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Gauss sums and the p -adic Γ -function

By BENEDICT H. GROSS and NEAL KOBLITZ

Introduction

Let p be an odd prime. Many authors have considered p -adic analogs of the classical gamma function [7], [15], [16]. The simplest construction, due to Morita, is to define for $z \in \mathbf{Z}_p$,

$$\Gamma_p(z) = \lim_{m \rightarrow z} (-1)^m \prod_{\substack{0 < j < m \\ (p, j) = 1}} j,$$

where m approaches z through positive integers. This gives a continuous function $\Gamma_p: \mathbf{Z}_p \rightarrow \mathbf{Z}_p^*$ which satisfies the functional equation:

$$\Gamma_p(z+1) = \begin{cases} -\Gamma_p(z) & \text{if } z \in p\mathbf{Z}_p \\ -z\Gamma_p(z) & \text{if } z \in \mathbf{Z}_p^*. \end{cases}$$

Morita [15] shows that $\Gamma_p(z)$ is analytic on $p\mathbf{Z}_p$ and gives a uniform analytic function on the set $\{z \in C_p: |z| \leq |p|\}$, where C_p is the completion of the algebraic closure of $\mathbf{Q}_p$. It is *not* analytic on the closed unit disc, or else the functional equation $\Gamma_p(z+1) = -\Gamma_p(z)$ would hold on all of $\mathbf{Z}_p$.

In this paper, we investigate the values of $\Gamma_p(z)$ at rational arguments. Let N be a positive integer prime to p , and let r be an integer between 0 and N . In Section 1 we show how the value $\Gamma_p(r/N)$ is related to a Gauss sum whose multiplicative character is the r^{th} -power of the standard power residue symbol on $\mathbf{Q}(\mu_N)$. The proof, which uses Stickelberger's theorem and a result of Katz [11] on the p -adic cohomology of the Fermat curve $x^N + y^N = 1$, is given in Section 2. As corollaries, we prove that $\Gamma_p(r/N)$ is algebraic when $p \equiv 1 \pmod{N}$, and derive a multiplication formula for the function $\Gamma_p(z)$. In Section 4, we give a new proof of some of Deligne's recent results [5], [6] on Kummer extensions of $\mathbf{Q}(\mu_N)$ whose Galois character may be identified with a Hecke character defined by Jacobi sums. These extensions are generated by products of values of the *classical* Γ -function at rational arguments, and our method yields a striking formula relating $\Gamma(z)$ to its p -adic analog. We also show how the classical and p -adic Γ -functions are

related via the Chowla-Selberg formula, which gives the periods of elliptic curves with complex multiplication.

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1. Gauss sums and the p -adic Γ -function

Let N be a positive integer, and let $K = \mathbf{Q}(\mu_N)$ be the cyclotomic field of N^{th} roots of unity. Let $\mathcal{P}$ be a prime of K which does not divide $2N$, with $N\mathcal{P} = q = p^f$, and let t be the map from N^{th} roots of unity in the residue field $k_{\mathcal{P}}$ to the N^{th} roots of unity in K which is the inverse to reduction mod $\mathcal{P}$.

Fix a non-trivial additive character

$$(1.1) \quad \Psi: \mathbf{F}_p \longrightarrow \mu_p.$$

For any $a \in (1/N)\mathbf{Z}/\mathbf{Z} - \{0\}$, define the Gauss sum (note the minus signs) [17]:

$$(1.2) \quad g(a, \mathcal{P}, \Psi \circ \text{Tr}) = - \sum t(x^{-a(q-1)}) \Psi(\text{Tr } x)$$

where the sum is taken over $x \in k_{\mathcal{P}}^*$ and the trace is from $k_{\mathcal{P}}$ to $\mathbf{F}_p$. Then $g(a, \mathcal{P}, \Psi \circ \text{Tr})$ is a non-zero element of the field $L = K(\mu_p)$; it depends on the choice of Ψ , but, as this character will be fixed throughout, we shall write it simply as $g(a, \mathcal{P})$.

If $\mathbf{a} = \sum m(a)\delta_a$ is an element of the free abelian group with basis $(1/N)\mathbf{Z}/\mathbf{Z} - \{0\}$, we may define the generalized Gauss sum:

$$(1.3) \quad g(\mathbf{a}, \mathcal{P}) = \prod g(a, \mathcal{P})^{m(a)}.$$

For $u \in (\mathbf{Z}/N\mathbf{Z})^*$ let $\mathbf{a}^{(u)} = \sum m(a)\delta_{ua}$. Then one has

$$(1.4) \quad g(\mathbf{a}^{(p)}, \mathcal{P}) = g(\mathbf{a}, \mathcal{P}),$$

as the additive character defining $g(a, \mathcal{P})$ comes from a character of the prime field.

Let $\mathfrak{B}$ be the unique prime over $\mathcal{P}$ in L , and let $L_{\mathfrak{B}}$ be the completion of L at $\mathfrak{B}$. Our aim in this section is to express $g(a, \mathcal{P})$ in $L_{\mathfrak{B}}$ in terms of Morita's p -adic gamma function. The field $L_{\mathfrak{B}}$ contains the $(p-1)$ -st roots of $-p$; if $\zeta = \Psi(1)$ let π be the unique $(p-1)$ -st root satisfying

$$(1.5) \quad \pi \equiv (\zeta - 1) \pmod{(\zeta - 1)^2}.$$

Then π gives a uniformizing element in $L_{\mathfrak{B}}$ which is completely determined by the choice of additive character Ψ . Furthermore, by (1.4) we see that $g(a, \mathcal{P})$ lies in the subfield $\mathbf{Q}_p(\pi) = \mathbf{Q}_p(\mu_p)$ of $L_{\mathfrak{B}}$.

