array}$$

be a bicartesian square in $C(\mathcal{A})$. There exists a canonical choice for a differential $\partial : c \rightarrow \Sigma a$ rendering the diagram into a special square in ${}^d\text{ho}C(\mathcal{A}) = {}^dD(\mathcal{A})$. Furthermore, this choice of differentials is coherent. That is, given a digram in $C(\mathcal{A})$ where all the squares are bicartesian

$$\begin{array}{ccccccc} d''' & \twoheadrightarrow & e'' & \twoheadrightarrow & f' & \twoheadrightarrow & g \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ c'' & \twoheadrightarrow & d'' & \twoheadrightarrow & e' & \twoheadrightarrow & f \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ b' & \twoheadrightarrow & c' & \twoheadrightarrow & d' & \twoheadrightarrow & e \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ a & \twoheadrightarrow & b & \twoheadrightarrow & c & \twoheadrightarrow & d \end{array}$$

we deduce two bicartesian squares, one contained in the other

$$\begin{array}{ccc}
 d'' & \longrightarrow & e' \\
 \uparrow & & \uparrow \\
 c' & \longrightarrow & d'
 \end{array}
 \qquad
 \begin{array}{ccc}
 d''' & \longrightarrow & g \\
 \uparrow & & \uparrow \\
 a & \longrightarrow & d
 \end{array}$$

The above tells us that there are canonical choices for two differentials

$$\delta_1 : e' \longrightarrow \Sigma c'$$

$$\delta_2 : g \longrightarrow \Sigma a .$$

The compatibility requirement, which we assert is automatic for the canonical choices of differentials, is that δ_1 should be the composite

$$e' \longrightarrow g \xrightarrow{\delta_2} \Sigma a \longrightarrow \Sigma c' .$$

Proof Let

$$\begin{array}{ccc}
 b' & \longrightarrow & c \\
 \beta \uparrow & & \uparrow \\
 a & \xrightarrow{\alpha} & b
 \end{array}$$

be a bicartesian square in $C(\mathcal{A})$. Let X be the mapping cone on the map

$$a \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} b \oplus b' .$$

We have maps

$$\Sigma a \xleftarrow{f} X \xrightarrow{g} c .$$

By Remark 38, the map $g : X \rightarrow c$ is invertible in $D(\mathcal{A})$. The canonical choice for the differential is fg^{-1} . The compatibility of these differentials comes from the commutative diagram

$$\begin{array}{ccccc}
 \Sigma a & \longleftarrow & \text{Cone}(a \rightarrow d \oplus d''') & \longrightarrow & g \\
 \downarrow & & \downarrow & & \parallel \\
 \Sigma c' & \longleftarrow & \text{Cone}(c' \rightarrow e \oplus e'') & \longrightarrow & g \\
 \parallel & & \uparrow & & \uparrow \\
 \Sigma c' & \longleftarrow & \text{Cone}(c' \rightarrow d' \oplus d'') & \longrightarrow & e' \quad .
 \end{array}$$

Corollary 40 Let $\mathcal{R} \subset \mathbb{Z} \times \mathbb{Z}$ be a region. Assume that $\mathcal{R}$ is *convex*. Suppose we have a functor $F : \mathcal{R} \rightarrow C(\mathcal{A})$. Any time we have four integers $i \leq i'$ and $j \leq j'$, these four integers define a commutative square in $\mathbb{Z} \times \mathbb{Z}$, namely

$$\begin{array}{ccc}
 (i, j') & \longrightarrow & (i', j') \\
 \uparrow & & \uparrow \\
 (i, j) & \longrightarrow & (i', j) \quad .
 \end{array}$$

Suppose that, whenever the square above happens to be contained in the region $\mathcal{R}$, the functor F takes it to a bicartesian square in $C(\mathcal{A})$

$$\begin{array}{ccc}
 F(i, j') & \longrightarrow & F(i', j') \\
 \uparrow & & \uparrow \\
 F(i, j) & \longrightarrow & F(i', j) \quad .
 \end{array}$$

Then there is a canonical way to associate to the functor F an augmented diagram for the pair $(\mathcal{R}, {}^dD(\mathcal{A}))$.

Proof We certainly have a functor

$$\mathcal{R} \xrightarrow{F} C(\mathcal{A}) \rightarrow D(\mathcal{A}) .$$

For any square lying in $\mathcal{R}$, the bicartesian square in $C(\mathcal{A})$

$$\begin{array}{ccc}
 F(i, j') & \longrightarrow & F(i', j') \\
 \uparrow & & \uparrow \\
 F(i, j) & \longrightarrow & F(i', j)
 \end{array}$$

permits us, using Lemma 39, to make the canonical choice of differential $F(i', j') \rightarrow \Sigma F(i, j)$. It only remains to check that the choices are coherent.

Suppose therefore that we have a diagram in $\mathbb{Z} \times \mathbb{Z}$

$$\begin{array}{ccccccc}
 (I, J') & \longrightarrow & (i, J') & \longrightarrow & (i', J') & \longrightarrow & (I', J') \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 (I, j') & \longrightarrow & (i, j') & \longrightarrow & (i', j') & \longrightarrow & (I', j') \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 (I, j) & \longrightarrow & (i, j) & \longrightarrow & (i', j) & \longrightarrow & (I', j) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 (I, J) & \longrightarrow & (i, J) & \longrightarrow & (i', J) & \longrightarrow & (I', J) \quad .
 \end{array}$$

If the large square

$$\begin{array}{ccc}
 (I, J') & \longrightarrow & (I', J') \\
 \uparrow & & \uparrow \\
 (I, J) & \longrightarrow & (I', J)
 \end{array}$$

lies in the region $\mathcal{R}$, then the convexity of $\mathcal{R}$ tells us that so does the entire diagram. We can therefore apply F to it, obtaining a diagram of bicartesian squares in $C(\mathcal{A})$

$$\begin{array}{ccccccc}
 F(I, J') & \longrightarrow & F(i, J') & \longrightarrow & F(i', J') & \longrightarrow & F(I', J') \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 F(I, j') & \longrightarrow & F(i, j') & \longrightarrow & F(i', j') & \longrightarrow & F(I', j') \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 F(I, j) & \longrightarrow & F(i, j) & \longrightarrow & F(i', j) & \longrightarrow & F(I', j) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 F(I, J) & \longrightarrow & F(i, J) & \longrightarrow & F(i', J) & \longrightarrow & F(I', J)
 \end{array}$$

Lemma 39 therefore applies, and tells us that the two special squares

$$\begin{array}{ccc}
 F(i, j') & \longrightarrow & F(i', j') \\
 \uparrow & & \uparrow \\
 F(i, j) & \longrightarrow & F(i', j)
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(I, J') & \longrightarrow & F(I', J') \\
 \uparrow & & \uparrow \\
 F(I, J) & \longrightarrow & F(I', J)
 \end{array}$$

have compatible differentials.

Remark 41 It is clear that proof of Corollary 40 uses less than the full strength of the convexity hypothesis. The corollary remains true for some non-convex regions. In this article, the main region of interest in $\mathcal{R}_n = \{(x, y) \mid 0 \leq x - y \leq n + 1\}$, and $\mathcal{R}_n$ is clearly convex. Hence we do not take the trouble to give the strongest version of the corollary.

41

Remark 42 Now we want to use Corollary 40 to construct simplices in $S_n(d\mathcal{T})$. As in Remark 36, we begin with the restriction of a putative simplex to $\mathcal{R}_n \cap \{\mathbb{Z} \times \{0\}\}$. In other words, we have sequence of composable maps in $\mathcal{T}$

42

$$0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x_n \rightarrow 0.$$

and we want to show that this sequence may be extended to a simplex.

Let $\mathcal{C}$ be any Waldhausen model for $\mathcal{T}$. The first observation is that we may choose a lifting of this sequence of composable maps to $\mathcal{C}$. We will define, by descending induction on i , a sequence of morphisms in $\mathcal{C}$

$$y_i \longrightarrow y_{i+1} \longrightarrow \cdots \longrightarrow y_{n-1} \longrightarrow y_n$$

isomorphic in $\mathcal{T}$ to the sequence

$$x_i \longrightarrow x_{i+1} \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_n.$$

Choose y_n to be any object of $\mathcal{C}$ isomorphic to x_n ; this defines the sequence for $i = n$. Suppose the sequence has been defined for i . The morphism $x_{i-1} \rightarrow x_i \simeq y_i$ is a map in $\mathcal{T}$, and by the calculus of fractions in biWaldhausen complicial categories (which we call Waldhausen model categories), it may be represented as $\alpha\beta^{-1}$, with α and β as below and β a weak equivalence

$$x_{i-1} \xleftarrow{\beta} y_{i-1} \xrightarrow{\alpha} y_i.$$

The map α can be used to extend our sequence to

$$y_{i-1} \longrightarrow y_i \longrightarrow y_{i+1} \longrightarrow \cdots \longrightarrow y_{n-1} \longrightarrow y_n.$$

This completes the induction. Replacing the x_i by y_i , we now assume our sequence lies in $\mathcal{C}$.

Now we need to construct the simplex. Choose in $\mathcal{C}$ a cofibration $x_1 \rightarrow y_1^1$, with y_1^1 contractible. (For example, y_1^1 could be the mapping cone on $1 : x_1 \rightarrow x_1$). Pushing out allows us to obtain a diagram of bicartesian squares

$$\begin{array}{ccccccccccc} y_1^1 & \rightarrow & y_1^2 & \rightarrow & \cdots & \rightarrow & y_1^{n-1} & \rightarrow & y_1^n & \rightarrow & y_1^{n+1} & \rightarrow & 0 \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & x_1 & \rightarrow & x_2 & \rightarrow & \cdots & \rightarrow & x_{n-1} & \rightarrow & x_n & \rightarrow & 0 \end{array}.$$

Choosing a cofibration $y_1^2 \twoheadrightarrow y_2^2$, with y_2^2 contractible, we can continue to

$$\begin{array}{ccccccccccc} & & y_2^2 & \rightarrow & \cdots & \rightarrow & y_2^n & \rightarrow & y_2^{n+1} & \rightarrow & y_2^{n+2} & \rightarrow & 0 \\ & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\ y_1^1 & \rightarrow & y_1^2 & \rightarrow & \cdots & \rightarrow & y_1^n & \rightarrow & y_1^{n+1} & \rightarrow & 0 \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & x_1 & \rightarrow & x_2 & \rightarrow & \cdots & \rightarrow & x_n & \rightarrow & 0 \end{array}.$$

Clearly, we can iterate this process, obtaining a commutative diagram where each square is bicartesian. We can also continue this diagram in the negative direction. Suppose y_{-1}^n is contractible, and suppose we have a fibration $y_{-1}^n \twoheadrightarrow x_n$. We can pull back to obtain

$$\begin{array}{ccccccccccc} 0 & \rightarrow & x_1 & \rightarrow & x_2 & \rightarrow & \cdots & \rightarrow & x_{n-1} & \rightarrow & x_n & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & y_{-1}^0 & \rightarrow & y_{-1}^1 & \rightarrow & y_{-1}^2 & \rightarrow & \cdots & \rightarrow & y_{-1}^{n-1} & \rightarrow & y_{-1}^n \end{array}.$$

By iterating this construction in both the negative and positive direction, we obtain a functor from the region $\mathcal{R}_n \subset \mathbb{Z} \times \mathbb{Z}$ to $\mathcal{C}$. In the category $\text{ho}\mathcal{C} = \mathcal{T}$, the object y_i^i and y_{-i}^{n+1-i} are isomorphic to zero. Consider the composite functor

$$\mathcal{R}_n \rightarrow \mathcal{C} \rightarrow \mathcal{T}.$$

It vanishes at the boundary of the region $\mathcal{R}_n$. Corollary 40 then tells us that we have a simplex in $S_n({}^d\mathcal{T})$.

Remark 43 We have shown how to construct elements of $S_n({}^d\mathcal{T})$ starting from diagrams of bicartesian squares in a Waldhausen model. An element $s \in S_n({}^d\mathcal{T})$ is called *Waldhausen liftable* if there exists some Waldhausen model $\mathcal{C}$ for $\mathcal{T}$, a diagram y of bicartesian squares in $\mathcal{C}$, and an isomorphism of augmented $(\mathcal{R}_n, {}^d\mathcal{T})$ diagrams $y \cong s$.

43

Definition 44 The simplicial subset ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ is defined to be the simplicial set of all Waldhausen liftable simplices.

44

Remark 45 Note that the simplicial subset ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ does not depend on a choice of model. A simplex is liftable if there exists some model $\mathcal{C}$ for $\mathcal{T}$, and a lifting to $\mathcal{C}$. The model $\mathcal{C}$ is not specified in advance.

45

Remark 46 If we let β be the inclusion map ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$, then what we have so far are four simplicial maps

46

$${}^wS_*(\mathcal{T}) \xrightarrow{\beta} S_*({}^d\mathcal{T}) \xrightarrow{\gamma} S_*({}^v\mathcal{T}) \xrightarrow{\delta} S_*(\mathrm{Gr}^b\mathcal{A}).$$

Next we define the fifth and last map.

Remark 47 For the remainder of this section, we will assume that the reader is familiar with t -structures in triangulated categories. For an excellent account, the reader is referred to Chapter 1 of Beilinson, Bernstein and Deligne's [8]. In this section, we will use the following facts. Given a triangulated category $\mathcal{T}$ with a t -structure, there is a full subcategory $\mathcal{A} \subset \mathcal{T}$, called the *heart* of $\mathcal{T}$. It satisfies

47

- (i) $\mathcal{A}$ is an abelian category.
- (ii) Given a monomorphism $f : a \rightarrow b$ in $\mathcal{A}$, there is a canonically unique way to complete it to a distinguished triangle

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a.$$

The object c lies in $\mathcal{A} \subset \mathcal{T}$, and

$$0 \longrightarrow a \xrightarrow{f} b \xrightarrow{g} c \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$ (this already makes the cokernel map $g : b \rightarrow c$ unique up to canonical isomorphism). What is being asserted is that, given $g : b \rightarrow c$, the map $h : c \rightarrow \Sigma a$ is unique. See [8, Corollaire 1.1.10(ii)].

- (iii) There is a canonical way to define a homological functor $H : \mathcal{T} \rightarrow \mathcal{A}$.
 (iv) The t -structure is called *bounded* if H is a bounded homological functor (see Example 16 for the definition of bounded homological functors), and if

$$\{\forall i \in \mathbb{Z}, H(\Sigma^i x) = 0\} \Rightarrow x = 0.$$

Example 48. Let $\mathcal{T} = D(\mathcal{A})$ be the derived category of an abelian category $\mathcal{A}$. There is a t -structure on $\mathcal{T} = D(\mathcal{A})$, called the *standard t -structure*. The heart of $\mathcal{T}$ is $\mathcal{A} \subset D(\mathcal{A})$, where $\mathcal{A}$ is embedded in $D(\mathcal{A})$ as the complexes which vanish in all degrees but zero. The homological functor $H : \mathcal{T} \rightarrow \mathcal{A}$ of part (iii) is just the functor taking a chain complex $X \in D(\mathcal{A})$ to $H^0(X)$. This t -structure is not bounded on $\mathcal{T} = D(\mathcal{A})$. Define a full subcategory $D^b(\mathcal{A}) \subset D(\mathcal{A})$ by

$$\text{Ob}(D^b(\mathcal{A})) = \left\{ X \in D(\mathcal{A}) \mid \begin{array}{l} H^n(X) = 0 \text{ for all but} \\ \text{finitely many } n \in \mathbb{Z} \end{array} \right\}.$$

Then $D^b(\mathcal{A})$ is a triangulated subcategory of $D(\mathcal{A})$. The standard t -structure on $\mathcal{T} = D(\mathcal{A})$ restricts to a standard t -structure on $D^b(\mathcal{A}) \subset D(\mathcal{A})$. The heart is still $\mathcal{A}$, and the t -structure on $D^b(\mathcal{A})$ is bounded, as in (iv) above.

49

Lemma 49 Suppose $\mathcal{T}$ is a triangulated category with a t -structure, and let $\mathcal{A}$ be the heart. Suppose $\mathcal{T}$ has at least one Waldhausen model. Then there is a simplicial map

$$\alpha : S_*(\mathcal{A}) \rightarrow {}^w S_*(\mathcal{T}).$$

Here, by $S_*(\mathcal{A})$ we mean the Waldhausen S_* -construction on the abelian category $\mathcal{A}$.

Proof An element in Waldhausen's $S_n(\mathcal{A})$ is a string of $(n-1)$ composable monomorphisms in $\mathcal{A}$, together with a (canonically unique) choice of the quotients. That is, maps

$$0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_n$$

together with a choice of the quotients x_j/x_i for all $i < j$. Choose any Waldhausen model $\mathcal{C}$ for $\mathcal{T}$. In Remark 42 we saw that the sequence

$$0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x_n \rightarrow 0$$

can be extended to a simplex in $S_*({}^d\mathcal{T})$, with a Waldhausen lifting to $\mathcal{C}$. That is, it can be extended to a simplex in ${}^w S_*(\mathcal{T})$.

But now the restriction of this simplex to the region $\mathcal{D}_n \subset \mathcal{R}_n$ gives us nothing other than the sequence of monomorphisms

$$0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_n$$

together with a choice of the quotients x_j/x_i . This choice of the quotients must be canonically isomorphic to the choice that comes from the simplex in $S_n(\mathcal{A})$. Remark 47(ii) tells us that even the differentials are canonically unique. But by Remark 32 the simplex is determined by its restriction to $\mathcal{D}_n \subset \mathcal{R}_n$. [The careful reader, mindful of Remark 35, will recall that not all augmented diagrams on $\mathcal{D}_n$ extend to $\mathcal{R}_n$. But here we know that the extension exists, and the uniqueness always holds].

The Main Theorems

4.10

Up until now, all we have produced is a string of definitions. Let $\mathcal{T}$ be a triangulated category. Assume $\mathcal{T}$ has at least one Waldhausen model. Assume it has a bounded t -structure, with heart $\mathcal{A}$. We have constructed simplicial maps

$$S_*(\mathcal{A}) \xrightarrow{\alpha} {}^w S_*(\mathcal{T}) \xrightarrow{\beta} S_*(d\mathcal{T}) \xrightarrow{\gamma} S_*(v\mathcal{T}) \xrightarrow{\delta} S_*(\mathrm{Gr}^b \mathcal{A}) .$$

Consider the loop spaces of the geometric realisations of these maps. Write them as

$$K(\mathcal{A}) \xrightarrow{\alpha} K({}^w \mathcal{T}) \xrightarrow{\beta} K(d\mathcal{T}) \xrightarrow{\gamma} K(v\mathcal{T}) \xrightarrow{\delta} K(\mathrm{Gr}^b \mathcal{A}) .$$

The main theorem is

Theorem 50 With the notation as above, we have

50

- (i) The composite $\delta\gamma\beta\alpha : K(\mathcal{A}) \rightarrow K(\mathrm{Gr}^b \mathcal{A})$ is a homotopy equivalence.
- (ii) The map $\alpha : K(\mathcal{A}) \rightarrow K({}^w \mathcal{T})$ is a homotopy equivalence.
- (iii) The space $K(v\mathcal{T})$ has a homotopy type which depends only on $\mathcal{A}$. That is, $K(v\mathcal{T}) \cong K({}^v D^b(\mathcal{A}))$.

Proof The proofs of these statements are, at least at the moment, long and very difficult. The proof of (i) may be found in [70] and [71], or in [66] and [67]. For the proof of (ii), see [72], or [68] and [69]. The detailed proof of (iii) does not yet exist in print. The idea is that it follows by a slight modification of the proof of (ii).

In the sections which follow, I will try to highlight the problems which naturally arise. The aim of this survey is to explain why the theorems we now know, that is Theorem 50(i), (ii) and (iii), are deeply unsatisfying and cry for improvement.

Before we launch into an exhaustive treatment of the defects in what we know, in this section I will give a brief discussion of the positive. Here are some remarkable consequences of the theorems.

51 Remark 51 From Theorem 50(i) we know that the spaces $K({}^w\mathcal{T})$, $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$ all contain Quillen's K -theory $K(\mathcal{A})$ as a retract. Far from losing all information about higher K -theory, the passage to the derived category has, if anything, added more information.

52 Remark 52 From Theorem 50(ii) we conclude the following. Suppose $\mathcal{T}$ is a triangulated category with at least one Waldhausen model. Suppose it admits two bounded t -structures, with hearts $\mathcal{A}$ and $\mathcal{B}$. Then the Quillen K -theories of $\mathcal{A}$ and $\mathcal{B}$ agree. In symbols, we have

$$K(\mathcal{A}) \cong K(\mathcal{B}) .$$

After all both are isomorphic, by Theorem 50(ii), with $K({}^w\mathcal{T})$.

This was unknown even for the “standard t -structures” of Example 48. In other words, a special case of the above is where we have two abelian categories $\mathcal{A}$ and $\mathcal{B}$, with $D^b(\mathcal{A}) \cong D^b(\mathcal{B})$. Put $\mathcal{T} = D^b(\mathcal{A}) = D^b(\mathcal{B})$. Then $\mathcal{T}$ certainly has at least one Waldhausen model, namely $C^b(\mathcal{A})$. It has two bounded t -structures, namely the standard one on $D^b(\mathcal{A})$ and the standard one on $D^b(\mathcal{B})$. The hearts of these two t -structures are $\mathcal{A}$ and $\mathcal{B}$ respectively. We conclude that $K(\mathcal{A}) \cong K(\mathcal{B})$.

53 Remark 53 In comparing the consequences of Theorem 50 with what was known earlier, it is helpful to recall some of the work of Waldhausen.

To each Waldhausen category $\mathcal{C}$, Waldhausen associates a K -theory. Let us call it $WK(\mathcal{C})$, for the Waldhausen K -theory of $\mathcal{C}$. Suppose we are given an exact functor of Waldhausen categories $\alpha : \mathcal{C} \rightarrow \mathcal{D}$. Suppose that

$$\mathrm{ho}(\alpha) : \mathrm{ho}\mathcal{C} \rightarrow \mathrm{ho}\mathcal{D}$$

is an equivalence of triangulated categories. From Waldhausen's Approximation Theorem, it is possible to deduce fairly easily that

$$WK(\alpha) : WK(\mathcal{C}) \rightarrow WK(\mathcal{D})$$

is a homotopy equivalence. For details see Thomason [89, Theorem 1.9.8]. It follows that, given any zigzag of exact functors of Waldhausen categories

$$\begin{array}{ccccccc}
 & \mathcal{C}_1 & & \cdots & & \mathcal{C}_{n-1} & \\
 \swarrow \alpha_1 & & \searrow \alpha_2 & & \swarrow \alpha_{n-2} & & \searrow \alpha_{n-1} \\
 \mathcal{C}_0 & & \mathcal{C}_2 & & \mathcal{C}_{n-2} & & \mathcal{C}_n
 \end{array}$$

if each $\mathrm{ho}(\alpha_i)$ is an equivalence of triangulated categories, then $WK(\mathcal{C}_0) \cong WK(\mathcal{C}_n)$.

Example 54. For example, let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and assume that the categories $D^b(\mathcal{A}) \cong D^b(\mathcal{B})$ are equivalent. Assume further that the equivalence can be lifted to models. This means there is a zigzag of exact functors of Waldhausen models from $C^b(\mathcal{A})$ to $C^b(\mathcal{B})$, as in Remark 53. Then it follows that $K(\mathcal{A}) \cong K(\mathcal{B})$. Already in this baby application there is an advantage to Theorem 50 over the older results. The advantage is that, in applying Theorem 50, there is no need to assume the equivalence $D^b(\mathcal{A}) \cong D^b(\mathcal{B})$ can be lifted to models.

While on the subject of comparing Theorem 50 with the earlier results, let us mention a question raised by Thomason. Thomason asked the following: Does there exist a pair of Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, with

$$\mathrm{ho}\mathcal{C} \cong \mathrm{ho}\mathcal{D} \quad \text{but} \quad WK(\mathcal{C}) \not\cong WK(\mathcal{D}) ?$$

By Remark 53, a pair of the sort Thomason asked for could not possibly be compared by a zigzag of maps of models as above. Not quite so obvious is the fact that, if no such pair exists, then the “standard t -structure” case of my theorem above becomes a consequence of Waldhausen’s work.

We now know that such a pair exists. The result may be found in Schlichting [85]. In this very precise sense, my result cannot be deduced from Waldhausen’s.

Remark 55 Quillen defined a K -theory space $K(\mathcal{A})$ for any abelian category $\mathcal{A}$. If we define $K'(\mathcal{A}) = K({}^\nu D^b(\mathcal{A}))$, we have a functor such that

- (i) By Theorem 50(i), there is a natural split inclusion $K(\mathcal{A}) \rightarrow K'(\mathcal{A})$.
- (ii) By Theorem 50(iii), if $\mathcal{A}$ is the heart of a bounded t -structure, on a triangulated category $\mathcal{T}$ with at least one Waldhausen model, then

$$K'(\mathcal{A}) \cong K({}^\nu \mathcal{T}) .$$

No information is lost if we replace $K(\mathcal{A})$ by $K'(\mathcal{A})$, and for all we know $K'(\mathcal{A})$ might be better.

4.11 Computational Problems

It is time to turn to the problems in the subject, which are very many. Let us begin with what ought to be the easiest. We should be able to compute the various maps, at least for low dimensions.

Theorem 50(i) tells us that the spaces $K({}^w\mathcal{T})$, $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$ all contain Quillen's K -theory $K(\mathcal{A})$ as a retract. It is easy to see that in K_0 , this is an isomorphism

$$K_0(\mathcal{A}) = K_0({}^w\mathcal{T}) = K_0({}^d\mathcal{T}) = K_0({}^v\mathcal{T}) .$$

Very embarrassingly, this is all we know. The first question would be

56 Problem 56 Is it true that

$$K_1(\mathcal{A}) = K_1({}^w\mathcal{T}) = K_1({}^d\mathcal{T}) = K_1({}^v\mathcal{T}) ?$$

If not, can one say anything about the other direct summands?

It goes without saying that the same problem is entirely open for K_i , for any $i > 1$. I stated Problem 56 as a problem about K_1 for two reasons.

- (i) In order to show how embarrassingly little we know.
- (ii) Because very recently, as a result of Vakinin [91, 92], we actually have a half-way usable description of K_1 of a triangulated category $\mathcal{T}$.

57 Remark 57 One way to compute K_1 is from the definition we gave. The K -theory spaces are the loop spaces of the simplicial sets ${}^wS_*(\mathcal{T})$, $S_*({}^d\mathcal{T})$ and $S_*({}^v\mathcal{T})$ respectively. This means the groups K_1 are the second homotopy groups

$$\pi_2|{}^wS_*(\mathcal{T})| , \quad \pi_2|S_*({}^d\mathcal{T})| , \quad \pi_2|S_*({}^v\mathcal{T})|$$

I do not consider this a computationally-friendly description. The comment (ii) above reminds the reader that, from the recent work by Vakinin [91, 92], we have a much more useful description. It is for this reason that the problem might be more manageable in K_1 .

So far we have looked only at hearts of t -structures, which are always abelian categories. One special case is $\mathcal{T} = D^b(\mathcal{A})$, with the standard t -structure as in Example 48. We know that there are maps in K -theory

$$K(\mathcal{A}) \xrightarrow{\alpha} K({}^wD^b(\mathcal{A})) \xrightarrow{\beta} K({}^dD^b(\mathcal{A})) \xrightarrow{\gamma} K({}^vD^b(\mathcal{A})) ,$$

and that the map $\gamma\beta\alpha : K(\mathcal{A}) \rightarrow K({}^vD^b(\mathcal{A}))$ is a monomorphism (it is even split injective). It is natural to wonder what happens if we replace the abelian category $\mathcal{A}$ by an exact category $\mathcal{E}$. There is a sensible way to define the derived category $D^b(\mathcal{E})$

for any exact category $\mathcal{E}$. The category $D^b(\mathcal{E})$ is definitely a triangulated category. This construction may be found in [64].

The general formalism, valid for any triangulated category, specialises in the case of $D^b(\mathcal{E})$ to give maps

$$K({}^wD^b(\mathcal{E})) \xrightarrow{\beta} K({}^dD^b(\mathcal{E})) \xrightarrow{\gamma} K({}^vD^b(\mathcal{E})) .$$

Not quite so immediate, but nevertheless true, is that there is also a continuous map $\alpha : K(\mathcal{E}) \rightarrow K({}^wD^b(\mathcal{E}))$. From Vaknin's direct computations [91], we have

Proposition 58 For certain choices of the exact category $\mathcal{E}$, the induced map $K_1(\alpha) : K_1(\mathcal{E}) \rightarrow K_1({}^wD^b(\mathcal{E}))$ has a non-trivial kernel. **58**

Note that this is quite unlike what happens when $\mathcal{E}$ is abelian; in the abelian case we know that $K(\mathcal{E})$ is a retract of each of $K({}^wD^b(\mathcal{E}))$, $K({}^dD^b(\mathcal{E}))$ and $K({}^vD^b(\mathcal{E}))$. This leads to:

Problem 59 For an exact category $\mathcal{E}$, compute the maps **59**

$$K(\mathcal{E}) \xrightarrow{\alpha} K({}^wD^b(\mathcal{E})) \xrightarrow{\beta} K({}^dD^b(\mathcal{E})) \xrightarrow{\gamma} K({}^vD^b(\mathcal{E})) .$$

Even an explicit computational understanding of what happens in K_1 would be a vast improvement over what we now know.

Functoriality Problems

4.12

Starting with any triangulated category $\mathcal{T}$, we have defined three possible candidates for its K -theory. They are the spaces $K({}^w\mathcal{T})$, $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$. Of the three, $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$ are functors in $\mathcal{T}$. Given a triangulated functor of triangulated categories $f : \mathcal{S} \rightarrow \mathcal{T}$, there are natural induced maps

$$K({}^df) : K({}^d\mathcal{S}) \rightarrow K({}^d\mathcal{T}) \quad \text{and} \quad K({}^vf) : K({}^v\mathcal{S}) \rightarrow K({}^v\mathcal{T}) .$$

Remark 60 The simplicial sets $S_*({}^d\mathcal{T})$ and $S_*({}^v\mathcal{T})$ have a very nice addition defined on them, allowing us to construct an infinite loop structure on $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$. From now on, we will view these as spectra. **60**

Theorem 50 tells us little about $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$. All we know is that, if $\mathcal{T}$ had a bounded t -structure with heart $\mathcal{A}$, then $K(\mathcal{A})$ is a retract of both $K({}^d\mathcal{T})$ and

$K({}^v\mathcal{T})$. The good theorem is about $K({}^w\mathcal{T})$. Suppose $\mathcal{T}$ has at least one Waldhausen model. Theorem 50(ii) tells us that $K(\mathcal{A}) = K({}^w\mathcal{T})$. This suggests that we define the K -theory of the triangulated category $\mathcal{T}$ to be $K({}^w\mathcal{T})$, and forget about the other options. Let me now point to all the faults of $K({}^w\mathcal{T})$. First we should remind the reader of the definition of $K({}^w\mathcal{T})$.

Given a triangulated category $\mathcal{T}$, there is a simplicial set $S_*({}^d\mathcal{T})$. The set $S_n({}^d\mathcal{T})$ has for its elements the augmented diagrams for the pair $(\mathcal{R}_n, {}^d\mathcal{T})$, which vanish on the boundary of the region $\mathcal{R}_n$. In Sect. 4.9, we defined what it means for an element of $S_n({}^d\mathcal{T})$ to have a Waldhausen lifting (see Remark 43). The simplicial subset ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ is defined to be the simplicial subset of all Waldhausen liftable simplices. The K -theory $K({}^w\mathcal{T})$ is the loop space of the geometric realisation of ${}^wS_*(\mathcal{T})$.

61 Remark 61 There is no obvious H -space structure on ${}^wS_*(\mathcal{T})$. Suppose we are given two n -simplices. Both are augmented diagrams for the pair $(\mathcal{R}_n, {}^d\mathcal{T})$. Each diagram has a lifting to some Waldhausen model. Suppose the first diagram lifts to a model $\mathcal{C}_1$ and the second lifts to a model $\mathcal{C}_2$. For all we know, the direct sum may not have a lifting to any model.

Since ${}^wS_*(\mathcal{T})$ is not obviously an H -space, it most certainly is not obviously an infinite loop space. Let us now be careful about what Theorem 50(ii) tells us. If $\mathcal{A}$ is the heart of a bounded t -structure on $\mathcal{T}$, the theorem asserts that $K(\mathcal{A}) \cong K({}^w\mathcal{T})$. This is only a homotopy equivalence of spaces. It is not an H -map of H -spaces, and most certainly not an infinite loop map of infinite loop spaces. In Remark 52 we observed that, if $\mathcal{A}$ and $\mathcal{B}$ are two hearts of two bounded t -structures on a single triangulated category $\mathcal{T}$, then $K(\mathcal{A}) \cong K(\mathcal{B})$. Both $K(\mathcal{A})$ and $K(\mathcal{B})$ are naturally infinite loop spaces, but the above isomorphism is only as spaces. It is not an infinite loop map.

62 Remark 62 Unlike the many open problems I am in the process of outlining, this problem is settled. Suppose we are in the situation above. That is, $\mathcal{T}$ is a triangulated category with at least one Waldhausen model, and $\mathcal{A}$ and $\mathcal{B}$ are two hearts of two bounded t -structures on $\mathcal{T}$. Then $K(\mathcal{A}) \cong K(\mathcal{B})$, even as infinite loop spaces. The proof is to introduce yet another simplicial set, which we can denote ${}^+S_*(\mathcal{T})$. We define ${}^+S_*(\mathcal{T})$ to be a subset of $S_*({}^d\mathcal{T})$. A simplex in $S_*({}^d\mathcal{T})$ belongs to ${}^+S_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ if it can be written as a direct sum of simplices, each with a Waldhausen lifting. In other words, we obtain ${}^+S_*(\mathcal{T})$ as the closure of ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ under direct sums. Define $K({}^+\mathcal{T})$ to be the loop space of the geometric realisation of ${}^+S_*(\mathcal{T})$.

It is now easy to see that $K({}^+\mathcal{T})$ is an infinite loop space. It turns out that the proof of Theorem 50(ii) works well for $K({}^+\mathcal{T})$. We conclude that the map $K(\mathcal{A}) \rightarrow K({}^+\mathcal{T})$ is a homotopy equivalence. Since it is an infinite loop map of infinite loop spaces, the problem posed by Remark 61 is solved.

There is something quite unappetising about the nature of the proof outlined in Remark 62. Surely we do not want to have to introduce a new simplicial set, and a new definition for the K -theory of the triangulated category $\mathcal{T}$, every time we wish to prove a new theorem. This method of proof by modification of the simplicial set is the best we know; presumably there is a good choice of the simplicial set, rendering such trickery unnecessary.

Remark 63 The most serious problem with $K(^w\mathcal{T})$ is that it is not a functor of $\mathcal{T}$. Let $f : \mathcal{S} \rightarrow \mathcal{T}$ be a triangulated functor of triangulated categories. I do not know how to construct an induced map $K(^w\mathcal{S}) \rightarrow K(^w\mathcal{T})$. The same problem also holds for the simplicial sets of Remark 62. Starting with a triangulated functor $f : \mathcal{S} \rightarrow \mathcal{T}$, I do not know how to construct an induced map $K(^+\mathcal{S}) \rightarrow K(^+\mathcal{T})$.

63

Problem 64 Find a simplicial set $K(^?\mathcal{T})$, which is a functor of $\mathcal{T}$ and for which the strong statement of Theorem 50(ii) holds.

64

It is quite possible that $K(^?\mathcal{T})$ is already on the list of possibilities we have considered, and that the problem is that we do not yet know how to prove enough about it.

Localisation

4.13

In order to turn the K -theory of triangulated categories into a powerful tool, one would need to have some theorems about the way $K(^?\mathcal{T})$ changes as $\mathcal{T}$ varies. Note that I have left it vague which particular K -theory one should consider. At this point our ignorance is so profound that we should do the unprejudiced thing and consider all the possibilities. When we know more, we will presumably know which of the simplicial sets can safely be forgotten.

