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KOSZUL DUALITY FOR OPERADS

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CONTENTS

0. Introduction

(0.1) The purpose of this paper is to relate two seemingly disparate developments. One is the theory of graph cohomology of Kontsevich [39], [40], which arose out of earlier works of Penner [54] and Kontsevich [38] on the cell decomposition and intersection theory on the moduli spaces of curves. The other is the theory of Koszul duality for quadratic associative algebras, which was introduced by Priddy [55] and has found many applications in homological algebra, algebraic geometry, and representation theory (see, e.g., [5], [6], [7], [30], [51]). The unifying concept here is that of an operad.

This paper can be divided into two parts consisting of Chapters 1, 3 and 2, 4, respectively. The purpose of the first part is to establish a relationship between operads, moduli spaces of stable curves, and graph complexes. To each operad we associate a collection of sheaves on moduli spaces. We introduce, in a natural way, the cobar complex of an operad and show that it is nothing but a (special case of the) graph complex, and that both constructions can be interpreted as the Verdier duality functor on sheaves.

In the second part we introduce a class of operads, called quadratic, and introduce a distinguished subclass of Koszul operads. The main reason for introducing Koszul operads (and in fact for writing this paper) is that most of the operads "arising from nature" are Koszul; cf. (0.8) below. We define a natural duality on quadratic operads (which is analogous to the duality of Priddy [55] for quadratic associative algebras) and show that it is intimately related to the cobar-construction, i.e., to graph complexes.

(0.2) Before going further into discussion of the results of the paper, let us make some comments for the reader not familiar with the notion of an operad. Operads were introduced by J. P. May [52] in 1972 for the needs of homotopy

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theory. Since then it has been gradually realized that this concept has in fact fundamental significance for mathematics in general. From an algebraic point of view, an operad is a system of data that formalizes properties of a collection of maps $X^n \to X$, a certain set for each $n = 1, 2, \ldots$, which are closed under permutations of arguments of the maps and under all possible superpositions. Such a collection may be generated by iterated compositions of some primary maps, called the generators of the operad, and the whole structure of the operad may be determined, in principle, by giving the list of relations among the generators. This is very similar to defining a group by generators and relations. As for groups, the consideration of the whole operad and not only generators and relations presents several obvious advantages. For instance, we can develop "homological algebra" for operations, i.e., study higher syzygies. Note, in particular, that all the types of algebras encountered in practice (associative, Lie, Poisson, etc.) are governed by suitable operads. (We take the binary operations involved as generators and the identities which they satisfy as relations.)

The place of operads among other structures may be illustrated (very roughly) by the following table.

	Algebra	Geometry	Linear Physics	Nonlinear Physics
Spin 1	Modules	Vector bundles	Maxwell equations	Yang–Mills theory
Spin 2	Algebras	Manifolds	Linear gravity equations	Einstein gravity
Spin 3	Operads	?(Moduli spaces)	Rarita–Schwinger equations	?(Conformal field theory)

Question marks indicate that the name put at the corresponding square of the table is just the first approximation to an unknown ultimate name. The relation of operads to algebras in the first column is similar to the relation of algebras to modules. Given an algebra A, one has a notion of an A-module. Similarly, given an operad $\mathcal{P}$, there is a notion of a $\mathcal{P}$-algebra. Further, for any $\mathcal{P}$-algebra A, there is a well-defined abelian category of A-modules; see §1.6 below. This explains the hierarchy of the first column of the above table. In the geometric column, vector bundles on a fixed manifold correspond to modules over the (commutative) algebra of functions on the manifold. Similarly, giving an operad is analogous to fixing the class of geometric objects we want to consider. This is equivalent to studying the moduli space of these geometric objects. The relevance of the notion of an operad to conformal field theory can be justified by the following fact, which plays a crucial role in the theory of operads and in the present paper in particular:

The collection $\mathcal{M} = \{\overline{\mathcal{M}}_{0,n+1}, n = 2, 3, \ldots\}$ *(Grothendieck-Knudsen moduli spaces of stable genus* 0 *curves with* $n+1$ *punctures) has a natural structure of an operad of smooth manifolds.*

(0.3) The paper is organized as follows. Chapter 1 begins with introducing a category of trees that plays the key role (cf. [11]) throughout the paper. We then recall the definition of an operad and produce a few elementary examples of operads. Next, the above-mentioned operad $\mathcal{M}$ formed by the Grothendieck-Knudsen moduli spaces is described in some detail. The significance of this operad for the whole theory of operads is explained: any operad can be described as a collection of sheaves on $\mathcal{M}$.

(0.4) Chapter 2 is devoted to *quadratic operads*, the ones generated by binary operations subject to relations involving three arguments only. Most of the structures that one encounters in algebra, e.g., associative, commutative, Lie, Poisson, etc. algebras, correspond to quadratic operads.

Given a quadratic operad $\mathcal{P}$, we define the *quadratic dual* operad $\mathcal{P}^!$ analogously to the definition of quadratic duality (Priddy) of quadratic associative algebras. In particular, the operads $\mathcal{C}om$ governing commutative and $\mathcal{L}ie$ (governing Lie) algebras are quadratic dual to each other. In some informal sense, as a correspondence between the categories of (differential graded) commutative and Lie algebras, this relation goes back at least to the work of Quillen [56], [57] and Moore [53]. Our theory exhibits a very simple and precise algebraic fact which is the reason for this relation. The operad $\mathcal{A}s$ describing associative (not necessarily commutative) algebras is self-dual in our sense.

There is a natural concept of a quadratic algebra over a quadratic operad, and for the dual quadratic operads $\mathcal{P}$ and $\mathcal{Q}$ we construct a duality between quadratic (super-) algebras over $\mathcal{P}$ and $\mathcal{Q}$. For the case of associative algebras (whose operad is self-dual) we recover the construction of Priddy.

Quadratic operads have several nice features. In §2.2 we introduce on the category of such operads the internal *hom* in the spirit of Manin [51]. We show that the quadratic duality can be interpreted as $hom(-, \mathcal{L}ie)$ where $\mathcal{L}ie$ is the Lie operad which therefore plays the role of a dualizing object in our theory.

(0.5) In Chapter 3 we introduce a contravariant duality functor $\mathbf{D}$ on the category of differential graded (*dg*-) operads (as opposed to the quadratic duality functor $\mathcal{P} \mapsto \mathcal{P}^!$ studied in the previous chapter). We present various approaches to duality. From the algebraic point of view, the duality $\mathbf{D}$ is an analog of the cobar construction and a generalization of the tree part of graph complexes. From the geometric point of view, the duality is an analog of the Verdier duality for sheaves. In more detail, let $\mathcal{W}_{n+1}$ be the moduli space of n-labelled trees with metric; cf. [54], [39]. This is a contractible topological space with a natural cell decomposition. This cell decomposition is dual, in a sense, to the canonical stratification of $\overline{\mathcal{M}}_{0,n+1}$. Moreover, the collection $\mathcal{W} = \{\mathcal{W}_{n+1}\}$ forms a *cooperad* which plays the role dual to that of the operad $\mathcal{M}$. We show that any *dg*-operad gives rise to a compatible collection of constructible complexes on the spaces $\mathcal{W}_{n+1}$, one for each n. Furthermore the collection arising from the $\mathbf{D}$-dual *dg*-operad turns out to be formed by the Verdier duals of the complexes corresponding to the original operad. The spaces $\mathcal{W}_{n+1}$ are similar in nature to Bruhat-Tits buildings, and there is yet another description of duality in terms of sheaf cohomology, which is reminiscent of the Deligne-Lusztig duality [15]–[17] for representations of finite Chevalley groups.

Next, to any dg-operad we associate its generating function which is in fact (see Definition 3.1.8) a certain formal map $\mathbf{C}^r \to \mathbf{C}^r$. It turns out (Theorem 3.3.2) that for dg-operads dual in our sense their generating maps are, up to signs, composition inverses to each other. Recall (see, e.g., [7], [46]) that for an associative algebra A over a field k its cobar-construction (i.e., a suitable dg-model for the Yoneda algebra $\mathrm{Ext}^{\bullet}_A(k, k)$) has a generating function which is multiplicative inverse to the generating function of A. The role of composition (change of coordinates on a manifold) versus multiplication (change of coordinates in a vector bundle) for generating functions of operads also fits nicely into the table above.

(0.6) Section 4 is devoted to *Koszul operads*, the quadratic operads whose quadratic dual is quasi-isomorphic (canonically) to the **D**-dual. We prove that the operads $\mathcal{A}s$, $\mathcal{C}om$, and $\mathcal{L}ie$ are Koszul. Associated to any quadratic operad is its Koszul complex. We show that a quadratic operad $\mathcal{P}$ is Koszul if and only if its Koszul complex is exact, which is also equivalent to vanishing of higher homology for free $\mathcal{P}$-algebras. Given a Koszul operad $\mathcal{P}$, we introduce the notion of a *Homotopy $\mathcal{P}$-algebra* which reduces in the special cases of Lie and commutative algebras to that introduced earlier by Schlessinger and Stasheff [59] and exploited in an essential way by Kontsevich [39]. In fact, Koszul operads provide the most natural framework for the "formal noncommutative geometry".

(0.7) The concept of an operad in its present form had several important precursors. One should mention the formalism of "theories" of Lawvere (see [11]) and the pioneering work of Stasheff [60] on homotopy associative H-spaces. A little earlier, in the 1955 paper [43], Lazard considered what we would now call formal groups in an operad. He used the notion of "analyseur" which is essentially equivalent (though formally different) to the modern notion of an operad. In (2.2.14) we give a natural interpretation of Lazard's "Lie theory" for formal groups in analyseurs in terms of Koszul duality. We are grateful to Y. I. Manin for pointing out to us the reference [43].

In late 1950s Kolmogoroff and Arnold [37], [2] studied, in connection with Hilbert's 13th problem, what in our present language is the operad of continuous operations $\mathbf{R}^n \to \mathbf{R}$. It was proved in these papers that any continuous function in $n \geqslant 3$ variables can be represented as a superposition of continuous functions in only one and two variables, i.e., the above operad is generated by unitary and binary operations. From the point of view of the present paper (0.2), this result raises the interesting question of whether all the *relations* among the binary generators follow from those provided by functions of three variables. It seems that this question has never been addressed. We would like to thank I. M. Gelfand for drawing our attention to Kolmogoroff's work.

(0.8) Our interest in this subject originated from an attempt to explain a striking similarity between combinatorics involved in the graph complex, introduced by Kontsevich in his work on Chern-Simons theory, and combinatorics of the Grothendieck-Knudsen moduli spaces used by A. Beilinson and the first author in their work on local geometry of moduli spaces of G-bundles [9]. In particular, the main motivation for the study of Koszul operads begun in this paper is the investigation of the operads formed by Clebsch-Gordan spaces (see (1.3.12) below) for representations of quantum groups and affine Lie algebras. In a future publication we plan to show that these operads are Koszul. In short, Koszul algebras should be replaced by Koszul operads whenever the category under consideration has a tensor-type (e.g., fusion) structure.

(0.9) We are very much indebted to Maxim Kontsevich, whose ideas stimulated most of our constructions. We are also very grateful to Ezra Getzler, who informed us about his work in progress with J. D. S. Jones [23] and was the first to suggest that the constructions we were working with were related to operads, the notion unknown to the first author at the time (June 1992). His remarks helped to clarify several important points. We much benefited from conversations with Sasha Beilinson, whose current joint work with the first author (see [9], [10]) is closely related to the subject. We would like to thank I. M. Gelfand, Y. I. Manin, J. P. May, V. V. Schechtman, and J. D. Stasheff for discussions of the results of this paper. Several people kindly responded to the call for comments to the preliminary version. In particular, we are indebted to Ph. Hanlon, A. A. Voronov, and D. Wright for valuable correspondence. Special thanks are due to J. L. Loday and J. D. Stasheff for pointing out numerous inaccuracies in the text.

1. Operads in algebra and geometry

1.1. Preliminaries on trees

(1.1.1) By a *tree* we mean in this paper a nonempty connected oriented graph T without loops (oriented or not) with the following property: there is at least one incoming edge and exactly one outgoing edge at each vertex of T; see Figure 1a.

Trees are viewed as abstract graphs (1-dimensional topological spaces), a plane picture being irrelevant. We allow some edges of a tree to be bounded by a vertex at one end only. Such edges will be called *external.* All other edges (those bounded by vertices at both ends) will be called *internal.* Any tree has a unique

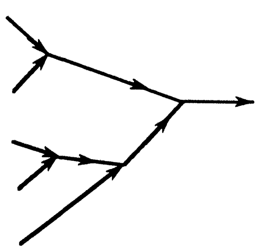

FIGURE 1a

FIGURE 1b

outgoing external edge, called the *output* or the *root* of the tree, and several incoming external edges, called *inputs* or *leaves* of the tree. Similarly, the edges going in and out of a vertex v of a tree will be referred to as inputs and outputs at v. A tree with possibly several inputs and a single vertex is called a *star*. There is also a tree (see Figure 1b) with a single input and without vertices called the *degenerate* tree.

We use the notion $\mathrm{In}(T)$ for the set of input edges of a tree T; for any vertex $v \in T$ we denote by $\mathrm{In}(v)$ the set of input edges at v. Similarly, we denote by $\mathrm{Out}(T)$ the unique output edge (root) of T and for every vertex $v \in T$ we denote by $\mathrm{Out}(v)$ the output edge at v.

Let I be a finite set. A tree T equipped with a bijection between I and the set $\mathrm{In}(T)$ will be referred to as an *I-labelled* tree or an I-tree for short. Two I-trees T, T' are called *isomorphic* if there exists an isomorphism of trees $T \to T'$ preserving orientations and the labellings of the inputs.

We denote by $[n]$ the finite set $\{1, 2, \ldots, n\}$ and call $[n]$-trees simply n-trees.

(1.1.2) *Composition.* Let I_1 be a set and let I_2 be another set with a marked element $i \in I_2$. Define the composition of I_1 and I_2 along i as $I = I_1 \circ_i I_2 = I_1 \cup (I_2 \backslash \{i\})$. Given an I_1-tree T_1 and an I_2-tree T_2, let $T = T_1 \circ_i T_2$ be the I-tree obtained by identifying the output of T_1 with the ith input of T_2 as depicted in Figure 2.

The tree T is called the *composition* of T_1 and T_2 (along i). Note that any tree can be obtained as an iterated composition of stars.

Let T be a tree and $v \xrightarrow{e} w$ an internal edge of T. Then we can form a new tree T/e by contracting e into a point. This new point is a vertex of T/e denoted by $\langle e \rangle$. Clearly, we have

$$\mathrm{In}(\langle e \rangle) = \mathrm{In}(w) \circ_e \mathrm{In}(v). \tag{1.1.3}$$

If T is I-labelled, then so is T/e. If T, T' are two I-labelled trees, then we write $T \xrightarrow{e} T'$ if the tree T' is isomorphic (as a labelled tree) to T/e.

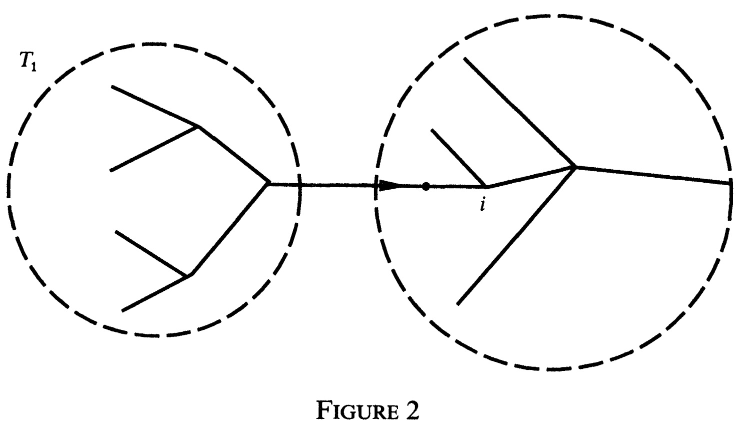

FIGURE 2

Write $T \geqslant T'$ if there is a sequence of edge contractions $T \to T_1 \to \cdots \to T_k \to T'$. Thus, $\geqslant$ is a partial order on the set of all I-trees. An I-star is the unique minimal element with respect to that order.

A tree T (with at least one vertex) is called a *binary tree* if there are exactly 2 inputs at each vertex of T. Binary trees are the maximal elements with respect to the partial order $\leqslant$. It can be shown that the number of nonisomorphic binary n-trees is equal to $(2n-3)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)$. A simple proof of this fact using generating functions will be given in Chapter 3.

(1.1.4) *The category of trees.* Let T, T' be trees (viewed as 1-dimensional topological spaces). By a *morphism* from T to T' we understand a continuous surjective map $f: T \to T'$ with the following properties:

(i) f takes each vertex to a vertex and each edge into an edge or a vertex.

(ii) f is monotone, i.e., preserves the orientation.

(iii) The inverse image of any point of T' under f is a connected subtree in T.

Thus any morphism is a composition of an isomorphism and several edge contractions. In this way we get a category which we denote Trees.

(1.1.5) Let Σ_n denote the symmetric group of order n. For any two sets I, J of the same cardinality we denote by $\mathrm{Iso}(I, J)$ the set of all bijections $I \to J$. We write Σ_I for $\mathrm{Iso}(I, I)$, so that $\Sigma_n = \mathrm{Iso}([n], [n])$. Clearly, $\mathrm{Iso}(I, J)$ is a principal homogeneous left Σ_I-set and a principal homogeneous right Σ_J-set.

Let W be a vector space (over some field k) with an action of Σ_n. There is a canonical way to construct (out of W) a functor $I \mapsto W(I)$ from the category of n-element sets and bijections to the category of k vector spaces. Namely, put

$$W(I) = \left(\bigoplus_{f \in \mathrm{Iso}([n], I)} W \right)_{\Sigma_n}, \tag{1.1.6}$$

the coinvariants with respect to the simultaneous action of Σ_n on $\mathrm{Iso}([n], I)$ and W. The original space W is recovered as the value of this functor on the set $[n]$. Similarly, if W is a set, or topological space with Σ_n-action, we can construct a functor $I \mapsto W(I)$ from the category of n-element sets and their bijections to the category of sets, or topological spaces as above, by the following analog of (1.1.6):

$$W(I) = \mathrm{Iso}([n], I) \times_{\Sigma_n} W.$$

1.2. k-linear operads

(1.2.1) Let k be a field of characteristic 0. A k-linear operad $\mathscr{P}$ is a collection $\{\mathscr{P}(n), n \geqslant 1\}$ of k-vector spaces equipped with the following set of data:

(i) An action of the symmetric group Σ_n on $\mathscr{P}(n)$ for each $n \geqslant 1$.

(ii) Linear maps (called compositions)

$$\gamma_{m_1, \ldots, m_l}: \mathscr{P}(l) \otimes \mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_l) \to \mathscr{P}(m_1 + \cdots + m_l)$$

for all $m_1, \ldots, m_l \geqslant 1$. We write $\mu(v_1, \ldots, v_l)$ instead of $\gamma_{m_1, \ldots, m_l}(\mu \otimes v_1 \otimes \cdots \otimes v_l)$.

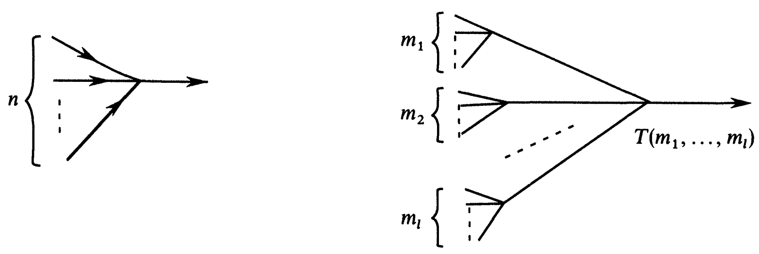

FIGURE 3

(iii) An element $1 \in \mathscr{P}(1)$, called the unit, such that $\mu(1, \ldots, 1) = \mu$ for any l and any $\mu \in \mathscr{P}(l)$.

It is required that these data satisfy the conditions (associativity and equivariance with respect to symmetric group actions) specified by May ([52], §1). These conditions are best expressed in terms of trees. Observe first that the datum (i) allows us to assign to any finite set I a vector space $\mathscr{P}(I)$ as in (1.1.5). Next, we associate to any tree T the vector space

$$\tilde{\mathscr{P}}(T) = \bigotimes_{v \in T} \mathscr{P}(\mathrm{In}(v)). \tag{1.2.2}$$

To the degenerate tree without vertices, we associate, by definition, the field k.

Note in particular that to the trees $T(n)$ and $T(m_1, \ldots, m_l)$ depicted in Figure 3 we associate the spaces $\mathscr{P}(n)$ and $\mathscr{P}(l) \otimes \mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_l)$.

Thus the datum (ii) gives a map

$$\tilde{\mathscr{P}}(T(m_1, \ldots, m_l)) \to \tilde{P}(T(m_1 + \cdots + m_l)) = \mathscr{P}(m_1 + \cdots + m_l). \tag{1.2.3}$$

For any n-tree T there exists a sequence of trees

$$T = T_0 \to T_1 \to \cdots \to T_r = T(n) \tag{1.2.4}$$

where each T_i is obtained from T_{i-1} by replacing a fragment of type $T(m_1, \ldots, m_l)$ by $T(m_1 + \cdots + m_l)$. So maps (1.2.3) give rise to a sequence of maps

$$\tilde{\mathscr{P}}(T) = \tilde{\mathscr{P}}(T_0) \to \tilde{\mathscr{P}}(T_1) \to \cdots \to \tilde{\mathscr{P}}(T_r) = \mathscr{P}(n).$$

The associativity condition is equivalent to the requirement that the composite map $\tilde{\mathscr{P}}(T) \to P(n)$ does not depend on the choice of a sequence (1.2.4).

(1.2.5) Observe that the tree $T(m_1 + \cdots + m_l)$ is obtained from $T(m_1, \ldots, m_l)$ by contracting all the l internal edges. The existence of unit $1 \in \mathscr{P}(1)$ makes it possible to decompose the map (1.2.3), corresponding to this contraction, into more elementary ones, each consisting of contracting a single edge.

Namely, let T be a tree, $v \xrightarrow{e} w$ an internal edge of T, and T/e the tree obtained by contracting e. Let $\langle e \rangle$ be the vertex of T/e obtained from the contracted edge. We define a map

$$\mathscr{P}(\mathrm{In}(v)) \otimes \mathscr{P}(\mathrm{In}(w)) \to \mathscr{P}(\mathrm{In}(v) \circ_e \mathrm{In}(w)) \overset{(1.1.3)}{=} \mathscr{P}(\mathrm{In}(\langle e \rangle))$$

by the formula $\mu \otimes \nu \mapsto \mu(1, \ldots, \nu, \ldots, 1)$ where ν is placed at the entry corresponding to the edge e. By tensoring this map with the identity elsewhere on T, we obtain a map

$$\tilde{\gamma}_{T,e}\colon \tilde{\mathscr{P}}(T) \to \tilde{\mathscr{P}}(T/e). \tag{1.2.6}$$

More generally, if I is a finite set and T, T' are two I-trees such that $T \geqslant T'$, then by composing maps of the type $\tilde{\gamma}_{T,e}$ we get a map

$$\tilde{\gamma}_{T,T'}\colon \tilde{\mathscr{P}}(T) \to \tilde{\mathscr{P}}(T') \tag{1.2.7}$$

which is well defined, due to the associativity condition.

Thus a k-linear operad $\mathscr{P}$ gives rise to a functor

$$\tilde{\mathscr{P}}\colon \mathrm{Trees} \to \mathrm{Vect}, \qquad T \mapsto \tilde{\mathscr{P}}(T)$$

equipped with the following additional structures:

(i) For any trees T_1 and T_2 and any $j \in \mathrm{In}(T_2)$, one has a functorial isomorphism

$$\Phi_{T_1,T_2}\colon \tilde{\mathscr{P}}(T_1) \otimes \tilde{\mathscr{P}}(T_2) \to \tilde{\mathscr{P}}(T_1 \circ_j T_2). \tag{1.2.8}$$

(ii) The isomorphisms in (i) satisfy the associativity constraint saying that for any trees T_1, T_2, T_3 and any $i \in \mathrm{In}(T_2)$, $j \in \mathrm{In}(T_3)$ the following diagram commutes:

$$\begin{array}{ccc}
\tilde{\mathscr{P}}(T_1) \otimes \tilde{\mathscr{P}}(T_2) \otimes \tilde{\mathscr{P}}(T_3) & \xrightarrow{\mathrm{Id} \otimes \Phi_{T_2,T_3}} & \tilde{\mathscr{P}}(T_1) \otimes \tilde{\mathscr{P}}(T_2 \circ_j T_3) \\
\Big\downarrow{\scriptstyle \Phi_{T_1,T_2} \otimes \mathrm{Id}} & & \Big\downarrow{\scriptstyle \Phi_{T_1,T_2 \circ T_3}} \\
\tilde{\mathscr{P}}(T_1 \circ_i T_2) \otimes \tilde{\mathscr{P}}(T_3) & \xrightarrow{\Phi_{T_1 \circ T_2, T_3}} & \tilde{P}(T_1 \circ_i T_2 \circ_j T_3).
\end{array}$$

(1.2.9) Let V be a k-vector space. Its *operad of endomorphisms*, $\mathscr{E}_V$, consists of vector spaces

$$\mathscr{E}_V(n) = \mathrm{Hom}(V^{\otimes n}, V).$$

with compositions and the Σ_n-action on $\mathscr{E}_V(n)$ being defined in an obvious way. We have $\mathscr{E}_V(1) = \mathrm{End}(V)$.

(1.2.10) Observe that for any k-linear operad $\mathscr{P}$, the space $K = \mathscr{P}(1)$ has a natural structure of an associative k-algebra with unit. Conversely, if K is a k-algebra, then the collection $\{\mathscr{P}(1) = K, \mathscr{P}(n) = \{0\}, n > 1\}$ forms an operad, in which the unique nontrivial map (1.2.1) (ii) is $\mathscr{P}(1) \otimes \mathscr{P}(1) \to \mathscr{P}(1)$, the multiplication in $\mathscr{P}(1) = K$. Furthermore, for an arbitrary k-linear operad $\mathscr{P}$, the space $\mathscr{P}(n)$ has several $\mathscr{P}(1)$-module structures. These structures are summarized in the following definition.

(1.2.11) *Definition.* Let K be an associative k-algebra with unit. A K-collection is a family $E = \{E(n), n \geqslant 2\}$ of k-vector spaces equipped with the following structures:

(i) a left Σ_n-action on $E(n)$, for each $n \geqslant 2$,

(ii) a structure of a left K-module and a right $K^{\otimes n}$-module on $E(n)$, $n \geqslant 2$.

These structures are required to satisfy the following compatibility conditions:

(a) For any $s \in \Sigma_n$ and any $\lambda_1, \ldots, \lambda_n \in K$, $a \in E(n)$, we have

$$s(a \cdot (\lambda_1 \otimes \cdots \otimes \lambda_n)) = s(a) \cdot (\lambda_{s(1)} \otimes \cdots \otimes \lambda_{s(n)}).$$

(b) For any $\lambda \in K$ and $a \in E(n)$, we have $s(\lambda \otimes a) = \lambda \otimes s(a)$.

If $\mathscr{P}$ is a k-linear operad and $K = \mathscr{P}(1)$, then $\{\mathscr{P}(n), n \geqslant 2\}$ is, clearly, a K-collection.

(1.2.12) It will be convenient for future purposes to give a reduced, in a certain sense, version of the tree formalism above. We call a tree T *reduced* if there are at least two inputs at each vertex $v \in T$. (The degenerate tree without vertices is also assumed to be reduced.)

Let K be an associative k-algebra and E a K-collection. As in (1.1.5) we extend E to a functor on finite sets and bijections. To each reduced tree T we associate a vector space $E(T)$ as follows. For the degenerate tree $\to$ we set $E(\to) = K$, and for a tree T with a nonempty set of vertices we set

$$E(T) = \bigotimes_{v \in T}{}_K\, E(\mathrm{In}(v)). \tag{1.2.13}$$

This tensor product is taken over the ring K by using the $(K, K^{\otimes \mathrm{In}(v)})$-bimodule structure on each $E(\mathrm{In}(v))$; see Figure 4.

Explicitly, this means that $E(T)$ is the quotient of the k-tensor product $\bigotimes_v E(\mathrm{In}(v))$ by associativity conditions described as follows. Suppose that $e \mapsto \lambda_e$ is any K-valued function on the set of all edges such that $\lambda_e = 1$ for all external edges. Then for any collection of $a_v \in E(\mathrm{In}(v))$ we impose the relation

$$\bigotimes_{v \in T} a_v \cdot \left(\bigotimes_{e \in \mathrm{In}(v)} \lambda_e \right) = \bigotimes_{v \in T} \lambda_{\mathrm{Out}(v)} \cdot a_v .$$

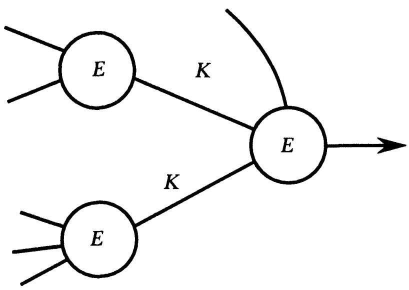

FIGURE 4

(1.2.14) Now let $\mathscr{P} = \{\mathscr{P}(n)\}$ be a k-linear operad and $K = \mathscr{P}(1)$. Since $\{\mathscr{P}(n), n \geqslant 2\}$ form a K-collection, we can assign to any reduced tree T a vector space $\mathscr{P}(T)$ by formula (1.2.13). This assignment has the following two fundamental structures:

(i) A linear map

$$\gamma_{T,T'}\colon \mathscr{P}(T) \to \mathscr{P}(T') \tag{1.2.15}$$

is defined whenever $T \geqslant T'$.

