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Algebraic Cohomology of Ring Spectra

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Abstract. Recent years have seen the development of new cohomology theories in p-adic algebraic geometry, including syntomic and prismatic cohomology. We promote these invariants of p-adic rings into invariants of p-adic ring spectra and try to understand what results.

Key words. homotopy, ring spectra, syntomic cohomology, topological cyclic homology

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Homotopy theorists understand the category of commutative rings to sit fully faithfully inside the larger category of $\mathbb{E}_\infty$-ring spectra. While the initial commutative ring is the integers $\mathbb{Z}$, the initial $\mathbb{E}_\infty$-ring is the sphere spectrum $\mathbb{S}$, which contains data about all the stable homotopy groups of spheres.

Given an invariant of commutative rings, we hope that it can be extended to an invariant of $\mathbb{E}_\infty$-rings and in the latter context that it might have meaning relevant to homotopy theory or differential topology. Such is certainly the case for the invariant known as topological cyclic homology, which is defined on all $\mathbb{E}_\infty$-rings and can be used to calculate information about diffeomorphism groups of manifolds. When restricted to p-adic commutative rings and considered up to p-adic completion, TC recovers étale sheafified algebraic K-theory [CM21, CMM21]. We will not attempt to do justice to the long and beautiful history of topological cyclic homology, but refer the reader to surveys such as [Rog14, HN19]. Instead, we begin around the year 2018.

An extraordinary tool for understanding the topological cyclic homology of p-adic commutative rings emerged in work of Bhatt, Morrow, and Scholze [BMS19]. They constructed, for any discrete quasisyntomic commutative ring R, a *motivic filtration* on $\mathrm{TC}(R)^\wedge_p$. We write $\mathrm{fil}^*_{\mathrm{mot}}\,\mathrm{TC}(R)^\wedge_p$ to denote this filtered $\mathbb{E}_\infty$-ring and view the associated spectral sequence as analogous to the Atiyah–Hizerburch spectral sequence for the topological K-theory of a topological space. The associated graded of the motivic filtration is known as *syntomic cohomology* (with various Breuil–Kisin twists).

Remark 0.1. A clear example of the motivic spectral sequences' effectiveness is found in work of Antieau, Krause, and Nikolaus [AKN22], who for each $n \geq 2$ developed an algorithm computing $\pi_k\,\mathrm{TC}(\mathbb{Z}/p^n)$. The algorithm terminates reasonably quickly for small integers k, including many values of k for which $\pi_k\,\mathrm{TC}(\mathbb{Z}/p^n)$ was previously completely inaccessible. By systematically analyzing their algorithm, they proved the striking fact that $\pi_k\,\mathrm{TC}(\mathbb{Z}/p^n)$ is trivial for all sufficiently large even integers k. On the other hand, also using the motivic spectral sequence but from a different point of view, we gave with Levy and Senger complete closed form calculations of $\pi_*(\mathrm{TC}(\mathbb{Z}/p^n)/(p,v_1))$ [HLS24]. Combining these ideas, it seems tantalizingly within reach to give a closed form computation of $\pi_*\,\mathrm{TC}(\mathbb{Z}/p^n)$, or at least of $\pi_*(\mathrm{TC}(\mathbb{Z}/p^n)/p)$.

Joint work with Raksit and Wilson generalized the Bhatt–Morrow–Scholze filtration, giving a construction of $\mathrm{fil}^*_{\mathrm{mot}}\,\mathrm{TC}(R)^\wedge_p$ for all "chromatically quasisyntomic" $\mathbb{E}_\infty$-rings R [HRW22]. In particular, it became possible to speak of the syntomic cohomology of a broad class of $\mathbb{E}_\infty$-ring spectra. Sections 1 and 2 of this survey will review the construction of the motivic filtration and some explicit calculations of syntomic cohomology.

Contemplating the syntomic cohomology of $\mathbb{E}_\infty$-rings can sometimes clarify features of the syntomic cohomology of discrete rings; two particularly nice examples of this can be found in [Man24] and [Wag25]. Perhaps more to the point, the syntomic cohomology of $\mathbb{E}_\infty$-rings has direct applications to stable homotopy theory:

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the motivic spectral sequence, when combined with many other spectacular recent advances, led to the resolution of Ravenel's telescope conjecture [BHLS23]. A concrete application of the disproof of the telescope conjecture, to appear in joint work with Burklund, Carmeli, Levy, Schlank, and Yanovski, is a new lower bound on the growth rate of the stable homotopy groups of spheres.

There remains a large gap between our understanding of syntomic cohomology of p-adic discrete rings and our understanding of syntomic cohomology of $\mathbb{E}_\infty$-rings. In particular, syntomic cohomology of discrete or animated $\mathbb{Z}$-algebras is often studied via prismatization and syntomification functors, using the formalism of ring stacks [BL22a, BL22b]. In the penultimate section of this survey, we will explain an approach to prismatization and syntomification of chromatically quasisyntomic $\mathbb{E}_\infty$-ring spectra that uses genuine S^1-equivariant homotopy theory. In the final section, we consider $\mathbb{E}_\infty$-rings that are not necessarily chromatically quasisyntomic and set up a spectral analogue of the category of animated $\mathbb{Z}$-algebras.

We aim only to give an idiosyncratic snapshot of rapidly evolving mathematics and to suggest approachable new directions of research. Hopefully, we will at least convey some excitement.

1 Constructing the motivic filtration. An $\mathbb{E}_\infty$-ring B is said to be *even* if $\pi_*(B)$ is concentrated in even degrees (in other words if $\pi_{2n-1}(B) \cong 0$ for all $n \in \mathbb{Z}$). Even $\mathbb{E}_\infty$-rings play an important organizational role in stable homotopy theory, thanks to a bundle of good properties summarized by the term "complex orientability." We will often view an even $\mathbb{E}_\infty$-ring B as filtered by its double-speed Postnikov filtration $\mathrm{fil}^*(B) = \tau_{\geq 2*}B$. The following definition, which first appeared in [HRW22], more interestingly filters an arbitrary $\mathbb{E}_\infty$-ring.

