Let $\{X_\alpha\}$ be an arbitrary decomposition into simple cells which satisfies condition (A) of Sec. 5 of [2]. Let $P = -(1/8\pi^2)\Pi_2$ be the form which corresponds to the first Pontryagin class of the manifold X. Then the following assertion holds.

THEOREM 5.8. The cohomology class $P(X) - \Phi(P, X)$ coincides with the class which is the dual to the Poincaré combinatorial cycle $\Gamma_{n-4}(X)$, defined in Sec. 10 of [2].

The proofs of Theorems 5.7 and 5.8 are rather long and we do not give them here.

Remark. If the decomposition $\{X_\alpha\}$ is linearizable then Theorem 2 of [2] follows from Theorem 5.7 (for the case where the decomposition is into simple cells); and Theorem 5.8 implies Theorem 1 of [2].

LITERATURE CITED

1. M. F. Atiyah, V. K. Patodi, and I. M. Singer, "Spectral asymmetry and Riemannian geometry," Bull. London Math. Soc., 5, 229-234 (1973).
2. A. M. Gabriélov, I. M. Gel'fand, and M. B. Locik, "Combinatorial calculation of characteristic classes," Funktsional'. Analiz i Ego Prilozhen., 9, No. 2, 12-28 (1975); No. 3, 5-26 (1975).
3. D. B. Fuchs, A. M. Gabriélov, and I. M. Gel'fand, "The Gauss–Bonnet formula and the Atiyah–Patodi–Singer functionals for characteristic classes of foliations," Topology (1976).

COVERINGS OF p-ADIC SYMMETRIC REGIONS

V. G. Drinfel'd

Let K be a local non-Archimedean field, d a natural number, and Ω^d the set of points $\mathbf{P}_K^{d-1}$, not lying on any hyperplane defined over K. Ω^d is a strict analytic space over K, on which PGL(d, K) acts (see [1]). As opposed to Archimedean symmetric regions, Ω^d is not simply connected. Our basic aim in this article is to construct some system of finite étale equivariant coverings of Ω^d (see §3). The Galois group of this system is the profinite completion of the multiplicative group of the central skew field over K with invariant 1/d. It is probable that the cohomologies of these coverings are a realization of all the cuspidal representations of GL(d, K).

In fact the analytic space Ω^d arises from some formal scheme $\hat{\Omega}^d$ over the ring of integers of K. The interpretation of $\hat{\Omega}^d$ as a variety of modules of certain formal groups plays a basic part in constructing a system of coverings of Ω^d (see §2). For d = 1, our method of constructing coverings of Ω^d becomes the Lyubin–Tate construction of the maximal Abelian extension of K.

In §4, by means of a local theorem in §2 (for $K = Q_p$, d = 2), we prove again a special case of I. V. Cherednik's theorem (see [3]). In [3] it is proved that certain curves over Q which admit complex uniformization, also admit p-adic uniformization; I. V. Cherednik uses an indirect method. The method of §4 is based on interpreting the curves over Q as varieties of modules of certain Abelian varieties; it turns out that these Abelian varieties have supersingular reduction by the module p; this allows us to connect the local theorem in §2 with the Serre–Tate theorem.

The only published proof known to the author of the Serre–Tate theorem (see [5]) uses the theory of crystals, associated with p-divisible groups. Therefore, at the end of this article there is a proof of the Serre–Tate theorem, which uses only commonly known facts about Abelian schemes and schemes of finite groups.

The ideas in this article belong to P. Deligne and D. A. Kazhdan; they explained how I. V. Cherednik's theorem indicates the existence of an equivariant system of coverings of Ω^d. Moreover, P. Deligne supplied a convenient definition of the formal scheme $\hat{\Omega}^d$.

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§1. THE DIEUDONNÉ – CARTIER THEORY FOR FORMAL MODULES

Let O be the ring of integers of a local non-Archimedean field and B be an algebra over O.

Definition. A formal O module over B is a formal group over B (in the sense of [4]) together with an action on it by O, which induces the natural action on the tangent space.

Let q be the order of the residue field of O, and let π be a prime element of O. We denote by A the cancelling functor from the category of O algebras to the category of sets, and by A^∞ the functor which sends B to the product of a countable number of copies of the set B (these copies are enumerated by the nonnegative integers). For any nonnegative integer n we define the morphism $w_n: A \to A$ as follows:

$$w_n(x_0, x_1, x_2, \ldots) = \sum_{k=0}^{n} \pi^k x_k^{q^{n-k}}.$$

Define the morphism V: $A^\infty \to A^\infty$ by the formula $V(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots)$. We introduce the notation $[a] = (a, 0, 0, \ldots)$.

Proposition 1.1. There exists a unique method of lifting A^∞ to a functor into the category of O algebras such that w_n: $A^\infty(B) \to B$ is a homomorphism of O algebras for any n and for any O algebra B. (The functor into the category of O algebras obtained in this way will henceforward be denoted by W_O or simply W.) There exists a unique morphism F: $W \to W$ such that $w_n \circ F = w_{n+1}$. In addition, we have the following equations: $[ab] = [a]\cdot[b]$, ${}^F[a] = [a^q]$ for $a, b \in B$; ${}^{FV}x = \pi x$, ${}^V x \cdot y = {}^V(x \cdot {}^F y)$ for x, y $\in$ W(B); the mapping V is O-linear. If the image of π in B is equal to zero, then ${}^F(x_0, x_1, x_2, \ldots) = (x_0^q, x_1^q, x_2^q, \ldots)$. If the image of π in B is equal to zero and B is a perfect field, then there exists precisely one complete discretely normed O algebra with residue field B, in which π is a prime element; namely, W(B).

For any O algebra B consider the (noncommutative) O algebra generated by the ring W(B) and by the two elements F and V, with the following relations: $VaF = {}^V a$, $Fa = {}^F a \cdot F$, $aV = V \cdot {}^F a$, and FV = π for $a \in$ W(B) (here Fa is the composition of the generator F onto the element $a \in$ W(B), whereas ${}^F a$ is the result of applying the mapping F to the element a). The completion of this O algebra in the topology given by the right ideals, spanned on V^n, will be denoted by $\mathrm{Cart}_O(B)$ or simply by Cart(B). Each element of Cart(B) may be uniquely written in the form $\sum_{m=0}^{\infty} V^m a_m + \sum_{n=1}^{\infty} b_n \cdot F^n$, where a_m, $b_n \in$ W(B), and $b_n = 0$ for all but a finite number of n. Moreover, each element of Cart(B) may also be written uniquely in the form $\sum_{n,m=0}^{\infty} V^n [x_{m,n}] F^m$, where $x_{m,n} \in$ B and for each n $x_{m,n} = 0$ for sufficiently large m.