There is one obvious conjecture. Suppose $\mathcal{S}$ is a triangulated category, and suppose that $\mathcal{R} \subset \mathcal{S}$ is a thick subcategory. This means that $\mathcal{R}$ is a full, triangulated subcategory of $\mathcal{S}$, and that if $y \in \mathcal{R}$ decomposes as $y = x \oplus x'$ in the category $\mathcal{S}$, then both x and x' lie in $\mathcal{R}$. That is, $\mathcal{R}$ is closed under the formation in $\mathcal{S}$ of direct summands of its objects. Verdier thesis [94] taught us how to form the quotient category $\mathcal{T} = \mathcal{S}/\mathcal{R}$. We have triangulated functors of triangulated categories

$$\mathcal{R} \rightarrow \mathcal{S} \rightarrow \mathcal{T},$$

and the composite $\mathcal{R} \rightarrow \mathcal{T}$ is naturally isomorphic to the zero map.

Problem 65 Find a suitable K -theory of triangulated categories $K(^? -)$ so that

(i) $K(^?\mathcal{T})$ is a functor of the triangulated category $\mathcal{T}$.

65

(ii) By (i) we know that the functor $K(^?\mathcal{R})$ yields continuous maps

$$K(^?\mathcal{R}) \rightarrow K(^?\mathcal{S}) \rightarrow K(^?\mathcal{T}).$$

The composite $K(^?\mathcal{R}) \rightarrow K(^?\mathcal{T})$ must be the null map, and there is a natural map from $K(^?\mathcal{R})$ to the homotopy fiber of $K(^?\mathcal{S}) \rightarrow K(^?\mathcal{T})$. We want this map to be a homotopy equivalence.

66

Remark 66 The natural candidates for the functor $K(^?\mathcal{R})$ are $K(^d\mathcal{R})$ and $K(^v\mathcal{R})$; what makes them natural is that we know they are functors. Unless we have a functor, the question makes no sense. Without a functor, the maps $\mathcal{R} \rightarrow \mathcal{S} \rightarrow \mathcal{T}$ will not, in general, induce maps in K -theory, and it would be meaningless to ask whether the induced sequence is a homotopy fibration. For $K(^d\mathcal{R})$ and $K(^v\mathcal{R})$, the problem is concrete enough. We are asking whether one or both of the sequences

$$\begin{array}{ccccc} K(^d\mathcal{R}) & \longrightarrow & K(^d\mathcal{S}) & \longrightarrow & K(^d\mathcal{T}) \\ K(^v\mathcal{R}) & \longrightarrow & K(^v\mathcal{S}) & \longrightarrow & K(^v\mathcal{T}) \end{array}$$

is a homotopy fibration.

I spent a long time working on this problem. It goes without saying that I do not know the answer; if I did, I would not keep it secret.

67

Remark 67 Suppose we succeed in finding a K -theory $K(^?\mathcal{R})$ of triangulated categories, so that

(i) As in Problem 65, when $\mathcal{T} = \mathcal{S}/\mathcal{R}$ we have a homotopy fibration

$$K(^?\mathcal{R}) \rightarrow K(^?\mathcal{S}) \rightarrow K(^?\mathcal{T}).$$

(ii) If $\mathcal{A}$ is the heart of a bounded t -structure on $\mathcal{T}$, then there is a natural isomorphism

$$K(\mathcal{A}) \rightarrow K(^?\mathcal{T}).$$

Then Quillen's localisation theorem [77, Theorem 5 of §5] follows easily. Given abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ with $\mathcal{C} = \mathcal{B}/\mathcal{A}$, we have triangulated categories

$D^b(\mathcal{C}) = \frac{D^b(\mathcal{B})}{D^b_{\mathcal{A}}(\mathcal{B})}$ where $D^b_{\mathcal{A}}(\mathcal{B})$ is the category of bounded chain complexes in $\mathcal{B}$, whose cohomology lies in $\mathcal{A} \subset \mathcal{B}$. Applying (i) and (ii) above to these triangulated categories with the obvious t -structures, Quillen's localisation theorem is immediate.

Bounded δ -Functors

Many people have tried to study K -theory using techniques from derived categories. The usual approach has been to rigidify. One passes from the derived category to an algebraic gadget with more structure (Waldhausen categories or Grothendieck derivators), one defines K -theory using the added structure, and then one proves theorems suggesting that the outcome is largely independent of the rigidification.

Theorem 50(i) at least hints that this might be the wrong direction to go. In this section we will study this theorem. We will see that it tells us how, using just a δ -functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$, it is possible to construct an induced map in higher K -theory. A δ -functor is much less than a derived functor between the derived categories. It is just possible that the way to make real progress is not by rigidifying (passing from the derived category to a model), but by passing to something less rigid. I do not have a candidate to propose; finding one is an open problem we discuss in this Section and the next.

Theorem 50(i) is very intriguing. We remind the reader. In this article we constructed maps

$$K(\mathcal{A}) \xrightarrow{\alpha} K({}^w\mathcal{T}) \xrightarrow{\beta} K({}^d\mathcal{T}) \xrightarrow{\gamma} K({}^v\mathcal{T}) \xrightarrow{\delta} K(\mathrm{Gr}^b\mathcal{A}).$$

Theorem 50(i) asserts that the composite $\delta\gamma\beta\alpha : K(\mathcal{A}) \rightarrow K(\mathrm{Gr}^b\mathcal{A})$ is a homotopy equivalence. What is quite surprising is that this composite is independent of the triangulated category $\mathcal{T}$.

For any abelian category $\mathcal{A}$, Example 15 constructs for us a category with squares $\mathrm{Gr}^b\mathcal{A}$, and we formally have a simplicial set $S_*(\mathrm{Gr}^b\mathcal{A})$. The space $K(\mathrm{Gr}^b\mathcal{A})$ is the loop space of the geometric realisation of $S_*(\mathrm{Gr}^b\mathcal{A})$. Quillen's K -theory $K(\mathcal{A})$ is the loop space of the geometric realisation of Waldhausen's simplicial set $S_*(\mathcal{A})$. The maps α, β, γ and δ are all the loops on the geometric realisations of explicit simplicial maps. It is not difficult to compute the composite; it amounts to remembering the definitions of the maps α, β, γ and δ . We leave the details to the reader; let us only state the conclusion. In the next paragraphs, we tell the reader what the map $\delta\gamma\beta\alpha$ does to a simplex in Waldhausen's $S_*(\mathcal{A})$.

Suppose $s \in S_n(\mathcal{A})$ is an n -simplex in Waldhausen's simplicial set $S_*(\mathcal{A})$. The simplex s is a sequence of monomorphisms in $\mathcal{A}$

$$0 \longrightarrow x_1 \rightrightarrows x_2 \rightrightarrows \cdots \rightrightarrows x_{n-1} \rightrightarrows x_n$$

together with choices for the cokernels γ_i^j of each monomorphism $x_i \rightarrow x_j$. Recall the region $\mathcal{D}_n \subset \mathcal{R}_n$ of Remark 32. The simplex $s \in S_n(\mathcal{A})$ is a functor

$$\mathcal{D}_n \rightarrow \mathcal{A} \subset \mathrm{Gr}^b\mathcal{A}.$$

To make it into an augmented diagram for the pair $(\mathcal{D}_n, \text{Gr}^b \mathcal{A})$ we only need to choose the coherent differentials; we choose them all to be zero. The region $\mathcal{D}_n \subset \mathcal{R}_n$ is a fundamental domain for augmented diagrams on $\mathcal{R}_n$. Any augmented diagram on $\mathcal{R}_n$ is uniquely determined by its restriction to $\mathcal{D}_n$. Furthermore, by the last paragraph of Remark 35, for the category with squares $\text{Gr}^b \mathcal{A}$ there is no extension problem; our augmented diagram on $\mathcal{D}_n$ extends (uniquely) to an augmented diagram on $\mathcal{R}_n$. The simplicial map $\delta\gamma\beta\alpha : S_*(\mathcal{A}) \rightarrow S_*(\text{Gr}^b \mathcal{A})$ takes $s \in S_n(\mathcal{A})$ to this augmented diagram for the pair $(\mathcal{R}_n, \text{Gr}^b \mathcal{A})$.

The next step is to generalise this to arbitrary δ -functors. We should begin by reminding the reader what a δ -functor is. I will only give a sketch here; much more detail may be found in Grothendieck [35]. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A δ -functor $f^* : \mathcal{A} \rightarrow \mathcal{B}$ is a functor taking short exact sequences in $\mathcal{A}$ to long exact sequences in $\mathcal{B}$. More precisely

68 Definition 68 A δ -functor $f^* : \mathcal{A} \rightarrow \mathcal{B}$ is

- (i) For each integer $i \in \mathbb{Z}$, an additive functor $f^i : \mathcal{A} \rightarrow \mathcal{B}$.
- (ii) For each integer $i \in \mathbb{Z}$ and each short exact sequence in $\mathcal{A}$

$$0 \rightarrow a' \rightarrow a \rightarrow a'' \rightarrow 0,$$

a map $\partial : f^i(a'') \rightarrow f^{i+1}(a')$.

- (iii) The maps ∂ are natural in the short exact sequences. Given an integer $i \in \mathbb{Z}$ and a map of short exact sequences in $\mathcal{A}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & a' & \longrightarrow & a & \longrightarrow & a'' & \longrightarrow & 0 \\ & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' & & \\ 0 & \longrightarrow & b' & \longrightarrow & b & \longrightarrow & b'' & \longrightarrow & 0 \end{array}$$

there is a commutative square

$$\begin{array}{ccc} f^i(a'') & \xrightarrow{\partial} & f^{i+1}(a') \\ \downarrow f^i(\alpha'') & & \downarrow f^{i+1}(\alpha') \\ f^i(b'') & \xrightarrow[\partial]{} & f^{i+1}(b') \end{array}.$$

- (iv) Every short exact sequence in $\mathcal{A}$

$$0 \rightarrow a' \rightarrow a \rightarrow a'' \rightarrow 0$$

goes to a long exact sequence in $\mathcal{B}$

$$\cdots \rightarrow f^{i-1}(a'') \xrightarrow{\partial} f^i(a') \rightarrow f^i(a) \rightarrow f^i(a'') \xrightarrow{\partial} f^{i+1}(a') \rightarrow \cdots.$$

A δ -functor $f^* : \mathcal{A} \rightarrow \mathcal{B}$ is called *bounded* if, for every object $a \in \mathcal{A}$, $f^i(a)$ vanishes for all but finitely many $i \in \mathbb{Z}$.

Now that we have recalled the definition, we make the observation

Lemma 69 Let $f^* : \mathcal{A} \rightarrow \mathcal{B}$ be a bounded δ -functor. Define a functor, which by abuse of notation we will write as

69

$$f^* : \mathcal{A} \rightarrow \text{Gr}^b(\mathcal{B}) .$$

It is the functor taking $a \in \mathcal{A}$ to the sequence $\{f^i(a) \mid i \in \mathbb{Z}\}$. Given a bicartesian square in $\mathcal{A}$

$$\begin{array}{ccc} & \delta & \\ b' & \longrightarrow & c \\ \beta \uparrow & & \uparrow \gamma \\ a & \xrightarrow{\alpha} & b \end{array}$$

we assert that the functor $f^* : \mathcal{A} \rightarrow \text{Gr}^b(\mathcal{B})$ takes it to a special square in $\text{Gr}^b(\mathcal{B})$. Furthermore, if we are given a diagram of bicartesian squares in $\mathcal{A}$

$$\begin{array}{ccccccc} d''' & \longrightarrow & e'' & \longrightarrow & f' & \longrightarrow & g \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ c'' & \longrightarrow & d'' & \longrightarrow & e' & \longrightarrow & f \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ b' & \longrightarrow & c' & \longrightarrow & d' & \longrightarrow & e \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & d \end{array}$$

we deduce two bicartesian squares, one contained in the other

$$\begin{array}{ccc} d'' & \longrightarrow & e' \\ \uparrow & & \uparrow \\ c' & \longrightarrow & d' \end{array} \qquad \begin{array}{ccc} d''' & \longrightarrow & g \\ \uparrow & & \uparrow \\ a & \longrightarrow & d \end{array} .$$

The functor f^* takes these to two special squares, with differentials

$$\begin{aligned}\partial_1 : f^*(e') &\longrightarrow \Sigma f^*(c') \\ \partial_2 : f^*(g) &\longrightarrow \Sigma f^*(a) \quad .\end{aligned}$$

These differentials are compatible; that is, ∂_1 is the composite

$$f^*(e') \rightarrow f^*(g) \xrightarrow{\partial_2} \Sigma f^*(a) \rightarrow \Sigma f^*(c').$$

Proof The commutative square

$$\begin{array}{ccc} & & \delta \\ b' & \xrightarrow{\quad} & c \\ \beta \uparrow & & \uparrow \gamma \\ a & \xrightarrow{\quad} & b \\ & & \alpha \end{array}$$

is bicartesian, and by Definition 37 this means that

$$0 \rightarrow a \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} b \oplus b' \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} c \rightarrow 0$$

is a short exact sequence in $\mathcal{A}$. But then the δ functor f^* gives us a map $\partial : f^*(c) \rightarrow \Sigma f^*(a)$, so that

$$f^*(a) \xrightarrow{\begin{pmatrix} f^*(\alpha) \\ -f^*(\beta) \end{pmatrix}} f^*(b) \oplus f^*(b') \xrightarrow{\begin{pmatrix} f^*(\gamma) & f^*(\delta) \end{pmatrix}} f^*(c) \xrightarrow{\partial} \Sigma f^*(a)$$

is a long exact sequence. In other words, the differential $\partial : f^*(c) \rightarrow \Sigma f^*(a)$ together with the commutative square

$$\begin{array}{ccc} & & f^*(\delta) \\ f^*(b') & \xrightarrow{\quad} & f^*(c) \\ f^*(\beta) \uparrow & & \uparrow f^*(\gamma) \\ f^*(a) & \xrightarrow{\quad} & f^*(b) \\ & & f^*(\alpha) \end{array}$$

give us a special square in $\text{Gr}^b(\mathcal{B})$. It remains to establish the coherence of the differentials.

Assume therefore that we are given a diagram of bicartesian squares in $\mathcal{A}$

$$\begin{array}{ccccccc}
 d''' & \longrightarrow & e'' & \longrightarrow & f' & \longrightarrow & g \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 c'' & \longrightarrow & d'' & \longrightarrow & e' & \longrightarrow & f \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 b' & \longrightarrow & c' & \longrightarrow & d' & \longrightarrow & e \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & d \quad .
 \end{array}$$

We deduce maps of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & a & \longrightarrow & d \oplus d''' & \longrightarrow & g \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & c' & \longrightarrow & e \oplus e'' & \longrightarrow & g \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & c' & \longrightarrow & d' \oplus d'' & \longrightarrow & e' \longrightarrow 0 \quad .
 \end{array}$$

The fact that f^* is a δ -functor gives us commutative squares

$$\begin{array}{ccc}
 f^*(g) & \xrightarrow{\partial_2} & \Sigma f^*(a) \\
 \parallel & & \downarrow \\
 f^*(g) & \xrightarrow{\partial_3} & \Sigma f^*(c') \\
 \uparrow & & \parallel \\
 f^*(e') & \xrightarrow{\partial_1} & \Sigma f^*(c')
 \end{array}$$

from which the coherence for the differentials immediately follows.

70

Proposition 70 Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Then any δ -functor $f^* : \mathcal{A} \rightarrow \mathcal{B}$ induces a simplicial map of simplicial sets

$$S_*(\mathcal{A}) \rightarrow S_*(\mathrm{Gr}^b \mathcal{B}) .$$

Proof As at the beginning of this section, a simplex $s \in S_n(\mathcal{A})$ is a sequence of monomorphisms in $\mathcal{A}$

$$0 \longrightarrow x_1 \rightrightarrows x_2 \rightrightarrows \cdots \rightrightarrows x_{n-1} \rightrightarrows x_n$$

together with choices for the cokernels y_i^j of each monomorphism $x_i \rightarrow x_j$. Applying f^* to the diagram as in Lemma 69, we deduce an augmented diagram for the pair $(\mathcal{D}_n, \mathrm{Gr}^b \mathcal{B})$. The region $\mathcal{D}_n \subset \mathcal{R}_n$ is a fundamental domain for $\mathcal{R}_n$ by Remark 32, and there is no extension problem by the last paragraph of Remark 35. Hence the diagram extends uniquely to an augmented diagram for the pair $(\mathcal{R}_n, \mathrm{Gr}^b \mathcal{B})$, vanishing on the boundary. That is, we have a simplex in $S_n(\mathrm{Gr}^b \mathcal{B})$. This defines the simplicial map.

71

Remark 71 The identity functor $1 : \mathcal{A} \rightarrow \mathcal{A}$ can always be viewed as a δ -functor. That is, we define a δ -functor $i^* : \mathcal{A} \rightarrow \mathcal{A}$ by putting $i^0 = 1$, and $i^j = 0$ if $j \neq 0$. The differential ∂ is zero for every short exact sequence in $\mathcal{A}$.

In terms of Proposition 70, the computation at the beginning of the section says that the map $\delta\gamma\beta\alpha : S_*(\mathcal{A}) \rightarrow S_*(\mathrm{Gr}^b \mathcal{A})$ is nothing other than the map induced by the the trivial δ -functor i^* . That is,

$$\delta\gamma\beta\alpha = i^* : S_*(\mathcal{A}) \rightarrow S_*(\mathrm{Gr}^b \mathcal{A}).$$

Theorem 50(i) asserts that i^* induces a homotopy equivalence.

Given any δ -functor $f^* : \mathcal{A} \rightarrow \mathcal{B}$, we can now define an induced map $K(f^*) : K(\mathcal{A}) \rightarrow K(\mathcal{B})$. Consider the diagram

$$\begin{array}{ccc} S_*(\mathcal{A}) & & S_*(\mathcal{B}) \\ & \searrow f^* & \swarrow i^* \\ & S_*(\mathrm{Gr}^b \mathcal{B}) & \end{array}$$

If we pass to loop spaces of geometric realisations, we have a diagram

$$\begin{array}{ccc}
 K(\mathcal{A}) & & K(\mathcal{B}) \\
 & \searrow K(f^*) & \swarrow K(i^*) \\
 & K(\mathrm{Gr}^b \mathcal{B}) &
 \end{array}$$

and the map $K(i^*)$ is a homotopy equivalence. The map induced by f^* is simply

$$K(i^*)^{-1}K(f^*) : K(\mathcal{A}) \rightarrow K(\mathcal{B}) .$$

What I find so puzzling about this theorem is

Problem 72 What happens to the composite of two δ -functors? Suppose we have three abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, and two δ -functors

72

$$\mathcal{A} \xrightarrow{f^*} \mathcal{B} \xrightarrow{g^*} \mathcal{C} .$$

The above tells us how to construct maps in K -theory

$$K(\mathcal{A}) \xrightarrow{K(f^*)} K(\mathcal{B}) \xrightarrow{K(g^*)} K(\mathcal{C}) .$$

There is a composite map $K(g^*)K(f^*) : K(\mathcal{A}) \rightarrow K(\mathcal{C})$. What is the homological algebra data inducing it?

Presumably the composite map $K(g^*)K(f^*)$ must be induced by the composite g^*f^* . But what is the composite of two δ -functors? A δ -functor is a strange beast, taking short exact sequences in $\mathcal{A}$ to long exact sequences in $\mathcal{B}$. What does the composite of two such things do? Does it take short exact sequences in $\mathcal{A}$ to spectral sequences in $\mathcal{B}$? If so, how?

It would already be interesting if someone could formulate a plausible conjecture for Problem 72.

Devissage

4.15

There are two theorems about the K -theory of abelian categories which are formally very similar. They are Quillen's resolution theorem [77, Theorem 3 and Corollary 1 of §4] and Quillen's devissage theorem [77, Theorem 4 of §5]. Let me remind the reader.

73 **Theorem 73** Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a fully faithful, exact embedding of exact categories. If either (i) or (ii) below holds, then the induced map

$$K(f) : K(\mathcal{A}) \rightarrow K(\mathcal{B})$$

is a homotopy equivalence. It remains to tell the reader what are the hypotheses (i) and (ii).

(i) *Resolution*: Whenever we have an exact sequence

$$0 \rightarrow b' \rightarrow b \rightarrow b'' \rightarrow 0$$

in $\mathcal{B}$, then

$$\{b, b'' \in \mathcal{A}\} \Rightarrow b' \in \mathcal{A} \quad \text{and} \quad \{b', b'' \in \mathcal{A}\} \Rightarrow b \in \mathcal{A}$$

Furthermore, every object $y \in \mathcal{B}$ admits a resolution

$$0 \rightarrow x_n \rightarrow x_{n-1} \rightarrow \cdots \rightarrow x_1 \rightarrow x_0 \rightarrow y \rightarrow 0,$$

with all the x_i 's in $\mathcal{A}$.

(ii) *Devissage*: The categories $\mathcal{A}$ and $\mathcal{B}$ are both abelian. Furthermore, every object $y \in \mathcal{B}$ admits a filtration

$$0 = x_n \subset x_{n-1} \subset \cdots \subset x_1 \subset x_0 = y,$$

with all the intermediate quotients x_i/x_{i+1} in $\mathcal{A}$.

We can wonder what these theorems mean in the K -theory of triangulated categories. For resolution, the following well-known lemma is suggestive.

74 **Lemma 74** Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a fully faithful, exact embedding of exact categories. Suppose the resolution hypothesis holds. Then the natural map

$$D^b(f) : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$$

is an equivalence of categories.

75 **Remark 75** In most of this article I have avoided all mention of exact categories, focusing instead on the special case of abelian categories. This is mostly because we know much more about the K -theory of derived categories of abelian categories. For the resolution theorem, it would be a mistake to try to state it only for abelian categories. The reason is simple. If $f : \mathcal{A} \rightarrow \mathcal{B}$ is a fully faithful, exact embedding of abelian categories, and if $D^b(f) : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$ is an equivalence of categories, then one can easily show that f must be an equivalence of categories. For an embedding of abelian categories $f : \mathcal{A} \rightarrow \mathcal{B}$, Quillen's resolution theorem is content-free.

As we mentioned in Remark 75 our main result, Theorem 50, is about abelian categories. Therefore Quillen's resolution theorem does not formally follow. But morally we have been learning that K -theory depends only on the derived category. In the light of Lemma 74, Quillen's resolution theorem is hardly surprising.

The devissage theorem, by contrast, has always been very puzzling. Since the statement is so similar to the resolution theorem, one has to wonder whether the two have a common generalisation. Let me try to propose one. In both cases, the theorem asserts that an inclusion $\mathcal{A} \subset \mathcal{B}$ induces a homotopy equivalence in K -theory. Let us, for simplicity, look at resolutions and filtrations of length 1. Conditions (i) and (ii), of the resolution and devissage theorems in the special case of length 1 resolutions and filtrations, are

- (i) *Resolution:* Every object $y \in \mathcal{B}$ admits an exact sequence

$$0 \rightarrow x \rightarrow x' \rightarrow y \rightarrow 0$$

with x, x' in $\mathcal{A}$.

- (ii) *Devissage:* Every object $y \in \mathcal{B}$ admits an exact sequence

$$0 \rightarrow x \rightarrow y \rightarrow x' \rightarrow 0$$

with x, x' in $\mathcal{A}$.

The point I want to make is that, in the derived category, these become indistinguishable. In other words, if the inclusion $\mathcal{A} \subset \mathcal{B}$ satisfies the hypothesis of devissage, then the natural map

$$D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$$

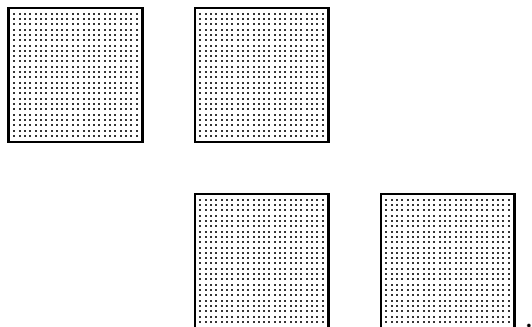
should satisfy a something analogous to the hypothesis of resolution. And morally resolution is the statement that K -theory of $\mathcal{A}$ is really a functor of $D^b(\mathcal{A})$.

This leads one to expect that there should be some construction, which we will call the derived category of a triangulated category. In fact, categories ought to be infinitely differentiable. Given a category $\mathcal{T}$, it should be possible to define its derived category $D^b(\mathcal{T})$, and this category should have a K -theory isomorphic to the K -theory of $\mathcal{T}$. Devissage is presumably the statement that the K -theory of an abelian category depends only on the derived category of its derived category.

Since this problem is so ill-posed, let me not try to say much more. The major thrust of the results in Theorem 50 is that K -theory is an invariant that captures relatively little of the homological structure we have been using. Perhaps the clearest evidence for this is the fact that even a δ -functor is enough to induce a map in higher K -theory; see Sect. 4.14. So perhaps the problem I am trying to pose in this section is: Find the right homological algebra gadget, which comes closer to being completely detected by K -theory.

About the Proofs

There are several ideas that come into the proofs which, as I have already said, are long and very difficult. One way to explain the strategy is the following. To define the K -theory of a triangulated category, we looked at the cosimplicial region $\mathcal{R}_n$ of Sect. 4.6. It turns out that there are many other cosimplicial regions. For example, we can look at regions in $\mathbb{Z} \times \mathbb{Z}$ which look like

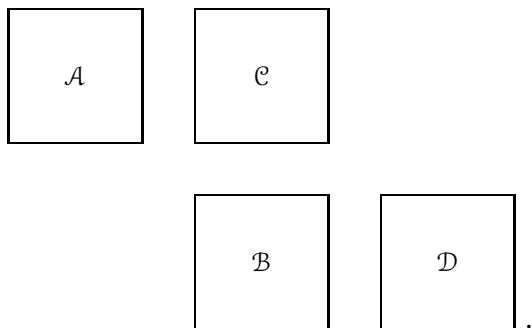


It turns out to be very easy to make this into a cosimplicial region. That is, there is a straight-forward way of finding a functor

$$\Theta : \Delta \rightarrow \{\text{Regions in } \mathbb{Z} \times \mathbb{Z}\},$$

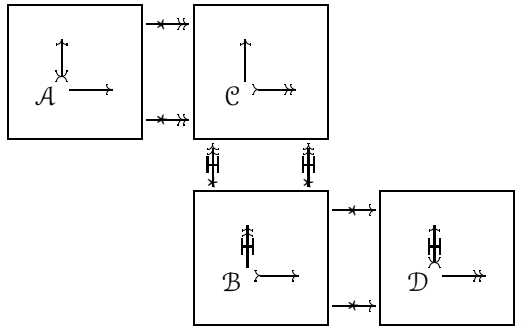
which takes an object $n \in \Delta$ to a region with the indicated shape. Let $\mathcal{T}$ be a category with squares. As in Sect. 4.7, we can take the functor sending $n \in \Delta$ to augmented diagrams for the pair $(\Theta(n), \mathcal{T})$. This is a functor $\Delta^{\text{op}} \rightarrow \{\text{Sets}\}$, that is a simplicial set. The idea is to study many such simplicial sets, for many choices of cosimplicial regions.

In fact, we can produce many variants. Our region is the disjoint union of four subregions, which I have drawn well separated from each other. One way to produce variants is by imposing different restrictions on each subregion. If we have four subcategories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{T}$, we can look at the simplicial subset



This just means that the augmented diagram takes the indicated subregions to the prescribed subcategories. We can also place restrictions on the horizontal and

vertical morphisms in each subregion, and on the morphisms connecting the subregions:



The reader will notice that the papers containing the proofs have many such simplicial sets and simplicial maps among them. At some level, the proofs amount to a combinatorial manipulation of the many possible simplicial sets that arise this way. Each of the main steps in the proofs shows that two regions, with all the adornment indicated above, give rise to homotopy equivalent simplicial sets.

This raises the obvious problem:

Problem 76 Are there more conceptual, less combinatorial proofs? Is it possible to give easier proofs of the main theorems?

76

For what it is worth, let me quote what Thomason had to say about this. When he came to believe that I really had a proof of the theorems I was claiming, his comment was: “There *has* to be a better proof.” What I have tried to explain in this manuscript is that, before looking for the optimal proof, perhaps we should search for improved theorems. Thomason was undoubtedly right about the existence of a better proof. All I wish to add to Thomason’s remark is: “There *has* to be a better theorem”.

Appendix: Examples of $D(\mathcal{A}) = D(\mathcal{B})$

4.17

In this appendix we outline the many examples now known, of pairs of abelian categories $\mathcal{A}$ and $\mathcal{B}$ with $D(\mathcal{A}) = D(\mathcal{B})$. Let me thank Bernhard Keller and Idun Reiten for much help with this appendix. However, all responsibility for mistakes rests with me.

The overwhelming majority of known examples fall into three types.

- (i) Both $\mathcal{A}$ and $\mathcal{B}$ are categories of modules, for different rings R and S .
- (ii) $\mathcal{A}$ is a category of modules over some ring R , and $\mathcal{B}$ is the category of (quasi)-coherent sheaves on some projective variety (or a non-commutative analog of a projective variety).

- (iii) Both $\mathcal{A}$ and $\mathcal{B}$ are categories of (quasi)-coherent sheaves, on some projective varieties X and Y .

The first example was probably Beilinson's 1978 article [11]. Beilinson produces three abelian categories with $D(\mathcal{A}) = D(\mathcal{B}) = D(\mathcal{C})$. In the example, $\mathcal{A}$ is the category of coherent sheaves on $\mathbb{P}^n$ (the n -dimensional projective space). For $\mathcal{B}$ and $\mathcal{C}$ Beilinson produced two rings R and S , and $\mathcal{B}$ and $\mathcal{C}$ are the categories of finite modules over R and S , respectively. Since the module categories for the rings R and S are not equivalent, Beilinson's example is simultaneously of types (i) and (ii).

The first example of type (iii) seems to be in Mukai's 1981 article [62]. In Mukai's example, $\mathcal{A}$ and $\mathcal{B}$ are the categories of coherent sheaves on an abelian variety X and on its dual $\hat{X}$, respectively.

Both Beilinson's and Mukai's example have been infinitely generalised and extended since. Let us first discuss type (iii). Let $\mathcal{A}$ and $\mathcal{B}$ be the abelian categories of coherent sheaves on smooth, projective varieties X and Y . Orlov's paper [74] gives a characterisation of all the equivalences $D(\mathcal{A}) = D(\mathcal{B})$. Bondal and Orlov [16] show that if the canonical bundle on X is ample or its negative is ample, then $D(\mathcal{A}) = D(\mathcal{B})$ implies $X = Y$. Kawamata shows [49] that if X is of general type or if the Kodaira dimension of $-K_X$ is the dimension of X , then $D(\mathcal{A}) = D(\mathcal{B})$ implies that X and Y are birational. Non-birational examples (where the Kodaira dimension is restricted by the above) may be found first of all in Mukai's original papers [62, 63], but more recently also in Bridgeland [20, 21], Bridgeland and Maciocia [24, 25], Orlov [74, 75] and Polishchuk [76]. But in some sense the case where X and Y are birational is most interesting, since it seems to be closely related to the minimal models program. It is conjectured that whenever X and Y are related by a sequence of flops then the derived categories should be the same. The first paper to prove such a theorem, for certain smooth flops, was Bondal and Orlov [15]. A particularly nice treatment for general smooth 3-fold flops, in terms of a certain moduli problem, may be found in Bridgeland [22]. For 3-fold flops with only terminal Gorenstein singularities this was done by Chen [30], and for flops with only quotient singularities by Kawamata [50]. One of the problems with more general singularities is that it is not quite clear what the precise statement should be. That is, just exactly which derived category of sheaves is right. The reader can find a brief discussion of the conjectured relationship between derived categories and birational geometry in Reid [78, §3.6]. Another algebro-geometric example of $D(\mathcal{A}) = D(\mathcal{B})$ comes from the McKay correspondence in Bridgeland, King and Reid [23]. It is slightly different from the above in that $\mathcal{A}$ is not just the abelian category of sheaves on some variety, but rather the category of sheaves with some compatible group action. The reader can find a much more thorough survey of all the algebro-geometric examples in Bondal and Orlov [17].

Next we mention more examples of type (ii). That is, $\mathcal{A}$ is a module category and $\mathcal{B}$ is a category of sheaves on some projective variety, and $D(\mathcal{A}) = D(\mathcal{B})$. The general case, of how such equivalences come about, was studied by Dagmar Baer [7] and by Alexei Bondal [14]. Baer applied it to coherent sheaves on weighted projective lines. Then the algebras R are *Ringel's canonical algebras*. Bondal studied

braid group actions on the collection of exceptional sequences in $D(\mathcal{B})$. Kapranov generalised Beilinson's example to other homogenous spaces; see [45–48]. In the realm of non-commutative algebraic geometry, see LeBruyn's [56] work on Weyl algebras, which was extended by Berest–Wilson [9] (note the appendix by Michel Van den Bergh). Kapranov–Vasserot's McKay equivalence [44] is also almost of type (ii).

The richest collection of known examples are the ones of type (i). It is probably fair to say that the subject began with Happel's Habilitationsschrift [36, 37]. Happel observes that, if (R, T, S) is a tilting triple (that is, R and S are rings and T is an $R - S$ -bimodule satisfying certain conditions), then there is an equivalence of categories $D(\mathcal{A}) = D(\mathcal{B})$. Here $\mathcal{A}$ and $\mathcal{B}$ are, respectively, the categories of R - and of S -modules.

Remark 77 We should make a historical note here. Tilting triples predate Happel's work. One of Happel's key contributions was to observe that they naturally give rise to equivalences of the form $D(\mathcal{A}) = D(\mathcal{B})$. For historical completeness we note

77

- Important precursors of tilting triples may be found in Gelfand–Ponomarev [33, 34], Bernstein–Gelfand–Ponomarev [10], Auslander–Platzek–Reiten [6] and Marmaridis [60]. One should note that Street independently developed similar ideas in his (unpublished) 1968 PhD thesis. See also his article [88].
 - The people (before Happel) who gave tilting theory its modern form: Brenner–Butler [19], who first proved the ‘tilting theorem’, Happel–Ringel [40], who improved the theorem and defined tilted algebras, Bongartz [18], who streamlined the theory, and Miyashita [61], who generalized it to tilting modules of projective dimension > 1 .
-

Remark 78 Happel found that the existence of a tilting module was sufficient to give an equivalence $D(\mathcal{A}) = D(\mathcal{B})$. A necessary and sufficient condition appeared soon after in Rickard's work [80].

78

Remark 79 For a concise introduction to tilting theory and its link with derived equivalences the reader is referred to Keller [53]. There is also Chapter XII in Gabriel–Roiter's book [31], the lecture notes edited by König–Zimmermann [54] and Assem's introduction [3].

79

There is a long list of applications of tilting theory (that is, of examples of rings R and S with $D(R) = D(S)$). If R is a hereditary algebra, the reader is referred to Happel–Rickard–Schofield [39] for a general theorem about the possible S 's. For certain specific R 's (precisely, for R the algebra of a quiver of Dynkin type) there is a complete classifications of all possible S 's. For type A , this is in Keller–Vossieck [51] and Assem–Happel [4]. For type D see Keller [52]. Type $\tilde{A}$ may be found in Assem–Skowronski [5], while types B and C are in Assem [2].

More examples of algebras R , for which all S 's with $D(R) = D(S)$ have been classified, are the Brauer tree algebras treated by Rickard [79], the representation-finite selfinjective algebras of Asashiba [1], the discrete algebras introduced by Dieter Vossieck [95], Brüstle's derived tame tree algebras in [29] (the main theorem was independently obtained by Geiss [32]), or Bocian–Holm–Skowronski's weakly symmetric algebras of Euclidean type [12] (the preprint is available at Thorsten Holm's homepage).

Another large source of examples comes from Broué's abelian defect group conjecture. Let me state the conjecture:

80

Conjecture 80 Let p be a prime, let O be a complete discrete valuation ring of characteristic zero with residue field k of characteristic p . Suppose that O and k are large enough.

Let G be a finite group, let R be a block algebra of the group algebra OG that has an abelian defect group D , and let S be the Brauer correspondent of R . We remind the reader that S is a block algebra of $ON_G(D)$, the group algebra of the normalizer of D in G . In any case R and S are rings. Their module categories will be $\mathcal{A}$ and $\mathcal{B}$.

Then Broué conjectured, in his 1990 paper [27], that there is an equivalence $D(\mathcal{A}) = D(\mathcal{B})$.

81

Remark 81 It might be helpful to give the reader a special case, which is already very interesting. Suppose k is an algebraically closed field of characteristic $p > 0$. Let G be a finite group, P a p -Sylow subgroup of G . Assume P is abelian. Let $N_G(P)$ be the normaliser of P in G . Let R and S be the principal blocks of kG and $kN_G(P)$, respectively. It follows from Conjecture 80 that the derived categories of R and S are equivalent.