DEFINITION 1.1. *Let A be an $\mathbb{E}_\infty$-ring spectrum. Then the even filtration $\mathrm{fil}^*_{\mathrm{ev}}A$ is the limit, over all maps of $\mathbb{E}_\infty$-rings $A \to B$ where B is even, of $\tau_{\geq 2*}B$.*

The even filtration is more formally defined as a right Kan extension of the double-speed Postnikov filtration, along the inclusion of the full subcategory of even $\mathbb{E}_\infty$-rings into the category of all $\mathbb{E}_\infty$-rings. It is computable in practice via a form of faithfully flat descent and in particular via the following proposition.

PROPOSITION 1.2. *Suppose that $A \to B$ is a map of $\mathbb{E}_\infty$-rings, that B is even, and that B admits an A-module cell decomposition with even-dimensional cells. Then $\mathrm{fil}^*_{\mathrm{ev}}A \simeq \lim_\Delta(\tau_{\geq 2*}(B^{\otimes_A \bullet+1}))$. In other words, the spectral sequence associated to the even filtration on A is exactly the usual descent spectral sequence computing $\pi_*(A)$ from $\pi_*(B^{\otimes_A \bullet+1})$.*

Example 1.3. The $\mathbb{E}_\infty$-ring MU of complex bordism has even homotopy groups and even $\mathbb{S}$-module cells, so Proposition 1.2 applies to the unit map $\mathbb{S} \to \mathrm{MU}$. Thus, the spectral sequence associated to $\mathrm{fil}^*_{\mathrm{ev}}\mathbb{S}$ is the Adams–Novikov spectral sequence computing $\pi_*(\mathbb{S})$.

Remark 1.4. The filtered $\mathbb{E}_\infty$-ring $\mathrm{fil}^*_{\mathrm{ev}}\mathbb{S}$ carries a great deal of structure beyond its associated spectral sequence and in fact plays an outsized role in modern stable homotopy theory. In particular, it was observed in [GIKR22] that p-completed $\mathrm{fil}^*_{\mathrm{ev}}\mathbb{S}$-modules form a category equivalent to the category of p-completed, cellular $\mathbb{C}$-motivic specta. Following Pstrągowski's works [Pst23, Pst24], we call the category of (not necessarily p-complete) $\mathrm{fil}^*_{\mathrm{ev}}\mathbb{S}$-modules the category of *synthetic spectra.*

Remark 1.5. If A is any $\mathbb{E}_\infty$-ring, then multiplicativity properties of the even filtration ensure that $\mathrm{fil}^*_{\mathrm{ev}}A$ is a $\mathrm{fil}^*_{\mathrm{ev}}\mathbb{S}$-module or, in other words, a synthetic spectrum. In particular, the p-completion of $\mathrm{fil}^*_{\mathrm{ev}}A$ may be viewed as a cellular $\mathbb{C}$-motivic spectrum. In fact, as recognized by Burklund and Senger, the p-completion of $\mathrm{fil}^*_{\mathrm{ev}}A$ is a very special sort of cellular $\mathbb{C}$-motivic spectrum. It carries the structure of a *coconnective normed $\mathbb{C}$-motivic spectrum,* as we will discuss in section 4.

For well-behaved $\mathbb{E}_\infty$-ring spectra R, the even filtration can be used to define a motivic filtration on $\mathrm{THH}(R)$.

DEFINITION 1.6 ([HRW22]). *A discrete commutative ring A is said to be* quasisyntomic *if it has bounded p-power torsion for each prime p and the algebraic cotangent complex $L^{\mathrm{alg}}_{A/\mathbb{Z}}$ is concentrated in Tor-amplitude* $[0,1]$. *A connective $\mathbb{E}_\infty$-ring R is said to be* chromatically quasisyntomic *if $\mathrm{MU}_* R$ is concentrated in even degrees and, when viewed as an ungraded commutative ring, is quasisyntomic in the sense of the previous sentence. If R is chromatically quasisyntomic, we define $\mathrm{fil}^*_{\mathrm{mot}}\,\mathrm{THH}(R)$ to be $\mathrm{fil}^*_{\mathrm{ev}}\,\mathrm{THH}(R)$.*

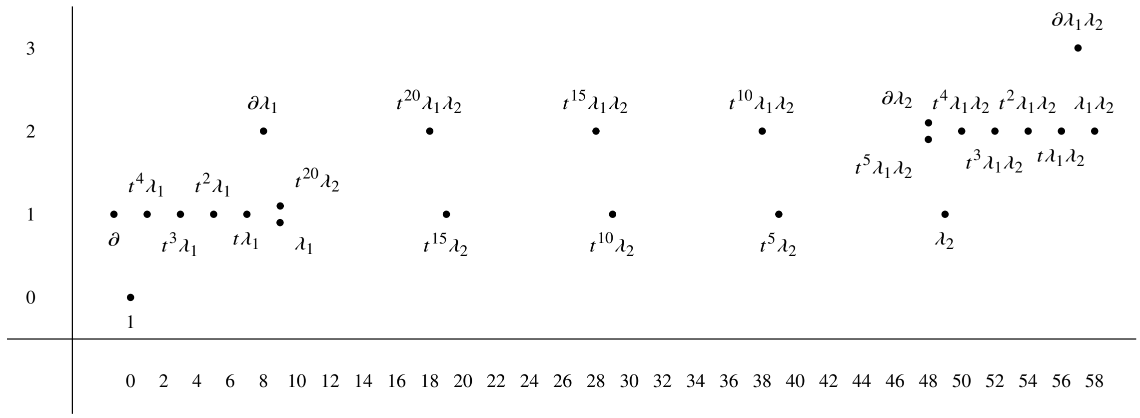
Attention is restricted to chromatically quasisyntomic R precisely because under that hypothesis $\mathrm{fil}^*_{\mathrm{ev}}\,\mathrm{THH}(R)$ can be understood using Proposition 1.2. In fact, it is proved in [HRW22] that a map

$$A = \mathrm{THH}(R) \to B$$

can be found, satisfying the hypotheses of Proposition 1.2, that also respects the circle action and Frobenii present on $\mathrm{THH}(R)$. This is enough to ensure that the motivic filtration on $\mathrm{THH}(R)$ is compatible with both circle action and p-primary Frobenius for each prime p [AR24] and in particular to define $\mathrm{fil}^*_{\mathrm{mot}}\,\mathrm{TC}(R)^\wedge_p$. One defines the *syntomic cohomology* of R, with various Breuil–Kisin twists, to be the associated graded $\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(R)^\wedge_p$.

