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ised version of letter to Waldhausen of July 10:

1. The cyclic nerve of $wS.\mathcal{C}$

call that the cyclic nerve of a category $\mathcal{C}$ is the cyclic set $N^{cy}_{.}\mathcal{C}$ given

$$N^{cy}_k\mathcal{C} = \{ \text{ circular diagrams } C_0 \overset{f_0}{-->} C_1 \overset{f_1}{-->} \cdots --> C_k \overset{f_k}{-->} C_0 \text{ in } \mathcal{C} \}$$

ace maps are given by composing morphisms, degeneracies are given by serting identity morphisms, and $\mathbb{Z}/(k+1)$ acts by rotating the diagram.) $\mathcal{C}$ is a groupoid (every morphism is an isomorphism) then the cyclic bset of $N^{cy}_{.}\mathcal{C}$ given by the condition $f_k f_{k-1} \cdots f_0 = 1$ is isomorphic (as a mplicial set) to the ordinary nerve $N.\mathcal{C}$. By realization we then get a ir of S^1-spaces $(|N^{cy}_{.}\mathcal{C}|, |N.\mathcal{C}|)$, and it is known that this is equivalent, the weak equivariant sense, to the pair $(\Lambda X, X)$ where $X=|N.\mathcal{C}|$, "Λ" notes free loopspace, X is the subspace of constant loops in ΛX, and S^1 ts in the obvious way.

t is also worth noting that if $\mathcal{C}$ is a groupoid then the cyclic nerve of $\mathcal{C}$ s isomorphic (as a simplicial set) to the ordinary nerve of the groupoid ut($\mathcal{C}$) whose objects are the automorphisms of objects in $\mathcal{C}$ and whose somorphisms are the obvious ones.

w let $\mathcal{C}$ be a category with cofibrations and weak equivalences w as in
1]. We can form the cyclic nerve of the simplicial category $wS.\mathcal{C}$ to get
cyclic simplicial set $N^{cy}_{.}wS.\mathcal{C}$. Let's call the loopspace $\Omega|N^{cy}_{.}wS.\mathcal{C}|$ the
cyclic K-theory" and denote it by $K^{cy}(w,\mathcal{C})$. In the minimal case when w=i,
that only the isomorphisms are weak equivalences, then $wS.\mathcal{C}$ is a
implicial groupoid and one has an inclusion $N.iS.\mathcal{C} \to N^{cy}_{.}iS.\mathcal{C}$ and so a
ap from K-theory to cyclic K-theory.

or general w one can still define the cyclic K-theory. In particular one
an define it in the maximal case when w is all of $\mathcal{C}$. In this case it is
$|N^{cy}_{.}S.\mathcal{C}|$. Of course in the maximal case we have nothing like a groupoid,
o we don't expect to see a map from K-theory. This is just as well, since
n this case the K-theory is contractible. However, the cyclic K-theory is
ot. In fact, I think that $K^{cy}(w,\mathcal{C})$ in the special case when w is all of $\mathcal{C}$
s going to play an important role in studying trace maps from K-theory to
Hochschild) homology. It seems to live somewhere between the two; it's
efined like K-theory but behaves like homology. I'm calling this new
object $W(\mathcal{C})$ (the "W" stands for "Witt vector"); it is a functor of the
category-with-cofibrations $\mathcal{C}$.

By the way, the additivity theorem seems to extend easily to this setting:
I claim that for any $\mathcal{C}$ and w the obvious map

$$N^{cy}_{.}wS.\mathcal{E}(\mathcal{C}) \dashrightarrow N^{cy}_{.}wS.\mathcal{C} \times N^{cy}_{.}wS.\mathcal{C}$$

an equivalence. To see this, just follow the proof of 1.4.2 from 1.4.3 [W1], but replacing the category $t(m,w)$ (the linear diagrams of length in w) by the category of circular diagrams

$$C_0 \dashrightarrow C_1 \dashrightarrow \dots \dashrightarrow C_m \dashrightarrow C_0$$

n w. One consequence of this is that $K^{cy}(w,\mathcal{C})$, and in particular $W(\mathcal{C})$, is ne 0-th space in a connective Ω-spectrum made up of the spaces $^{cy}(w,S^{(n)}_{\cdot}\mathcal{C})$, with S^1 acting on the entire spectrum. In general, for any , one has a diagram (of ∞-loopspaces with S^1-action)

inclusion?

$$K(i,\mathcal{C}) \dashrightarrow K^{cy}(i,\mathcal{C}) \dashrightarrow W(\mathcal{C})$$

n which the action of S^1 on $K(i,\mathcal{C})$ is essentially trivial.