Note. The construction of the functors W and Cart depended on the choice of π, but in fact these functors (to within canonical isomorphism) do not depend on the choice of π.

Let $O \to O'$ be a homomorphism, and let f be the degree of extension of the residue field.

Proposition 1.2. There exists a unique morphism of functors (on the category of O' algebras) u: $W_O \to W_{O'}$ such that $w'_n \circ u = w_{fn}$. For any O' algebra B we have the relations $u([a]) = [a]$ for $a \in$ B, $u(F^f x) = F'(u(x))$, $u({}^V x) = (\pi/\pi') \cdot {}^{V'}(u(F^{f-1}x))$ for $x \in W_0(B)$; all objects relating to O' are written with a dash.

Definition. Let B be an algebra over O. The Cart(B)-module M is called reduced if the action of V on M is injective, $M = \lim M/V^k M$, and M/VM is a free B module.

THEOREM. The category of formal O modules over B is equivalent to the category of reduced Cart(B) modules.

Proof. In the case when $O = Z_p$, the theorem is proved by Cartier (see [4]). Therefore, it is sufficient to prove that if $O \to O'$ is a homomorphism and the theorem is true for O, then it is also true for O'.

a) Let O' be nonramified over O, $f = [O' : O]$. Let B be an algebra over O'. A formal O' module over B is a reduced $\mathrm{Cart}_O(B)$-module M with an action by O', inducing the trivial action on M/VM. The natural homomorphism $O' \to B$ induces a homomorphism $O' \to W_O(B)$ [viz., the composition $O' \to W_O(O') \to W_O(B)$]. Therefore, $O' \otimes_O \mathrm{Cart}_O(B)$ is isomorphic to a produce of f copies of $\mathrm{Cart}_O(B)$. Therefore, we have a Z/dZ grading on M with the following properties: deg $a = 0$ for $a \in \mathrm{Cart}_O(B)$, deg $V = 1$, deg $F = -1$, V: $M^i \to M^{i+1}$ is an isomorphism for $i \neq -1$, and on M^0 both actions of O' coincide. Define the map $U: M^0 \to M^0$ by the formula $U = V^{1-d}F$. Associate with $\sum_{n,m=0}^{\infty} V'^n u(x_{m,n}) \cdot F'^m \in \mathrm{Cart}_{O'}(B)$ (where $x_{m,n} \in W_O(B)$, and $u: W_O(B) \to W_{O'}(B)$ is the homomorphism constructed above) the operator $\sum_{n,m=0}^{\infty} V^{fn} x_{m,n} U^m$. It is easily verified that this is well

defined. Starting with a formal O' module, we have constructed a reduced $\mathrm{Cart}_{O'}(B)$-module M^0. This functor is easily seen to be an equivalence.

b) Let the residue fields of O and O' have the same order, and let B be an algebra over O'. It is easily seen that there exists a unique continuous homomorphism $\mathrm{Cart}_O(B) \to \mathrm{Cart}_{O'}(B)$, whose limit on $W_O(B)$ coincides with the homomorphism u constructed above, taking V into V', and F into $(\pi/\pi') \cdot F'$. Using this homomorphism we may construct a functor from the category of reduced $\mathrm{Cart}_{O'}(B)$-modules into the category of formal O' modules. On the other hand, let M be a reduced $\mathrm{Cart}_O(B)$ module with an action by O', inducing the trivial action on M/VM. Then we may define a map Φ: $M \to M$ such that $V\Phi = \pi' - [\pi']$. Let $\varphi \in \mathrm{Cart}_{O'}(B)$ be an element such that $V'\varphi = \pi' - [\pi']$. Then each element of $\mathrm{Cart}_{O'}(B)$ may be written in the form $\sum_{n,\, m=0}^{\infty} V'^n u\,(x_{m,n})\,\varphi^m$, where $x_{m,n} \in W_O(B)$. Associate with the element $\sum V'^n u\,(x_{m,n}) \cdot \varphi^m$ the operator $\sum V^n x_{m,n} \Phi^m$. It can be verified that this is well defined and that the two functors thus constructed are inverses.

Henceforward, by a formal group over B we shall mean not only a formal group in the sense of [4], but also an object identified from formal groups over localizations of B. Correspondingly, in the definition of a reduced Cart(B) module we shall replace the word "free" with the word "projective."

§2. THE BASIC THEOREM

A) Formulation of the Basic Theorem. Let K be a fixed local non-Archimedean field. Let $O \subset K$ be the ring of integers, v: $K^* \to \mathbf{Z}$ a normalization, O^{nr} a strict Henselization of O, and $\hat{O}^{\mathrm{nr}}$ the completion of O^{nr}. Let D be the central skew field over K with invariant 1/d, and $O_D \subset D$ the ring of integers.

By a quasi-isogeny from the formal O module X into the formal O module Y we shall mean an element of $\mathrm{Hom}\,(X, Y) \otimes_O K$, for which there exists an inverse element in $\mathrm{Hom}\,(Y, X) \otimes_O K$. It is easily seen that if B is an algebra over O, and $I \subset B$ is an ideal such that $I^2 = 0$, $\pi I = 0$, and X, Y are formal O modules over B with X_0, Y_0 their reductions by the module I, then $\mathrm{Hom}\,(X, Y) \otimes_O K = \mathrm{Hom}\,(X_0, Y_0) \otimes_O K$.

Definition. A formal O_D module over the O algebra B is a formal O module together with an action by O_D which extends the action of O. A formal O_D module of dimension d, such that a representation of the maximal nonramified subring of O_D on the tangent space is a sum of d different characters in each geometrical point of Spec B, is called special.