Recall that the function $\Gamma_p: \mathbf{Z}_p \rightarrow \mathbf{Z}_p$ is defined by

$$(1.6) \quad \Gamma_p(z) = \lim_{m \rightarrow z} (-1)^m \prod_{\substack{0 < j < m \\ (p, j) = 1}} j,$$

where m approaches z through positive integers. If a is any element of $(1/N)\mathbf{Z}/\mathbf{Z}$, let $\langle a \rangle$ be the unique rational number with $0 < \langle a \rangle \leq 1$ and $a \equiv \langle a \rangle \pmod{\mathbf{Z}}$. We shall prove:

THEOREM 1.7.

$$g(a, \mathcal{P}) = \pi^{(p-1)\sum_{j=0}^{f-1}\langle p^j a \rangle} \prod_{j=0}^{f-1} \Gamma_p(\langle p^j a \rangle) \quad \text{in } L_{\mathbb{B}}.$$

Note that the exponent of π is an integer.

This theorem yields the algebraicity of certain Γ_p -values:

COROLLARY 1.8. *If $p \equiv 1 \pmod{N}$ then $\Gamma_p(r/N) \in \mathbf{Q}(\mu_{Np}, \sqrt[N]{-p})$.*

Indeed, for $0 < r/N < 1$ this follows directly from Theorem 1.7. Since $\Gamma_p(z+1)/\Gamma_p(z)$ is always rational when z is rational, the corollary holds in general.

Theorem 1.7 is easily generalized to the Gauss sums $g(a, \mathcal{P})$. For $a = \sum m(a)\delta_a$, define

$$(1.9) \quad n(a) = \sum m(a)\langle a \rangle,$$

$$(1.10) \quad \Gamma_p(a) = \prod \Gamma_p(\langle a \rangle)^{m(a)}.$$

Then we obtain:

COROLLARY 1.11.

$$g(a, \mathcal{P}) = \pi^{(p-1)\sum_{j=0}^{f-1}n(a^{(p^j)})} \prod_{j=0}^{f-1} \Gamma_p(a^{(p^j)}) \quad \text{in } L_{\mathbb{B}}.$$

In the special case when $n(a) \in \mathbf{Z}$, the Gauss sum $g(a, \mathcal{P})$ is independent of the choice of additive character Ψ and lies in $K = \mathbf{Q}(\mu_N)$. These Jacobi sums were studied by Weil in connection with the zeta-functions of Fermat hypersurfaces; as a function of $\mathcal{P}$, $g(a, \cdot)$ gives a Hecke character of type A_0 for K with values in K [14], [2]. This character has algebraic part

$$\Theta = \sum_{u \in (\mathbf{Z}/N\mathbf{Z})^*} n(a^{(u)}) \sigma_u^{-1}.$$

If we consider $g(a, \mathcal{P})$ in the completion $K_{\mathcal{P}}$, then by (1.4) $g(a, \mathcal{P})$ lies in the subfield $\mathbf{Q}_p$. As a special case of Corollary 1.11 we have:

THEOREM 1.12. *If $n(a) \in \mathbf{Z}$ then*

$$g(a, \mathcal{P}) = (-p)^{\sum_{j=0}^{f-1}n(a^{(p^j)})} \prod_{j=0}^{f-1} \Gamma_p(a^{(p^j)}) \quad \text{in } K_{\mathcal{P}}.$$

Note that both sides are now independent of Ψ .

Finally, suppose a satisfies the condition:

$$(1.13) \quad n(a^{(u)}) \text{ is an integer which is independent of } u \in (\mathbf{Z}/N\mathbf{Z})^*.$$

The Hecke character

$$(1.14) \quad \chi_a(\mathcal{P}) = g(a, \mathcal{P}) N(\mathcal{P})^{-n(a)}$$

is then of finite order.

COROLLARY 1.15. *If $\mathbf{a}$ satisfies (1.13) then*

$$\chi_{\mathbf{a}}(\mathcal{P}) = (-1)^{n(\mathbf{a})f} \prod_{j=0}^{f-1} \Gamma_p(\mathbf{a}^{(p^j)}) .$$

This implies, in particular, that the product of gamma values on the right is a root of unity in K^* .

2. The proof of Theorem 1.7

We begin with a proof of Theorem 1.12: the corresponding result for Jacobi sums. Here are two important examples:

$$(2.1) \quad \mathbf{a} = \delta_{r/N} + \delta_{(N-r)/N} \quad \text{with} \quad 0 < r < N .$$

Then

$$g(\mathbf{a}, \mathcal{P}) = q(-1)^{r(p-1)/N} .$$

$$(2.2) \quad \mathbf{a} = \delta_{r/N} + \delta_{s/N} - \delta_{(r+s)/N} \quad \text{with} \quad 0 < r, s < N, r + s \neq N .$$

Then

$$g(\mathbf{a}, \mathcal{P}) = - \sum_{k_{p-\{0,1\}}} x^{-r(q-1)/N} t((1-x)^{-s(q-1)/N}) .$$

The latter Jacobi sum arises as an eigenvalue of Frobenius on the Fermat curve $x^N + y^N = 1$ over $\mathbf{Q}(\mu_N)$. As any $\mathbf{a}$ with $n(\mathbf{a}) \in \mathbf{Z}$ is a $\mathbf{Z}$ -linear combination of the elements in examples (2.1) and (2.2), it is sufficient to prove Theorem 1.12 in these special cases (Lemmas 2.5 and 2.6).