2. Groups and fixpoints

The simplest example of all this is when $\mathcal{C}$ is the category of (based) finite sets. In this case let us try to identify the spaces and maps

$$K(i,\mathcal{C}) \dashrightarrow K^{cy}(i,\mathcal{C}) \dashrightarrow W(\mathcal{C})$$

Of course we know the first space: every filtered object in $\mathcal{C}$ is canonically the sum of its subquotients, so the category $iS_k\mathcal{C}$ is equivalent to the product of k copies of it and $K(i,\mathcal{C})$ is equivalent to Segal's group completion of $\mathcal{C}$, in other words QS^0.

e can apply the same method to $K^{cy}(i,t)$ by viewing $N.^{cy}iS_k t$ as $.Aut(iS_k t)=N.iS_k Aut(t)$. Since $Aut(t)$ is the weak product over all $n>0$ of he category $t_{Z/nZ}$ of finite (based) free (Z/nZ)-sets, and the group ompletion of $t_{Z/nZ}$ is $QB(Z/nZ)_+$, we obtain

$$K^{cy}(i,\mathcal{E}) = K(i,Aut(\mathcal{E})) \simeq \prod_n{}' K(i,\mathcal{E}_{Z/nZ}) \simeq \prod_n{}' QB(Z/nZ)_+$$

his doesn't give the S^1-action, but it is clear that S^1 acts diagonally n the product and I'm quite sure that if we look more closely we will ind that the action on the nth factor comes from the action of S^1 on $(\mathbb{R}/nZ)/(Z/nZ)\simeq B(Z/nZ)$ given by the short exact sequence

$$Z/nZ \dashrightarrow \mathbb{R}/nZ \dashrightarrow \mathbb{R}/Z=S^1$$

? this gives trivial action of S^1 on $\mathbb{Z}/n\mathbb{Z}$?

inally, I believe that the third term $W(\mathcal{E})$ (where $\mathcal{E}$ is still the category of finite sets) contains $K^{cy}(i,\mathcal{E})$ as a deformation retract. Define a map v from $W(\mathcal{E})$ to $W(S_2\mathcal{E})$ as follows: Given an element $C_0\to\ldots\to C_j\to C_0$ of $N_j^{cy}\mathcal{E}$, nake the element $C_0^{\cdot}\to\ldots\to C_j^{\cdot}\to C_0^{\cdot}$ of $N_j^{cy}S_2\mathcal{E}$ where $C_i^{\cdot}$ is the pair $(C_i,f(C_i))$, f being the composed map once around the circular diagram from C_i to itself. This is a cyclic map from $N.^{cy}\mathcal{E}$ to $N.^{cy}S_2\mathcal{E}$. It extends in the other simplicial direction to give a map from $N.^{cy}S.\mathcal{E}$ to $N.^{cy}S.S_2\mathcal{E}$. Now use the three exact functors "total set, subset, quotient set" from $S_2\mathcal{E}$ to $\mathcal{E}$. The additivity theorem says that tv (which is the identity) is homotopic to sv+qv. Note that sv is the identity on $N.^{cy}iS.\mathcal{E}$ and that any finite subcomplex of $N.^{cy}S.\mathcal{E}$ is carried into $N.^{cy}iS.\mathcal{E}$ by some iterate of sv. Thus I

because $f(C_i)=C_i$?

ll have what I want if qv is nullhomotopic. I believe that it is. In any se it's clear that there is a retraction from $W(\mathcal{C})$ to $K^{cy}(\mathcal{C})$

ie map $W(\mathcal{C})\text{-->}K(i,\mathcal{C}_{Z/nZ})$ given by the retraction followed by the nth ojection can be described in terms of counting periodic orbits of ertain maps, and I will therefore denote it by Per_n. The description is s follows. Given a circular diagram $C_0\text{->}C_1\text{->}\ldots\text{->}C_k\text{->}C_0$ in $\mathcal{C}$ we select rom each set C_j the subset consisting of those elements which have period xactly n for the composed map $C_j\text{->}C_j$ (and also the basepoint). These form circular diagram in $i\mathcal{C}_{Z/nZ}$. The same construction applies more generally or filtered objects, and so gives a cyclic map from $N^{cy}_{.}S.\mathcal{C}$ to a "twisted" opy of $N.iS.\mathcal{C}_{Z/nZ}$ in $N^{cy}_{.}iS.\mathcal{C}_{Z/nZ}$.

ore generally if we start from the category $\mathcal{C}_G$ of finite free G-sets here G is some discrete group we can analyze the sequence

$$K(i,\mathcal{C}_G) \text{ -> } K^{cy}(i,\mathcal{C}_G) \text{ -> } W(\mathcal{C}_G)$$

e obtain

$$K(i,\mathcal{C}_G) \simeq Q(BG)_+$$

You know this, and I have already mentioned it in the case $G=Z/nZ$.)