(ii) An isomorphism

$$\mathscr{P}(T_1 \circ_i T_2) \to \mathscr{P}(T_1) \otimes_K \mathscr{P}(T_2)$$

is given for any $i \in \mathrm{In}(T_2)$.

The maps in (i) and (ii) are induced by (1.2.7) and (1.2.8), respectively.

Observe further that the assignment $T \to \mathscr{P}(T)$ extends to a functor on the (sub-) category of reduced trees; see (1.1.4).

(1.2.16) *Convention.* In the rest of this paper we shall consider only reduced trees, unless specified otherwise.

1.3. Algebraic operads

(1.3.1) Let $\mathscr{P}$, $\mathscr{Q}$ be two k-linear operads. A *morphism of operads* $f\colon \mathscr{P} \to \mathscr{Q}$ is a collection of linear maps which are equivariant with respect to Σ_n-action, commute with compositions $\gamma_{m_1,\ldots,m_l}$ in $\mathscr{P}$ and $\mathscr{Q}$, and take the unit of $\mathscr{P}$ to the unit of $\mathscr{Q}$; see [52].

(1.3.2) *Definition.* Let $\mathscr{P}$ be a k-linear operad. A $\mathscr{P}$-algebra is a k-vector space A equipped with a morphism of operads $f\colon \mathscr{P} \to \mathscr{E}_A$ where $\mathscr{E}_A$ is the operad of endomorphisms of A; see (1.2.9).

Clearly, giving a structure of a $\mathscr{P}$-algebra on A is the same as giving a collection of linear maps

$$f_n\colon \mathscr{P}(n) \otimes A^{\otimes n} \to A \tag{1.3.3}$$

satisfying natural associativity, equivariance and unit conditions. We write $\mu(a_1, \ldots, a_n)$ for $f_n(\mu \otimes (a_1 \otimes \cdots \otimes a_n))$, $\mu \in \mathscr{P}(n)$, $a_i \in A$.

(1.3.4) *Free algebras.* Let $\mathscr{P}$ be a k-linear operad and $K = \mathscr{P}(1)$. As noted in (1.2.10), K is an associative k-algebra and every $\mathscr{P}(n)$ is a left K-module and a right $K^{\otimes n}$-module.

Let V be any left K-module. Form the graded vector space

$$F_{\mathscr{P}}(V) = \bigoplus_{n \geqslant 1} \left(\mathscr{P}(n) \otimes_{K^{\otimes n}} V^{\otimes n} \right)_{\Sigma_n} \tag{1.3.5}$$

where $V^{\otimes n}$ is the nth tensor power of V over k and the group Σ_n is acting diagonally.

(1.3.6) Lemma. *Compositions in $\mathscr{P}$ induce natural maps $\mathscr{P}(n) \otimes F_{\mathscr{P}}(V)^{\otimes n} \to F_{\mathscr{P}}(V)$. These maps make $F_{\mathscr{P}}(V)$ into a $\mathscr{P}$*-algebra.

We call $F_{\mathscr{P}}(V)$ the *free $\mathscr{P}$-algebra* generated by V. Note that $F_{\mathscr{P}}(V)$ has a natural grading given by the decomposition (1.3.5). Here are some other examples of operads and algebras.

(1.3.7) *Associative algebras.* For any n let $\mathscr{A}s(x_1, \ldots, x_n)$ denote the free associative k-algebra on generators $x_1, \ldots, x_n$ (i.e., the algebra of noncommutative polynomials). Let $\mathscr{A}s(n) \subset \mathscr{A}s(x_1, \ldots, x_n)$ be the subspace spanned by monomials containing each x_i exactly once. There are exactly $n!$ such monomials, namely $x_{s(1)} \cdot \cdots x_{s(n)}$, $s \in \Sigma_n$. Clearly, $\mathscr{A}s(n)$ has a natural Σ_n-action and as a Σ_n-module it is isomorphic to the regular representation of Σ_n. The collection $\mathscr{A}s = \{\mathscr{A}s(n)\}$ forms an operad called the *associative operad.* The composition maps (see (1.2.1))

$$\mathscr{A}s(l) \otimes \mathscr{A}s(m_1) \otimes \cdots \otimes \mathscr{A}s(m_l) \to \mathscr{A}s(m_1 + \cdots + m_l)$$

are given by substituting the monomials $\phi_1 \in \mathscr{A}s(m_1), \ldots, \phi_l \in \mathscr{A}s(m_l)$ in place of generators $x_1, \ldots, x_l$ into a monomial $\psi \in \mathscr{A}s(l)$.

We leave to the reader to verify that an $\mathscr{A}s$-algebra is nothing but an associative algebra in the usual sense (possibly without unit).

(1.3.8) *Commutative (associative) algebras.* For any n, let $\mathscr{C}om(x_1, \ldots, x_n) = k[x_1, \ldots, x_n]$ be the free commutative algebra on generators $x_1, \ldots, x_n$ (i.e., the algebra of polynomials). There exists precisely one monomial containing each x_i exactly once, namely $x_1 \cdot \cdots \cdot x_n$. Let $\mathscr{C}om(n) \subset \mathscr{C}om(x_1, \ldots, x_n)$ be the 1-dimensional subspace spanned by this monomial. The collection $\mathscr{C}om = \{\mathscr{C}om(n)\}$ forms an operad with respect to the trivial actions of Σ_n on $\mathscr{C}om(n)$ and compositions defined similarly to (1.3.7). We call $\mathscr{C}om$ the *commutative operad.*

Again, it is straightforward to see that a $\mathscr{C}om$-algebra is nothing but a commutative associative algebra in the usual sense (possibly without unit).

(1.3.9) *Lie algebras.* For any n, let $\mathscr{L}ie(x_1, \ldots, x_n)$ be the free Lie algebra over k generated by $x_1, \ldots, x_n$. Let $\mathscr{L}ie(n) \subset \mathscr{L}ie(x_1, \ldots, x_n)$ be the subspace spanned by all bracket monomials containing each x_i exactly once. Note that such monomials are not all linearly independent due to the Jacobi identity. The subspace $\mathscr{L}ie(n)$ is invariant under the action of Σ_n on $\mathscr{L}ie(x_1, \ldots, x_n)$ by permutations of x_i. It is known that

$$\dim \mathscr{L}ie(n) = (n-1)!. \tag{1.3.10}$$

Moreover, if k is algebraically closed, then A. Klyachko [35] constructed an isomorphism of Σ_n-modules

$$\mathscr{L}ie(n) \cong \mathrm{Ind}_{\mathbf{Z}/n}^{\Sigma_n}(\chi) \tag{1.3.11}$$

where χ is the 1-dimensional representation of the cyclic group $\mathbf{Z}/n$ sending the generator into a primitive nth root of 1. (The induced module does not depend on the choice of such a χ.)

The collection $\mathscr{L}ie = \{\mathscr{L}ie(n)\}$ forms an operad with respect to the composition operations defined similarly to the ones described in (1.3.4). An algebra over the operad $\mathscr{L}ie$ is the same as a Lie algebra in the usual sense.

It should be clear to the reader at this point how to construct operads governing other types of algebras encountered in practice: Poisson algebras, Jordan algebras etc.

(1.3.12) A collection of finite-dimensional k-vector spaces

$$\{E^j(a_1, \ldots, a_r), a_1, \ldots, a_r \in \mathbf{Z}_+, \textstyle\sum a_i \geqslant 1, j = 1, \ldots, r\}$$

is called an r-fold collection if the following holds:

(i) For any $a_1, \ldots, a_r \in \mathbf{Z}_+$ the space $E^j(a_1, \ldots, a_r)$ is equipped with an action of the group $\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}$.

(ii)

$$E^j(0, \ldots, 0, 1, 0, \ldots, 0)\ (1 \text{ on the } j\text{th place}) = \begin{cases} 0 & \text{if } i \neq j \\ k & \text{if } i = j \end{cases}.$$

Now let K be a semisimple k-algebra, and $\{M_1, \ldots, M_r\}$ a complete collection of (the isomorphism classes of) simple left K-modules. Given a K-collection $E = \{E(n),\ n \geqslant 2\}$ (1.2.11), we regard $E(n)$ as a left module over the algebra $\Pi_n = K^{\otimes n} \otimes K^{op}$. For any $a_1, \ldots, a_r \in \mathbf{Z}_+$ such that $\sum a_i = n$, we define

$$E^i(a_1, \ldots, a_r) = \mathrm{Hom}_{\Pi_n}(M_1^{\otimes a_1} \otimes \cdots \otimes M_r^{\otimes a_r} \otimes M_i^*, E(n)). \tag{1.3.13}$$

The k-vector spaces $E^j(a_1, \ldots, a_r)$, thus defined, clearly form an r-fold collection. It will be called the r-fold collection *associated* to be K-collection E.

(1.3.14) *Clebsch-Gordan spaces and operads.* Let $\mathscr{A}$ be a semisimple abelian k-linear category equipped with a symmetric monoidal structure $\otimes$ (for example, the tensor category of finite-dimensional representations of a finite, or an algebraic, group).

Fix any integer $r \geqslant 1$ and any set $X_1, \ldots, X_r$ of pairwise nonisomorphic simple objects of $\mathscr{A}$. Let $X = \bigoplus_{i=1}^r X_i$. Associated to X is the operad $\mathscr{P}_X$ defined by collection of vector spaces

$$\mathscr{P}_X(n) = \operatorname{Hom}_{\mathscr{A}}(X^{\otimes n}, X). \tag{1.3.15}$$

Observe that $\mathscr{P}_X(1) = \bigoplus \operatorname{End}(X_i)$ is a semisimple k-algebra. We put $K = \mathscr{P}_X(1)$ and view the collection (1.3.15) as a K-collection. Clearly, the r-fold collection associated to that K-collection via (1.3.13) is formed by the Clebsch-Gordan spaces

$$\mathscr{P}^i(a_1, \ldots, a_r) = \operatorname{Hom}_{\mathscr{A}}(X_1^{\otimes a_1} \otimes \cdots \otimes X_r^{\otimes a_r}, X_i). \tag{1.3.16}$$

(1.3.17) *Operads in monoidal categories.* Let Vect be the category of k-vector spaces. Note that the concept of a k-linear operad appeals only to the category Vect with its symmetric monoidal structure given by the tensor product. Clearly, one can define operads in any symmetric monoidal category, i.e., a category $\mathscr{A}$ (not necessarily additive or abelian) equipped with a bifunctor $\otimes: \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ and natural associativity and commutativity constraints for this functor [49].

All the preceding constructions (e.g., the notion of a $\mathscr{P}$-algebra) can be carried over to the setup of operads in any symmetric monoidal category $(\mathscr{A}, \otimes)$. Given such an operad, one defines, as in (1.1.5), for any finite set I, an object $\mathscr{P}(I) \in \mathscr{A}$, and for tree T an object $\tilde{\mathscr{P}}(T) \in \mathscr{A}$.

In this section we concentrate on algebraic examples of categories $\mathscr{A}$.

(1.3.18) *Two categories of graded vector spaces.* Let $g\,\mathrm{Vect}^+$ be the category whose objects are graded vector spaces $V^\bullet = \bigoplus_{i \in \mathbf{Z}} V^i$ and morphisms are linear maps preserving the grading. For $v \in V^i$ we write $\deg(v) = i$. We introduce on $g\,\mathrm{Vect}^+$ a symmetric monoidal structure defining with the tensor product

$$(V^\bullet \otimes W^\bullet)^m = \bigoplus_{i+j=m} V^i \otimes W^j. \tag{1.3.19}$$

The associativity morphism $V^\bullet \otimes (W^\bullet \otimes X^\bullet) \to (V^\bullet \otimes W^\bullet) \otimes X^\bullet$ takes $v \otimes (w \otimes x) \mapsto (v \otimes w) \otimes x$. The commutativity isomorphism $V^\bullet \otimes W^\bullet \to W^\bullet \otimes V^\bullet$ takes $v \otimes w \mapsto w \otimes v$.

The symmetric monoidal category $g\,\mathrm{Vect}^-$ has the same objects, morphisms, tensor product, and associativity isomorphism as $g\,\mathrm{Vect}^+$, but the commutativity isomorphism is changed to

$$v \otimes w \mapsto (-1)^{\deg(v) \cdot \deg(w)} w \otimes v. \tag{1.3.20}$$

Any k-linear operad $\mathscr{P}$ can be regarded as an operad in either of the categories $g\,\mathrm{Vect}^{\pm}$ and so we can speak about $\mathscr{P}$-algebras in these categories. For example, if $\mathscr{P} = \mathscr{C}om$ is the commutative operad, then a $\mathscr{C}om$-algebra in $g\,\mathrm{Vect}^{+}$ is a commutative associative algebra equipped with a grading compatible with the algebra structure. A $\mathscr{C}om$-algebra in $g\,\mathrm{Vect}^{-}$ is an associative graded algebra which is graded (or super-) commutative.

(1.3.21) *The determinant operad* Λ. In the operad $\mathscr{C}om$, each space $\mathscr{C}om(n)$ is 1-dimensional and is equipped with the trivial action of Σ_n. We now introduce an operad Λ in the category $g\,\mathrm{Vect}^{-}$ which is an "odd" analog of $\mathscr{C}om$.

We define $\Lambda(n)$ to be the 1-dimensional vector space $\bigwedge^n(k^n)$ (the sign representation of Σ_n) placed in degree $(1-n)$. In order to describe compositions in Λ we first describe what is to be a Λ-algebra in $g\,\mathrm{Vect}^{-}$.

We consider graded vector spaces $A = \bigoplus A_i$ with one binary operation $(a, b) \mapsto ab$ satisfying the following identities:

(i) $\deg(ab) = \deg(a) + \deg(b) - 1$,

(ii) $ab = (-1)^{\deg(a)+\deg(b)+1} ba$,

(iii) $a(bc) = (-1)^{\deg(a)+1}(ab)c$.

The operation ab corresponds to the generator $\mu \in \Lambda(2)$. The condition (i) means that $\deg(\mu) = -1$ and the condition (ii) means that Σ_2 acts on μ by the sign representation.

We fix integers $d_1, \ldots, d_n$ and let $\Lambda(x_1, \ldots, x_n)$ be the free algebra with the above identities generated by symbols x_i, of degree d_i.

(1.3.22) LEMMA. *The subspace $E_{d_1,\ldots,d_n}$ in $\Lambda(x_1, \ldots, x_n)$, spanned by nonassociative monomials containing each x_i exactly once, has dimension* 1. *Moreover, any two such monomials are proportional with coefficients* ± 1.

The meaning of this lemma is that the identities (i)–(iii) above are consistent, i.e., do not lead to the contradiction $1 = 0$. In other words, any two ways of comparing the signs of the monomials lead to the same result.

Proof of the lemma. As in Mac Lane's axiomatics of symmetric monoidal categories [49], it is enough to check the three elementary ambiguities, i.e., the two ways of comparing $a(bc)$ and $(bc)a$, of $(ab)c$ and $c(ab)$, and of $a(b(cd))$ and $((ab)c)d$. We leave this to the reader.

Another way to see Lemma 1.3.22 can be based on the following remark (made to us by E. Getzler). If A is a graded commutative algebra (i.e., a $\mathscr{C}om$-algebra in the category $g\mathrm{Vect}^{-}$), then the same algebra with the grading shifted by one and new multiplication $a * b = (-1)^{\deg(a)} ab$ will satisfy the identities (i)–(iii) above.

(1.3.23) To finish the construction of the operad Λ, we take, in the situation of Lemma 1.3.22, n generators $x_1, \ldots, x_n$ of degree 0. We denote by $\Lambda'(n) = E_{0,\ldots,0}$

the 1-dimensional subspace from this lemma. The space $\Lambda'(n)$ is Σ_n-invariant, and the action of Σ_n on $\Lambda'(n)$ is given by the sign representation. To see the last assertion it is enough to show that any transposition (ij) acts by (-1). But we can take the basis vector of $\Lambda'(n)$ given by any product $\cdots(x_i x_j)\cdots$ in which x_i and x_j are bracketed together. By (ii) we have $x_i x_j = -x_j x_i$, whence the assertion.

So we can identify $\Lambda'(n)$ with $\Lambda(n) = \bigwedge^n(k^n)$, and the substitution of monomials in place of generators, as in (1.3.7), defines the operad structure on Λ.

1.4. Geometric operads

(1.4.1) The category of topological spaces has an obvious symmetric monoidal structure given by the Cartesian product. Operads in this category will be called *topological operads*. (This was the original context of [52].)

Given a topological operad $\mathscr{P}$, the total homology spaces $H_\bullet(\mathscr{P}(n), k)$ form an operad in the category $g\mathrm{Vect}^-$ (by the Künneth formula). Furthermore, for any $q \geqslant 0$, the subspaces $H_{q(n-1)}(\mathscr{P}(n), k)$ form a suboperad.

(1.4.2) An important example of a topological operad is given by the little m-cubes operad $\mathscr{C}_m$ of Boardman-Vogt-May [11], [52]. By definition, $\mathscr{C}_m(n)$ is the space of numbered n-tuples of nonintersecting m-dimensional cubes inside the standard cube I^m, with faces parallel to those of I^m. The operad $\mathscr{C}_2$ of little squares has an algebro-geometric analog which will be particularly interesting for us and which we proceed to describe.

(1.4.3) *The moduli space $\mathscr{M}(n)$.* Let $M_{0,n+1}$ be the moduli space of $(n+1)$-tuples $(x_0, \dots, x_n)$ of distinct points on the complex projective line $\mathbf{C}P^1$ modulo projective automorphisms. Choose a point $\infty \in \mathbf{C}P^1$ so that $\mathbf{C}P^1 = \mathbf{C} \cup \{\infty\}$. Letting $x_0 = \infty$, one gets an isomorphism of $M_{0,n+1}$ with the moduli space of n-tuples of distinct points on $\mathbf{C}$ modulo affine automorphisms.

The space $M_{0,n+1}$ has a canonical compactification $\mathscr{M}(n) \supset M_{0,n+1}$ introduced by Grothendieck and Knudsen [14], [36]. The space $\mathscr{M}(n)$ is the moduli space of stable $(n+1)$-pointed curves of genus 0, i.e., systems $(C, x_0, \dots, x_n)$ where C is a possibly reducible curve (with at most nodal singularities) and $x_0, \dots, x_n \in C$ are distinct smooth points such that:

(i) Each component of C is isomorphic to $\mathbf{C}P^1$.

(ii) The graph of intersections of components of C (i.e., the graph whose vertices correspond to the components and edges to the intersection points of the components) is a tree.

(iii) Each component of C has at least 3 special points where a special point means either one of the x_i or a singular point of C.

The space $M_{0,n+1}$ forms an open dense part of $\mathscr{M}(n)$ consisting of $(C, x_0, \dots, x_n)$ such that C is isomorphic to $\mathbf{C}P^1$. The space $\mathscr{M}(n)$ is a smooth complex projective variety of dimension $n-2$. It has the following elementary construction; see [9], [21], [32].

Let $\mathrm{Aff} = \{x \mapsto ax + b\}$ be the group of affine transformations of $\mathbf{C}$. The group Aff acts diagonally on $\mathbf{C}^n$ preserving the open part $\mathbf{C}^n_*$ formed by points with pairwise distinct coordinates. As we noted before, we have an isomorphism $M_{0,n+1} \cong \mathbf{C}^n_*/\mathrm{Aff}$. Denote by $\Delta \subset \mathbf{C}^n$ the principal diagonal, i.e., the space of points $(x, x, \ldots, x)$. Then we have an embedding

$$M_{0,n+1} = \mathbf{C}^n_*/\mathrm{Aff} \subset (\mathbf{C}^n - \Delta)/\mathrm{Aff} = \mathbf{C}P^{n-2}.$$

The coordinate axes in $\mathbf{C}^n$ give n distinguished points $p_1, \ldots, p_n \in \mathbf{C}P^{n-2}$. Let us blow up all the points p_i, then blow up the proper transforms (=closures of the preimages of some open parts) of the lines $\langle p_i, p_j \rangle$ then blow up the proper transforms of the planes $\langle p_i, p_j, p_k \rangle$, and so on. It can be shown (see the references above) that the resulting space is isomorphic to $\mathcal{M}(n)$.

(1.4.4) *The configuration operad $\mathcal{M}$.* The family of spaces $\mathcal{M} = \{\mathcal{M}(n), n \geqslant 1\}$ forms a topological operad. The symmetric group action on $\mathcal{M}(n)$ is given by

$$(C, x_0, \ldots, x_n) \mapsto (C, x_0, x_{s(1)}, \ldots, x_{s(n)}), \qquad s \in \Sigma_n.$$

The composition map

$$\mathcal{M}(l) \times \mathcal{M}(m_1) \times \cdots \times \mathcal{M}(m_l) \to M(m_1 + \cdots + m_l)$$

is defined by

$$(C, y_0, \ldots, y_l), (C^{(1)}, x_0^{(1)}, \ldots, x_{m_1}^{(1)}), \ldots, (C^{(l)}, x_0^{(l)}, \ldots, x_{m_1}^{(l)})$$

$$\mapsto (C', y_0, x(1)_1, \ldots, x_{m_1}^{(1)}, \ldots, x_1^{(l)}, \ldots, x_{m_l}^{(l)})$$

where C' is the curve obtained from the disjoint union $C \sqcup C^{(1)} \sqcup \ldots \sqcup C^{(l)}$ by identifying $x_0^{(i)}$ with y_i, $i = 1, \ldots, l$ (see Figure 5).

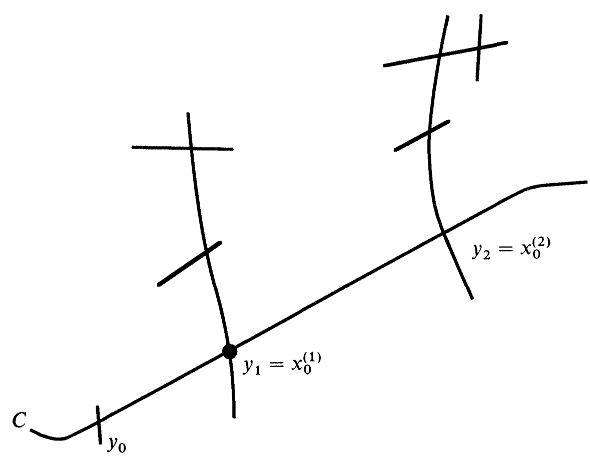

FIGURE 5

We call $\mathcal{M}$ the *configuration operad*, since $\mathcal{M}(n)$ can be regarded as (compactified) configuration spaces of points on $\mathbf{C}P^1$.

(1.4.5) *The stratification of the space* $\mathcal{M}(n)$. Given a point $(C, x_0, \ldots, x_n) \in \mathcal{M}(n)$, we associate to it an n-tree $T = T(C, x_0, \ldots, x_n)$ as follows. The vertices of T correspond to the irreducible components of C. The vertices corresponding to two components C_1, C_2 are joined by an (internal) edge if $C_1 \cap C_2 \neq \emptyset$. An external edge is assigned to each of the marked points $x_0, \ldots, x_n$. The input edge labelled by $i \neq 0$ is attached to the vertex correponding to the component containing x_i. The output edge is attached to the vertex corresponding to the component containing x_0. The property (iii) of stable curves ensures that $T(C, x_0, \ldots, x_n)$ is a reduced tree.

For any reduced n-tree T, let $\mathcal{M}(T) \subset \mathcal{M}(n)$ be the subset consisting of points $(C, x_0, \ldots, x_n)$ such that $T(C, x_0, \ldots, x_n) = T$. In this way we obtain an algebraic stratification $\mathcal{M}(n) = \bigcup \mathcal{M}(T)$. This stratification has the following properties (cf. [9]):

$$\text{codim}\, \mathcal{M}(T) = \#\ \text{internal edges of } T. \tag{1.4.6}$$

$$\mathcal{M}(T) \subset \overline{\mathcal{M}(T')} \quad \Leftrightarrow \quad T \geqslant T'. \tag{1.4.7}$$

In particular, 0-dimensional strata are labelled by binary trees. Codimension-1 strata correspond to trees with two vertices. Their closures are precisely the irreducible components of $\mathcal{M}(n) - M_{0,n+1}$, which is a normal crossing divisor. Moreover, the entire stratification above can be recovered by intersecting these components in all possible ways. In addition, we have the following result.

(1.4.8) PROPOSITION. *There are canonical direct product decompositions*

$$\mathcal{M}(T) = \prod_{v \in T} M_{0, |\text{In}(v)|+1},$$

$$\overline{\mathcal{M}(T)} = \prod_{v \in T} \mathcal{M}(\text{In}(v)).$$

In particular, the closure of each stratum is smooth.

Proof. The first equality means that a pointed curve $(C, x_0, \ldots, x_n) \in \mathcal{M}(T)$ is uniquely determined, up to isomorphism, by the projective equivalence classes of the configurations formed on each component by the marked points x_i and the double points which happen to lie on this component. This is obvious. The second equality follows from the first one once we note that $\overline{\mathcal{M}}(T) = \bigcup_{\Gamma \geqslant T} \mathcal{M}(\Gamma)$.

Let $T(m_1, \ldots, m_l)$ be the tree in Figure 3.

(1.4.9) COROLLARY. *The structure maps (1.2.1) (ii) of the operad* $\mathcal{M}$ *can be identified with the embedding of the stratum*

$$\mathcal{M}(l) \times \mathcal{M}(m_1) \times \cdots \times \mathcal{M}(m_l) = \mathcal{M}(T(m_1, \ldots, m_l)) \hookrightarrow \mathcal{M}(n).$$

1.5. Operads and sheaves

(1.5.1) Let $\mathscr{P}$ be a k-linear operad. The compositions in $\mathscr{P}$ make it possible to construct, for any n, a sheaf $\mathscr{F}_{\mathscr{P}}(n)$ on the moduli space $\mathscr{M}(n)$, as we now proceed to explain.

Let X be any CW-complex, and $S = \{X_\alpha\}$ a Whitney stratification [26] of X into connected strata X_α. We say that a sheaf $\mathscr{F}$ of k-vector spaces on X is *S-combinatorial* if the restriction of $\mathscr{F}$ to each stratum X_α is a constant sheaf. Thus "S-combinatorial" is more restrictive than "S-constructible" [26], [34], which means that $\mathscr{F}|_{X_\alpha}$ are only locally constant.

It is immediate to see that giving an S-combinatorial sheaf $\mathscr{F}$ is equivalent to giving the following linear algebra data:

(i) vector spaces $F_\alpha = H^0(X_\alpha, \mathscr{F})$, one for each stratum X_α;

(ii) *generalization maps* $g_{\alpha\beta}\colon F_\alpha \to F_\beta$ defined whenever $X_\alpha \subset \overline{X_\beta}$ and satisfying the transitivity condition:

$$g_{\alpha\gamma} = g_{\beta\gamma} \circ g_{\alpha\beta} \qquad \text{if } X_\alpha \subset \overline{X_\beta} \subset \overline{X_\gamma}.$$

(1.5.2) We now consider the moduli space $\mathscr{M}(n)$ defined in §1.4 together with the stratification $\mathscr{M}(n) = \bigcup_T \mathscr{M}(T)$ labelled by the set of n-trees. A sheaf on $\mathscr{M}(n)$ combinatorial with respect to this stratification will be referred to as a "combinatorial sheaf of $\mathscr{M}(n)$".

Let $\mathscr{P}$ be a k-linear operad. To any n-tree T we have associated in (1.2.13) a vector space $\mathscr{P}(T)$ and to any pair of n-trees $T' \leqslant T$ we have associated a linear map $\gamma_{T,T'}\colon \mathscr{P}(T) \to \mathscr{P}(T')$. Note that we have an inclusion $\mathscr{M}(T) \subset \overline{\mathscr{M}(T')}$ precisely when $T' \leqslant T$. Thus associating to a stratum $\mathscr{M}(T)$ the space $\mathscr{P}(T)$, and using the $\gamma_{T,T'}$ as generalization maps, we get a combinatorial sheaf on $\mathscr{M}(n)$ which we denote by $\mathscr{F}_{\mathscr{P}}(n)$.

The sheaves $\mathscr{F}_{\mathscr{P}}(n)$ for different n enjoy certain compatibility with the operad structure on $\{\mathscr{M}(n)\}$. Those compatibility properties are formalized in the next subsection. Before proceeding to do this, let us agree on the following shorthand notation. If X, Y are topological spaces, $\mathscr{F}$ is a sheaf on X, and $\mathscr{G}$ is a sheaf on Y, then the sheaf $p_X^*\mathscr{F} \otimes p_Y^*\mathscr{G}$ on $X \times Y$, where $p_X\colon X \times Y \to X$, $p_Y\colon X \times Y \to Y$ are the natural projections, will be denoted by $\mathscr{F} \otimes \mathscr{G}$. We use similar notation for more spaces.