LEMMA 2.3. *If $z \in \mathbf{Z}_p$ then $\Gamma_p(z)\Gamma_p(1-z) = (-1)^{\hat{z}}$, where $\hat{z}$ is the unique integer with $0 < \hat{z} \leq p$ and $\hat{z} \equiv z \pmod{p}$.*

Proof. Write $z = \sum_{n=0}^{\infty} a_n p^n$ with $0 \leq a_n < p$ and let $m_\nu = \sum_{n=0}^{\nu-1} a_n p^n$. Then

$$\begin{aligned} \Gamma_p(z)\Gamma_p(1-z) &= \lim_{\nu \rightarrow \infty} ((-1)^{m_\nu} \prod_{\substack{j < m_\nu \\ (p,j)=1}} j) ((-1)^{p^\nu+1-m_\nu} \prod_{\substack{j < p^\nu+1-m_\nu \\ (p,j)=1}} j) \\ &= \lim_{\nu \rightarrow \infty} (-1)^{\text{Card } \{j < m_\nu: p \nmid j\}} \prod_{\substack{j < p^\nu \\ p \nmid j}} j \\ &= (-1)^{\hat{z}-1} (\lim_{\nu \rightarrow \infty} \prod_{\substack{j < p^\nu \\ p \nmid j}} j) = (-1)^{\hat{z}} \end{aligned}$$

by Wilson's Theorem.

LEMMA 2.4. *Assume $0 < r/N < 1$ and write*

$$(p^f - 1)(r/N) = z_f + z_1 p + z_2 p^2 + \cdots + z_{f-1} p^{f-1} \quad \text{with} \quad 0 \leq z_j < p .$$

Then

$$\langle \widehat{p^j r/N} \rangle = p - z_{f-j} .$$

Proof. This is clear for $j = 0$. For $j > 0$ write

$$p^j r/N = b_j + \langle p^j r/N \rangle \quad \text{with} \quad b_j \in \mathbf{Z} .$$

Then one has the formula

$$b_j = z_{f-j} + pz_{f-j+1} + \cdots + p^{j-1}z_{f-1}.$$

Consequently

$$\langle p^j r/N \rangle \equiv -b_j \equiv -z_{f-j} \pmod{p}$$

which gives the lemma.

LEMMA 2.5. *If $\mathbf{a} = \delta_{r/N} + \delta_{(N-r)/N}$, the formula for $g(\mathbf{a}, \mathcal{P})$ in (1.12) is true.*

Proof. We know $g(\mathbf{a}, \mathcal{P}) = q(-1)^{r(q-1)/N}$. As $n(\mathbf{a}^{(u)}) = 1$ for $u \in (\mathbf{Z}/N\mathbf{Z})^*$:

$$\begin{aligned} (-p)^{\sum n(\mathbf{a}^{(p^j)})} \prod_{j=0}^{f-1} \Gamma_p(\mathbf{a}^{(p^j)}) &= q(-1)^f \prod_{j=0}^{f-1} \Gamma_p \langle p^j r/N \rangle \Gamma_p(1 - \langle p^j r/N \rangle) \\ &= q(-1)^{f + \sum_{j=0}^{f-1} \langle p^j r/N \rangle} \quad \text{by (2.3).} \\ &= q(-1)^{f + \sum_{j=0}^{f-1} (p - z_{f-j})} \quad \text{by (2.4).} \\ &= q(-1)^{\sum_{j=1}^f z_j}. \end{aligned}$$

As $\sum_{j=1}^f z_j \equiv (q-1)r/N \pmod{p-1}$, this gives the required identity.

LEMMA 2.6. *If $\mathbf{a} = \delta_{r/N} + \delta_{s/N} - \delta_{(r+s)/N}$, the formula for $g(\mathbf{a}, \mathcal{P})$ in (1.12) is true.*

Proof. We must show

$$g(\mathbf{a}, \mathcal{P}) = \prod_{j=0}^{f-1} (-p)^{\varepsilon_j} \frac{\Gamma_p(\langle p^j r/N \rangle) \Gamma_p(\langle p^j s/N \rangle)}{\Gamma_p(\langle p^j(r+s)/N \rangle)},$$

where $\varepsilon_j = \langle p^j r/N \rangle + \langle p^j s/N \rangle - \langle p^j(r+s)/N \rangle$ is either 0 or 1. Set $t = (1 + \varepsilon_0)N - r - s$.