$$K^{cy}(i,\mathcal{C}_G) = K(i,Aut(\mathcal{C}_G)) \simeq \prod_n' W_n(G)$$

ere $W_n(G)$ is the Segal group completion of the
oupoid-with-sum-operation whose objects are automorphisms of finite free
-sets such that the induced automorphism of the G-orbit set generates a
ee $(\mathbf{Z}/n\mathbf{Z})$-action. This can be analyzed and shown to be equivalent to
$E(\mathbf{Z}/n\mathbf{Z}) \times_{\mathbf{Z}/n\mathbf{Z}} \wedge BG)_+$. Again I believe that the correct action of S^1 on $W_n(G)$
s given by writing $E(\mathbb{R}/n\mathbf{Z})$ instead of $E(\mathbf{Z}/n\mathbf{Z})$. Concerning $W(\mathcal{C}_G)$, which I
ill abbreviate $W(G)$, there is a retraction $W(G) \rightarrow K^{cy}(i,\mathcal{C}_G) \simeq \Pi' W_n(G)$,
nd just as in the case G=1 I believe that it is an equivalence. Again I
enote the composed map $W(G) \rightarrow \Pi' W_n(G) \rightarrow W_n(G)$ by Per_n. Just as in the
ase G=1 it can be described in terms of counting periodic orbits.

3. Rings and traces

et R be a discrete ring and let $\mathcal{C}=\mathcal{M}_R$ be the category of finite(ly
enerated) projective modules. In the diagram

$$K(R) \underset{def}{:=} K(i,\mathcal{M}_R) \rightarrow K^{cy}(i,\mathcal{M}_R) \rightarrow W(\mathcal{M}_R) \underset{def}{=:} W(R)$$

he first term is algebraic K-theory of R. The second term is the
algebraic K-theory of the exact category of
finite-projective-modules-with-automorphism. Thus for example if R is a
field then this is the category of finitely generated torsion modules for
the ring $R[x,x^{-1}]$. This is the weak product of the K-theory of all of the
residue fields. What can we say about the third term W(R)?

W(R) is the group given by the following generators and relations: There a generator [M,f] for every endomorphism $f:M\to M$ of a finite projective module. There are two kinds of relations: the Grothendieck relation for ort exact sequences, and the relation [M,gf]=[N,fg] for $M\xrightarrow{f}N\xrightarrow{g}M$. When R commutative there is a homomorphism to the multiplicative group of rmal power series $1+a_1T+a_2T^2+\ldots$, given by

$$[A] \longrightarrow \det(I-AT)$$

is easily seen to be a split surjection onto its image, which is of urse the subgroup generated by polynomials. In the case of a field the rnel is easily seen to be trivial. Actually this turns out to be true r every commutative ring: According to a theorem of Almkvist [A][Gr] the rnel is trivial even when the relation [M,gf]=[N,fg] is replaced by the parently weaker relation [M,0]=0. Note that the map from $\pi_0 W(R)$ to power ries is a ring homomorphism. Here the ring structure on $\pi_0 W(R)$ is given tensor product of endomorphisms of modules and the target group is a ıbring of the ring of Witt vectors, namely the group of all quotients of lynomials with constant terms 1. Let's call it the ring of rational Witt ctors.

related construction sends $\pi_0 W(R)$ to the product $\prod_{n>0} R$ by

$$[A] \longrightarrow (\mathrm{trace}(A),\mathrm{trace}(A^2),\mathrm{trace}(A^3),\ldots)$$

ere are several things to say about this. First, it is determined by the her construction because trace(A^n) can be expressed in terms of the rst n coefficients of det(I-AT) by the universal formulas which express ower sums in terms of elementary symmetric polynomials. (One cannot go e other way unless R is a Q-algebra.) Second, this map really goes into product, not a weak product. Third, this map generalizes to the ncommutative case; one has only to take the traces to lie in the ochschild homology group R/[R,R]. Fourth, in the case of a group ring =ZG this is closely related to a construction which I mentioned in onnection with $K^{cy}(\mathcal{C}_G)$. There is in fact a diagram

$$\begin{array}{ccccc} K(i,\mathcal{C}_G) & \to & K^{cy}(i,\mathcal{C}_G) & \xrightarrow{\cong} & W(G) \\ \downarrow & & \downarrow & & \downarrow \\ K(i,\mathcal{M}_R) & \to & K^{cy}(i,\mathcal{M}_R) & \to & W(R) \end{array}$$

nduced by the exact functor "linearization" from $\mathcal{C}_G$ to $\mathcal{M}_R$, and one can asily express the maps

$$\pi_0 W(G) \to \pi_0 W(R) \xrightarrow{\text{"trace}(A^n)\text{"}} R/[R,R]$$

n terms of the maps

$$\pi_0 W(G) \xrightarrow{\pi_0 Per_n} \pi_0^S(F(\mathbb{Z}/n\mathbb{Z}) \underset{\mathbb{Z}/n\mathbb{Z}}{\times} \Lambda BG)_+ = R/[R,R]$$

y considering traces of powers of matrices like

$$\begin{matrix} 0 & 0 & g \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \quad , \; g \in G$$

think about this in the following way: trace(A) corresponds to counting he fixpoints of a map. Therefore trace(A^n) corresponds to counting the ixpoints of the nth iterate. A certain linear combination of trace(A^j) or $1 \le j \le n$ corresponds to counting the points of period exactly n. To count he orbits of such points one should divide this linear combination by n. his can be done when the ring is $\mathbb{Z}$. (It can't be done for a general ring, even in the commutative case.)