For more on the conjecture see Broué [26–28], König–Zimmermann [54] and Rickard [81, 82]. For us the relevance is that the cases where the conjecture has been verified give equivalences $D(\mathcal{A}) = D(\mathcal{B})$. In the cases where $\mathcal{A}$ and $\mathcal{B}$ are not equivalent (and there are many of these), this gives examples of type (i).

A list of three of the large classes of known examples so far is:

1. All blocks with cyclic defect groups. See Rickard [79], Linckelmann [59] and Rouquier [83, 84].
2. All blocks of symmetric groups with abelian defect groups of order at most p^5 . (Preprint by Chuang and Kessar).
3. The non-principal block with full defect of $SL_2(p^2)$ in characteristic p . The defect group is $C_p \times C_p$. (Preprint by Holloway).

A much more complete and up-to-date list may be found on Jeremy Rickard's home page, at

<http://www.maths.bris.ac.uk/~majcr/adgc/which.html>

Remark 82 It is perhaps worth noting that the original evidence, which led Broué to formulate his conjecture, was obtained by counting characters. In other words, the evidence was mostly K_0 computations.

We should say a little bit about examples not of the three types (i), (ii) and (iii). The first to find a technique to produce such examples were Happel, Reiten and Smalø [38]. For a different approach see Schneiders [86]. A discussion of both approaches, the relation between them and improvements to the theorem may be found in Bondal and van den Bergh [13, Section 5.4 and Appendix B].

In all of the above I have said nothing about the uniqueness of an equivalence $D(\mathcal{A}) \simeq D(\mathcal{B})$. Any such equivalence is unique up to an automorphism of $D(\mathcal{A})$. If X is a Calabi–Yau manifold and $\mathcal{A}$ the category of coherent sheaves on it, then $D(\mathcal{A})$ is expected to have a large automorphism group, and this is expected to be related to the mirror partner of X . The reader can find more about this in Kontsevich [55] or Seidel and Thomas [87]. There has been some beautiful work on this, but our survey must end at some point.

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Appendix

**Bourbaki Articles
on the Milnor Conjecture**

Motivic Complexes of Suslin and Voevodsky

A

Eric M. Friedlander*

A.1	<i>Introduction: Connections with K-Theory</i>	1082
A.2	<i>Algebraic Singular Complexes</i>	1083
A.3	<i>Nisnevich Sheaves with Transfers</i>	1086
A.4	<i>Formalism of the Triangulated Category DM_k</i>	1090
A.5	<i>Motivic Cohomology and Homology</i>	1093
A.6	<i>Duality with Applications</i>	1097
A.7	<i>Conjecture of Beilinson–Lichtenbaum</i>	1100
	<i>References</i>	1103

* Partially supported by the N.S.F. and the Humboldt Foundation.

Summary. In this report we sketch some of the insights and consequences of recent work by Andrei Suslin and Vladimir Voevodsky concerning algebraic K-theory and motivic cohomology. We can trace these developments to a lecture at Luminy by Suslin in 1987 and to Voevodsky's Harvard thesis in 1992. What results is a powerful general theory of sheaves with transfers on schemes over a field, a theory developed primarily by Voevodsky with impressive applications by Suslin and Voevodsky.

This paper was originally delivered in the Séminaire Bourbaki and published in *Astérisque* [10] by the Société Mathématique de France and is reprinted here with permission.

A.1

Introduction: Connections with K -Theory

Criteria for a good motivic cohomology theory originate in topology. This should be a theory which plays some of the same role in algebraic geometry as singular cohomology plays in algebraic topology. One important aspect of singular cohomology is its relationship to (complex, topological) K-theory as formalized by the Atiyah–Hirzebruch spectral sequence for a topological space T [1]

$$E_2^{p,q} = H^p(X, K_{top}^q) \Rightarrow K_{top}^{p+q}(T)$$

where K_{top}^q is the q^{th} coefficient of the generalized cohomology theory given by topological K-theory (equal to $\mathbb{Z}$ if $q \leq 0$ is even and 0 otherwise). Indeed, when tensored with the rational numbers, this spectral sequence collapses to give $K_{top}^n(T) \otimes \mathbb{Q} = \bigoplus_{p+q=n, p \geq 0, q \leq 0} H^p(T, K_{top}^q) \otimes \mathbb{Q}$. This direct sum decomposition can be defined intrinsically in terms of the weight spaces of Adams operations acting upon $K_n^{top}(T)$. This becomes particularly suggestive when compared to the well known results of Alexander Grothendieck [18] concerning algebraic K_0 of a smooth scheme X :

$$K_0(X) \otimes \mathbb{Q} = \bigoplus CH^d(X) \otimes \mathbb{Q},$$

where $CH^d(X)$ is the Chow group of codimension d cycles on X modulo rational equivalence; moreover, this decomposition is once again given in terms of weight spaces for Adams operations.

Working now in the context of schemes (typically of finite type over a field k), William Dwyer and Friedlander [8] developed a topological K-theory for schemes (called étale K-theory) which also has such an Atiyah–Hirzebruch spectral sequence with E_2 -term the étale cohomology of the scheme. In [5], Spencer Bloch introduced complexes $Z_*^d(X)$ for X quasi-projective over a field which consist of certain algebraic cycles of codimension d on the product of X and affine spaces of varying dimensions. The homology of $Z_*^d(X)$ is closely related to the (higher Quillen) algebraic K-theory of X . If $CH^d(X, n)$ denotes the n -th homology group of the Bloch complex $Z_*^d(X)$ and if X is a smooth scheme, then

$$K_n(X) \otimes \mathbb{Q} = \bigoplus_d CH^d(X, n) \otimes \mathbb{Q}$$

(see also [21]); this decomposition is presumably given in terms of weight spaces for Adams operations on K-theory. Together with Stephen Lichtenbaum, Bloch has

moreover established a spectral sequence [6] converging to algebraic K-theory in the special case that X is the spectrum of a field F

$$E_2^{p,q} = CH^{-q}(\operatorname{Spec} F, -p - q) \Rightarrow K_{-p-q}(\operatorname{Spec} F) .$$

As anticipated many years ago by Alexander Beilinson [2], there should be such a spectral sequence for a quite general smooth scheme

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

converging to algebraic K-theory whose E_2 -term is *motivic cohomology*. Moreover, Beilinson [4] and Lichtenbaum [23] anticipated that this motivic cohomology should be the cohomology of *motivic chain complexes*. Although such a spectral sequence still eludes us (except in the case of the spectrum of a field), the complexes $\mathbb{Z}(n)$ of Voevodsky and Suslin (see §4) satisfy so many of the properties required of motivic complexes that we feel comfortable in calling their cohomology motivic cohomology. The first sections of this exposition are dedicated to presenting some of the formalism which leads to such a conclusion. As we see in §5, a theorem of Suslin [28] and duality established by Friedlander and Voevodsky [15] imply that Bloch's higher Chow groups $CH^d(X, n)$ equal motivic cohomology groups of Suslin–Voevodsky for smooth schemes X over a field k “which admits resolution of singularities.”

The Beilinson–Lichtenbaum Conjecture (cf. [2, 3, 19]) predicts that the conjectural map of spectral sequences from the conjectured spectral sequence converging to algebraic K-theory mod- ℓ to the Atiyah–Hirzebruch spectral sequence converging to étale K-theory mod- ℓ should be an isomorphism on E_2 -terms (except for a fringe effect whose extent depends upon the mod- ℓ étale cohomological dimension of X) for smooth schemes over a field k in which ℓ is invertible. This would reduce the computation of mod- ℓ K-theory of many smooth schemes to a question of computing “topological invariants” which in many cases has a known solution. In §6, we sketch the proof by Suslin and Voevodsky that the “Bloch–Kato Conjecture” for a field k and a prime ℓ invertible in k implies this Beilinson–Lichtenbaum Conjecture for k and ℓ . As discussed in the seminar by Bruno Kahn, Voevodsky has proved the Bloch–Kato Conjecture for $\ell = 2$ (in which case it was previously conjectured by John Milnor and thus is called the Milnor Conjecture.) Recent work by B. Kahn and separately by Charles Weibel and John Rognes establishes that computations of the 2-primary part of algebraic K-theory for rings of integers in number fields can be derived using special arguments directly from the Beilinson–Lichtenbaum Conjecture and the Bloch–Lichtenbaum spectral sequence.

Algebraic Singular Complexes

A.2

The elementarily defined *Suslin complexes* $Sus_*(X)$ provide a good introduction to many of the fundamental structures underlying the general theory developed by Voevodsky. Moreover, the relationship between the mod- n cohomology of $Sus_*(X)$

and the étale cohomology mod- n of X stated in Theorem 1 suggests the close relationship between étale motivic cohomology mod- ℓ and étale cohomology mod- ℓ .

As motivation, we recall from algebraic topology the following well known theorem of A. Dold and R. Thom [7]. If T is a reasonable topological space (e.g., a C.W. complex) and if $SP^d(T)$ denotes the d -fold symmetric product of T , then the homotopy groups of the group completion $(\coprod_d \text{Sing.}(SP^d(T)))^+$ of the simplicial abelian monoid $\coprod_d \text{Sing.}(SP^d(T))$ are naturally isomorphic to the (singular) homology of T . Here, $\text{Sing.}(SP^d(T))$ is the (topological) singular complex of the space $SP^d(T)$, whose set of n -simplices is the set of continuous maps from the topological n -simplex $\Delta[n]$ to $SP^d(T)$.

Suppose now that X is a scheme of finite type over a field k ; each $SP^d(X)$ is similarly a scheme of finite type over k . Let Δ^n denote $\text{Spec } k[t_0, \dots, t_n]/\sum_i t_i - 1$ and let Δ^* denote the evident cosimplicial scheme over k which in codimension n is Δ^n . We define the Suslin complex $Sus_*(X)$ of X to be the chain complex associated to the simplicial abelian group $(\coprod_d \text{Hom}_{\text{Sch}/k}(\Delta^*, SP^d(X)))^+$.

Various aspects of $Sus_*(X)$ play an important role in our context. First, $Sus_*(X)$ equals $c_{\text{equi}}(X, 0)(\Delta^*)$, where $c_{\text{equi}}(X, 0)$ is a sheaf in the *Nisnevich topology* on the category Sm/k of smooth schemes over the field k . Second, the sheaf $c_{\text{equi}}(X, 0)$ is a *presheaf with transfers*. Third, if we denote by $\underline{C}_*(c_{\text{equi}}(X, 0))$ the complex of Nisnevich sheaves with transfers (sending a smooth scheme U to $c_{\text{equi}}(X, 0)(U \times \Delta^*)$), then this complex of sheaves has homology presheaves which are *homotopy invariant*: the natural pull-back

$$c_{\text{equi}}(X, 0)(U \times \Delta^*) \rightarrow c_{\text{equi}}(X, 0)(U \times \mathbb{A}^1 \times \Delta^*)$$

induces an isomorphism on homology groups.

1

Theorem 1 ([29]). Let X be a quasi-projective scheme over an algebraically closed field k and let n be a positive integer relatively prime to the exponential characteristic of k . Then the mod- n cohomology of $Sus_*(X)$ (i.e., the cohomology of the complex $R\text{Hom}(Sus_*(X), \mathbb{Z}/n)$) is given by

$$H^*(Sus_*(X), \mathbb{Z}/n) \simeq H_{\text{et}}^*(X, \mathbb{Z}/n),$$

where the right hand side is the étale cohomology of the scheme X with coefficients in the constant sheaf $\mathbb{Z}/n$.

Quick sketch of proof

This theorem is proved using the rigidity theorem of Suslin and Voevodsky stated below as Theorem 8. We apply this to the (graded) homotopy invariant (cf. Lemma 7) presheaves with transfers

$$\Phi_i(-) = H_i(c_{\text{equi}}(X, 0)(- \times \Delta^*)) \otimes \mathbb{Z}[1/p]$$

where p is the exponential characteristic of k . An auxiliary topology, the “qfh topology” is introduced which has the property that the free $\mathbb{Z}[1/p]$ sheaf in this topology represented by X equals $c_{\text{equi}}(X, 0) \otimes \mathbb{Z}[1/p]$. Since $\text{Sus}_*(X) = \Phi_*(\text{Spec } k)$, Theorem 8 and the comparison of cohomology in the qfh and étale topologies provides the following string of natural isomorphisms.

$$\begin{aligned} \text{Ext}_{\text{Ab}}^*(\text{Sus}_*(X), \mathbb{Z}/n) &= \text{Ext}_{\text{EtShv}/X}^*((\Phi_*)_{\text{et}}, \mathbb{Z}/n) = \text{Ext}_{\text{qfhShv}/X}^*((\Phi_*)_{\text{qfh}}, \mathbb{Z}/n) \\ &= \text{Ext}_{\text{qfhShv}(k)}^*(\mathbb{Z}[1/p](X), \mathbb{Z}/n) = H_{\text{qfh}}^*(X, \mathbb{Z}/n) \\ &= H_{\text{et}}^*(X, \mathbb{Z}/n). \end{aligned}$$

These concepts of presheaves with transfers, Nisnevich sheaves, and homotopy invariant presheaves will be explained in the next section. Even before we investigate their definitions, we can appreciate their role from the following theorem of Voevodsky.

Theorem 2 [32, 5.12] Assume that k is a perfect field. Let

2

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$$

be a short exact sequence of Nisnevich sheaves on Sm/k with transfers. Then the resulting triple of chain complexes of abelian groups

$$F_1(\Delta^*) \rightarrow F_2(\Delta^*) \rightarrow F_3(\Delta^*) \rightarrow F_1(\Delta^*)[1]$$

is a distinguished triangle (i.e., determines a long exact sequence in homology groups).

Quick sketch of proof

Let P denote the presheaf cokernel of $F_1 \rightarrow F_2$. Then the kernel and cokernel of the natural map $P \rightarrow F_3$ have vanishing associated Nisnevich sheaves. The theorem follows from an acyclicity criterion for $Q(\Delta^*)$ in terms of the vanishing of $\text{Ext}^*(Q_{\text{Nis}}, -)$ for any presheaf with transfers Q on Sm/k (with associated Nisnevich sheaf Q_{Nis}). A closely related acyclicity theorem is stated as Theorem 15 below.

One consequence of Theorem 2 (and Proposition 5 below) is the following useful property. The resulting long exact sequence in Suslin homology is far from evident if one works directly with the definition of the Suslin complex.

Corollary 3 [32, 5.17] Let k be a perfect field and X a scheme of finite type over k . Then for any open covering $X = U \cup V$ of X

3

$$\text{Sus}_*(U \cap V) \rightarrow \text{Sus}_*(U) \oplus \text{Sus}_*(V) \rightarrow \text{Sus}_*(X) \rightarrow \text{Sus}_*(U \cap V)[1]$$

is a distinguished triangle.

Nisnevich Sheaves with Transfers

Let Sm/k denote the category of smooth schemes over a field k . (In particular, such a scheme is of finite type over k .) Then the *Nisnevich topology* on Sm/k (cf. [24]) is the Grothendieck topology (finer than the Zariski topology and less fine than the étale topology) whose coverings $\{U_i \rightarrow U\}_{i \in I}$ are étale coverings with the property that for each point $u \in U$ there exists some $i \in I$ and some point $\tilde{u} \in U_i$ mapping to u such that the induced map of residue fields $k(u) \rightarrow k(\tilde{u})$ is an isomorphism. A key property of this topology is that its points are Hensel local rings.

In order to consider singular schemes which admit resolutions by smooth schemes, we shall also consider the stronger *cdh topology* on the category Sch/k of schemes of finite type over k . This is defined to be the minimal Grothendieck topology for which Nisnevich coverings are coverings as are proper, surjective morphisms of the following type:

$$W \coprod U_1 \xrightarrow{p \amalg i} U,$$

where $i : U_1 \rightarrow U$ is a closed embedding and $p^{-1}(U - U_1) \rightarrow U - U_1$ is an isomorphism.

We shall often have need to assume that the field “admits resolution of singularities” as formulated in the following definition. At this time, this hypothesis is only known to hold for fields of characteristic 0. As one can see, the cdh topology is designed to permit the study of singular schemes over a field which admits resolution of singularities by employing coverings by smooth schemes.

4

Definition 4 A field k is said to admit resolution of singularities provided that

1. For any scheme of finite type X over k there is a proper, birational, surjective morphism $Y \rightarrow X$ such that Y is a smooth scheme over k .
2. For any smooth scheme X over k and any proper, birational, surjective map $q : X' \rightarrow X$, there exists a sequence of blow-ups $p : X_n \rightarrow \cdots \rightarrow X_1 = X$ with smooth centers such that p factors through q .

We define the presheaf of abelian groups

$$c_{equi}(X, 0) : (Sm/k)^{op} \rightarrow Ab$$

to be the evident functor whose values on a smooth connected scheme U is the free abelian group on the set of integral closed subschemes on $X \times U$ finite and surjective over U . This is a sheaf for the étale topology and hence also for the Nisnevich topology; indeed, as mentioned following the statement of Theorem 1, $c_{equi}(X, 0)$ can be constructed as the sheaf in the qfh-topology (stronger than the étale topology) associated to the presheaf sending U to the free abelian group on $Hom_{Sch/k}(U, X)$.

We shall have occasion to consider other Nisnevich sheaves defined as follows:

$$z_{equi}(X, r) : (Sm/k)^{op} \rightarrow Ab$$

sends a connected smooth scheme U to the group of cycles on $U \times X$ equidimensional of relative dimension r over U . In particular, if X is proper over k , then $c_{\text{equi}}(X, 0) = z_{\text{equi}}(X, 0)$.

One major advantage of our Nisnevich and cdh topologies when compared to the Zariski topology is the existence of Mayer–Vietoris, localization, and blow-up exact sequences as stated below.

Proposition 5 (cf. [30, 4.3.7; 4.3.1; 4.3.2]) For any smooth scheme X over k and any Zariski open covering $X = U \cup V$, the sequence of sheaves in the Nisnevich topology

$$0 \rightarrow c_{\text{equi}}(U \cap V, 0) \rightarrow c_{\text{equi}}(U, 0) \oplus c_{\text{equi}}(V, 0) \rightarrow c_{\text{equi}}(X, 0) \rightarrow 0$$

of Mayer–Vietoris type is exact.

For any scheme X of finite type over k , any open covering $X = U \cup V$, and any closed scheme $Y \subset X$, the sequences of sheaves in the cdh topology

$$0 \rightarrow c_{\text{equi}}(U \cap V, 0)_{\text{cdh}} \rightarrow c_{\text{equi}}(U, 0)_{\text{cdh}} \oplus c_{\text{equi}}(V, 0)_{\text{cdh}} \rightarrow c_{\text{equi}}(X, 0)_{\text{cdh}} \rightarrow 0$$

$$0 \rightarrow z_{\text{equi}}(Y, r)_{\text{cdh}} \rightarrow z_{\text{equi}}(X, r)_{\text{cdh}} \rightarrow z_{\text{equi}}(X - Y, r)_{\text{cdh}} \rightarrow 0$$

of Mayer–Vietoris and localization type are exact.

For any scheme X of finite type over k , any closed subscheme $Z \subset X$, and any proper morphism $f : X' \rightarrow X$ whose restriction $f^{-1}(X - Z) \rightarrow X - Z$ is an isomorphism, the sequences of sheaves in the cdh topology

$$\begin{aligned} 0 \rightarrow c_{\text{equi}}(f^{-1}(Z), 0)_{\text{cdh}} &\rightarrow c_{\text{equi}}(X', 0)_{\text{cdh}} \oplus c_{\text{equi}}(Z, 0)_{\text{cdh}} \\ &\rightarrow c_{\text{equi}}(X, 0)_{\text{cdh}} \rightarrow 0 \end{aligned}$$

$$\begin{aligned} 0 \rightarrow z_{\text{equi}}(f^{-1}(Z), r)_{\text{cdh}} &\rightarrow z_{\text{equi}}(X', r)_{\text{cdh}} \oplus z_{\text{equi}}(Z, r)_{\text{cdh}} \\ &\rightarrow z_{\text{equi}}(X, r)_{\text{cdh}} \rightarrow 0 \end{aligned}$$

of blow-up type are exact.

Remarks on the proof

The only issue is exactness on the right. We motivate the proof of the exactness of the localization short exact sequences using Chow varieties, assuming that X is quasi-projective. Let W be a smooth connected scheme and $Z \subset (X - Y) \times W$ a closed integral subscheme of relative dimension r over W . Such a Z is associated to a rationally defined map from W to the Chow variety of some projective closure of X . The projection to W of the graph of this rational map determines a cdh-covering $W' \rightarrow W$ restricted to which the pull-back of Z on $(X - Y) \times W'$ extends to a cycle on $X \times W'$ equidimensional of relative dimension r over W' .

We next introduce the important notion of transfers (i.e., functoriality with respect to finite correspondences).

6

Definition 6 The category of smooth correspondences over k , $SmCor(k)$, is the category whose objects are smooth schemes over k and for which

$$Hom_{SmCor(k)}(U, X) = c_{equi}(X, 0)(U) ,$$

the free abelian group of finite correspondences from U to X . A presheaf with transfers is a contravariant functor

$$F : (SmCor(k))^{op} \rightarrow Ab .$$

The structure of presheaves with transfers on $c_{equi}(X, 0)$ and $z_{equi}(X, r)$ is exhibited using the observation that if Z is an equidimensional cycle over a smooth scheme X and if $W \rightarrow X$ is a morphism of schemes of finite type, then the pull-back of Z to W is well defined since the embedding of the graph of $W \rightarrow X$ in $W \times X$ is a locally complete intersection morphism [16]. Consequently, if $U \leftarrow W \rightarrow X$ is a finite correspondence in $SmCor(k)$, then we obtain transfer maps by first pulling back cycles of X to W and then pushing them forward to U . The reader should be forewarned that earlier papers of Voevodsky, Suslin, and Friedlander use the condition on a presheaf that it be a “pretheory of homological type” which is shown in [33, 3.1.10] to be implied by the existence of transfers.

One can easily prove the following lemma which reveals the key property of homotopy invariance possessed by the algebraic singular complex used to define Suslin homology. For any presheaf F on Sm/k , we employ the notation $\underline{C}_*(F)$ for the complex of presheaves on Sm/k sending U to the complex $F(U \times \Delta^*)$.

7

Lemma 7 Let $F : (Sm/k)^{op} \rightarrow Ab$ be a presheaf on Sm/k and consider $\underline{h}^{-i}(F) : (Sm/k)^{op} \rightarrow Ab$ sending U to the i -th homology of $\underline{C}_*(F)$ (for some non-negative integer i). Then $\underline{h}^{-i}(F)$ is homotopy invariant:

$$\underline{h}^{-i}(F)(U) = \underline{h}^{-i}(F)(U \times \mathbb{A}^1) .$$

As we saw in our sketch of proof of Theorem 1, the following rigidity theorem of Suslin and Voevodsky, extending the original rigidity theorem of Suslin [27] is of considerable importance.

8

Theorem 8 [29, 4.4] Let Φ be a homotopy invariant presheaf with transfers satisfying $n\Phi = 0$ for some integer n prime to the residue characteristic of k . Let S_d be the henselization of $\mathbb{A}^d$ (i.e., affine d -space) at the origin. Then

$$\Phi(S_d) = \Phi(Spec k) .$$

Idea of Proof

In a now familiar manner, the theorem is reduced to an assertion that any two sections of a smooth relative curve $X \rightarrow S$ with good compactification which coincide at the closed point of S induce the same map $\Phi(X) \rightarrow \Phi(S)$. The difference Z of these sections is a finite correspondence from S to X . Since Φ is a homotopy invariant presheaf with transfers, to show that the map induced by Z is 0 it suffices to show that the difference is 0 in the relative Picard group $\text{Pic}(\bar{X}, Y)/n \subset H_{\text{et}}^2(\bar{X}, j_!(\mu_n))$, where $\bar{X} \rightarrow S$ is a good compactification, $Y = \bar{X} - X$, and $j : X \subset \bar{X}$. The proper base change theorem implies that it suffices to show that the image of Z is 0 upon base change to the closed point of S . This is indeed the case since the two sections were assumed to coincide on the closed point.

The following theorem summarizes many of the results proved by Voevodsky in [32] and reformulated in [33]. In particular, this theorem enables us to replace consideration of cohomology in the Nisnevich topology by cohomology in the Zariski topology for smooth schemes.

Theorem 9 [33, 3.1.11] If $F : (\text{SmCor}(k))^{op} \rightarrow \text{Ab}$ is a homotopy invariant presheaf with transfers, then its associated Nisnevich sheaf F_{Nis} is also a homotopy invariant presheaf with transfers and equals (as a presheaf on Sm/k) the associated Zariski sheaf F_{Zar} .

Moreover, if k is perfect, then

$$H_{\text{Zar}}^i(-, F_{\text{Zar}}) = H_{\text{Nis}}^i(-, F_{\text{Nis}})$$

for any $i \geq 0$, and these are homotopy invariant presheaves with transfer.

To complete the picture relating sheaf cohomology for different topologies we mention the following result which tells us that if we consider the cdh topology on schemes of finite type over k then the resulting cohomology equals Nisnevich cohomology whenever the scheme is smooth.

Proposition 10 [15, 5.5] Assume that k is a perfect field admitting resolution of singularities. Let F be a homotopy invariant presheaf on Sm/k with transfers. Then for any smooth scheme of finite type over k

$$H_{\text{cdh}}^*(X, F_{\text{cdh}}) = H_{\text{Nis}}^*(X, F_{\text{Nis}}) = H_{\text{Zar}}^*(X, F_{\text{Zar}}).$$

Remark on Proof

The proof uses the techniques employed in the proof of Theorem 15 below applied to the cone of $\mathbb{Z}(\mathcal{U}) \rightarrow \mathbb{Z}(U)$, where $\mathcal{U}$ is an arbitrarily fine hypercovering of U for the cdh topology consisting of smooth schemes.

Formalism of the Triangulated Category DM_k

A.4

Voevodsky's approach [33] to motives for smooth schemes and for schemes of finite type over a field admitting resolution of singularities entails a triangulated category $DM_{gm}^{eff}(k)$ of effective geometric motives. Roughly speaking, $DM_{gm}^{eff}(k)$ is obtained by adjoining kernels and cokernels of projectors to the localization (to impose homotopy invariance) of the homotopy category of bounded complexes on the category of smooth schemes and finite correspondences. Voevodsky then inverts the "Tate object" $\mathbb{Z}(1)$ in this category to obtain his triangulated category $DM_{gm}(k)$ of geometric motives. (See [22] for another approach to the triangulated category of mixed motives by Marc Levine.

In this section, we focus our attention upon another triangulated category introduced by Voevodsky which we denote by DM_k for notational convenience. (Voevodsky's notation is $DM_{-}^{eff}(k)$.) Voevodsky proves [33, 3.2.6] that his category $DM_{gm}^{eff}(k)$ of effective geometric motives embeds as a full triangulated subcategory of DM_k . Furthermore, as we see in Theorem 34 below, under this embedding the Tate motive is quasi-invertible so that DM_{gm} is also a full triangulated subcategory of DM_k .

11

Definition 11 Let X be a scheme over a field k . Assume either that X is smooth or that X is of finite type and k admits resolution of singularities. We define the motive of X to be

$$M(X) \equiv \underline{C}_*(c_{equi}(X, 0)) : (Sm/k)^{op} \rightarrow C_*(Ab) .$$

Similarly, we define the motive of X with compact supports to be

$$M^c(X) \equiv \underline{C}_*(z_{equi}(X, 0)) : (Sm/k)^{op} \rightarrow C_*(Ab) .$$

We shall use the usual (but confusing) conventions when working with complexes. Our complexes will have cohomological indexing, meaning that the differential increases degree by 1. We view this differential of degree +1 as shifting 1 position to the right. If K is a complex, then $K[1]$ is the complex obtained from K by shifting 1 position to the left. This has the convenience when working with (hyper-) cohomology that $H^i(X, K[1]) = H^{i+1}(X, K)$.

We now introduce the triangulated category DM_k designed to capture the Nisnevich cohomology of smooth schemes over k and the cdh cohomology of schemes of finite type over k .

12

Definition 12 Denote by $Shv_{Nis}(SmCor(k))$ the category of Nisnevich sheaves with transfers and let $D_-(Shv_{Nis}(SmCor(k)))$ denote the derived category of complexes of $Shv_{Nis}(SmCor(k))$ which are bounded above. We define

$$DM_k \subset D_-(Shv_{Nis}(SmCor(k)))$$

to be the full subcategory of those complexes with homotopy invariant cohomology sheaves.

By Lemma 7 and Theorem 9, $M(X)$ and $M^c(X)$ are objects of the triangulated category DM_k .

We obtain the following relatively formal consequence of our definitions.

Proposition 13 [33, 3.1.8, 3.2.6] If X is smooth over k , then for any $K \in DM_k$,

13

$$H_{Zar}^n(X, K) = Hom_{DM_k}(M(X), K[n]) ;$$

in particular, if X is smooth, then

$$Hom_{DM_k}(M(X), M(Y)[i]) = H_{Zar}^i(X, \underline{C}_*(Y)) .$$

If X is of finite type over k and k admits resolution of singularities, then

$$H_{cdh}^n(X, K_{cdh}) = Hom_{DM_k}(M(X), K[n]_{cdh}) .$$

Taking $X = Spec\ k$, we obtain an interpretation of $Sus_*(Y)$ in terms of DM_k .

Corollary 14 If Y is a scheme of finite type over k , then the homology of $Sus_*(Y)$ is given by $Hom_{DM_k}(\mathbb{Z}[*], M(Y))$.

14

The machinery of presheaves with transfers and the formulation of the cdh topology permits the following useful vanishing theorem. This is an extension of an earlier theorem of Voevodsky asserting the equivalence of the conditions on a homotopy invariant presheaf with transfers that the homology sheaves of $\underline{C}_*(F)_{Zar}$ vanish and that $Ext_{NisShv}^*(F_{Nis}, -) = 0$ [32, 5.9].

Theorem 15 [15, 5.5.2] Assume F is a presheaf with transfers on Sm/k where k is a perfect field which admits resolution of singularities. If $F_{cdh} = 0$, then $\underline{C}_*(F)_{Zar}$ is quasi-isomorphic to 0.

15

Idea of Proof

If $\underline{C}_*(F)_{Zar}$ is not quasi-isomorphic to 0, let $\underline{h}_n(F)_{Zar}$ be the first non-vanishing cohomology sheaf. Using Theorem 9 and techniques of [32], we conclude that a non-zero element of this group determines a non-zero element of

$$Hom_{D(Sm/k)_{Nis}}(\underline{C}_*(F)_{Nis}, \underline{h}_n(F)_{Nis}[n]) = Ext_{NisShv}^n(F_{Nis}, \underline{h}_n(F)_{Nis}) .$$

On the other hand, using a resolution of F by Nisnevich sheaves which are the free abelian sheaves associated to smooth schemes, we verify that the vanishing of F_{cdh} together with [32, 5.9] implies that

$$Ext_{NisShv}^*(F_{Nis}, G_{Nis}) = 0$$

for any homotopy invariant presheaf G with transfers.

In conjunction with Proposition 5, Theorem 15 leads to the following distinguished triangles for motives and motives with compact support.

16

Corollary 16 Assume that the field k admits resolution of singularities and that X is a scheme of finite type over k . If $X = U \cup V$ is a Zariski open covering, then we have the following distinguished triangles of Mayer–Vietoris type

$$M(U \cap V) \rightarrow M(U) \oplus M(V) \rightarrow M(X) \rightarrow M(U \cap V)[1]$$

$$M^c(X) \rightarrow M^c(U) \oplus M^c(V) \rightarrow M^c(U \cap V) \rightarrow M^c(X)[1] .$$

If $Y \subset X$ is a closed subscheme with Zariski open complement U , then we have the following distinguished triangle of localization type

$$M^c(Y) \rightarrow M^c(X) \rightarrow M^c(U) \rightarrow M^c(Y)[1] .$$

Finally, if $f : X' \rightarrow X$ is a proper morphism and $Z \subset X$ is a closed subscheme such that the restriction of f above $X - Z$, $f|_1 : X' - f^{-1}(Z) \rightarrow X - Z$ is an isomorphism, then we have the following distinguished triangles for abstract blow-ups:

$$M(f^{-1}(Z)) \rightarrow M(X') \oplus M(Z) \rightarrow M(X) \rightarrow M(f^{-1}(Z))[1]$$

$$M^c(f^{-1}(Z)) \rightarrow M^c(X') \oplus M^c(Z) \rightarrow M^c(X) \rightarrow M^c(f^{-1}(Z))[1] .$$

Armed with these distinguished triangles, one can obtain results similar to those of Henri Gillet and Christophe Soulé in [17].

We next introduce the *Tate motive* $\mathbb{Z}(1)[2]$ in DM_k and define the *Tate twist* of motives.

17

Definition 17 We define the Tate motive $\mathbb{Z}(1)[2]$ to be the cone of $M(\text{Spec } k) \rightarrow M(\mathbb{P}^1)$.

We define the Tate twist by

$$M(X)(1) = \text{cone}\{M(X) \rightarrow M(X \times \mathbb{P}^1)[-2]\} ,$$

$$M^c(X)(1) = \text{cone}\{M^c(X) \rightarrow M^c(X \times \mathbb{P}^1)[-2]\} .$$

Thus, if X is projective and k admits resolution of singularities,

$$M(X)(1) = M^c(X \times \mathbb{A}^1)[-2] .$$

We briefly introduce the analogous triangulated category for the étale site.

Definition 18 Denote by $\mathcal{Shv}_{\text{ét}}(\text{SmCor}(k))$ the category of presheaves with transfers which are sheaves on the étale site of (Sm/k) and let

18

$$D_{-}(\mathcal{Shv}_{\text{ét}}(\text{SmCor}(k)))$$

denote the derived category of complexes of $\mathcal{Shv}_{\text{ét}}(\text{SmCor}(k))$ which are bounded above. We define $DM_{k,\text{ét}} \subset D_{-}(\mathcal{Shv}_{\text{ét}}(\text{SmCor}(k)))$ to be the full subcategory of those complexes with homotopy invariant cohomology sheaves.

Observe that the exact functor

$$\pi^{*} : \mathcal{Shv}_{\text{Nis}}(\text{SmCor}(k)) \rightarrow \mathcal{Shv}_{\text{ét}}(\text{SmCor}(k))$$

induces a natural map

$$\text{Hom}_{DM_k}(K, L) \rightarrow \text{Hom}_{DM_{k,\text{ét}}}(\pi^{*}K, \pi^{*}L) .$$

Voevodsky observes that

$$\text{Hom}_{DM_{k,\text{ét}}}(M(X), K[n]) = H_{\text{ét}}^n(X, K)$$

for any $K \in DM_{k,\text{ét}}$.

Motivic Cohomology and Homology

A.5

Having introduced the triangulated category DM_k , we now proceed to consider the motivic complexes $\mathbb{Z}(n) \in DM_k$ whose cohomology and homology is motivic cohomology and homology. Other authors (e.g., Lichtenbaum and Friedlander–Gabber) have considered similar complexes; the importance of the approach of Suslin and Voevodsky is the context in which these complexes are considered. The many properties established for DM_k enable many good formal properties to be proved.

Definition 19 For a given positive integer n , let F_n be the sum of the images of the n embeddings

19

$$c_{\text{equi}}((\mathbb{A}^1 - \{0\})^{n-1}, 0) \rightarrow c_{\text{equi}}((\mathbb{A}^1 - \{0\})^n, 0)$$

determined by the embeddings $(t_1, \dots, t_{n-1}) \mapsto (t_1, \dots, t_{i-1}, 1, t_i, \dots, t_{n-1})$. We define

$$\mathbb{Z}(n) = \underline{C}_{*}(c_{\text{equi}}((\mathbb{A}^1 - \{0\})^n, 0)/F_n)[-n] .$$

For any positive integer m , we define

$$\mathbb{Z}/m(n) = \underline{C}_{*}(c_{\text{equi}}((\mathbb{A}^1 - \{0\})^n, 0)/F_n) \otimes \mathbb{Z}/m[-n] .$$

Observe that Mayer–Vietoris implies that $\mathbb{Z}(1)$ defined as in Definition 19 agrees with (i.e., is quasi-isomorphic to) $\mathbb{Z}(1)$ as given in Definition 17; similarly, for any $n > 0$,

$$\mathbb{Z}(n) = \underline{C}_* (c_{\text{equi}}(\mathbb{P}^n, 0)/c_{\text{equi}}(\mathbb{P}^{n-1}))[-2n] .$$

Moreover, if k admits resolution of singularities, then localization implies that

$$\mathbb{Z}(n) = \underline{C}_* (z_{\text{equi}}(\mathbb{A}^n, 0))[-2n] .$$

We obtain the following determination of $\mathbb{Z}(0)$ and $\mathbb{Z}(1)$ which we would require of any proposed definition of motivic complexes.