Remark 1.7. With care, a motivic filtration on $\mathrm{THH}(R)$ can often be defined when R admits only part of an $\mathbb{E}_\infty$-ring structure [Pst24, Wag25, AKHW25]. This remains an interesting area of ongoing research.

2 Arithmetic duality in examples. Some $\mathbb{E}_\infty$-rings, such as the p-completions of ku, ko, tmf or the Adams summand ℓ, behave very much like rings of integers in p-adic local fields. Specific computations of syntomic cohomology in these instances show striking patterns, originally anticipated in visionary conjectures by Rognes and by Ausoni and Rognes, which beg for general explanations (see, e.g., [Rog14, section 5]). To demonstrate these patterns, consider the following image of the reduced syntomic cohomology of the Adams summand ℓ at the prime $p=5$.



The above image is reproduced from [HRW22], where a calculation of $\pi_*\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\ell)/(5,v_1,v_2)$ is explained in great detail. The rotational symmetry reflects the fact that $\pi_a\mathrm{gr}^b_{\mathrm{mot}}\,\mathrm{TC}(\ell)/(5,v_1,v_2)$ is a finite Poincaré duality algebra with top class $\partial\lambda_1\lambda_2$. The horizontal axis here records a, while the vertical axis records $2b-a$.

The situation is reminiscent of local Tate duality in the mod p étale cohomology of $\mathbb{Q}_p$ and of the Lagrangian refinement thereof proved by Bhatt and Lurie [Bha23, Theorem 2.18]. Ongoing joint work with Devalapurkar and Rognes identifies a large class of ring spectra B for which the syntomic cohomology of B admits similar arithmetic duality, though the optimal set of hypotheses to put on B remains unclear. When proving duality, a key and independently interesting step is to understand compatibility between transfer maps and the motivic filtration.

Example 2.1. The fact that $\mathbb{F}_p$ is a perfect ℓ-module gives rise to a restriction functor from the category of perfect $\mathbb{F}_p$-modules to the category of perfect ℓ-modules and therefore to a transfer map

$$\mathrm{TC}(\mathbb{F}_p)\to\mathrm{TC}(\ell)$$

of $\mathrm{TC}(\ell)$-modules. Supposing that the prime p is large enough that a Smith–Toda complex exists, we can form a reduced transfer map

$$\pi_*(\mathrm{TC}(\mathbb{F}_p)/(p,v_1,v_2))\to\pi_*(\mathrm{TC}(\ell)/(p,v_1,v_2)),$$

and a key point is that a class detected by $\partial\lambda_1\lambda_2$ is in the image of this reduced transfer. More generally, one can use the transfer

$$\pi_*\,\mathrm{TC}(\mathbb{F}_p)/(p^a,v_1^b,v_2^c)\to\pi_*\,\mathrm{TC}(\ell)/(p^a,v_1^b,v_2^c)$$

to define compatible dualizing classes in the codomain as $a,b,$ and c tend to ∞.

Up to a shift in motivic weights, it turns out that the transfer map $\mathrm{TC}(\mathbb{F}_p)\to\mathrm{TC}(\ell)$ is compatible with the motivic filtrations on both sides, but this requires nontrivial work to establish.

Question 2.2. *In what generality can it be proved that transfer maps in topological cyclic homology are compatible with motivic filtrations?*

Remark 2.3. When working with discrete rings, Tang has produced transfer maps for syntomic cohomology and used them to prove various duality statements [Tan24]. Further study of arithmetic duality appears, for example, in [CF25]. The question above concerns generalizing these results to ring spectra.

Remark 2.4. Let $B \to k$ be a connective $\mathbb{E}_\infty$-ring mapping to a finite field k, and suppose that $\pi_*(k \otimes_B k)$ is finite. An avatar of the arithmetic duality in syntomic cohomology that we have been discussing already can be seen in the homotopy groups of $\mathrm{THH}(B;k)$, where it was investigated and explained by Greenlees [Gre16]. Greenlees proves that, in many instances, the $\mathbb{E}_\infty$-ring map $\mathrm{THH}(B;k) \to k$ is relatively Gorenstein.

Remark 2.5. The fact that the bigraded homotopy groups $\pi_*\mathrm{gr}^*\,\mathrm{TC}(\ell)/(p,v_1,v_2)$ are concentrated in a finite range of degrees is enough to deduce the Lichtenbaum–Quillen property for $\mathrm{TC}(\ell)$, which is the statement that

$$\mathrm{TC}(\ell)_{(p)} \to L_2^f\,\mathrm{TC}(\ell)_{(p)}$$

has bounded above fiber. This is the key feature of $\mathrm{TC}(\ell)$ that is used in the disproof of the height 2 telescope conjecture at the prime p. It was proved for primes $p \geq 5$ in visionary work of Ausoni and Rognes [AR02] and first proved for $p = 2,3$ using an early prototype of the motivic spectral sequence [HW22]. To disprove the telescope conjecture at a general prime p and height n, a crucial input is to bound $V_*\,\mathrm{TC}(\mathrm{BP}\langle n\rangle)$ and $V_*\,\mathrm{TR}(\mathrm{BP}\langle n\rangle)$ whenever V is a type $n+2$ complex. Outside of a select few heights and primes, all known arguments for such a bound proceed via the motivic filtration.