I believe that the homotopy type of W(R) is unchanged if instead of projective modules we use free or based modules; this is easy to check on π_0.

I will now show that the usual trace map from K-theory to Hochschild homology factors through W(R) by giving an explicit map

$$W(R) \xrightarrow{\text{Trace}} \Omega|N.HH.(R)| \simeq |HH.(R)| = HH(R)$$

to (a space whose homotopy groups are) the Hochschild homology of R. I use the definition of W(R) in terms of based modules. Thus an element in $N_j^{cy} S_1 M_R$ determines a (j+1)-tuple of matrices $(A_0, A_1, \ldots, A_j)$ whose shapes nd sizes are such that it is possible to multiply any two consecutive matrices (including A_j and A_0). An element in $N^{cy}_j S_k \mathcal{M}_R$ determines a (j+1)-tuple of matrices in kxk block triangular form which are similarly ready to multiply. To go to Hochschild homology, just carry out the matrix multiplication: send a (j+1)-tuple to the product matrix $A_0A_1\ldots A_j$; but don't use the multiplication in R, just leave the matrix entries as elements of the tensor product R^{j+1}. The matrix is square, in kxk triangular block form, with square blocks on the diagonal. Take the trace within each diagonal block. This gives an element of the direct sum of tensor powers $\oplus_k R^{j+1}$. The map that I have just described is a map of bisimplicial sets. Its source is $N^{cy}_. S.\mathcal{M}_R$ and its target is the nerve (with respect to addition) of the simplicial abelian group HH.(R) which is the standard model for Hochschild homology. Thus after realizing and looping it becomes a map from W(R) to $\Omega|N.HH.(R)|$. Of course this is also an S^1-equivariant map because it comes from a map of cyclic objects. It even extends to give an equivariant map of Ω-spectra. On $\pi_0 W(R)$ it gives the map "[A]->trace(A)" to $R/[R,R]=HH_0(R)$ which we saw before.

In the same way we can define "trace power maps" generalizing the construction "[A]->trace(A^n)" from π_0 to the space level. Namely, instead of taking the trace of $A_0A_1\ldots A_j$ in R^{j+1} take the trace of $(A_0A_1\ldots A_j)^n$ in $R^{n(j+1)}$. Again this is a map from $N^{cy}_. S.\mathcal{M}_R$ into the nerve of a simplicial abelian group. However, this time the simplicial abelian group is not HH.(R) but rather a different simplicial model for Hochschild homology; call it $HH^{(n)}_.(R)$. It is the diagonal of a certain n-simplicial abelian oup. (You can find this multisimplicial object on page 383 of [Gol]; in ur case the bimodules I_j should all be taken to be R.) $HH^{(n)}_{\cdot}(R)$ is a odel for Hochschild homology which has a simplicial $(\mathbb{Z}/n\mathbb{Z})$-action, given y cyclically permuting the simplicial directions. I believe that $|HH^{(n)}_{\cdot}(R)|$ is actually isomorphic as a topological abelian group with $\mathbb{Z}/n\mathbb{Z}$)-action to $|HH.(R)|$ where $\mathbb{Z}/n\mathbb{Z}$ acts on the latter as a subgroup of S^1. The map from $N^{cy}_{\cdot}S_k M_R$ actually goes into the fixed-point set $HH^n_{\cdot}(R)^{\mathbb{Z}/n\mathbb{Z}}$ or this action so that the result of our efforts is a map

$$W(R) \xrightarrow{\text{Trace}_n} \Omega|N.(HH^{(n)}_{\cdot}(R)^{\mathbb{Z}/n\mathbb{Z}})| = \Omega|N.(HH(R)^{\mathbb{Z}/n\mathbb{Z}})| \simeq HH(R)^{\mathbb{Z}/n\mathbb{Z}}$$

'm a little surprised to be in a fixpoint set $X^{\mathbb{Z}/n\mathbb{Z}}$ rather than in a omotopy fixpoint set $X^{h(\mathbb{Z}/n\mathbb{Z})}$. In any case we can consider the latter as ell. We get weak maps of spaces (or better Ω-spectra with S^1-action)

$$K(R) \to W(R) \to HH(R)^{\mathbb{Z}/n\mathbb{Z}} \to HH(R)^{h(\mathbb{Z}/n\mathbb{Z})} \to HH(R)$$

In particular on the level of homotopy groups we get maps

$$(3.1) \quad K_*(R) \to \pi_* W(R) \to \pi_*(HH(R)^{\mathbb{Z}/n\mathbb{Z}}) \to H^{-*}(\mathbb{Z}/n\mathbb{Z};HH(R)) \to HH_*(R)$$

Here I have interpreted the homotopy groups of the homotopy fixpoint spectrum as (hyper)cohomology groups of $\mathbb{Z}/n\mathbb{Z}$ with coefficients in a topological abelian group. (I don't know a good interpretation for the actual fixpoint set.) Note that if n happens to be invertible in R then the last two maps in 3.1 are isomorphisms.