Let us fix a special formal O_D module Φ over $O^{\mathrm{nr}}/(\pi)$ with dimension d and height d^2. [It is easy to see that up to isogeny there is precisely one such module, and $(\mathrm{End}\,\Phi) \otimes_O K$ is isomorphic to the algebra of matrices of order d over K.] Fix an isomorphism $\mathrm{Mat}\,(d, K) \simeq (\mathrm{End}\,\Phi) \otimes_O K$.

Let Nilp be the category of O algebras in which the image of π is nilpotent. We now define a functor G from Nilp into the category of sets. Let $B \in \mathrm{Ob\ Nilp}$; G(B) is by definition the set of isomorphism classes of triples consisting of 1) a homomorphism $\psi : O^{\mathrm{nr}}/(\pi) \to B/(\pi)$, 2) a special formal O_D-module Y over B with height d^2, and 3) a quasi-isogeny ρ of height 0 from $\psi_*(\Phi)$ into the reduction of Y by the module π. The group $GL(d, K) \times D^*$ acts on G(B) according to the formula

$$(g_1, g_2) \cdot (\psi, Y, \rho) = (\psi \circ \mathrm{Fr}^{-n},\ {}^{g_2}Y,\ \rho g_1^{-1} g_2^{-1}\,\mathrm{Frob}^n),$$

where $g_1 \in \mathrm{GL}\,(d, K)$, $g_2 \in D^*$, $n = v\,(\det g_1) + v\,(\det g_2)$, $\mathrm{Fr} : O^{nr}/(\pi) \to O^{nr}/(\pi)$ is the Frobenius morphism, and Frob^n is the natural quasi-isogeny $(\psi \cdot \mathrm{Fr}^{-n})_*\,(\Phi) \to \psi_*\,(\Phi)$; ${}^{g_2}Y$ is a formal O_D module coinciding with Y as an O module, but with a different O_D action; viz., $a \in O_D$ acts on ${}^{g_2}Y$ in the same way as $g_2^{-1} a g_2$ acts on Y. O_D^* and elements of the form (a, a^{-1}), $a \in K^*$, act on G trivially.

The aim of this paragraph is the proof of the following theorem.

THEOREM. We may represent the functor G by the formal scheme $\hat{\Omega}^d \hat{\otimes} \hat{O}^{nr}$. The action of $GL(d, K) \times D^*$ on the functor G corresponds to the natural action of PGL(d, K) on $\hat{\Omega}^d$ and the following action of $GL(d, K) \times D^*$ on O^{nr}: $(g_1, g_2) \mapsto (\mathrm{Fr})^{-v(\det g_1) - v(\det g_2)}$.

Here $\hat{\Omega}^d$ is the formal scheme over O defined by P. Deligne. We shall introduce a variant of his definition which will be useful in the future. We shall denote by $B[\Pi]$ the ring $B[\Pi]/(\Pi^d - \pi)$ together with the following $\mathbf{Z}/d\mathbf{Z}$ grading: $\deg b = 0$ for $b \in B$, $\deg \Pi = 1$. Let S be a scheme over O on which π is nilpotent. An S point of the formal scheme $\hat{\Omega}^d$ is by definition an object consisting of 1) a $\mathbf{Z}/d\mathbf{Z}$-graded planar $O[\Pi]$-sheaf η on S, 2) a $\mathbf{Z}/d\mathbf{Z}$-graded sheaf T of $O_S[\Pi]$ modules, whose homogeneous components are invertible, 3) a homogenous (degree 0) $O[\Pi]$-linear morphism $u : \eta \to T$ such that $\eta \otimes O_S \to T$ is an epimorphism, and 4) a K-linear

isomorphism $r : \underline{K}^d \xrightarrow{\sim} \eta^0 \otimes K$, for which the following conditions hold; a) let $S_i \subset S$ be the set of zeros of the morphism $\Pi : T^i \to T^{i+1}$; then $\eta^i |_{S_i}$ is a fixed sheaf with fiber O^d, b) for any $s \in S$ the map $\eta_s/\Pi\eta_s \to (T_s/\Pi T_s) \otimes \varkappa_s$ is injective, where $\varkappa_S$ is the residue field of s, and c) $\Lambda^d(\eta^i)|_{S_i} = \pi^{-i}\Lambda^d(\Pi^i r \underline{O}^d)|_{S_i}$ for $i \in \mathbf{Z}$.

B) Proof of the Basic Theorem. Let $O_d^{nr} \subset O^{nr}$ be a nonramified subring of degree d over O. Then O_D is generated by O_d^{nr} plus the element Π with the following relations: $\Pi^d = \pi$, $\Pi a = \mathrm{Fr}(a)\cdot\Pi$ for $a \in O_d^{nr}$. Therefore, the category of formal O_D modules over the O^{nr}-algebra B is equivalent to the category of $\mathbf{Z}/d\mathbf{Z}$-graded reduced Cart(B) modules [the ring Cart(B) is graded such that deg $a = 0$ for $a \in W(B)$, deg $V = 1$, deg $F = -1$], together with an endomorphism Π of degree 1 which commutes with the action of Cart(B) and satisfies $\Pi^d = \pi$. A formal O_D module is special if and only if all homogeneous components of the tangent space have dimension 1.

Proposition 2.1. Let B be an algebra over O^{nr}. Given a set of elements $a_{m,i} \in B$, where $m \in \mathbf{Z}$, $i \in \mathbf{Z}/d\mathbf{Z}$, $m \geq 0$, and $\prod_{i \in \mathbf{Z}/d\mathbf{Z}} a_{0,i} = \pi$, there exists a special formal O_D module over B and a homogeneous basis system $\{\gamma_i,\ i \in \mathbf{Z}/d\mathbf{Z}\}$ (γ_i lies in the i-th component of the Dieudonné module) such that $\Pi\gamma_i = \sum_{m,i} V^m [a_{m,i}]\gamma_{i+1-m}$. The pair, consisting of the formal O_D module and the basis system, possessing the properties enumerated above, is defined uniquely up to isomorphism.