3 Prismatization via genuine equivariance. We now discuss in more detail how the circle action and Frobenius on $\mathrm{THH}(R)$ interact with the even filtration. This has been explored from multiple directions (cf. [AR24]), the most basic of which from [HRW22] suffices to define $\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(R)$ and thus define syntomic cohomology. However, the approach of [HRW22] does not suffice to understand the *syntomification, prismatization,* or *Nygaardification* of R, and we will sketch here an approach to those stacks that will appear in joint work with Devalapurkar, Raksit, and Yuan.

As a warm-up, consider the circle action on $\mathrm{THH}(R)$ when R is the sphere spectrum. In this case, $\mathrm{THH}(R)$ is the S^1-equivariant sphere spectrum, and experience suggests that we should consider descent along the S^1-equivariant $\mathbb{E}_\infty$-ring map

$$\mathbb{S}_{\mathrm{S}^1} \to \mathrm{MU}_{\mathrm{S}^1}.$$

By means of analogy with Proposition 1.2, note that $\mathrm{MU}_{\mathrm{S}^1}$ has complex representation cells as an $\mathbb{S}_{\mathrm{S}^1}$-module and that $\pi_{V-1}\,\mathrm{MU}_{\mathrm{S}^1} \cong 0$ for every $V \in \mathrm{RU}(\mathrm{S}^1)$ (see, e.g., [Hau22]).

Remark 3.1. So far, we have been purposefully ambiguous regarding whether we view $\mathrm{THH}(R)$ as a genuine S^1-spectrum or (following the developments of Nikolaus and Scholze [NS18]) as a Borel S^1-equivariant spectrum. In the case that R is the sphere spectrum, the Segal conjecture tells us that after p-completion there is minimal difference between these points of view [Lin80, AGM85, Car84]. However, even after p-completion, $\mathrm{MU}_{\mathrm{S}^1}$ is far from Borel. Though ultimately interested in Borel S^1-equivariant spectra, we will want to resolve them via S^1-equivariant spectra that are not Borel.

Remark 3.2. Somewhat in conflict with the above remark, a technical but important point is that $\mathrm{THH}(R)$ should always be viewed as cyclonic, i.e., with genuine S^1 fixed points determined as the limit

$$\mathrm{THH}(R)^{\mathrm{S}^1} = \lim_{C_m}(\mathrm{THH}(R)^{C_m})^{\mathrm{hS}^1/C_m},$$

where the limit ranges over finite subgroups $C_m \subset \mathrm{S}^1$. Cyclonic spectra sit inside all S^1-spectra as a localization, and when we write any S^1-spectrum such as $\mathrm{MU}_{\mathrm{S}^1}$ we often implicitly mean its cyclonic completion. Let us not belabor this point here.

Of course, THH of any ring also carries Frobenii, and continuing with our warm-up one can ask what sort of Frobenii are carried by $\mathrm{MU}_{\mathrm{S}^1}$. An answer comes in the form of S^1-equivariant morphisms

$$\varphi_p : \mathrm{MU}_{\mathrm{S}^1} \to \Phi^{C_p}\,\mathrm{MU}_{\mathrm{S}^1}$$

associated to prime numbers p. For each p, φ_p may be constructed by applying the Thom construction to a map of S^1-spaces $\mathrm{BU}_{\mathrm{S}^1} \to \mathrm{BU}_{\mathrm{S}^1}^{C_p}$, which classifies the construction taking a virtual S^1-equivariant vector bundle V to the tensor product $V \otimes (1 \oplus \lambda \oplus \cdots \oplus \lambda^{\otimes(p-1)})$ of the pullback of V with a direct sum of line bundles. Here, λ refers to the standard action of S^1 on the complex numbers.

Remark 3.3. Nikolaus and Scholze made a careful study of the symmetric monoidal category of S^1-spectra X equipped with compatible maps $X \to \Phi^{C_p} X$ [NS18, II.5]. They called this category $\mathrm{CoAlg}_{(\Phi^{C_p})_{p\in\mathbb{P}}}(\mathbb{T}\mathrm{Sp}_{\mathcal{F}})$, but we prefer to call it the category of *lax cyclotomic spectra.* The reader familiar with the usual category of cyclotomic spectra should note that we do not require these maps $X \to \Phi^{C_p} X$ to be equivalences, nor do we require X to be Borel. In some literature, lax cyclotomic spectra are called op-pre cyclotomic spectra.

Remark 3.4. Suppose that R is a bounded below $\mathbb{E}_\infty$-ring. Then $\mathrm{THH}(R)$ viewed as a Borel S^1-spectrum is lax cyclotomic, with structure maps given by the Nikolaus–Scholze Frobenii

$$\mathrm{THH}(R) \to \Phi^{C_p}\,\mathrm{THH}(R) \cong \mathrm{THH}(R)^{tC_p}.$$

On the other hand, we will write $\mathrm{THH}^{\mathrm{gen}}(R)$ to refer to the traditional (and often not Borel) S^1-equivariant enhancement of $\mathrm{THH}(R)$, which is lax cyclotomic via equivalences $\mathrm{THH}^{\mathrm{gen}}(R) \xrightarrow{\simeq} \Phi^{C_p}\,\mathrm{THH}^{\mathrm{gen}}(R)$.

As promised, we now connect this discussion with the theory of prismatization. The following result should be reminiscent of Proposition 1.2.