4. Simplicial groups and simplicial rings

Up to now I have only been able to describe a map from K to K^{cy} in cases where the isomorphisms are the weak equivalences. This is not adequate for dealing with the K-theory of spaces, or simplicial rings, or rings up to homotopy. I will try to remedy this by using your trick of "blowing up the weak equivalences to a simplicial category".

For example, if G is a simplicial loop group for a space X then one of your ways of defining A(X) is $K(w,\mathcal{R})$, where $\mathcal{R}=\mathcal{R}_{hf}(*,G)$ is the category of those based simplicial G-sets which are free (in the sense of attaching cells) and finite up to weak equivalence. The weak equivalences are the maps which after realization are weak equivalences of spaces. You also have a blown-up version in which the category w is replaced by a simplicial category w. having the same objects, and you show ([W1]) that the trisimplicial object $N.w.S.\mathcal{R}$ is equivalent to the bisimplicial object $N.wS.\mathcal{R}$ and so also yields A(X). The reason why this works is that for each k $N.w.S_k\mathcal{R}$ is equivalent to $N.wS_k\mathcal{R}$. For each k the simplicial category $wS_k\mathcal{R}$ is "groupoid-like" in the sense that after replacing morphisms by $\pi_0|\text{morphisms}|$ it becomes a groupoid. It ought to be true, and probably is known, that groupoid-like simplicial categories (with discrete sets of objects) can be functorially replaced by weakly naturally equivalent simplicial groupoids. (Equivalent here means related by a simplicial functor which is bijective on objects and a weak equivalence on morphisms.) So another definition of A(X) is as the nerve of the simplicial category obtained by replacing each $w.S_k\mathcal{R}$ by a simplicial oupoid. Now since we have a groupoid we can map the nerve to the cyclic rve. We thus obtain a weak map from $K(w.,\mathcal{R})$ (which is $A(X)$) to $K^{cy}(w.,\mathcal{R})$ hich I will now name $A^{cy}(X)$). Note: I am using the obvious fact that placing the simplicial categories by the simplicial groupoids does not ange the homotopy type of the nerve (resp. cyclic nerve). This is true ready when we fix the degree in both the "S." simplicial direction and e nerve (resp. cyclic nerve) direction. I am also using your fact that $w.,\mathcal{R})\simeq K(w,\mathcal{R})$. But I am not claiming that $K^{cy}(w.,\mathcal{R})\simeq K^{cy}(w,\mathcal{R})$; I am simply oosing to use only the blown-up version $K^{cy}(w.,\mathcal{R})$ as my definition of $^{cy}(X)$ and not the other.

e next step is to consider the version in which all maps are declared to e equivalences. We map $K^{cy}(w.,\mathcal{R})$ into a blown-up version of $W(\mathcal{R})$, which ight be called $W(\mathcal{R}.)$. This just means including "function spaces" actually simplicial categories) of weak G-equivalences into larger function spaces" of arbitrary G-maps. Again I am not claiming that this $(\mathcal{R}.)$ is equivalent to $W(\mathcal{R})$. I will return later to the question of what his $W(\mathcal{R}.)$ might actually be. It is some new and interesting homotopy unctor of the simplicial group G or of the space X. It is related to A(X) n the same way that (for discrete rings R) W(R) is related to K(R). Let's all it T(X).

he same approach works in various other settings, too, as long as there are simplicial catories of weak equivalences with good properties. For example, again for a simplicial group G we could have used a topological version of $\mathcal{R}$ such as your $\mathcal{R}_f(*,|G|)$. I believe that up to natural weak equivalence this will give not only the same $A(X)$ (by taking nerves with or without blowing up the equivalences) but also the same $A^{cy}(X)$ (by taking cyclic nerves, and this time definitely blowing up). I also believe the same for $T(X)$.

One could also introduce the finitely dominated version $\mathcal{R}_{fd}(*,G)$. Just as this gives the correct $A(X)$ (except that π_0 is now the full class group), I think that the cyclic nerve of the blown-up version of this gives the correct $A^{cy}(X)$ (except for π_0), and that the cyclic nerve of the blown-up version with all maps as equivalences gives the correct $T(X)$ (this time even with the correct π_0).

Of course I do not assert, in the case of a discrete group G, that the constructions $K^{cy}(i,\mathcal{E}_G)$ and $W(G)$ based on finite free G-sets are the same as the corresponding constructions $A^{cy}(BG)$ and $T(BG)$ based on (homotopically) finite free G-spaces. (This is already false in the non-cyclic version: $K(i,\mathcal{E}_G)\simeq Q(X_+)$ is not the same as $A(X)$.) However, I think it is instructive to compare them.