Definition. Let Y be a special formal O_D module over the O^{nr}-algebra B, and M_Y its Dieudonné module. We call the factor of $(M_Y)^d$ by the subgroup generated by elements of the form $(0, 0, \ldots, 0, Vu, -\Pi u, 0, \ldots, 0)$ the modified Dieudonné module; we denote it by N_Y. Define the map $\lambda_Y : N_Y \to M_Y$ by the formula $(x_1, x_2, \ldots, x_d) \mapsto \Pi^{d-1}x_1 + \Pi^{d-2}Vx_2 + \ldots + V^{d-1}x_d$.

All operators acting on M_Y also act on N_Y, if they commute with λ_Y (but, as opposed to M_Y, the Cart(B)-module N_Y is not in general reduced). There exist canonical maps $N_Y \to M_Y/VM_Y$ (viz., $x_1, \ldots, x_d) \to x_1$ mod VM_Y), and $M_Y \hookrightarrow N_Y$ [viz., $x \to (0, 0, \ldots, x)$]. We introduce a grading on N_Y: $\deg(x_1, \ldots, x_d) = i$, if $\deg x_1 = \deg x_2 = \ldots = \deg x_d = i$. It is easily seen that if B is an algebra over K, then λ_Y is an isomorphism.

Proposition 2.2. There exists a unique functorial (with respect to B) method of defining, for each O^{nr} algebra B and each special formal O_D-module Y over B, a map $L_Y : M_Y \to N_Y$ such that $\lambda_Y L_Y = F$.

Proof. It is sufficient to consider those formal modules for which all the homogeneous components of the tangent space are free. Proposition 2.1 shows that it is sufficient to prove the inclusion $F(M_Y) \subset \lambda_Y(N_Y) = \Pi^{d-1}M_Y + \Pi^{d-2}VM_Y + \ldots + V^{d-1}M_Y$ (since in the universal case λ is injective). Let $\{\gamma_i,\ i \in \mathbf{Z}/d\mathbf{Z}\}$ be a homogeneous basis system in M, deg $\gamma_i = i$. Since $FVM_Y = \pi M_Y \subset \Pi^{d-1}M_Y$, it is sufficient to prove that $F\gamma_i \in \lambda_Y(N_Y)$. Since $\pi - [\pi] = {}^V\varepsilon$, where ε is an invertible element of $W(O)$, it is sufficient to prove that $(\pi - [\pi])\gamma_i \in V\lambda_Y(N_Y)$. Let $\Pi\gamma_i = [a_i]\gamma_{i+1} + Vx_i$. Then $(\pi - [\pi])\gamma_i = (\Pi^d - [\pi])\gamma_i = \Pi^{d-1}Vx_i + [a_i]\Pi^{d-2}Vx_{i+1} + \ldots + [a_i a_{i+1} \ldots a_{i-2}]Vx_{i-1}$, and it remains only to use the trivial relation $[a_j]M_Y^{i+1} \subset \Pi M_Y^i + VM_Y^i$.

Define the map $\varphi_Y : N_Y \to N_Y$ by the formula $\varphi_Y(x_1, \ldots, x_d) = L_Y(x_1) + (x_2, x_3, \ldots, x_d, 0)$ (this is easily seen to be well defined). Clearly, φ_Y preserves the grading and commutes with the action of $O[\Pi]$.

Proposition 2.3. Let B be an algebra over O^{nr}, $I \subset B$ an ideal such that $I^2 = 0$, $\pi I = 0$. Let Y be a special formal O_D module over B, and Y_0 the reduction of Y by the module I. Then the action of φ_Y on $\mathrm{Ker}(N_Y \to N_{Y_0})$ is nilpotent.

Now let B be an algebra over O^{nr} and let the image of π in B be nilpotent. Let Y be a special formal O_D module over B of height d^2. Define a sheaf η_Y on Spec B in the following way: If Spec $B' \subset$ Spec B is an open set, then $H^0(\mathrm{Spec}\ B',\ \eta_Y) \overset{\mathrm{def}}{=} \{z \in N_{Y\otimes B'} | \varphi_Y(z) = z\}$. Denote by T_Y the tangent sheaf to Y. The canonical maps $N_{Y\otimes B'} \to M_{Y\otimes B'}/VM_{Y\otimes B'}$ define a morphism $u_Y : \eta_Y \to T_Y$. Finally, defining a quasi-isogeny of degree 0 from Φ to the reduction of Y by the module π induces a morphism $r_Y : \underline{K}^d \to \psi_Y^0 \otimes K$.

Proposition 2.4. (η_Y, T_Y, u_Y, r_Y) satisfies the conditions a), b), and c) at the end of §2A). If B' is an algebra over B, then $(\eta_{Y\otimes B'}, T_{Y\otimes B'}, u_{Y\otimes B'}, r_{Y\otimes B'})$ is the inverse image of (η_Y, T_Y, u_Y, r_Y).

Proof. a) If $I \subset B$ is a nilpotent ideal and the proposition is proved for B/I, then it can be proved for B by means of Proposition 2.3.

b) If I_1, I_2 are ideals in B, $I_1 \cap I_2 = 0$ and the proposition is true for B/I_1 and B/I_2, then it is true for B as well.

c) It remains to consider the case when for some $i \in \mathbf{Z}/d\mathbf{Z}$ the morphism $\Pi: T_Y^i \to T_Y^{i+1}$ is equal to zero. It is easily seen that the subgroup $V^jM_Y^i \subset N_Y^{i+j}$ is invariant with respect to φ for $j > 0$, and moreover, that $\Pi V^jM_Y^i \subset V^{j+1}M_Y^i$. Clearly, for each $j > 0$ there exists an affine scheme Z_j over B such that $Z_j(B') = \{z \in N_{Y\otimes B'}^{i+j}/V^jM_{Y\otimes B'}^i \mid \varphi(z) = z\}$. From Proposition 2.3 it follows that the schemes are étale over B. If B is an algebraically closed field, then the first part of Proposition 2.4 may be immediately verified, for in this case $Z_{dj} = \eta_Y^i/\pi^j\eta_Y^i$, and for $0 < k < d$, Z_{dj-k} is the image of η_Y^{i-k} in Z_{dj}. Hence, it follows that even if B is not a field, Z_{dj} is an étale covering of Spec B with fiber $(O/(\pi)^j)^d$, and for $0 < k < d$, Z_{dj-k} is an open subset in Z_{dj}. r_Y induces a morphism of some d-dimensional O-submodule K^d into $\varprojlim_j Z_{dj}(B)$, which induces in each geometrical point of Spec E → Spec B an isomorphism $K^d \simeq K \otimes_O \varprojlim Z_{dj}(E)$. Hence, it follows that $Z_{dj} \simeq$ Spec B × $(O/(\pi)^j)^d$. Proposition 2.4 is now evident.