THEOREM 3.5. *Let R denote an F-smooth discrete quasisyntomic ring, and view $\mathrm{THH}(R)$ as a Borel lax cyclotomic $\mathbb{E}_\infty$-ring. Then there is a map of lax cyclotomic $\mathbb{E}_\infty$-rings $\mathrm{THH}(R) \to B$, such that $\pi^{S^1}_{\star-1} B \cong 0$ for each $\star \in \mathrm{RU}(S^1)$, and such that B has complex representation cells as a $\mathrm{THH}(R)$-module. Descent along this map recovers the prismatization, Nygaardification, and syntomification of R.*

To be more precise about the final sentence in the theorem statement, the p-completion of the descent computing $\pi^{\mathrm{S}^1}_\star\,\mathrm{THH}(R)$ from $\pi^{\mathrm{S}^1}_\star(B^{\otimes_{\mathrm{THH}(R)}\bullet+1})$ records the Nygaardification R^{Nyg}. The p-completion of the descent computing $\pi^{\mathrm{S}^1/C_p}_\star(\Phi^{C_p}\,\mathrm{THH}(R))$ from $\pi^{\mathrm{S}^1/C_p}_\star(\Phi^{C_p} B^{\otimes_{\mathrm{THH}(R)}\bullet+1})$ records the prismatization R^{Δ}. The lax cyclotomic structure then accounts for the natural maps

$$\mathrm{can}, \varphi : R^{\Delta} \to R^{\mathrm{Nyg}}$$

that are required to define syntomification.

Remark 3.6. A similar statement to Theorem 3.5 holds for arbitrary discrete quasisyntomic rings R but with the caveat that one must equip $\mathrm{THH}(R)$ with a slightly more sophisticated lax cyclotomic $\mathbb{E}_\infty$-ring structure than the Borel one.

Of course, underlying Theorem 3.5 is an S^1-equivariant variant of the even filtration, based on representation graded homotopy groups.

QUESTION 3.7. *What can be said about the equivariant even filtration associated to a general finite group G or compact Lie group G? In particular, does the equivariant even filtration of the sphere give rise to a well-behaved spectral sequence? Is the equivariant even filtration of the sphere related to G-equivariant $\mathbb{C}$-motivic homotopy theory?*

Forthcoming work of Allen and Piessevaux answers these questions in the case that G is a finite abelian group, by proving a connection between the G-equivariant Adams–Novikov spectral sequence and the pre-existing notion of G-equivariant, cellular $\mathbb{C}$-motivic homotopy theory [Hoy17]. It would be particularly interesting if the questions had affirmative answers for groups G such that MU_G is not well-behaved.

Generalizing Theorem 3.5, one can ask about stacks associated to other lax cyclotomic $\mathbb{E}_\infty$-ring spectra.

Example 3.8. Using the map $\mathrm{THH}^{\mathrm{gen}}(\mathbb{S}) = \mathbb{S}_{\mathrm{S}^1} \to \mathrm{MU}_{\mathrm{S}^1}$ mentioned above, we can calculate stacks that deserve the names $\mathbb{S}^{\mathrm{Nyg}}$ and $\mathbb{S}^{\Delta}$, as well as Frobenius and canonical maps between them. For example, at a fixed prime p we calculate the Nygaardification of the sphere spectrum to be the moduli stack of C_{p^∞}-equivariant formal groups.

We also calculate stacks associated to $\mathrm{THH}^{\mathrm{gen}}(\mathrm{MU})$ using the following result, which may be summarized by saying that $\mathrm{MU}_{\mathrm{S}^1}$ admits the structure of a lax cyclotomic base.

Theorem 3.9. *There exists a lax cyclotomic $\mathbb{E}_\infty$-ring structure on* $\mathrm{MU}_{\mathrm{S}^1}$*, together with a map of lax cyclotomic $\mathbb{E}_\infty$-rings*

$$\mathrm{THH}^{\mathrm{gen}}(\mathrm{MU}) \to \mathrm{MU}_{\mathrm{S}^1}.$$

As an S^1-equivariant $\mathrm{THH}^{\mathrm{gen}}(\mathrm{MU})$*-module,* $\mathrm{MU}_{\mathrm{S}^1}$ *admits a cell decomposition with complex representation cells.*

Remark 3.10. The Segal conjecture of [LeNR11] implies that $\mathrm{THH}^{\mathrm{gen}}(\mathrm{MU})$ agrees with the Borel $\mathrm{THH}(\mathrm{MU})$ after p-completion at any prime p.

One upshot of our investigation of the syntomification of MU is a completely algebraic description of the E_2-page of the motivic spectral sequence computing $\pi_* \mathrm{TC}(\mathrm{MU})$, though explicitly calculating that E_2-page remains challenging in practice. It would be nice to see explicit computations in low degrees.

Question 3.11. *Does the motivic spectral sequence converging to* $\pi_* \mathrm{TC}(\mathrm{MU})$ *collapse at its* E_2*-page?*

Question 3.12. *Can one characterize the Hurewicz image of* $\pi_*\mathbb{S}$ *inside of* $\pi_* \mathrm{TC}(\mathrm{MU})$ *or the image of the Adams–Novikov* E_2*-page inside of the* E_2*-page of the motivic spectral sequence computing* $\pi_* \mathrm{TC}(\mathrm{MU})$?

This question is of course just the beginning of an oft-suggested program to compute the spectral sequence associated to the cosimplicial object $\mathrm{TC}(\mathrm{MU}^{\otimes \bullet+1})$.

We would also like to better understand the Hurewicz images of $\pi_*\mathbb{S}$ in the topological cyclic homologies of rings that are not complex-orientable (indeed, much of our knowledge of telescopic homotopy is now gleaned by contemplating the unit map from $T(2)_*\mathbb{S}$ to $T(2)_* \mathrm{TC}(j_\zeta)$). Motivic spectral sequences should make it approachable, for instance, to compare the Hurewicz images of $\pi_*\mathbb{S}$ in $\pi_*\mathrm{TC}(\mathrm{ko})$ and $\pi_*\mathrm{TC}(\mathrm{ku})$ (cf. [AKAR23]).