20 **Proposition 20** [33, 3.4.3]

- (a.) $\mathbb{Z}(0)$ is the constant sheaf $\mathbb{Z}$.
- (b.) $\mathbf{G}_m \simeq \mathbb{Z}(1)[1]$, where $\mathbf{G}_m$ is viewed as a sheaf of abelian groups.

We now introduce motivic cohomology.

21 **Definition 21** For any scheme of finite type over a field k , we define the motivic cohomology of X by

$$H^i(X, \mathbb{Z}(j)) = H_{\text{cdh}}^i(X, \mathbb{Z}(j)_{\text{cdh}}) .$$

For any positive integer m , we define the mod- m motivic cohomology of X by

$$H^i(X, \mathbb{Z}/m(j)) = H_{\text{cdh}}^i(X, \mathbb{Z}/m(j)_{\text{cdh}}) .$$

Thus, if X is smooth and k is perfect, then Theorem 9 and Proposition 13 imply that motivic cohomology is Zariski hypercohomology (where the complex $\mathbb{Z}(j)$ of Nisnevich sheaves is viewed as a complex of Zariski sheaves by restriction):

$$H^i(X, \mathbb{Z}(j)) = H_{\text{Zar}}^i(X, \mathbb{Z}(j)) = \text{Hom}_{\text{DM}_k}(M(X), \mathbb{Z}(j)[i]) .$$

Similarly, if k admits resolution of singularities, then for any X of finite type over k

$$H^i(X, \mathbb{Z}(j)) = \text{Hom}_{\text{DM}_k}(M(X), \mathbb{Z}(j)[i]) .$$

If d denotes the dimension of X , then

$$H^i(X, \mathbb{Z}(j)) = 0 \quad \text{whenever } i > d + j .$$

The following theorem relating Milnor K-theory to motivic cohomology appears in various guises in [5] and [25]. The reader is referred to [31] for a direct proof given in our present context.

Theorem 22 For any field k and any non-negative integer n , there is a natural isomorphism

22

$$K_n^M(k) \simeq H^n(\operatorname{Spec} k, \mathbb{Z}(n))$$

where $K_*^M(k)$ is the Milnor K-theory of k .

So defined, motivic cohomology is cohomology with respect to the Zariski site for smooth schemes (and with respect to the cdh site for more general schemes of finite type) as anticipated by Beilinson. One can also consider the analogous cohomology with respect to the étale site following the lead of Lichtenbaum.

As usual, we let μ_ℓ denote the sheaf of ℓ -th roots of unity on $(Sm/k)_{\text{ét}}$.

Theorem 23 [33, 3.3] Define the étale motivic cohomology $H_{\text{ét}}^i(X, \mathbb{Z}(j))$ of a scheme X of finite type over k by

23

$$H_{\text{ét}}^i(X, \mathbb{Z}(j)_{\text{ét}}) = \operatorname{Hom}_{DM_{k,\text{ét}}} (M(X)_{\text{ét}}, \mathbb{Z}(j)_{\text{ét}}[i]) ;$$

similarly for any positive integer relatively prime to the residue characteristic of k , define

$$H_{\text{ét}}^i(X, \mathbb{Z}/m(j)) \equiv H_{\text{ét}}^i(X, \mathbb{Z}/m(j)_{\text{ét}}) = \operatorname{Hom}_{DM_{k,\text{ét}}} (M(X)_{\text{ét}}, \mathbb{Z}/m(j)_{\text{ét}}[i]) .$$

Then there is a natural quasi-isomorphism

$$\mu_m^{\otimes j} \rightarrow \mathbb{Z}/m(j)_{\text{ét}}$$

In particular, this gives an isomorphism

$$H_{\text{ét}}^*(X, \mu_m^{\otimes j}) \xrightarrow{\sim} H_{\text{ét}}^*(X, \mathbb{Z}/m(j)) .$$

Sketch of proof

By Proposition 20b, μ_m is quasi-isomorphic to $\mathbb{Z}/m(1)$. More generally, we construct an explicit map $\mu_m^{\otimes j}(F(\zeta_m)) \rightarrow \mathbb{Z}/m(j)(F(\zeta_m))$ where F is a field extension of k and ζ_m is a primitive m -th root of unity and show that this map is $\operatorname{Gal}(F(\zeta_m)/F)$ -invariant. This determines a map of étale sheaves with transfers $\mu_m^{\otimes j} \rightarrow \mathbb{Z}/m(j)_{\text{ét}}$. By Theorems 1 and 8, this map is a quasi-isomorphism.

Because the étale cohomology of a Hensel local ring is torsion, we readily conclude the following proposition using Proposition 2.7.

Proposition 24 For any smooth scheme,

24

$$H^*(X, \mathbb{Z}(j)) \otimes \mathbb{Q} = H_{\text{ét}}^*(X, \mathbb{Z}(j)) \otimes \mathbb{Q} .$$

As we shall see in the next section, motivic cohomology is dual to motivic locally compact homology for smooth schemes over a field admitting resolution of singularities. This locally compact homology was initially formulated in [15] (essentially following the definition in [11]) using $\underline{C}_*(z_{\text{equi}}(X, r))$. To rephrase this in terms of our triangulated category DM_k , we need the following proposition.

25 Proposition 25 [33, 4.2.8] Let X be a scheme of finite type over a field k and let r be a non-negative integer. Then there is a natural isomorphism in DM_k

$$\underline{C}_*(z_{\text{equi}}(X, r)) \simeq \underline{\text{Hom}}_{DM_k}(\mathbb{Z}(r)[2r], M^c(X)) ,$$

where $\underline{\text{Hom}}$ denotes internal Hom in the derived category of unbounded complexes of Nisnevich sheaves with transfers.

We now define three other theories: motivic cohomology with compact supports, motivic homology, and motivic homology with locally compact supports. We leave implicit the formulation of these theories with mod- m coefficients.

26 Definition 26 Let X be a scheme of finite type over a field k which admits resolution of singularities. Then we define

$$H_c^i(X, \mathbb{Z}(j)) = \text{Hom}_{DM_k}(M^c(X), \mathbb{Z}(j)[i])$$

$$H_i^{lc}(X, \mathbb{Z}(j)) = \text{Hom}_{DM_k}(\mathbb{Z}(j)[i], M^c(X))$$

$$H_i(X, \mathbb{Z}(j)) = \text{Hom}_{DM_k}(\mathbb{Z}(j)[i], M(X)) .$$

Since $c_{\text{equi}}(-, 0)$ is covariantly functorial (using push-forward of cycles), we conclude that $H^*(X, \mathbb{Z}(j))$ is contravariantly functorial and $H_*(X, \mathbb{Z}(j))$ is covariantly functorial for morphisms of schemes of finite type over k . Similarly, the functoriality of $z_{\text{equi}}(-, 0)$ implies that $H_c^*(X, \mathbb{Z}(j))$ (respectively, $H_*^{lc}(X, \mathbb{Z}(j))$) is contravariant (resp. covariant) for proper maps and covariant (resp. contravariant) for flat maps.

We recall the bivariate theory introduced in [15], which is closely related to a construction in [11] and which is an algebraic version of the bivariate motivic cohomology introduced by Friedlander and Lawson in [12]:

$$A_{r,i}(Y, X) \equiv H_{cdh}^{-i}(Y, \underline{C}_*(z_{\text{equi}}(X, r))_{cdh}) .$$

This bivariate theory is used in §5 when considering the duality relationship between motivic cohomologies and homologies.

We conclude this section with a proposition, proved by Voevodsky, which interprets this bivariate theory in the context of the triangulated category DM_k and the Tate twist of Definition 17.

27 Proposition 27 [33, 4.2.3] Let k be a field admitting resolution of singularities and X, Y schemes of finite type over k . There is a natural isomorphism

$$A_{r,i}(Y, X) = \text{Hom}_{DM_k}(M(Y)(r)[2r + i], M^c(X)) .$$

As special cases of $A_{r,i}(Y, X)$, we see that

$$A_{0,i}(Y, \mathbb{A}^j) = H^{2j-i}(Y, \mathbb{Z}(j))$$

(since localization implies that $\mathbb{Z}(j)[2j]$ is quasi-isomorphic to $M^c(\mathbb{A}^j)$) and

$$A_{r,i}(\text{Spec } k, X) = H_{2r+i}^{lc}(X, \mathbb{Z}(r))$$

(since $M(\text{Spec } k)(r) = \mathbb{Z}(r)$).

Duality with Applications

A.6

In [14], Friedlander and H.B. Lawson prove a moving lemma for families of cycles on a smooth scheme which enables one to make all effective cycles of degree bounded by some constant to intersect properly all effective cycles of similarly bounded degree. This was used to establish duality isomorphisms [13], [9] between Lawson homology (cf. [19]) and morphic cohomology (cf. [12]), topological analogues of motivic homology with locally compact supports and motivic cohomology.

Theorem 30 presents the result of adapting the moving lemma of [14] to our present context of DM_k . As consequences of this moving lemma, we show that a theorem of Suslin implies that Bloch's higher Chow groups of a smooth scheme over a field which admits resolution of singularities equals motivic cohomology as defined in §4. We also prove that applying Tate twists is fully faithful in DM_k .

We first translate the moving lemma of [14] into a statement concerning the presheaves $z_{\text{equi}}(X, *)$. The moving lemma enables us to move cycles on $U \times W \times X$ equidimensional over a smooth W to become equidimensional over $U \times W$ provided that U is also smooth. (In other words, cycles are moved to intersect properly each of the fibres of the projection $U \times W \times X \rightarrow U \times W$.)

Theorem 28 [15, 7.4] Assume that k admits resolution of singularities, that U is a smooth, quasi-projective, equidimensional scheme of dimension n over k , and that X is a scheme of finite type over k . For any $r \geq 0$, the natural embedding of presheaves on Sm/k

28

$$\mathcal{D} : z_{\text{equi}}(X, r)(U \times -) \rightarrow z_{\text{equi}}(X \times U, r + n)$$

induces a quasi-isomorphism of chain complexes

$$\mathcal{D} : z_{\text{equi}}(X, r)(U \times \Delta^*) \rightarrow z_{\text{equi}}(X \times U, r + n)(\Delta^*) .$$

As shown in [15, 7.1], the hypothesis that k admits resolution of singularities may be dropped provided that we assume instead that X and Y are both projective and smooth.

Applying Theorem 28 to the map of presheaves

$$z_{\text{equi}}(X, r)(\Delta^* \times \mathbb{A}^1 \times -) \rightarrow z_{\text{equi}}(X \times \mathbb{A}^1, r+1)(\Delta^* \times -)$$

and using Lemma 7, we obtain the following homotopy invariance property.

29 **Corollary 29** Assume that k admits resolution of singularities. Then the natural map of presheaves induced by product with $\mathbb{A}^1$

$$z_{\text{equi}}(X, r) \rightarrow z_{\text{equi}}(X \times \mathbb{A}^1, r+1)$$

induces a quasi-isomorphism

$$\underline{C}_*(z_{\text{equi}}(X, r)) \xrightarrow{\cong} \underline{C}_*(z_{\text{equi}}(X \times \mathbb{A}^1, r+1)) .$$

Massaging Theorem 28 into the machinery of the previous sections provides the following duality theorem.

30 **Theorem 30** [15, 8.2] Assume that k admits resolution of singularities. Let X, Y be schemes of finite type over k and let U be a smooth scheme of pure dimension n over k . Then there are natural isomorphisms

$$\begin{aligned} A_{r,i}(Y \times U, X) &\equiv H_{\text{cdh}}^{-i}(Y \times U, \underline{C}_*(z_{\text{equi}}(X, r))_{\text{cdh}}) \\ &\xrightarrow{\cong} H_{\text{cdh}}^{-i}(Y, \underline{C}_*(z_{\text{equi}}(X \times U, r+n))_{\text{cdh}}) \equiv A_{r+n,i}(Y, X \times U) . \end{aligned}$$

Setting $Y = \text{Spec } k$, $X = \mathbb{A}^j$, and $r = 0$, we obtain the following duality relating motivic cohomology to motivic homology with locally compact supports.

31 **Corollary 31** Assume that k admits resolution of singularities and that U is a smooth scheme of pure dimension n over k . Then there are natural isomorphisms

$$H^m(U, \mathbb{Z}(j)) \xrightarrow{\cong} H_{2n-m}^{\text{lc}}(U, \mathbb{Z}(n-j))$$

provided $n \geq j$.

Proof We obtain the following string of equalities provided $n \geq j$:

$$\begin{aligned} H^m(U, \mathbb{Z}(j)) &= H^m(U, \underline{C}_*(z_{\text{equi}}(\mathbb{A}^j, 0))[-2j]) \\ &\xrightarrow{\cong} H^m(\text{Spec } k, \underline{C}_*(z_{\text{equi}}(U \times \mathbb{A}^j, n))[-2j]) \\ &= H^m(\text{Spec } k, \underline{C}_*(z_{\text{equi}}(U, n-j))[-2j]) \\ &= \text{Hom}_{\text{DM}_k}(\mathbb{Z}(n-j)[2n-2j], M^c(U)[m-2j]) = H_{2n-m}^{\text{lc}}(X, \mathbb{Z}(j)) . \end{aligned}$$

The following theorem was proved by Suslin in [28] using a different type of moving argument which applies to cycles over affine spaces. The content of this theorem is that Bloch's complex (consisting of cycles over algebraic simplices which meet the pre-images of faces properly) is quasi-isomorphic to complex of cycles equidimensional over simplices.

Theorem 32 Let X be a scheme of finite type of pure dimension n over a field k and assume that either X is affine or that k admits resolution of singularities. Let $Z_*^j(X)$ denote the Bloch complex of codimension j cycles (whose cohomology equals Bloch's higher Chow groups $CH^j(X, *)$). Then whenever $0 \leq j \leq n$, the natural embedding

$$\underline{C}_*(z_{\text{equi}}(X, n-j))(Spec\ k) \rightarrow Z_*^j(X)$$

is a quasi-isomorphism.

Combining Corollary 31 and Theorem 32, we obtain the following comparison of motivic cohomology and Bloch's higher Chow groups.

Corollary 33 Let X be a smooth scheme of finite type of pure dimension n over a field k and assume that k admits resolution of singularities. Then there is a natural isomorphism

$$H^{2j-i}(X, \mathbb{Z}(j)) \simeq CH^j(X, i).$$

Another important consequence of Theorem 30 is the following theorem.

Theorem 34 [33, 4.3.1] Let X, Y be schemes of finite type over a field k which admits resolution of singularities. Then the natural map

$$Hom_{DM_k}(M(X), M(Y)) \rightarrow Hom_{DM_k}(M(X)(1), M(Y)(1))$$

is an isomorphism.

Sketch of proof

We use the following identification (cf. [33, 4.23.])

$$A_{r,i}(X, Y) = Hom_{DM_k}(\underline{C}_*(c_{\text{equi}}(X, 0))(r)_{cdh}[2r+i], \underline{C}_*(z_{\text{equi}}(Y, 0))_{cdh}).$$

Using localization, we reduce to the case that X, Y are projective. Then,

$$Hom_{DM_k}(M(X)(1), M(Y)(1)) = Hom_{DM_k}(M(X)(1), M^c(Y \times \mathbb{A}^1)[-2])$$

equals $A_{1,0}(X, Y \times \mathbb{A}^1)$ by Proposition 27 which is isomorphic to $A_{0,0}(X, Y) = Hom_{DM_k}(X, Y)$ by Theorem 30.

Conjecture of Beilinson–Lichtenbaum

In this section, we sketch a theorem of Suslin and Voevodsky which permits K-theoretic conclusions provided that one can prove the Bloch–Kato Conjecture. Since this conjecture for the prime 2 is precisely the Milnor Conjecture recently proved by Voevodsky [34], the connection established by Suslin and Voevodsky has important applications to the 2-primary part of algebraic K-theory.

Throughout this section ℓ is a prime invertible in k and k is assumed to admit resolution of singularities. We recall the Bloch–Kato Conjecture.

35

Conjecture 35 (Bloch–Kato conjecture in weight n over k) For any field F over k , the natural homomorphism

$$K_n^M(F)/\ell \rightarrow H_{\text{et}}^n(F, \mu_\ell^{\otimes n})$$

is an isomorphism. In other words,

$$H^n(\text{Spec } F, \mathbb{Z}/\ell(n)) \xrightarrow{\sim} H_{\text{et}}^n(\text{Spec } F, \mathbb{Z}/\ell(n)) .$$

If K is a complex of sheaves on some site, we define $\tau_{\leq n}(K)$ to be the natural subcomplex of sheaves such that

$$\underline{H}^i(\tau_{\leq n}(K)) = \begin{cases} \underline{H}^i(K) & i \leq n \\ 0 & i > n . \end{cases}$$

36

Definition 36 Let $\pi : (Sm/k)_{\text{et}} \rightarrow (Sm/k)_{\text{Zar}}$ be the evident morphism of topologies on smooth schemes over k . Let $\mathbf{R}\pi_*(\mu_\ell^{\otimes n})$ denote the total right derived image of the sheaf $\mu_\ell^{\otimes n}$. We denote by $B/\ell(n)$ the complex of sheaves on $(Sm/k)_{\text{Zar}}$ given by

$$B/\ell(n) = \tau_{\leq n} \mathbf{R}\pi_*(\mu_\ell^{\otimes n}) .$$

As shown in [31, 5.1], $B/\ell(n)$ is a complex of presheaves with transfers with homotopy invariant cohomology sheaves. By Propositions 2.7 and 3.3, this implies the natural isomorphism for any smooth scheme X over k

$$H_{\text{Zar}}^i(X, B/\ell(n)) \xrightarrow{\sim} \text{Hom}_{DM_k}(M(X), B/\ell(n)[i]) ,$$

where the cohomology is Zariski hypercohomology.

The following conjecture of Beilinson [2], related to conjectures of Lichtenbaum [23], is an intriguing generalization of the Bloch–Kato conjecture. We use the natural quasi-isomorphism $\mu_\ell^{\otimes n} \simeq \mathbb{Z}/\ell(n)_{\text{et}}$ of Theorem 23 plus the acyclicity of $\mathbb{Z}/\ell(n)$ in degrees greater than n to conclude that the natural maps

$$\mathbb{Z}/\ell(n) \rightarrow \mathbf{R}\pi_* \mathbb{Z}/\ell(n)_{\text{et}} \simeq \mathbf{R}\pi_* \mu_\ell^{\otimes n} \leftarrow B/\ell(n)$$

determine a natural map (in the derived category of complexes of sheaves in the Zariski topology)

$$\mathbb{Z}/\ell(n) \rightarrow B/\ell(n).$$

Conjecture 37 (Beilinson–Lichtenbaum Conjecture in weight n over k) The natural morphism

37

$$\mathbb{Z}/\ell(n) \rightarrow B/\ell(n)$$

is a quasi-isomorphism of complexes of sheaves on $(Sm/k)_{\text{Zar}}$.

Remark 38 A well known conjecture of Beilinson [2], [3] and Christophe Soulé [26] asserts that $H^i(X, \mathbb{Z}(n))$ vanishes for $i < 0$. Since $H_{\text{et}}^i(X, \mu_\ell^{\otimes n}) = 0$ for $i < 0$, Conjecture 37 incorporates the mod- ℓ analogue of the Beilinson–Soulé Conjecture.

38

We now state the theorem of Suslin and Voevodsky. M. Levine provided a forerunner of this theorem in [20].

Theorem 39 [31, 5.9] Let k be a field which admits resolution of singularities and assume that the Bloch–Kato conjecture holds over k in weight n . Then the Beilinson–Lichtenbaum conjecture holds over k in weight n .

39

Sketch of Proof

One readily verifies that the validity of the Bloch–Kato Conjecture in weight n implies the validity of this conjecture in weights less than n . Consequently, proceeding by induction, we may assume the validity of the Beilinson–Lichtenbaum Conjecture in weights less than n . Moreover, since both $\mathbb{Z}/\ell(n)$ and $B/\ell(n)$ have cohomology presheaves which are homotopy invariant presheaves with transfers annihilated by multiplication by n , we may apply the rigidity theorem (Theorem 8) to conclude that to prove the asserted quasi-isomorphism $\mathbb{Z}/\ell(n) \rightarrow B/\ell(n)$ it suffices to prove for all extension fields F over k that the induced map

$$H^*(\text{Spec } F, \mathbb{Z}/\ell(n)) \rightarrow H^*(\text{Spec } F, B/\ell(n))$$

is an isomorphism. By construction, $H^i(\text{Spec } F, \mathbb{Z}/\ell(n)) = 0$ for $i > n$, so that it suffices to prove

$$H^i(\text{Spec } F, \mathbb{Z}/\ell(n)) \xrightarrow{\sim} H_{\text{et}}^i(\text{Spec } F, \mu_\ell^{\otimes n}) \quad i \leq n.$$

Suslin and Voevodsky easily conclude that it suffices to prove that

$$H^i(\text{Spec } F, \mathbb{Z}/\ell(n)) \rightarrow H_{\text{et}}^i(\text{Spec } F, \mu_\ell^{\otimes n}) \quad i < n$$

is injective (assuming the validity of the Bloch–Kato Conjecture in weight n). This in turn is implied by the assertion that

$$H^n(\partial\Delta_F^j, \mathbb{Z}/\ell(n)_{cdh}) \rightarrow H^n(\partial\Delta_F^j, B/\ell(n)_{cdh})$$

is injective for all j , where $\partial\Delta_F^j$ is the (singular) boundary of the j -simplex over F whose cohomology fits in Mayer–Vietoris exact sequence for a covering by two contractible closed subschemes whose intersection is $\partial\Delta_F^{j-1}$.

We denote by S^1 the scheme obtained from $\mathbb{A}^1$ by gluing together $\{0\}, \{1\}$. We have natural embeddings

$$H^n(\partial\Delta_F^j, \mathbb{Z}/\ell(n)_{cdh}) \rightarrow H^{n+1}(\partial\Delta_F^j \times S^1, \mathbb{Z}/\ell(n)_{cdh})$$

$$H^n(\partial\Delta_F^j, B/\ell(n)_{cdh}) \rightarrow H^{n+1}(\partial\Delta_F^j \times S^1, B/\ell(n)_{cdh}).$$

Any cohomology class in $H^n(\partial\Delta_F^j, \mathbb{Z}/\ell(n))$ which does not arise from

$$H^n(\operatorname{Spec} F, \mathbb{Z}/\ell(n))$$

vanishes on some open subset $U \subset \partial\Delta_F^j \times S^1$ containing all the points of the form $p_i \times \infty$ where $\infty \in S^1$ is the distinguished point. In other words, all such cohomology lies in the image of $H_{\mathbb{Z}}^{n+1}(\partial\Delta_F^j \times S^1, \mathbb{Z}/\ell(n)_{cdh})$, the direct limit of cohomology with supports in closed subschemes missing each of the points $p_i \times \infty$.

The localization distinguished triangle of Corollary 16 gives us long exact sequences in cohomology with coefficients $\mathbb{Z}/\ell(n)_{cdh}$ and $B/\ell(n)_{cdh}$ and a map between these sequences; the terms involve the cohomology of S (the semi-local scheme of the set $\{p_i \times \{\infty\}\}$), of $\Delta_F^j \times S^1$ with supports in $\mathbb{Z}$, and of $\Delta_F^j \times S^1$ itself. Although S is not smooth, one can conclude that our Bloch–Kato hypothesis implies that $H^n(S, \mathbb{Z}/\ell(n)_{cdh}) \rightarrow H^n(S, B/\ell(n)_{cdh})$ is surjective. Another application of the localization distinguished triangle plus induction (on n) implies that the map on cohomology with supports in $\mathbb{Z}$ is an isomorphism. The required injectivity now follows by an easy diagram chase.

An important consequence of Theorem 39 is the following result of Suslin and Voevodsky.

40

Proposition 40 [31, 7.1] The Bloch–Kato conjecture holds over k in weight n if and only if for any field F of finite type over k the Bockstein homomorphisms

$$H_{\text{et}}^n(F, \mu_{\ell^m}^{\otimes n}) \rightarrow H_{\text{et}}^{n+1}(F, \mu_{\ell}^{\otimes n})$$

are zero for all $m > 0$.

Comment about the Proof

If the Bloch–Kato conjecture holds, then

$$H_{\text{et}}^n(F, \mu_{\ell^m}^{\otimes n})$$

consists of sums of products of elements of $H_{et}^1(F, \mu_{\ell^m})$. The vanishing of the Bockstein homomorphism on classes of cohomology degree 1 follows immediately from Hilbert's Theorem 90.

The proof of the converse is somewhat less direct.

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La conjecture de Milnor (d'après V. Voevodsky)

Bruno Kahn

<i>B.1</i>	<i>Introduction</i>	1107
<i>B.2</i>	<i>Résultats antérieurs et premières réductions</i>	1110
	Résultats connus antérieurement	1110
	Nettoyages	1111
<i>B.3</i>	<i>Cohomologie motivique</i>	1113
<i>B.4</i>	<i>Corps dont la K-théorie de Milnor est divisible</i>	1117
<i>B.5</i>	<i>Variétés de déploiement</i>	1119
	Corps de déploiement	1119
	Variétés de déploiement	1121
<i>B.6</i>	<i>Homotopie des variétés algébriques</i>	1126
	Topologie de Nisnevich	1126
	Catégorie homotopique	1126
	Deux cercles	1127
	T -spectres	1128
	Théories cohomologiques et homologiques	1128
	Spectres d'Eilenberg-Mac Lane et cohomologie motivique	1129
<i>B.7</i>	<i>Opérations de Steenrod en cohomologie motivique</i>	1131
<i>B.8</i>	<i>Démonstration du théorème 50</i>	1133
	Réalisation topologique	1133
	Espaces de Thom et cobordismes algébriques	1134
	Le théorème principal	1135

B.9 Démonstration du théorème 37 1140

 Le motif de Rost 1140

 Zéro-cycles à coefficients dans les unités..... 1142

B.10 Compléments..... 1144

 Le motif de Rost-Voevodsky 1144

(v_n, l) -variétés et variétés de déploiement génériques 1145

Bibliographie..... 1146

Résumé. This paper was originally delivered in the Séminaire Bourbaki and published in Astérisque [14] by the Société Mathématique de France and is reprinted here with permission. It gives an outline of the first version of Voevodsky's proof of the Milnor conjecture, which was later modified and simplified by the author [49]. There is no difference with the original text, except that a few misprints have been corrected, some references have been updated, and the layout has been adapted.

Introduction

B.1

Soit F un corps commutatif. La K -théorie de Milnor de F est l'anneau gradué $K_*^M(F)$ défini par générateurs et relations de la manière suivante:

- Générateurs: $\{a\}, a \in F^*$.
- Relations: $\{ab\} = \{a\} + \{b\}$ ($a, b \in F^*$), $\{a\} \cdot \{1 - a\} = 0$ ($a \in F^* - \{1\}$).

En d'autres termes, $K_*^M(F)$ est le quotient de l'algèbre tensorielle du $\mathbf{Z}$ -module F^* par l'idéal bilatère engendré par les $a \otimes (1 - a)$ pour $a \neq 1$. On a $K_0(F) = \mathbf{Z}$, $K_1(F) = F^*$. Pour $a_1, \dots, a_n \in F^*$, le produit $\{a_1\} \cdots \{a_n\} \in K_n^M(F)$ est noté $\{a_1, \dots, a_n\}$. Pour $a \neq 1$, les relations

$$\{a, 1 - a\} = 0$$

$$\{a^{-1}, 1 - a^{-1}\} = 0$$

et la bilinéarité entraînent

$$\{a, -a\} = 0$$

d'où, encore par bilinéarité

$$\{a, b\} = -\{b, a\} \quad \text{pour } a, b \in F^*.$$

L'anneau gradué $K_*^M(F)$ est donc *commutatif*.

Les groupes $K_n^M(F)$ ont été introduits dans [23] par Milnor, qui était motivé par le fait que $K_2^M(F) = K_2(F)$ (théorème de Matsumoto).

Soit m un entier premier à l'exposant caractéristique de F , et soit F_s une clôture séparable de F . La suite exacte de Kummer

$$1 \rightarrow \mu_m \rightarrow F_s^* \xrightarrow{m} F_s^* \rightarrow 1$$

fournit un homomorphisme

$$F^* \rightarrow H^1(F, \mu_m) \tag{B.1}$$

$$a \mapsto (a)$$

vers la cohomologie galoisienne de F .

1 **Lemme 1:** (Tate) L'homomorphisme (B.1) se prolonge par le cup-produit en une famille d'homomorphismes

$$K_n^M(F)/m \xrightarrow{u_{n,m}(F)} H^n(F, \mu_m^{\otimes n}).$$

Cela revient à voir que $(a) \cup (1 - a) = 0$ dans $H^2(F, \mu_m^{\otimes 2})$, pour tout $a \in F^*$. Pour cela, considérons l'algèbre étale $E = F[t]/t^m - a$. Si α est l'image de t dans E , on a

$$\alpha^m = a$$

$$N_{E/F}(1 - \alpha) = 1 - a.$$

En utilisant la formule de projection en cohomologie étale, il en résulte:

$$(a) \cup (1 - a) = \text{Cor}_{E/F}((a) \cup (1 - \alpha)) = \text{Cor}_{E/F}(m(\alpha) \cup (1 - \alpha)) = 0. \quad \square$$

Les homomorphismes $u_{n,m}(F)$ sont parfois appelés, pour des raisons historiques, *homomorphismes de résidu normique*. Notons $K(n, m, F)$ l'énoncé suivant:

$$\text{L'homomorphisme } u_{n,m}(F) \text{ du lemme 1 est bijectif.} \quad (K(n, m, F))$$

Kato a proposé la conjecture suivante:

2 **Conjecture 2:** ([15, conj. 1]) $K(n, m, F)$ est vrai pour tout (n, m, F) .

Pour $n = 2$, cette conjecture avait été indiquée par Milnor lui-même [23, p. 540], et Bloch [6, lecture 5] avait posé la question de la surjectivité des $u_{n,m}$ lorsque F est un corps de fonctions sur $\mathbb{C}$ (notons que, dans ce cas, la surjectivité équivaut au fait que l'algèbre de cohomologie $H^*(F, \mathbb{Z}/m)$ est engendrée en degré 1).

La conjecture de Kato a été démontrée dans un grand nombre de cas particuliers (voir B.2.1). Elle vient d'être démontrée dans le cas 2-primaire par Voevodsky:

3 **Théorème 3:** ([49]) $K(n, m, F)$ est vrai pour tout (n, F) lorsque m est une puissance de 2.

La démonstration de Voevodsky est par récurrence sur n : elle est exposée dans les prochaines sections¹. Contrairement aux démonstrations précédentes, qui utilisaient la K -théorie algébrique, elle n'utilise "que" la *cohomologie motivique*, qu'il a contribué à développer (voir à ce sujet l'exposé de E. Friedlander dans ce sémi-

¹ Le rédacteur ne prétend pas avoir vérifié les moindres ramifications de cette démonstration, qui s'appuie sur un travail antérieur considérable (notamment [12, 43, 45, 46]). Il a par contre vérifié les arguments de [49] dans un détail suffisant pour juger que son contenu mérite d'être exposé dans ce séminaire. Néanmoins, il doit souligner que la démonstration de [49] ne sera complète que lorsque les articles [27] et [48], sur lesquels elle repose, seront achevés et rendus publics.

naire). Malgré cela, la topologie algébrique y joue un rôle essentiel, sous la forme de la catégorie homotopique (et de la catégorie homotopique stable) des variétés, introduite par Morel et Voevodsky [26, 27, 47]. Les arguments de Voevodsky n'utilisent pas non plus de réduction aux corps de nombres, comme c'était le cas pour certaines des démonstrations antérieures.

Pour le lecteur qui ne souhaiterait pas se plonger dans les détails, nous en donnons ici un résumé. Il est facile de voir qu'on peut se limiter au cas où F est de caractéristique 0, voire un sous-corps de $\mathbb{C}$ (corollaire 7 et proposition 8). On suppose la conjecture connue en degré $n - 1$. La première étape, largement inspirée de (mais non identique à) la stratégie antérieure de Merkurjev-Suslin, réduit le problème à démontrer un "théorème 90 de Hilbert en degré n " (corollaire 14): celui-ci est exprimé en termes de cohomologie motivique. La deuxième étape, toujours inspirée par Merkurjev-Suslin, consiste à réduire ce théorème 90 à l'existence d'une variété de déploiement convenable pour un symbole $a \in K_n^M(F)/2$, c'est-à-dire une variété intègre X_a telle que a s'annule par extension des scalaires de F à $F(X_a)$: on prend pour X_a la quadrique projective définie par une voisine de dimension $2^{n-1} + 1$ de la forme de Pfister associée à a . Ici la stratégie diverge de celle de Merkurjev-Suslin: Voevodsky montre qu'il suffit d'établir la nullité d'un certain groupe de cohomologie motivique d'un schéma simplicial $\check{C}(X_a)$ associé à X_a (proposition 25 et théorème 30). Cette approche simplifie grandement celle de Merkurjev et Suslin, qui étaient obligés de démontrer une multitude d'énoncés parasites.

Toutes les démonstrations antérieures de cas particuliers de la conjecture de Kato utilisent le fait que, pour une variété de déploiement convenable X associée comme ci-dessus à un symbole, le groupe des "zéros-cycles à coefficients dans les unités modulo l'équivalence rationnelle"

$$A_0(X, K_1)$$

s'injecte dans F^* par l'intermédiaire de la norme (voir section B.9). Pour $l = 2$, ce résultat est démontré par M. Rost en tout degré (théorème 65). Grâce à une décomposition du motif de Chow de X_a , également due à Rost (théorème 61), Voevodsky montre que cette injectivité est équivalente à la nullité d'un *autre* groupe de cohomologie motivique de $\check{C}(X_a)$ (théorème 37). Sa contribution essentielle est alors de relier le premier groupe au deuxième par une opération cohomologique α , qu'il va montrer être injective.

Pour définir α , Voevodsky utilise la *catégorie homotopique des F -variétés*, qu'il construit conjointement avec F. Morel. Elle lui permet de définir des opérations de Steenrod en cohomologie motivique, analogues à celles existant en topologie algébrique, et α est une version entière de l'une de ces opérations. Supposant F plongé dans $\mathbb{C}$, l'injectivité de α sur la cohomologie motivique de la variété X_a résulte d'une part de l'existence d'une classe fondamentale dans le bordisme algébrique de X_a , et d'autre part du fait que la classe de $X_a(\mathbb{C})$ en $(n - 1)$ -ième K -théorie de Morava est un générateur périodique, ce qui établit un lien mystérieux entre la démonstration de Voevodsky et des objets intervenant dans des propriétés profondes de la catégorie homotopique stable classique ([10, 32, 33])...

Une partie considérable de l'argument de Voevodsky s'applique au cas d'un nombre premier quelconque. Nous nous sommes efforcé de mettre en évidence cette généralité; on en trouvera les fruits dans la section B.10.1. Pour avoir tous les détails de la démonstration, le lecteur devra naturellement consulter [49], ainsi que les articles dont il dépend. Nous l'encourageons également à lire [47], ancêtre direct de [49], qui contient des commentaires éclairants ayant disparu de ce dernier article.

Supposons F de caractéristique différente de 2. Soient $W(F)$ l'anneau de Witt de F , classifiant les formes quadratiques non dégénérées sur F , IF son idéal d'augmentation et, pour tout $n > 0$, $I^n F = (IF)^n$. Le groupe abélien IF est engendré par les classes des formes binaires $\langle 1, -a \rangle$ pour $a \in F^*$; le groupe $I^n F$ est donc engendré par les classes des n -formes de Pfister $\ll a_1, \dots, a_n \gg := \langle 1, -a_1 \rangle \otimes \dots \otimes \langle 1, -a_n \rangle$. L'application $(a_1, \dots, a_n) \mapsto \ll a_1, \dots, a_n \gg$ induit un homomorphisme surjectif

$$K_n^M(F)/2 \rightarrow I^n F / I^{n+1} F. \quad (\text{B.2})$$

En collaboration avec D. Orlov et A. Vishik, Voevodsky a annoncé une démonstration du fait que (B.2) est *bijectif* pour tout (n, F) ; cela avait été également conjecturé par Milnor. Nous n'aborderons pas ici cet aspect de son travail, qui utilise essentiellement les mêmes méthodes (cf. [29]).