(1.5.3) *Sheaves on an operad.* Let Q be a topological operad, and

$$\gamma_{m_1,\ldots,m_l}\colon Q(l) \times Q(m_1) \times \cdots \times Q(m_l) \to Q(m_1 + \cdots + m_l)$$

its structure maps. A sheaf on Q is, by definition, a collection $\mathscr{F} = \{\mathscr{F}(n)\}$ where $\mathscr{F}(n)$ is a sheaf of k-vector spaces on $Q(n)$ together with the following data:

(i) a structure of Σ_n-equivariant sheaf on $\mathscr{F}(n)$;

(ii) for each $m_1, \ldots, m_l$, a homomorphism of sheaves on $Q(l) \times Q(m_1) \times \cdots \times Q(m_l)$:

$$r_{m_1,\ldots,m_l}\colon \gamma^*_{m_1\cdots m_l}\mathscr{F}(m_1 + \cdots + m_l) \to \mathscr{F}(l) \otimes \mathscr{F}(m_1) \otimes \cdots \otimes \mathscr{F}(m_l);$$

(iii) a homomorphism $\varepsilon\colon \mathscr{F}(1)_1 \to k$ where $\mathscr{F}(1)_1$ is the stalk of the sheaf $\mathscr{F}(1)$ at the point $1 \in Q(1)$.

These data should satisfy three conditions of coassociativity, equivariance, and counit which we now explain.

(1.5.4) *The coassociativity condition.* Observe that the associativity condition for the topological operad Q amounts to the commutativity, for all $l, m_1, \dots, m_l, m_{ij}, i = 1, \dots, l, j = 1, \dots, v_i$, of the diagrams

$$\begin{array}{ccc}
Q(l) \times \prod Q(m_i) \times \prod_{i,j} Q(m_{ij}) & \xrightarrow{1\times\gamma} & Q(l) \times \prod_i Q\left(\sum_j m_{ij}\right) \\
\big\downarrow{\scriptstyle \gamma\times 1} & & \big\downarrow{\scriptstyle \gamma} \\
Q\left(\sum_i m_{ij}\right) \times \prod_{i,j} Q(m_{ij}) & \xrightarrow{\gamma} & Q\left(\sum_{i,j} m_{ij}\right)
\end{array}$$

Correspondingly, the structure data (ii) for a sheaf $\mathscr{F}$ give rise to the following diagram of sheaves on $Q(l) \times \prod_i Q(m_i) \times \prod_{i,j} Q(m_{ij})$:

$$\begin{array}{ccc}
(\gamma \circ (1 \times \gamma))^* \mathscr{F}\left(\sum_{i,j} m_{ij}\right) & \xrightarrow{r} & (1 \times \gamma)^*\left(\mathscr{F}(l) \otimes \bigotimes_i \mathscr{F}\left(\sum_j m_{ij}\right)\right) \\
\big\downarrow{\scriptstyle r} & & \big\downarrow{\scriptstyle 1\times r} \\
(\gamma \times 1)^*\left(\mathscr{F}\left(\sum m_i\right) \otimes \bigotimes_{i,j} \mathscr{F}(m_{ij})\right) & \xrightarrow{r\otimes 1} & \mathscr{F}(l) \otimes \bigotimes_i \mathscr{F}(m_i) \otimes \bigotimes_{i,j} \mathscr{F}(m_{ij}) .
\end{array}$$

It is required that all such diagrams commute.

(1.5.5) *The equivariance condition.* It is required, first of all, that the map $r_{m_1,\dots,m_l}$ commute with the natural action of $\Sigma_{m_1} \times \cdots \times \Sigma_{m_l}$ on $\mathscr{F}(m_1 + \cdots + m_l)$ and $\mathscr{F}(m_1) \otimes \cdots \otimes \mathscr{F}(m_l)$.

Second, for any permtuation $s \in \Sigma_l$ we denote by $\tilde{s} \in \Sigma_{m_1+\cdots+m_l}$ the block permutation which moves segments of lengths $m_1, \dots, m_l$ according to s. Note that the equivariance condition for the topological operad Q implies the commutativity of the diagram

$$\begin{array}{ccc}
Q(l) \times Q(m_1) \times \cdots \times Q(m_l) & \xrightarrow{\gamma} & Q(m_1 + \cdots + m_l) \\
\big\downarrow{\scriptstyle \mathrm{Id}\times s} & & \big\downarrow{\scriptstyle \tilde{s}^*} \\
Q(l) \times Q(m_{s(1)}) \times \cdots \times Q(m_{s(l)}) & \xrightarrow{\gamma} & Q(m_{s(1)} + \cdots + m_{s(l)}).
\end{array}$$

It is now required that the following diagram of sheaves on $Q(l) \times Q(m_1) \times \cdots \times$

$Q(m_l)$ commute:

$$\begin{array}{ccc}
\begin{pmatrix}(\gamma \circ (\tilde{s} \times \mathrm{Id}))^* \mathscr{F}(m_{s(1)} + \cdots + m_{s(l)}) \\ = (\tilde{s}^* \circ \gamma)^* \mathscr{F}(m_{s(1)} + \cdots + m_{s(l)})\end{pmatrix} & \xrightarrow{r} & (s \times \mathrm{Id})^*[\mathscr{F}(l) \otimes \mathscr{F}(m_{s(1)}) \otimes \cdots \otimes \mathscr{F}(m_{s(l)})] \\
\Big\downarrow \text{eq.} & & \Big\downarrow \text{eq.} \\
\gamma^* \mathscr{F}(m_1 + \cdots + m_l) & \xrightarrow{r} & \mathscr{F}(l) \otimes \mathscr{F}(m_1) \otimes \cdots \otimes \mathscr{F}(m_l).
\end{array}$$

where "eq." denotes morphisms of equivariance.

(1.5.6) *The counit condition.* Recall that the element $1 \in Q(1)$ is such that the composition

$$Q(l) \xrightarrow{j} Q(l) \times Q(1) \times \cdots \times Q(1) \xrightarrow{\gamma} Q(l),$$

where $j(q) = (q, 1, \ldots, 1)$, is the identity map. It is required, in addition, that the composition

$$\mathscr{F}(l) = (\gamma \circ j)^* \mathscr{F}(l) \xrightarrow{\quad r \quad} j^*(\mathscr{F}(l) \otimes \mathscr{F}(1) \otimes \cdots \otimes \mathscr{F}(1)) \xrightarrow{\mathrm{Id} \otimes \varepsilon \otimes \cdots \otimes \varepsilon} \mathscr{F}(l)$$

be the identity homomorphism. This completes the definition of a sheaf on a topological operad.

(1.5.7) *Example.* Let Pt be the topological operad having $Pt(n) = \{pt\}$ (1-point space) for any n and all the structure maps being the identities. We call Pt the *trivial operad.* (It defines associative and commutative H-spaces; see [52].) Let $\mathscr{F}$ be a sheaf on Pt. For any n the sheaf $\mathscr{F}$ on the space $Pt(n)$ is just a vector space $F(n)$. The structure data (i)–(iii) of (1.5.3) amount to Σ_n-action on the space $F(n)$, linear maps

$$F(m_1 + \cdots + m_l) \to F(l) \otimes F(m_1) \otimes \cdots \otimes F(m_l),$$

and a linear functional $\varepsilon \colon F(1) \to k$.

It is immediate to see that these data make the collection of the dual vector spaces $F(n)^*$ into a k-linear operad. Conversely, every k-linear operad $\mathscr{P}$ with finite-dimensional spaces $\mathscr{P}(n)$ defines a sheaf on the operad Pt.

(1.5.8) *Remark.* One might say that the $F(n)$ in the above example form a *cooperad*, a structure dual to that of an operad. We prefer to postpone the discussion of this structure until §3.

The considerations of (1.5.7) can be generalized as follows.

(1.5.9) PROPOSITION. *Let Q be a topological operad, and $\mathscr{F}$ a sheaf on Q. Then the graded spaces $H^\bullet(Q(n), \mathscr{F}(n))^*$ (the duals to the total cohomology spaces) form an operad in the category $g\mathrm{Vect}^-$ of graded vector spaces. Moreover, for any $q \geqslant 0$, the subspaces $H^{q(n-1)}(Q(n), \mathscr{F}(n))^*$ form a suboperad.*

The proof is obvious from the axioms and is left to the reader.

The fact (1.4.1) that the homology spaces $H_\cdot(Q(n), k)$ form an operad is a particular case of this proposition. Indeed, for any topological operad Q, the constant sheaves $\underline{k}_{Q(n)}$ form a sheaf on Q.

(1.5.10) Let Q be a topological operad, and $\mathscr{F}$ a sheaf on Q. We say that $\mathscr{F}$ is an *isosheaf* if all the maps $r_{m_1,\ldots,m_l}$ (data (ii) of a sheaf) are isomorphisms. Now we can formulate the precise relation between k-linear operads and sheaves on the configuration operad $\mathscr{M}$.

(1.5.11) THEOREM. (a) *If $\mathscr{P}$ is a k-linear operad, then the sheaves $\mathscr{F}_{\mathscr{P}}(n)$ on the spaces $\mathscr{M}(n)$ introduced in (1.5.2) form a sheaf $\mathscr{F}_{\mathscr{P}}$ on the operad $\mathscr{M}$.*

(b) *If $\mathscr{P}(1) = k$, then $\mathscr{F}_P$ is an isosheaf.*

(c) *Any combinatorial isosheaf $\mathscr{F}$ on $\mathscr{M}$ has the form $\mathscr{F}_P$ for some k-linear operad $\mathscr{P}$ with $\mathscr{P}(1) = k$.*

The proof is straightforward and left to the reader.

(1.5.12) *Remark.* If one replaces in (1.5.11) (c) the adjective "combinatorial" by "constructible", then one obtains a notion of a *braided operad* introduced by Fiedorowicz [20].

1.6. *Modules over an algebra over an operad*

(1.6.1) Let $\mathscr{P}$ be a k-linear operad, and A a $\mathscr{P}$-algebra. An *A-module* (or a $(\mathscr{P}, A)$-module, if $\mathscr{P}$ is to be specified explicitly) is a k-vector space M together with a collection of linear maps

$$q_n\colon \mathscr{P}(n) \otimes A^{\otimes(n-1)} \otimes M \to M, \qquad n \geqslant 1$$

satisfying the following conditions:

(i) (Associativity) For any natural numbers $n, r_1, \ldots, r_n$, any elements $\lambda \in \mathscr{P}(n)$, $\mu_i \in \mathscr{P}(r_i)$, $a_{ij} \in A$, $j = 1, \ldots, r_i$, $i = 1, \ldots, n-1$, and also $a_{n1}, \ldots, a_{n,r_n-1} \in A$, $m \in M$, we have the equality

$$q_{m_1+\cdots+m_n}(\lambda(\mu_1, \ldots, \mu_n) \otimes a_{11} \otimes \cdots \otimes a_{1,r_1} \otimes a_{21} \otimes \cdots \otimes a_{n,r_n-1} \otimes m)$$

$$= q_n(\lambda \otimes \mu_1(a_{11}, \ldots, a_{1,r_1}) \otimes \cdots \otimes \mu_{n-1}(a_{n-1,1}, \ldots, a_{n-1,r_{n-1}})$$

$$\otimes\, q_{r_n}(\mu_n \otimes a_{n1} \otimes \cdots \otimes a_{n,r_n-1} \otimes m)).$$

(ii) (Equivariance) The map q_n is equivariant with respect to the action of $\Sigma_{n-1} \subset \Sigma_n$ (the subgroup of permutations preserving n) on $\mathscr{P}(n) \otimes A^{\otimes(n-1)} \otimes M$ given by

$$s(\lambda \otimes (a_1 \otimes \cdots \otimes a_{n-1}) \otimes m) = s^*(\lambda) \otimes (a_{s(1)} \otimes \cdots \otimes a_{s(n-1)}) \otimes m.$$

(iii) (Unit) We have $q_1(\mathbf{1} \otimes 1 \otimes m) = m$ for any $m \in M$ where $\mathbf{1} \in \mathscr{P}(1)$ is the unit of $\mathscr{P}$ and 1 is the unit element of $k = A^{\otimes 0}$.

We often write $\lambda(a_1, \ldots, a_{n-1}, m)$ for $q_n(\lambda \otimes (a_1 \otimes \cdots \otimes a_{n-1}) \otimes m)$.

(1.6.2) *Examples.* (a) Every $\mathscr{P}$-algebra is a module over itself.

(b) Let $\mathscr{P} = \mathscr{A}s$, and let A be a $\mathscr{P}$-algebra, i.e., an associative algebra. An $(\mathscr{A}s, A)$-module is the same as an A-bimodule in the usual sense. Indeed, let us realize $\mathscr{A}s(n)$ as the space of noncommutative polynomials in $x_1, \ldots, x_n$ spanned by the monomials $x_{s(1)} \cdot \cdots \cdot x_{s(n)}$, $s \in \Sigma_n$; see (1.3.6). Given an A-bimodule M, we define the map $q_n\colon \mathscr{A}s(n) \otimes A^{\otimes(n-1)} \otimes M \to M$ by the rule

$$q_n(x_{s(1)} \cdot \cdots \cdot x_{s(n)} \otimes (a_1 \otimes \cdots \otimes a_{n-1}) \otimes m) = a_{s(1)} \cdot \cdots \cdot a_{s(n)}$$

where we set $a_n = m$.

(c) Let $\mathscr{P} = \mathscr{C}om$, and let A be a $\mathscr{P}$-algebra, i.e., a commutative algebra. A $(\mathscr{C}om, A)$-module is the same as an A-module in the usual sense.

(d) Let $\mathscr{P} = \mathscr{L}ie$, and let A be a Lie algebra. A $(\mathscr{L}ie, A)$-module is the same as an A-module (representation of the Lie algebra A) in the usual sense.

(1.6.3) Let M, M' be two $(\mathscr{P}, A)$-modules. A morphism of modules $f\colon M \to M'$ is a k-linear map such that $f(\lambda(a_1, \ldots, a_{n-1}, m)) = \lambda(a_1, \ldots, a_{n-1}, f(m))$ for any n, any $\lambda \in \mathscr{P}(n)$, $a_i \in A$, $m \in M$. Clearly, all $(\mathscr{P}, A)$-modules form an abelian category.

(1.6.4) *The universal enveloping algebra.* As for every abelian category, it is natural to expect that the category of $(\mathscr{P}, A)$-modules can be described in terms of left modules over a certain associative algebra. Such an algebra indeed exists and is called the *universal enveloping algebra* of A, to be denoted $U(A)$ or $U_{\mathscr{P}}(A)$. Here is the construction:

By definition, $U(A)$ is generated by symbols $X(\lambda; a_1, \ldots, a_{n-1})$, for every n, every $\lambda \in \mathscr{P}(n)$, and every $a_1, \ldots, a_{n-1} \in A$. (In particular, when $n = 1$, then no a_i should be specified and we have generators $X(\lambda)$, $\lambda \in \mathscr{P}(1)$.) These symbols are subject to conditions of polylinearity with respect to each argument and to the identifications

(1.6.5)
$$X(\lambda; \mu_1(a_{11}, \ldots, a_{1,r_1}), \ldots, \mu_{n-1}(a_{n-1,1}, \ldots, a_{n-1,r_{n-1}}))$$
$$= X(\lambda(\mu_1, \ldots, \mu_{n-1}, \mathbf{1}); a_{11}, \ldots, a_{n-1,r_{n-1}})$$

for any $\lambda \in \mathscr{P}(n)$, $\mu_i \in \mathscr{P}(m_i)$, $a_{ij} \in A$.

The multiplication in the algebra $U(A)$ is given by the formula

$$X(\lambda; a_1, \ldots, a_{n-1})X(\mu, b_1, \ldots, b_{l-1}) = X(\lambda(\mathbf{1}, \ldots, \mathbf{1}, \mu); a_1, \ldots, a_{n-1}, b_1, \ldots, b_{l-1}).$$

It is immedate to verify that we get in this way an associative algebra $U(A)$, and the image of $X(\mathbf{1})$ is the unit of this algebra.

Heuristically, $X(\lambda; a_1, \ldots, a_{n-1})$ corresponds to the operator

$$m \mapsto \lambda(a_1, \ldots, a_{n-1}, m),$$

which acts on every $(\mathscr{P}, A)$-module M.

The following fact is obvious and its proof is left to the reader.

(1.6.6) PROPOSITION. *Let A be a $\mathscr{P}$-algebra. The category of A-modules is equivalent to the category of left modules over the associative algebra $U_{\mathscr{P}}(A)$.*

(1.6.7) *Examples.* (a) If $\mathscr{G}$ is a Lie algebra, then $U_{\mathscr{L}ie}(\mathscr{G})$ defined above is the ordinary universal enveloping algebra of $\mathscr{G}$.

(b) If A is an associative algebra and we regard it as an $\mathscr{A}s$-algebra, then $U_{\mathscr{A}s}(A) = A \otimes A^{op}$.

(c) If A is an associative commutative algebra and we regard it as a $\mathscr{C}om$-algebra, then $U_{\mathscr{C}om}(A) = A$.

(1.6.8) By construction, the universal enveloping algebra $U_{\mathscr{P}}(A)$ has the form

$$U_P(A) = \bigoplus \mathscr{P}(n) \otimes A^{\otimes(n-1)}/\equiv$$

where $\equiv$ is the equivalence relation described in (1.6.5). The nth summand above is just the space of generators $X(\lambda; a_1, \ldots, a_{n-1})$.

Observe that $U(A) = U_{\mathscr{P}}(A)$ has a natural (multiplicative) filtration $F_\bullet$ where $F_n U(A)$ is the subspace consisting of images of generators $X(\lambda; a_1, \ldots, a_{r-1})$ for $r \leqslant n$.

2. Quadratic operads

2.1. Description of operads by generators and relations

(2.1.1) Let K be an assoicative k-algebra. Given an aribtrary K-collection $E = \{E(n), n \geqslant 2\}$ (1.2.11), we define an operad $F(E)$ called the *free operad* generated by E. By definition

$$F(E)(n) = \bigoplus_{n-\text{trees } T} E(T)$$

where T runs over isomorphism classes of n-trees and $E(T)$ is defined as in (1.2.13). The composition maps (1.2.1) (ii) for $F(E)$,

$$F(E)(l) \otimes F(E)(m_1) \otimes \cdots \otimes F(E)(m_l) \to F(E)(m_1 + \cdots + m_l),$$

are defined by means of maps

$$E(T) \otimes E(T_1) \otimes \cdots \otimes E(T_l) \to E(T(T_1, \ldots, T_l)) \tag{2.1.2}$$

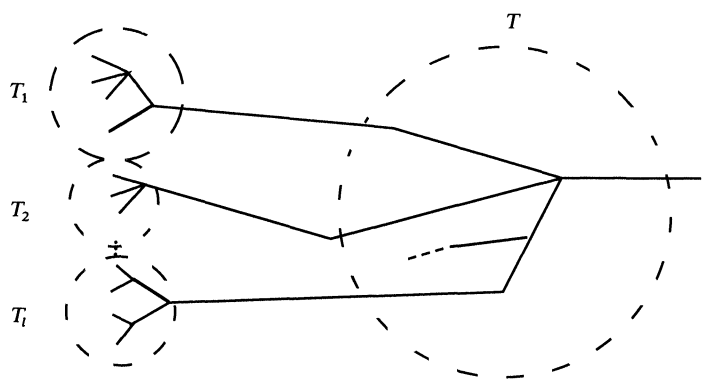

FIGURE 6

where T is an l-tree, T_i, $i = 1, \ldots, l$, is an m_i-tree, and $T(T_1, \ldots, T_l)$ is their composition; see Figure 6.

The definition of the map (2.1.2) is obvious, keeping in mind that both the left-hand side and right-hand side are tensor products of the same spaces but over different rings (some over k and some over K).

(2.1.3) *Ideals.* Let $\mathcal{P} = \{\mathcal{P}(n)\}$ be a k-linear operad. A (2-sided) ideal in $\mathcal{P}$ is a collection $\mathcal{I}$ of vector subspaces $\mathcal{I}(n) \subset \mathcal{P}(n)$ satisfying the following three conditions:

(i) For each n, the space $\mathcal{I}(n)$ is preserved by the action of Σ_n on $\mathcal{P}(n)$.

(ii) If $\lambda \in \mathcal{P}(n)$, $\mu_1 \in \mathcal{P}(m_1), \ldots, \mu_n \in \mathcal{P}(m_n)$, and for at least one j we have $\mu_j \in \mathcal{I}(m_j)$, then the composition $\lambda(\mu_1, \ldots, \mu_n)$ belongs to $\mathcal{I}(m_1 + \cdots + m_n)$.

(iii) If $\lambda \in \mathcal{I}(n)$ and $\mu_i \in \mathcal{P}(m_i)$, $i = 1, \ldots, n$, then $\lambda(\mu_1, \ldots, \mu_n) \in \mathcal{I}(m_1 + \cdots + m_n)$.

If $\mathcal{I}$ is an ideal in an operad $\mathcal{P}$, then we can construct the quotient operad $\mathcal{P}/\mathcal{I}$ with components $(\mathcal{P}/\mathcal{I})(n) = \mathcal{P}(n)/\mathcal{I}(n)$ (quotient linear spaces). The conditions (ii) and (iii) above imply that compositions in $\mathcal{P}$ induce well-defined compositions in $\mathcal{P}/\mathcal{I}$.

It is straightforward to see that the kernel of a morphism of k-linear operads $f: \mathcal{P} \to \mathcal{Q}$ is an ideal in $\mathcal{P}$.

(2.1.4) Let V be a finite dimensional k-vector space, and let $\mathcal{L}ie(V)$, $\mathcal{A}s(V)$, $\mathcal{C}om(V)$ be respectively the free Lie algebra, free associative (tensor) algebra and free commutative (polynomial) algebra generated by V. There are canonical linear maps

$$\mathcal{L}ie(V) \overset{\varepsilon}{\hookrightarrow} \mathcal{A}s(V) \overset{\pi}{\to} \mathcal{C}om(V). \tag{2.1.5}$$

Both maps are the identity on V and are uniquely determined by the requirement that π be a homomorphism of associative algebras and ε be a homomorphism of Lie algebras (with the structure of a Lie algebra on $\mathcal{A}s(V)$ given by $[a, b] = ab -$

ba). The maps (2.1.5) give rise to a collection of linear maps

$$\mathscr{L}ie(n) \overset{\varepsilon_n}{\hookrightarrow} \mathscr{A}s(n) \overset{\pi_n}{\to} \mathscr{C}om(n), \qquad n = 1, 2, \ldots$$

such that $\pi_n \circ \varepsilon_n = 0$ for $n \geqslant 2$.

These maps give morphisms of operads $\mathscr{L}ie \overset{\varepsilon}{\hookrightarrow} \mathscr{A} \overset{\pi}{\to} \mathscr{C}om$. Let $\mathscr{L}ie^+$ be the collection of spaces $\{\mathscr{L}ie(n), n \geqslant 2\}$ and $(\mathscr{L}ie^+)$ be the minimal ideal in $\mathscr{A}s$ containing $\mathscr{L}ie^+$. The proof of the following result is left to the reader.

(2.1.6) PROPOSITION. $(\mathscr{L}ie^+) = \mathrm{Ker}(\pi)$, *that is*, $\mathscr{C}om \cong \mathscr{A}s/(\mathscr{L}ie^+)$.

(2.1.7) *Quadratic operads.* Let K be a semisimple k-algebra. Let E be a $(K, K^{\otimes 2})$-bimodule with an involution $\sigma: E \to E$ such that

$$\sigma(\lambda e) = \lambda\sigma(e), \qquad \sigma(e \cdot (\lambda_1 \otimes \lambda_2)) = \sigma(e) \cdot (\lambda_2 \otimes \lambda_1), \qquad \forall \lambda, \lambda_1, \lambda_2 \in K, e \in \mathrm{E}.$$

We form the space $E \otimes_K E$, the tensor product with respect to the right K-module structure on the first factor given by $e \cdot \lambda = e \cdot (\lambda \otimes 1)$. This space has two structures:

(i) a Σ_2-action given by the action of σ on the second factor E,

(iii) a structure of $(K, K^{\otimes 3})$-bimodule.

Therefore the induced Σ_2-module $\mathrm{Ind}_{\Sigma_2}^{\Sigma_3}(E \otimes_K E)$ inherits the $(K, K^{\otimes 3})$-bimodule structure. Let $R \subset \mathrm{Ind}_{\Sigma_2}^{\Sigma_3}(E \otimes_K E)$ be a Σ_3-stable $(K, K^{\otimes 3})$-sub-bimodule. To any such data (E, R) we associate an operad $\mathscr{P}(K, E, R)$ in the following way. We form the K-collection $\{E(2) = E, E(n) = 0, n > 2\}$ (denoted also by E) and the corresponding free operad $F = F(E)$. Observe that $F(E)(3) = \mathrm{Ind}_{\Sigma_2}^{\Sigma_3}(E \otimes_K E)$; see Figure 7.

More generally,

$$F(E)(n) = \bigoplus_{\substack{\text{binary} \\ n\text{-trees } T}} E(T). \tag{2.1.8}$$

Let (R) be the ideal in $F(E)$ generated by the subspace $R \subset F(E)(3)$. We put $\mathscr{P} = \mathscr{P}(K, E, R) = F(E)/(R)$. An operad of type $\mathscr{P}(K, E, R)$ is called a *quadratic*

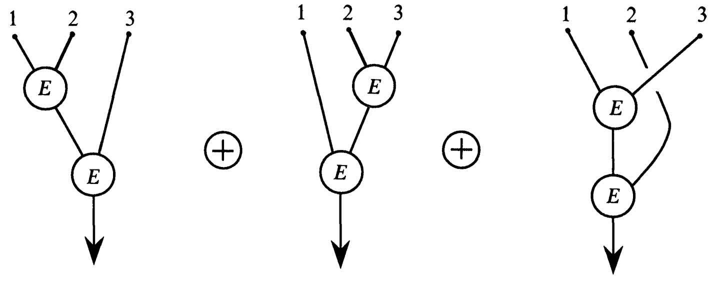

FIGURE 7. $F(E)(3)$

operad with the space of generators E and the space of relations R. Note that K, E, R can be recovered from $\mathscr{P}$ as $K = \mathscr{P}(1)$, $E = \mathscr{P}(2)$, and $R = \mathrm{Ker}\{F(E)(3) \to \mathscr{P}(3)\}$.

(2.1.9) *The quadratic duality.* Given a semisimple k-algebra K and a finite-dimensional left K-module V with a Σ_n-action, we define $V^\vee = \mathrm{Hom}_K(V, K)$. This is a right K-module, i.e., a left module over the opposite algebra K^{op}. We always equip $V^\vee$ with the transposed action of Σ_n twisted by the sign representation.

Let $\mathscr{P} = \mathscr{P}(K, E, R)$ be a quadratic operad. The space $E^\vee$ has a natural structure of $(K^{op}, K^{op} \otimes K^{op})$-bimodule. Observe that $F(E^\vee)(3) = F(E(3))^\vee$. Let $R^\perp \subset F(E^\vee)(3)$ be the orthogonal complement of R. It is stable under the Σ_3-action and the three K^{op}-actions on $F(E^\vee)(3)$. We define the *dual quadratic operad* $\mathscr{P}^!$ to be

$$\mathscr{P}^! = \mathscr{P}(K^{op}, E^\vee, R^\perp).$$

(2.1.10) *Examples.* Suppose that $K = k$ and $\mathscr{P} = \mathscr{P}(k, E, R)$ is a quadratic operad. A $\mathscr{P}$-algebra is a vector space A with several *binary* operations (parametrized by E) which are subject to certain identities (parametrized by R) each involving three arguments. This is precisely the way of defining the types of algebras most commonly encountered in practice. In particular, the operads $\mathscr{A}s$, $\mathscr{C}om$, $\mathscr{L}ie$ governing, respectively, associative, commutative, and Lie algebras, are quadratic.

Note that the structure constants of an algebra A over a quadratic operad satisfy quadratic relations. For example, if $\{e_i\}$ is a basis of A, define $\mu\colon A \times A \to A$ by $\mu(e_i \otimes e_j) = \sum c_{ij}^k e_k$. The condition that μ is associative means that

$$\sum_k c_{ij}^k c_{kl}^m = \sum_k c_{jk}^k c_{ik}^m \qquad \forall i, j, m.$$

Other examples are similar. This explains the name "quadratic operad."

(2.1.11) THEOREM. *We have isomorphisms of operads*

$$\mathscr{C}om^! = \mathscr{L}ie, \qquad \mathscr{L}ie^! = \mathscr{C}om, \qquad \mathscr{A}s^! = \mathscr{A}s.$$

The idea that the "words" of commutative, Lie, and associative algebras are, in some sense, dual to each other, as described above, was promoted by Drinfeld [18] and Kontsevich [39]. (The $\mathscr{C}om$-$\mathscr{L}ie$ duality was implicit already in Quillen's paper [56] and in Moore's Nice talk [53].) The concept of quadratic operads and their Koszul duality allows us to make this idea into a theorem. Observe also that the isomorphism $\mathscr{C}om \cong \mathscr{A}s/(\mathscr{L}ie^+)$ of Proposition 2.1.6 is, in a sense, "self-dual".

Proof of the theorem. The group Σ_3 has three irreducible representations which we denote **1** (the identity representation), Sgn (the sign representation) and V_2 (the 2-dimensional representation in the hyperplane $\sum x_i = 0$ in the 3-dimensional space of (x_1, x_2, x_3)). Both $\mathscr{C}om$ and $\mathscr{L}ie$ have 1-dimensional spaces of generators with Σ_2 acting trivially on $\mathscr{C}om(2)$ and by sign on $\mathscr{L}ie(2)$. An elementary calculation of group characters shows that we have isomorphism of Σ_3-modules

$$F(\mathscr{C}om(2))(3) = \mathbf{1} \oplus V_2, \qquad F(\mathscr{L}ie(2))(3) = \mathrm{Sgn} \oplus V_2.$$

This implies that we have

$$\mathscr{C}om(3) = \mathbf{1}, \qquad R_{\mathscr{C}om} = V_2, \qquad \mathscr{L}ie(3) = V_2, \qquad R_{\mathscr{L}ie} = \mathrm{Sgn}.$$

The duality between $\mathscr{C}om$ and $\mathscr{L}ie$ follows.