Question 3.13. *Are there clear algebraic descriptions of the prismatizations and syntomifications of* ku *or* ko *along the lines of the known descriptions of* $\mathbb{Z}_p^{\Delta}$ *and* $\mathbb{Z}_p^{\mathrm{syn}}$?

The above question is substantially deeper than asking about the prismatic or syntomic cohomologies of ku and ko, which we currently have a better handle on.

4 Synthetic $\mathbb{E}_\infty$-rings and synthetic THH**.** So far, our discussion has been largely restricted to the case of chromatically quasisyntomic $\mathbb{E}_\infty$-ring spectra. It is desirable to have more robust definitions moving beyond this setting, especially when aiming to study formal properties of the syntomification functor as a whole. In the case of discrete p-adic rings, a key point is that prismatization is defined on all *animated rings,* rather than merely discrete rings, and many constructions are defined and analyzed via the operation of animation.

In this final section we will briefly indicate work in progress, jointly with Devalapurkar, Raksit, and Senger, that defines appropriate analogues of animated rings in the setting of spectral algebraic geometry (since parts of the story remain work in progress, this section may be viewed as speculative). Our analogues will be *synthetic* $\mathbb{E}_\infty$*-rings,* which are algebras over a monad on synthetic spectra. Forthcoming results of Burklund and Senger prove that the category of synthetic $\mathbb{E}_\infty$-rings agrees with the Bachmann–Hoyois category of cellular, normed $\mathbb{C}$-motivic spectra after completion at any prime p [BH21]. In general, our conception of synthetic $\mathbb{E}_\infty$-rings was heavily influenced by forthcoming motivic results due variously to Bachmann, Burklund, Hopkins, Hoyois, and Senger. However, the theory we now present is phrased entirely without reference to motivic homotopy theory and so is in several aspects more technically straightforward.

To begin, recall that animated rings embed fully faithfully in the larger category of *derived commutative rings,* which is the category of algebras for the derived symmetric algebra monad LSym on $\mathbb{Z}$-module spectra. Animated rings are precisely those derived rings whose underlying spectra are connective. On the other hand, the category of coconnective derived rings embeds fully faithful into the category of prestacks, with essential image the so-called affine stacks [MM25]. In spectral algebraic geometry, one can offer the following analogue of prestacks.

Definition 4.1. *An even prestack is a functor from the full subcategory of even* $\mathbb{E}_\infty$*-rings to* Spaces.

Remark 4.2. The choice to consider even $\mathbb{E}_\infty$-rings as test objects in spectral algebraic geometry, as opposed to all $\mathbb{E}_\infty$-rings or connective $\mathbb{E}_\infty$-rings, may seem a priori unmotivated. At the end of the day, we believe this variant of spectral algebraic geometry is justified by its connections to both the Bhatt–Morrow–Scholze motivic filtration and the Bachmann–Hoyois category of normed $\mathbb{C}$-motivic spectra.

Remark 4.3. Any $\mathbb{E}_\infty$-ring R gives rise to an even prestack $A \mapsto \mathrm{Hom}_{\mathbb{E}_\infty-\mathrm{rings}}(R, A)$. We call this even prestack $\mathrm{Spev}(R)$. The Yoneda embedding implies that Spev is fully faithful when restricted to even $\mathbb{E}_\infty$-rings, but a more surprising fact is that Spev is also fully faithful when restricted to connective $\mathbb{E}_\infty$-rings. This is a consequence of forthcoming work of Burklund and Krause, as explained in [Gre25, 4.4.1–4.4.2] in the closely related setting of even periodic prestacks.

A guiding principle, first crystallized by Raksit, is that coconnective synthetic $\mathbb{E}_\infty$-rings should embed fully faithfully into even prestacks. To make sense of that, we need to explain what sort of synthetic spectra we call coconnective.

Definition 4.4. *A* $\mathrm{fil}^*_{\mathrm{ev}}\mathbb{S}$*-module*

$$M = \cdots \to \mathrm{fil}^2 M \to \mathrm{fil}^1 M \to \mathrm{fil}^0 M \to \mathrm{fil}^{-1} M \to \mathrm{fil}^{-2} M \to \cdots$$

in filtered spectra is said to be coconnective *if the following hold:*

1. *It is a complete filtered object, meaning* $\lim_n \mathrm{fil}^n M \cong 0$.
2. *For each* $i \in \mathbb{Z}$*, the* i*th associated graded piece* $\mathrm{gr}^i M$ *is* $(2i)$*-coconnective.*

This is a variant of the Beilinson coconnectivity described in [Ant24, Definition 3.25].

Remark 4.5. If A is any $\mathbb{E}_\infty$-ring, then $\mathrm{fil}^*_{\mathrm{ev}} A$ is coconnective. If A is even, then $\mathrm{fil}^*_{\mathrm{ev}} A = \tau_{\geq 2*} A$ has the additional property that its ith associated graded piece is concentrated exactly in degree $2i$. Coconnective synthetic spectra with this additional property are said to be *pure.* Another guiding principle of the theory (proved in the motivic context by Burklund and Senger) is that the full subcategory of pure synthetic $\mathbb{E}_\infty$-rings should be equivalent to the full subcategory of even $\mathbb{E}_\infty$-rings. In other words, representable even prestacks correspond precisely to pure synthetic $\mathbb{E}_\infty$-rings.

Suppose now that we wish to understand the free synthetic $\mathbb{E}_\infty$-ring on a bigraded sphere $S^{a,b}$, and for simplicity let us assume that $a > 0$ and $2b - a \geq 0$. These conditions ensure that the free synthetic $\mathbb{E}_\infty$-ring will be coconnective and so can be understood as an even prestack. Indeed, as an even prestack it is simply given by the functor, from even $\mathbb{E}_\infty$-rings to spaces, that sends R to $B^{2b-a}\Omega^\infty \Sigma^{-2b} R$. From this point of view, maps between free synthetic $\mathbb{E}_\infty$-rings are just natural transformations of functors.