Je remercie Fabien Morel pour son aide dans la préparation de ce texte.

Notation Si A est un foncteur sur la catégorie des extensions de F , si $a \in A(F)$ et si E est une extension de F , on note a_E l'image de a dans $A(E)$.

Résultats antérieurs et premières réductions

B.2

B.2.1 Résultats connus antérieurement

Notons $K(n, m)$ l'énoncé $\{K(n, m, F) \text{ pour tout corps } F \text{ de caractéristique ne divisant pas } m\}$.

L'énoncé $K(0, m, F)$ dit que le groupe de cohomologie galoisienne $H^0(F, \mathbb{Z}/m)$ est isomorphe à $\mathbb{Z}/m$: c'est trivial.

L'énoncé $K(1, m, F)$ dit que le groupe $H^1(F, \mu_m)$ est isomorphe à F^*/F^{*m} . Ce résultat, classique, est connu sous le nom de *théorie de Kummer*. Lorsque $\mu_m \subset F$, il équivaut au fait que les caractères d'ordre divisant m du groupe de Galois $G_F = \text{Gal}(F_s/F)$ correspondent bijectivement aux éléments de F^*/F^{*m} , après le choix d'une racine primitive m -ième de l'unité. L'injectivité de $u_{1,m}(F)$ résulte immédiatement de sa définition; sa surjectivité résulte du théorème 90 de Hilbert (ou plutôt de la version d'Emmy Noether de ce théorème):

$$H^1(F, F_s^*) = 0.$$

La démonstration de $K(2, m, F)$ est due à Tate pour les corps globaux [44]; Tate utilise la théorie du corps de classes. La démonstration de $K(2, m)$ est due à Merkurjev pour $m = 2$ [19] et à Merkurjev-Suslin pour m quelconque [21]. Elle utilise la K -théorie algébrique de Quillen; voir l'exposé Bourbaki de Soulé à ce sujet [40].

La démonstration de $K(3, 2)$ est due indépendamment à Rost [34] et à Merkurjev-Suslin [22]. La démonstration de Merkurjev-Suslin utilise la K -théorie algébrique, alors que celle de Rost ne l'utilise pas. Une démonstration de $K(4, 2)$ a été annoncée par Rost vers 1988, mais celui-ci ne l'a jamais rédigée.

Rost et Voevodsky ont récemment annoncé une démonstration de $K(3, 3)$ et de $K(4, 3)$ (voir section B.10.1).

Enfin, en dehors des cas cités ci-dessus, $K(n, m, F)$ est connu pour des corps F particuliers:

- *Corps globaux*: $K(n, m, F)$ est connu pour tout (n, m) avec $n \geq 3$ (Bass-Tate [5]). Bass et Tate démontrent plus: pour $n \geq 3$, le groupe $K_n^M(F)$ est isomorphe à $(\mathbb{Z}/2)^{r_1}$, où r_1 est le nombre de places réelles de F .
- *Corps henséliens*: soit F un corps de caractéristique 0, hensélien pour une valuation discrète, à corps résiduel de caractéristique $p > 0$. Alors $K(n, p, F)$ est connu pour tout n (Bloch-Gabber-Kato [7]).

Nettoyages

B.2.2

Proposition 4 a) Soient m_1, m_2 deux entiers premiers entre eux. Alors, pour tout corps F de caractéristique première à $m_1 m_2$ et pour tout $n \geq 0$, $K(n, m_1 m_2, F) \iff \{K(n, m_1, F) \text{ et } K(n, m_2, F)\}$.

b) (Tate) Soient $m \geq 1$, F un corps de caractéristique première à m et E/F une extension de degré premier à m . Soit $n \geq 0$. Alors $K(n, m, E) \implies K(n, m, F)$.

c) (Tate) Soit l un nombre premier. Alors, pour tout corps F de caractéristique différente de l , $\{K(n-1, l, F) \text{ et } K(n, l, F)\} \implies \{K(n, l^v, F) \text{ pour tout } v \geq 1\}$.

Démonstration a) est clair. Pour démontrer b), on remarque que les deux foncteurs $F \mapsto K_n^M(F)$ et $F \mapsto H^n(F, \mu_m^{\otimes n})$ sont munis de transferts

$$N_{E/F} : \begin{cases} K_n^M(E) & \rightarrow K_n^M(F) \\ H^i(E, \mu_m^{\otimes n}) & \rightarrow H^n(F, \mu_m^{\otimes n}) \end{cases}$$

pour toute extension finie E/F , vérifiant la formule de projection et tels que $N_{E/F} \circ i_{E/F}$ soit la multiplication par le degré $[E : F]$, où $i_{E/F}$ correspond à la fonctorialité (c'est classique pour la cohomologie galoisienne, cf. [39]; voir [5, §5] et [15, §1.7] pour la K -théorie de Milnor), et que ces transferts commutent à l'homomorphisme $u_{n,m}$.

Pour démontrer c), on se réduit via b) au cas où F contient une racine primitive l -ième de l'unité ζ : en effet, le degré $[F(\mu_l) : F]$ divise $l - 1$, donc est premier à l . On raisonne par récurrence sur v , en considérant le diagramme

$$\begin{array}{ccccccc}
 K_{n-1}^M(F)/l & \xrightarrow{\{\zeta\} \cdot} & K_n^M(F)/l^v & \longrightarrow & K_n^M(F)/l^{v+1} & \longrightarrow & K_n^M(F)/l \longrightarrow 0 \\
 u_{n-1,l} \downarrow & & u_{n,l} \downarrow & & u_{n,l^{v+1}} \downarrow & & u_{n,l} \downarrow \\
 H^{n-1}(F, \mu_l^{\otimes(n-1)}) & \xrightarrow{\varphi} & H^n(F, \mu_{l^v}^{\otimes n}) & \longrightarrow & H^n(F, \mu_{l^{v+1}}^{\otimes n}) & \longrightarrow & H^n(F, \mu_l^{\otimes n})
 \end{array}$$

où φ est le cup-produit par la classe $[\zeta]$ de ζ dans $H^0(F, \mu_l) = \mu_l$ suivi du Bockstein ∂ associé à la suite exacte de coefficients

$$0 \rightarrow \mu_{l^v}^{\otimes n} \rightarrow \mu_{l^{v+1}}^{\otimes n} \rightarrow \mu_l^{\otimes n} \rightarrow 0. \quad (A_n)$$

Dans ce diagramme, la ligne inférieure est exacte, et la ligne supérieure est exacte sauf peut-être en $K_n^M(F)/l^v$. La commutativité du diagramme est évidente, sauf celle du carré de gauche. Pour vérifier cette dernière, on remarque que, si $x \in K_{n-1}^M(F)$, son image y par $u_{n-1,l}$ provient de $\tilde{y} = u_{n-1,l^{v+1}}(x) \in H^{n-1}(F, \mu_{l^{v+1}}^{\otimes(n-1)})$, et donc que

$$\varphi(y) = \partial([\zeta] \cup y) = \partial([\zeta]) \cup \tilde{y} = (\zeta) \cup \tilde{y} = u_{n,l^v}(\{\zeta\} \cdot x).$$

L'énoncé résulte alors d'une chasse aux diagrammes. $\square$

5

Proposition 5 Soit E un corps complet pour une valuation discrète, de corps résiduel F . Alors, pour tout m premier à la caractéristique de F et tout $n \geq 1$, on a $K(n, m, E) \iff \{K(n, m, F) \text{ et } K(n-1, m, F)\}$.

En effet, on a un diagramme

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_n^M(F)/m & \longrightarrow & K_n^M(E)/m & \longrightarrow & K_{n-1}^M(F)/m \longrightarrow 0 \\
 & & u_{n,m}(F) \downarrow & & u_{n,m}(E) \downarrow & & u_{n-1,n}(F) \downarrow \\
 0 & \longrightarrow & H^n(F, \mu_m^{\otimes n}) & \longrightarrow & H^n(E, \mu_m^{\otimes n}) & \longrightarrow & H^{n-1}(F, \mu_m^{\otimes(n-1)}) \longrightarrow 0.
 \end{array}$$

La ligne supérieure est exacte scindée par le choix d'une uniformisante de E [23, lemma 2.6], ainsi que la ligne inférieure, cf. [39, p. 121, (2.2)]. On vérifie facilement que ce diagramme est commutatif [23, p. 341] et que les deux scindages sont compatibles. La proposition résulte alors du lemme des 5. $\square$

6

Corollaire 6 $K(n, m) \Rightarrow K(n-1, m)$.

Démonstration On applique la proposition 5 avec $E = F((t))$. $\square$

Corollaire 7 $K(n, m)$ en caractéristique 0 implique $K(n, m)$ en toute caractéristique première à n .

7

Démonstration Soit F un corps de caractéristique $p > 0$ première à m . D'après la proposition 4 b), pour démontrer $K(n, m, F)$, on peut supposer F parfait. On applique alors la proposition 5 en prenant pour E le corps des fractions de l'anneau des vecteurs de Witt de F . $\square$

Pour démontrer le théorème 3, on peut donc supposer que $m = 2$ et que F est un corps de caractéristique 0. Cela servira à disposer non seulement de la résolution des singularités, mais aussi d'un "foncteur de réalisation" de la catégorie homotopique des F -schémas vers la catégorie homotopique classique, associé à un plongement de F dans $\mathbb{C}$, si par exemple F est de type fini sur $\mathbb{Q}$ (voir section B.8.1). Cette hypothèse supplémentaire est innocente en vertu de la

Proposition 8 Soit F un corps. Si $K(n, m, k)$ est vrai pour tout sous-corps $k \subset F$ de type fini sur son sous-corps premier, alors $K(n, m, F)$ est vrai.

8

C'est clair, puisque la K -théorie de Milnor et la cohomologie galoisienne commutent aux limites inductives filtrantes. $\square$

Cohomologie motivique

B.3

Soit F un corps. A tout F -schéma lisse de type fini X , Suslin et Voevodsky [43, §2] associent une famille de complexes de groupes abéliens $\mathbf{Z}(n, X)_{n \geq 0}$, contravariants en X et commutant aux limites projectives à morphismes de transition affines. Pour chaque $n \geq 0$, les $\mathbf{Z}(n, X)$ définissent donc un complexe de faisceaux $\mathbf{Z}(n)$ sur le grand site zariskien de $\text{Spec } F$ restreint à la sous-catégorie pleine des F -schémas lisses. Ces complexes de faisceaux ont les propriétés suivantes:

- (A) $\mathbf{Z}(0) = \mathbb{Z}$, concentré en degré 0.
- (B) $\mathbf{Z}(1)$ est quasi-isomorphe à $\mathbb{G}_m[-1]$ (le faisceau des unités placé en degré cohomologique 1).
- (C) Pour tout $n \geq 0$, $\mathbf{Z}(n)$ est acyclique en degré $> n$.
- (D) Pour $m, n \geq 0$, il existe un produit $\mathbf{Z}(m) \otimes^L \mathbf{Z}(n) \rightarrow \mathbf{Z}(m+n)$. Ce produit est commutatif et associatif à homotopie près.
- (E) Pour tout m premier à l'exposant caractéristique de F , $\alpha^* \mathbf{Z}(n) \otimes^L \mathbb{Z}/m$ est quasi-isomorphe à $\mu_m^{\otimes n}$, où α est la projection du grand site étale de $\text{Spec } F$ sur son grand site zariskien.
- (F) $\mathbf{Z}(n)$ est un complexe de faisceaux avec transferts, à faisceaux de cohomologie invariants par homotopie, au sens de [45].
- (G) $\mathbb{H}^n(\text{Spec } F, \mathbf{Z}(n)) = K_n^M(F)$ [43, §3].

Pour plus de détails, voir l'exposé de Friedlander.

Pour tout groupe abélien A , on note $A(n)$ le complexe $\mathbf{Z}(n) \overset{L}{\otimes} A$. On note $H_B^*(X, A(n))$ (resp. $H_L^*(X, A(n))$) les groupes d'hypercohomologie $\mathbb{H}_{\text{Zar}}^*(X, A(n))$ (resp. $\mathbb{H}_{\text{ét}}^*(X, \alpha^* A(n))$). Pour $X = \text{Spec } F$, on convient de noter simplement ces groupes $H_B^*(F, A(n))$ et $H_L^*(F, A(n))$.

Comme on ne sait pas pour $n \geq 2$ si $\mathbf{Z}(n)$ est cohomologiquement borné à gauche (c'est une conjecture), il est bon de rappeler la définition de l'hypercohomologie d'un complexe non borné et de vérifier quelques propriétés des groupes ci-dessus (ces points sont quelque peu passés sous silence dans [43] et [49]). Si X est un site et C un complexe de faisceaux sur X , à valeurs dans les groupes abéliens, $\mathbb{H}^*(X, C)$ est la cohomologie d'un complexe $\mathcal{F}$ - K -injectif au sens de [41], quasi-isomorphe à C . Si X est de dimension cohomologique finie, on a des suites exactes:

$$0 \rightarrow \varprojlim^1 \mathbb{H}^{q-1}(X, \tau_{\geq n} C) \rightarrow \mathbb{H}^q(X, C) \rightarrow \varprojlim \mathbb{H}^q(X, \tau_{\geq n} C) \rightarrow 0. \quad (\text{B.3})$$

Si C est cohomologiquement borné à gauche ou si X est de dimension cohomologique finie d , on a des suites spectrales d'hypercohomologie fortement convergentes:

$$I_1^{p,q} = H^q(X, C^p) \Rightarrow \mathbb{H}^{p+q}(X, C) \Leftarrow H^p(X, \mathcal{H}^q(C)) = I_2^{p,q}. \quad (\text{B.4})$$

On aura aussi besoin de la cohomologie motivique de certains schémas simpliciaux. Si $x_\bullet$ est un objet simplicial de X , on définit $\mathbb{H}^*(x_\bullet, C)$ comme étant la cohomologie du complexe total associé au complexe cosimplicial $\mathcal{F}(x_\bullet)$, où $\mathcal{F}$ est comme ci-dessus. Si C est cohomologiquement borné à gauche ou si $x_\bullet$ est de dimension finie, on a une suite spectrale fortement convergente:

$$E_1^{p,q} = \mathbb{H}^q(x_p, C) \Rightarrow \mathbb{H}^{p+q}(x_\bullet, C). \quad (\text{B.5})$$

Si $x_\bullet^{(r)}$ est un système inductif d'objets simpliciaux, de limite inductive $x_\bullet$, on a des suites exactes analogues à (B.3):

$$0 \rightarrow \varprojlim^1 \mathbb{H}^{q-1}(x_\bullet^{(r)}, C) \rightarrow \mathbb{H}^q(x_\bullet, C) \rightarrow \varprojlim \mathbb{H}^q(x_\bullet^{(r)}, C) \rightarrow 0. \quad (\text{B.6})$$

Pour tout nombre premier l , notons $\mathbf{Z}_{(l)}$ le localisé de $\mathbf{Z}$ en l .

9

Proposition 9 Pour tout corps F et tout nombre premier $l \neq \text{car } F$,

- a) L'application naturelle $H_B^q(F, \mathbf{Q}(n)) \rightarrow H_L^q(F, \mathbf{Q}(n))$ est un isomorphisme pour tout $q \in \mathbf{Z}$.
- b) Le foncteur $F \mapsto H_L^q(F, \mathbf{Z}_{(l)}(n))$, où $l \neq \text{car } F$, commute aux limites inductives filtrantes pour tout $q \in \mathbf{Z}$.
- c) $H_L^q(F, \mathbf{Z}_{(l)}(n))$ est de torsion pour $q > n$.

Démonstration a) Plus généralement, pour tout corps F et tout complexe de faisceaux de $\mathbf{Q}$ -espaces vectoriels K sur le grand site zariskien de $\text{Spec } F$, munis de transferts au sens de [45], l'application $H^q(K(F)) \rightarrow \mathbb{H}_{\text{ét}}^q(F, \alpha^* K)$ est un isomor-

phisme. Si K est réduit à un faisceau, c'est dû à l'existence de transferts et au fait que la cohomologie galoisienne d'un G_F -module est de torsion en degré > 0 . En général, on note qu'un corps est de $\mathbf{Q}$ -dimension cohomologique étale 0, et qu'on peut donc appliquer la suite spectrale II de (B.4).

b) Il suffit de démontrer l'énoncé analogue pour les groupes de cohomologie à coefficients $\mathbf{Q}(n)$ et $\mathbf{Q}_l/\mathbf{Z}_l(n)$. Dans le premier cas, cela résulte de a); dans le deuxième, cela résulte de la propriété (E) de $\mathbf{Z}(n)$ et de la commutation bien connue de la cohomologie étale d'un faisceau aux limites inductives filtrantes.

c) Cela résulte de a) et des propriétés (C) et (E) de $\mathbf{Z}(n)$. $\square$

Soit l un nombre premier différent de $\text{car } F$. Considérons l'énoncé suivant:

$$\text{Pour tout } i \leq n, \quad H_L^{i+1}(F, \mathbf{Z}_{(l)}(i)) = 0. \quad (H90(n, l, F))$$

Exemples 10.

1. $n = 0$: l'énoncé se traduit en $H_{\text{ét}}^1(F, \mathbf{Z}_{(l)}) = 0$. C'est clair, puisque le groupe de Galois G_F , profini, n'a pas de caractères continus d'ordre infini.
2. $n = 1$: l'énoncé se traduit en le précédent et $H_{\text{ét}}^1(F, \mathbb{G}_m) \otimes \mathbf{Z}_{(l)} = 0$. C'est la version d'Emmy Noether du théorème 90 de Hilbert.

Notons $H90(n, l)$ l'énoncé $\{H90(n, l, F) \text{ pour tout corps } F \text{ de caractéristique } 0\}$. Par ailleurs, notons $B(n)$ le complexe de faisceaux zariskiens $\tau_{\leq n+1} R\alpha_* \alpha^* \mathbf{Z}(n)$: on a un morphisme naturel

$$\mathbf{Z}(n) \rightarrow B(n) \quad (B_n)$$

sur le grand site zariskien de $\text{Spec } \mathbf{Q}$.

Théorème 11: ([49, th. 2.11]) Les conditions suivantes sont équivalentes:

11

- (i) $H90(n, l)$ est vrai.
- (ii) Pour tout $i \leq n$, le morphisme $(B_i) \otimes \mathbf{Z}_{(l)}$ est un quasi-isomorphisme.

Démonstration (ii) $\Rightarrow$ (i) est clair, d'après la propriété (C) de $\mathbf{Z}(n)$. Pour voir la réciproque, introduisons le cône $K(i)$ du morphisme $(B_i) \otimes \mathbf{Z}_{(l)}$: c'est un complexe de faisceaux sur le grand site zariskien de $\text{Spec } F$, et il faut montrer qu'il est acyclique. Pour tout anneau local A d'une F -variété lisse X , il résulte de la propriété (F) de $\mathbf{Z}(i)$ que $\mathbb{H}^*(\text{Spec } A, K(i)) \rightarrow \mathbb{H}^*(\text{Spec } E, K(i))$ est injectif ("conjecture de Gersten", [45, cor. 4.17]). Cela ramène à démontrer, sans perte de généralité, que $\mathbb{H}^*(F, K(i)) = 0$. D'après la proposition 9 c), $\mathbb{H}^*(F, K(i) \otimes \mathbf{Q}) = 0$. Il reste à voir que $\mathbb{H}^*(F, K(i) \otimes_L^L \mathbf{Q}_l/\mathbf{Z}_l) = 0$, c'est-à-dire que $H_B^q(F, \mathbf{Q}_l/\mathbf{Z}_l(i)) \rightarrow H_L^q(F, \mathbf{Q}_l/\mathbf{Z}_l(i))$ est bijectif pour $q \leq i$. Or:

12 **Lemme 12** Si $H90(n, l, F)$ est vrai, alors, pour tout $i \leq n$, le Bockstein

$$H^i(F, \mu_p^{\otimes i}) \rightarrow H^{i+1}(F, \mu_l^{\otimes i})$$

associé à la suite exacte (A_i) est nul.

En effet, la propriété (E) du complexe $Z(i)$ implique que ce Bockstein se factorise par le Bockstein

$$H^i(F, \mu_p^{\otimes i}) \rightarrow H_B^{i+1}(F, Z_{(l)}(i)) = 0$$

associé au triangle distingué

$$\alpha^* Z_{(l)}(i) \xrightarrow{l^\nu} \alpha^* Z_{(l)}(i) \rightarrow \alpha^* Z[l^\nu](i) \rightarrow \alpha^* Z_{(l)}(i)[1] .$$

La conclusion résulte maintenant de [43, prop. 7.1 et th. 5.9] (voir l'exposé de Friedlander, prop. 40). $\square$

13 **Remarque 13** L'hypothèse de caractéristique 0 intervient dans la démonstration de [43, th. 5.9], qui utilise la résolution des singularités.

14 **Corollaire 14** $H90(n, l) \Rightarrow K(n, l)$.

Vu les propriétés (E) et (G) de $Z(n)$, il suffit d'appliquer le foncteur $C \mapsto \mathbb{H}_{\text{Zar}}^n(F, C \otimes^L Z/l)$ au morphisme (B_n) et de tenir compte du corollaire 7. $\square$

Dans la section suivante, on aura également besoin du

15 **Corollaire 15:** (*théorème 90 de Hilbert pour K_n^M , [49, Cor. 2.14]*) Supposons que $H90(n, l)$ soit vrai. Soient F un corps de caractéristique 0, E/F une extension cyclique de degré l^ν ($\nu \geq 1$) et σ un générateur de son groupe de Galois. Alors la suite

$$K_n^M(E) \xrightarrow{1-\sigma} K_n^M(E) \xrightarrow{N_{E/F}} K_n^M(F) \quad (\text{B.7})$$

est exacte.

Démonstration Soit $G = \text{Gal}(E/F)$. On a une suite exacte de G_F -modules

$$0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}[G] \xrightarrow{1-\sigma} \mathbf{Z}[G] \rightarrow \mathbf{Z} \rightarrow 0$$

que l'on considère comme un complexe K de faisceaux sur le petit site étale de $\text{Spec } F$. On a donc

$$\text{Ext}_{F,\text{ét}}^q(K, \alpha^* \mathbf{Z}_{(l)}(n-1)) = 0 \quad \text{pour tout } q \in \mathbf{Z}.$$

Notons que $\text{Ext}_{F,\text{ét}}^q(\mathbf{Z}, \alpha^* \mathbf{Z}_{(l)}(n)) = H_L^q(F, \mathbf{Z}_{(l)}(n))$ et que $\text{Ext}_{F,\text{ét}}^q(\mathbf{Z}[G], \alpha^* \mathbf{Z}_{(l)}(n)) = H_L^q(E, \mathbf{Z}_{(l)}(n))$. D'après le théorème 11, ces groupes coïncident respectivement avec $H_B^q(F, \mathbf{Z}_{(l)}(n))$ et $H_B^q(E, \mathbf{Z}_{(l)}(n))$ pour $q \leq n+1$. En utilisant une suite spectrale d'hypercohomologie convergeant vers $\text{Ext}_{F,\text{ét}}^*(K, \alpha^* \mathbf{Z}_{(l)}(n))$ et la propriété (G) de $\mathbf{Z}(n)$, on en déduit que la suite (B.7) est exacte après tensorisation par $\mathbf{Z}_{(l)}$. Mais l'homologie de (B.7) est de l^v -torsion, en vertu de la formule $N_{E/F}(x)_E = \sum_{k=0}^{v-1} \sigma^k x$; le corollaire 15 en résulte. $\square$

Remarque 16 Dans le cas $l = 2$, Merkurjev a démontré indépendamment que la propriété du lemme 12 pour $v = 1$ entraîne la conjecture de Milnor, sans utiliser la résolution des singularités [20].

Vu le corollaire 14, le théorème 3 résulte maintenant du théorème suivant:

Théorème 17: ([49, th. 4.1]) $H90(n, 2)$ est vrai pour tout $n \geq 0$.

Corps dont la K -théorie de Milnor est divisible

Le but de cette section est de démontrer:

Théorème 18 Soit l un nombre premier, et soit F un corps de caractéristique 0, sans extensions finies de degré premier à l , tel que $K_n^M(F) = lK_n^M(F)$. Alors $H90(n-1, l) \Rightarrow H90(n, l, F)$.

Démonstration On a besoin de quelques lemmes:

Lemme 19: (cf. [49, lemma 2.20]) Supposons $H90(n-1, l)$ vrai. Soit F un corps de caractéristique 0, sans extensions de degré premier à l . Soit E/F une extension cyclique de degré l telle que la norme $K_{n-1}^M(E) \xrightarrow{N_{E/F}} K_{n-1}^M(F)$ soit surjective. Alors la suite

$$K_n^M(E) \xrightarrow{1-\sigma} K_n^M(E) \xrightarrow{N_{E/F}} K_n^M(F) \rightarrow 0$$

est exacte.

Démonstration L'exactitude en $K_n^M(F)$ résulte facilement de l'hypothèse. Pour démontrer l'exactitude en $K_n^M(E)$, on définit un homomorphisme

$$K_n^M(F) \xrightarrow{\varphi} K_n^M(E)/(1-\sigma)K_n^M(E)$$

par la formule $\varphi(\{a_1, \dots, a_n\}) = b \cdot \{a_n\}$ où $b \in K_{n-1}^M(E)$ est tel que $N_{E/F}(b) = \{a_1, \dots, a_{n-1}\}$. Le corollaire 15 implique que

$$b \cdot \{a_n\} \in K_n^M(E)/(1-\sigma)K_n^M(E)$$

ne dépend pas du choix de b . Pour voir que φ est bien défini, il faut vérifier que $b \cdot \{a_n\}$ dépend multilinéairement de $(a_1, \dots, a_n)$, ce qui est immédiat, et que cet élément est nul si $a_1 + a_n = 1$. Pour simplifier, supposons $a_1 \notin F^{*l}$ (l'autre cas est plus facile), et soit $K = F(\sqrt[l]{a_1})$. Soit $c \in K^*$ tel que $c^l = a_1$. Notons que

$$N_{KE/K}(b_{KE}) = N_{E/F}(b)_K = \{a_1, \dots, a_{n-1}\}_K = l\{c, a_2, \dots, a_{n-1}\}$$

donc que $N_{KE/K}(b_{KE} - \{c, a_2, \dots, a_{n-1}\}) = 0$; en appliquant de nouveau le corollaire 15, on obtient un élément $d \in K_{n-1}^M(KE)$ tel que $(1-\sigma)d = b_{KE} - \{c, a_2, \dots, a_{n-1}\}$. Notons aussi que $1 - a_1 = N_{K/F}(1 - c)$. On a alors:

$$\begin{aligned} b \cdot \{a_n\} &= b \cdot \{1 - a_1\} = N_{KE/E}(b_{KE} \cdot \{1 - c\}) \\ &= N_{KE/E}((b_{KE} - \{c, a_2, \dots, a_{n-1}\}) \cdot \{1 - c\}) \\ &= N_{KE/E}((1 - \sigma)d \cdot \{1 - c\}) \\ &= (1 - \sigma)N_{KE/E}(d \cdot \{1 - c\}) \\ &\in (1 - \sigma)K_n^M(E). \end{aligned}$$

Il est clair que φ est une section de l'homomorphisme $K_n^M(E)/(1-\sigma)K_n^M(E) \xrightarrow{\nu} K_n^M(F)$ induit par la norme. Reste à voir qu'il est surjectif. Or, d'après Bass-Tate [5, cor. 5.3], $K_n^M(E)$ est engendré par les symboles de la forme $\{b, a_2, \dots, a_n\}$ avec $b \in E^*$ et $a_2, \dots, a_n \in F^*$. On vérifie facilement sur ces symboles que $\varphi \circ \nu$ est l'identité. $\square$

20

Lemme 20: (cf. [49, lemma 2.17]) Soit F un corps de caractéristique 0, sans extensions de degré premier à l . Supposons $H90(n-1, l)$ vrai. Alors, pour toute extension cyclique E/F de degré l , la suite

$$H^{n-1}(E, \mathbf{Z}/l) \xrightarrow{N_{E/F}} H^{n-1}(F, \mathbf{Z}/l) \xrightarrow{\cup \chi} H^n(F, \mathbf{Z}/l) \rightarrow H^n(E, \mathbf{Z}/l),$$

où $\chi \in H^1(F, \mathbf{Z}/l)$ est un caractère définissant E , est exacte.

Nous renvoyons à [49] pour la démonstration: en effet, pour $l = 2$, ce résultat est vrai sans l'hypothèse $H90(n-1, l)$ (ni d'ailleurs celle que F n'ait pas d'extensions premières à l), cf. par exemple [1, Cor. 4.6].

Lemme 21: ([49, lemma 2.22]) Sous l'hypothèse du théorème 18, on a $K_n^M(E) = lK_n^M(E)$ pour tout extension finie E/F .

21

Démonstration Il suffit de traiter le cas où E/F est cyclique de degré l . Montrons d'abord que la norme $K_{n-1}^M(E) \xrightarrow{N_{E/F}} K_{n-1}^M(F)$ est surjective: comme son conoyau est de l -torsion, cela résulte de la surjectivité de $K_{n-1}^M(E)/l \xrightarrow{N_{E/F}} K_{n-1}^M(F)/l$. Comme F n'a pas d'extensions de degré premier à l , il contient une racine primitive l -ième de l'unité dont le choix identifie le module galoisien μ_l à $\mathbf{Z}/l$; de plus, on a $E = F(\sqrt[l]{a})$ pour un $a \in F^*$ convenable. On a alors un diagramme commutatif

$$\begin{array}{ccccccc}
 K_{n-1}^M(E)/l & \xrightarrow{N_{E/F}} & K_{n-1}^M(F)/l & \xrightarrow{\cdot\{a\}} & K_n^M(F)/l = 0 & \longrightarrow & K_n^M(E)/l \\
 u_{n-1,l} \downarrow & & u_{n-1,l} \downarrow & & u_{n,l} \downarrow & & u_{n,l} \downarrow \\
 H^{n-1}(E, \mathbf{Z}/l) & \xrightarrow{N_{E/F}} & H^{n-1}(F, \mathbf{Z}/l) & \xrightarrow{\cup(a)} & H^n(F, \mathbf{Z}/l) & \longrightarrow & H^n(E, \mathbf{Z}/l)
 \end{array}
 \tag{B.8}$$

où les deux flèches verticales de gauche sont des isomorphismes par le corollaire 14 et dont la ligne inférieure est exacte par le lemme 20. La surjectivité en résulte.

Soit σ un générateur de $\text{Gal}(E/F)$. L'égalité $K_n^M(F) = lK_n^M(F)$ et le lemme 19 entraînent facilement que l'endomorphisme $1 - \sigma$ de $K_n^M(E)/l$ est surjectif. La conclusion en résulte, puisque $(1 - \sigma)^l = 0$. $\square$

Démonstration du théorème 18.

Montrons que $H^n(F, \mathbf{Z}/l) = 0$: c'est suffisant vu la proposition 9 c) et la propriété (E) de $\mathbf{Z}(n)$. Soit $\alpha \in H^n(F, \mathbf{Z}/l)$. Il existe une extension finie galoisienne E/F telle que $\alpha_E = 0$. Grâce au lemme 21, on peut supposer par récurrence sur $[E : F]$ que E/F est cyclique de degré l . Soit $E = F(\sqrt[l]{a})$ pour $a \in F^*$. En réutilisant le diagramme (B.8), on voit facilement que $\alpha = 0$. $\square$

Variétés de déploiement

B.5

Corps de déploiement

B.5.1

Définition 22 Soient F un corps, $n > 0$ et $x \in K_n^M(F)/l$. On dit qu'une extension K/F est un *corps de déploiement* (resp. un *corps de déploiement générique*) pour x si $x_K = 0$ (resp. si, pour toute extension E/F , $x_E = 0 \iff$ il existe une F -place de

22

K vers E). On dit qu'une F -variété intègre X est une *variété de déploiement* (resp. une *variété de déploiement générique*) pour x si $F(X)$ est un corps de déploiement (resp. un corps de déploiement générique) pour x .

23

Remarque 23 Si la variété X est de plus *propre*, la condition de généricité se traduit sous la forme suivante: pour toute extension E/F , $x_E = 0$ si et seulement si $X \otimes_F E$ a un point rationnel. Cela résulte du critère valuatif de propreté. Si Y est une autre variété de déploiement pour x , il existe donc un F -morphisme d'un ouvert de Y vers X . La pertinence de cette notion apparaîtra dans la section B.10.2.

Exemples 24.

1. F_ζ est un corps de déploiement pour tout x : en effet, la K -théorie de Milnor de F_ζ est l -divisible. Cela prouve l'existence de corps de déploiement (et même de corps de déploiement de degré fini sur F).
2. Pour la démonstration du théorème 17, on utilisera des corps de déploiement *génériques* dans le cas où x est un symbole. En voici des exemples:
 - a) $n = 2$. Supposons $\mu_l \subset F$ et choisissons une racine primitive l -ième de l'unité ζ . Pour $a, b \in F^*$, l'algèbre centrale simple $A = \left(\begin{smallmatrix} a & b \\ & F \end{smallmatrix}\right)_\zeta$ admet une *variété de Severi-Brauer* X : c'est une F -variété projective, lisse, géométriquement intègre, isomorphe à $\mathbf{P}^{l-1}$ si et seulement si A n'est pas à division ([8], [2]). On montre que X est une variété de déploiement générique pour $\{a, b\} \in K_2^M(F)/l$ (Bass-Tate, [24, th. 15.7 et 15.12]).
 - b) $n = 3$. Avec les mêmes hypothèses et notations que ci-dessus, soit c un troisième élément de F^* . Notons U la variété affine d'équation $\text{Nrd}_A(x) = c$, où Nrd_A est la norme réduite associée à A : c'est une "forme tordue" de SL_l . Il résulte de [21, th. 12.1] que U est une variété de déploiement générique pour $\{a, b, c\} \in K_3^M(F)/l$. Une complétion projective de U est donnée par $X = \{[x, y, t] \in \mathbf{P}(A \oplus A \oplus F) \mid xy = t^2, x^* = yt^{l-2}c, y^* = xt^{l-2}c^{-1}\}$, où $x \mapsto x^* \in A$ est une fonction polynomiale (bien définie!) telle que $xx^* = \text{Nrd}_A(x)$, l'immersion ouverte $U \rightarrow X$ étant donnée par $x \mapsto [x, x^{-1}, 1]$ (Rost). La variété X n'est toutefois lisse que pour $l = 3$.
 - c) $l = 2$. Pour $\underline{a} = (a_1, \dots, a_n) \in (F^*)^n$ la quadrique projective $X_{\underline{a}}$ définie par la n -forme de Pfister $\varphi = \ll a_1, \dots, a_n \gg$ est une variété de déploiement générique pour $\{a_1, \dots, a_n\} \in K_n^M(F)/2$ [11, cor. 3.3]. *Variante*: on remplace φ par une de ses voisines (sous-forme de dimension $> 2^{n-1}$) [18, ex. 4.1].

Pour $x \in K_n^M(F)/l$, notons $D(x)$ la propriété suivante:

$D(x)$ Il existe un corps de déploiement K pour x , de type fini sur F et tel que $H_L^{n+1}(F, \mathbf{Z}_l(n)) \rightarrow H_L^{n+1}(K, \mathbf{Z}_l(n))$ soit injectif.

Proposition 25 Supposons $H90(n-1, l)$ vrai. Supposons de plus que, pour tout corps E de caractéristique 0 et tout $(a_1, \dots, a_n) \in (E^*)^n$, $D(\{a_1, \dots, a_n\})$ soit vrai. Alors $H90(n, l)$ est vrai.