Introduce the notation: $H(B) \overset{\text{def}}{=} (\hat{\Omega}^d \hat{\otimes} \hat{O}^{nr})(B)$, $\mathrm{Nilp}_n \subset \mathrm{Nilp}$ is the complete subcategory consisting of those O algebras in which the image of π^n is equal to zero. We have constructed a morphism ξ: G → H (Proposition 2.4); it remains to prove that ξ is an isomorphism.

Proposition 2.5. If $\xi|_{\mathrm{Nilp}_1}$ is an isomorphism then ξ is an isomorphism.

Proof. Let $n \geq 1$ and $\xi|_{\mathrm{Nilp}_n}$ be an isomorphism. We prove that $\xi|_{\mathrm{Nilp}_{n+1}}$ is an isomorphism. Let $B \in \mathrm{Ob}\ \mathrm{Nilp}_{n+1}$. Fix $\alpha \in G(B/(\pi^n))$. For any $C \in \mathrm{Ob}\ \mathrm{Nilp}$ and any morphism $C \to B/(\pi^n)$, we denote by $G_\alpha(C)$ the inverse image of α in G(C), and by $H_\alpha(C)$ the inverse image of $\xi(\alpha)$ in H(C). It is necessary to prove that $G_\alpha(B) \simeq H_\alpha(B)$. If $H_\alpha(B) = \phi$, there is nothing to prove. If $H_\alpha(B) \neq \phi$ then $G_\alpha(B) \neq \phi$ (this follows from Propostion 2.1). It is easily seen that there exists a natural group structure on $G_\alpha(B/(\pi^n) \oplus (\pi^n))$ and on $H_\alpha(B/(\pi^n) \oplus (\pi^n))$, and that on $G_\alpha(B)$ and $H_\alpha(B)$ there exists a structure of principal homogeneous spaces over them, and that ξ is consistent with these structures. But $B/(\pi^n) \oplus (\pi^n) \in \mathrm{Ob}\ \mathrm{Nilp}_n$, and therefore, $G_\alpha(B/(\pi^n) \oplus (\pi^n)) \simeq H_\alpha(B/(\pi^n) \oplus (\pi^n))$.

It remains to prove that $\xi|_{\mathrm{Nilp}_1}$ is an isomorphism. It is not hard to prove that $G|_{\mathrm{Nilp}_1}$ may be represented by a scheme which is projective over $\hat{\Omega}^d \otimes O^{nr}/(\pi)$. Therefore, it is sufficient to prove the bijectivity of the map G(B) → H(B) in the case when $B = (O^{nr}/(\pi)[\varepsilon])/(\varepsilon^2)$, and this can be done immediately.

§3. THE BASIC CONSTRUCTION

Let X be a universal formal O_D module over $\hat{\Omega}^d \hat{\otimes} \hat{O}^{nr}$. It is easily seen that the action of (GL(d, K) × D*)/K* on $\hat{\Omega}^d \hat{\otimes} \hat{O}^{nr}$ lifts in a natural way to the action of this same group on the formal group X; in view of this the action of O_D^* coincides with the action arising from the O_D-module structure. For any natural number n we denote the kernel of the homomorphism $X \xrightarrow{\pi^n} X$ by Γ_n. It is easily seen that Γ_n is a finite planar scheme of $O_D/(\pi^n)$ modules over $\hat{\Omega}^d \hat{\otimes} \hat{O}^{nr}$, of order q^{d^2n}.

We introduce the notation $K^{nr} \overset{\text{def}}{=} O^{nr} \otimes_O K$, $\hat{K}^{nr} \overset{\text{def}}{=} \hat{O}^{nr} \otimes_O K$. Clearly, $\Gamma_n \otimes_{\hat{O}^{nr}} \hat{K}^{nr}$ is an étale covering of $\hat{\Omega}^d \hat{\otimes}_O \hat{K}^{nr}$ whose geometrical fibers are isomorphic to $O_D/(\pi^n)$. (Here $\Gamma_n \otimes_{\hat{O}^{nr}} \hat{K}^{nr}$ and $\hat{\Omega}^d \hat{\otimes}_O \hat{K}^{nr}$ are strict analytic spaces over $\hat{K}^{nr}$, corresponding to the formal schemes Γ_n and $\hat{\Omega}^d \hat{\otimes} \hat{O}^{nr}$.) Set $\Sigma^{d,n} \overset{\text{def}}{=} (\Gamma_n \otimes_{\hat{O}^{nr}} \hat{K}^{nr}) - (\Gamma_{n-1} \otimes_{\hat{O}^{nr}} \hat{K}^{nr})$ for $n > 0$, and $\Sigma^{d,0} \overset{\text{def}}{=} \hat{\Omega}^d \hat{\otimes} \hat{K}^{nr}$. (GL(d, K) × D*)/K* acts on $\Sigma^{d,n}$, and $\Sigma^{d,n}$ is a principal homogeneous $(O_D/(\pi^n))^*$ space over $\hat{\Omega}^d \hat{\otimes} \hat{K}^{nr}$. Multiplication by π induces equivariant epimorphisms $\Sigma^{d,n+1} \to \Sigma^{d,n}$.

Note. If B is a Tate algebra over K, the categories of finite étale algebras over $B \otimes K^{nr}$ and $B \hat{\otimes} \hat{K}^{nr}$ are mutually equivalent. Therefore, the projective system of étale coverings of $\hat{\Omega}^d \hat{\otimes}_{\hat{O}^{nr}} \hat{K}^{nr}$ constructed above defines a projective system of étale coverings of $\hat{\Omega}^d \otimes K$.