We obtain the category of synthetic $\mathbb{E}_\infty$-rings by an animation process applied to the full subcategory of free synthetic $\mathbb{E}_\infty$-rings on coconnective bigraded spheres, where we understand that full subcategory according to the principles of the preceding paragraph. The animation process has a mild subtlety, in that we must enforce relations relating bar constructions on free algebras to other free algebras. The relations can be enforced as in [Bal25, section 3], and a detailed discussion will appear in forthcoming work of Raksit.

Construction 4.6. *There is a category of synthetic* $\mathbb{E}_\infty$*-rings that is both monadic and comonadic over the category of* $\mathbb{E}_\infty$*-algebra objects in synthetic spectra. A full subcategory of coconnective* $\mathbb{E}_\infty$*-rings embeds fully faithfully inside of the category of even prestacks. The even filtration is left adjoint to the Betti realization functor from coconnective synthetic* $\mathbb{E}_\infty$*-rings to* $\mathbb{E}_\infty$*-rings.*

Several of the facts outlined in Construction 4.6 are consequences of very general results regarding animation by Raksit but require key calculations specific to this setting. In particular, we crucially use the classification by Landweber of finitely generated $\mathrm{MU}_* \mathrm{MU}$ comodules [Lan76], Wilson's work on the infinite loop spaces in MU [Wil73], and a Künneth theorem for the even filtration on a tensor product of $\mathbb{E}_\infty$-rings. The idea that the even filtration can be characterized as a left adjoint was first noticed in the motivic setting by Burklund and Senger.

Once the category of synthetic $\mathbb{E}_\infty$-rings is at hand, one can define synthetic THH of synthetic $\mathbb{E}_\infty$-rings via animation (with its values on coconnective free algebras given by the even filtration). As an endofunctor of synthetic $\mathbb{E}_\infty$-rings, there is no general reason that synthetic THH should send a coconnective object to a coconnective object. In other words, when applying synthetic THH we may very well leave the realm of even prestacks. From this point of view, the role of quasisyntomicity assumptions becomes clearer: when a discrete ring A is quasisyntomic, $\mathrm{THH}^{\mathrm{syn}}(A)$ is coconnective. It would be valuable to more fully understand the collection of coconnective synthetic $\mathbb{E}_\infty$-rings A for which $\mathrm{THH}^{\mathrm{syn}}(A)$ remains coconnective.

Given $\mathrm{THH}^{\mathrm{syn}}$, it becomes plausible to develop a ring stack approach to syntomic cohomology. There remains, however, much work to be done, including carefully connecting the theory sketched here to the ideas of section 3 and [AR24]. Let us end by mentioning a few large scale questions about the syntomic cohomology of synthetic

$\mathbb{E}_\infty$-rings that we would like to see answered. The importance of the first of these has been repeatedly emphasized by Senger.

Question 4.7. *Suppose R is a synthetic $\mathbb{E}_\infty$-ring. Does the synthetic $T(n)$ localization of $\mathrm{TC}^{\mathrm{syn}}(R)$ depend only on the synthetic $T(n-1)\oplus T(n)$-localization of R?*

Remark 4.8. Comparison theorems between the syntomic and étale cohomologies of discrete p-adic rings appear consistent with an affirmative answer. Of course, this question is motivated by [LMMT24].

Question 4.9. *For R a synthetic $\mathbb{E}_\infty$-ring, what structure on R/τ determines what structure on $\pi_{*,*}(\mathrm{TC}^{\mathrm{syn}}(R)/\tau)$?*

Remark 4.10. By means of analogy, note that if A is an $\mathbb{E}_\infty$-ring, then $\mathrm{TC}(A)$ is also an $\mathbb{E}_\infty$-ring, but the homotopy ring $\pi_*\,\mathrm{TC}(A)$ depends only on A as an $\mathbb{E}_2$-algebra.

As a final point, taking seriously the idea that $\pi_*\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(R)$ records a form of étale motivic cohomology with Tate twists, we may ask the following question.

Question 4.11. *What natural transformations exist between syntomic cohomology of different bigradings? In other words, what is the syntomic Steenrod algebra?*

One Steenrod operation can be defined using the differential in the motivic (i.e., τ-Bockstein) spectral sequence converging from $\pi_{*,*}(\mathrm{TC}^{\mathrm{syn}}(R)/\tau)$ to $\pi_*\,\mathrm{TC}^{\mathrm{syn}}(R)[\tau^{-1}]$. The question may admit better answers when restricted to full subcategories of synthetic $\mathbb{E}_\infty$-rings. For example, as we understand it, the conjectured prismatic stable homotopy theory of Lurie predicts that many Steenrod operations exist between the syntomic cohomologies of discrete $\mathbb{Z}_p$-algebras.

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References

[AGM85] J. F. Adams, J. H. Gunawardena, and H. Miller, *The Segal conjecture for elementary abelian p-groups*, Topology, 24 (1985), pp. 435–460.

[AKAR23] G. Angelini-Knoll, C. Ausoni, and J. Rognes. *Algebraic K-Theory of Real Topological K-Theory*, preprint, arXiv:2309.11463, 2023.

[AKHW25] G. Angelini-Knoll, J. Hahn, and D. Wilson, *Syntomic Cohomology of Morava K-Theory*, preprint, arXiv:2410.07048, 2025.

[AKN22] B. Antieau, A. Krause, and T. Nikolaus, *On the K-Theory of $\mathbb{Z}/p^n$*, preprint, https://arxiv.org/abs/2405.04329, 2024.

[Ant24] B. Antieau, *Spectral Sequences, Décalage, and the Beilinson T-Structure*, preprint, arXiv:2411.09115, 2024.

[AR02] C. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory*, Acta Math., 188 (2002), pp. 1–39.

[AR24] B. Antieau and N. Riggenbach, *Cyclotomic Synthetic Spectra*, preprint, arXiv:2411.19929, 2024.