To prove that $\mathscr{A}s^! = \mathscr{A}s$, we consider the space $F(\mathscr{A}s(2))(3)$. This space has dimension 12 and is spanned by the 12 expressions of the form $x_{s(1)}(x_{s(2)}x_{s(3)})$ and $(x_{s(1)}x_{s(2)})x_{s(3)}$, $s \in \Sigma_3$.

We introduce on this space a scalar product $\langle\ ,\ \rangle$ by setting all these products orthogonal to each other and putting

$$\langle x_i(x_jx_k), x_i(x_jx_k)\rangle = \mathrm{sgn}\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}, \qquad \langle (x_ix_j)x_k, (x_ix_j)x_k\rangle = -\mathrm{sgn}\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}.$$

This product is sign-invariant with respect to the Σ_3-action, i.e., $\langle s(\mu), s(\nu)\rangle = \mathrm{sgn}(s)\langle \mu, \nu\rangle$. The (6-dimensional) space of relations $R = R_{\mathscr{A}s}$ is spanned by all the associators $x_i(x_jx_k) - (x_ix_j)x_k$. This space coincides with its own annihilator with respect to the described scalar product. This shows that $R^\perp = R$ and so $\mathscr{A}s^! = \mathscr{A}s$.

2.2. The analogs of Manin's tensor products. The Lie operad as a dualizing object.

(2.2.1) In this section we consider only quadratic operads $\mathscr{P}$ with $\mathscr{P}(1) = k$. Such an operad is defined by a vector space of generators $E = \mathscr{P}(2)$ with an involution σ (action of Σ_2) and a Σ_3-invariant subspace R of relations inside $F(E)(3)$ where $F(E)$ is the free operad generated by E. Observe that $F(E)(3)$ is the direct sum of three copies of $E \otimes E$ (see Figure 6). So we refer to this space as $3(E \otimes E)$. Our aim is to present an operad-theoretic version of [51].

(2.2.2) Let $(\mathscr{A}, \otimes)$ be any symmetric monoidal category and let $\mathscr{P}$, $\mathscr{Q}$ be two operads in $\mathscr{A}$. The collection of objects $\mathscr{P}(n) \otimes \mathscr{Q}(n)$ forms a new operad in $\mathscr{A}$ denoted by $\mathscr{P} \otimes \mathscr{Q}$. The main property of this operad is that if A is a $\mathscr{P}$-algebra and B is a $\mathscr{Q}$-algebra (in $\mathscr{A}$), then $A \otimes B$ is a $\mathscr{P} \otimes \mathscr{Q}$-algebra. We are interested in the case of k-linear operads, i.e., $\mathscr{A} = \mathrm{Vect}$.

(2.2.3) Suppose that $\mathscr{P}$, $\mathscr{Q}$ are quadratic operads with $\mathscr{P}(1) = \mathscr{Q}(1) = k$ so that $\mathscr{P}(2)$ and $\mathscr{Q}(2)$ are their spaces of generators. We denote by $\mathscr{P} \circ \mathscr{Q}$ the suboperad in $\mathscr{P} \otimes \mathscr{Q}$ generated by $\mathscr{P}(2) \otimes \mathscr{Q}(2)$. Let $R_{\mathscr{P}} \subset 3(\mathscr{P}(2) \otimes \mathscr{P}(2))$ and $R_{\mathscr{Q}} \subset 3(\mathscr{Q}(2) \otimes \mathscr{Q}(2))$ be the spaces of relations of $\mathscr{P}$ and $\mathscr{Q}$. Then the operad $\mathscr{P} \circ \mathscr{Q}$ is described as follows.

The space of generators of $\mathcal{P} \circ \mathcal{Q}$ is, by definition, $\mathcal{P}(2) \otimes \mathcal{Q}(2)$. The third component of the free operad generated by $\mathcal{P}(2) \otimes \mathcal{Q}(2)$ is $3(\mathcal{P}(2) \otimes \mathcal{Q}(2) \otimes \mathcal{P}(2) \otimes \mathcal{Q}(2))$, which can be regarded in two ways as:

1. $(\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes 3(\mathcal{Q}(2) \otimes \mathcal{Q}(2))$ and, regarded as such, it contains the subspace $(\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes R_{\mathcal{Q}}$.
2. $3(\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes (\mathcal{Q}(2) \otimes \mathcal{Q}(2))$ and, regarded as such, it contains the subspace $R_{\mathcal{P}} \otimes (\mathcal{Q}(2) \otimes \mathcal{Q}(2))$.

(2.2.4) PROPOSITION. *If $\mathcal{P}$, $\mathcal{Q}$ are quadratic, then $\mathcal{P} \circ \mathcal{Q}$ is also quadratic with the space of generators $\mathcal{P}(2) \otimes \mathcal{Q}(2)$ and the space of relations*

$$((\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes R_{\mathcal{Q}}) + (R_{\mathcal{P}} \otimes (\mathcal{Q}(2) \otimes \mathcal{Q}(2))).$$

(The sum is not necessarily direct.)

The proof is straightforward.

Thus $\mathcal{P} \circ \mathcal{Q}$ is the analog of the white circle product $A \circ B$ for associative algebras considered by Manin [51].

(2.2.5) Given two quadratic operads $\mathcal{P}$, $\mathcal{Q}$ as before, we define the quadratic operad $\mathcal{P} \bullet \mathcal{Q}$ (the black circle product; cf. [51]) to have the same space of generators $\mathcal{P}(2) \circ \mathcal{Q}(2)$ as $\mathcal{P} \otimes \mathcal{Q}$ but the space of relations

$$((\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes R_{\mathcal{Q}}) \cap (R_{\mathcal{P}} \otimes (\mathcal{Q}(2) \otimes \mathcal{Q}(2))).$$

(2.2.6) THEOREM. *Each of the products $\circ$, $\bullet$ defines on the category of quadratic operads (and morphisms defined in (1.3.1)) a symmetric monoidal strucutre. Moreover, we have:*

(a) $(\mathcal{P} \circ \mathcal{Q})^! = \mathcal{P}^! \bullet \mathcal{Q}^!$.

(b) $\mathrm{Hom}(\mathcal{P} \bullet \mathcal{Q}, \mathcal{R}) = \mathrm{Hom}(\mathcal{P}, \mathcal{Q}^! \circ \mathcal{R})$. *In particular, the commutative operad is a unit object with respect to $\circ$ and the Lie operad is a unit object with respect to $\bullet$.*

Proof. (a) It is similar to the argument of Manin [51] for quadratic algebras: we should use the obvious relations between sums, intersections, and orthogonal complements of subspaces in a vector space.

(b) The fact that $\mathcal{C}om$ is a unit object with respect to $\circ$ follows because $\mathcal{C}om(n) = k$ for any n. The fact that $\mathcal{L}ie$ is a unit object with respect to $\bullet$ follows from part (a) and the fact that $\mathcal{L}ie^! = \mathcal{C}om$.

(2.2.7) Given quadratic operads $\mathcal{P}$ and $\mathcal{Q}$, we define the quadratic operad $hom(\mathcal{P}, \mathcal{Q})$ as follows. Its space of generators is set to be

$$hom(\mathcal{P}, \mathcal{Q})(2) = \mathrm{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2)).$$

The space of relations $R_{hom(\mathcal{P},\mathcal{Q})}$ is defined to be the minimal subspace in $F(\mathrm{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2)))(3)$ such that the canonical map

$$\mathrm{can}\colon \mathcal{Q}(2) \to \mathrm{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2)) \otimes \mathcal{P}(2) = \mathcal{P}(2)^* \otimes \mathcal{Q}(2) \otimes \mathcal{P}(2)$$

extends to a morphism of operads $\mathcal{Q} \mapsto hom(\mathcal{P}, \mathcal{Q}) \circ \mathcal{P}$. More precisely, we define $R_{hom(\mathcal{P},\mathcal{Q})}$ to be minimal among subspaces J with the property that

$$J \otimes (\mathcal{P}(2) \otimes \mathcal{P}(2)) \subset F(\mathrm{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2)) \otimes \mathcal{P}(2))(3)$$

contains the image of the embedding

$$\mathrm{can}_*: F(\mathcal{Q}(2))(3) \hookrightarrow F(\mathrm{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2)) \otimes \mathcal{P}(2))(3)$$

induced by can.

If $\mu_1, \ldots, \mu_m$ is a basis of $\mathcal{P}(2)$, $v_1, \ldots, v_n$ is a basis of $\mathcal{Q}(2)$, and a_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ is the corresponding basis of $\mathrm{Hom}(\mathcal{P}(2), \mathcal{Q}(2))$, then the elements

$$\tilde{v}_i = \sum_j a_{ij} \otimes \mu_j \in (hom(\mathcal{P}, \mathcal{Q}) \circ \mathcal{P})(2)$$

satisfy all the quadratic relations holding for actual v_i.

(2.2.8) THEOREM. *We have a natural isomorphism of operads*

$$hom(\mathcal{P}, \mathcal{Q}) \cong \mathcal{P}^! \bullet \mathcal{Q}.$$

Proof. Simple linear algebra.

(2.2.9) COROLLARY. (a) *For any quadratic operad $\mathcal{P}$, we have*

$$\mathcal{P}^! = hom(\mathcal{P}, \mathcal{L}ie).$$

(b) *There exists a morphism of operads $\mathcal{L}ie \to \mathcal{P} \otimes \mathcal{P}^!$ which takes the generator of the* 1-*dimensional space $\mathcal{L}ie(2)$ into the identity operator in $\mathcal{P}(2) \otimes \mathcal{P}^!(2) = \mathcal{P}(2) \otimes \mathcal{P}(2)^*$. In particular, for any $\mathcal{P}$-algebra A and $\mathcal{P}^!$-algebra B, the vector space $A \otimes_k B$ has a natural Lie algebra structure.*

(2.2.10) Let us give another interpretation of the operad $hom(\mathcal{P}, \mathcal{Q})$ in terms of algebras. Suppose that $V_1, \ldots, V_m$, W are k-vector spaces. By a *$\mathcal{P}$-multilinear map* $V_1 \times \cdots \times V_m \to W$, we understand an element

$$\Phi \in F_{\mathcal{P}}(V_1^* \oplus \cdots \oplus V_m^*) \otimes W$$

which is homogeneous of degree 1 with respect to dilations of any of V_i. (Recall (1.3.6) that $F_{\mathcal{P}}$ means the free $\mathcal{P}$-algebra generated by a vector space.) Such maps form a vector space which we denote by $\mathrm{Mult}_{\mathcal{P}}(V_1, \ldots, V_m | W)$. For example, $\mathcal{P}(n) = \mathrm{Mult}_{\mathcal{P}}(k, \ldots, k|k)$ (m copies of k before the bar), cf. (1.3.7)–(1.3.9). The space of ordinary multilinear maps, i.e., $\mathrm{Hom}_k(V_1 \otimes \cdots \otimes V_m, W)$ is nothing but $\mathrm{Mult}_{\mathcal{C}om}(V_1, \ldots, V_m | W)$ where $\mathcal{C}om$ is the commutative operad.

Similar to usual multilinear maps, $\mathscr{P}$-multilinear maps can be composed. In particular, for any vector space V, the spaces

$$\mathscr{E}_{\mathscr{P},V}(n) = \mathrm{Mult}_{\mathscr{P}}(V, \dots, V|V) \qquad (n \text{ copies of } V) \tag{2.2.11}$$

form a k-linear operad which we call the *operad of endomorphisms of V in $\mathscr{P}$.*

(2.2.12) Let $\mathscr{Q}$ be another k-linear operad. By a $\mathscr{Q}$-algebra in $\mathscr{P}$ we mean a k-vector space A together with a morphism of operads $f\colon \mathscr{Q} \to \mathscr{E}_{\mathscr{P},A}$. When $\mathscr{P} = \mathscr{C}om$ is the commutative operad, we get the usual notion of a $\mathscr{Q}$-algebra.

(2.2.13) THEOREM. *Let $\mathscr{P}$ and $\mathscr{Q}$ be quadratic operads with $\mathscr{P}(1) = \mathscr{Q}(1) = k$. Then $\mathscr{Q}$-algebras in $\mathscr{P}$ are the same as $hom(\mathscr{P}, \mathscr{Q})$-algebras (in the usual sense).*

Proof. Let A be a $\mathscr{Q}$-algebra in $\mathscr{P}$. The corresponding morphism $f\colon \mathscr{Q} \to \mathscr{E}_{\mathscr{P},A}$ is defined by its second component

$$f_2\colon \mathscr{Q}(2) \to \mathscr{E}_{\mathscr{P},A}(2) = (\mathscr{P}(2) \otimes (A^*)^{\otimes 2})_{\Sigma_2} \otimes A.$$

By taking a partial transpose of f_2, we get a Σ_2-equivariant map

$$f_2^{\dagger}\colon \mathrm{Hom}_k(\mathscr{P}(2), \mathscr{Q}(2)) \to \mathrm{Hom}_k(A \otimes A, A) = \mathscr{E}_A(2).$$

In order that a given linear map f_2 come from a morphism of operads $f\colon Q \to \mathscr{E}_{\mathscr{P},A}$, the quadratic relations among the generators of Q should be satisfied. By the definition of relations in $hom(\mathscr{P}, \mathscr{Q})$, this is equivalent to the condition that $f_2^{\dagger}$ extend to a morphism $hom(\mathscr{P}, \mathscr{Q}) \to \mathscr{E}_A$, i.e., that we have on A a structure of a $hom(\mathscr{P}, \mathscr{Q})$-algebra.

(2.2.14) *Lazard-Lie theory for formal groups in operads and Koszul duality.* The idea of considering formal groups in operads goes back to an important paper of Lazard [43]. He used the concept of "analyseur" which is essentially equivalent to the modern notion of operad. The main result of Lazard is a version of Lie theory (=correspondence between (formal) Lie groups and Lie algebras) for his generalized formal groups. It turns out that Lazard-Lie theory has a very transparent interpretation in terms of Koszul duality for operads.

We start with formulating basic definitions in the modern language. Let $\mathscr{P}$ be a k-linear operad. We assume $\mathscr{P}(1) = k$. Let W be a k-vector space and

$$\hat{F}_{\mathscr{P}}(W) = \prod_{n \geqslant 1} (\mathscr{P}(n) \otimes W^{\otimes n})_{\Sigma_n}, \tag{2.2.15}$$

the completed free $\mathscr{P}$-algebra on W. If $z_1, \dots, z_r$ form a basis of W, then elements of $\hat{F}_{\mathscr{P}}(W)$ can be regarded as formal series in $z_1, \dots, z_r$ whose terms are products of z_i with respect to operations in $\mathscr{P}$. Thus, for example, if $\mathscr{P} = \mathscr{C}om$, we get the usual power series algebra; if $\mathscr{P} = \mathscr{A}s$, we get the algebra of noncommutative power series etc. Note that our series do not have constant terms since all the free algebras are without unit.

There is a natural projection (on the first factor)

$$\hat{F}_{\mathscr{P}}(W) \to W. \tag{2.2.16}$$

It can be thought of as the differential at zero.

Let V, W be two k-vector spaces. We define the space of *formal $\mathscr{P}$-maps* from V to W as

$$\mathrm{FHom}_{\mathscr{P}}(V, W) = V \otimes \hat{F}_{\mathscr{P}}(W^*).$$

For $\Phi \in \mathrm{FHom}_{\mathscr{P}}(W)$, we denote by $d_0\Phi \in \mathrm{Hom}_k(V, W)$ its differential at 0, i.e., the image of Φ under the natural projection of $V \otimes W^*$ induced by (2.2.16). There are obvious composition maps

$$\mathrm{FHom}_{\mathscr{P}}(V, W) \times \mathrm{FHom}_{\mathscr{P}}(W, X) \to \mathrm{FHom}_{\mathscr{P}}(V, X)$$

(given by inserting of power series into arguments of other power series) which make the collection of vector spaces and formal $\mathscr{P}$-maps into a category. A simple generalization of the classical inverse function theorem shows that $\Phi \in \mathrm{FHom}_{\mathscr{P}}(V, W)$ is invertible if and only if $d_0\Phi \in \mathrm{Hom}_k(V, W)$ is invertible.

(2.2.17) *Definition.* Let $\mathscr{P}$ be a k-linear operad. A $\mathscr{P}$-formal group is a pair (V, Φ) where V is a k-vector space and $\Phi \in \mathrm{FHom}_{\mathscr{P}}(V \oplus V, V)$ is a formal $\mathscr{P}$-map satisfying two conditions:

(i) $d_0\Phi\colon V \oplus V \to V$ is the addition map $(v, v') \mapsto v + v'$.

(ii) Φ is associative, i.e., we have the equality $\Phi(x, \Phi(y, z)) = \Phi(\Phi(x, y), z)$ of formal maps $V \oplus V \oplus V \to V$.

A formal homomorphism of formal $\mathscr{P}$-groups (V, Φ) and (W, Ψ) is a formal $\mathscr{P}$-map $f \in \mathrm{FHom}_{\mathscr{P}}(V, W)$ such that $f(\Phi(x, y)) = \Psi(f(x), f(y))$.

(2.2.18) If (V, Φ) is a $\mathscr{P}$-formal group, then we define, following Lazard, its Lie bracket by

$$[x, y] = \Phi(x, y) - \Phi(y, x) \qquad \text{(mod. cubic terms)}.$$

This construction makes V into a Lie algebra in $\mathscr{P}$ in the sense of (2.2.12). Furthermore, we obtain, from Theorem 2.2.13, Corollary 2.2.9, and the result of Lazard ([43], Theorem 7.1), the following theorem.

(2.2.19) THEOREM. *Let $\mathscr{P}$ be a quadratic operad, and $\mathscr{P}^!$ its quadratic dual. The category of finite-dimensional $\mathscr{P}$-formal groups (and their formal homomorphisms) is equivalent to the category of finite-dimensional $\mathscr{P}^!$-algebras.*

A special case of this theorem corresponding to $\mathscr{P} = \mathscr{A}s$, the associative operad, was pointed out to us earlier by M. Kontsevich. Since $\mathscr{A}s^! = \mathscr{A}s$, the theorem in this case says that, for formal groups defined by means of power series with noncommuting variables, the role of Lie algebras is played by associative algebras (possibly without unit).

2.3. Quadratic algebras over a quadratic operad

(2.3.1) Let $\mathscr{P}$ be a k-linear operad and A a $\mathscr{P}$-algebra. An *ideal* in A is a linear subspace $I \subset A$ such that, for any n, any $\mu \in \mathscr{P}(n)$, and any $a_1, \ldots, a_{n-1} \in A$, $i \in I$, we have $\mu(a_1, \ldots, a_{n-1}, i) \in I$. Given any ideal $I \subset A$, the quotient vector space A/I has a natural structure of a $\mathscr{P}$-algebra.

(2.3.2) *Quadratic algebras.* Let $\mathscr{P} = \mathscr{P}(K, E, R)$ be a quadratic operad (2.1.7) so that $K = \mathscr{P}(1)$ is a semisimple k-algebra and $E = \mathscr{P}(2)$ is a $(K, K^{\otimes 2})$-bimodule. Let also V be a K-bimodule. The tensor product $E \otimes_{K^{\otimes 2}} V^{\otimes 2}$ has a natural structure of a $(K, K^{\otimes 2})$-bimodule. (The left K-action comes from that on E and the right $K^{\otimes 2}$-action comes from that on $V^{\otimes 2}$.) Moreover, there is a Σ_2-action on $E \otimes_{K^{\otimes 2}} V^{\otimes 2}$ given by

$$\sigma(e \otimes (v_1 \otimes v_2)) = \sigma(e) \otimes (v_2 \otimes v_1). \tag{2.3.3}$$

Observe that the space of coinvariants $(E \otimes_{K^{\otimes 2}} V^{\otimes 2})_{\Sigma_2}$ inherits a K-bimodule structure. Let

$$S \subset (E \otimes_{K^{\otimes 2}} V^{\otimes 2})_{\Sigma_2} \tag{2.3.4}$$

be a K-sub-bimodule. Given V and S, we construct a $\mathscr{P}$-algebra $A = A(V, S)$ as follows.

Let $F_{\mathscr{P}}(V)$ be the free $\mathscr{P}$-algebra generated by V; see (1.3.5). Observe that $F_{\mathscr{P}}(V)_2$, the degree-2 graded component of $F_{\mathscr{P}}(V)$, is $(E \otimes_{K^{\otimes 2}} V^{\otimes 2})_{\Sigma_2}$. Let $(S) \subset \mathscr{F}_{\mathscr{P}}(V)$ be the ideal generated by S (= the minimal ideal containing S). Put

$$A(V, S) = F_{\mathscr{P}}(V)/(S). \tag{2.3.5}$$

An algebra of this type will be called a *quadratic $\mathscr{P}$-algebra* (with the space of generators V and the space of relations S).

Observe that $A(V, S)$ has a natural grading $A(V, S) = \bigoplus_{i \geq 1} A_i(V, S)$. Furthermore, $\mathscr{P}$, as any operad in the category Vect, can be regarded as an operad in the category $g\mathrm{Vect}^+$ of graded vector spaces (1.4.2) (c) and $A(V, S)$ is a $\mathscr{P}$-algebra in this category.

(2.3.6) *Quadratic superalgebras.* Let $\mathscr{P} = \mathscr{P}(K, E, R)$ be a quadratic operad as before. We now view $\mathscr{P}$ as an operad in the other category of graded vector spaces (1.4.2), namely $g\mathrm{Vect}^-$. A $\mathscr{P}$-algebra in this category will be referred to as a (graded) *$\mathscr{P}$-superalgebra.*

Let V be a K-bimodule. We replace the Σ_2-action on $E \otimes_{K^{\otimes 2}} V^{\otimes 2}$ given by (2.3.3) by the following one:

$$\sigma(e \otimes (v_1 \otimes v_2)) = -\sigma(e) \otimes (v_2 \otimes v_1). \tag{2.3.7}$$

Let $S \subset (E \otimes_{K^{\otimes 2}} V^{\otimes 2})_{\Sigma_2}$ be a K-sub-bimodule where the coinvariants are taken with respect to the new action (2.3.7). Given such data (V, S), we define a $\mathscr{P}$-

superalgebra $A(V, S)^-$ as the quotient of the free $\mathcal{P}$-superalgebra generated by V (placed in degree 1) by the ideal generated by S.

(2.3.8) *Quadratic duality.* Let $\mathcal{P} = \mathcal{P}(K, E, R)$ be a quadratic operad and $\mathcal{P}^! = \mathcal{P}(K^{op}, E^\vee, R^\perp)$ the dual operad (2.1.9). Given a quadratic $\mathcal{P}$-algebra $A = A(V, S)$ we define the quadratic $\mathcal{P}^!$-superalgebra $A^! = A(V^\vee, S^\perp)^-$. Here $V^\vee = \mathrm{Hom}_K(V, K)$ and

$$S^\perp \subset ((E \otimes_{K^{\otimes 2}} V^{\otimes 2})_{\Sigma_2})^\vee = ((V^\vee)^{\otimes 2} \otimes_{K^{\otimes 2}} E^\vee)^{\Sigma_2} = (E^\vee \otimes_{(K^{op})^{\otimes 2}} (V^\vee)^{\otimes 2})_{\Sigma_2}$$

is the annihilator of S.

In a similar way, given a quadratic $\mathcal{P}$-superalgebra B, we define the dual $\mathcal{P}^!$-algebra (in the category $g\mathrm{Vect}^+$) $B^!$. The assignment $A \mapsto A^!$ gives a 1-1 correspondence between quadratic $\mathcal{P}$-algebras (resp. superalgebras) and quadratic $\mathcal{P}^!$-superalgebras (resp. algebras).

(2.3.9) *Examples.* (a) Let $\mathcal{P} = \mathcal{A}s$ be the associative operad. Then $\mathcal{A}s^! = \mathcal{A}s$. Note that an $\mathcal{A}s$-superalgebra (i.e., an $\mathcal{A}s$-algebra in $g\mathrm{Vect}^-$) is the same as an $\mathcal{A}s$-algebra (in $g\mathrm{Vect}^+$). Both concepts give the usual notion of a graded associative algebra. In this case the quadratic duality (2.3.8) reduces to the well-known Koszul duality for quadratic associative algebras introduced by Priddy [55].

(b) Let $\mathcal{P} = \mathcal{C}om$ be the commutative algebra, and let A be a quadratic $\mathcal{C}om$-algebra i.e., a quadratic commutative (associative) algebra. Let $A^!_{as}$ be the Koszul dual of A as of an associative algebra. One can show that $A^!_{as}$ has a structure of a graded-commutative Hopf algebra (in fact, it is a subalgebra in $\mathrm{Ext}^*_A(k, k)$, for which see [57]). Hence $A^!_{as}$ is the enveloping algebra of a certain graded Lie superalgebra (i.e., a $\mathcal{L}ie$-algebra in the category $g\mathrm{Vect}^-$), which we denote $\mathcal{G} = \mathrm{Prim}(A^!_{as})$. We have $A^! = \mathcal{G}$.

(2.3.10) *Quadratic duality and enveloping algebras.* Let $\mathcal{P} = \mathcal{P}(K, E, R)$ be a quadratic operad. Let A be a quadratic $\mathcal{P}$-algebra. Then we have (1.6.4) the universal enveloping algebra $U_{\mathcal{P}}(A)$, which is an associative algebra in ordinary sense. If A is a quadratic $\mathcal{P}$-superalgebra, the same constructon as in (1.6.4) defines its universal enveloping superalgebra $U^-_{\mathcal{P}}(A)$, which is an $\mathcal{A}s$-algebra in the category $g\mathrm{Vect}^-$. As noted in (2.3.9) (a), we can regard $U^-_{\mathcal{P}}(A)$ as an ordinary graded associative algebra.

(2.3.11) THEOREM. *If A is a quadratic $\mathcal{P}$-algebra and $A^!$ the dual quadratic $\mathcal{P}^!$-superalgebra, then the universal enveloping algebras $U_{\mathcal{P}}(A)$, $U^-_{\mathcal{P}^!}(A^!)$ are quadratic associative algebras in the ordinary sense and*

$$(U_{\mathcal{P}}(A))^! = U^-_{\mathcal{P}^!}(A^!).$$

Proof. Let $\mathcal{P} = \mathcal{P}(K, E, R)$ and $A = A(V, S)$. The algebra $U_{\mathcal{P}}(A)$ has an obvious grading in which the generator $X(\lambda; a_1, \ldots, a_{n-1})$, $\lambda \in \mathcal{P}(n)$, $a_i \in A_{m_i}$, has degree $m_1 + \cdots + m_{n-1}$. The degree-1 component of $U_{\mathcal{P}}(A)$ is linearly spanned by $X(\lambda, a)$, $\lambda \in \mathcal{P}(2) = E$, $a \in A_1 = V$ and is isomorphic to $E \otimes_K V$. Obviously this component generates $U_{\mathcal{P}}(A)$ as an algebra.

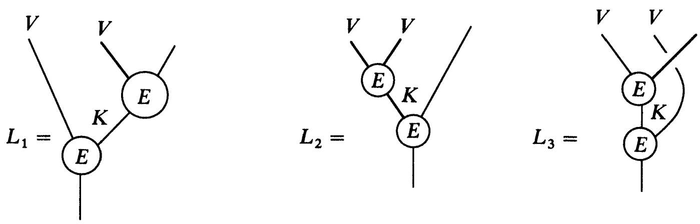

FIGURE 8

To describe the relations among these generators, consider the space $Y = F(E)(3) \otimes V^{\otimes 2}$. (Here and in the remainder of this seciton all tensor products are taken over K.) This space splits into the direct sum of three components $Y = L_1 \oplus L_2 \oplus L_3$ depicted in Figure 8.

The letter "K" on some of the edges in Figure 8 means that the tensor product over K is taken with respect to the structures of left/right K-module represented by the ends of this edge. Note that every L_i and hence X is a K-bimodule with respect to the actions corresponding to the edges not marked "K". Elements of L_1 can be viewed as formal expressions $\sum X(\lambda_i; a_i)X(\mu_i; b_i)$, as can elements of L_3. The group Σ_2, permuting the left two inputs of the trees in Figure 8, maps L_1 isomorphically to L_3 and preserves L_2. Thus, denoting by $X = Y_{\Sigma_2}$ the space of coinvariants, we have

$$X = E \otimes V \otimes E \otimes V \quad \oplus \quad E \otimes (E \otimes V^{\otimes 2})_{\Sigma_2}.$$

Let $W \subset X$ be the image, under the canonical projection $Y \to X$, of the subspace $R \otimes V^{\otimes 2}$, where $R \subset F(E)(3)$ is the space of relations of $\mathscr{P}$.

(2.3.12) PROPOSITION. *The algebra $U_{\mathscr{P}}(A)$ is defined by the space of generators $E \otimes V$ and the space of quadratic relations*

$$E \otimes V \otimes E \otimes V \cap (W + E \otimes S) \tag{2.3.13}$$

where $S \subset (E \otimes V^{\otimes 2})_{\Sigma_2}$ is the space of relations in A and the intersection is taken inside X.

The proof is straightforward and left to the reader.

Similarly, the algebra $U_{\mathscr{P}^!}(A^!)$ has the space of generators $E^\vee \otimes V^\vee$ and the space of relations

$$E^\vee \otimes V^\vee \otimes E^\vee \otimes V^\vee \cap (W^\perp + E^\vee \otimes S^\perp). \tag{2.3.14}$$

We want to show that (2.3.13) and (2.3.14) are the orthogonal complement to each other. This is a particular case of the following general lemma.