We shall now describe the connection between the spaces Σ constructed for different fields. Let K_1 be an extension of K of degree m: All objects relating to K_1 will have index 1 in the following. Let d_1 be a natural number, $d = d_1m$, D and D_1 the central skew fields over K and K_1 with invariants $1/d$ and $1/d_1$, and let $D_1 \to D$ be an embedding. For any formal O_{D_1}-module Y_1 we may construct a formal O_D module, which we shall denote by $O_D \otimes_{O_{D_1}} Y_1$ (the construction is the same as for ordinary modules). It is easily verified that if Y_1 is special, then so is $O_D \otimes_{O_{D_1}} Y_1$. Fix an embedding $O^{nr} \to O_1^{nr}$ and a special formal O_{D_1}-module Φ_1 over $O_1^{nr}/(\pi_1)$ of height d_1^2. Set $\Phi = O_D \otimes_{O_{D_1}} \Phi_1$. Fix isomorphisms $(\mathrm{End}\ \Phi_1) \otimes K \simeq \mathrm{Mat}(d_1, K_1)$ and $(\mathrm{End}\ \Phi) \otimes K \simeq \mathrm{Mat}(d, K)$; they induce an embedding $\mathrm{Mat}(d_1, K_1) \to \mathrm{Mat}(d, K)$. For any O_1-algebra $B \in \mathrm{Ob}\ \mathrm{Nilp}_1$, define a map $G_1(B) \to G(B)$ as follows: with the triple (ψ_1, Y_1, ρ_1) (where ψ, Y, and ρ are the same as in §2A)) we associate the triple (ψ, Y, ρ) where a) ψ is defined by the commutative diagram

$$\begin{array}{ccc} O_1^{nr}/(\pi_1) & \xrightarrow{\psi_1} & B/(\pi_1) \\ \| & & \uparrow \\ O^{nr}/(\pi) & \xrightarrow{\psi} & B/(\pi), \end{array}$$

b) $Y = O_D \otimes_{O_{D_1}} Y_1$, and c) ρ is chosen such that the quasi-isogeny $(\psi_1)_*(\Phi) \to Y \bmod (\pi_1)$, induced by ρ_1, coincides with that which comes from ψ.

The functor $G_1 \to G|_{\mathrm{Nilp}_1}$ is consistent with the action of $GL(d_1, K_1) \times D_1^*$. We have thus obtained the equivariant map ν: $\hat{\Omega}_1^{d_1} \hat{\otimes}_{O_1} \hat{O}_1^{nr} \to \hat{\Omega}^d \hat{\otimes}_O \hat{O}^{nr}$ which is easily seen to be a closed embedding. Let X be a universal formal O_D module over $\hat{\Omega}^d \hat{\otimes}_O \hat{O}^{nr}$, and let X_1 be the same for K_1: then $\nu^*(X) = O_D \otimes_{O_{D_1}} X_1$. The natural embedding $X_1 \to \nu^*(X)$ allows us to define, for any natural number n, equivariant embeddings $\Sigma_1^{d_1, en} \to \Sigma^{d, n}$, where e is the ramification index of K over K_1.

Proposition 3.1. a) $(\hat{\Omega}^l)^{L^*} = \hat{\Omega}_1^{d_1}$,

b) $\hat{\Omega}_1^{d_1} \times_{(\hat{\Omega}^d \otimes K)} \Sigma^{d, n} = (D^*/(1 + \pi^n O_D) \times \Sigma^{d_1, en})/D_1^*$.

Proof. The proof of a) is immediate. The identification of $(\hat{\Omega}^d)^{L^*}$ with $\hat{\Omega}_1^{d_1}$ and the embedding $O_1^{\mathrm{nr}} \to O^{\mathrm{nr}}$, which we fixed above, allow us to define an embedding $\hat{\Omega}^{d_1} \hat{\otimes}_{O_1} \hat{O}_{1r}^n \hookrightarrow \hat{\Omega}^d \hat{\otimes}_O \hat{O}^{nr}$. We will prove that the images of i and ν coincide; from this it is not hard to derive b). It is sufficient to prove that the image of ν is contained in the image of i. In fact, since the elements (a, a^{-1}), $a \in K_1^*$, act trivially on $\hat{\Omega}_1^{d_1}, \hat{\otimes}_{O_1} \hat{O}_1^{nr}$, we have

$$\nu(\hat{\Omega}_1^{d_1} \hat{\otimes}_{O_1} \hat{O}_1^{nr}) \subset (\hat{\Omega}^d)^{L^*} \hat{\otimes}_O \hat{O}^{nr} = \hat{\Omega}_1^{d_1} \hat{\otimes}_O \hat{O}^{nr} = \hat{\Omega}_1^{d_1} \hat{\otimes}_{O_1} (\hat{O}^{nr} \otimes_O O_1).$$

From the commutative diagram

$$\begin{array}{ccc} \hat{\Omega}_1^{d_1} \hat{\otimes}_{O_1} \hat{O}_1^{nr} & \to & \hat{\Omega}^d \hat{\otimes}_O \hat{O}^{nr} \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, \hat{O}_1^{nr} & \to & \mathrm{Spf}\, \hat{O}^{nr} \end{array}$$

it follows that the image of ν is contained in the image of i.

Note. From the equivariance of ν and i and from the fact that their images coincide, it is easy to derive the equation $\nu = i$.

4. THE THEOREM OF I. V. CHEREDNIK

Let Δ be an algebra of quaternions over $\mathbf{Q}$ and T a set of points of $\mathbf{Q}$, over which Δ is ramified; suppose $\infty \notin T$. Let L be a maximal order in Δ (the choice of L plays no part, since all maximal orders in Δ are mutually conjugate; this follows from the strong approximation theorem).

Definition. An Abelian L scheme is an Abelian scheme with an action by L. A two-dimensional Abelian L-scheme A over S is called special if for any $p \in T$ and any point of s of characteristic p, the action of $L \otimes \mathbf{Z}_p$ on the tangent space to A at this point is special in the sense of §2.

Proposition 4.1. a) Any two-dimensional Abelian L variety over a field of characteristic $p \in T$ is supersingular.

b) Any two-dimensional Abelian L variety over a discretely normed field has potentially good reduction.

Proof. a) follows from the fact that the dimension (over $\mathbf{Z}_p$) of a $L \otimes \mathbf{Z}_p$ module cannot equal 1 or 2. b) follows from the fact that the dimension (over $\mathbf{Z}$) of an L module cannot equal 1 or 2.