[Bal25] W. Balderrama, *Deformations of homotopy theories via algebraic theories*, Adv. Math., 480 (2025), 110496, https://doi.org/10.1016/j.aim.2025.110496.

[BH21] T. Bachmann and M. Hoyois, *Norms in motivic homotopy theory*, Astérisque, 425 (2021).

[Bha23] B. Bhatt, *Algebraic geometry in mixed characteristic*, in ICM—International Congress of Mathematicians, Vol. 2, EMS Press, Berlin, 2023, pp. 712–748.

[BHLS23] R. Burklund, J. Hahn, I. Levy, and T. M. Schlank. *K-Theoretic Counterexamples to Ravenel's Telescope Conjecture*, preprint, arXiv:2310.17459, 2023.

[BL22a] B. Bhatt and J. Lurie, *Absolute Prismatic Cohomology*, preprint, arXiv:2201.06120, 2022.

[BL22b] B. Bhatt and J. Lurie, *The Prismatization of p-Adic Formal Schemes*, preprint, arXiv:2201.06124, 2022.

[BMS19] B. Bhatt, M. Morrow, and P. Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci., 129 (2019), pp. 199–310.

[Car84] G. Carlsson, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Ann. of Math. (2), 120 (1984), pp. 189–224, https://doi.org/10.2307/2006940.

[CF25] S. Carmeli and T. Feng, *Prismatic Steenrod Operations and Arithmetic Duality on Brauer Groups*, preprint, arXiv:2507.13471, 2025.

[CM21] D. Clausen and A. Mathew, *Hyperdescent and étale K-theory*, Invent. Math., 225 (2021), pp. 981–1076.

[CMM21] D. Clausen, A. Mathew, and M. Morrow, *K-theory and topological cyclic homology of henselian pairs*, J. Amer. Math. Soc., 34 (2021), pp. 411–473.

[GIKR22] B. Gheorghe, D. C. Isaksen, A. Krause, and N. Ricka, *$\mathbb{C}$-motivic modular forms*, J. Eur. Math. Soc. (JEMS), 24 (2022), pp. 3597–3628.

[Gre16] J. P. C. Greenlees, *Ausoni-Bökstedt duality for topological Hochschild homology*, J. Pure Appl. Algebra, 220 (2016), pp. 1382–1402, https://doi.org/10.1016/j.jpaa.2015.09.007.

[Gre25] R. Gregoric, *Even Periodization of Spectral Stacks*, preprint, arXiv:2504.16410, 2025.

[Hau22] M. Hausmann, *Global group laws and equivariant bordism rings*, Ann. of Math. (2), 195 (2022), pp. 841–910, https://doi.org/10.4007/annals.2022.195.3.2.

[HLS24] J. Hahn, I. Levy, and A. Senger, *Crystallinity for Reduced Syntomic Cohomology and the mod $(p, v_1^{p^{n-2}})$ K-Theory of $\mathbb{Z}/p^n$*, preprint, arXiv:2409.20543, 2024.

[HN19] L. Hesselholt and T. Nikolaus, *Topological Cyclic Homology*, preprint, arXiv:1905.08984, 2019.

[Hoy17] M. Hoyois, *The six operations in equivariant motivic homotopy theory*, Adv. Math., 305 (2017), pp. 197–279, https://doi.org/10.1016/j.aim.2016.09.031.

[HRW22] J. Hahn, A. Raksit, and D. Wilson, *A Motivic Filtration on the Topological Cyclic Homology of Commutative Ring Spectra*, preprint, arXiv:2206.11208, 2022.

[HW22] J. Hahn and D. Wilson, *Redshift and multiplication for truncated Brown-Peterson spectra*, Ann. Math., 196 (2022), pp. 1277–1351, https://doi.org/10.4007/annals.2022.196.3.6.

[Lan76] P. S. Landweber, *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*, Amer. J. Math., 98 (1976), pp. 591–610.

[LeNR11] S. L. Nielsen and J. Rognes, *The Segal conjecture for topological Hochschild homology of complex cobordism*, J. Topol., 4 (2011), pp. 591–622, https://doi.org/10.1112/jtopol/jtr015.

[Lin80] W. H. Lin, *On conjectures of Mahowald, Segal and Sullivan*, Math. Proc. Cambridge Philos. Soc., 87 (1980), pp. 449–458.

[LMMT24] M. Land, A. Mathew, L. Meier, and G. Tamme, *Purity in chromatically localized algebraic K-theory*, J. Amer. Math. Soc., 37 (2024), pp. 1011–1040.

[Man24] D. Manam, *On the Drinfeld Formal Group*, preprint, arXiv:2403.02555, 2024.

[MM25] A. Mathew and S. Mondal, *Affine Stacks and Derived Rings*, preprint, to appear.

[NS18] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Math., 221 (2018), pp. 203–409, https://doi.org/10.4310/ACTA.2018.v221.n2.a1.

[Pst23] P. Pstrągowski, *Synthetic spectra and the cellular motivic category*, Invent. Math., 232 (2023), pp. 553–681, https://doi.org/10.1007/s00222-022-01173-2.

[Pst24] P. Pstrągowski, *Perfect Even Modules and the Even Filtration*, preprint, arXiv:2304.04685, 2023.

[Rog14] J. Rognes, *Algebraic K-theory of strict ring spectra*, in Proceedings of the International Congress of Mathematicians—Seoul 2014, Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 1259–1283.

[Tan24] L. Tang, *Syntomic cycle classes and prismatic Poincaré duality*, Compos. Math., 160 (2024), pp. 2322–2365, https://doi.org/10.1112/S0010437X24007334.

[Wag25] F. Wagner, *q-Hodge Filtrations, Habiro Cohomology, and Ku*, preprint, arXiv:2510.06057, 2025.

[Wil73] W. S. Wilson, *The Ω-spectrum for Brown-Peterson cohomology*, I, Comment. Math. Helv., 48 (1973), pp. 45–55; corrigendum, ibid. 48 (1973), 194.