(2.3.15) LEMMA. *Let X be any K-bimodule decomposed into a direct sum of bimodules $X = M \oplus N$. Let $X^\vee = M^\vee \oplus N^\vee$ be the corresponding decomposition of $X^\vee = \mathrm{Hom}_K(X, K)$. Let $W \subset X$ and $L \subset N$ be any sub-bimodules. Then the orthogonal complement (in $M^\vee$) of $M \cap (W + L)$ coincides with $M^\vee \cap (W_X^\perp + L_N^\perp)$, where $W_X^\perp$ and $L_N^\perp$ are the orthogonal complements in X and N, respectively.*

Proof of the lemma. Since K is semisimple, we can write $N = L \oplus P$ where P is another sub-bimodule, so $X = M \oplus L \oplus P$ and $X^\vee = M^\vee \oplus L^\vee \oplus P^\vee$. If $Z \subset X$ is any sub-bimodule, then

$$(2.3.16) \quad (M \cap Z)_M^\perp = \mathrm{Im}\{(M \cap Z)_X^\perp \xrightarrow{\pi} M^\vee\} = M^\vee \cap ((M \cap Z)_X^\perp + L^\vee + P^\vee),$$

where $\pi: X^\vee \to M^\vee$ is the projection dual to the embedding $M \hookrightarrow X$, i.e., the projection along $L^\vee \oplus P^\vee$. Let us apply this to $Z = W + L$ and note that

$$(M \cap (W + L))_X^\perp = M_Z^\perp + (W_X^\perp \cap L_X^\perp) = L^\vee + P^\vee + (W_X^\perp \cap (M^\vee + P^\vee)).$$

We get that the right-hand side of (2.3.16) is equal to

$$(2.3.17) \qquad M^\vee \cap (W_X^\perp \cap (M^\vee + P^\vee) + L^\vee + P^\vee).$$

To finish the proof of the lemma, it remains to show that (2.3.17) coincides with $M^\vee \cap (W_X^\perp + P^\vee)$. (Note that $P^\vee$ is the same as $L_N^\perp$.) To show this, suppose that $m = w + p \in M^\vee \cap (W_X^\perp + P^\vee)$, so that $w \in W_X^\perp$, $p \in P^\vee$. Then $w = m - p \in M^\vee + P^\vee$, so writing $m = w + 0 + p$, we get that m belongs to (2.3.17). Conversely, let m belong to (2.3.17), so $m = w + l + p$ with $l \in L^\vee$, $p \in P^\vee$ and $w \in W_X^\perp \cap (M^\vee + P^\vee)$, so $w = m' + p'$, $m' \in M^\vee$, $p' \in P^\vee$. Then we have $m = m' + p' + l + p$ and, since $X^\vee = M^\vee \oplus L^\vee \oplus P^\vee$, we get $m = m'$, $p + p' = 0$, $l = 0$. Thus $m = w + p$, so $m \in M^\vee \cap (W_X^\perp + P^\vee)$. Lemma 2.3.15 and (hence) Theorem 2.3.11 are proven.

3. Duality for *dg*-operads

3.1. dg-operads. Generating maps

(3.1.1) Let dg Vect be the symmetric monoidal category of differential graded (dg-) vector spaces, i.e., of complexes over the base field k. By definition, an object of dg Vect is a graded vector space $V^\bullet$ together with a linear map (differential) $d: V^\bullet \to V^\bullet$ of degree 1 such that $d^2 = 0$. Morphisms are linear maps preserving gradings and differentials. The tensor product of two complexes $V^\bullet$ and $W^\bullet$ is defined by (1.4.3) with the differential given by the Leibniz rule

$$d(v \otimes w) = d(v) \otimes w + (-1)^i v \otimes d(w), \qquad v \in V^i, w \in W^j.$$

The symmetry isomorphism $V^\bullet \otimes W^\bullet \to W^\bullet \otimes V^\bullet$ is the same as in the category $g\mathrm{Vect}^-$; see (1.4.2 (c)). As usual, for a complex $V^\bullet \in dg$ Vect we define the dual complex V^* by

$$(V^*)^i = (V^{-i})^*, \qquad d_{V^*} = (d_V)^* \tag{3.1.2}$$

and the shifted complex $V^\cdot[i]$, $\subset \in \mathbf{Z}$, by

$$(V^\cdot[i])^j = V^{i+j}, \qquad d_{V^\cdot[i]} = (-1)^i d_V. \tag{3.1.3}$$

An operad in the category *dg* Vect is called a *dg-operad* (over k). An algebra over a *dg*-operad in the category *dg* Vect will be called a *dg*-algebra. Note that any k-linear operad $\mathscr{P}$ can be regarded as a *dg*-operad (each $\mathscr{P}(n)$ is placed in degree 0).

(3.1.4) For any *dg*-operad $\mathscr{P}$, the collection of cohomology vector spaces $H^\cdot\mathscr{P}(n)$ forms an operad $H^\cdot(\mathscr{P})$ in the category $g\mathrm{Vect}^-$. A morphism of *dg*-operads $f\colon \mathscr{P} \to \mathscr{Q}$ is called a *quasi isomorphism* if the induced morphism $H^\cdot(f)\colon H^\cdot(\mathscr{P}) \to H^\cdot(\mathscr{Q})$ is an isomorphism. Similarly for *dg*-algebras over *dg*-operads.

(3.1.5) A *dg*-operad $\mathscr{P}$ will be called *admissible* if the following conditions hold:

(i) Each $\mathscr{P}(n)$ is a finite-dimensional *dg*-vector space.

(ii) The space $\mathscr{P}(1)$ is concentrated in degree 0, and is a semisimple k-algebra.

Given a semisimple k-algebra K, we denote by $dg\ OP(K)$ the category of admissible *dg*-operads $\mathscr{P}$ with $\mathscr{P}(1) = K$ and with morphisms equal to the identity on first components.

(3.1.6) Our next aim is to define a kind of "generating function" for an admissible *dg*-operad $\mathscr{P}$. It will be convenient for the future to work in a slightly greater generality. Let K be a semisimple k-algebra. By a K–*dg*-collection we mean a collection $E = \{E(n), n \geqslant 2\}$ of finite-dimensional *dg*-vector spaces $E(n)$ together with a left Σ_n-action and a structure of $(K, K^{\otimes n})$-bimodule on each $E(n)$ which satisfy the compatibility condition identical to the one given in (1.2.11).

Clearly, if $\mathscr{P}$ is an admissible *dg*-operad and $K = \mathscr{P}(1)$, then $\mathscr{P}(n)$, $n \geqslant 2$, form a K–*dg*-collection. Given any K–*dg*-collection E, one defines the *free dg-operad* $F(E)$ as in (2.1.1).

By an *r-fold dg-collection*, we mean, similarly to (1.3.15), a collection of finite-dimensional complexes

$$E^i(a_1, \ldots, a_r), \qquad a_i \in \mathbf{Z}_+, i = 1, \ldots, r, \qquad \sum a_i \geqslant 1$$

of $\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}$-modules such that

$$E^i(0, \ldots, 0, \underbrace{1}_{j}, 0, \ldots, 0) = \begin{cases} k & \text{for } i = j; \\ 0 & \text{for } i \neq j. \end{cases}$$

Given a semisimple k-algebra K with r simple summands and a K–*dg*-collection E, we define the r-fold *dg*-collection associated to E by the formula identical to (1.3.13).

(3.1.8) *Definition.* Let K be a semisimple k-algebra, r the number of simple summands in K, and E a K–dg-collection. The generating map of E is the r-tuple of formal power series

$$(3.1.9)\quad g_E^{(i)}(x_1, \dots, x_r) = \sum_{a_1, \dots, a_r = 0}^{\infty} \chi[E^i(a_1, \dots, a_r)] \frac{x_1^{a_1}}{a_1!} \cdot \dots \cdot \frac{x_r^{a_r}}{a_r!}, \qquad i = 1, \dots, r$$

where $\{E^i(a_1, \dots, a_r)\}$ is the r-fold dg-collection associated to E and χ stands for the Euler characteristic.

The r-tuple $g_E(x) = (g_E^{(1)}(x), \dots, g_E^{(r)}(x))$ will be regarded as a formal map

$$(3.1.10)\qquad g_E\colon \mathbf{C}^r \to \mathbf{C}^r, \qquad x = (x_1, \dots, x_r) \mapsto (g_E^{(1)}(x), \dots, g_E^{(r)}(x)).$$

Note that by (3.1.7) we have

$$(3.1.11)\qquad g_E^{(i)}(x) = x_i + (\text{higher-order terms}).$$

In particular, for any admissible dg-operad $\mathscr{P}$, we have its generating map $g_{\mathscr{P}}$.

Special case. If $\mathscr{P}$ *is a* k-linear operad with $\mathscr{P}(1) = k$, then its generating map is a single power series

$$g_{\mathscr{P}}(x) = \sum_{n=1}^{\infty} \dim \mathscr{P}(n) \frac{x^n}{n!}.$$

(3.1.12) *Examples.* (a) The operads $\mathscr{A}s$, $\mathscr{C}om$, and $\mathscr{L}ie$ (see (1.3.7)–(1.3.9)) are admissible operads with $K = k$ and trivial dg-structure. Therefore $r = 1$ and the generating maps of these operads are the following power series in one variable:

$$g_{\mathscr{A}s}(x) = \frac{1}{1-x} - 1, \qquad g_{\mathscr{C}om}(x) = e^x - 1, \qquad g_{\mathscr{L}ie}(x) = -\log(1-x).$$

(b) Let $\mathscr{A}$ be the symmetric monoidal category of representations of the group $\mathbf{Z}/2$. It has two (1-dimensional) irreducible objects: the trivial representation I and the sign representation J. Taking $X = I \oplus J$, we define an admissible k-linear operad $\mathscr{P}$ with $\mathscr{P}(n) = \mathrm{Hom}_{\mathscr{A}}(X^{\otimes n}, X)$; see (1.3.12). We have $\mathscr{P}(1) = k \oplus k$. As explained in (1.3.16), the 2-fold collection associated to $\mathscr{P}$ consists of Clebsch-Gordan spaces

$$\mathrm{Hom}_{\mathscr{A}}(I^{\otimes a} \otimes J^{\otimes b}, I) = \begin{cases} 0 & \text{for } b \text{ odd} \\ k & \text{for } b \text{ even} \end{cases}$$

$$\mathrm{Hom}_{\mathscr{A}}(I^{\otimes a} \otimes J^{\otimes b}, J) = \begin{cases} 0 & \text{for } b \text{ even} \\ k & \text{for } b \text{ odd}. \end{cases}$$

Therefore the generating map of $\mathcal{P}$ is given by the formulas

$$g_{\mathcal{P}}^{(1)}(x_1, x_2) = \sum_{a,b} \dim \operatorname{Hom}_{\mathcal{A}}(I^{\otimes a} \otimes J^{\otimes b}, I) \frac{x_1^a}{a!} \frac{x_2^b}{b!} = e^{x_1} \cosh(x_2) - 1,$$

$$g_{\mathcal{P}}^{(2)}(x_1, x_2) = \sum_{a,b} \dim \operatorname{Hom}_{\mathcal{A}}(I^{\otimes a} \otimes J^{\otimes b}, J) \frac{x_1^a}{a!} \frac{x_2^b}{b!} = e^{x_1} \sinh(x_2).$$

(3.1.13) It is possible to refine the generating map of a K–dg-collection so as to take into account not only the dimensions of the graded components of the complexes $E^i(a_1, \ldots, a_r)$ but also the symmetric groups actions.

Let $\mathcal{R}_n$ be the Grothendieck group of the category of finite-dimensional representations of the symmetric group Σ_n over k. For any such representation V we denote by $[V]$ its class in $\mathcal{R}_n$. If $V^{\bullet}$ is a complex of S_n-representations, then we set $[V^{\bullet}] = \sum (-1)^i [V^i]$. The correspondence $[V] \mapsto \dim(V)$ extends to a group homomorphism $\mathcal{R}_n \to \mathbf{Z}$ also denoted by dim. The maps

$$\text{(3.1.14)} \qquad \mathcal{R}_m \otimes \mathcal{R}_n \to \mathcal{R}_{m+n}, \qquad [V] \otimes [W] \mapsto [\operatorname{Ind}_{\Sigma_m \times \Sigma_n}^{\Sigma_{m+n}}(V \otimes W)]$$

make $\mathcal{R} = \prod_{n \geqslant 0} \mathcal{R}_n$ into a commutative graded ring. For $n = 0$, we set $\mathcal{R}_0 = \mathbf{Z}$, and $1 \in \mathbf{Z}$ is the unit of $\mathcal{R}$. The class of trivial representation of Σ_1 will be denoted I. Elements of $\mathcal{R}$ will be written as formal infinite sums $\sum_{n=0}^{\infty} v_n$ with $v_n \in \mathcal{R}_n$. For every k–dg-collection $E = \{E^{\bullet}(n)\}$, we denote by $[E] = \sum_{n=0}^{\infty} [E(n)]$ the corresponding element of $\mathcal{R}$. It is well known [50] that $\mathcal{R}$ is isomorphic to the completion of the ring $\mathbf{Z}[e_1, e_2, \ldots]$ of symmetric functions in infinitely many variables (that latter ring is the direct sum $\bigoplus_{n \geqslant 0} \mathcal{R}_n$). There is a natural ring homomorphism $h: \mathcal{R} \to \mathbf{Z}[[x]]$ defined by

$$\text{(3.1.15)} \qquad h\left(\sum v_n\right)(x) = \sum \dim(v_n) \frac{x^n}{n!}, \qquad a_n \in \mathcal{R}_n.$$

Note that for the element $I \in \mathcal{R}$ above we have $h(I) = x$. For every k–dg-collection $E = \{E^{\bullet}(n)\}$ such that $E^{\bullet}(n) = 0$ for $n \gg 0$, we have a polynomial functor

$$\Phi_E: W \mapsto \bigoplus (E(n) \otimes W^{\otimes n})_{\Sigma_n}$$

on the category of complexes; cf. [50], Chapter 1, Appendix. If E and F are two such collections, then the composition $\Phi_E \Phi_F$ is a polynomial functor corresponding to a new collection $E \circ F$ called the *plethysm* of E and F, see loc. cit. and [23], n.1.2. The element $[E \circ F] \in \mathcal{R}$ is determined by $[E]$ and $[F]$ alone; in other words, the plethysm descends to an operation on $\mathcal{R}$ which will still be denoted by $u \circ v$; see [50], Chapter I, §8.

(3.1.16) The tensor product $\mathscr{R}_{a_1} \otimes \cdots \otimes \mathscr{R}_{a_r}$ is naturally identified with the Grothendieck group of representations of the Cartesian product $\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}$. If $V^\bullet$ is any finite-dimensional complex of $\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}$-modules, then by $[V^\bullet] \in \mathscr{R}^{\otimes r}$ we denote the alternating sum of classes of V^i in $\mathscr{R}_{a_1} \otimes \cdots \otimes \mathscr{R}_{a_r} \subset \mathscr{R}^{\otimes r}$. Let $\hat{\mathscr{R}}^{\otimes r} = \prod \mathscr{R}_{a_1} \otimes \cdots \otimes \mathscr{R}_{a_r}$ be the completed rth tensor power of the ring $\mathscr{R}$. Its elements will be written as formal r-fold sums $\sum v_{a_1,\ldots,a_r}$ with $v_{a_1,\ldots,a_r} \in \mathscr{R}_{a_1} \otimes \cdots \otimes \mathscr{R}_{a_r}$. We introduce for later use the elements

$$I_\nu = 1 \otimes \cdots \otimes 1 \otimes I \otimes 1 \otimes \cdots \otimes 1 \in \mathscr{R}_0 \otimes \cdots \otimes \mathscr{R}_1 \otimes \cdots \otimes \mathscr{R}_0$$

where $I \in \mathscr{R}_1$ (see (3.1.14)) is on the νth place.

If $E = \{E^i(a_1, \ldots, a_r)\}$ is an r-fold dg-collection, then we get a vector of length r over $\hat{\mathscr{R}}^{\otimes r}$, namely

$$(3.1.17)\quad [E] = \left(\sum_{a_1,\ldots,a_r} [E^1(a_1, \ldots, a_r)], \ldots, \sum_{a_1,\ldots,a_r} [E^r(a_1, \ldots, a_r)] \right) \in \hat{\mathscr{R}}^{\otimes r} \times \cdots \times \hat{\mathscr{R}}^{\otimes r}.$$

If E is an r-fold dg-collection with only finitely many nonzero terms, we have a polynomial functor Φ_E on the category dg Vect^r of r-tuples of complexes. Namely, if $W_1, \ldots, W_r$ are complexes, then the ith component of the r-tuple $\Phi_E(W_1, \ldots, W_r)$ is

$$\bigoplus_{a_1,\ldots,a_r} (E^i(a_1, \ldots, a_r) \otimes W_1^{\otimes a_1} \otimes \cdots \otimes W_r^{\otimes a_r})_{\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}}.$$

As before, composition of polynomial functors on dg Vect^r induces the plethysm $(E, F) \mapsto E \circ F$ on r-fold dg-connections. The vector $[E \circ F]$ can be expressed through $[E]$ and $[F]$ alone by means of the binary operation on the r-fold product $\hat{\mathscr{R}}^{\otimes r} \times \cdots \times \hat{\mathscr{R}}^{\otimes r}$ which we also call the plethysm and denote $(u, v) \mapsto u \circ v$.

(3.1.18) Let now K be a semisimple k-algebra with r simple summands, and let E be a K–dg-collection. We define the *refined generating map* of E to be the r-tuple

$$G_E = (G_E^1, \ldots, G_E^r) = [E']$$

where E' is the r-fold dg-collection associated to E; see (3.1.6) and (1.3.13).

Let $h_r\colon \hat{\mathscr{R}}^{\otimes r} \to \mathbf{Z}[[x_1, \ldots, x_r]]$ be the rth tensor power of the homomorphism (3.1.15), so that, for instance, $h_r(I_\nu) = x_\nu$. Then the numerical generating map of E can be recovered as

$$(3.1.19)\qquad g_E^{(i)}(x_1, \ldots, x_r) = h_r(G_E).$$

Let $\omega\colon \mathscr{R} \to \mathscr{R}$ be the ring homomorphism which for every n takes $[\mathbb{1}_n]$, the class of the trivial representation of Σ_n into $[\mathrm{sgn}_n]$, the class of the sign representation; see [50]. Let $w^{\otimes r}\colon \hat{\mathscr{R}}^{\otimes r} \to \hat{\mathscr{R}}^{\otimes r}$ be the tensor power of ω.

For an element $v = \sum v_{a_1,\ldots,a_r} \in \hat{\mathscr{R}}^{\otimes r}$, we denote

$$\varepsilon(v) = \sum (-1)^{a_1+\cdots+a_r} \omega^{\otimes r} v_{a_1,\ldots,a_r}.$$

If $f(x) = f(x_1, \ldots, x_r) = h_r(v) \in \mathbf{Z}[[x_1, \ldots, x_r]]$, then $h_r(\varepsilon(v))$ is the series $f(-x)$.

3.2. The cobar-duality

(3.2.0) For a finite-dimensional k-vector space V, we denote by $\mathrm{Det}(V)$ the top exterior power of V.

Let T be a tree (1.1.1). We denote by $\mathrm{Ed}(T)$ the set of all edges of T except the output edge $\mathrm{Out}(T)$. We denote by $\mathrm{Det}(T)$ the 1-dimensional vector space $\mathrm{Det}(k^{\mathrm{Ed}(T)})$. Similarly, let $\mathrm{ed}(T)$ be the set of internal edges of T and let $\det(T) = \mathrm{Det}(k^{\mathrm{ed}(T)})$. The number of internal edges of T will be denoted by $|T|$.

(3.2.1) Let $\mathscr{P}$ be an admissible dg-operad (3.1.5), so that $K = \mathscr{P}(1)$ is a semi-simple k-algebra. For any $n \geqslant 2$, we construct a complex

$$\mathscr{P}(n)^* \otimes \det(k^n) \xrightarrow{\delta} \bigoplus_{\substack{n\text{-trees } T\\ |T|=1}} \mathscr{P}(T)^* \otimes \det(T) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigoplus_{\substack{n\text{-trees } T\\ |T|=n-2}} \mathscr{P}(T)^* \otimes \det(T) \tag{3.2.2}$$

where the sums are over isomorphism classes of (reduced) n-trees, $\mathscr{P}(T)$ was defined in (1.2.13), and $\mathscr{P}(T)^*$ means the dual vector space. The differential δ is defined by its matrix elements

$$\delta_{T',T}: \mathscr{P}(T')^* \otimes \det(T') \to \mathscr{P}(T)^* \otimes \det(T) \tag{3.2.3}$$

where T, T' are n-trees, $|T| = i$, $|T'| = i - 1$. By definition, $\delta_{T',T} = 0$ unless $T' = T/e$ is obtained from T by contracting an internal edge e. If this is the case, then we set

$$\delta_{T',T} = (\gamma_{T,T'})^* \otimes l_e \tag{3.2.4}$$

where $\gamma_{T,T'}$ is the composition map from (1.2.15) and the map $l_e: \det(T') \to \det(T)$ is defined by the formula

$$l_e(f_1 \wedge \cdots \wedge f_m) = e \wedge f_1 \wedge \cdots \wedge f_m.$$

In this formula we use the natural identification $\mathrm{ed}(T) = \mathrm{ed}(T') \cup \{e\}$ and regard e as a basis vector of $k^{\mathrm{ed}(T)}$.

(3.2.5) It is straightforward to verify that $\delta^2 = 0$, i.e., (3.2.2) is a complex. We normalize the grading of this complex by placing the sum over T with $|T| = i$ in degree $i + 1$.

Observe that, for a dg-operad $\mathscr{P}$, each term of (3.2.2) is a dg-vector space whose differential we denote by d. Clearly d commutes with δ, so (3.2.2) is a complex of dg-vector spaces, i.e., a double complex.

(3.2.6) We now define a collection of *dg*-vector spaces $C(\mathscr{P})(n)$, $n \geqslant 1$. For $n = 1$ we put $C(\mathscr{P})(1) = K^{op}$, placed in degree 0 (with trivial differential). For $n > 1$ we define $C(\mathscr{P})(n)$ to be the total complex (= *dg*-vector space) associated to the double complex (3.2.2).

(3.2.7) THEOREM. (a) *The collection* $C(\mathscr{P}) = \{C(\mathscr{P})(n), n \geqslant 1\}$ *has a natural structure of an admissible dg-operad.*

(b) *The correspondence* $\mathscr{P} \mapsto C(\mathscr{P})$ *extends to the contravariant functor* C: $dg\, OP(K) \to dg\, OP(K^{op})$ *(notation of (3.1.5)). This functor takes quasi isomorphisms to quasi isomorphisms.*

Proof. (a) The complexes $\mathscr{P}(n)$, $n \geqslant 2$, obviously form a K–*dg*-collection (3.1.6). Therefore the shifted dual complexes $\mathscr{P}(n)^*[-1]$ form a K^{op}–*dg*-collection (shift and duality are defined by (3.1.2)–(3.1.3)). We denote this collection by $\mathscr{P}^*[-1]$. Thus we can form the free *dg*-operad $F(\mathscr{P}^*[-1])$.

(3.2.8) LEMMA. *For any n, the complex* $F(\mathscr{P}^*[-1])(n)$ *is isomorphic to the total complex of the double complex (3.2.2), the latter taken with the internal differential d (induced by that on* $\mathscr{P}$*) only.*

The lemma is proved by immediate inspection. The only point that needs explanation is the appearance of vector spaces $\det(T)$ in (3.2.2). The reason for this is part (b) of the following lemma.

(3.2.9) LEMMA. (a) *Let I be a finite set of m elements, and let* $W_i^\bullet$, $i \in I$ *be dg-vector spaces. Then there is a canonical isomorphism*

$$\phi: \bigotimes_{i \in I} (W_i^\bullet[-1]) \xrightarrow{\approx} \left(\bigotimes_{i \in I} W_i^\bullet\right)[-m] \otimes \mathrm{Det}(k^I).$$

(b) *Let* $E = \{E(n), n \geqslant 2\}$, *be a K–dg-collection, let* $E[-1]$ *be the collection of shifted dg-vector spaces, and let T be a tree with m vertices. Then there is a canonical isomorphism*

$$E[-1](T) \cong E(T)[-m] \otimes \det(T).$$

Proof. (a) To define $\bigotimes_{i \in I} W_i^\bullet$, we should choose some ordering $(i_1, \ldots, i_m)$ on I and consider the product $W_{i_1}^\bullet \otimes \cdots \otimes W_{i_m}^\bullet$. Any other ordering will give the product related to this one by a uniquely defined isomorphism. The same is true for $\bigotimes W_i^\bullet[-1]$. Let now $w_i \in W_i^\bullet[-1]$ be some elements and let w_i' denote the same elements but regarded as elements of $W_i^\bullet$. We define

$$\phi(w_{i_1} \otimes \cdots \otimes w_{i_m}) = w_{i_1}' \otimes \cdots \otimes w_{i_m}' \otimes (i_1 \wedge \cdots \wedge i_m).$$

The proof that ϕ is independent on the choice of ordering $(i_1, \ldots, i_m)$ follows from the definition of the symmetric monoidal structure in *dg* Vect and is left to the reader.

(b) Let $\mathrm{Vert}(T)$ be the set of vertices of T. Let $\Gamma \subset T$ be the subtree consisting of all vertices and all internal edges. This is a contractible 1-dimensional simplicial complex, so taking its chain complex we get the exact sequence

$$0 \to k^{\mathrm{ed}(T)} \to k^{\mathrm{Vert}(T)} \to k \to 0$$

whence the space $\mathrm{Det}(k^{\mathrm{Vert}(T)})$ is canonically identified with $\det(T) = \mathrm{Det}(k^{\mathrm{ed}(T)})$. Now part (b) of the lemma follows from part (a), since $E(T) = \bigotimes_{v \in \mathrm{Vert}(T)} E(\mathrm{In}(v))$.

Lemmas 3.2.9 and 3.2.8 are proven.

(3.2.10) By Lemma 3.2.8, the compositions in the free operad $F(\mathscr{P}^*[-1])$ give rise to maps of graded vector spaces

$$C(\mathscr{P})(l) \otimes C(\mathscr{P})(m_1) \otimes \cdots \otimes C(\mathscr{P})(m_l) \to C(\mathscr{P})(m_1 + \cdots + m_l) \tag{3.2.11}$$

which satisfy the Leibniz rule with respect to internal differentials d in $C(\mathscr{P})(n)$. To complete the proof of Theorem 3.2.7 (a), it remains to check that the maps (3.2.11) also satisfy the Leibniz rule with respect to the differentials δ in $C(\mathscr{P})(n)$ defined in (3.2.3). We leave this straightforward checking to the reader. This completes the proof of part (a) of Theorem 3.2.7. Part (b) is straightforward.

(3.2.12) We call the *dg*-operad $C(\mathscr{P})$ the *cobar-construction* of $\mathscr{P}$. We define the *dual dg-operad* $\mathbf{D}(\mathscr{P})$ by

$$\mathbf{D}(\mathscr{P}) = C(\mathscr{P}) \otimes \Lambda = \{C(\mathscr{P})(n) \otimes \Lambda(n)\}, \tag{3.2.13}$$

where Λ is the determinant operad (1.3.21) and the product $\otimes$ is defined in (2.2.2). Recall that $\Lambda(n)$ is a 1-dimensional vector space in degree $(1 - n)$ with the sign action of Σ_n. Hence $\mathbf{D}(\mathscr{P})(n)$ is the *dg*-vector space associated to the following complex which differs from (3.2.2) by shifting the grading by $(1 - n)$ and by replacing det by Det:

$$\cdots \to \bigoplus_{\substack{n\text{-trees } T\\ |T|=n-3}} \mathscr{P}(T)^* \otimes \mathrm{Det}(T) \to \bigoplus_{\substack{n\text{-trees } T\\ |T|=n-2}} \mathscr{P}(T)^* \otimes \mathrm{Det}(T). \tag{3.2.14}$$

The grading in (3.2.14) is arranged so that the rightmost term is placed in degree 0.

It follows from the definitions that the correspondence $\mathscr{P} \mapsto \mathbf{D}(\mathscr{P})$ extends to a contravariant functor $\mathbf{D}$: $dg\ OP(K) \to dg\ OP(K^{op})$ taking quasi isomorphisms to quasi isomorphisms.

(3.2.15) *Example.* Let $\mathscr{P} = \mathscr{C}om$ be the commutative operad so that $\mathscr{C}om(n) = k$ for every n; see (1.3.8). The nth component of $\mathbf{D}(\mathscr{P})$ is the *tree complex*

$$\bigoplus_{|T|=0} \mathrm{Det}(T) \to \bigoplus_{|T|=1} \mathrm{Det}(T) \to \cdots \to \bigoplus_{|T|=n-2} \mathrm{Det}(T).$$

This is a special case of more general graph complexes considered by Kontsevich [39]. In the paper [9] it was proven by using Hodge theory that for $k = \mathbf{C}$ this complex is exact everywhere except the rightmost term, and the cokernel of the rightmost differential is naturally isomorphic to $\mathscr{L}ie(n)$, the nth space of the Lie operad. Thus we have an isomorphism $\mathbf{D}(\mathscr{C}om) \cong \mathscr{L}ie$. We give a purely algebraic proof of this later in §4.