By a comparison of the action of Frobenius on the crystalline cohomology of the Fermat curve $x^N + y^N = 1$ over $k_{\mathcal{P}}$ with its action on the eigendifferential

$$\omega_{r,s} = x^{r-1} y^{s-1} \frac{dx}{y^{N-1}} = x^{r-1} (1 - x^N)^{(s-N)/N} dx$$

in the de Rham cohomology of the curve over the Witt vectors $W(k_{\mathcal{P}})$, Katz ([11], [12]) obtained the formula

$$(2.7) \quad g(\mathbf{a}, \mathcal{P}) = \prod_{j=0}^{f-1} \lim_{k \rightarrow -\langle p^j r/N \rangle} \frac{\left(pk + (p \langle p^j r/N \rangle - \langle p^{j+1} r/N \rangle) \right)}{\left(\frac{\langle p^j t/N \rangle - 1}{k} \right)}.$$

We want to interpret the limits on the right side of (2.7) as products of p -adic Γ -values.

Let h and k be positive even integers with

$$\begin{aligned} h &\equiv -t/N \pmod{p^a}, \\ k &\equiv -r/N \pmod{p^b}, \end{aligned}$$

with a and b large. Let $[x]$ be the greatest integer less than or equal to x . Since

$$\begin{aligned} ph + [pt/N] &\equiv -\langle pt/N \rangle \pmod{p^{a+1}}, \\ pk + [pr/N] &\equiv -\langle pr/N \rangle \pmod{p^{b+1}}, \end{aligned}$$

we see the $j = 0$ term in (2.7) is approximated by

$$(2.8) \quad (-1)^{[pr/N]} \frac{(p(h+k) + [pt/N] + [pr/N])!}{(ph + [pt/N])! (pk + [pr/N])!} \frac{(h+k)! p^{h+k}}{h! p^h k! p^k}.$$

If $\varepsilon_0 = 0$, then $[pt/N] + [pr/N] < p$, and this ratio approaches

$$(-1)^{[pr/N] + \varepsilon_1} \frac{\Gamma_p(1 - \langle pr/N \rangle - \langle pt/N \rangle)}{\Gamma_p(1 - \langle pr/N \rangle) \Gamma_p(1 - \langle pt/N \rangle)}.$$

Using Lemma 2.3 and the functional equation for $\Gamma_p(z)$, we see this is equal to

$$(2.9) \quad (\varepsilon_0 = 0) \begin{cases} \Gamma_p(\langle pr/N \rangle) \Gamma_p(\langle ps/N \rangle) / \Gamma_p(\langle p(r+s)/N \rangle) & \text{if } \varepsilon_1 = 0 \\ \frac{1}{(1 - \langle pr/N \rangle - \langle pt/N \rangle)} \Gamma_p(\langle pr/N \rangle) \Gamma_p(\langle ps/N \rangle) / \Gamma_p(\langle p(r+s)/N \rangle) & \text{if } \varepsilon_1 = 1. \end{cases}$$

If $\varepsilon_0 = 1$, then we find that the $j = 0$ term in (2.7) is equal to

$$(2.10) \quad (\varepsilon_0 = 1) \begin{cases} -p(1 - r/N - t/N) \Gamma_p(\langle pr/N \rangle) \Gamma_p(\langle ps/N \rangle) / \Gamma_p(\langle p(r+s)/N \rangle) & \text{if } \varepsilon_1 = 0 \\ \frac{-p(1 - r/N - t/N)}{(1 - \langle pr/N \rangle - \langle pt/N \rangle)} \Gamma_p\left(\left\langle \frac{pr}{N} \right\rangle\right) \Gamma_p(\langle ps/N \rangle) / \Gamma_p(\langle p(r+s)/N \rangle) & \text{if } \varepsilon_1 = 1. \end{cases}$$

Now repeat the argument for each term in the product of (2.7), replacing r/N by $\langle p^j r/N \rangle$, etc. Multiplying these results for $j = 0, 1, \dots, f-1$ and noting the telescoping of the $1 - r/N - s/N$ terms, we obtain the lemma, and hence Theorem 1.12.

We can obtain Theorem 1.7 from Theorem 1.12 as follows. For any $a \in (1/N)\mathbf{Z}/\mathbf{Z} - \{0\}$ let $\mathbf{a} = (p^f - 1)\delta_a$. Theorem 1.12 then gives

$$\begin{aligned} g(\mathbf{a}, \mathcal{P}) &= g(\mathbf{a}, \mathcal{P})^{p^f-1} = (-p)^{\sum_{j=0}^{f-1} n(\mathbf{a}^{(p^j)})} \prod_{j=0}^{f-1} \Gamma_p(\mathbf{a}^{(p^j)}) \\ &= (\pi^{(p-1)\sum \langle p^j \mathbf{a} \rangle} \prod \Gamma_p \langle p^j \mathbf{a} \rangle)^{p^f-1}. \end{aligned}$$

Therefore the quantities $g(\mathbf{a}, \mathcal{P})$ and $\pi^{(p-1)\sum \langle p^j \mathbf{a} \rangle} \prod \Gamma_p \langle p^j \mathbf{a} \rangle$ differ by a $(p^f - 1)$ -st root of unity. As they both lie in $\mathbf{Q}_p(\pi) = \mathbf{Q}_p(\mu_p)$, they must differ by a $(p-1)$ -st root of unity. To show they are equal, we shall prove that they are congruent (mod π).

Write $(p^f - 1)\langle \mathbf{a} \rangle = z_f + z_1 p + \dots + z_{f-1} p^{f-1}$ with $0 \leq z_i < p$ as in

(2.4).