25

Démonstration Par l'absurde.² Soit $\alpha \in H_L^{n+1}(F, \mathbf{Z}_{(l)}(n)) - \{0\}$. Choisissons un domaine universel pour F , c'est-à-dire une extension $\tilde{F}/F$, algébriquement close et de degré de transcendance infini. D'après la proposition 9 b), l'ensemble des sous-extensions K/F telles que $\alpha_K \neq 0$ est inductif; il a donc un élément maximal E . Ce corps E n'a pas d'extensions finies de degré premier à l (argument de transfert). D'après le théorème 18, on a donc $K_n^M(E)/l \neq 0$. Soit $x = \{a_1, \dots, a_n\} \in K_n^M(E)/l - \{0\}$. En appliquant $D(x)$, on trouve une extension K/E de type fini, telle que $x_K = 0$ (donc $K \neq E$) et $\alpha_K \neq 0$. Comme K/E est de type fini, K se plonge dans $\tilde{F}$, ce qui contredit la maximalité de E . $\square$

Variétés de déploiement

B.5.2

Pour toute F -variété intègre X , notons $\check{C}(X)$ le schéma simplicial tel que $\check{C}(X)_n = X^{n+1}$, les faces et dégénérescences étant données par les projections et diagonales partielles. On a une chaîne de morphismes de schémas simpliciaux

$$\mathrm{Spec} F(X) \rightarrow X \rightarrow \check{C}(X) \rightarrow \mathrm{Spec} F$$

où les objets autres que $\check{C}(X)$ sont considérés comme des schémas simpliciaux constants.

Lemme 26 a) Si X a un point rationnel, les homomorphismes

26

$$H_B^*(F, \mathbf{Z}(n)) \rightarrow H_B^*(\check{C}(X), \mathbf{Z}(n))$$

sont des isomorphismes.

b) Les homomorphismes

$$H_L^*(F, \mathbf{Z}(n)) \rightarrow H_L^*(\check{C}(X), \mathbf{Z}(n))$$

sont des isomorphismes.

Démonstration a) C'est classique: le choix d'un point rationnel de X définit une rétraction r de l'application naturelle

$$H_B^*(F, \mathbf{Z}(n)) \xrightarrow{\alpha} H_B^*(\check{C}(X), \mathbf{Z}(n)) .$$

² As Deligne kindly pointed out, this argument should be replaced by the original transfinite argument due to Merkurjev.

Pour prouver que $\alpha \circ r$ est l'identité, on construit comme d'habitude une homotopie de l'identité à l'application correspondant à $\alpha \circ r$ sur un complexe calculant $H_B^*(\check{C}(X), \mathbf{Z}(n))$.

b) C'est clair par le même raisonnement qu'en a) si X a un point rationnel, par exemple si F est algébriquement clos. En général, cela résulte de la comparaison des suites spectrales convergentes

$$H^p(F, \mathbb{H}_{\text{ét}}^q(F_s, K)) \Rightarrow \mathbb{H}_{\text{ét}}^{p+q}(F, K)$$

$$H^p(F, \mathbb{H}_{\text{ét}}^q(\check{C}(X_s), K)) \Rightarrow \mathbb{H}_{\text{ét}}^{p+q}(\check{C}(X), K)$$

où $X_s = X \otimes_F F_s$ et $K = \mathbf{Q}(n)$ ou $\mathbf{Q}/\mathbf{Z}(n)$, cf. la démonstration de la proposition 9 b). $\square$

27 Remarque 27 Une démonstration du lemme 26 a) plus naturelle d'un point de vue homotopique pourra être obtenue à partir de l'exemple 38 et du théorème 46 ci-dessous.

28 Définition 28 Soit $x \in K_n^M(F)/l$. Une variété de déploiement X pour x est *bonne* si les conditions suivantes sont vérifiées:

- (i) X est lisse.
- (ii) $X_{F(X)}$ est rétracte rationnelle.
- (iii) $H_B^{n+1}(\check{C}(X), \mathbf{Z}(l)(n)) = 0$.

Rappelons qu'une F -variété intègre X est *rétracte rationnelle* s'il existe un ouvert non vide $U \subset X$ tel que Id_U se factorise par un ouvert d'un espace affine. Cette notion est due à D. Saltman [38].

Exemples 29.

1. Soit X une *variété projective homogène* sur F : il existe donc un groupe semi-simple G , défini sur F , tel que $X \otimes_F F_s$ soit F_s -isomorphe à $G \otimes_F F_s/P$ pour un F_s -sous-groupe parabolique P convenable de $G \otimes_F F_s$ [9, prop. 4]. Alors X vérifie les hypothèses (i) et (ii) de la définition 28. Pour (i), c'est classique; pour (ii) on utilise la décomposition de Bruhat généralisée qui montre que $X_{F(X)}$ est même $F(X)$ -rationnelle [3, th. 21.20] (je remercie Philippe Gille de m'avoir indiqué cette référence). Ceci s'applique aux exemples 24 (2) (a) et (c).
2. La variété U de l'exemple 24 (2) (b) vérifie également les hypothèses (i) et (ii) de la définition 28: pour (ii), on remarque que si U a un point rationnel, on peut se ramener à $c = 1$ par multiplicativité de la norme réduite. Il faut donc montrer que $SL_{1,A}$ est rétracte rationnelle. Comme l'indice de A est premier, le théorème de Wang [51] implique que tout élément de norme réduite 1 est produit de commutateurs. En appliquant ceci au point générique η de $SL_{1,A}$, on obtient une factorisation

$$\begin{array}{ccc}
 \eta & \hookrightarrow & SL_{1,A} \\
 & \searrow & \nearrow \\
 & (GL_{1,A})^{2m} &
 \end{array}$$

pour $m \geq 1$ convenable. Cette factorisation s'étend en un triangle commutatif

$$\begin{array}{ccc}
 V & \hookrightarrow & SL_{1,A} \\
 & \searrow & \nearrow \\
 & (GL_{1,A})^{2m} &
 \end{array}$$

où V est un ouvert convenable de $SL_{1,A}$, ce qui entraîne facilement la propriété cherchée. (Je remercie Jean-Louis Colliot-Thélène de m'avoir expliqué cette démonstration.)

Théorème 30: (cf. [49, th. 2.25]) Supposons que $H90(n-1, l)$ soit vrai. Soit $x \in K_n^M(F)/l$; supposons que x admette une bonne variété de déploiement. Alors $D(x)$ est vrai.

30

Démonstration On a encore besoin de quelques lemmes:

Lemme 31: ([49, th. 2.15]) Supposons que $H90(n-1, l)$ soit vrai; soit $K(n)$ le cône du morphisme (B_n) ci-dessus, localisé en l . Alors $X \mapsto \mathbb{H}^*(X, K(n))$ est un invariant birationnel lorsque X décrit les F -variétés lisses et intègres.

31

Démonstration Il faut montrer que, pour tout ouvert non vide $U \subset X$, $\mathbb{H}^*(X, K(n)) \rightarrow \mathbb{H}^*(U, K(n))$ est bijectif. Soit $Z = X - U$. Par récurrence sur $\dim Z$, on se ramène au cas où Z est lisse (considérer son lieu singulier). De la pureté de la cohomologie motivique [43, prop. 2.4] et de la cohomologie étale à coefficients racines de l'unité tordues, on déduit alors que

$$\mathbb{H}_Z^q(X, K(n)) \simeq \mathbb{H}^{q-2c}(Z, K(n-c))$$

où $c = \text{codim}_X(Z)$. La conclusion résulte maintenant du théorème 11. $\square$

Lemme 32 Avec les hypothèses et notations du lemme 31, soit $Y \xrightarrow{f} X$ un morphisme dominant de F -variétés lisses et intègres, dont la fibre générique est une variété rétracte rationnelle. Alors $\mathbb{H}^*(X, K(n)) \xrightarrow{f^*} \mathbb{H}^*(Y, K(n))$ est un isomorphisme.

32

Démonstration Grâce au lemme 31, on se ramène au cas où X est un corps. En réutilisant si besoin est le lemme 31, le lemme 32 résulte alors de l'invariance par homotopie de la cohomologie motivique (propriété (F) de $\mathbf{Z}(n)$) et de la cohomologie étale à coefficients racines de l'unité tordues. $\square$

33

Lemme 33 Supposons $H90(n-1, l)$ vrai. Soit X une F -variété intègre vérifiant les propriétés (i) et (ii) de la définition 28. Alors,

a) Les homomorphismes

$$\mathbb{H}^*(\check{C}(X), K(n)) \rightarrow \mathbb{H}^*(X, K(n)) \rightarrow H^*(F(X), K(n))$$

sont des isomorphismes.

b) On a une suite exacte

$$H_B^{n+1}(\check{C}(X), \mathbf{Z}_{(l)}(n)) \rightarrow H_L^{n+1}(F, \mathbf{Z}_{(l)}(n)) \rightarrow H_L^{n+1}(F(X), \mathbf{Z}_{(l)}(n)) .$$

Démonstration a) C'est clair pour l'homomorphisme de droite, en vertu du lemme 31. Si ∂ est une face de $\check{C}(X)_{r+1}$ vers $\check{C}(X)_r$, il résulte du lemme 32 que l'application induite par ∂ en $K(n)$ -hypercohomologie est un isomorphisme. Pour tout $r \geq 0$, soit $\check{C}(X)^{(r)}$ le r -ième squelette de $\check{C}(X)$. D'après la remarque ci-dessus, les différentielles d_1 sont alternativement nulles et bijectives dans la suite spectrale (B.5) associée à $\check{C}(X)^{(2r)}$. Il en résulte que cette suite spectrale dégénère et induit des isomorphismes

$$\mathbb{H}^*(\check{C}(X)^{(2r)}, K(n)) \xrightarrow{\sim} \mathbb{H}^*(X, K(n)) , \quad r \geq 0 .$$

En particulier, les systèmes projectifs $(\mathbb{H}^*(\check{C}(X)^{(2r)}, K(n)))_{r \geq 0}$ sont constants, et il résulte des suites exactes (B.6) que les homomorphismes

$$\mathbb{H}^*(\check{C}(X), K(n)) \rightarrow \mathbb{H}^*(\check{C}(X)^{(2r)}, K(n))$$

sont des isomorphismes pour tout $r \geq 0$.

b) Cela résulte de a) et du diagramme commutatif aux lignes exactes

$$\begin{array}{ccccc} 0 = H_B^{n+1}(F(X), \mathbf{Z}_{(l)}(n)) & \longrightarrow & H_L^{n+1}(F(X), \mathbf{Z}_{(l)}(n)) & \longrightarrow & \mathbb{H}^{n+1}(F(X), K(n)) \\ & \uparrow & \uparrow & & \wr \uparrow \\ H_B^{n+1}(\check{C}(X), \mathbf{Z}_{(l)}(n)) & \longrightarrow & H_L^{n+1}(\check{C}(X), \mathbf{Z}_{(l)}(n)) & \longrightarrow & \mathbb{H}^{n+1}(\check{C}(X), K(n)) \\ & & \wr \uparrow & & \\ & & H_L^{n+1}(F, \mathbf{Z}_{(l)}(n)) & & \end{array}$$

où l'isomorphisme du haut résulte de a), et celui du bas du lemme 26 b). $\square$

Le théorème 30 résulte immédiatement de la définition 28 et du lemme 33 b). $\square$

Vu la proposition 25 et le théorème 30, le théorème 17 résulte maintenant du

Théorème 34: (cf. [49, prop. 4.10]) Supposons $H90(n-1, 2)$ vrai. Soient $\underline{a} = (a_1, \dots, a_n) \in (F^*)^n$ et $Q_{\underline{a}}$ la quadrique projective de dimension $2^{n-1} - 1$ définie par la forme quadratique

34

$$\ll a_1, \dots, a_{n-1} \gg \perp \langle -a_n \rangle.$$

Alors $Q_{\underline{a}}$ est une bonne variété de déploiement pour $\{a_1, \dots, a_n\} \in K_n^M(F)/2$.

Remarque 35 Soient X, Y deux F -variétés lisses et intègres qui sont *stablement équivalentes*, par exemple $X_{F(Y)}$ est $F(Y)$ -rationnelle et $Y_{F(X)}$ est $F(X)$ -rationnelle. Alors, pour $x \in K_n^M(F)/l$, X est une bonne variété de déploiement pour x si et seulement si Y l'est. Cela résulte facilement du lemme 33 b) (voir aussi exemple 38).

35

Dans l'énoncé du théorème 34, on pourrait donc remplacer la quadrique $Q_{\underline{a}}$ par la quadrique $X_{\underline{a}}$ associée à la n -forme de Pfister $\ll a_1, \dots, a_n \gg$. Toutefois, l'existence du modèle $Q_{\underline{a}}$ est cruciale pour la démonstration de Voevodsky, comme le montre l'énoncé du théorème 36 ci-dessous.

Le théorème 34 résulte formellement de la conjonction des deux énoncés suivants, le premier de nature "topologique", le deuxième de nature "arithmétique":

Théorème 36 Supposons $H90(n-1, l)$ vrai. Soient F un sous-corps de $\mathbb{C}$ et X une F -variété projective lisse de dimension $d = l^{n-1} - 1$ telle que $s_d(X(\mathbb{C})) \not\equiv 0 \pmod{l^2}$, où $s_d(X(\mathbb{C}))$ est le nombre de Chern de $X(\mathbb{C})$ associé au d -ième polynôme de Newton (cf. [25, §16]). Alors il existe une injection

36

$$H_B^{n+1}(\check{C}(X), \mathbf{Z}_{(l)}(n)) \xrightarrow{\alpha} H_B^{2^{\frac{l^{n-1}-1}{l-1}+1}}\left(\check{C}(X), \mathbf{Z}_{(l)}\left(\frac{l^{n-1}-1}{l-1} + 1\right)\right).$$

Le théorème 36 sera démontré dans les sections B.7 et B.8.3.

Théorème 37 Supposons $H90(n-1, 2)$ vrai. Soient F et $\underline{a}, Q_{\underline{a}}$ comme dans le théorème 34. Alors $s_d(Q_{\underline{a}}(\mathbb{C})) \not\equiv 0 \pmod{4}$ et $H_B^{2^{n-1}-1}(\check{C}(Q_{\underline{a}}), \mathbf{Z}(2^{n-1})) = 0$.

37

Le théorème 37 sera démontré dans la section B.9. Notons tout de suite que sa première conclusion résulte d'un calcul élémentaire (on trouve $s_d(Q_{\underline{a}}(\mathbb{C})) = 2(2^{2^{n-1}-1} - 2^{n-1} - 1)$, cf. [25, problem 16-D]).

Pour la démonstration du théorème 36, Voevodsky utilise des opérations de Steenrod en cohomologie motivique. Pour les définir, il faut introduire la *catégorie homotopique stable des schémas*: c'est fait dans la prochaine section.

B.6

Homotopie des variétés algébriques

B.6.1

Topologie de Nisnevich

Soit X un schéma. Un *recouvrement de Nisnevich* de X est une famille $(U_i \xrightarrow{f_i} X)_{i \in I}$ de morphismes étales telle que, pour tout $x \in X$, il existe $i \in I$ et $u \in f_i^{-1}(x)$ tel que $\kappa(x) \rightarrow \kappa(u)$ soit un isomorphisme. Les recouvrements de Nisnevich définissent une topologie de Grothendieck sur la catégorie Sm/F des F -schémas lisses: la *topologie de Nisnevich* [28]. Les anneaux locaux de cette topologie sont les anneaux locaux henséliens.

B.6.2

Catégorie homotopique

Soient $Shv_{Nis}(Sm/F)$ le topos associé (faisceaux d'ensembles) et $\Delta^{op}Shv_{Nis}(Sm/F)$ la catégorie des objets simpliciaux de ce topos. On identifiera, sans plus de commentaires, les objets suivants à des objets de $\Delta^{op}Shv_{Nis}(Sm/F)$: ensembles simpliciaux (faisceaux constants), F -schémas simpliciaux (faisceaux représentables), faisceaux d'ensembles, F -schémas.

Pour $U \in Sm/F$ et $u \in U$, notons $\mathcal{O}_{U,u}^h$ le hensélisé de l'anneau local de U en u . Alors $\text{Spec } \mathcal{O}_{U,u}^h$ est limite projective de F -schémas lisses U_α ; pour tout $\mathcal{X} \in \Delta^{op}Shv_{Nis}(Sm/F)$, on définit sa fibre $\mathcal{X}_u$ en u comme la limite inductive des ensembles simpliciaux $\mathcal{X}(U_\alpha)$. On dit qu'un morphisme $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ de $\Delta^{op}Shv_{Nis}(Sm/F)$ est une *équivalence faible simpliciale* si, pour tout $U \in Sm/F$ et tout $u \in U$, $\varphi_u = \mathcal{X}_u \rightarrow \mathcal{Y}_u$ est une équivalence faible d'ensembles simpliciaux.

Exemple 38. (cf. [49, lemma 3.8]) Soit $f : X \rightarrow S$ un morphisme de Sm/F . Considérons le schéma simplicial $\check{C}_S(X)$ tel que $\check{C}_S(X)_n = \underbrace{X \times_S \cdots \times_S X}_{n+1}$,

les faces et dégénérescences étant données par les projections et diagonales partielles (pour $S = \text{Spec } F$, on retrouve l'objet $\check{C}(X)$ considéré ci-dessus). Si f_s a une section pour tout $s \in S$, la projection $\check{C}_S(X) \rightarrow S$ est une équivalence faible simpliciale: c'est évident. En particulier, supposons $X = Y \times_F S$ pour un F -schéma lisse Y ; alors, si $\text{Hom}_F(S, Y) \neq \emptyset$, la projection $\check{C}(Y) \times_F S \rightarrow S$ est une équivalence faible simpliciale.

Prenons par exemple $S = \text{Spec } F$, et pour Y une variété de déploiement générique pour un élément $x \in K_n^M(F)/I$ (cf. définition 22). Supposons Y propre. Alors le faisceau simplicial $\check{C}(Y)$ est faiblement simplicialement équivalent au faisceau d'ensembles Φ_x défini par

$$\Phi_x(U) = \begin{cases} \emptyset & \text{si } x_{F(U)} \neq 0 \\ pt & \text{si } x_{F(U)} = 0 \end{cases}$$

où U décrit les F -schémas lisses intègres et pt désigne un ensemble à 1 élément: cela résulte de la remarque 23. Ceci montre que l'objet $\check{C}(Y)$, vu à homotopie près, ne dépend que de x .

Notons $\mathcal{H}_s(\Delta^{op} Shv_{Nis}(Sm/F))$ la localisation de $\Delta^{op} Shv_{Nis}(Sm/F)$ par rapport aux équivalences faibles simpliciales. On dit qu'un objet $\mathcal{X} \in \Delta^{op} Shv_{Nis}(Sm/F)$ est A^1 -local si, pour tout $\mathcal{Y} \in \Delta^{op} Shv_{Nis}(Sm/F)$, $Hom_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X}) \rightarrow Hom_{\mathcal{H}_s}(\mathcal{Y} \times A^1, \mathcal{X})$ est bijective, et qu'un morphisme $f : \mathcal{Y} \rightarrow \mathcal{Y}'$ de $\Delta^{op} Shv_{Nis}(Sm/F)$ est une A^1 -équivalence faible si pour tout objet A^1 -local $\mathcal{X}$, l'application correspondante

$$Hom_{\mathcal{H}_s}(\mathcal{Y}', \mathcal{X}) \xrightarrow{f^*} Hom_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X})$$

est bijective. Disons qu'un morphisme φ de $\Delta^{op} Shv_{Nis}(Sm/F)$ est une *cofibration* (resp. une *équivalence faible*) si φ est un monomorphisme (resp. une A^1 -équivalence faible). D'après [27], ceci munit $\Delta^{op} Shv_{Nis}(Sm/F)$ d'une structure de catégorie de modèles fermée au sens de Quillen [30]. La catégorie homotopique correspondante $\mathcal{H}(F)$ est appelée *catégorie homotopique des F -schémas*.

On a une version pointée $\mathcal{H}_\bullet(F)$ de $\mathcal{H}(F)$, en partant de la catégorie $\Delta^\bullet Shv_{Nis}(Sm/F)$ des faisceaux d'ensembles simpliciaux pointés, et un foncteur

$$\mathcal{H}(F) \rightarrow \mathcal{H}_\bullet(F)$$

induit par le foncteur $\mathcal{X} \mapsto \mathcal{X}_+ = \mathcal{X} \coprod pt$. Notons que le faisceau simplicial pointé constant réduit à un point est représenté par $\text{Spec } F$. Si $\mathcal{X}, \mathcal{Y}$ sont deux faisceaux simpliciaux pointés, on définit leur *smash produit* $\mathcal{X} \wedge \mathcal{Y}$ comme étant le faisceau associé au préfaisceau $U \mapsto \mathcal{X}(U) \wedge \mathcal{Y}(U)$. Ceci munit $\Delta^\bullet Shv_{Nis}(Sm/F)$ et $\mathcal{H}_\bullet$ d'une structure monoïdale symétrique, l'objet unité étant S^0 (faisceau constant, que l'on peut décrire comme $(\text{Spec } F)_+$).

Deux cercles

B.6.3

On définit deux "cercles" $S_s^1, S_t^1 \in \Delta^\bullet Shv_{Nis}(Sm/F)$:

- S_s^1 est le cercle simplicial, vu comme faisceau constant.
- S_t^1 est le F -schéma $A_F^1 - \{0\}$ pointé par 1, vu comme faisceau représentable (constant comme objet simplicial).

On note également T le faisceau simplicial pointé donné par le carré cocartésien

$$\begin{array}{ccc} A_F^1 - \{0\} & \longrightarrow & A_F^1 \\ \downarrow & & \downarrow \\ \text{Spec } F & \longrightarrow & T. \end{array}$$

Dans $\mathcal{H}_\bullet(F)$, on a des isomorphismes $S_s^1 \wedge S_t^1 \approx T \approx (\mathbf{P}_F^1, 0)$.

Soit $f : \mathcal{X} \rightarrow \mathcal{Y}$ un morphisme de $\Delta_{\bullet}^{op} Shv_{Nis}(Sm/F)$. Le *cône* de f , $\text{cône}(f)$, est le faisceau associé au préfaisceau $U \mapsto \text{cône}(f(U))$. On a un morphisme canonique $\mathcal{Y} \rightarrow \text{cône}(f)$ qui s'étend comme d'habitude en une suite

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \rightarrow \text{cône}(f) \rightarrow S_s^1 \wedge \mathcal{X}$$

appelée *suite cofibrante* associée à f .

B.6.4 T -spectres

39 **Définition 39** Un T -spectre sur F est une suite $\mathbf{E} = (E_i, e_i : T \wedge E_i \rightarrow E_{i+1})_{i \in \mathbb{Z}}$, où $E_i \in \Delta_{\bullet}^{op} Shv_{Nis}(Sm/F)$ pour tout i . Soient $\mathbf{E} = (E_i, e_i)$, $\mathbf{F} = (F_i, f_i)$ deux T -spectres. Un morphisme $\varphi : \mathbf{E} \rightarrow \mathbf{F}$ est la donnée, pour tout i , d'un morphisme $\varphi_i : E_i \rightarrow F_i$, avec $\varphi_{i+1} \circ e_i = f_i \circ \varphi_i$.

On note $\text{Spect}_T(F)$ la catégorie des T -spectres sur F . En utilisant la structure de modèles fermée sur $\Delta_{\bullet}^{op} Shv_{Nis}(Sm/F)$, on définit comme dans [4] des structures de modèles fermées stable et stricte. On note $\mathcal{SH}(F)$ la catégorie homotopique associée à la structure stable: c'est la *catégorie homotopique stable des F -schémas*.

Soit $\mathcal{X} \in \Delta_{\bullet}^{op} Shv_{Nis}(Sm/F)$. On a le *spectre des suspensions* de $\mathcal{X}$:

$$\Sigma_T^{\infty} \mathcal{X} = (T^{\wedge i} \wedge \mathcal{X}, Id) .$$

Par abus de notation, on notera parfois $\mathcal{X}$ au lieu de $\Sigma_T^{\infty} \mathcal{X}$. Cette construction induit un foncteur $\mathcal{H}_{\bullet}(F) \rightarrow \mathcal{SH}(F)$.

40 **Théorème 40:** ([49, th. 3.10]) Il existe une structure de catégorie triangulée tensorielle sur $\mathcal{SH}(F)$ ayant les propriétés suivantes:

- (i) Le foncteur de décalage $\mathbf{E} \mapsto \mathbf{E}[1]$ est donné par $\mathbf{E}[1] = S_s^1 \wedge \mathbf{E}$.
- (ii) Le foncteur Σ_T^{∞} transforme suites cofibrantes en triangles distingués.
- (iii) Le foncteur Σ_T^{∞} est un foncteur monoïdal symétrique de $(\mathcal{H}_{\bullet}(F), \wedge)$ vers $(\mathcal{SH}(F), \wedge)$.
- (iv) L'objet T de $\mathcal{SH}(F)$ est inversible.

B.6.5 Théories cohomologiques et homologiques

Fixons des objets S_s^{-1}, S_t^{-1} de $\text{Spect}_T(F)$ et des isomorphismes

$$S_s^1 \wedge S_s^{-1} \cong S^0$$

$$S_t^1 \wedge S_t^{-1} \cong S^0$$

dans $\mathcal{SH}(F)$. Notons, pour $n \in \mathbb{Z}$,

$$S_s^n = \begin{cases} (S_s^1)^{\wedge n} & \text{pour } n \geq 0 \\ (S_s^{-1})^{\wedge (-n)} & \text{pour } n \leq 0 \end{cases} \quad S_t^n = \begin{cases} (S_t^1)^{\wedge n} & \text{pour } n \geq 0 \\ (S_t^{-1})^{\wedge (-n)} & \text{pour } n \leq 0 \end{cases}$$

et, pour $p, q \in \mathbb{Z}$

$$S^{q,p} = S_t^p \wedge S_s^{q-p}.$$

Pour $E \in \mathcal{SH}(F)$, on note $E(p)[q] = S^{q,p} \wedge E$.

Définition 41 Soit $E \in \mathcal{SH}(F)$. La *théorie cohomologique associée à E* est le foncteur

41

$$\widetilde{E}^{p,q} : \mathcal{SH}(F) \rightarrow (Ab)^{\mathbb{Z} \times \mathbb{Z}}$$

$$X \mapsto \text{Hom}_{\mathcal{SH}(F)}(X, E(q)[p]).$$

La *théorie homologique associée à E* est le foncteur

$$\widetilde{E}_{p,q}(X) : \mathcal{SH}(F) \rightarrow (Ab)^{\mathbb{Z} \times \mathbb{Z}}$$

$$X \mapsto \text{Hom}_{\mathcal{SH}(F)}(S^{q,p}, E \wedge X).$$

Si $\mathcal{X} \in \mathcal{H}(F)$, on note

$$E^{p,q}(\mathcal{X}) = \widetilde{E}^{p,q}(\Sigma_T^\infty(\mathcal{X}_+))$$

$$E_{p,q}(\mathcal{X}) = \widetilde{E}_{p,q}(\Sigma_T^\infty(\mathcal{X}_+)).$$

Spectres d'Eilenberg-Mac Lane et cohomologie motivique

B.6.6

Pour toute F -variété lisse X , notons $L(X)$ le faisceau pour la topologie de Nisnevich qui associe à un schéma lisse connexe U le groupe abélien libre engendré par les fermés irréductibles de $U \times_F X$ qui sont finis et surjectifs sur U (c'est le faisceau $c_{\text{equi}}(X, 0)$ de l'exposé de Friedlander, §2). Soit A un groupe abélien. Pour $n \geq 0$, on note

$$K(A(n), 2n)$$

le faisceau de groupes abéliens quotient $L(A^n)/L(A^n - \{0\}) \otimes A$, considéré comme faisceau d'ensembles pointés. On a des morphismes de faisceaux d'ensembles pointés

$$T \wedge K(A(n), 2n) \xrightarrow{e_n} K(A(n+1), 2n+2).$$

Pour $n < 0$, on pose $K(A(n), 2n) = \{*\}$.

42 Définition 42 Le spectre d'Eilenberg-Mac Lane $\mathbf{H}_A$ est le T -spectre $(K(A(n), 2n), e_n)$.

Pour $X \in \mathcal{SH}(F)$ (resp. $\mathcal{X} \in \mathcal{H}(F)$), on note $\tilde{H}^{p,q}(X, A) = \tilde{\mathbf{H}}_A^{p,q}(X)$ (resp. $H^{p,q}(\mathcal{X}, A) = \mathbf{H}_A^{p,q}(\mathcal{X})$) (cf. définition 41): c'est la cohomologie motivique de X (resp. de $\mathcal{X}$). Cette terminologie est justifiée par le

43 Théorème 43: ([27]) Soit F un corps de caractéristique 0, et soit $\mathcal{X}$ un F -schéma simplicial lisse. Alors, pour tout groupe abélien A , on a

$$H^{p,q}(\mathcal{X}, A) = H_B^p(\mathcal{X}, A(q)) .$$

Indications sur la démonstration

(d'après F. Morel). Il résulte de la quasi-invertibilité de $\mathbf{Z}(1)$ dans la catégorie triangulée $DM^{eff}(F)$ des F -motifs effectifs ([46, th. 4.3.1], voir aussi l'exposé de Friedlander, th. 5.7) que le spectre $\mathbf{H}_A$ est un Ω_T -spectre. Il suffit donc de montrer que, pour tous $m, n, i \geq 0$, l'ensemble des morphismes dans $\mathcal{H}_\bullet(F)$

$$[\Sigma_s^m \Sigma_t^n(\mathcal{X}_+), K(A(i), 2i)]$$

s'identifie naturellement au groupe $H_B^{2i-m-n, i-n}(\mathcal{X}, A)$. Cela résulte d'une adjonction essentiellement formelle. $\square$

44 Remarque 44 On peut montrer que le foncteur $A \mapsto \mathbf{H}_A$ se “prolonge” en un foncteur

$$\mathbf{H} : DM(F) \rightarrow \mathcal{SH}(F)$$

où $DM(F)$ est la catégorie triangulée des F -motifs, convenablement complétée, où l'on a inversé le motif de Tate. Ce foncteur a pour adjoint à gauche un foncteur

$$M : \mathcal{SH}(F) \rightarrow DM(F)$$

qui “prolonge” le foncteur “motif” $Sm/F \xrightarrow{M} DM(F)$ (cf. l'exposé de Friedlander, définition 3.1). Ce résultat généralise le théorème 43.

Opérations de Steenrod en cohomologie motivique

B.7

Définition 45 L'algèbre de Steenrod motivique modulo l sur F est l'algèbre $\mathcal{A}^{*,*}(F, \mathbb{Z}/l)$ des endomorphismes du T -spectre $\mathbf{H}_{\mathbb{Z}/l}$ dans $\mathcal{SH}(F)$.

45

Par définition, on a

$$\mathcal{A}^{p,q}(F, \mathbb{Z}/l) = \text{Hom}_{\mathcal{SH}(F)}(\mathbf{H}_{\mathbb{Z}/l}, \mathbf{H}_{\mathbb{Z}/l}(q)[p]) = \tilde{H}^{p,q}(\mathbf{H}_{\mathbb{Z}/l}, \mathbb{Z}/l) .$$

Théorème 46: ([49, th. 3.14], [48]) On a

46

- (i) $\mathcal{A}^{p,q}(F, \mathbb{Z}/l) = 0$ pour $q < 0$
- (ii) $\mathcal{A}^{0,0}(F, \mathbb{Z}/l) = \mathbb{Z}/l$, engendré par l'identité.

Théorème 47: ([49, th. 3.15], [48]) L'homomorphisme de Künneth

47

$$\tilde{H}^{*,*}(\mathbf{H}_{\mathbb{Z}/l}, \mathbb{Z}/l) \otimes_{\tilde{H}^{*,*}(S^0, \mathbb{Z}/l)} \tilde{H}^{*,*}(\mathbf{H}_{\mathbb{Z}/l}, \mathbb{Z}/l) \rightarrow \tilde{H}^{*,*}(\mathbf{H}_{\mathbb{Z}/l} \wedge \mathbf{H}_{\mathbb{Z}/l}, \mathbb{Z}/l)$$

est un isomorphisme.

On va avoir besoin d'opérations $Q_i \in \mathcal{A}^{2l^i-1, l^i-1}(F, \mathbb{Z}/l)$, analogues aux opérations de Milnor. Pour les définir, on procède comme en topologie algébrique: on définit des opérations $P^i \in \mathcal{A}^{2i(l-1), i(l-1)}(F, \mathbb{Z}/l)$ analogues aux puissances de Steenrod, et on définit inductivement

$$Q_0 = \beta$$

$$Q_{i+1} = [Q_i, P^{l^i}]$$

où β est le Bockstein modulo p . Pour cet exposé, les propriétés principales des Q_i sont:

Théorème 48: ([49, th. 3.17], [48])

48

- (i) $Q_i^2 = 0$.
- (ii) Pour tout $i > 0$, il existe des opérations q_i telles que $Q_i = [\beta, q_i]$.

Corollaire 49 Soient $X \in \mathcal{SH}(F)$ et $p, q \in \mathbb{Z}$. Pour tout $x \in \tilde{H}^{p,q}(X, \mathbb{Z}_{(l)})$ et tout $i > 0$, posons

49

$$\tilde{Q}_i(x) = \tilde{\beta} q_i(\bar{x})$$

où q_i est comme dans le théorème 48 (ii), $\tilde{\beta}$ est le Bockstein entier et $-$ désigne la réduction modulo l . Alors le diagramme

$$\begin{array}{ccc} \tilde{H}^{p,q}(X, \mathbf{Z}_{(l)}) & \xrightarrow{\tilde{Q}_i} & \tilde{H}^{p+2\tilde{l}^i-1, q+\tilde{l}^i-1}(X, \mathbf{Z}_{(l)}) \\ \downarrow & & \downarrow \\ \tilde{H}^{p,q}(X, \mathbf{Z}/l) & \xrightarrow{Q_i} & \tilde{H}^{p+2\tilde{l}^i-1, q+\tilde{l}^i-1}(X, \mathbf{Z}/l) \end{array}$$

est commutatif. $\square$

Vu la propriété (i) des Q_i , on a pour tout objet $\mathcal{X} \in \mathcal{SH}(F)$ des complexes

$$\begin{aligned} \dots \xrightarrow{Q_i} \tilde{H}^{p-2(\tilde{l}^i-1), q-\tilde{l}^i+1}(\mathcal{X}, \mathbf{Z}/l) \xrightarrow{Q_i} \tilde{H}^{p,q}(\mathcal{X}, \mathbf{Z}/l) \xrightarrow{Q_i} \\ \tilde{H}^{p+2(\tilde{l}^i-1), q+\tilde{l}^i-1}(\mathcal{X}, \mathbf{Z}/l) \xrightarrow{Q_i} \dots \quad (\text{B.9}) \end{aligned}$$

50

Théorème 50 Soient F et X comme dans l'énoncé du théorème 36. Alors les complexes (B.9) sont acycliques pour $i \leq n-1$ et $\mathcal{X} = \text{fibre}(\Sigma_T^\infty(\check{C}(X)_+) \rightarrow S^0)$.

Le théorème 50 sera démontré dans la section B.8.3. Déduisons-en tout de suite le théorème 36 avec $\alpha = \tilde{Q}_{n-2} \dots \tilde{Q}_1$, où les $\tilde{Q}_i$ sont les opérations cohomologiques du corollaire 49. D'après le théorème 43, l'algèbre de Steenrod motivique opère sur la cohomologie motivique de $\check{C}(X)$, de telle façon que l'on ait un diagramme commutatif:

$$\begin{array}{ccc} H_B^{n+1}(\check{C}(X), \mathbf{Z}_{(l)}(n)) & \xrightarrow{\tilde{Q}_{n-2} \dots \tilde{Q}_1} & H_B^{2\frac{n-1}{l-1}+1}(\check{C}(X), \mathbf{Z}_{(l)}(\frac{n-1}{l-1} + 1)) \\ \downarrow & & \downarrow \\ \tilde{H}^{n+1,n}(\mathcal{X}, \mathbf{Z}_{(l)}) & \xrightarrow{\tilde{Q}_{n-2} \dots \tilde{Q}_1} & H^{2\frac{n-1}{l-1}+1, \frac{n-1}{l-1}+1}(\mathcal{X}, \mathbf{Z}_{(l)}) \\ \downarrow & & \downarrow \\ \tilde{H}^{n+1,n}(\mathcal{X}, \mathbf{Z}/l) & \xrightarrow{Q_{n-2} \dots Q_1} & H^{2\frac{n-1}{l-1}+1, \frac{n-1}{l-1}+1}(\mathcal{X}, \mathbf{Z}/l) \end{array}$$

La propriété (B.3) des $\mathbf{Z}(i)$ implique que les flèches verticales supérieures sont des isomorphismes. Par ailleurs, le lemme 26 a) et un argument de transfert impliquent que $\tilde{H}^{*,*}(\mathcal{X}, \mathbf{Z}_{(l)})$ est d'exposant l ; les flèches verticales inférieures sont donc injectives. Par conséquent, pour démontrer le théorème 36, il suffit de prouver que la flèche horizontale inférieure est injective.