(3.2.16) THEOREM. *For any admissible dg-operad $\mathscr{P}$ there is a canonical quasi isomorphism* $\mathbf{D}(\mathbf{D}(\mathscr{P})) \to \mathscr{P}$.

Proof. Let us write $\mathbf{D}(\mathbf{D}(\mathscr{P}))(n)$ explicitly. By definition (3.2.14),

$$\mathbf{D}(\mathbf{D}(\mathscr{P}))(n) = \bigoplus_{n\text{-trees } S} \left[\bigotimes_{v \in S} \mathbf{D}(\mathscr{P})(\mathrm{In}(v))^* \otimes \mathrm{Det}(S) \right].$$

Substituting here the definitions (1.1.6) of the value on a set of the functor corresponding to $\mathbf{D}(\mathscr{P})$, and keeping track of cancellation of some Det-factors, we get

$$\mathbf{D}(\mathbf{D}(P))(n) = \bigoplus_{T \geqslant T'} \frac{\bigotimes_{v \in T} \mathscr{P}(\mathrm{In}(v))}{\bigotimes_{w \in T'} \mathrm{Det}(T_w)}. \tag{3.2.17}$$

Here the summation is over the isomorphism classes of pairs T, T' of n-trees such that $T \geqslant T'$, i.e., T' can be obtained from T by contracting some (possibly empty) set of edges. For $w \in T'$ we denote by T_w the subtree of T contracted into w. Division by a 1-dimensional vector space is understood as tensoring with the dual space.

The construction may be understood better using Figure 9a where vertices w of the tree T' are indicated as big circles ("regions") containing inside them the corresponding trees T_w. Let T_n denote the unique n-tree with a single vertex. The summand in (3.2.17) corresponding to the pair $T_n \geqslant T_n$ is nothing but $\mathscr{P}(n)$ (Figure 9b).

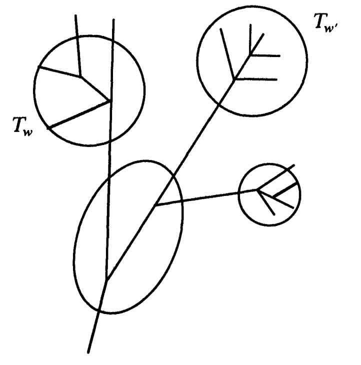

FIGURE 9a

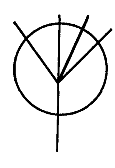

FIGURE 9b

It is straightforward to verify that the summand $\mathscr{P}(n)$ is in fact a quotient complex of the whole complex $\mathbf{D}(\mathbf{D}(\mathscr{P}))(n)$, i.e. the projection to this summand along all other summands is a (surjective) morphism of complexes which we denote by f_n. It is also easily verified that we get in this way a morphism of *dg*-operads $f\colon \mathbf{D}(\mathbf{D}(\mathscr{P})) \to \mathscr{P}$. Let us show that f is a quasi isomorphism, i.e., that each subcomplex $\operatorname{Ker}(f_n) \subset \mathbf{D}(\mathbf{D}(\mathscr{P}))(n)$, is acyclic. Note that $\operatorname{Ker}(f_n)$, as part of $\mathbf{D}(\mathbf{D}(\mathscr{P}))(n)$, is actually a triple complex, and so its differential is a sum of three partial differentials $d_1 + d_2 + d_3$. The differential d_1 is induced by the differential in $\mathscr{P}$; the differential d_2 is induced by the composition in $\mathscr{P}$ (which induces the second differential in $\mathbf{D}(\mathscr{P})$). Finally, d_3 is induced by the composition in $\mathbf{D}(\mathscr{P})$, i.e., by the grafting of trees.

It is enough to show that $\operatorname{Ker} f_n$ is acyclic with respect to d_3. If T is an n-tree with more than one vertex, then the summands in (3.2.17) with all $T' \leqslant T$ form a subcomplex $K_T^{\bullet} \subset (\operatorname{Ker}(f_n), d_3)$ and $\operatorname{Ker}(f_n)$ is the direct sum of such $K_T^{\bullet}$. We shall prove that each $K_T^{\bullet}$ is acyclic.

The differential d_3 in $K_T^{\bullet}$ consists purely in redrawing the boundaries among regions in Figure 9 (a). More precisely, $K_T^{\bullet}$ is the tensor product of the vector space $\mathscr{P}(T)$ and a purely combinatorial complex $C_T^{\bullet}$ where

$$C_T^i = \bigoplus_{\substack{T' \leqslant T \\ |T| - |T'| = i}} \bigotimes_{w \in T'} \operatorname{Det}(T_w)^* .$$

Observe that specification of a tree $T' \leqslant T$ is equivalent to a specification of a subset of internal edges of T which are contracted in T'. We see that $C_T^{\bullet}$ is isomorphic to the augmented chain complex of a simplex whose vertices correspond to internal edges of T. Thus, $C_T^{\bullet}$ and $K_T^{\bullet}$ are acyclic. Theorem 3.2.16 follows.

(3.2.18) PROPOSITION. *For any admissible dg-operad $\mathscr{P}$ with $\mathscr{P}(1) = k$, there exists a natural morphism of dg-operads $\lambda\colon \mathbf{D}(\mathscr{C}om) \to \mathscr{P} \otimes \mathbf{D}(\mathscr{P})$. In particular, for any dg-algebra A over $\mathscr{P}$ and any dg-algebra B over $\mathbf{D}(\mathscr{P})$, the tensor product $A \otimes_k B$ has a natural structure of a $\mathbf{D}(\mathscr{C}om)$-algebra.*

This result is analogous to Corollary 2.2.9 (b). The precise meaning of the analogy will be explained in §4.

Proof. Since $\mathscr{C}om(n) = k$ for every n, we find that $\mathbf{D}(\mathscr{C}om)(n)$ is the *dg*-vector space whose $(-i)$th graded piece is the direct sum of spaces $\operatorname{Det}(T)$ corresponding to n-trees T with $n - 2 - i$ interior edges. We define the required maps λ_n: $\mathbf{D}(\mathscr{C}om)(n) \to \mathscr{P}(n) \otimes \mathbf{D}(\mathscr{P})(n)$ by prescribing their restrictions on the summands $\operatorname{Det}(T)$. We define

$$\lambda_n|_{\operatorname{Det}(T)}\colon \operatorname{Det}(T) \to \mathscr{P}(n) \otimes \mathscr{P}^*(T) \otimes \operatorname{Det}(T) \subset \mathbf{D}(\mathscr{P})(n)$$

to be $\gamma_T^* \otimes \operatorname{Id}_{\operatorname{Det}(T)}$, where the element $\gamma_T^{\dagger} \in \mathscr{P}(n) \otimes \mathscr{P}^*(T)$ is the transpose of the map $\gamma_T\colon \mathscr{P}(T) \to \mathscr{P}(n)$ given by the composition in $\mathscr{P}$; see (1.2.4).

This defines the maps λ_n. The proof that these maps form a morphism of operads is straightforward and left to the reader.

3.3. The generating map of the dual dg-operad

(3.3.1) Let $\mathscr{P}$ be an admissible dg-operad, and $Q = \mathbf{D}(\mathscr{P})$ its dual (3.2.13). Let r be the number of simple summands of the semisimple algebra $\mathscr{P}(1) = \mathscr{Q}(1)^{op}$, and let $g_{\mathscr{P}}, g_{\mathscr{Q}}: \mathbf{C}^r \to \mathbf{C}^r$ be the generating maps of $\mathscr{P}$, $\mathscr{Q}$; see (3.1.8). Let also $G_{\mathscr{P}}$, $G_{\mathscr{Q}}$ be the refined generating maps; see (3.1.16).

The following is the main result of this section.

(3.3.2) THEOREM. (a) *We have the following identity of formal maps* $\mathbf{C}^r \to \mathbf{C}^r$:

$$g_{\mathscr{Q}}(-g_{\mathscr{P}}(-x)) = x, \qquad x = (x_1, \ldots, x_r).$$

(b) *For the refined generating maps we have the similar plethystic identity (notations from (3.1.16)–(3.1.19))*

$$G_{\mathscr{P}} \circ (-\varepsilon(G_Q)) = (I_1, \ldots, I_r) \in \hat{\mathscr{R}}^{\otimes r} \times \cdots \times \hat{\mathscr{R}}^{\otimes r}.$$

Clearly, part (b) generalizes part (a). This generalization is due to Ph. Hanlon, who kindly communicated it to the authors.

(3.3.3) *Example.* Let $\mathscr{P} = \mathscr{C}om$ be the commutative operad. As we saw in (3.1.12), $g_{\mathscr{P}}(x) = e^x - 1$. By (3.2.15), $\mathscr{Q} = \mathbf{D}(\mathscr{P})$ is quasi isomorphic to the operad $\mathscr{L}ie$, so $g_{\mathscr{Q}} = g_{\mathscr{L}ie}(x) = -\log(1 - x)$. One sees that these two series satisfy Theorem 3.3.2.

(3.3.4) Let $K = \mathscr{P}(1)$. As a first step towards the proof of our theorem, we describe the r-fold dg-collection associated to the K^{op}–dg-collection $\mathbf{D}(\mathscr{P}) = \{\mathbf{D}(\mathscr{P})(n)\}$. By an r-colored tree, we understand a tree T together with a function c ("coloring") from the set of all edges of T to $\{1, \ldots, r\}$. For such a tree T and a vertex $v \in T$, let $\mathrm{In}_i(v)$ denote the set of input edges at v of color i. By an $(a_1, \ldots, a_r)$-tree we mean an r-colored tree T with a_i inputs of color i, which are labelled by the numbers $1, 2, \ldots, a_i$.

(3.3.5) Let $\{\mathscr{P}^i(a_1, \ldots, a_r)\}$ be the r-fold dg-collection associated to $\mathscr{P}$. For any i we use the $\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}$-action on each $\mathscr{P}^i(a_1, \ldots, a_r)$ to define, similarly to (1.1.6), a functor

$$(I_1, \ldots, I_r) \mapsto \mathscr{P}^i(I_1, \ldots, I_r)$$

on the category of r-tuples of finite sets and bijections. Let also $\{\mathscr{Q}^i(a_1, \ldots, a_r)\}$ be the r-fold dg-collection associated to the dual dg-operad $\mathscr{Q} = \mathbf{D}(\mathscr{P})$. Recall that $|T|$ denotes the number of internal edges of a tree T.

(3.3.6) PROPOSITION. *Each $\mathcal{Q}^i(a_1, \ldots, a_r)$ is isomorphic, as a graded vector space with $\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}$-action, to*

$$\bigoplus_{\substack{(a_1,\ldots,a_r)\text{-trees } T\\ c(\mathrm{Out}(T))=i}} \bigotimes_{v \in T} \mathcal{P}^{c(\mathrm{Out}(v))}(\mathrm{In}_1(v), \ldots, \mathrm{In}_r(v))^* \left[|T| - \sum a_i - 2 \right] \otimes \det(T).$$

The proof is straightforward.

(3.3.7) To prove Theorem 3.3.2, we invoke a purely algebraic formula describing the inversion of a formal (or analytic) map $g\colon \mathbf{C}^r \to \mathbf{C}^r$. This formula is due to J. Towber, see [62], Theorem 3.10 or [48], Theorem 2.13. (We are grateful to D. Wright for pointing out the references to this formula, which was rediscovered by us.)

Suppose we have r formal power series

$$g^{(i)}(x_1, \ldots, x_r) = x_i + \sum_{\substack{a_1,\ldots,a_r \geq 0\\ a_1+\cdots+a_r \geq 2}} p^{(i)}(a_1, \ldots, a_r) \frac{x_1^{a_1}}{a_1!} \cdot \cdots \cdot \frac{x_r^{a_r}}{a_r!},$$

where $p^{(i)}(a_1, \ldots, a_r)$ are some complex coefficients, and let

$$y_i = g^{(i)}(x_1, \ldots, x_r) \tag{3.3.8}$$

be a formal change of variables given by these series. Since $g^{(i)}(x) = x_i +$ terms of order ≥ 2, this change of variables is formally invertible, i.e., we can express x_i as power series in $y_1, \ldots, y_n$. The question is to find the coefficients of these power series provided the coefficients $p^{(i)}(a_1, \ldots, a_r)$ are known.

(3.3.9) THEOREM. *The inverse of the formal map (3.3.8) is given by*

$$x_i = h_i(y_1, \ldots, y_r) = y_i + \sum_{\substack{a_1,\ldots,a_r \geq 0\\ a_1+\cdots+a_r \geq 2}} q^{(i)}(a_1, \ldots, a_r) \frac{y_1^{a_1}}{a_1!} \cdot \cdots \cdot \frac{y_r^{a_r}}{a_r!} \tag{3.3.10}$$

where

$$q^{(i)}(a_1, \ldots, a_r) = \sum_{m=0}^{a_1+\cdots+a_r-1} (-1)^{a_1+\cdots+a_r-m} \times \left\{ \sum_{\substack{(a_1,\ldots,a_r)\text{-trees } T\\ |T|=m,\, c(\mathrm{Out}(T))=i}} \prod_{v \in T} p^{c(\mathrm{Out}(v))}(|\mathrm{In}_1(v)|, \ldots, |\mathrm{In}_r(v)|) \right\}. \tag{3.3.11}$$

Several other inversion formulas for analytic maps can be found in [4], [25], [29], [48], [61].

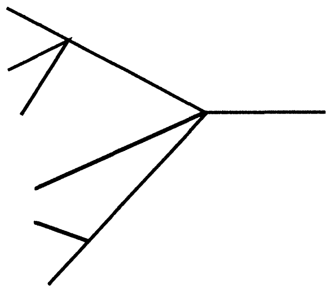

FIGURE 10

(3.3.12) We prefer to give here a simple direct proof of Theorem 3.3.9, partly for the convenience of the reader, partly because an intermediate lemma in the proof will be used later. The proof proceeds by direct substitution of the proposed answer into the claimed equation. Suppose first that the series h_i (i.e., the coefficients $q^{(i)}(a_1, \ldots, a_r)$) in (3.3.10) are arbitrary. Let us form the composition

$$f^{(i)}(y) = g^{(i)}(h^{(1)}(y), \ldots, h^{(r)}(y)), \qquad y = (y_1, \ldots, y_r)$$

and write

$$f^{(i)}(y) = y_i + \sum_{\substack{a_1, \ldots, a_r \geqslant 0 \\ a_1 + \cdots + a_r \geqslant 2}} u^{(i)}(a_1, \ldots, a_r) \frac{x_1^{a_1}}{a_1!} \cdot \cdots \frac{x_r^{a_r}}{a_r!}.$$

It is immediate to get a general formula for the coefficients $u^{(i)}(a_1, \ldots, a_r)$. Call a rooted tree T *short* if T has no consecutive internal edges (see Figure 10).

The lowest vertex of a tree T (the one which is adjacent to the output edges $\mathrm{Out}(T)$) will be denoted by $\mathrm{Lw}(T)$.

(3.3.13) LEMMA. *We have*

$$u^{(i)}(a_1, \ldots, a_r) = \sum_{\substack{\text{short } (a_1, \ldots, a_r)\text{-trees } T \\ c(\mathrm{Out}(T)) = i}} \Big\{ p^{(c(\mathrm{Out}(T)))}(|\mathrm{In}_1(\mathrm{Lw}(T))|, \ldots, |\mathrm{In}_r(\mathrm{Lw}(T))|)$$

$$\times \prod_{v \in T, v \neq \mathrm{Lw}(T)} q^{(c(\mathrm{Out}(v)))}(|\mathrm{In}_1(v)|, \ldots, |\mathrm{In}_r(v)|) \Big\}$$

where T runs over short $(a_1, \ldots, a_r)$-trees such that the vertex $\mathrm{Lw}(T)$ *may have just one input edge but all other vertices have $\geqslant 2$ input edges.*

The lemma is proved by explicit substitution of power series.

Let us now substitute into Lemma 3.3.13 the particular values of the coefficients $q^{(i)}(a_1, \ldots, a_r)$ from (3.3.11). We get a formula for $u^{(i)}(a_1, \ldots, a_r)$ which involves a summation over all $(a_1, \ldots, a_r)$-labelled trees T, not necessarily short, such that the lowest vertex $\mathrm{Lw}(T)$ is allowed to have one input (whose color coincides with the color of $\mathrm{Out}(T)$), but no other vertex is allowed to have one input. That means that any summand in the arising formula for $u^{(i)}(a_1, \ldots, a_r)$ will enter twice: it will correspond once to a tree whose lowest vertex has $\geqslant 2$ inputs and it will correspond for the second time to the same tree but with a one-input vertex appended at the bottom. These two summands will enter with opposite signs, and hence they will cancel each other. Therefore all the coefficients $u^{(i)}(a_1, \ldots, a_r)$ are equal to zero. So $f_i(y) = y_i$ and Theorem 3.3.9 is proven.

This completes the proof of part (a) of Theorem 3.3.2. The proof of part (b) is quite similar and consists of repeating the same arguments in the new context of representations of symmetric group. The main step is that Lemma 3.1.13, being intepreted at the level of r-fold dg-collections, gives the plethystic composition. We omit further details.

(3.3.14) *Example.* Let b_n be the number of nonisomorphic binary n-trees. Let $\mathscr{P}$ be the k-linear operad with $\mathscr{P}(1) = \mathscr{P}(2) = k$, $\mathscr{P}(n) = 0$, $n \geqslant 3$. Applying the definition, we see that the complex $\mathbf{D}(\mathscr{P})(n)$ consists of one vector space $V(n)$, of dimension b_n, placed in degree 0. The generating maps of $\mathscr{P}$ and $\mathbf{D}(\mathscr{P})$ are therefore $g(x) = x + x^2/2$ and $h(x) = x + \sum_{n=2}^{\infty} b_n x^n/n!$, respectively. The identity $g(-h(-x)) = x$ yields $h(x) = 1 - \sqrt{1-2x}$ whence $b_n = (2n-3)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)$.

3.4. *The configuration operad and duality*

(3.4.1) Let X be a topological space. A complex of sheaves of k-vector spaces on X can be regarded as a sheaf with values in the abelian category dg Vect of dg-vector spaces. Such objects will be referred to as *dg-sheaves.*

For a dg-sheaf $\mathscr{F}^\bullet$ on X we denote by $R\Gamma(X, \mathscr{F}^\bullet)$ the dg-vector space of global sections of the canonical Godement resolution of $\mathscr{F}^\bullet$; see [24], [34]. The cohomology spaces of $R\Gamma(X, \mathscr{F}^\bullet)$ are $H^i(X, \mathscr{F}^\bullet)$, the usual topological hypercohomology with coefficients in $\mathscr{F}^\bullet$. Given two topological spaces X_1, X_2 and dg-sheaves $\mathscr{F}_i^\bullet$ on X_i, there is a natural morphism

$$R\Gamma(X_1, \mathscr{F}_1^\bullet) \otimes R\Gamma(X_2, \mathscr{F}_2^\bullet) \to R\Gamma(X_1 \times X_2, \mathscr{F}_1^\bullet \otimes \mathscr{F}_2^\bullet) \tag{3.4.2}$$

which is a quasi isomorphism.

(3.4.3) Let $\mathscr{Q}$ be a topological operad. The notion of a dg-sheaf on $\mathscr{Q}$ is completely analogous to the notion of a sheaf on $\mathscr{Q}$ in (1.5.3). As in (1.5.9) we have the following.

(3.4.4) Proposition. *If $\mathscr{Q}$ is a topological operad and $\mathscr{F}^\bullet$ is a dg-sheaf on $\mathscr{Q}$, then the collection*

$$R\Gamma(\mathscr{Q}, \mathscr{F}^\bullet) = \{R\Gamma(\mathscr{Q}(n), \mathscr{F}^\bullet(n))^*, n \geqslant 1\}$$

forms a dg-operad, so that the cohomology spaces $H^\cdot(\mathcal{Q}(n), \mathcal{F}^\cdot(n))^*$ *form an operad on the category* $g\mathrm{Vect}^-$.

(3.4.5) *Example: logarithmic forms of the configuraton operad.* Let $k = \mathbf{C}$ be the field of complex numbers and let $\mathcal{M}$ be the configuration operad (1.4.4). For any $n \geqslant 1$ let $j_n\colon M_{0,n+1} \hookrightarrow \mathcal{M}(n)$ be the embedding of the open stratum and $\underline{\mathbf{C}}_{M_{0,n+1}}$ be the constant sheaf on $M_{0,n+1}$. Let $D(n) = \mathcal{M}(n) - M_{0,n+1}$. This is a divisor with normal crossings in $\mathcal{M}(n)$, consisting of $2^{n-1} - 1$ smooth components $\overline{\mathcal{M}(T)}$, where T is an n-tree with exactly one internal edge. Let $\Omega^\cdot_{\mathcal{M}(n)}(\log D(n))$ be the corresponding logarithmic de Rham complex [13]. Consider the collection of shifted and twisted complexes

$$\Omega^\cdot_{\mathcal{M}}(\log D) = \{\Omega^\cdot_{\mathcal{M}(n)}(\log D(n))[n-2] \otimes \mathrm{Det}(\mathbf{C}^n), n \geqslant 1\}. \tag{3.4.6}$$

This collection has a natural structure of a dg-sheaf on $\mathcal{M}$, whose structure maps (1.5.3) (ii) are given by the *Poincaré residue* maps ([13], n. 3.1.5.2). Let us briefly recall this notion in the generality we need. Let X be a smooth variety, and let $Y \subset X$ be a divisor with normal crossings which we assume to consist of smooth components $Y_1, \ldots, Y_M$. Let $Z \subset Y$ be a codimension-m subvariety given by intersection of some m of these components: $Z = \bigcap_{i \in I} Y_i$, $|I| = m$, and let $\gamma\colon Z \hookrightarrow X$ be the embedding. The Poincaré residue is the map

$$\mathrm{res}\colon \gamma^*\Omega^\cdot_X(\log Y)[m] \otimes \mathrm{Det}(\mathbf{C}^I) \to \Omega^\cdot_Z\left(\log\left(Z \cap \bigcap_{i \notin I} Y_i\right)\right). \tag{3.4.7}$$

The appearance of $\mathrm{Det}(\mathbf{C}^I)$ stems from the fact that the operations of taking the residues along individual hypersurfaces Y_i, $i \in I$ anticommute with each other due to the residue theorem.

Returning to our situation, let $m_1, \ldots, m_l \geqslant 1$ be given. The structure map of the operad $\mathcal{M}$

$$\gamma_{m_1,\ldots,m_l}\colon \mathcal{M}(l) \times \mathcal{M}(m_1) \times \cdots \times \mathcal{M}(m_l) \to \mathcal{M}(m_1 + \cdots + m_l),$$

described in (1.4.4), is a closed embedding whose image is the intersection of several components of the divisor $D(m_1 + \cdots + m_l)$. If we denote the set of these components by I, then we have a canonical identification

$$\mathrm{Det}(\mathbf{C}^I) \cong \left(\mathrm{Det}(\mathbf{C}^l) \otimes \bigotimes \mathrm{Det}(\mathbf{C}^{m_i})\right)^* \otimes \mathrm{Det}(\mathbf{C}^{m_1 + \cdots + m_l}).$$

Thus the maps (3.4.6) indeed supply the necessary structure.

(3.4.8) *The gravity operad* $\mathcal{G}$. We denote the dg-operad $R\Gamma(\mathcal{M}, \Omega^\cdot_{\mathcal{M}}(\log D))^*$ simply by $\tilde{\mathcal{G}}$, and by $\mathcal{G}$ we denote the cohomology operad of $\tilde{\mathcal{G}}$, i.e., the operad in the category of graded vector spaces $g\mathrm{Vect}^-$ given by $\mathcal{G}(n) = H^\cdot(\tilde{\mathcal{G}}(n))$. Following a suggestion of E. Getzler, we call $\mathcal{G}$ the *gravity operad*. Since $\Omega^\cdot_{\mathcal{M}(n)}(\log D(n))$ is quasi-isomorphic to the direct image $Rj_{n*}\underline{\mathbf{C}}_{M_{0,n+1}}$, we have

$$H^i(\mathcal{M}(n), \Omega^\cdot(\log D(n))) = H^i(M_{0,n+1}, \mathbf{C}).$$

Thus the ith component of the graded vector space $\mathcal{G}(n)$ is $H_{n-2-i}(M_{0,n+1}, \mathbf{C}) \otimes \mathrm{Det}(\mathbf{C}^n)$.

(3.4.9) PROPOSITION. *There is a quasi isomorphism of dg-operads $\tilde{\mathcal{G}} \to \mathcal{G}$ (where $\mathcal{G}$ is considered with zero differential).*

Proof. Let $L^i(n)$ be the space of global logarithmic i-forms on $\mathcal{M}(n)$. It is known [19] that $L^i(n)$ consists of closed forms and is naturally identified with $H^i(M_{0,n+1}, \mathbf{C})$. Thus the embedding of the graded vector space $L^\cdot(n) = \bigoplus L^i(n)$ with zero differential into $R\Gamma(\mathcal{M}(n), \Omega^\cdot_{\mathcal{M}(n)}(\log D(n)))$ is a quasi isomorphism. Denote this embedding by ϕ_n. Clearly the Poincaré residue homomorphisms preserve global logarithmic forms so they make the collection of graded vector spaces $L^\cdot(n)^*[-n+2] \otimes \mathrm{Det}(\mathbf{C}^n)$ into an operad which is isomorphic to $\mathcal{G}$. Taking duals of the embeddings ϕ_n above, we get the required quasi isomorphism $\tilde{\mathcal{G}} \to \mathcal{G}$.

(3.4.10) Let $\mathcal{P}$ be an admissible dg-operad. As in (1.5.2) we associate to $\mathcal{P}$ a dg-sheaf $\mathcal{F}_{\mathcal{P}}$ on the configuration operad $\mathcal{M}$. By (3.4.4) the complexes $R\Gamma(\mathcal{M}(n), \mathcal{F}_{\mathcal{P}}(n))^*$ form another dg-operad. It is described as follows.

(3.4.11) THEOREM. *Suppose that $k = \mathbf{C}$. Then the operad $R\Gamma(\mathcal{M}, \mathcal{F}_{\mathcal{P}})^*$ is quasi-isomorphic to $\mathbf{D}(\mathcal{P} \otimes \mathcal{G})$ where $\mathbf{D}$ is the duality for dg-operads introduced in (3.2.13).*

Proof. Let X be a finite (compact) CW-complex and let $S = \{X_\alpha\}$ be its Whitney stratification by locally closed CW-subcomplexes. We denote by $j_\alpha\colon X_\alpha \hookrightarrow X$ the embeddings of the strata. Let $X_{\geqslant m}$ be the union of strata of dimensions $\geqslant m$. This is an open subset in X and we denote by $j_{\geqslant m}\colon X_{\geqslant m} \hookrightarrow X$ the embedding. Let $\mathcal{F}$ be any (dg-) sheaf on X. Let $\mathcal{F}_\alpha = j_\alpha^* \mathcal{F}$ be the restrictions of $\mathcal{F}$ to the strata. Note that $\mathcal{F}$ has a decreasing filtration

$$\mathcal{F} = (j_{\geqslant 0})_! j_{\geqslant 0}^* \mathcal{F} \supset (j_{\geqslant 1})_! j_{\geqslant 1}^* \mathcal{F} \supset \cdots \tag{3.4.12}$$

with quotients

$$\bigoplus_{\dim X_\alpha = m} j_{\alpha !} j_\alpha^* \mathcal{F}.$$

Here $j_{\alpha !}$ are the direct images with proper support (extensions by zero). We can regard this filtration as a Postnikov system realizing $\mathcal{F}$ as a convolution [33] of the following complex of objects of $D^b(Sh_X)$, the derived category of sheaves on X:

$$\bigoplus_{\dim X_\alpha = 0} j_{\alpha !} j_\alpha^* \mathcal{F} \to \bigoplus_{\dim X_\alpha = 1} j_{\alpha !} j_\alpha^* \mathcal{F}[1] \to \bigoplus_{\dim X_\alpha = 2} j_{\alpha !} j_\alpha^* \mathcal{F}[2] \to \cdots. \tag{3.4.13}$$

By replacing $j_{\alpha!}j_\alpha^*\mathscr{F}$ with appropriate injective resolutions, we can realize (3.4.13) as an actual double complex of sheaves on X. By applying the functor $R\Gamma(X, -)$ to (3.4.13), we get that $R\Gamma(X, \mathscr{F})$ is the total complex of the double complex

$$(3.4.14) \qquad \bigoplus_{\dim X_\alpha = 0} R\Gamma_c(X_\alpha, j_\alpha^*\mathscr{F}) \to \bigoplus_{\dim X_\alpha = 1} R\Gamma_c(X_\alpha, j_\alpha^*\mathscr{F})[1] \to \cdots$$

where $R\Gamma_c$ is the derived functor of global sections with compact support. The horizontal grading in this complex is so normalized that the sum with $\dim X_\alpha = 0$ has horizontal degree 0.