- LEMMA 2.11. 1) $(p-1)\sum_{j=0}^{f-1}\langle p^j a \rangle = \sum_{j=1}^f z_j$.
 2) $g(a, \mathcal{P})\pi^{-\sum z_j} = u \in \mathbf{Z}_p^*$.
 3) $u \equiv \prod_{j=0}^{f-1} \Gamma_p(\langle p^j a \rangle) \pmod{p}$.

Proof. 1) is immediate from the formula ($j \geq 1$):

$$\langle p^j a \rangle = \frac{z_f p^j + \cdots + z_{f-j-1} p^{f-1} + z_{f-j} + \cdots + z_{f-1} p^{j-1}}{(p^f - 1)}.$$

Thus $u = g(a, \mathcal{P})\pi^{-\sum z_j}$ is a product of Γ_p -values times a $(p-1)$ -st root of unity, by our preceding remarks. Hence u is a p -adic unit.

3) is essentially the content of Stickelberger's famous theorem on Gauss sums [17], [14]. This states that

$$g(a, \mathcal{P}) \equiv \left(\frac{\pi^{\sum z_j}}{\prod_{j=1}^f z_j!} \right) \pmod{\pi^{\sum z_j + 1}},$$

which gives

$$u \equiv \left(\prod_{j=1}^f z_j! \right)^{-1} \pmod{p}.$$

On the other hand, since $\langle \widehat{p^j a} \rangle = p - z_{f-j}$ by (2.4), we have

$$\begin{aligned} \Gamma_p(\langle p^j a \rangle) &\equiv (p - z_{f-j} - 1)! (-1)^{p-z_{f-j}} \\ &\equiv (z_{f-j}!)^{-1} \pmod{p}. \end{aligned}$$

Therefore

$$u \equiv \prod_{j=0}^{f-1} \Gamma_p(\langle p^j a \rangle) \pmod{p}$$

as claimed.

This completes the proof of Theorem 1.7.

3. A multiplication formula for $\Gamma_p(z)$

Let p be an odd prime, and $q = p^f$. Let ω be the Teichmüller character on $\mathbf{Z}_p^*$, and let m be a positive integer prime to p .

THEOREM 3.1. Assume $0 < x < 1$ and $(q-1)x \in \mathbf{Z}$.

1) If $\mathbf{a} = \delta_x + \delta_{1-x} - 2\delta_{1/2}$ then $\prod_{j=0}^{f-1} \Gamma_p(\mathbf{a}^{(p^j)}) = (-1)^{(q-1)(x-1/2)}$.

2) If $\mathbf{a} = \delta_{x/m} + \delta_{(x+1)/m} + \cdots + \delta_{(x+m-1)/m} - \delta_x - \delta_{1/m} - \delta_{2/m} - \cdots - \delta_{m-1/m}$

then

$$\prod_{j=0}^{f-1} \Gamma_p(\mathbf{a}^{(p^j)}) = \omega(m^{(q-1)x}).$$

Proof. Part 1) is a restatement of Lemma 2.5 for $x = r/(q-1)$, as $\Gamma_p(1/2)^2 = (-1)^{(p+1)/2}$.

To prove 2), let N be the common denominator of the terms $(x+h)/m$ for $h = 0, 1, \dots, m-1$. Let g be the order of $p \pmod{N}$. Since $n(\mathbf{a}^{(u)}) = 0$ for all $u \in (\mathbf{Z}/N\mathbf{Z})^*$, Corollary 1.16 implies that $\prod_{j=0}^{g-1} \Gamma_p(\mathbf{a}^{(p^j)})$ is a root of

unity. But

$$\prod_{j=0}^{g-1} \Gamma_p(\mathbf{a}^{(p^j)}) = (\prod_{j=0}^{f-1} \Gamma_p(\mathbf{a}^{(p^j)}))^{g/f}$$

as $\mathbf{a}^{(p^f)} = \mathbf{a}$. Hence $\prod_{j=0}^{f-1} \Gamma_p(\mathbf{a}^{(p^j)})$ must also be a root of unity; as it lies in $\mathbf{Z}_p^*$ we are reduced to a proof of the congruence:

$$(3.2) \quad \prod_{j=0}^{f-1} \Gamma_p(\mathbf{a}^{(p^j)}) \stackrel{?}{=} m^{(q-1)x} \pmod{p}.$$

Writing $(q-1)x = z_f + z_1p + \cdots + z_{f-1}p^{f-1}$ as in (2.4), we have $\sum_{k=1}^f z_k \equiv (q-1)x \pmod{p-1}$, so

$$m^{(q-1)x} \equiv \prod_{k=1}^f m^{z_k} \equiv \prod_{j=0}^{f-1} m^{z_{f-j}} \pmod{p}.$$

Therefore (3.2) follows from the congruence

$$\Gamma_p(\mathbf{a}) \equiv m^{z_f} \pmod{p}.$$

The proof of this congruence is elementary, and we omit it.

Notes: 1) A more general multiplication formula has been obtained by Dwork [1], [11]. His proof uses the functional equation of $\Gamma_p(z)$.