There are analogous definitions and statements concerning rings. If R is a simplicial ring then one can consider various categories of free simplicial R-modules (finite, homotopically finite, or finitely dominated) and in each case one can either blow up the weak equivalences or not.

heorems of yours say that for K-theory (except for π_0) it doesn't matter hich version you use. I think that the same is true for cyclic K-theory s long as you always blow up. I think this is also true for W(R) (cyclic -theory of simplicial modules when all maps are declared to be quivalences), and that in this case even π_0 is unaffected by the choice f category of modules.

'or rings, as opposed to groups, there is one difference. Namely when a iscrete ring is viewed as a simplicial ring the K^{cy} based on discrete odules and isomorphisms is the same as the K^{cy} for simplicial modules and a simplicial category of) equivalences. The same should be true for W: he K^{cy} based on discrete modules and homomorphisms should be the same as he K^{cy} based on simplicial modules and (a simplicial category of) omomorphisms. This would not be surprising, since the non-cyclic version is a theorem of yours.

Actually it seems pretty clear that W(R) for a simplicial ring R (defined in terms of simplicial modules and a simplicial category of module maps) can equivalently be defined in terms of spaces of matrices, as in the description of W(R) for discrete R which was used in defining the trace map above. This has several good consequences. One is the fact which I just mentioned, that W(R) for R discrete is the same as W(R) for R viewed as simplicial. Another is that the construction of the trace power maps on the space level for discrete R can be generalized to simplicial R. Another is that W(R) for simplicial R can be defined degreewise. This is a strong statement. Of course the corresponding statement for K-theory is false, even though K(R) can also be defined in terms of spaces of matrices. (The problem is that the matrices which are invertible up to homotopy cannot be identified degreewise.) This property of W(R) should be a great aid in computing it. As a simple example, suppose one wanted to give another proof of the statement that for any discrete commutative ring R the map from $\pi_0 W(R)$ to the ring of rational Witt vectors is an isomorphism. It would be enough to prove the more general statement for commutative simplicial rings: $\pi_0 W(R)$ = rational Witt vectors of $\pi_0(R)$. But F respects weak equivalences, and every commutative simplicial ring has a free resolution. It follows easily that it would be enough to prove the statement in the case of polynomial rings over **Z**.

On a related subject, I want to point out that both $K^{cy}(R)$ and W(R) are equipped with an external pairing, and hence in the commutative case also an internal pairing. The argument is just a variant of your argument in [W1] for K(R), using a biexact functor. The maps $K(R) \to K^{cy}(R) \to W(R)$ are ring maps. It's interesting to note that W(R) is not an R-module (although HH(R) is), even in the commutative case. (Look at $\pi_0 W(Z/p)$!)

Some of these comments about rings have analogues for groups. Thus it seems to me that T(X) must also have a description in terms of "spaces of matrices" and that as a consequence it can be defined degreewise (as a functor of the simplicial group G).

resumably much of what I am doing here can be generalized to "rings up to omotopy" in various senses. In that setting HH(R) should be replaced by a ersion of Bokstedt's THH(R) [B]. Even without leaving the world of implicial rings I think that my trace W(R)->HH(R) can probably be actored through THH(R). I don't quite know what to expect as the target f trace power maps when THH is substituted for HH. For some vague ideas bout this, see the next two sections.

5. Fixpoints again

hen G was a discrete group it turned out that W(G) was completely etected by periodic points: We looked at circular diagrams of finite free -sets. We composed the maps around the diagram to get a self-map of a ree G-set. We counted the periodic orbits of length n for the action of his map on the G-orbit set. Doing this systematically we found that we ere mapping $N^{cy}_{\cdot}S.\mathcal{E}_G$ into another cyclic object and thus mapping W(G) into a certain space $W_n(G)$. Assembling these for all n>0 we obtained an equivalence to the weak product $\Pi' W_n(G)$.

would like to do something similar for simplicial groups and thereby btain a good target for a map from A(X). In fact these ideas seem to lead o a map which goes from T(X) to the product (not the weak product) of the spaces $W_n(G)$. (What I mean by $W_n(G)$ here is W_n extended from groups to simplicial groups by degreewise extension. Note that we still have, as in he case of discrete groups, $W_n(G) \simeq Q(E(\mathbb{Z}/n\mathbb{Z}) \times_{\mathbb{Z}/n\mathbb{Z}} \Lambda BG)_+$.)