Le théorème 50 implique que, pour $1 \leq i \leq n-2$, la suite

$$\begin{aligned} \tilde{H}^{n+1-2l \frac{i-2i^{i-1}+1}{i-1} - i+2, n - \frac{i-2i^{i-1}+1}{i-1} - i+2}(\mathcal{X}, \mathbf{Z}/l) &\xrightarrow{Q_i} \\ \tilde{H}^{n+1+2l \frac{i-1}{i-1} - i+1, n+l \frac{i-1}{i-1} - i+1}(\mathcal{X}, \mathbf{Z}/l) &\xrightarrow{Q_i} \\ \tilde{H}^{n+1+2l \frac{i-1}{i-1} - i, n+l \frac{i-1}{i-1} - i}(\mathcal{X}, \mathbf{Z}/l) \end{aligned}$$

est exacte. Mais $\tilde{H}^{n+1-2l \frac{i-2i^{i-1}+1}{i-1} - i+2, n - \frac{i-2i^{i-1}+1}{i-1} - i+2}(\mathcal{X}, \mathbf{Z}/l) = 0$: si $n - \frac{i-2i^{i-1}+1}{i-1} - i+2 < 0$, c'est trivial, et sinon cela résulte du théorème 11 (ii) et du lemme 26 b). $\square$

Démonstration du théorème 50

B.8

Réalisation topologique

B.8.1

Soit F un sous-corps de $\mathbb{C}$. Soit $\Delta^{op}Ens$ la catégorie des ensembles simpliciaux. On a un foncteur

$$\begin{aligned} Sm/F &\rightarrow \Delta^{op}Ens \\ X &\mapsto Sing(X(\mathbb{C})), \end{aligned}$$

où $Sing(M)$ désigne l'ensemble simplicial singulier associé à une variété complexe. D'après [27], on peut "prolonger" ce foncteur en un foncteur *réalisation topologique*

$$t_C : \Delta^{op}Shv_{Nis}(Sm/F) \rightarrow \Delta^{op}Ens$$

tel que, pour tout $X \in Sm/F$, $t_C(X)$ soit naturellement isomorphe à $Sing(X(\mathbb{C}))$. Ce foncteur transforme les $\mathbf{A}_F^1$ -équivalences faibles en équivalences faibles, donc induit un foncteur sur les catégories homotopiques:

$$\begin{aligned} \mathcal{H}(F) &\xrightarrow{t_C} \mathcal{H} \\ \mathcal{H}\mathcal{H}(F) &\xrightarrow{t_C} \mathcal{H}\mathcal{H}. \end{aligned}$$

On a

$$t_C(S_s^1) \cong t_C(S_t^1) \cong S^1$$

donc

$$t_C(T) \cong S^2.$$

De plus, le théorème de Dold-Thom implique:

$$t_C(\mathbf{H}_{\mathbb{Z}}) \cong \mathbf{H}_{\mathbb{Z}}.$$

Finalement:

$$\begin{aligned} t_{\mathbf{C}}(P^i) &= P^i \\ t_{\mathbf{C}}(Q_i) &= Q_i . \end{aligned}$$

B.8.2 Espaces de Thom et cobordismes algébriques

Soient $X \in \mathbf{Sm}/F$, $\mathcal{E}$ un fibré vectoriel sur X et s la section nulle de $\mathcal{E}$. On définit l'espace de Thom de $\mathcal{E}$ comme étant le faisceau pointé $Th(\mathcal{E})$ donné par le carré cocartésien

$$\begin{array}{ccc} \mathcal{E} - s(X) & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathrm{Spec} F & \longrightarrow & Th(\mathcal{E}) \end{array}$$

généralisant le carré qui définit T [27]. Si $\mathcal{E} = 0$, on a évidemment:

$$Th(\mathcal{E}) = X_+ .$$

Si $\mathcal{F}$ est un fibré sur une autre variété Y et $\mathcal{E} \boxplus \mathcal{F}$ est leur somme externe sur $X \times_F Y$, on a un isomorphisme canonique de faisceaux pointés [27]

$$Th(\mathcal{E} \boxplus \mathcal{F}) = Th(\mathcal{E}) \wedge Th(\mathcal{F})$$

en particulier, pour $Y = \mathrm{Spec} F$ et $\mathcal{F} = \mathcal{O}^n$:

$$Th(\mathcal{E} \oplus \mathcal{O}^n) = T^{\wedge n} \wedge Th(\mathcal{E}) .$$

les $Th(\mathcal{E} \oplus \mathcal{O}^n)$ forment donc un spectre isomorphe au spectre des suspensions de $Th(\mathcal{E})$. Dans $\mathcal{H}_{\bullet}(F)$, on a un isomorphisme [27]

$$Th(\mathcal{E}) \simeq \mathbf{P}(\mathcal{E} \oplus \mathcal{O})/\mathbf{P}(\mathcal{E})$$

d'où l'on déduit des isomorphismes, avec $d = \dim \mathcal{E}$:

$$\tilde{H}^{p,q}(Th(\mathcal{E}), A) \simeq H_B^{p-2d}(X, A(q-d)) , \quad p, q \in \mathbf{Z} , \quad (\text{B.10})$$

pour tout A , à l'aide du théorème 43 et du calcul de la cohomologie motivique d'un fibré projectif [43, prop. 2.5].

51

Théorème 51: (*théorème de pureté homotopique*, [27]) Soit $i : Z \rightarrow X$ une immersion fermée de F -variétés lisses. Notons U l'ouvert complémentaire et $v_i : Z \rightarrow X$ le fibré normal de i . Alors on a un isomorphisme canonique dans $\mathcal{H}_{\bullet}(F)$:

$$X/U \approx Th(v_i) .$$

Nous noterons $\mathbf{M}(\mathcal{E})$ la désuspension $T^{-d}\Sigma_T^\infty Th(\mathcal{E})$, où $d = \dim \mathcal{E}$: c'est le *spectre de Thom* de $\mathcal{E}$. En remplaçant au besoin X par une F -variété affine par le procédé de Jouanolou [13, lemme 1.5], on peut étendre cette construction en une fonction

$$\mathbf{M} : K_0(X) \rightarrow \mathcal{SH}(F) .$$

Le foncteur t_C envoie $Th(\mathcal{E})$ (*resp.* $\mathbf{M}(\mathcal{E})$) sur l'espace (*resp.* le spectre) de Thom classique $Th(\mathcal{E}(C))$ (*resp.* $\mathbf{M}(\mathcal{E}(C))$).

Soit $G(m, n)$ la grassmannienne standard, munie de son fibré canonique $\mathcal{E}_{m,n}$. En lui appliquant la construction précédente, on obtient un spectre $\mathbf{M}(\mathcal{E}_{m,n})$. La limite inductive de ces spectres est notée **MGL**: c'est le *spectre des (F -)cobordismes algébriques*. La formule (B.10) et la propriété (C) des $\mathbf{Z}(n)$ entraînent:

Théorème 52: ([49, th. 3.21]) Pour tout groupe abélien A , on a $\tilde{H}^{p,q}(\mathbf{MGL}, A) = 0$ pour $p > 2q$ et $\tilde{H}^{0,0}(\mathbf{MGL}, A) = A$.

52

On note τ le générateur canonique de $\tilde{H}^{0,0}(\mathbf{MGL}, \mathbf{Z})$.

Le foncteur t_C envoie **MGL** sur le spectre du cobordisme complexe **MU**. En particulier, pour tout F -schéma simplicial lisse $\mathcal{X}$, on a des homomorphismes

$$\mathbf{MGL}_{p,q}(\mathcal{X}) \rightarrow \mathbf{MU}_p(\mathcal{X}(C))$$

naturels en $\mathcal{X}$.

Définition 53 Soit $X \in Sm/F$. On note I_X l'image de l'homomorphisme composé

53

$$\bigoplus_{i \geq 0} \mathbf{MGL}_{2i,i}(X) \rightarrow \bigoplus_{i \geq 0} \mathbf{MGL}_{2i,i}(\mathrm{Spec} F) \rightarrow \mathbf{MU}_*(pt) .$$

On vérifie facilement que I_X est un *idéal* de $\mathbf{MU}_*(pt)$.

Le théorème principal

B.8.3

Nous commençons par énoncer le théorème principal de Voevodsky. Pour cela, nous avons besoin d'une définition:

Définition 54 a) Un (v_n, l) -élément de $\mathbf{MU}_*(pt)$ est une classe de bordisme complexe v_n représentée par une variété compacte Y telle que

54

1. $d := \dim Y = l^n - 1$
2. $s_d(Y) \not\equiv 0 \pmod{l^2}$.

b) Soit F un sous-corps de C . Une F -variété X , propre et lisse, est une (v_n, l) -variété si $X(C)$ définit un (v_n, l) -élément de $\mathbf{MU}_*(pt)$.

Dans a), il revient au même de dire que v_n définit un générateur multiplicatif de $\pi_*(BP)$ (resp. de $\pi_*(K(n))$), où BP (resp. $K(n)$) est le spectre de Brown-Peterson en l (resp. la n -ième K -théorie de Morava en l) [31, ch. 4].

Exemples 55.

1. L'espace projectif $\mathbf{P}_F^d$ est une (v_n, l) -variété si et seulement si $n = 1$ et $d = l - 1$ [25, exemple 16-6].
2. Une hypersurface projective lisse $X \subset \mathbf{P}_F^{d+1}$ de degré l est une (v_n, l) -variété si et seulement si $d = l^n - 1$ [25, problème 16-D].
3. On peut montrer que, pour $l = 3$, la variété X de l'exemple 24 (2) (b) est une $(v_2, 3)$ -variété (Rost).

56

Théorème 56: ([49, th. 3.25]) Soit F un sous-corps de $\mathbf{C}$, et soit $X \in Sm/F$ telle que I_X (cf. définition 53) contienne un (v_n, l) -élément. Alors les complexes (B.9) sont acycliques pour $i = n$ et $\mathcal{X} = \text{fibre}(\Sigma_T^\infty(\check{C}(X)_+) \rightarrow S^0)$.

Démonstration Notons Φ_n la fibre homotopique de $Q_n : \mathbf{H}_{Z/l} \rightarrow S_s^1 \wedge T^{l^n-1} \wedge \mathbf{H}_{Z/l}$. On a donc un triangle distingué

$$T^{l^n-1} \wedge \mathbf{H}_{Z/l} \xrightarrow{u} \Phi_n \xrightarrow{v} \mathbf{H}_{Z/l} \xrightarrow{Q_n} S_s^1 \wedge T^{l^n-1} \wedge \mathbf{H}_{Z/l}.$$

D'autre part, notons $\tilde{\tau}$ le composé

$$\mathbf{MGL} \wedge \mathbf{H}_{Z/l} \xrightarrow{\tau \wedge Id} \mathbf{H}_{Z/l} \wedge \mathbf{H}_{Z/l} \xrightarrow{m} \mathbf{H}_{Z/l}$$

où τ est défini après l'énoncé du théorème 52 et m est le produit en cohomologie motivique. Du théorème 52 et d'une formule donnant $\Delta(Q_n)$, où Δ est le coproduit de l'algèbre de Steenrod associé à m via le théorème 47, on déduit l'existence d'un morphisme φ_n tel que le diagramme

$$\begin{array}{ccc} \mathbf{MGL} \wedge T^{l^n-1} \wedge \mathbf{H}_{Z/l} & \xrightarrow{\tilde{\tau}} & T^{l^n-1} \wedge \mathbf{H}_{Z/l} \\ \downarrow & & \downarrow u \\ \mathbf{MGL} \wedge \Phi_n & \xrightarrow{\varphi_n} & \Phi_n \\ \downarrow & & \downarrow v \\ \mathbf{MGL} \wedge \mathbf{H}_{Z/l} & \xrightarrow{\tilde{\tau}} & \mathbf{H}_{Z/l} \end{array}$$

soit commutatif dans $\mathcal{H}(F)$.

Fixons $\mathcal{Y} \in \mathcal{SH}(F)$, un morphisme $\mathcal{Y} \xrightarrow{p} S^0$, un entier d et $\varphi \in \mathbf{MGL}_{2d,d}(\mathcal{Y})$. Pour tout $\mathcal{X} \in \mathcal{SH}$, on a un homomorphisme

$$\pi(\varphi) : \tilde{\Phi}^{*,*}(\mathcal{Y} \wedge \mathcal{X}) \rightarrow \tilde{\Phi}^{*-2d,*-d}(\mathcal{X})$$

qui envoie l'élément de $\tilde{\Phi}^{*,*}(\mathcal{Y} \wedge \mathcal{X})$ donné par le morphisme

$$\alpha : \mathcal{Y} \wedge \mathcal{X} \rightarrow \Phi_n(*)[*]$$

sur l'élément de $\tilde{\Phi}^{*-2d,*-d}(\mathcal{X})$ donné par la composition

$$T^d \wedge \mathcal{X} \xrightarrow{\varphi \wedge Id} \mathbf{MGL} \wedge \mathcal{Y} \wedge \mathcal{X} \xrightarrow{Id \wedge \alpha} \mathbf{MGL} \wedge \Phi_n(*)[*] \xrightarrow{\varphi_n(*)[*]} \Phi_n(*)[*]$$

où φ_n est comme dans le diagramme ci-dessus. On a :

Proposition 57: ([49, prop. 3.24]) Avec les notations ci-dessus, supposons que $d = l^n - 1$ et que $t_C(p_*\varphi) \in \mathbf{MU}_{2(l^n-1)}(pt)$ soit un (v_n, l) -élément. Alors il existe $c \in (Z/l)^*$ tel que, pour tout $\mathcal{X} \in \mathcal{SH}(F)$, le diagramme

57

$$\begin{array}{ccc} \tilde{\Phi}_n^{*,*}(\mathcal{X}) & \xrightarrow{p^*} & \tilde{\Phi}_n^{*,*}(\mathcal{Y} \wedge \mathcal{X}) \\ v \downarrow & & \pi(\varphi) \downarrow \\ \tilde{H}^{*,*}(\mathcal{X}, Z/l) & \xrightarrow{cu} & \tilde{\Phi}^{*-2(l^n-1),*-(l^n-1)}(\mathcal{X}) \end{array}$$

soit commutatif.

Démonstration En introduisant les “spectres fonctionnels” $RHom(\mathcal{Y}, \Phi_n)$ et $RHom(T^{l^n-1}, \Phi_n)$, l'énoncé peut être réinterprété de la manière suivante: le diagramme

$$\begin{array}{ccc} \Phi_n & \xrightarrow{p^*} & RHom(\mathcal{Y}, \Phi_n) \\ v \downarrow & & \pi(\varphi) \downarrow \\ \mathbf{H}_{Z/l} & \xrightarrow{cu} & RHom(T^{l^n-1}, \Phi_n) \end{array} \quad (\text{B.11})$$

est commutatif à homotopie près. Les deux composés de ce diagramme définissent des éléments de

$$Hom_{\mathcal{SH}(F)}(\Phi_n, RHom(T^{l^n-1}, \Phi_n)) = Hom_{\mathcal{SH}(F)}(T^{l^n-1} \wedge \Phi_n, \Phi_n).$$

D'après le théorème 46 et la définition de Φ_n , ce groupe est cyclique d'ordre l et s'injecte dans le groupe correspondant $Hom_{\mathcal{SH}}(S^{2(l^n-1)} \wedge t_C(\Phi_n), t_C(\Phi_n))$. Il suffit

donc de démontrer la commutativité du diagramme (B.11) après lui avoir appliqué le foncteur $t_{\mathbb{C}}$, et ceci résulte d'un calcul facile (cf. [49, lemme 3.6]). $\square$

Fin de la démonstration du théorème 56.

On applique la proposition 57 à $\mathcal{X}$, avec $\mathcal{Y} = \Sigma_T^\infty(X_+)$, p la projection naturelle et φ l'antécédent d'un (v_n, l) -élément de I_X . On a $\Sigma_T^\infty(X_+) \wedge \mathcal{X} = 0$: cela résulte de l'exemple 38. La commutativité du diagramme implique donc que le composé

$$\tilde{\Phi}^{p,q}(\mathcal{X}) \xrightarrow{\nu} \tilde{H}^{p,q}(\mathcal{X}, \mathbb{Z}/l) \xrightarrow{cu} \tilde{\Phi}^{*-2(l^{m-1}), *-(l^m-1)}(\mathcal{X})$$

est identiquement nul pour tout (p, q) , ce qui est équivalent à l'énoncé du théorème 56. $\square$

Démonstration du théorème 50.

Il faut voir que I_X contient un (v_i, l) élément pour tout $i \leq n-2$ dès que X satisfait les hypothèses du théorème 36. En utilisant les opérations de Landweber-Novikov sur $\mathbf{MU}_*$, on peut montrer que si I_X contient un (v_i, l) -élément, il contient un (v_j, l) -élément pour tout $j \leq i$. Comme X est par hypothèse une (v_{n-1}, l) -variété, il suffit de voir que la classe de bordisme de $X(\mathbb{C})$ est dans I_X . Cela résulte du théorème plus précis suivant:

58 **Théorème 58:** ([49, th. 3.22]) Soit X une variété projective et lisse de dimension d sur un sous-corps F de $\mathbb{C}$. Il existe un élément $\varphi_X \in \mathbf{MGL}_{2d,d}(X)$ tel que l'image $t_{\mathbb{C}}(\varphi_X)$ de φ_X dans $\mathbf{MU}_{2d}(X(\mathbb{C}))$ soit la classe fondamentale de $X(\mathbb{C})$ en $\mathbf{MU}$ -homologie.

59 **Remarque 59** Dans le cas où $l = 2$, on peut éviter le recours aux opérations de Landweber-Novikov mentionnées juste avant l'énoncé: en effet, le théorème 58 implique que l'idéal $I_{Q_{\underline{a}}}$ contient les classes de toutes les variétés Y telles que $\mathrm{Hom}_F(Y, Q_{\underline{a}}) \neq \emptyset$. Or l'exemple 55 montre qu'une section plane de dimension $2^i - 1$ de $Q_{\underline{a}}$ est une (v_i, l) -variété.

Indications sur la démonstration

(F. Morel): il faut construire un morphisme

$$T^d \rightarrow \mathbf{MGL} \wedge X_+$$

associé à X . On procède comme en topologie algébrique, avec quelques complications dues à la géométrie algébrique. Soit ν_X le fibré normal de X , vu comme l'opposé de son fibré tangent dans $K_0(X)$. Rappelons qu'il lui est associé canoniquement une classe de spectre $\mathbf{M}(\nu_X) \in \mathcal{SH}(F)$. Il suffit de construire des morphismes dans $\mathcal{SH}(F)$

$$T^d \rightarrow \mathbf{M}(\nu_X)$$

$$\mathbf{M}(\nu_X) \rightarrow \mathbf{M}(\nu_X) \wedge X_+$$

et

$$\mathbf{M}(\nu_X) \wedge X_+ \rightarrow \mathbf{MGL} \wedge X_+.$$

Le dernier morphisme provient d'un morphisme $\mathbf{M}(\nu_X) \rightarrow \mathbf{MGL}$, lui-même obtenu à partir d'un morphisme $X \rightarrow Gr$ classifiant le fibré (virtuel) ν_X , où Gr est la grassmannienne infinie [27]. Le deuxième est simplement le morphisme de spectres de Thom associé au pull-back par la diagonale du fibré $\nu_X \boxplus 0$ sur $X \times_F X$.

Enfin, pour définir le premier morphisme, on se ramène d'abord au cas où $X = \mathbf{P}_F^n$. On a le lemme suivant, qui généralise le théorème 51 (et s'en déduit):

Lemme 60 Soient $i : Z \rightarrow X$ une immersion fermée de F -variétés lisses, U l'ouvert complémentaire, ν_i le fibré normal de i et $\mathcal{E}$ un fibré vectoriel sur X . Alors la cofibre homotopique du morphisme évident

60

$$Th(\mathcal{E}|_U) \rightarrow Th(\mathcal{E})$$

s'identifie canoniquement dans $\mathcal{H}_\bullet(F)$ à $Th(i^*\mathcal{E} \oplus \nu_i)$.

En appliquant ce lemme à l'immersion fermée $\mathbf{P}_F^n \xrightarrow{\Delta} \mathbf{P}_F^n \times_F \mathbf{P}_F^n$ et à $\mathcal{E} = p_1^* \nu_{\mathbf{P}_F^n}$, où p_1 est la première projection, on obtient un morphisme $Th(p_1^* \nu_{\mathbf{P}_F^n}) \rightarrow Th(\nu_{\mathbf{P}_F^n} \oplus \nu_\Delta)$, qui se traduit après projection de $\mathbf{P}_F^n$ sur le point en un morphisme dans $\mathcal{SH}(F)$

$$D : \mathbf{M}(\nu_{\mathbf{P}_F^n}) \wedge (\mathbf{P}_F^n)_+ \rightarrow T^n.$$

On montre alors par dévissage que l'adjoint de D

$$M(\nu_{\mathbf{P}_F^n}) \rightarrow RHom((\mathbf{P}_F^n)_+, T^n)$$

est un isomorphisme, en filtrant $\mathbf{P}_F^n$ par les $\mathbf{P}_F^i$, $i \leq n$. Le morphisme cherché correspond maintenant au morphisme $T^n \rightarrow RHom((\mathbf{P}_F^n)_+, T^n)$ induit par la projection de $(\mathbf{P}_F^n)_+$ sur S^0 . Le lecteur au courant aura reconnu au passage la S -dualité... $\square$

B.9

Démonstration du théorème 37

Dans cette section, on suppose $l = 2$.

B.9.1

Le motif de Rost

Soient $\underline{a} = (a_1, \dots, a_n) \in (F^*)^n$, $\varphi = \ll a_1, \dots, a_n \gg$ la n -forme de Pfister associée, et soient $X_{\underline{a}}$ (resp. $Q_{\underline{a}}$) la quadrique projective d'équation $\varphi = 0$ (resp. d'équation $\ll a_1, \dots, a_{n-1} \gg \perp < -a_n > = 0$). On a $\dim X_{\underline{a}} = 2d$ (resp. $\dim Q_{\underline{a}} = d$), avec

$$d = 2^{n-1} - 1.$$

L'énoncé qui suit est une réinterprétation par Voevodsky d'un théorème de Rost, dans le langage de la catégorie $DM^{eff}(F)$:

61

Théorème 61: ([49, th. 4.5]) Il existe un facteur direct $M_{\underline{a}}$ de $M(Q_{\underline{a}})$, muni de deux morphismes

$$\psi^* : \mathbf{Z}(d)[2d] \rightarrow M_{\underline{a}}$$

$$\psi_* : M_{\underline{a}} \rightarrow \mathbf{Z}$$

tel que:

(i) Les composés

$$\mathbf{Z}(d)[2d] \xrightarrow{\psi^*} M_{\underline{a}} \rightarrow M(Q_{\underline{a}})$$

$$M(Q_{\underline{a}}) \rightarrow M_{\underline{a}} \xrightarrow{\psi_*} \mathbf{Z}$$

sont respectivement la classe fondamentale et le morphisme canonique $M(Q_{\underline{a}}) \rightarrow M(\text{Spec } F) = \mathbf{Z}$.

(ii) Pour toute extension K/F telle que $Q_{\underline{a}}(K) \neq \emptyset$, la suite

$$\mathbf{Z}(d)[2d] \xrightarrow{\psi^*} M_{\underline{a}} \otimes_F K \xrightarrow{\psi_*} \mathbf{Z} \xrightarrow{0} \mathbf{Z}(d)[2d+1]$$

est un triangle distingué scindé dans $DM^{eff}(F)$.

Dans l'énoncé originel de Rost [36, th. 3], ces propriétés sont énoncées de la façon suivante: a) le morphisme canonique $\mathbf{Z} \rightarrow CH^0(M_{\underline{a}})$ est un isomorphisme; b) le degré induit une injection $CH^d(M_{\underline{a}}) \rightarrow \mathbf{Z}$, d'image $2\mathbf{Z}$ (resp. $\mathbf{Z}$) si $Q_{\underline{a}}$ n'a pas de point rationnel (resp. a un point rationnel); c) si $Q_{\underline{a}}$ a un point rationnel, $M_{\underline{a}}$ se décompose canoniquement en $L^0 \oplus L^d$ (en tant que motif de Chow), où L est le motif de Lefschetz.

Rost construit $M_{\underline{a}}$ par récurrence sur n . Sa démonstration repose sur les techniques de [37]: il est impossible de l'exposer ici en détail. Nous nous bornerons à en donner le principe.³

Définir $M_{\underline{a}}$ revient à construire un projecteur dans $\text{End}(M(Q_{\underline{a}}))$. Supposons construit le motif M' correspondant au symbole $\{a_1, \dots, a_{n-1}\}$ et notons

$$\tilde{M} = \bigoplus_{i=1}^{d'} M' \otimes L^i,$$

où $d' = 2^{n-2} - 1$. Rost construit des morphismes

$$\tilde{M} \begin{matrix} \xrightarrow{f} \\ \xleftarrow{g} \end{matrix} M(Q_{\underline{a}})$$

tels que $g \circ f$ soit inversible dans $\text{End}(\tilde{M})$. Le point clé de cette construction est:

Lemme 62: (Rost) Il existe $\theta \in CH_{2d'}(M(Q_{\underline{a}}) \otimes M')$ tel que

62

$$\theta \otimes_F F_s \equiv h \times P + u \times (M \otimes_F F_s)$$

où P est un point fermé de $Q_{\underline{a}} \times_F F_s$, h est une section hyperplane de $Q_{\underline{a}}$ et $u = \frac{1}{2}h^{d'+1}$.

Rost pose alors

$$g_i = h^{i-1}\theta \in CH_{2d'+1-i}(M(Q_{\underline{a}}) \otimes M') = \text{Hom}(M' \otimes L^i, M(Q_{\underline{a}}))$$

$$f_i = {}^t g_{d'+1-i} \in CH_{2d'+1-i}(M' \otimes M(Q_{\underline{a}})) = \text{Hom}(M(Q_{\underline{a}}), M' \otimes L^i),$$

et enfin $f = (f_i), g = (g_i)$. □

Du théorème 61, Voevodsky déduit, de manière essentiellement formelle:

Théorème 63: ([49, th. 4.4]) Avec les notations ci-dessus, on a un triangle distingué dans $DM^{\text{eff}}(F)$

63

$$M(\check{C}(Q_{\underline{a}}))(d)[2d] \rightarrow M_{\underline{a}} \rightarrow M(\check{C}(Q_{\underline{a}})) \xrightarrow{\gamma} M(\check{C}(Q_{\underline{a}}))(d)[2d+1].$$

Remarque 64 On trouvera dans la section B.10.1 une description du morphisme γ .

64

³ See N. Karpenko, *Characterization of minimal Pfister neighbors via Rost projectors*, J. Pure Appl. Algebra **160** (2001), 195–227, §5 for a published construction of the Rost motive. This construction relies on Rost's nilpotence theorem: a published proof may be found in P. Brosnan, *A short proof of Rost nilpotence via refined correspondences*, Doc. Math. **8** (2003), 69–78.

En prenant la cohomologie motivique de ce triangle, on obtient une suite exacte

$$\begin{aligned} H_B^0(\check{C}(Q_{\underline{a}}), \mathbf{Z}(1)) &\rightarrow H_B^{2d+1}(\check{C}(Q_{\underline{a}}), \mathbf{Z}(d+1)) \\ &\rightarrow H_B^{2d+1}(M_{\underline{a}}, \mathbf{Z}(d+1)) \xrightarrow{N} H_B^1(\check{C}(Q_{\underline{a}}), \mathbf{Z}(1)) . \end{aligned}$$

Le premier groupe à partir de la gauche est nul, le quatrième s'identifie canoniquement à F^* et le troisième est facteur direct de $H_B^{2d+1}(Q_{\underline{a}}, \mathbf{Z}(d+1))$. Ce dernier groupe s'identifie, par la conjecture de Gersten pour la cohomologie motivique et les propriétés (C) et (G) de $\mathbf{Z}(d+1)$, au conoyau $A_0(Q_{\underline{a}}, K_1)$ de l'homomorphisme

$$\coprod_{x \in (Q_{\underline{a}})_{(1)}} K_2^M(F(x)) \xrightarrow{\partial} \coprod_{x \in (Q_{\underline{a}})_{(0)}} K_1(F(x))$$

où $(Q_{\underline{a}})_{(p)}$ désigne l'ensemble des points de $Q_{\underline{a}}$ de dimension p et ∂ est une collection d'homomorphismes résidus [16]. Pour $x \in (Q_{\underline{a}})_{(0)}$, l'extension $F(x)/F$ est finie; la norme induit un homomorphisme

$$A_0(X, K_1) \xrightarrow{N} F^* \quad (\text{B.12})$$

(cela résulte de la “réciprocité de Weil”), compatible avec celui de la suite exacte ci-dessus. Pour démontrer le théorème 37, on est donc ramené à démontrer:

65 **Théorème 65:** (Rost [35]) L'homomorphisme (B.12) est injectif.

Pour $n = 2$, ce résultat est dû à Suslin [42]; pour $n = 3$, il avait été obtenu, antérieurement à [35], indépendamment par Rost [34] et Merkurjev-Suslin [22, prop. 2.2]. Sa démonstration est esquissée dans la section suivante.

B.9.2 Zéro-cycles à coefficients dans les unités

Le cas $n = 2$: $Q_{\underline{a}}$ est une conique. Comme indiqué ci-dessus, le théorème 65 est alors dû à Suslin: il utilise la K -théorie de Quillen. Une démonstration élémentaire, due à Merkurjev, n'utilise que le théorème de Riemann-Roch sur $Q_{\underline{a}}$ (une courbe de genre 0!) [50, th. 2.5].

Le cas $n > 2$. La stratégie est de se ramener au cas $n = 2$. Pour toute F -variété projective et lisse X , notons $A_0(X, K_1)$ le conoyau de l'application analogue à (B.12). On montre que $A_0(X, K_1)$ est un invariant birationnel stable de X (stable signifie que $A_0(X \times_{\mathbf{P}^1_F}, K_1) \rightarrow A_0(X, K_1)$ est un isomorphisme). On peut donc remplacer $Q_{\underline{a}}$ par $X_{\underline{a}}$ dans la démonstration du théorème 61. De plus, par un argument de transfert, on peut supposer que F n'a pas d'extensions de degré impair.

On commence par montrer que $\text{Im}(N) \subset D(\varphi)$, où $D(\varphi)$ est l'ensemble des valeurs non nulles de φ : cela résulte de la multiplicativité des formes de Pfister [17, ex. 10.2.4] et du principe de norme de Knebush (*ibid.*, th. 7.5.1). On construit alors une application

$$\sigma : D(\varphi) \rightarrow A_0(X_{\underline{a}}, K_1)$$

qui est une section surjective de N , ce qui termine la démonstration (et décrit du même coup l'image de N).

Pour construire σ , on note V l'espace sous-jacent à φ , on choisit $v_0 \in V$ tel que $\varphi(v_0) = 1$ et on écrit

$$V = Fv_0 \oplus V'$$

où V' est le supplémentaire orthogonal de v_0 . Soit $b \in D(\varphi)$. Ecrivons $b = \varphi(xv_0 + yv')$ avec $x, y \in F$ et $v' \in V' - \{0\}$. On a donc

$$b = x^2 - ay^2$$

avec

$$a = -\varphi(v') .$$

Ainsi

$$b \in N_{E|F}(E^*)$$

où $E = F(\sqrt{a})$.

Comme $\langle 1, -a \rangle$ est une sous-forme de φ , on a $\text{Spec } E \in (X_{\underline{a}})_{(0)}$. On peut maintenant poser

Définition 66 $\sigma(b) = i_*(x + \sqrt{a}y) \in A_0(X_{\underline{a}}, K_1)$, où i_* est induit par le plongement $E^* \hookrightarrow \coprod_{x \in (Q_{\underline{a}})_{(0)}} K_1(F(x))$.

66

Pour que cette définition ait un sens, il faut voir que $\sigma(b)$ ne dépend pas du choix de v', x, y . On note que, de toute façon,

$$N(\sigma(b)) = b . \tag{B.13}$$

Si on a une autre écriture

$$b = \varphi(\tilde{x}v_0 + \tilde{y}\tilde{v}') ,$$

on note W le sous-espace de V engendré par $v_0, \tilde{v}$ et $\tilde{v}'$ et $\tilde{\sigma}(b)$ l'élément correspondant de $A_0(X_{\underline{a}}, K_1)$. Pour simplifier, supposons W de dimension 3 (l'autre cas est plus facile). Si Y est la conique correspondant à la restriction de φ à W , on a

$$\sigma(b), \tilde{\sigma}(b) \in \text{Im}(A_0(Y, K_1) \rightarrow A_0(X_{\underline{a}}, K_1)) .$$

D'après (B.13) et le cas $n = 2$, il en résulte bien que $\sigma(b) = \tilde{\sigma}(b)$.

Pour voir que σ est surjective, soient $x \in X_{\underline{a}}$ un point fermé, $E = F(x)$ et $\lambda \in E^*$; notons λ_x l'image de λ dans $A_0(X_{\underline{a}}, K_1)$. Alors λ_x est (tautologiquement!) la norme de λ_x vu dans $A_0(X_{\underline{a}} \times_F E, K_1)$. Comme φ_E est isotrope, $A_0(X_{\underline{a}} \times_F E, K_1) \xrightarrow{N_E} D(\varphi_E)$ est bijective, ainsi que σ_E . La conclusion résulte donc du fait que les normes commutent à σ . Pour le voir, on remarque que par hypothèse toute extension finie

de F est filtrée par des extensions quadratiques successives; on se ramène donc au cas d'une extension quadratique E/F . On remarque alors que $D(\varphi_E) = \bigcup D(\alpha_E)$ où α décrit les sous-formes ternaires de φ contenant le vecteur v_0 , ce qui ramène de nouveau au cas connu $n = 2$. $\square$

B.10

Compléments

Dans cette section, nous indiquons certains résultats annoncés par Voevodsky, qui réduisent la démonstration de la conjecture de Kato en général à un problème très spécifique.

B.10.1

Le motif de Rost-Voevodsky

Voevodsky a annoncé une construction indépendante du motif du théorème 61, qui offre l'intérêt de se généraliser au cas d'un nombre premier l quelconque. Ce qui suit est extrait de messages à Rost et au rédacteur, et reproduit avec son autorisation.

Commençons par décrire le morphisme γ du théorème 63. En raisonnant comme dans la démonstration du lemme 33, on établit facilement une suite exacte (sous les hypothèses de ce lemme):

$$0 \rightarrow H^n(\check{C}(X), \mathbb{Z}/l(n-1)) \rightarrow H_{\text{ét}}^n(F, \mu_l^{\otimes(n-1)}) \rightarrow H_{\text{ét}}^n(F(X), \mu_l^{\otimes(n-1)}) .$$

En identifiant $\mu_l^{\otimes(n-1)}$ à $\mu_l^{\otimes n}$ par le choix d'une racine primitive l -ième de l'unité de F (supposé en contenir), on en déduit un élément

$$\xi \in H^n(\check{C}(X_{\underline{a}}), \mathbb{Z}/l(n-1))$$

correspondant à $(a_1) \cdot \dots \cdot (a_n) \in H_{\text{ét}}^n(F, \mu_l^{\otimes n})$. Pour $l = 2$, on montre que γ est donné par le cup-produit par $\tilde{Q}_{n-2} \dots \tilde{Q}_1 \tilde{\beta}(\xi)$, où $\tilde{\beta}$ et les $\tilde{Q}_i$ sont comme dans le corollaire 49.

Dans le cas général, le même opérateur donne un triangle distingué dans $DM^{\text{eff}}(F)$

$$M'_{\underline{a}} \rightarrow M(\check{C}(X_{\underline{a}})) \xrightarrow{\gamma} M(\check{C}(X_{\underline{a}})(l^{n-1}-1)[2l^{n-1}-1]) \rightarrow M'_{\underline{a}}[1]$$

où $X_{\underline{a}}$ est une variété de déploiement pour $\{a_1, \dots, a_n\}$ vérifiant les hypothèses (i) et (ii) de la définition 28 et $M'_{\underline{a}}$ est simplement défini comme la fibre de γ . Voevodsky a annoncé:

67

Théorème 67 Avec les hypothèses et notations ci-dessus, supposons que F soit un sous-corps de $\mathbb{C}$, sans extensions de degré premier à l . Supposons de plus que X soit une (v_{n-1}, l) -variété. Posons

$$M_{\underline{a}} = S^{l-1}(M'_{\underline{a}}) ,$$

où S^{l-1} dénote la puissance symétrique $(l-1)$ -ième dans $DM^{eff}(F)$. Alors $M_{\underline{a}}$ est facteur direct autodual de $M(X_{\underline{a}})$.