2) If we translate Theorem 3.1 into the language of Gauss sums, using Theorem 1.12, we obtain a result due to Hasse and Davenport [3]:

$$(3.3) \quad \frac{\prod_{h=0}^{m-1} g\left(\frac{a+h}{m}, \mathcal{P}\right)}{\prod_{h=1}^{m-1} g\left(\frac{h}{m}, \mathcal{P}\right)} = g(a, \mathcal{P}) \cdot t(m^{(q-1)a}).$$

Where a is an element in $(1/N)\mathbf{Z}/\mathbf{Z} - \{0\}$ with $\langle a \rangle =$ the x of Theorem 3.1, t is the Teichmüller map (see beginning of §1), and $\mathcal{P}$ is a prime dividing p in $\mathbf{Q}(\mu_N)$.

3) Lang [14] has suggested interpreting the Hasse-Davenport theorem as a distribution relation for Gauss sums; one can also state this relation using $\Gamma_p(z)$.

Let $(\mathbf{Q}/\mathbf{Z})_{p'} = \coprod_{l \neq p} \mathbf{Q}_l/\mathbf{Z}_l$ and define the function

$$\varphi: (\mathbf{Q}/\mathbf{Z})_{p'} \longrightarrow p\mathbf{Z}_p$$

by setting

$$\varphi(a) = \log_p \prod_{j=0}^{f-1} \Gamma_p(\langle p^j a \rangle),$$

where f is the least integer such that $(p^f - 1)a \equiv 0 \pmod{\mathbf{Z}}$. Then by (3.1) the function φ satisfies the relations

$$(3.4) \quad \varphi(-a) = -\varphi(a),$$

$$(3.5) \quad \prod_{h=0}^{m-1} \varphi\left(\frac{a+h}{m}\right) = \varphi(a) \quad \text{for all } m \text{ prime to } p;$$

$$(3.6) \quad \varphi(pa) = \varphi(a).$$

Thus φ is an odd distribution on $(\mathbf{Q}/\mathbf{Z})_p$, with values in the group $p\mathbf{Z}_p$; in fact, it is the *universal* odd distribution satisfying (3.6) with values in this (torsion-free) group [13]. Note that the values of φ are all p -adic logarithms of algebraic numbers.

4. Relation to the classical Γ -function

In this section we shall fix N and consider those $\mathbf{a}$ in the free abelian group on $(1/N)\mathbf{Z}/\mathbf{Z} - \{0\}$ which satisfy condition (1.13): $n(\mathbf{a}^{(u)})$ is an integer which is independent of $u \in (\mathbf{Z}/N\mathbf{Z})^*$. These elements form a subgroup A ; for any $\mathbf{a} \in A$ we define

$$(4.1) \quad \Omega_{\mathbf{a}} = \frac{\Gamma(\mathbf{a})}{(2\pi i)^{n(\mathbf{a})}}.$$

When $\mathbf{a} \in A$, the Hecke character $\chi_{\mathbf{a}}$ defined in (1.14) is trivial on the connected component of the group of idele classes of K , so may be viewed as a character of $\text{Gal}(\bar{\mathbf{Q}}/K)$. (We normalize the isomorphism of class field theory to take a uniformizing parameter to a geometric Frobenius.)

THEOREM 4.2 (Deligne [5]). *If $\mathbf{a} \in A$ then $\Omega_{\mathbf{a}}$ is algebraic and*

$$\sigma\Omega_{\mathbf{a}} = \chi_{\mathbf{a}}(\sigma) \cdot \Omega_{\mathbf{a}} \quad \text{for any } \sigma \in \text{Gal}(\bar{\mathbf{Q}}/K).$$

Deligne's proof involves the Hodge theory of Fermat hypersurfaces. Our methods give this result on a large subgroup of A . Namely, if m is a positive integer dividing N , write $N = mk$ and let $x = r/k$. Then

$$(4.3) \quad \mathbf{a} = \delta_{x/m} + \delta_{x+1/m} + \cdots + \delta_{x+(m-1)/m} - \delta_x - \delta_{1/m} - \delta_{2/m} - \cdots - \delta_{m-1/m}$$

is an element of A with $n(\mathbf{a}) = 0$. Let B be the subgroup generated by all these "basic relations", along with those of the form

$$(4.4) \quad \mathbf{a} = \delta_x + \delta_{1-x}$$

with $x = r/N$ and $n(\mathbf{a}) = 1$.

THEOREM 4.5. *If $\mathbf{a} \in B$ then $\sigma\Omega_{\mathbf{a}} = \chi_{\mathbf{a}}(\sigma)\Omega_{\mathbf{a}}$ for all $\sigma \in \text{Gal}(\bar{\mathbf{Q}}/K)$.*

Note. Kubert [13] has shown that the quotient group A/B is killed by 2. Therefore Theorem 4.5 gives Deligne's full result "up to sign".

Proof. Let $w = |\mu(K^*)|$, so $w = \text{l.c.m.}(2, N)$. First we show that $\Omega_{\mathbf{a}}^w \in K^*$. It is enough to check this on basic relations. For $\mathbf{a}$ of type (4.3) we have

$$\Omega_{\mathbf{a}} = m^{1-x}.$$

For $\mathbf{a}$ of type (4.4) we have

$$\Omega_{\mathbf{a}} = \frac{e^{\pi i x}}{e^{2\pi i x} - 1}.$$

In both cases $\Omega_{\mathbf{a}}^w \in K^*$.