Proposition 4.2. Let $p \in T$. There exists precisely one (up to L-isogeny) two-dimensional Abelian L variety over $\overline{\mathbf{F}}_p$. Let A be such a variety and $\Delta' = (\mathrm{End}_L A) \otimes \mathbf{Q}$. Then Δ' is an algebra of quaternions for which $\mathrm{inv}_l \Delta' = \mathrm{inv}_l \Delta$ for $l \neq p, \infty$, $\mathrm{inv}_p \Delta' = 0$, and $\mathrm{inv}_\infty \Delta' = 1/2$.

Proof. It is sufficient to prove that any two-dimensional supersingular Abelian variety over $\overline{\mathbf{F}}_p$ is isogenic to the direct sum of two supersingular elliptic curves. In fact, such a variety is defined over some finite field $\mathbf{F}_q$. Let α be the Frobenius characteristic of $\mathbf{F}_q$. Since the formal group of a two-dimensional supersingular Abelian variety is isogenic to the direct sum of two one-dimensional formal groups of height 2, we see that α^2/q is the reciprocal of a whole algebraic number. From Weil's theorem it follows that α^2/q is a root of unity. It now remains to use Tate's theorem (see [2]).

Let δ be a product of prime numbers in T; then there exists an element $a \in L$ such that $a^2 = -\delta$. Define an involution σ: $L \to L$ by the formula $\sigma(x) = ax^*a^{-1}$, where $*$ is the standard involution.

Proposition 4.3. Let A be a special L variety. Then there exists a unique principal polarization on A, for which the restriction of Rosatti's involution on L coincides with σ.

Proof. a) It may be immediately verified that if S is the spectrum of an algebraically closed field, then there exists a unique (up to multiplication by a positive number) polarization, for which the restriction of Rosatti's involution on L coincides with σ.

b) Let S be the spectrum of an algebraically closed field of characteristic p. Then we may choose the polarization in a) such that its degree is of the form p^k (here we are using $a^2 = -\delta$).

c) LEMMA. Let F be a p-divisible group of dimension 2 and height 4 over a local Artin ring with residue field $\bar{F}_p$, on which $L \otimes Z_p$ acts in a special way. We define the action of $L \otimes Z_p$ on the dual group $\hat{F}$ by the composition $L \otimes Z_p \xrightarrow{\sigma} (L \otimes Z_p)^0 \to (\mathrm{End}\ F)^0 \simeq \mathrm{End}\ \hat{F}$. Then there exists a unique (up to a constant) cosymmetric homomorphism $F \to \hat{F}$, which commutes with L. Moreover, there is an isomorphism between these homomorphisms.

From the lemma we deduce the proposition; first, when S = Spec $\bar{F}_p$, and then when S is the spectrum of a local Artin ring with residue field $\bar{F}_p$, (by means of the Serre–Tate theorem), then when S is a scheme of finite type over Z, and lastly, for any S.

Proof of the Lemma. 1) Let $p \notin T$. Then it is sufficient to prove the following statement: For any one-dimensional p-divisible group G of height 2 over a local Artin ring with residue field $\bar{F}_p$, there exists a unique (up to a constant) cosymmetric homomorphism $G \to \hat{G}$, and between these homomorphisms there is an isomorphism. This follows from the Serre–Tate theorem and from the fact that all one-dimensional p-divisible groups of height 2 over $\bar{F}_p$ come from elliptic curves.

2) Let $p \in T$. Then F is a special formal $L \otimes Z_p$ module of height 4. Let Φ be a special formal module of height 4 over $\bar{F}_p$, which is the direct sum of two one-dimensional formal $L \otimes Z_p$ modules. The lemma can be immediately verified for Φ. It remains to prove the following: Let Y be a special formal $L \otimes Z_p$ module of height 4 over a local Artin ring with residue field $\bar{F}_p$, Y_0 its reduction by the module of the maximal ideal, and $\Phi \to Y_0$ a quasi-isogeny of height 0; then there exists an isomorphism $Y \simeq \hat{Y}$ such that the isomorphism $Y_0 \simeq \hat{Y}_0$ induced by it is contained in the commutative diagram

$$\begin{array}{ccc} Y_0 & \simeq & \hat{Y}_0 \\ \uparrow & & \downarrow \\ \Phi & \simeq & \hat{\Phi}. \end{array}$$

In other words, we need to prove that any involutory isomorphism of the scheme $\hat{\Omega}^d \hat{\otimes} \hat{Z}_p^{nr}$ is the identity. For this it is sufficient to prove that it commutes with the action of $GL(2, Q_p)$ and acts trivially on $\hat{Z}_p^{nr}$.

Let n be a natural number, S a scheme over Z[1/n], A a two-dimensional Abelian L scheme over S, and A_n the kernel of the homomorphism $A \xrightarrow{n} A$. We shall call the L-linear isomorphism $n^{-1} L/L \times S \simeq A_n$ over S a structure of level n on A.

Proposition 4.4. The functor, associating with the scheme S over Z[1/n], the set of isomorphism classes of special Abelian L schemes over S with structure of level n, may be represented by a projective scheme over Z[1/n] if $n \geq 3$.

Proof. The representability and quasiprojectivity of the representative scheme follow from Proposition 4.3 and Mumford's theorem (see [6]). Projectivity follows from Proposition 4.1 b).

Denote by S_n the scheme whose existence is stated in Proposition 4.4. Fix a prime number $p \in T$. For any n such that $n \geq 3$, $p \nmid n$, we denote by $\hat{S}_n$ the completion of S_n by the ideal (p). Denote by A^f the ring of finite adeles of the field Q: by A_p^f the ring of finite adeles of Q without the p components. The group $(\Delta \otimes A')^*/Q^*$ acts on $\varprojlim_{p \nmid n} \hat{S}_n$, and this action lifts in a natural way to an action on the formal group of the universal Abelian variety.