There is a little problem in choosing the right version of the category of free G-spaces. The category must be capable of being blown up and made groupoid-like. This rules out finite simplicial G-sets. It must have some sort of finiteness if one is to be able to count fixpoints and periodic points in a meaningful way. This rules out, say, homotopically finite fibrant simplicial G-sets. It must be a category with cofibrations! This rules out any PL approach. However, I believe the thing can be made to work somehow. I base this partly on the thinking that I did about periodic points when I was trying to work out a version of "higher traces" for smooth pseudoisotopy theory, and partly on the remark that if we ignore the point about nonexistence of cofibers in the PL category then things seem to work beautifully. Thus take G=1 for simplicity (so X=*). Suppose we have a circular diagram of compact polyhedra and PL maps. Let f be the map which sends each of the polyhedra to itself by running around the diagram once. The fixpoints of f^n form a subpolyhedron. Divide out by the fixpoints of f^m for all m<n. This yields a pointed finite complex with free $(\mathbb{Z}/n\mathbb{Z})$-action, in fact a circular diagram of isomorphisms of such things, so it seems to get us into $W_n(1)$. Taken all together it seems to give a map $T(*) \to \prod_n W_n(1)$. Of course we left the PL category (and we will also leave it as soon as we introduce filtered objects and try to define face maps in the "S." direction). But this still seems encouraging.

I emphasize that the map from T(X) to $W_n(G)$ which I am trying to define is something which should not make sense for rings or rings up to homotopy in general. When n=1 it should be the "trace", which does make sense in general and which has been defined (on K-theory) in various degrees of generality by various people (such as you [W2], Bokstedt [B], and Cohen and Jones [CJ])). In general the only "higher traces" that I know are the trace power maps. My feeling is that for group rings a more refined construction should be possible, and that "counting periodic points" should be the guiding idea.

By the way, if this works then the space T(X) is actually caught between the weak product and the full product of the spaces $W_n(G)$: we can map the weak product into T(X) by using the obvious maps

$$|[m] \to K^{cy}(i, \mathcal{C}_{G_m})| \to K^{cy}(i, \mathcal{R}) \to K^{cy}(w., \mathcal{R}) \to K^{cy}(\mathcal{R}.) = T(X)$$

Thus T(X) is no smaller than the weak product of the $W_n(G)$. It seems plausible that in some sense it is no bigger than the full product. As a first test of this one might try it for π_0. Is it true that for an arbitrary discrete group G the "trace power" map $[A] \to \{trace(A^n) | n>0\}$ injects $\pi_0 W(\mathbb{Z}G) = \pi_0 T(BG)$ into $\prod_n \mathbb{Z}G/[\mathbb{Z}G, \mathbb{Z}G]$?

I'll mention two other questions while I think of them: (1) What is the correct analogue of the ring of Witt vectors for a non-commutative ring? (2) For the class of commutative rings the characteristic polynomial det(I-TA) gave a sort of "higher trace" more refined than the trace power map. Does this generalize somehow from $\pi_0 W(R)$ to W(R) ? (My vague feeling is that for commutative rings W(R) is trying to be something like "the Witt vectors over THH(R)".)

6. Epicyclic objects

he cyclic nerve of a category is more than a cyclic object; there is one ther thing you can do to a circular diagram besides (1) composing two aps to make the diagram shorter, (2) inserting an identity map to make it onger, and (3) rotating it. Namely, (4) you can make "covering spaces" of uch diagrams. I will formalize this idea by introducing the idea of an epicyclic object". (The terminology was suggested by Andrew Ranicki.)

he category of epicyclic operators $\Lambda^{\sim}$ contains Connes' category Λ. It has he same objects as Λ. Index them by the integers $n>0$ and call the n^{th} bject (n). A map from (m) to (n) is a class (f) of functions f from $\mathbb{Z}$ to . The functions f are required to satisfy

$$f(x) \leq f(x+1)$$

$$f(x+m)-f(x) = dn$$

for all $x\in\mathbb{Z}$, where d is a positive integer (independent of x) called deg(f). The equivalence relation among the functions is given by

$$(f)=(g) \iff f(x)-g(x) \text{ is a multiple of } n \text{ independent of } x$$

(Note that this is a little stronger than saying that f and g define the same map of sets $\mathbb{Z}/m \to \mathbb{Z}/n$.) Note that degree of maps and composition of maps are well-defined on equivalence classes.

A map $(f):(m)\to(n)$ in $\Lambda^{\sim}$ may or may not have a representative f such that

$$1 \leq f(1) \leq f(m) \leq n$$

If so, it has a unique one. These maps (f) are the simplicial operators; they form a subcategory of $\Lambda^{\sim}$ isomorphic to Δ. The category Λ of cyclic operators is intermediate between Δ and $\Lambda^{\sim}$; it consists of the maps of degree one in $\Lambda^{\sim}$. It is generated by Δ together with the translations $(t):(n)\to(n)$ given by $t(x)=x+1$. The category $\Lambda^{\sim}$ is generated by Λ together with the standard coverings $(p):(dn)\to(n)$ given by $p(x)=x$. In fact, every map $(f):(m)\to(n)$ of degree d can be factored as $(m)\xrightarrow{(g)}(dn)\xrightarrow{(p)}(n)$ where (g) has degree one. Here (g) is determined up to a d-fold ambiguity. ($(t^n g)$ is as good as (g).)