Now let $\mathbf{a}$ be an element of B and $\mathcal{P}$ a prime not dividing N in K . If $F_{\mathcal{P}}$ is a geometric Frobenius at $\mathcal{P}$ we must prove

$$(4.6) \quad \frac{F_{\mathcal{P}}\Omega_{\mathbf{a}}}{\Omega_{\mathbf{a}}} = \chi_{\mathbf{a}}(\mathcal{P}) .$$

Since both sides are roots of unity in K^* it is enough to check this (mod $\mathcal{P}$). Furthermore we may restrict to primes $\mathcal{P}$ of degree 1 in K , as such $F_{\mathcal{P}}$ are dense in the Galois group. But then

$$\frac{F_{\mathcal{P}}\Omega_{\mathbf{a}}}{\Omega_{\mathbf{a}}} \equiv \Omega_{\mathbf{a}}^{1-p} \pmod{\mathcal{P}}$$

and, by Corollary 1.15,

$$\chi_{\mathbf{a}}(\mathcal{P}) = (-1)^{n(\mathbf{a})} \Gamma_p(\mathbf{a}) .$$

It is now an easy matter to check (4.6) for basic relations $\mathbf{a}$ of type (4.3) and (4.4) using Theorem 3.1. Since both sides of (4.6) are multiplicative in $\mathbf{a}$, we obtain the theorem for all $\mathbf{a}$ in B .

In the proof of Theorem 4.5 we used only (mod p) information. The following result however employs the full p -adic theory.

THEOREM 4.6. *Let $\mathcal{P}$ be a prime of K with $N\mathcal{P} = p$, let $\text{Frob}(\mathcal{P})$ be a geometric Frobenius at $\mathcal{P}$ in $\text{Gal}(\bar{\mathbf{Q}}/K)$, and let $i_{\mathcal{P}}$ be the inclusion of K into $K_{\mathcal{P}}$. If $\mathbf{a} \in A$ then*

$$\left(\frac{\Gamma(\mathbf{a})}{(2\pi i)^{n(\mathbf{a})}} \right)^{\text{Frob}(\mathcal{P})-1} = \left(\frac{\Gamma_p(\mathbf{a})}{(-1)^{n(\mathbf{a})}} \right)^{i_{\mathcal{P}}^{-1}} .$$

Indeed, this is merely a combination of Theorem 4.2 and Corollary 1.15. Deligne interprets this theorem in terms of periods of Hodge classes of type $(n(\mathbf{a}), n(\mathbf{a}))$ on products of Fermat hypersurfaces. We can obtain a similar theorem for the periods of one-forms on certain abelian varieties with complex multiplication by an abelian field; here we give the result for elliptic curves.

Let k be an imaginary quadratic field of discriminant $-N$; let h be the class number of k , w the number of roots of unity, and ε the quadratic Dirichlet character (mod N) associated to this extension. Define the element

$$(4.7) \quad \mathbf{a} = \frac{w}{2} \sum_{\varepsilon(\mathbf{a})=1} \delta_{\mathbf{a}} .$$

Then $n(\mathbf{a}) + n(\mathbf{a}^{(-1)}) = w\varphi(N)/4$ and, by Dirichlet's class-number formula: $n(\mathbf{a}^{(-1)}) - n(\mathbf{a}) = h$. Therefore $n(\mathbf{a})$ is half-integral; by genera theory $n(\mathbf{a})$ is integral whenever N is not a power of 2.

Let E be an elliptic curve with complex multiplication by the integers

of k ; suppose E is defined over H , the Hilbert classfield of k . If ω is a differential of the first kind on E and v is an infinite place of H , let

$$\int \omega_v \text{ in } H_v^*/k^*$$

be the integral of ω around any 1-cycle in the rational homology of $E_v(H_v)$. If v_∞ is a complex place of k , put

$$(4.8) \quad \text{Per}_{v_\infty}(E) = \prod_{v|v_\infty} \int \omega_v \text{ in } k_{v_\infty}^*/k^* .$$

Modulo k^* this period depends only on E , and not on the differential or cycles chosen. An application of the Chowla-Selberg formula, as stated in [10], gives

$$(4.9) \quad \text{Per}_{v_\infty}(E) \sim_{\bar{Q}^*} \frac{\Gamma(a)}{(2\pi i)^{n(a)}}$$

where $\sim_{\bar{Q}^*}$ means that the ratio is an algebraic number. To determine this algebraic number up to an element of k^* we shall use the p -adic gamma-function and the zeta-function of the curve.

In fact, the L -function of E can be expressed as the product of abelian L -series (see, e.g., [9]):

$$L(H^1(E), s) = L(\chi, s)L(\bar{\chi}, s) ,$$

where χ is an algebraic Hecke character of H with values in k . If q is a prime of H where E has good reduction, we can normalize so that $(\chi(q)) = \overline{N_{H/k}q}$ as ideals in k . If $v_{\mathcal{P}}$ is a finite place of k over which E has good reduction, we can imitate (4.8) by defining

$$(4.10) \quad \text{Per}_{v_{\mathcal{P}}}(E) = \chi(\mathcal{P}) = \prod_{q|v_{\mathcal{P}}} \chi(q) .$$

By so restricting χ to k , we obtain a Hecke character which was related to a Jacobi sum on $\mathbf{Q}(\mu_N)$ by Weil [17], [18]. If $N\mathcal{P} = p$ we can use this expression and Theorem 1.7 to show

$$(4.11) \quad \text{Per}_{v_{\mathcal{P}}}(E) \sim_{\mu} \Gamma_p(a) \text{ in } K_{v_{\mathcal{P}}}^*$$

where $\sim_{\mu}$ means the ratio is a root of unity. Since $\text{Per}_{v_{\mathcal{P}}}(E) = \chi(\mathcal{P})$ generates the principal ideal $(\bar{\mathcal{P}})^h = (\bar{\alpha})$, we can take the p -adic logarithm of (4.11) to obtain the identity

$$(4.12) \quad \sum_{0 < a < N} \varepsilon(a) \log_p \Gamma_p(a/N) = \frac{4}{w} \log_p \bar{\alpha} \text{ in } K_{v_{\mathcal{P}}} .$$

This evaluates the derivative at zero of the p -adic L -series with character ε [8], much as the Chowla-Selberg formula interprets the derivative of the classical Dirichlet L -function [2].