En particulier, $M_{\underline{a}}$ est un motif pur, canoniquement associé à $\underline{a}$ d'après l'exemple 38. En utilisant ce fait et le théorème 36, Voevodsky obtient alors:

Théorème 68 Supposons $H90(n-1, l)$ vrai. Supposons que, pour tout sous-corps F de $\mathbf{C}$ et tout $\underline{a} = (a_1, \dots, a_n) \in (F^*)^n$, il existe une variété de déploiement $X_{\underline{a}}$ pour $\{a_1, \dots, a_n\} \in K_n^M(F)/l$ telle que

- (i) $X_{\underline{a}}$ soit une (ν_{n-1}, l) -variété;
- (ii) la norme $A_0(X_{\underline{a}}, K_1) \xrightarrow{N} F^*$ soit injective.

68

Alors $H90(n, l)$ est vrai.

Cet énoncé donne une nouvelle démonstration du théorème de Merkurjev-Suslin (le cas $n = 2, l$ quelconque) modulo (ii), qui est démontré dans [21, cor. 8.7.2] pour la variété de Severi-Brauer d'une algèbre centrale simple de degré l . Dans le cas $l = 3$, Rost a annoncé une démonstration de (ii) pour la variété de l'exemple 24 (2) (b) (sa démonstration utilise une F -forme du plan projectif de Cayley), ce qui donne $K(3, 3)$, ainsi que pour une variété convenable de dimension 26, ce qui donne $K(4, 3) \dots$

Dans les autres cas, on ne dispose pas pour $X_{\underline{a}}$ de candidats aussi géométriques que précédemment. Voevodsky en a proposé une construction récursive, mais il n'a pour l'instant que des résultats partiels sur les variétés obtenues.

(ν_n, l) -variétés et variétés de déploiement génériques

B.10.2

Voevodsky a également annoncé des résultats qualitatifs sur les variétés de déploiement, qui clarifient grandement la situation et que nous ne résistons pas à l'envie d'exposer.

Si X, Y sont deux F -variétés, notons $X \leq_l Y$ s'il existe un morphisme $\varphi : M(X) \rightarrow M(Y)$ dans $DM^{eff}(F, \mathbf{Z}_{(l)})$ (motifs à coefficients dans $\mathbf{Z}_{(l)}$) tel que le diagramme

$$\begin{array}{ccc} M(X) & \xrightarrow{\varphi} & M(Y) \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(l)} & \xrightarrow{c} & \mathbf{Z}_{(l)} \end{array}$$

soit commutatif pour un $c \in \mathbf{Z}_{(l)}^*$ convenable. C'est une relation de préordre; si X et Y sont propres et lisses, $X \leq_l Y$ si et seulement si il existe un revêtement $Z \rightarrow X$ de degré premier à l et un morphisme $Z \rightarrow Y$. Notons $\cong_l$ la relation d'équivalence associée: c'est la l -équivalence. Voevodsky a alors annoncé:

-
- 69** **Théorème 69** Soient X une (v_n, l) -variété et Y une variété non l -triviale (c'est-à-dire non l -équivalente à $\text{Spec } F$). Alors tout morphisme $M(X) \rightarrow M(Y)$ est non trivial sur la classe fondamentale de X à coefficients $\mathbb{Z}/l$. En particulier, on a :
- (i) $X \leq_l Y \Rightarrow \dim X \leq \dim Y$;
 - (ii) Si $X \leq_l Y$ et $\dim X = \dim Y$, alors $X \cong_l Y$.
-
- 70** **Théorème 70** Supposons $H90(n-1, l)$ vrai. Soit $\underline{a} = (a_1, \dots, a_n) \in (F^*)^n$, et soit X une variété de déploiement pour $\{a_1, \dots, a_n\} \in K_n^M(F)/l$ qui est une (v_{n-1}, l) -variété. Alors, toute autre variété de déploiement Y vérifie $Y \leq_l X$.
-
- 71** **Corollaire 71** Sous les hypothèses du théorème 70, toute (v_{n-1}, l) -variété de déploiement pour un symbole $\{a_1, \dots, a_n\} \in K_n^M(F)/l$ est générique (cf. définition 22). Deux (v_{n-1}, l) -variétés de déploiement pour $\{a_1, \dots, a_n\}$ sont l -équivalentes.
-
- 72** **Corollaire 72** Supposons que $\{a_1, \dots, a_n\} \in K_n^M(F)/l$ admette une (v_{n-1}, l) -variété de déploiement. Alors toute variété de déploiement générique pour $\{a_1, \dots, a_n\}$ est de dimension $\geq l^{n-1} - 1$; si elle est propre et lisse de dimension $l^{n-1} - 1$, c'est une (v_{n-1}, l) -variété.
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Index

- + -construction *see* plus construction
- Γ -spaces 6
- ΩA 685
- δ -functor 1060–1065
- $\varepsilon \mathcal{A}$ 682
- γ -filtration 41
- γ -operations on K_0 258
- $\hat{\mathcal{A}}$ -class 762
- $\mathcal{A}$ -filtered 24
- $\hat{A}$ -genus
 - higher 762
- A-regular 781
- A-theory 597–599
- Abel’s five term relation 301
- Adams operations 280
- Adams spectral sequence 970
- additive category 11
- additivity theorem 22
- admissible epimorphism 11
- admissible monomorphism 11
- AF algebra 849
- Alexander polynomial 590
- Alexander trick 711
- algebraic cycle 41
- algebraic equivalence 878, 883
- algebraic K -theory
 - of commutative Banach algebra 854
 - of rings of integers 140
 - of spaces 597–599
 - of the integers 127, 141
- algebraic n -simplex 440
- amenable group action 777
- analytic torsion 590
- approximation lemma 75
- Arakelov motivic complex 312
- Artin’s 751
- Artin–Verdier duality 980
- aspherical 722
- assembly 30
- assembly map 735
 - analytic 804
 - relative 741
- asymptotic covering 33
- asymptotic dimension 778
- asymptotic morphism 666
- Atiyah’s Real K -theory 892
- Atiyah–Conjecture for Torsion Free Groups
 - 729
 - Strong 729
- Atiyah–Hirzebruch spectral sequence 43, 211, 457, 1082
- augmented diagrams 1025–1027
 - constructing them 1045–1048
- B 686
- Bökstedt’s JK -construction 1003
- balanced product
 - of $\mathcal{C}$ -spaces 795
- balanced smash product
 - of pointed $\mathcal{C}$ -spaces 795
- Banach algebra
 - commutative 846
 - semisimple 847

- bar construction 494
- based L-groups 619
- Bass Conjecture 386
 - Strong Bass Conjecture for $K_0(\mathbb{C}G)$ 753
 - Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ 753
 - Weak Bass Conjecture 754
- Bass–Heller–Swan-decomposition 716
- Baum–Connes Conjecture 673
 - fibered version 771
 - for Torsion Free Groups 724
 - real version 726
 - status of 775
 - status of the Coarse Baum–Connes Conjecture 779
 - with coefficients 765
 - with coefficients and a-T-menability groups 773
- BDF *see* Brown–Douglas–Fillmore theory
- Beilinson conjecture 396
- Beilinson–Lichtenbaum conjecture 202, 377, 433, 439, 449, 916, 1101
- Beilinson–Parshin conjecture 396
- Beilinson–Soulé conjecture 363
- Beilinson–Soulé vanishing conjectures 491
- Bernoulli numbers 144, 151, 180, 184, 998, 1002
- bicartesian square 1042
- Birch–Tate conjecture 152, 170
- bivariant Chern characters 694
- bivariant periodic cyclic homology 689
- Bloch complex 1082
- Bloch’s cycle algebra 492, 493
- Bloch’s cycle complex 307, 441
- Bloch’s cycle Lie algebra 493
- Bloch’s formula 276
 - singular varieties 284
- Bloch–Beilinson–Murre filtration conjecture 370
- Bloch–Kato conjecture 126, 202, 377, 449, 1100, 1108
- Bloch–Ogus cohomology 453
 - de Rham cohomology 455
 - Deligne cohomology 455
 - étale cohomology 455
- Bloch–Suslin complex 437
- Bloch–Wigner function 301
- blow-up 1087
- blow-up formula 201, 206
- Borel class 320
- Borel conjecture 722
- Borel regulator 321
- Bost conjecture 765
- Bott element 149, 169, 857
 - inverse 857
- Bott manifold 761
- Bott map 120, 149
- Bott periodic K -theory 125
- Bott periodicity 114, 117, 659, 846
- Bott periodicity theorem 851
- bounded geometry 767
- Bounded h-Cobordism Theorem 714
- bounded K -theory 27
- bounded map 713
- Brauer group 145
- Brauer lift 1003
- Bredon homology 740
- Brown–Comenetz duality 973
- Brown–Douglas–Fillmore theory 847
- Brown–Gersten spectral sequence 210
- Burnside ring 816
- C^* -algebra 845
 - AF 849
 - G - C^* -algebra 804
 - maximal complex group C^* -algebra 763
 - nuclear 849
 - reduced complex group C^* -algebra 725
 - reduced real group C^* -algebra 726
 - stable 849, 859
 - tensor product 849, 860
- C^* -theoretic Novikov conjecture and groups of finite asymptotic dimension 778
- $c\text{-}equi(X, 0)$ 1086
- calculus of functors 72
- Calkin algebra 846, 858
- cancellation theorem 52, 53
- categories with squares 1020–1023
- category
 - of compactly generated spaces 794
 - of weak equivalences 16

- with cofibrations 16
- with cofibrations and weak equivalences 597
- category with duality 540
 - exact category with duality 543
 - triangulated category with duality 559
- cdga 492
 - Adams graded 493
 - connected 493
 - motivic 493
- cdh topology 1086
- centralizer 816
- Chapman’s theorem 588
- Chapman–Ferry Theorem 592
- characteristic classes 583
 - secondary 584
- Chern character 42
 - in algebraic geometry 361
 - in topology 355
- Chern class 42, 153, 155, 583, 586
 - coefficients in Milnor K -theory 265
- Chern–Connes character 695
- Chern–Simons class 584
- Chern–Weil theory 583
- Chow groups 1082
 - graded by (co-)dimension 247
 - higher 59
 - singular surfaces 287
 - ungraded 246
- Chow ring 41
- Chow’s Moving Lemma 250
- class group 143, 977
 - ray 174
- classical polylogarithms 302
- classification of nuclear C^* -algebras 676
- classification of quadratic forms 556
- classifying map 681
- classifying space
 - for a family 739
 - for proper G -actions 739
 - of a group 714
- classifying variety 950
- Clifford algebras 118
- coarse invariance 27
- cobordism 710
 - bounded 713
 - diffeomorphic relative M^- 710
 - trivial 710
- cocompact 738
- coflasque torus 944
- cohomology
 - continuous étale 398
 - l -adic 391
 - Lichtenbaum (Weil-étale) 404
 - motivic 361, 362, 371, 1113
 - with compact supports 28
- commuting automorphisms 47
- completed K -theory 164
- concordance 599
- cone
 - reduced cone 794
- coniveau filtration 919
- connection 581
 - flat 582, 583
- Connes’ operator 88
- continuous cohomology 214
- control
 - bounded 591
 - epsilon 591
- controlled K -theory 591
- correspondence 53
 - direct sum Grothendieck group 53
- cosimplicial regions 1024–1025, 1068
- crystalline cohomology 91
- crystalline cycle map 206
- cubical hyperresolutions 461
- curves on $K(A)$ 106
- CW-complex
 - finite 712
 - finitely dominated 712
- cycle 240
 - associated to a module 242
 - degenerate 803
 - equivalent 803
 - even 803
 - homotopic 803
 - odd 803
 - prime cycle 241
- cycle classes 506
 - regulators by 506

- relative 506
- cycle complexes 440–442
 - and Milnor K -theory 446
 - Chern character isomorphism 445
 - cubical version 450
 - alternating complexes 452
 - comparison with simplicial version 451
 - products 452
- cycle complex associated to a cycle module 262
- functoriality 443
- homotopy 443
- localization 443
- Mayer–Vietoris 443
- products 444
- projective bundle formula 445
- properties 442
- cycle map 202
- cycle module 262
 - cycle complex 262
 - with ring structure 262
- cyclic bar-construction 78, 86, 101
- cyclic bicomplex 687
- cyclic homology 72, 87, 687
- cyclic polytopes 87
- cyclotomic tower 976
- cyclotomic trace 72, 599
- cyclotomic trace map 217
- cylinder functor 19

- de Rham–Witt complex 204
- decoration 719
- Dedekind zeta function 998
- deformation to the normal cone 288
- degree of a map 581, 597
- Deligne’s complex 307
- derived categories 1017–1018
- descent homomorphism 805
- determinant
 - Kadison–Fuglede 591
- devissage 1065–1067
- de Rham–Witt complex 73, 91
 - of a polynomial algebra 92
- diffotopic 679
- dilogarithm 301, 437
- dimension
 - birational invariance 242
 - relative to a base 242
- dimension group 850
- Dirac element 805
 - dual 805
- Dirac-dual Dirac method 805
- disjoint union axiom 740
- distinguished polynomial 961
- distinguished triangles 1018, 1035–1036
- divisor
 - Cartier 243
 - cap product by 245
 - Weil 241
- dominated 579
 - epsilon 592
- duality theorem 1098
- dualizable object 972

- e -invariant 147, 159, 164
- E -theory 661
- edge-wise subdivision 83
- Eilenberg swindle 858
- Eilenberg–MacLane spaces 5
- elementary module 961
- elementary spectrum 974
- elliptic operator
 - generalized elliptic G -operator 802
- embedding conjecture 729
- embedding theorem 487
- envelope 288
- equivalence of categories 796
- equivariant K -theory 122
- equivariant KK -and E -theory 667
- equivariant mapping space 80
- eta-invariant 730, 861
- étale K -theory 213, 978, 1082
 - for fields 125
- Euler characteristic 598
- Euler class 583
- exact category 11
- exact functor of categories with cofibrations and weak equivalences 16
- excision 27, 33

- for algebraic K -theory 857, 861–862
- for K_0 846
- for K_1 849
- for mod p K -theory 862
- for topological K -theory 857
- exotic sphere 712
- Ext* 660
- extension axiom 20
- extension of length n 681
- $\mathcal{F}$ -equivalence 73, 82
- factorial algebraic group 948
- family of subgroups 738
- Farrell–Jones Conjecture
 - and subgroups of Lie groups 780
 - fibered version 771
 - for $K_0(\mathbb{Q}G)$ 750
 - for $K_n(\mathbb{Z}G)$ for $n \leq -1$ 749
 - for low dimensional K -theory and torsion free groups 708
 - for pseudoisotopies of aspherical spaces 769
 - for torsion free groups and K -theory 714
 - for torsion free groups and L -theory 719
 - implies the Borel Conjecture 723
 - status of the Farrell–Jones Conjecture for torsionfree groups 781
 - status of the fibered Farrell–Jones Conjecture 779
 - vanishing of low dimensional K -theory for torsionfree groups and integral coefficients 710
 - with coefficients 766
- Fermat’s Last Theorem 144
- fiber functor 455
- field
 - 2-regular 177
 - exceptional 148, 165
 - non-exceptional 148, 154
- filtration
 - Brown 268
 - Brown, multiplicativity of 275
 - codimension of supports 256
 - coniveau 256
 - multiplicativity conjecture 257, 259, 277, 281
 - coniveau, on cohomology of presheaf of spectra 269
 - γ - 258
 - hypercohomology 268
 - of a space 46
 - weight filtration on K -cohomology 285
- filtrations
 - γ and coniveau compared 281
- finite correspondences 483
 - the category of 483
- finite fields 148
- finiteness obstruction 712
- fit set of primes 975
- flasque 25
- Fréchet algebra 861, 862
- Fredholm index 846
- Fredholm module 861, 863
- Fredholm operator 121, 846
- Free L -groups 619
- freeness defect 962, 984
- Friedlander–Suslin cohomology 447
 - comparison with higher Chow groups 447
- functional equations for the trilogarithm 303
- Fundamental Theorem of Algebraic K -Theory 716
- Fundamental Theorem of Hermitian K -Theory 132
- Fundamental Theorem of K -Theory 868
- G -CW-complex 79, 738
 - finite 738
 - proper 739
- G -morphism 928
- G -spectra 77
- G -stable category 77
- G -variety 928
- G -vector bundle on X 928
- Gelfand transform 846
- generalized cross-ratio 324

- generalized cross-ratio of 6 points in P^2 303
- generalized cycle map 912
- Gersten complex 273
- Gersten conjecture 276, 279, 1142
 - cycle complexes 264
 - for Witt groups 567, 569
- Gersten resolution 199, 201, 205, 211
- Goncharov's complexes 438
- Goodwillie's Theorem 862
- Grassmannian polylogarithms 315
- Greenberg conjecture 994, 996
- Gromov–Lawson–Rosenberg Conjecture
 - Homological Gromov–Lawson–Rosenberg Conjecture 762
 - Stable Gromov–Lawson–Rosenberg Conjecture 761
- Gross conjecture 974, 991, 996
- Gross–Sinnott kernel 983
- Grothendieck group
 - direct sum 49
- Grothendieck–Knebusch Conjecture for Witt groups 567
- Grothendieck–Riemann–Roch theorem 41
- Grothendieck–Witt group 557
- group
 - a-T-menable 774
 - having property (RD) 776
 - having property (T) 774
 - having the Haagerup property 774
 - K -amenable 764
 - linear 777
 - virtually cyclic 738
 - virtually cyclic of the first kind 738
 - virtually poly-cyclic 751
- group action 592, 594
 - locally linear 595
- h -cobordism 588, 615, 624, 640, 710
 - bounded 713
- h -cobordism theorem 588
- H -spaces 1056
- H -unital 861
- handle decomposition
 - equivariant 595
- Harris–Segal summand 149, 153, 159, 181
- Hattori–Stallings-rank 753
- heart 1049–1050
- Hecke algebra 773
 - isomorphism conjecture for the Hecke algebra 773
- Hensel's Lemma 856
- Henselian pair 856
- Hermitian forms 617
- Hermitian K -theory 129
 - of the integers 134
- Hermitian K -Theory Theorem 615
- higher Chow group sheaves 142
- higher Chow groups 196, 433, 440–442
 - Arakelov Chow groups 312
 - comparison with Friedlander–Suslin cohomology 447
 - cycle classes for 508
 - relative 508
- higher Hermitian K -theory 625
- Hilbert 90 1116
- Hochschild homology 74
- Hodge Conjecture 918
 - generalized 919
- Hodge filtration 919
- homogeneous variety 936
- homology sphere 582
- homology theory
 - equivariant G -homology theory 791
 - G -homology theory 739
 - proper G -homology theory 791
- homotopy category of schemes 1126
- homotopy cofiber 62
- homotopy colimit problem 626
- homotopy coniveau filtration 64
- homotopy equivalence
 - controlled 592
 - simple 587, 593
- homotopy fiber 59
- homotopy fixed point space 32
- homotopy Fixed Spectrum 625
- homotopy groups of $\text{Diff}(M)$ 724
- homotopy groups of $\text{Top}(M)$ 724
- homotopy invariance of $\tau^{(2)}(M)$ and $\varphi^{(2)}(M)$ 730

-
- homotopy invariance of the L^2 -Rho-invariant for torsionfree groups 730
 - homotopy limit problem 626
 - homotopy Orbit Spectrum 625
 - homotopy-theoretic group completion 884
 - homotopy-theoretic isomorphism conjecture for $(G, E, \mathcal{F})$ 800
 - hyper-cohomology spectrum 213
 - hyperbolic functor 130, 618
 - hyperbolic space, $H(M)$ 542
 - hypercohomology spectrum 986, 1007
 - hyperenvelope 289
 - idempotent conjecture 728
 - Igusa Stability Theorem 615
 - index
 - secondary 861
 - index theorems 670
 - induction
 - of equivariant spaces 791
 - induction structure 791
 - infinite loop space machine 6
 - infinite loop spaces 1055–1057
 - injectivity of the Baum–Connes assembly map 777
 - integral $K_0(\mathbb{Z}G)$ -to- $K_0(\mathbb{Q}G)$ -conjecture 756
 - integral operator
 - singular 844
 - intersection
 - multiplicity 250
 - – positivity conjecture 254
 - – Serre’s tor formula 253
 - – vanishing conjecture 254
 - proper 249
 - intersection product
 - commutativity for Cartier divisors 261
 - invertible spectrum 973
 - irregular prime 144, 145, 180, 182, 183
 - isometry 541
 - isomorphism conjecture for NK -groups 772
 - isomorphism conjecture for the Hecke-algebra 773
 - Iwasawa algebra 960
 - Iwasawa module 162
 - Iwasawa’s Main Conjecture 995
 - Iwasawa’s μ -invariant 993
 - $\mathcal{K}$ (compact operators) 661
 - $\mathcal{K}$ (smooth compact operators) 679
 - K - and L -theoretic Novikov conjecture and groups of finite asymptotic dimension 785
 - K - and L -theory Spectra over Groupoids 798
 - K_1 -chain 246
 - K -groups
 - algebraic K -groups of a ring 708
 - negative 851, 868
 - reduced algebraic K -groups of a ring 709
 - relative 860
 - topological K -groups of a C^* -algebra 725
 - topological K -homology groups of a space 725
 - K -groups with coefficients 209
 - K -homology 660
 - K -regularity 865–868
 - K -theory
 - negative 591
 - of $\mathbb{Z}/p^n\mathbb{Z}$ 107
 - Volodin 862
 - K -theory mod n 124
 - K -theory of algebraically closed fields 124
 - K -theory of Number Fields 143
 - K -theory of Quadratic Forms 617
 - K -theory of stable C^* -algebras 129
 - K -theory, product on 271
 - K -theory spectrum 121
 - algebraic 851
 - algebraic K -theory spectrum of a ring 714
 - mod n 853
 - of an exact category 270
 - over groupoids 798
 - topological 851
 - – connective 855

- K_0 with supports 254
- Kadison Conjecture 728
- Kahn map 153, 155, 161
- Kaplansky Conjecture 729
- Karoubi Conjecture 859, 862–865
 - unstable 865
- Karoubi Periodicity 619, 628, 632
- Karoubi’s forgetful trick 629
- Karoubi’s hyperbolic trick 630
- Kasparov module 662
- Kasparov product 804
- Kato Conjecture 415
- Kimura–O’Sullivan Conjecture 412
- KK -theory 661
 - equivariant 804
- knot 590
- Kobal’s Forgetful Theorem 629
- Kobal’s Hyperbolic Theorem 630
- KR -theory 121
- Kummer–Vandiver Conjecture 999

- L theory spectrum 637
- ℓ -adic completion 966
- ℓ -adic L -function 993
- L -class 732
- L -groups 719
- L -theory of Quadratic Forms 619
- L -theory spectrum
 - algebraic L -theory spectrum of a ring 719
 - over groupoids 798
- L^2 -eta-invariant 730
- L^2 -Rho-invariant 730
- lagrangian 544, 559
- λ -operations on K_0 258
- Laurent series 121
- Lawson homology 896
- layer
 - of a filtration 46
- length function 775
- lens space 590
- Leopoldt Conjecture 991, 996
- Leopoldt defect 980, 985
- Lichtenbaum Conjecture 393, 397
- Lichtenbaum–Quillen Conjecture 125, 215, 358, 986
 - 2-adic 1004, 1006
 - for $\mathbb{Z}$ 999
 - generalized 996
- Lie groups
 - made discrete 854
- liftable simplices 1049
- linear varieties 914
- linearization map 597
- local cyclic cohomology 692
- local cyclic homology 693
- local fields 150, 980
- localisation 1013, 1057–1059
- localization 198, 210, 1087
- localization theorem 19
- locally convex algebra 679
- locally finite homology 29
- logarithmic de Rham–Witt sheaf 205
- long exact sequence in KK 664
- lower K -groups 23
- lower L -groups 620

- m -algebra 679
- Main Conjecture of Iwasawa Theory 157, 170
- manifold
 - parallelizable 583
- manifold parametrized over $\mathbb{R}^k$ 713
- Manifold structure 621, 639
- mapping cylinder neighborhood 595
- Mayer–Vietoris 1087
- Merkurjev–Suslin conjecture 365
- Meta-Conjecture 735
- metabolic symmetric space 544, 559
- metrically proper 774
- Milnor conjecture 1108
- Milnor K -groups 356
- Milnor K -theory 201, 1095, 1107
 - of a field 261
- mixed Tate motives 450, 458, 488
 - Bloch–Kriz 500
 - Bloch–Tate 494
 - $K(\pi, 1)$ -conjecture 502
 - Kriz–May 500

- Spitzweck representation theorem 501, 502
- triangulated category of 489
- t -structure 491
- vanishing conjectures 491
- weight filtration 490
- Voevodsky 501
- model categories 1015–1017
- modules over a cdga 496
 - cell-modules 496
 - derived category of 497
 - structural results 499
 - t -structure 498
 - weight filtration 498
- moduli space of manifold structures 613
- Moore’s Theorem 163
- morphic cohomology 897, 898
- motive of a scheme 1090
- motives
 - effective geometric 484
 - geometric 484
 - mixed 453, 456
 - by compatible realizations 458–461
 - by Tannakian formalism 462
 - Nori’s construction 462, 468
 - triangulated categories of 469
 - comparison results 488
 - constructions 473
 - duality in 472
 - Hanamura’s construction 475
 - Huber’s construction 474
 - Levine’s construction 478
 - structure of 470
 - Voevodsky’s construction 483
- motivic
 - cdga 493
 - complexes 433
 - Galois group 458
 - Lie algebra 493
- motivic cohomology 195, 1094
- motivic complex 1093
- motivic spectral sequence 40, 142, 908
 - for fields 126
- moving lemma 60, 65
 - Chow’s 250
- multi-relative K -theory 60
- multicategory 13
- multiple polylogarithms on curves 342
- multiplier algebra 857
- negative homotopy groups of
 - pseudoisotopies 719
- negative K -theory 120
- neutral symmetric space 559
- Nil-groups 105, 716
- Nisnevich sheaf 485
- Nisnevich sheaf with transfer 485
- Nisnevich topology 1086, 1126
- norm fibration sequence 625
- normal cone
 - deformation to the 288
- normalizer 816
- Novikov Conjecture 26, 673, 733
 - for K and L -theory 732
- obstruction theory 580, 586
- octahedra 1039
- operad 10
- operations
 - Landweber–Novikov 1138
 - motivic Steenrod 1131
- operator
 - determinant-class 848
 - essentially normal 847
 - trace-class 848, 860
- operatorial homotopy 863
- orbit category 740
 - restricted 799
- orthogonal of a subobject 543
- orthogonal sum 542
- P 686
- p -chain 823
- P -exact category 15
- periodic cyclic homology 688
- periodicity isomorphisms 143
- permutative category 10
- Pfister form 1120
- Pfister neighbor 1120
- Pfister quadric 1120

- Pic-regularity 865
- Picard group 143, 182, 183
 - narrow 174
 - of a curve 160
- plus construction 208, 582, 597, 598
- Poincaré Complexes 621
- Poincaré Conjecture 711
- pointed monoid 81
 - algebra 81
- polylogarithmic motivic complexes 325
- polylogarithms on curves 341
- Pontrjagin class 583
- presheaf with transfers 1088
- presheaves with transfer 485
- primes over ℓ 977
- principal orbit 592
- principle
 - separation of variables 715, 819
- pro-abelian group 73
- product 662, 683
- projective bundle formula 201, 206
- projective L-Groups 619
- projective systems of C^* -algebras 668
- propagation
 - of group actions 595
- proper
 - G - C^* -algebra 805
 - G -space 802
 - metric space 777
- properties of the finiteness obstruction 712
- pseudo-equivalence 974
- pseudo-isomorphism 961
- pseudo-isotopy 599
- pseudoisotopy 717
- purely infinite 677

- Q-construction 208
- Q-construction 11
- qfh topology 1085
- quadratic chain complexes 635
- quasi-small object 968
- quasi-trivial torus 944
- Quillen Conjecture 357
- Quillen–Lichtenbaum Conjecture 917
 - semi-topological 917

- Ranicki periodicity 636
- rational computation of Algebraic K -Theory for Infinite Groups 818, 819
- rational computation of Topological K -Theory for Infinite Groups 817
- Rational $K_0(\mathbb{Z}G)$ -to- $K_0(\mathbb{Q}G)$ -Conjecture 754
- rational contribution of finite subgroups to $\mathrm{Wh}(G)$ 784
- rational equivalence 246
- rational injectivity of the Baum–Connes Assembly Map 776
- rational injectivity of the Classical K -Theoretic Assembly Map 783
- rational injectivity of the Farrell–Jones Assembly Map for Connective K -Theory 783
- real algebraic equivalence 892
- real space 892
- real vector bundle 893
- realization functor 456
- realizations 510
 - Bloch–Kriz 514
 - Huber’s method 513
 - via cycle classes 510, 512
- Recognition Principle 890
- refined cycle map 912
- regions in $\mathbb{Z} \times \mathbb{Z}$ 1023–1025
 - cosimplicial 1024–1025, 1068
- regular prime 144, 181, 182
- regular section 243
- regulators 506
- Reidemeister torsion 589
 - higher 602
- relative assembly map 741
- relative K -theory 72
- representation ring 813
- residue homomorphism 327
- resolution of singularities 867, 1086
- Riemann–Roch theorem 40
- rigid tensor category 455
- rigidity theorem 1088
- ring
 - integral domain 728
 - Noetherian 708

- regular 708
- ring with involution 797
- Rips complex 30
- Rost motive 1140
- Rothenberg sequence 614, 620, 632, 720
- $\mathbb{S}^1$ 49
- s -cobordism 583, 588
- s -cobordism theorem 588, 624, 711
 - controlled 592
 - equivariant 595
 - proper 593
 - stratified 596
- S -construction of Waldhausen 209, 270
- s operation 905
- saturation axiom 20
- Schatten ideals 860
- Segre class
 - total 901
- semi-discrete
 - module 962
 - prime 962, 975, 988
- semi- s -cobordism 583
- semi-topological K -theory
 - groups 884
 - real 891, 892
 - singular 888
 - space 884
- semi-topological spectral sequence 908–909
- seminormal 865
- Serre/Swan theorem 115
- sesquilinear forms 617
- Shaneson product formula 614, 620, 633, 637
- Shaneson-splitting 722
- sheaves with transfer 485
- shift
 - one-sided 602
 - two-sided 602
- shift equivalence 603
 - strong 603
- shift equivalence problem 603
- sign map 174
- signature 732
 - higher 732
- signature defect 174
- singular semi-topological functor 890
- singular set 592
- slice filtration 64
- slope spectral sequence 205
- small object 967
- smash product 794
 - of a space with a spectrum 794
- Soulé conjecture 391
- Soulé’s theorem 990, 996
- space
 - $\mathcal{C}$ -space 795
 - compactly generated 578
 - finitely dominated 579
 - ϵ -epsilon 592
 - Smith acyclic 594
 - stratified 592, 595
 - Browder-Quinn 595–597
- specialization homomorphism 252
- spectra 1055–1057
- spectral sequence
 - associated to filtered spectrum 268
 - Atiyah–Hirzebruch 270, 355
 - Bloch–Lichtenbaum et al. 360
 - Brown and Coniveau compared 275, 277
 - Brown, multiplicativity of 275
 - equivariant Atiyah–Hirzebruch spectral sequence 823
 - motivic 142, 143
 - p -chain spectral sequence 823
 - Quillen 272
 - Quillen and Coniveau compared 274
- spectrum 794
 - algebraic cobordism 1135
 - homotopy groups of a spectrum 794
 - map of spectra 794
 - motivic 63
 - motivic Eilenberg–Mac Lane 1129
 - motivic Thom 1135
 - structure maps of a spectrum 794
- spherical space form problem 580
- Spivak fibration 621
- split metabolic 545

- Splitting variety 1119
- stable homotopy groups 112
- stacks
 - intersection theory on 287
- standard conjecture 368
- sublagrangian 544
 - sublagrangian construction 559
 - sublagrangian reduction 548
- subshift of finite type 602
- surgery exact sequence 638
- Surgery Theorem 622
- surgery theory 620
- Suslin complex 446, 447, 486, 1083, 1091
- Suslin homology 446
- Suslin's Conjecture 916
- Suslin–Wodzicki Theorem 862
- suspension 660
- suspension extension 682
- suspension of a ring 120
- Swan homomorphism 594
- symbolic dynamics 602
- symmetric complexes 634
- symmetric form or space 541
- symmetric monoidal category 9
- symmetric ring spectrum 74
- symmetric signature 635
- Syntomic complex 207

- T -spectra 1128
- t -structures 1049–1050
- tame kernel 144
- tame symbol 144
- Tannakian category 455
- Tate conjecture 396
- Tate module 164
- Tate motive 1092
- Tate primes 963
 - extended 963
- Tate spectrum 626
- Tate twist 963
- Tate–Beilinson conjecture 400
- tensor algebra 680
- tensor product
 - of pointed $\mathbb{C}$ -space with a $\mathbb{C}$ -spectrum 795
- thick subcategory 972, 975
- Thom isomorphism in K -theory 122
- Toeplitz algebra 845, 847
- Toeplitz extension 682
- Toeplitz matrix 844
- Toeplitz operator 844
 - symbol of 844, 845
- topological cyclic homology 72, 90, 94, 216, 599
- topological filtration
 - K -theory 920
 - singular cohomology 919
- topological Hochschild homology 74, 76, 88
- topological Hochschild spectrum 77, 78
- topological K -theory 114, 659
- topological realisation functor 1133
- toric T -variety 938
- total quotient rings, sheaf of 244
- totally real field 993
- trace
 - standard trace of $C_r^*(G)$ 727
 - universal $\mathbb{C}$ -trace 753
- Trace Conjecture
 - for Torsion Free Groups 728
 - Modified Trace Conjecture 760
- Transitivity Principle 742
- transport groupoid 796
- triangles
 - distinguished 1018, 1035–1036
 - virtual 1019–1020
- triangulated categories 669, 1017–1018
- truncated polynomial algebra 72, 92, 94, 98, 103, 106
- twists and dimension shifting 631

- uniform embedding into Hilbert space 777
- unit groups 977
 - local 978
- Unit-Conjecture 730
- universal coefficient theorem 667
- unramified away from ℓ -extensions 978

- Vandiver's Conjecture 183
- vanishing conjecture 1101

-
- variety
 - complex algebraic 592, 595
 - projective homogeneous 1122
 - Severi-Brauer 1120
 - virtual triangles 1019–1020
 - virtually cyclic 738
 - Voevodsky conjecture 369
 - Voevodsky’s localization theorem 486

 - $W(-)$ 546, 560
 - $W^n(-)$ 560
 - $w_i(F)$ 140, 148, 169
 - $w_i^{(\ell)}(F)$ 147
 - Waldhausen K -theory 597, 598, 1013–1015, 1050, 1052–1053
 - Wall finiteness obstruction 578–581
 - epsilon-controlled 592
 - equivariant 593–595
 - Wall Realization Theorem 625
 - weak equivalence 578, 597
 - of spectra 794
 - weakly dualizable 973
 - weight n Deligne cohomology 307
 - weight filtration 457
 - on Borel–Moore homology 910
 - on K -cohomology 285
 - weight-two complexes 434
 - Whitehead group 588, 709
 - generalized 709
 - higher 820
 - of finite group 589
 - of torsion-free group 589
 - Whitehead torsion 586–589
 - equivariant 595
 - proper 593
 - stratified 596
 - Witt 540
 - shifted Witt groups of a triangulated category 560
 - Witt equivalence 546
 - Witt group of an exact category 546
 - Witt complex 91
 - Witt groups 132
 - Witt ring 1110

 - $z\text{-equi}(X, r)$ 1087
 - Zero-Divisor-Conjecture 729
 - Zero-in-the-spectrum Conjecture 733
 - zeta function 151, 152, 157, 161, 170, 390
 - Zink site 1008