An epicyclic object in a category is by definition a functor from $\Lambda^{\sim op}$. It is easy to see that cyclic nerves are not just cyclic sets but epicyclic sets. On the other hand, for a ring R the cyclic abelian group HH.(R)) is not an epicyclic abelian group, because for this one would need linear "diagonal maps" from R to tensor powers of R. (Some rings, such as group rings, do have an epicyclic structure on their Hochschild homology, and this fact is probably useful, but it is also confusing. In particular the trace map $F(R)\to HH(R)$ for a group ring R is not an epicyclic map, but only a cyclic map between epicyclic objects.)

e objects and maps

$$A(X) \dashrightarrow A^{cy}(X) \dashrightarrow T(X)$$
$$K(R) \dashrightarrow K^{cy}(R) \dashrightarrow W(R)$$

e all epicyclic (in the category of Ω-spectra with S^1-action).

should say what I think epicyclic objects really are. Just as simplicial ets are really spaces and (by [DHK]) cyclic sets are really spaces with say) right S^1-action, I think that epicyclic sets are really spaces with right action of the topological monoid M of all maps from the circle to tself with constant positive derivative. (It is important to say "right" ere since M is not isomorphic to M^{op}.) Notice that ΛX has such an action. otice also that the product $\prod_n HH(R)^{Z/n}$ which was the target of our "trace ower map" looks very much like it might be the universal epicyclic object ver the cyclic object HH(R).

7. Applications (?)

All of this originated in my efforts to understand the fiber of A(X)->A(Y) for 2-connected maps X->Y, or the fiber of K(R)->K(S) for 1-connected ring maps R->S, in terms of something simpler, something like the free loopspace and its S^1-action or (topological) Hochschild homology with its S^1-action (in other words cyclic theory). I now think that one should use the epicyclic theory instead. I'll save the details for a later letter, but my current thinking is this: We have maps

$$A(X) \to T(X) \to W(\Omega X) =_{def} T_1(X).$$

Because the maps are suitably equivariant the situation is actually better than this: we have maps

$$A(X) \to T(X)^{hM} \to T_1(X)^{hS^1}$$

into homotopy fixpoint sets (or rather spectra). Here M is the monoid which acts on the realization of an epicyclic object. Differentiating in the sense of [Go2], we obtain maps

$$A^{\cdot}(X) \to (T^{\cdot}(X))^{hM} \to (T_1^{\cdot}(X))^{hS^1}$$

I have known for some time that the composed map is an equivalence. I now believe that the second map may also be an equivalence. It would follow that the first one is. But I think there's a chance that $(T^{\cdot})^{hM} \simeq (T^{hM})^{\cdot}$. (It was not true that $(T_1^{\cdot})^{hS^1} \simeq (T_1^{hS^1})^{\cdot}$.) If so, and if T^{hM} is in some sense "analytic", then my map $A \to T^{hM}$, because it induces an equivalence of derivatives, will induce equivalences $fiber(A(X) \to A(Y)) \to fiber(T(X)^{hM} \to T(Y)^{hM})$. This should hold for 2-connected maps $X \to Y$ of finite complexes, I guess; to get a statement for arbitrary 2-connected maps one should take direct limits. (The functor A commutes with filtered direct limits, but T^{hM} does not.)

tually a more confident conjecture would be that all of what I have just id about $T(X)$ is true about $\prod_n W_n(\Omega X)$. My idea is that the latter object ould have an epicyclic structure and an epicyclic map from $T(X)$ (given r each n by a "Per_n" map). Based on my knowledge of the "Taylor tower" $W(\Omega X)$ I have the impression that the following should be true: when $\to Y$ is a 2-connected finite CW pair then the fiber of $A(X) \to A(Y)$ should the same as the fiber of $(\prod_n W_n(\Omega X))^{hM} \to (\prod_n W_n(\Omega Y))^{hM}$ by a yer-by-layer comparison of two towers. (It may also be true that the map $\to \prod_n W_n \Omega$ is an equivalence, but this is something I'm much less confident out.)

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References

[A] G. Almkvist, The Grothendieck ring of the category of endomorphisms, J. Algebra 28, 375-388 (1974)

[B] M, Bökstedt, Topological hochschild homology, preprint (1987)

[CJ] R. Cohen and J. Jones, Trace Maps and Chern characters in K-theory, preprint (1987).

[DHK] W. Dwyer, M. Hopkins, and D. Kan, The homotopy theory of cyclic sets, AMS Trans. 291, 281-289 (1985)

[Go1] T. Goodwillie, Relative algebraic K-theory and cyclic homology, Annals of Math., 124, 347-402 (1986)

[Go2] T. Goodwillie, Calculus I, in preparation.

[Gr] D. Grayson, Grothendieck rings and Witt vectors, Comm. in Alg., 6 (3), 249-255 (1978)

[W1] F. Waldhausen, Algebraic K-theory of spaces, LNM #1126, 318-419 (1985)

[W2] F. Waldhausen, Algebraic K-theory of topological spaces II, LNM #763 (1979)