But the relationship between (4.9) and (4.11) is more than formal: the

root of unity in (4.11) controls the algebraic number in (4.9) via Kummer theory. Here is the simplest case.

THEOREM 4.13. *Let E be an elliptic curve with complex multiplication by the integers of k which is defined over H , the Hilbert classfield of k . Assume the discriminant $-N$ of k is prime, and let $\mathcal{P}$ be a prime of k with $N\mathcal{P} = p$, $p \nmid N$. Assume E has good reduction at all places q of H dividing $\mathcal{P}$. Then*

$$\left(\frac{\text{Per}_{v_\infty}(E)}{\Gamma(\mathfrak{a})/(2\pi i)^{n(\mathfrak{a})}} \right)^{\text{Frob}(\mathcal{P})-1} = \left(\frac{\text{Per}_{v_{\mathcal{P}}}(E)}{\Gamma_p(\mathfrak{a})/(-1)^{n(\mathfrak{a})}} \right)^{i_{\mathcal{P}}^{-1}} \quad \text{in } \mu(k^*).$$

Proof. Assume, for simplicity, that $N > 3$ so $w = 2$. Let j be the modular invariant of E . There is a canonical curve $E(N)$ over H with this j -invariant. In fact, $E(N)$ can be defined over the subfield $\mathbf{Q}(j)$ and is characterized by the properties [9]:

- 1) $E(N)$ is isogenous over H to its conjugates by $\text{Gal}(H/\mathbf{Q})$.
- 2) $E(N)$ has good reduction at all primes q of H not dividing N .

If ω is a differential on $E(N)$ with associated discriminant ideal $\Delta = (-N^3)$, one can derive the formula,

$$(4.14) \quad \left| \prod_{v|v_\infty} \int_{E(H_v)} \omega \wedge \bar{\omega} \right| = \left(\frac{2\pi}{N} \right)^h \prod_{0 < \alpha < N} \Gamma(\alpha/N)^{\varepsilon(\alpha)},$$

directly from the identity of Chowla and Selberg [2]. From this identity and an analysis of $E(N)$ at the real infinite place of $\mathbf{Q}(j)$, one shows that [9]

$$\text{Per}_{v_\infty}(E(N)) \sim_k \Gamma(\mathfrak{a})/(2\pi i)^{n(\mathfrak{a})}.$$

To establish the theorem for $E = E(N)$ we must show that

$$(4.15) \quad \text{Per}_{v_{\mathcal{P}}}(E(N)) = \Gamma_p(\mathfrak{a})/(-1)^{n(\mathfrak{a})} \quad \text{in } k_{\mathcal{P}}$$

for all primes $\mathcal{P}$ of degree 1 not dividing N .

Choose a triple (r, s, t) with $0 < r, s, t < N$, $r + s + t = N$ and $\varepsilon(r) + \varepsilon(s) + \varepsilon(t) = 1$ (this is always possible). Then the character χ of $E(N)$, when restricted to ideals $\mathcal{P}$ of k , is given by the formula

$$g(\delta_{r/N} + \delta_{s/N} - \delta_{(r+s)/N}, \mathcal{P}) = \chi(\mathcal{P})N\mathcal{P}^{n(\mathfrak{a})}.$$

Indeed, the quotient character $g/\chi \cdot N^{n(\mathfrak{a})}$ would be a quadratic Galois character of k , as it has trivial infinite part. Furthermore, it is ramified only at $(\sqrt{-N})$, so must be trivial by classfield theory. Theorem (1.12) and (1.4) now give an expression for g in the completion $k_{\mathcal{P}}$, which in turn yields (4.15).

Since E and $E(N)$ have the same j -invariant, they become isomorphic over some quadratic extension $H(\sqrt{\alpha})$. One then finds

$$\text{Per}_{v_\infty}(E) \sim_{k^*} \sqrt{\beta} \frac{\Gamma(\mathfrak{a})}{(2\pi i)^{n(\mathfrak{a})}}$$

with $\beta = N_{H/k}(\alpha)$. On the other hand, restricting the Hecke characters to k , we find

$$\chi_E = \chi_{E(N)} \cdot \psi_\beta$$

where ψ_β is the quadratic Galois character associated to the extension $k(\sqrt{\beta})$. Theorem 4.13 thus reduces to the identity:

$$(\sqrt{\beta})^{\text{Frob}(\mathfrak{f})-1} = \psi_\beta(\mathcal{P}) \quad \text{in} \quad \mu(k^*).$$

The case $N = 3$ is similar, but ψ may have values in 6th roots of unity.

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