Assume now that $\mathscr{F}$ is S-combinatorial (1.5.1) and given by (dg-) vector spaces F_α. Then we have the equalities

$$(3.4.15) \qquad R\Gamma_c(X_\alpha, j_\alpha^*\mathscr{F}) = R\Gamma_c(X_\alpha, \mathbf{C}) \otimes F_\alpha .$$

We specialize now to the case when $X = \mathscr{M}(n)$, the stratification S consists of $\mathscr{M}(T)$, and $\mathscr{F} = \mathscr{F}_{\mathscr{P}}(n)$ is the sheaf corresponding to an admissible dg-operad $\mathscr{P}$. For an n-tree $\mathscr{T}$, let $D(T) \subset \overline{\mathscr{M}(T)}$ be the complement to $\mathscr{M}(T)$. This is a divisor with normal crossing. As usual, we denote by $|T|$ the number of internal edges in T, so that $\dim_{\mathbf{C}} \mathscr{M}(T) = n - 2 - |T|$. Note that we have the quasi isomorphisms

$$(3.4.16) \quad R\Gamma_c(\mathscr{M}(T), \mathbf{C}) \cong R\Gamma(\mathscr{M}(T), \mathbf{C})^*[-2(n-2-|T|)]$$
$$\cong R\Gamma(\overline{\mathscr{M}(T)}, \Omega^{\bullet}_{\overline{\mathscr{M}(T)}}(\log D(T)))^*[-2(n-2-|T|)].$$

Here the first quasi isomorphism is the Poincaré duality. The second quasi isomorphism is the consequence of the fact [13] that the logarithmic de Rham complex is quasi-isomorphic to the full direct image of the constant sheaf. Note that in view of the product decompositions (1.4.8) and the definition of the dg-operad $\tilde{\mathscr{G}}$ we have a quasi isomorphism

$$R\Gamma(\overline{\mathscr{M}(T)}, \Omega^{\bullet}_{\overline{\mathscr{M}(T)}}(\log D(T)))^* \cong \tilde{\mathscr{G}}(T)[n-2-|T|] \otimes \mathrm{Det}(T),$$

where $\mathrm{Det}(T)$ was introduced in (3.2.0). Taking into account Proposition 3.4.9, we see that the double complex (3.4.14), i.e., $R\Gamma(\mathscr{M}(n), \mathscr{F}_{\mathscr{P}}(n))$, can be replaced by the double complex

$$\bigoplus_{|T|=0} \mathscr{P}(T) \otimes \mathscr{G}(T) \otimes \mathrm{Det}(T) \to \bigoplus_{|T|=1} \mathscr{P}(T) \otimes \mathscr{G}(T) \otimes \mathrm{Det}(T) \to \cdots .$$

So the dual complex will be $\mathbf{D}(\mathscr{P} \otimes \mathscr{G})(n)$ as claimed.

(3.4.17) *Example.* Taking $\mathscr{P} = \mathscr{C}om$, the commutative operad, we get that every $\mathscr{F}_{\mathscr{P}}(n)$ is the constant sheaf $\mathbf{C}$ on $\mathscr{M}(n)$. Thus the operad formed by the total homology spaces $H_\bullet(\mathscr{M}(n), \mathbf{C})$ is quasi-isomorphic to $\mathbf{D}(\mathscr{G})$.

3.5. The building cooperad and duality

(3.5.1) By a *cooperad* in a symmetric monoidal category $(\mathcal{A}, \otimes)$ we mean an operad in the opposite category $\mathcal{A}^{op}$. Explicitly, a cooperad $\mathcal{B}$ is a collection of objects $\mathcal{B}(n) \in \mathcal{A}$, $n \geqslant 1$ with Σ_n-action on each $\mathcal{B}_n$ and morphisms

$$\rho_{m_1,\ldots,m_l}: \mathcal{B}(m_1 + \cdots + m_l) \to \mathcal{B}(l) \otimes \mathcal{B}(m_1) \otimes \cdots \otimes \mathcal{B}(m_l) \tag{3.5.2}$$

satisfying the conditions dual to the corresponding conditions for an operad.

If a collection $\mathcal{C} = \{\mathcal{C}(n)\}$ forms a cooperad in the category of (*dg*-) vector spaces, then the dual *dg*-vector spaces $\mathcal{C}(n)^*$ form a *dg*-operad $\mathcal{C}^*$. A *dg*-cooperad $\mathcal{C}$ is called *admissible* if $\mathcal{C}^*$ is an admissible operad in the sense of (3.1.5). Cooperads in the category of topological spaces will be called *topological cooperads.*

(3.5.2) *The building* $\mathcal{W}(n)$. Let T be a n-tree. By a *metric* on T we mean an assignment of a positive number (length) to each internal edge of T. The external edges can be thought of as having length 1. Let $\mathcal{W}(n)$ be the set of isometry classes of all n-trees with metrics. This set has a natural topology. In plain words, when the length of some edge is going to 0, we say that the limit tree is obtained by contracting this edge into a point. For any n-tree T (without metric), let $\mathcal{W}^0(T)$ denote the subset in $\mathcal{W}(T)$ consisting of all trees with metric isomorphic to T. This subset is clearly a cell (a product of several copies of $\mathbf{R}_+$, corresponding to internal edges of T). Thus $\mathcal{W}(n)$ is a union of these noncompact cells. For example, the space $\mathcal{W}(3)$ is the union of three half-lines glued along a common end (Figure 11).

The minimal cell in $\mathcal{W}(n)$ has dimension 0 and corresponds to the n-tree T_n with a single vertex (and no internal edges). Maximal cells of $\mathcal{W}(n)$ correspond to binary trees. Thus, the cells in $\mathcal{W}(n)$ are parametrized by the same set as the strata in the moduli space $\mathcal{M}(n)$; see (1.4.5). But the closure relations among the cells are *dual* to those among the strata in $\mathcal{M}(n)$, where the maximal stratum corresponds to the tree with one vertex and 0-dimensional strata correspond to binary trees.

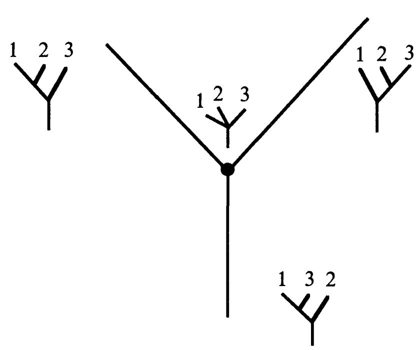

FIGURE 11

We call $\mathcal{W}(n)$ the *building* of n-trees since it is in many respects similar to the Bruhat-Tits building.

(3.5.3) *The building cooperad.* Suppose we have natural numbers $m_1, \ldots, m_l$. We define a map

$$\rho_{m_1,\ldots,m_l}: \mathcal{W}(m_1 + \cdots + m_l) \to \mathcal{W}(l) \times \mathcal{W}(m_1) \times \cdots \times \mathcal{W}(m_l)$$

as follows. Let $T(m_1, \ldots, m_l)$ be the $(m_1 + \cdots + m_l)$-tree drawn in Figure 3. Let (T', μ') be any other metrized $(m_1 + \cdots + m_l)$-tree. Let T'' be the maximal tree which is obtained from both T' and $T(m_1, \ldots, m_l)$ by contracting edges. The tree T'' is naturally divided into blocks $T_1'', \ldots, T_l'', T_\infty''$ where T_i'' has m_i inputs and T_∞'' has l inputs; see Figure 12.

Each block T_ν'', $\nu = 1, \ldots, \infty$, is naturally equipped with a metric μ_ν'' on its internal edges. We redefine the lengths of edges which become loose edges in T_ν'' by letting their lengths be equal to 1. The map $\rho_{m_1,\ldots,m_l}$ takes

$$(T', \mu') \mapsto ((T_\infty'', \mu_\infty''), (T_1'', \mu_1''), \ldots, (T_n'', \mu_n'')).$$

(3.5.4) PROPOSITION. *The collection of maps $\rho_{m_1,\ldots,m_l}$ and the natural actions of Σ_n on $\mathcal{W}(n)$, $n \geqslant 1$, define on the collection of $\mathcal{W}(n)$ the structure of a topological cooperad which will be denoted by $\mathcal{W}$.*

We call $\mathcal{W}$ the *building cooperad.* Note that the structure maps $\rho_{m_1,\ldots,m_l}$ for $\mathcal{W}$ are surjective, whereas the structure maps for the operad $\mathcal{M}$ are, dually, injective.

Just as any operad gives rise to a sheaf on the topological operad $\mathcal{M}$ (Theorem 1.5.11), any cooperad gives a sheaf on the cooperad $\mathcal{W}$. Let us give the corresponding definitions.

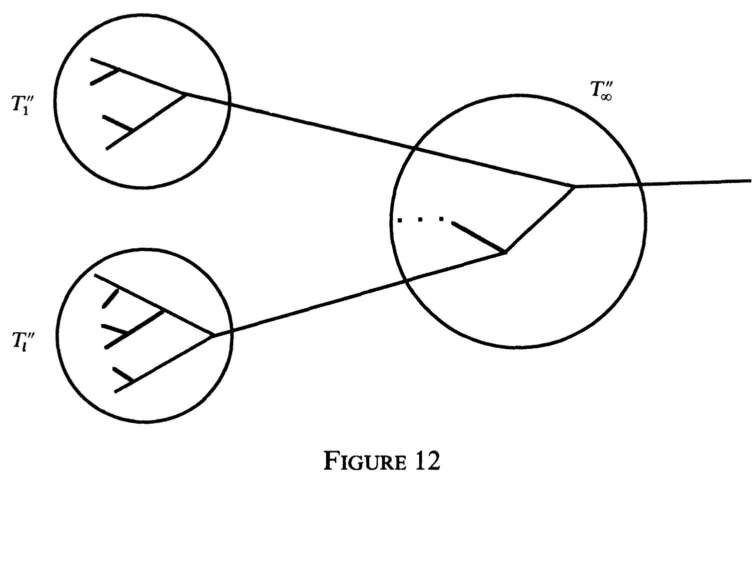

FIGURE 12

(3.5.5) *Definition.* Suppose $\mathscr{B}$ is a topological cooperad and let

$$\rho_{m_1,\ldots,m_l}: \mathscr{B}(m_1 + \cdots + m_l) \to \mathscr{B}(l) \times \mathscr{B}(m_1) \times \cdots \times \mathscr{B}(m_l)$$

be its structure maps (3.5.2). A sheaf (resp. a *dg*-sheaf) $\mathscr{F}$ on $\mathscr{B}$ consists of the following data:

(i) a collection of Σ_n-equivariant sheaves (resp. *dg*-sheaves) $\mathscr{F}(n)$ on $\mathscr{B}(n)$, one for each $n \geqslant 1$;

(ii) homomorphisms of sheaves on $\mathscr{B}(m_1 + \cdots + m_l)$

$$\mu_{m_1,\ldots,m_l}: \rho^*_{m_1,\ldots,m_l}(\mathscr{F}(l) \otimes \mathscr{F}(m_1) \otimes \cdots \otimes \mathscr{F}(m_l)) \to \mathscr{F}(m_1 + \cdots + m_l).$$

These data should satisfy the compatibility conditions dual to those given in (1.5.4)–(1.5.6).

If, for example, all the spaces $\mathscr{B}(n)$ consist of a single point, then a *dg*-sheaf on $\mathscr{B}$ is the same as a *dg*-opead. More generally, we have the following.

(3.5.6) PROPOSITION. *If $\mathscr{B}$ is a topological cooperad and $\mathscr{F}$ is a dg-sheaf on $\mathscr{B}$, then the collection of complexes $R\Gamma(\mathscr{B}(n), \mathscr{F}(n))$ forms a dg-operad. Similarly, the complexes $R\Gamma_c(\mathscr{B}(n), \mathscr{F}(n))$ (the derived functors of sections with compact support) form a dg-operad.*

Let $R\Gamma(\mathscr{B}, \mathscr{F})$ and $R\Gamma_c(\mathscr{B}, \mathscr{F})$ denote the *dg*-operads thus obtained.

(3.5.7) *Definition.* A sheaf $\mathscr{F}$ on a topological cooperad $\mathscr{B}$ is called an isosheaf if the morphisms

$$\mathscr{F}(l) \otimes \mathscr{F}(m_1) \otimes \cdots \otimes \mathscr{F}(m_l) \to (\rho_{m_1,\ldots,m_l})_* \mathscr{F}(m_1 + \cdots + m_l),$$

obtained from $\mu_{m_1,\ldots,m_l}$ by adjunction, are isomorphisms and if higher direct images $R^j(\rho_{m_1,\ldots,m_l})_* \mathscr{F}(m_1 + \cdots + m_l)$ vanish for $j \geqslant 1$.

(3.5.8) Let $\mathscr{Q}$ be any k-linear cooperad. Then $\mathscr{Q}(1)$ is a k-coalgebra and each $\mathscr{Q}(n)$ is a $(\mathscr{Q}(1)^{\otimes n}, \mathscr{Q}(1))$-bi-comodule. As in (1.1.5), we extend the collection of Σ_n-modules $\mathscr{Q}(n)$ to a functor $I \mapsto \mathscr{Q}(I)$ on finite sets and bijections. For any tree T we define, similarly to (1.2.13), the space

$$\mathscr{Q}(T) = \bigodot_{v \in T} {}_{\mathscr{Q}(1)} \mathscr{Q}(\mathrm{In}(v))$$

where $\bigodot$ is the cotensor product of comodules over a coalgebra. If $\mathscr{Q}(1) = k$, then $\mathscr{Q}(T) = \bigotimes_{v \in T} \mathscr{Q}(\mathrm{In}(v))$ is the usual tensor product over k. If all $\mathscr{Q}(n)$ are finite-dimensional over k, then we can use the operad $\mathscr{Q}^*$ to define $\mathscr{Q}(T) = (\mathscr{Q}^*(T))^*$ where $\mathscr{Q}^*(T)$ was defined by (1.2.13).

If $T \leqslant T'$ then the cooperad structure on $\mathcal{Q}$ defines a linear map $\rho_{T,T'}\colon \mathcal{Q}(T) \to \mathcal{Q}(T')$. The condition $T \leqslant T'$ means that the cell $\mathcal{W}^0(T)$ is contained in the closure of $\mathcal{W}^0(T')$. Therefore the maps $\rho_{T,T'}$ give rise to a combinatorial (1.5.1) sheaf $\mathcal{G}_{\mathcal{Q}}(n)$ on $\mathcal{W}(n)$ with fiber $\mathcal{Q}(T)$ on $\mathcal{W}^0(T)$. Here is the "dual" of Theorem 1.5.11.

(3.5.9) THEOREM. *Let $\mathcal{Q}$ be a k-linear operad. Then:*

(a) *The sheaves $\mathcal{G}_{\mathcal{Q}}(n)$ on the spaces $\mathcal{W}(n)$ form a sheaf $\mathcal{G}_{\mathcal{Q}}$ on the cooperad $\mathcal{W}$.*

(b) *If $\mathcal{Q}(1) = k$, then $\mathcal{G}_{\mathcal{Q}}$ is an isosheaf.*

(c) *Any isosheaf $\mathcal{G}$ on $\mathcal{W}$, such that each $\mathcal{G}(n)$ is constant on each cell $\mathcal{W}^0(T)$, has the form $\mathcal{G}_{\mathcal{Q}}$ for some k-linear cooperad $\mathcal{Q}$ with $\mathcal{Q}(1) = k$.*

The proof is straightforward and left to the reader. In a similar way, for any *dg*-operad $\mathcal{Q}$ we construct a *dg*-sheaf $\mathcal{G}^{\bullet}_{\mathcal{Q}}$ on $\mathcal{W}$.

(3.5.10) Let X be a CW-complex, and S its stratification into strata which are topological manifolds. Recall [34] that the Verdier duality gives a contravariant functor $\mathcal{F}^{\bullet} \mapsto \mathbf{V}(\mathcal{F}^{\bullet})$ from the derived category of *dg*-sheaves on X with S-constructible cohomology into itself.

The following result gives two different sheaf-theoretic interpretations of the duality $\mathbf{D}$ on *dg*-operads introduced in §3.2.

(3.5.11) THEOREM. *Let $\mathcal{Q}$ be an admissible dg-cooperad (3.5.1) so that $\mathcal{Q}^*$ is an admissible dg-operad. Then:*

(a) *There is a natural quasi isomorphism of dg-operads*

$$\mathbf{D}(\mathcal{Q}^*) \cong R\Gamma_c(\mathcal{W}, \mathcal{G}^{\bullet}_{\mathcal{Q}}) \otimes \Lambda$$

where Λ is the determinant dg-operad (1.3.21).

(b) *For any n, the dg-sheaves $\mathcal{G}^{\bullet}_{\mathcal{Q}}(n)$ and $\mathcal{G}_{\mathbf{D}(\mathcal{Q}^*)^*}$ on $\mathcal{W}(n)$ are Verdier dual to each other.*

Proof. Let (X, S) be any space stratified into cells. Giving an S-combinatoial (*dg*-) sheaf $\mathcal{F}$ on X is the same (1.5.11) as giving (*dg*-) vector spaces F_σ $(=R\Gamma(\sigma, \mathcal{F}))$ together with generalization maps $g_{\sigma\tau}\colon F_\sigma \to F_\tau$ for $\sigma \subset \bar{\tau}$ satisfying transitivity conditions. For any cell σ, let $\mathrm{OR}(\sigma) = H_c^{\dim(\sigma)}(\sigma, k)$ be its (1-dimensional) orientation space. Note that for the cell $\mathcal{W}^0(T)$ in the building $\mathcal{W}(n)$ corresponding to an n-tree T, the space $\mathrm{OR}(\mathcal{W}^0(T))$ can be naturally identified with the space $\det(T)$; see (3.2.0). Theorem 3.5.11 is a consequence of the following well-known combinatorial construction of the Verdier duality and hypercohomology functors.

(3.5.12) PROPOSITION. *Let (X, S) be as above and let $\mathcal{F}$ be an S-combinatorial dg-sheaf on X given by dg-vector spaces F_σ and maps $g_{\sigma\tau}$. Then:*

(a) *The dg-vector space $R\Gamma_c(X, \mathcal{F})$ is quasi-isomorphic naturally to (the total dg-vector space arising from) the complex*

$$\bigoplus_{\dim(\sigma)=0} F_\sigma \otimes \mathrm{OR}(\sigma) \to \bigoplus_{\dim(\sigma)=1} F_\sigma \otimes \mathrm{OR}(\sigma) \to \cdots$$

where the sum over σ with $\dim(\sigma) = m$ is placed in degree m.

(b) *The Verdier dual dg-sheaf* $\mathbf{V}(\mathscr{F})$ *is represented by the collection of dg-vector spaces associated to complexes*

$$\mathbf{V}(\mathscr{F})_\sigma = \left\{ \cdots \to \bigoplus_{\substack{\tau\supset\sigma \\ \dim(\tau)=\dim(\sigma)+2}} F_\tau^* \otimes \mathrm{OR}(\tau) \to \bigoplus_{\substack{\tau\supset\sigma \\ \dim(\tau)=\dim(\sigma)+1}} F_\tau^* \otimes \mathrm{OR}(\tau) \right.$$
$$\left. \to F_\sigma^* \otimes \mathrm{OR}(\sigma) \right\},$$

where the sum over τ *with* $\dim(\tau) = \dim(\sigma) + m$ *is placed in degree* $(-m)$.

This completes the proof of Theorem 3.5.11.

The isomorphism $\mathbf{D}(\mathscr{C}om) \cong \mathscr{L}ie$ (see (3.2.15)) yields the following.

(3.5.13) Corollary. *The cohomology of* $\mathscr{W}(n)$ *with compact support (and constant coefficients) are as follows:*

$$H_c^i(\mathscr{W}(n), k) = \begin{cases} 0, & i \neq n-2 \\ \mathscr{L}ie(n), & i = n-2. \end{cases}$$

This shows that the space $\mathscr{L}ie(n)$ is analogous to the Steinberg representation of the group $GL_n(\mathbf{F}_q)$. Note that the standard Steinberg representation of this group (see [47]) has dimension $q^{n(n-1)/2} = q^1 \cdot q^2 \cdot \cdots \cdot q^{n-1}$. The modified Steinberg (discrete series) representation introduced by Lusztig [47] has dimension $(q-1)(q^2-1)\cdots(q^{n-1}-1)$. Both numbers can be seen as q-analogs of $(n-1)! = \dim \mathscr{L}ie(n)$. The role of the Lie operad as the dualizing module in our theory (2.2.9) is analogous to the role of the Steinberg representation in the Deligne-Lusztig theory [15]–[17].

4. Koszul operads

4.1. Koszul operads and Koszul complexes

(4.1.1) Let K be a semisimple k-algebra, and $\mathscr{P} = \mathscr{P}(K, E, R)$ a quadratic operad. Let $\mathscr{P}^! = \mathscr{P}(K^{op}, E^\vee, R^\perp)$ be the dual quadratic operad, and $\mathbf{D}(\mathscr{P})$ the dual dg-operad. Observe that for every n, the degree-0 part, $\mathbf{D}(P)(n)^0$, of the dg-vector space $\mathbf{D}(\mathscr{P})(n)$ is equal to $F(E^\vee)(n)$, the nth space of the free operad generated by the single space E; i.e.,

$$\mathbf{D}(\mathscr{P})(n)^0 = \bigoplus_{\substack{\text{binary} \\ n\text{-trees } T}} E^\vee(T) \otimes \mathrm{Det}(T) = F(E^\vee)(n).$$

We define a morphism of dg-operads γ_P: $\mathbf{D}(\mathscr{P}) \to \mathscr{P}^!$ (here $\mathscr{P}^!$ is equipped with the trivial dg-structure) to be given by compositions

$$\mathbf{D}(\mathscr{P})(n) \to \mathbf{D}(\mathscr{P})(n)^0 = F(E^\vee)(n) \to F(E^\vee)(n)/(R^\perp) = \mathscr{P}^!(n).$$

(4.1.2) LEMMA. *For every n, the morphism $\gamma_{\mathcal{P}}$ induces an isomorphism $H^0(\mathbf{D}(\mathcal{P})(n)) \to \mathcal{P}^!(n)$.*

Proof. Observe that the penultimate term

$$\mathbf{D}(\mathcal{P})(n)^{-1} = \bigoplus_T \mathcal{P}^*(T) \otimes \mathrm{Det}(T)$$

is the sum over the n-trees T such that all but one vertices of T are binary and just one vertex is ternary. Note also that $\mathcal{P}(3)^\vee$ is dual to the space of relations of the quadratic operad $\mathcal{P}^!$. Thus, the image of the last differential in $\mathbf{D}(\mathcal{P})(n)$ is precisely the space of consequences of the relations in $\mathcal{P}^!$, whence the statement of the lemma.

(4.1.3) *Definition.* A quadratic operad $\mathcal{P}$ is called Koszul if the morphism $\gamma_{\mathcal{P}}: \mathbf{D}(\mathcal{P}) \to \mathcal{P}^!$ is a quasi isomorphism, i.e., each complex $\mathbf{D}(\mathcal{P})(n)$ is exact everywhere but the right end.

(4.1.4) PROPOSITION. (a) *A quadratic operad $\mathcal{P}$ is Koszul if and only if $\mathcal{P}^!$ is also.*

(b) *Let r be the number of simple summands of the algebra $\mathcal{P}(1) = \mathcal{P}^!(1)^{op}$ and let $g_{\mathcal{P}}, g_{\mathcal{P}^!}: \mathbf{C}^r \to \mathbf{C}^r$ be the generating maps of $\mathcal{P}$ and $\mathcal{P}^!$ respectively. If $\mathcal{P}$ is Koszul, we have the formal power series identity*

$$g_{\mathcal{P}^!}(-g_{\mathcal{P}}(-x)) = x, \qquad x = (x_1, \ldots, x_r).$$

Proof. (a) Form the composition

$$\mathbf{D}(\mathcal{P}^!) \xrightarrow{\mathbf{D}(\gamma_{\mathcal{P}})} \mathbf{D}(\mathbf{D}(\mathcal{P})) \xrightarrow{f_{\mathcal{P}}} \mathcal{P},$$

where $f_{\mathcal{P}}$ is the quasi isomorphism constructed in Theorem 3.2.16. It is immediate that this composition coincides with $\gamma_{\mathcal{P}^!}$. If $\mathcal{P}$ is Koszul, then $\gamma_{\mathcal{P}}$ and hence $\mathbf{D}(\gamma_{\mathcal{P}})$ are quasi isomorphisms, whence the statement.

(b) Follows from Theorem 3.3.2 and the observation that quasi-isomorphic dg-operads have the same generating maps.

(4.1.5) Let K be a semisimple k-algebra, and let $P = \{P(n)\}$ be a K-collection (1.2.11). We extend the collection P to a functor $I \mapsto P(I)$ on the category of finite sets and their bijections as in (1.1.5). For any surjection $f: I \to J$ of finite sets we put

$$P(f) = \bigotimes_{j \in J} P(f^{-1}(j)).$$

If, in addition, P is an operad, then for any composable pair of surjections $I \xrightarrow{f} J \xrightarrow{g} H$ the compositions in P give rise to a map

$$\mu_{g,f}: P(g) \otimes P(f) \to P(gf).$$

Let P, Q be two K-collections. We define a new K-collection $P(Q)$, called the *composition* of P and Q, as follows:

$$(4.1.6)\quad P(Q)(n) = \bigoplus_{m=1}^{n} P(Q)(n)^m, \quad \text{where} \quad P(Q)(n)^m = \bigoplus_{f:[n]\to[m]} [P(m) \otimes_A Q(f)]_{\Sigma_m}$$

and the last sum is taken over all surjections $[n] \to [m]$.

(4.1.7) PROPOSITION. *The generating map (3.1.8) of the collection $P(Q)$ is given by the composition*

$$g_{P(Q)}(x) = g_P(g_Q(x)), \qquad x = (x_1, \ldots, x_r).$$

Proof. This follows from Lemma 3.3.13.

(4.1.8) *Koszul complexes.* Recall [55] that the Koszul complex of a quadratic K-algebra A is the vector space $A \otimes_K (A^!)^\vee$ (where $(A^!)^\vee = \mathrm{Hom}_{K^{op}}(A^!, K^{op})$) equipped with the differential defined in a certain natural way.

Let now $\mathscr{P} = \mathscr{P}(K, E, R)$ be a quadratic operad, and $\mathscr{P}^!$ the dual operad. We define the nth Koszul complex of $\mathscr{P}$ to be the nth space of the composition $\mathscr{P}((\mathscr{P}^!)^\vee)(n)$. Here $(\mathscr{P}^!)^\vee$ is the K-collection consisting of $\mathscr{P}^!(n)^\vee = \mathrm{Hom}_{K^{op}}(\mathscr{P}^!(n), K^{op})$ with the Σ_n-action being the transposed one twisted by sign. We put a grading on the space $\mathscr{P}((\mathscr{P}^!)^\vee)(n)$ by (4.1.6) and define the differential d in the following way.

For $X \in \mathscr{P}(2)$ and a surjection g: $[m+1] \to [m]$, let

$$\mu_{g,X}: \mathscr{P}(m) \to \mathscr{P}(m+1)$$

be the operator of composition with X along g. For $\Xi \in \mathscr{P}^!(2) = \mathscr{P}(2)^\vee$ and surjections g: $[m+1] \to [m]$, h: $[n] \to [m+1]$ let

$$\mu_{g,h,\Xi}: \mathscr{P}^!(h) \to \mathscr{P}^!(hg)$$

be the operator of composition with Ξ induced by $\mu_{g,h}$.

Let us write the identity element

$$\mathrm{Id}_{\mathscr{P}(2)} \in \mathrm{Hom}_K(\mathscr{P}(2), \mathscr{P}(2)) = \mathscr{P}(2) \otimes_K \mathscr{P}^!(2)$$

in the form $\sum X_i \otimes \Xi_i$ where $X_1, \ldots, X_d \in \mathscr{P}(2)$ and $\Xi_1, \ldots, \Xi_d \in \mathscr{P}^!(2)$. For a surjection f: $[n] \to [m]$, we define a map

$$d_f: \mathscr{P}(m) \otimes (\mathscr{P}^!)^\vee(f) \to \bigoplus_{h:[n]\to[m+1]} \mathscr{P}(m+1) \otimes (\mathscr{P}^!)^\vee(h)$$

by

$$d_f = \sum_{\substack{g:[m+1]\to[m]\\ gh=f}} \sum_{i=1}^{d} \mu_{h,X_i} \otimes \mu^*_{g,h,\Xi_i}. \tag{4.1.9}$$

Clearly d_f is Σ_n-equivariant, and hence it factors through the spaces of Σ_n-coinvariants. Therefore the formula $d = \sum_{f:n\to m} d_f$ defines a linear map

$$d: \mathcal{P}((\mathcal{P}^!)^\vee)(n)^m \to \mathcal{P}((\mathcal{P}^!)^\vee)(n)^{m+1}.$$

(4.1.10) PROPOSITION. *The morphisms d satisfy, for various m, the condition $d^2 = 0$, thus making each $\mathcal{P}((\mathcal{P}^!)^\vee)(n)$ into a complex.*

(4.1.11) To prove Proposition 4.1.10, we introduce the following notation. Let $E = \mathcal{P}(2)$ and $F(E)$ be the corresponding free operad. By definition (2.1.1), $F(E)(n) = \bigoplus E(T)$ (sum over binary n-trees). Let $R \subset F(E)(3)$ be the space of relations of $\mathcal{P}$. Call an n-tree T a *1-ternary tree* if T contains exactly one ternary vertex $v = v(T)$, all other vertices being binary. For such a tree T there are exactly three binary trees T', T'', T''' such that T can be obtained from each of them by contracting an edge. We denote by

$$R_T \subset E(T') \otimes E(T'') \oplus E(T''') \subset F(E)(n)$$

the result of substituting R at the place v. Thus,

$$\mathcal{P}(n) = \frac{F(E)(n)}{\sum_{\substack{1\text{-ternary}\\ n\text{-trees } T}} R_T}, \qquad \mathcal{P}^!(n)^\vee = \bigcap_{\substack{1\text{-ternary}\\ n\text{-trees } T}} R_T.$$

Looking at any summand of $\mathcal{P}((\mathcal{P}^!)^\vee)(n)^m$ we find that this is a subquotient of $F(E)(n)$. More precisely, for a surjection $f: [n] \to [m]$, let $T(f)$ be the reduced tree obtained by depicting the map f (and ignoring vertices with one input (Figure 13).

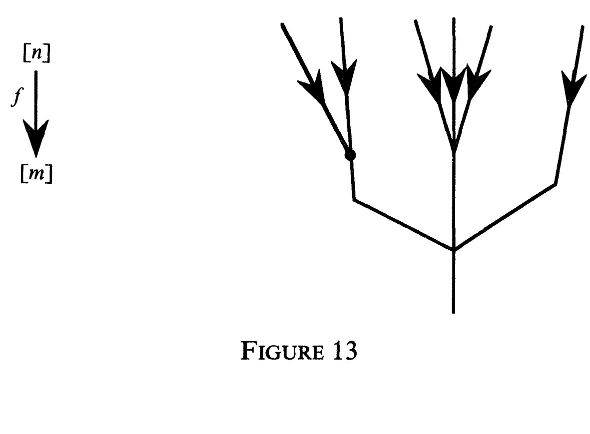

FIGURE 13

This tree has one vertex (root) at the bottom and several vertices at the top. We have

$$(4.1.12)\qquad \mathscr{P}(m)\otimes\mathscr{P}^!(f)=\frac{\bigcap_{\substack{\text{1-ternary trees } T:\\ T\geqslant T(f),\, v(T)\to\text{top}}} R_T}{\left(\sum_{\substack{\text{1-ternary trees } T:\\ T\geqslant T(f),\, v(T)\to\text{bottom}}} R_T\right)\cap\left(\bigcap_{\substack{\text{1-ternary trees } T:\\ T\geqslant T(f),\, v(T)\to\text{top}}} R_T\right)}.$$

Applying d amounts to deleting some R_T from the intersection in the numerator of (4.1.12) and adding some other R_T to the sum in the denominator, according to surjections $g: [m+1]\to[m]$. Hence applying d^2 to (4.1.12) moves some term R_T from the numerator to the denominator. Therefore $d^2=0$.

Now we state the main result of this section.

(4.1.13) THEOREM. *Let $\mathscr{P}$ be a quadratic operad. The following conditions are equivalent:*

(i) *$\mathscr{P}$ is Koszul.*

(ii) *The Koszul complexes $\mathscr{P}((\mathscr{P}^!)^\vee)(n)$ are exact for all $n\geqslant 2$.*

(4.1.14) We prove Theorem 4.1.13 by reducing it to a similar result about quadratic associative algebras or, rather, quadratic categories, a result which was proven in the required generality in [7].

For any operad $\mathscr{P}$ we construct, following Mac Lane, a certain PROP [1]. By definition, this is a category $\mathrm{Cat}(\mathscr{P})$ whose objects are symbols $[n]$, $n=0, 1, 2, \ldots$. The morphisms are defined by

$$\mathrm{Hom}_{\mathrm{Cat}(\mathscr{P})}([n],[m])=\bigoplus_{f:[n]\to[m]}\mathscr{P}(f),\qquad \text{for } n\geqslant m$$

where f runs over all surjections $[n]\to[m]$. For $n<m$ we set $\mathrm{Hom}_{\mathrm{Cat}(\mathscr{P})}([n],[m])=0$. The composition in the category $\mathrm{Cat}(\mathscr{P})$ is induced by maps $\mu_{g,f}$; see (4.1.5). It is well known [1] that $\mathrm{Cat}(\mathscr{P})$ has a natural structure of a symmetric monoidal category given on objects by $[m]\otimes[n]=[m+n]$. We write $\mathscr{P}(n,m)=\mathrm{Hom}_{\mathrm{Cat}(\mathscr{P})}([n],[m])$. Thus the space $\mathscr{P}(n)$ of our operad can be written as $\mathscr{P}(n,1)$.

If $\mathscr{P}$ is a quadratic operad, then $\mathrm{Cat}(\mathscr{P})$ is a quadratic category ($\mathbf{Z}_+$-algebra) in the sense of [7], §3, and $\mathrm{Cat}(\mathscr{P}^!)$ is the dual quadratic category. Let us compare the cobar-duals and the Koszul duals for $\mathscr{P}$ and $\mathrm{Cat}(\mathscr{P})$. The cobar-dual category of $\mathrm{Cat}(\mathscr{P})$ is the category $\mathbf{D}(\mathrm{Cat}(\mathscr{P}))$ with the same objects $[m]$ as $\mathrm{Cat}(\mathscr{P})$. The complex $\mathrm{Hom}_{\mathbf{D}(\mathrm{Cat}(\mathscr{P}))}([n],[m])$ is given by

$$\mathscr{P}(n,m)^*\to\bigoplus_{n>r>m}\mathscr{P}(n,r)^*\otimes_{\mathscr{P}(r,r)}\mathscr{P}(r,m)^*$$

$$\to\bigoplus_{n>r_1>r_2>m}\mathscr{P}(n,r_1)^*\otimes_{\mathscr{P}(r_1,r_1)}\mathscr{P}(r_1,r_2)^*\otimes_{\mathscr{P}(r_2,r_2)}\mathscr{P}(r_2,m)^*\to\cdots.$$

The category $\mathrm{Cat}(\mathscr{P})$ is Koszul if and only if each of these complexes is exact off the rightmost term. Observe that

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Cat}(\mathcal{P}))}([n],[m]) = (\mathrm{Hom}_{\mathbf{D}(\mathrm{Cat}(\mathcal{P}))}([n],[1]))^{\otimes m}$$

and

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Cat}(\mathcal{P}))}([n],[1]) = \mathbf{D}(\mathcal{P})(n)$$

is the nth complex of the cobar-operad $\mathbf{D}(\mathcal{P})$. This gives the following.

(4.1.15) LEMMA. *A quadratic operad $\mathcal{P}$ is Koszul if and only if* $\mathrm{Cat}(\mathcal{P})$ *is a Koszul quadratic category.*

Let us now recall the interpretation of Koszulness for categories in terms of Koszul complexes. Let $\mathcal{Q} = \mathcal{P}^!$. For the category $\mathrm{Cat}(\mathcal{P})$ we have, according to [7], n. 4.4, the complexes in the form

$$\{\mathcal{Q}(n,m)^* \to \mathcal{Q}(n,m+1)^* \otimes_{K[\Sigma_{m+1}]} \mathcal{P}(m+1,m) \to \mathcal{Q}(n,m+2)^* \otimes_{K[\Sigma_{m+2}]} \mathcal{P}(m+2,m)$$

$$\to \cdots \mathcal{Q}(n,n-1)^* \otimes_{K[\Sigma_{n-1}]} \mathcal{P}(n-1,m) \to \mathcal{P}(n,m)\}.$$

Let us denote the above complex by $\mathbf{K}^\bullet(n,m)$. Again, $\mathbf{K}^\bullet(n,m)$ is the mth tensor power of the complex $\mathbf{K}^\bullet(n,1)$ and $\mathbf{K}^\bullet(n,1)$ is the nth Koszul complex of the quadratic operad $\mathcal{P}$. Thus Koszulness of $\mathcal{P}$ is equivalent to the exactness of all the Koszul complexes for $\mathrm{Cat}(\mathcal{P})$ and hence it is equivalent to exactness of the complexes $\mathbf{K}^\bullet(n,1)$ only, i.e., of Koszul complexes of $\mathcal{P}$. This concludes the proof.

4.2. *Homology of algebras over a quadratic operad and Koszulness*

(4.2.1) Given a quadratic operad $\mathcal{P} = \mathcal{P}(K, E, R)$ and a $\mathcal{P}$-algebra A, we define a chain complex $(C^{\mathcal{P}}_\bullet(A), d)$ as follows. We put

$$C_n(A) = C^{\mathcal{P}}_n(A) = (A^{\otimes n} \otimes_{K^{\otimes n}} \mathcal{P}^!(n)^\vee)_{\Sigma_n}.$$

Recall (2.1.9) that the superscript "$\vee$" means the dual vector space with the Σ_n-action obtained from the dual one by twisting with the sign representation. To define $d_n\colon C_n(A) \to C_{n-1}(A)$, we first define a map

$$\bar{d}_n\colon \Bigg(A^{\otimes n} \otimes \Bigg(\bigoplus_{\substack{\text{binary}\\ n\text{-trees } T}} E(T) \otimes \mathrm{Det}(T)\Bigg)\Bigg)_{\Sigma_n}$$

$$\to \Bigg(A^{\otimes(n-1)} \otimes \Bigg(\bigoplus_{\substack{\text{binary}\\ (n-1)\text{-trees } S}} E(S) \otimes \mathrm{Det}(S)\Bigg)\Bigg)_{\Sigma_{n-1}},$$

where $E(T)$ is as in (1.2.13). If T is an n-tree, we call a vertex $v \in T$ *extremal* if all inputs of v are inputs of T. For such a v we define the set $[n]/v$ obtained by replacing the subset $\mathrm{In}(v) \subset [n]$ by a single element. Let T/v be the $[n]/v$-tree obtained by erasing v and replacing it by an external edge.

Let T be a binary n-tree and $v \in T$ an extremal vertex. The $\mathscr{P}$-action on A defines a map

$$\bar{d}_{T,v}\colon A^{\otimes n} \otimes E(T) \otimes \mathrm{Det}(T) \to A^{\otimes [n]/v} \otimes E(T/v) \otimes \mathrm{Det}(T/v).$$

We define $\bar{d}_n$ to be given by the matrix with entries $\bar{d}_{T,v}$. (Note that the ambiguity in numbering the set $[n]/v$ by $1, 2, \ldots, n-1$ will disappear after we take coinvariants of Σ_{n-1}.) Observe that $C_n(A) \subset (A^{\otimes n} \otimes (\bigoplus E(T) \otimes \mathrm{Det}(T)))_{\Sigma_n}$, because $\mathscr{P}^!(n)^\vee$ is a subspace in $\bigoplus E(T) \otimes \mathrm{Det}(T)$.

(4.2.2) PROPOSITION. *For any n the map $\bar{d}_n$ takes the subspace $C_n(A)$ into $C_{n-1}(A)$. The maps d_n, defined as restrictions of $\bar{d}_n$ to $C_n(A)$, satisfy $d_{n-1} \circ d_n = 0$.*

Proof. Let X be an element of $A^{\otimes n} \otimes R_T$ for some 1-ternary tree T (here R_T stands for the space of relations at the ternary vertex of T). Let $v \in T$ be the ternary vertex, and let T_1, T_2, T_3 be the binary trees obtained by splitting this vertex. Suppose first that the vertex $v \in T$ is extremal. Let e_i be the edge of T_i which is contracted into v. Let also $v_i \in T_i$ be the source of the edge e_i. Then v_i is an extremal vertex. Therefore, after erasing v_i, the edge e_i will become external and cannot be contracted.

If v is not extremal, then all the extremal vertices of T_i come from extremal binary vertices of T. If $w \in T$ is such a vertex, then T/w is 1-ternary and $\sum_{i=1}^3 \bar{d}_{T_i,w}(X)$ belongs to $R_{T/w}$. Clearly all 1-ternary $(n-1)$-trees can be represented as T/w. So if $X \in A^{\otimes n} \otimes R_T$ for all 1-ternary n-trees T, then $\bar{d}_n(X) \in A^{\otimes(n-1)} \otimes R_S$ for all 1-ternary $(n-1)$-trees S. This shows that $\bar{d}_n(C_n(A)) \subset C_{n-1}(A)$.

Let us prove that $\bar{d}_{n-1} \circ \bar{d}_n = 0$ on $C_n(A)$. Let T be a 1-ternary tree T whose 1-ternary vertex v is extremal. Suppose that $X \in A^{\otimes n} \otimes R_T \subset \bigoplus_{i=1}^3 A^{\otimes n} \otimes E(T_i)$ where T_i are as before. Denote by w_i the end of the edge e_i. Then w_i will become extremal after erasing v_i. Thus

$$\sum_{i=1}^{3} \bar{d}_{T_i/e_i, w_i}(\bar{d}_{T_i, v_i}(X))$$

is a sum of quantities like $r(a, b, c)$ with $r \in R \subset F(E)(3)$ and $a, b, c \in A$. Since A is a P-algebra and R is the space of relations for P, every such quantity is equal to 0. This implies that $\bar{d}_{n-1}(\bar{d}_n(X)) = 0$ if $X \in \bigcap_T A^{\otimes n} \otimes R_T$. The proposition follows.

(4.2.3) *Definition.* The complex $C_\bullet(A) = \bigoplus C_n(A)$ with the differential d defined above is called the chain complex of the $\mathscr{P}$-algebra A. Its homology will be denoted $H_n(A)$ or $H_n^{\mathscr{P}}(A)$.

(4.2.4) *Examples.* (a) Let $\mathscr{P} = \mathscr{A}s$ be the associative operad. The dual operad $\mathscr{P}^!$ is isomorphic to $\mathscr{P}$ itself. Let A be an associative algebra. Since $\mathscr{P}^!(n)$ is the regular representation of Σ_n, we find that the complex $C_\bullet(A)$ has the form

$$\cdots \to A \otimes A \otimes A \to A \otimes A \to A.$$

A straightforward calculation of the differential shows that it is the standard Hochschild complex of A with coefficients in k. Thus, $H_i^{\mathscr{P}}(A)$ is the Hochschild homology $HH_i(A, k)$.

(b) Let $\mathscr{P} = \mathscr{L}ie$ be the Lie operad. Then $\mathscr{P}^! = \mathscr{C}om$ is the commutative operad; in particular, $\mathscr{P}^!(n)$ is the trivial 1-dimensional Σ_n-module and $\mathscr{P}^!(n)^\vee$ is the sign representation. Let $\mathscr{G}$ be a Lie algebra. By the above, its chain complex, as a $\mathscr{P}$-algebra, has the form

$$\cdots \to \Lambda^3\mathscr{G} \to \Lambda^2\mathscr{G} \to \mathscr{G}.$$

Again, a straightforward calculation of the differential shows that this is the standard Chevalley-Eilenberg chain complex of $\mathscr{G}$, so that $H_i^{\mathscr{P}}(\mathscr{G}) = H_i(\mathscr{G}, k)$ is the Lie algebra homology of $\mathscr{G}$ with constant coefficients.

(c) Similarly, if $\mathscr{P} = \mathscr{C}om$ is the commutative operad, and A is a commutative algebra, one finds that $H_i^{\mathscr{P}}(A)$ is the Harrison homology of A; see, e.g., [45].

(4.2.5) Theorem. *Let $\mathscr{P}$ be a quadratic operad. Then $\mathscr{P}$ is Koszul if and only if for any free $\mathscr{P}$-algebra $F = F_{\mathscr{P}}(V)$ we have $H_i^{\mathscr{P}}(F) = 0$ for $i > 0$.*

Proof. Let F_n denote the free $\mathscr{P}$-algebra on generators $x_1, \ldots, x_n$. This algebra has an obvious $(\mathbf{Z}_+)^n$-grading such that $\deg(x_i) = (0, \ldots, 1, \ldots, 0)$ (the unit on the ith place) and $\deg(\mu(a, b)) = \deg(a) + \deg(b)$ for any $a, b \in F_n$, $\mu \in \mathscr{P}(2)$.

The chain complex $C_\bullet^{\mathscr{P}}(F_n)$ also inherits this grading. The following result, which is immediate from the definitions, implies the "if" part of Theorem 4.2.5.

(4.2.6) Proposition. *The nth Koszul complex $K_\bullet^{\mathscr{P}}(n)$ of $\mathscr{P}$ is isomorphic to the multihomogeneous part of $C_\bullet^{\mathscr{P}}(F_n)$ of multidegree $(1, \ldots, 1)$.*

Before proving the "only if" part of (4.2.5), let us mention the following corollary from what we have already done.

(4.2.7) Corollary. *The operads $\mathscr{A}s$, $\mathscr{C}om$, $\mathscr{L}ie$ are Koszul.*

Proof. It is well known that a free associative algebra without unit has higher Hochschild homology trivial and a free Lie algebra has higher homology with constant coefficients trivial. This proves that $\mathscr{A}s$ and $\mathscr{L}ie$ are Koszul. The case of $\mathscr{C}om$ follows by duality (4.1.4).

Another situation of applicability of Theorem 4.2.5 is provided by the work of E. Getzler and J. D. S. Jones [23]. They proved, in the context of their work on iterated integrals, the vanishing of the homology of free algebras over the operads associated to homology of configuration space.

(4.2.8) *End of the proof of Theorem 4.2.5.* Let A be any $\mathscr{P}$-algebra. Along with the chain complex $C_\bullet^{\mathscr{P}}(A)$, we introduce the "big" chain complex $BC_\bullet^{\mathscr{P}}(A)$. By definition,

$$BC_n^{\mathscr{P}}(A) = \bigoplus_{\substack{i+j=n \\ j\leqslant 0}} \left(\bigoplus_{\substack{i\text{-trees } T \\ |T|=i-2+j}} A^{\otimes i} \otimes \mathscr{P}(T) \otimes \mathrm{Det}(T) \right)_{\Sigma_i}. \tag{4.2.9}$$

The differential $d\colon BC_n^{\mathscr{P}}(A) \to BC_{n-1}^{\mathscr{P}}(A)$ is given by the following two types of matrix elements:

$$d_{T,e}\colon A^{\otimes i} \otimes \mathscr{P}(T) \otimes \mathrm{Det}(T) \to A^{\otimes i} \otimes \mathscr{P}(T/e) \otimes \mathrm{Det}(T/e)$$

defined for any i-tree T and any internal edge $e \in \mathrm{Ed}(T)$, and

$$d_{T,v}\colon A^{\otimes i} \otimes \mathscr{P}(T) \otimes \mathrm{Det}(T) \to A^{\otimes [i]/v} \otimes \mathscr{P}(T/v) \otimes \mathrm{Det}(T/v)$$

defined for any i-tree T and any extremal vertex $v \in T$.

The operator $d_{T,e}$ is $1 \otimes \mu_{T,e} \otimes l_e^*$ where $\mu_{T,e}\colon \mathscr{P}(T) \to \mathscr{P}(T/e)$ is induced by the composition in $\mathscr{P}$ and the map $l_e^*\colon \mathrm{Det}(T) \to \mathrm{Det}(T/e)$ is dual to the exterior multiplication by e. The operator $d_{T,v}$ is induced, in a similar way, by the $\mathscr{P}$-action on A.

It is immediate to verify that $d^2 = 0$. The embedding $\mathscr{P}^!(n)^\vee \subset \bigoplus_{\text{binary trees } T} R(T)$ gives rise to an embedding of complexes

$$j\colon C_\bullet^{\mathscr{P}}(A) \to BC_\bullet^{\mathscr{P}}(A). \tag{4.2.10}$$

The complex $BC_\bullet^{\mathscr{P}}(A)$ has an increasing filtration F with the mth term $F_m BC_n^{\mathscr{P}}(A)$ being the sum of the summands in (4.2.9) with $i \leqslant m$. The associated graded complex has the form

$$\mathrm{gr}_m^F BC_\bullet^{\mathscr{P}}(A) = A^{\otimes m} \otimes \mathbf{D}(\mathscr{P})(m)^\vee$$

where D is the duality for dg-operads (3.2.13). This implies the following result.

(4.2.11) PROPOSITION. *If the operad $\mathscr{P}$ is Koszul, then the embedding (4.2.10) is a quasi isomorphism for any $\mathscr{P}$-algebra A.*

To complete the proof of Theorem 4.2.5, it is enough, in view of Proposition 4.2.11, to establish the following.

(4.2.12) PROPOSITION. *Let $\mathscr{P}$ be any k-linear operad, $K = \mathscr{P}(1)$, and $A = F_{\mathscr{P}}(V)$ the free $\mathscr{P}$-algebra generated by a finite-dimensional K-module V. Then*

$$H_i(BC_\bullet^{\mathscr{P}}(A)) = \begin{cases} 0, & i > 0 \\ V, & i = 0. \end{cases}$$

Proof. To each tree T and a function $\nu\colon \mathrm{In}(T) \to \{1, 2, 3, \ldots\}$ we associate the vector space

$$C(T, \nu) = \mathrm{Det}(T) \otimes \bigotimes_{v \in T} \mathscr{P}(\mathrm{In}(v)) \otimes \bigotimes_{i \in \mathrm{In}(T)} A_{\nu(i)}$$

where A_n, $n = 1, 2, 3, \ldots$, denotes the degree-n component of A. By construction, the graded vector space $BC_{\bullet}^{\mathscr{P}}(A)$ is the direct sum of spaces $C(T, \nu)$ over all (isomorphism classes of) pairs (T, ν).

Given a pair (T, ν), we call an input edge $e \in \mathrm{In}(T)$ *nondegenerate* if $\nu(e) > 1$. Given a nondegenerate edge $e \in \mathrm{In}(T)$, we define the *degeneration* of (T, ν) along e as the pair $(\hat{T}, \hat{\nu})$ where $\hat{T}$ is obtained by attaching at e a star (1.1.1) with $\nu(e)$ inputs. The function $\hat{\nu}$ is set equal to 1 on the new inputs and remains unchanged on the other inputs. A pair (T, ν) is called *nondegenerate* if for any extremal vertex $v \in T$ there is an edge $e \in \mathrm{In}(v)$ such that $\nu(e) > 1$. A nondegenerate pair cannot be obtained by degeneration.

Let $\mathrm{Nd}(T, \nu)$ denote the number of nondegenerate input edges of T. We introduce an increasing filtration G on $BC_{\bullet}^{\mathscr{P}}(A)$ by putting

$$G_i BC_{\bullet}^{\mathscr{P}}(A) = \bigoplus_{\#(\text{vertices of } T) + \mathrm{Nd}(T, \nu) \leqslant i} C(T, \nu).$$

The differential preserves the filtration. Furthermore, the complex $\mathrm{gr}_i^G\, BC_{\bullet}(A)$ splits into a direct sum of complexes $E_{\bullet}(T, \nu)$ labelled by nondegenerate pairs (T, ν). These complexes have the form

$$E_{\bullet}(T, \nu) = \bigoplus_{(\hat{T}, \hat{\nu})} C(\hat{T}, \hat{\nu})$$

where the sum runs over all possible pairs $(\hat{T}, \hat{\nu})$ obtained from (T, ν) by (iterated) degeneration. Such pairs $(\hat{T}, \hat{\nu})$ are parametrized by all possible subsets of the set S of nondegenerate inputs of T. It is clear that the corresponding spaces $C(\hat{T}, \hat{\nu})$ are all the same. Observe that subsets of S correspond to faces of a simplex with $|S|$ vertices (denote this simplex by Δ). Moreover, we find that $E_{\bullet}(T, \nu)$ is the tensor product of the fixed vector space $C(T, \nu)$ and the augmented chain complex of Δ. This complex is acyclic unless $S = \varnothing$. The only nondegenerate pair (T, ν) with $S = \varnothing$ consists of $T = \{\to\}$ (the tree with no vertices) and $\nu = 1$. This gives $H_0(BC_{\bullet}(A)) = V$, and the theorem follows.

(4.2.13) *Homotopy $\mathscr{P}$-algebras.* Let $\mathscr{P} = \mathscr{P}(K, E, R)$ be a Koszul quadratic operad and $\mathscr{P}^!$ its quadratic dual. Then the canonical morphism of operads (4.1.1) $\gamma_{\mathscr{P}}\colon \mathbf{D}(\mathscr{P}^!) \to \mathscr{P}$ is a quasi isomorphism. Hence, any $\mathscr{P}$-algebra can be viewed as a $\mathbf{D}(\mathscr{P}^!)$-algebra. This motivates the following.

(4.2.14) *Definition.* A *dg*-algebra over $\mathbf{D}(\mathscr{P}^!)$ is called a homotopy $\mathscr{P}$-algebra.

(4.2.15) PROPOSITION. *Let $A = \bigoplus A_n$ be a graded K-bimodule with* $\dim(A_n) < \infty$ *for all n. Giving a structure of a homotopy $\mathscr{P}$-algebra on A is the same as giving a nonhomogeneous differential d on the free algebra $F_{\mathscr{P}^!}(A^*[1])$ satisfying the conditions:*

(i) $d^2 = 0$;

(ii) *d is a derivation with respect to any binary operation in $\mathscr{P}^!$, i.e., we have*

$$d(\mu(a, b)) = \mu(d(a), b) + (-1)^{\deg(a)}\mu(a, d(b)), \qquad \mu \in \mathscr{P}(2),\ a, b \in \mathscr{F}_{\mathscr{P}^!}(A^*[1]).$$

Proof. Let d be a differential in the free algebra $F = F_{\mathscr{P}^!}(A^*[1])$ satisfying (i) and (ii). Recall that the free algebra has a natural grading $F = \bigoplus_{n \geqslant 1} F_n$ where (1.3.5)

$$F_n = (\mathscr{P}^!(n) \otimes_{K^{\otimes n}} (A^*)^{\otimes n})_{\mathrm{sgn}}. \tag{4.2.16}$$

(The subscript "sgn" stands for the "anti-invariants" of the symmetric group action.) We decompose the differential d into homogeneous components $d = d_1 + d_2 + \cdots$ where the component d_i shifts degree by $i - 1$, i.e., $d_i\colon F_j \to F_{j+i-1}$, for all j. The equality $d^2 = 0$ yields, in particular, $d_1^2 = 0$.

Observe that, with the operad $\mathscr{P}^!$ being quadratic, the algebra F is generated by its degree-1 component $F_1 = \mathscr{P}^!(1) \otimes_K A^* = A^*$. Hence the differential d is completely determined, due to the property (ii), by its restriction to F_1. Separating the degrees, we see that this restriction is given by a collection of maps $d_m\colon A^* = F_1 \to F_m$, $m = 1, 2, \ldots$. In view of (4.2.16), we may view d_m as a Σ_m-invariant element

$$d_m \in \mathrm{Det}(k^m) \otimes \mathscr{P}^!(m) \otimes_{K^{\otimes m}} (A^*)^{\otimes m} \otimes_K A. \tag{4.2.17}$$

Next, we replace, using (1.1.6), the integer m in (4.2.17) by any m-element set. Further, for any n-tree T, we form the tensor product of the elements d_m over all vertices of T to get an element $\bigotimes_{v \in T} d_{\mathrm{In}(v)}$. Rearranging the factors in the tensor product, we get

$$\bigotimes_{v \in T} d_{\mathrm{In}(v)} \in \mathrm{Det}(T) \otimes \bigotimes_{v \in T} \left(\mathscr{P}^!(\mathrm{In}(v)) \otimes_K \left(\bigotimes_{e \in \mathrm{In}(v)} A^* \right) \otimes_K A \right) \tag{4.2.18}$$

where the factor A on the right corresponds to the output edge of v and $\mathrm{Det}(T)$ was introduced in (3.2.0). Each internal edge of T occurs in the above tensor product twice, once as an input at some vertex, contributing to the factor A^* and once as an output at some vertex, contributing to the factor A. Contracting the pairs of factors A^* and A corresponding to each internal edge and using the notation (1.2.13), we obtain from (4.2.18) a well-defined element

$$d(T) \in \mathrm{Det}(k^n) \otimes \mathscr{P}^!(T) \otimes_K (A^*)^{\otimes n} \otimes A.$$

This element can be regarded a morphism

$$d(T)\colon \mathscr{P}^!(T)^\vee = \mathscr{P}^!(T)^* \otimes \mathrm{Det}(k^n) \to \mathrm{Hom}(A^{\otimes n}, A). \tag{4.2.19}$$

The morphisms (4.2.19), assembled for various n-trees together, define, for each n, a morphism

$$\mathbf{D}(\mathscr{P}^!)(n) \to \mathrm{Hom}(A^{\otimes n}, A) = \mathscr{E}_A(n).$$

One can check by a direct calculation that these morphisms give rise to a morphism of operads $\mathbf{D}(\mathscr{P}^!) \to \mathscr{E}_A$ if and only if the original differential d on the free algebra F satisfies the property (i) of Proposition 4.2.15. That makes A a $\mathbf{D}(\mathscr{P}^!)$-algebra. The opposite implication is proved by reversing the above argument.

(4.2.20) *Examples.* (a) Let $\mathscr{P} = \mathscr{L}ie$ be the Lie operad so $\mathscr{P}^! = \mathscr{C}om$. Since $\mathscr{C}om(n) = k$ for any n, a structure of a $\mathbf{D}(\mathscr{C}om)$-algebra on a dg-vector space A is determined by specification of n-ary antisymmetric operations $[x_1, \dots, x_n]$ for any $n \geqslant 2$. (These operations correspond to the basis vectors of $\mathscr{C}om(n)$.) Proposition 4.2.15 in this case gives the equivalence of two definitions [59] of a homotopy Lie algebra: the first as a vector space with brackets $[x_1, \dots, x_n]$ satisfying the generalized Jacobi identity, and the second as a vector space A with a differential on the exterior algebra $\bigwedge^\bullet (A^*)$, satisfying the Leibniz rule.

(b) If we take $\mathscr{P} = \mathscr{C}om$, we get a notion of a homotopy $\mathscr{C}om$-algebra which is a dg-algebra over $\mathbf{D}(\mathscr{L}ie)$. Such algebras are particular cases of algebras "associative and commutative up to all higher homotopies" (May algebras) [27], [41], [42]. More precisely, a homotopy $\mathscr{C}om$-algebra is strictly commutative and equipped with a system of natural homotopies which ensure, in particular, the associativity of the cohomology algebra. This structure is not the same as a homotopy between ab and ba in an associative dg-algebra, the data often referred to as "homotopy commutativity".

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