Let Δ' be an algebra of quaternions over Q such that $\mathrm{inv}_\infty \Delta' = 1/2$, $\mathrm{inv}_p \Delta' = 0$, $\mathrm{inv}_l \Delta' = \mathrm{inv}_l \Delta$ for $l \neq p, \infty$. Fix isomorphisms $\Delta' \otimes Q_p \simeq \mathrm{Mat}\ (2, Q_p)$, $\Delta' \otimes A_p^f \simeq \Delta^0 \otimes A_p^f$, where Δ^0 is the inverse algebra to Δ. We identify $(\Delta \otimes A_p^f)^*$ and $(\Delta^0 \otimes A_p^f)^*$ by means of the isomorphism $g \to g^{-1}$. Set $X_n = U_n \setminus (\Delta' \otimes A^f)^*/\Delta'^*$, where $U_n \subset \prod_{l \neq p} (L \otimes Z_l)^*$ is the principal congruence-subgroup of level n.

Theorem of I. V. Cherednik. There exists a $(\Delta \otimes A^f)^*$-equivariant isomorphism $\hat{S}_n \simeq GL\ (2, Q_p) \setminus ((\hat{\Omega}^2 \hat{\otimes} \hat{Z}_p^{nr}) \times X_n)$, which may be lifted to an equivariant isomorphism of the formal groups on these formal schemes.

Note. It is easily verified that $\mathrm{GL}(2, \mathbf{Q}_p) \setminus ((\hat{\Omega}^2 \hat{\otimes} \hat{\mathbf{Z}}_p^{nr}) \times X_n)$ is the union of a finite number of components, each of which is of the form $\Gamma \setminus \hat{\Omega}^2 \hat{\otimes} \hat{\mathbf{Z}}_p^{nr}$, where Γ is a discrete subgroup of $\mathrm{GL}(2, \mathbf{Q}_p)$ with a compact factor and with no elements of finite order, which contains some power of p.

Proof. Let Φ be the special formal $L \otimes \mathbf{Z}_p$ module over $\overline{\mathbf{F}}_p$, which we fixed in §2A. We call a two-dimensional Abelian L variety whose formal group is identified (as an $L \otimes \mathbf{Z}_p$ module) with Φ, an algebraization of Φ. Denote by X the set of algebraizations of Φ, which have mutually consistent structures of all levels not divisible by p. X is acted on in a natural way by the group $\mathrm{GL}(2, \mathbf{Q}_p) \times (\Delta \otimes \mathbf{A}_p^f)^*$ or, which is the same thing, by $(\Delta' \otimes \mathbf{A}^f)^*$. It follows from Proposition 4.2 that X and $(\Delta' \otimes \mathbf{A}^f)^* / \Delta'^*$ are isomorphic as $\Delta' \otimes \mathbf{A}^f$ sets. Therefore, X_n is the set of algebraizations of Φ which have structure of level n.

We shall construct a morphism $(\hat{\Omega}^2 \otimes \overline{\mathbf{F}}_p) \times X_n \to S_n \otimes \mathbf{F}_p$. For this we need to construct a special Abelian L scheme with structure of level n over $(\hat{\Omega}^2 \otimes \overline{\mathbf{F}}_p) \times X_n$. Denote by Y a universal formal $L \otimes \mathbf{Z}_p$ module over $\hat{\Omega}^2 \otimes \overline{\mathbf{F}}_p$, and by $\tilde{Y}$ its inverse image over $(\hat{\Omega}^2 \otimes \overline{\mathbf{F}}_p) \times X_n$. Denote by $\tilde{\Phi}$ the inverse image of Φ over $(\hat{\Omega}^2 \otimes \overline{\mathbf{F}}_p) \times X_n$. The canonical quasi-isogeny $\tilde{\Phi} \to \tilde{Y}$ allows us to define, for any algebraization of $\tilde{\Phi}$ with structure of level n, an algebraization of $\tilde{Y}$ with structure of level n. Therefore, in order to define a morphism $(\hat{\Omega}^2 \otimes \overline{\mathbf{F}}_p) \times X_n \to S_n \otimes \mathbf{F}_p$ it remains to construct an algebraization of $\tilde{\Phi}$ with structure of level n; the interpretation of X_n as a set of algebraizations of Φ with structure of level n allows us to do this canonically.

It is easily verified that the morphism constructed in this way induces a morphism $\mathrm{GL}(2, \mathbf{Q}_p) \setminus ((\hat{\Omega}^2 \otimes \overline{\mathbf{F}}_p) \times X_n) \to S_n \otimes \mathbf{F}_p$. It is easy to show that this morphism in turn induces a bijection of sets of $\mathbf{F}_p$ points. It follows from the Serre–Tate theorem that it also induces a bijection of sets of B points, where B is any local Artin ring with residue field $\mathbf{F}_p$. Since $\mathrm{GL}(2, \mathbf{Q}_p) \setminus ((\hat{\Omega}^2 \otimes \overline{\mathbf{F}}_p) \times X_n)$ is a projective scheme, the morphism thus constructed is an isomorphism. It follows from the Serre–Tate theorem that the isomorphism $S_n \otimes \mathbf{F}_p \simeq \mathrm{GL}(2; \mathbf{Q}_p) \setminus ((\hat{\Omega}^2 \otimes \overline{\mathbf{F}}_p) \times X_n)$ can be extended to an isomorphism $\hat{S}_n \simeq \mathrm{GL}(2, \mathbf{Q}_p) \setminus ((\hat{\Omega}^2 \hat{\otimes} \hat{\mathbf{Z}}_p^{nr}) \times X_n)$.

COROLLARY. Let k be a nonnegative integer. Then $(S_{np^k} \otimes \mathbf{Q}_p)_{an} = \mathrm{GL}(2, \mathbf{Q}_p) \setminus \Sigma^{2,k} \times X_n$ (here $(S_{np^k} \otimes \mathbf{Q}_p)_{an}$ is a strict analytic space over $\mathbf{Q}_p$, corresponding to $S_{np^k} \otimes \mathbf{Q}_p$).

APPENDIX

In this appendix we shall prove the Serre–Tate theorem.

Theorem of Serre and Tate. Let B be a ring, $I \subset B$ a nilpotent ideal, p a prime number, and let some power of p annihilate I. Let A_0 be an Abelian scheme over B/I, and Γ_0 its p-divisible group. Then the natural map {isomorphism classes of lifts of A_0 to an Abelian scheme over B} → {isomorphism classes of lifts of Γ_0 to p-divisible groups over B} is bijective.

The Serre–Tate theorem is a result of the following two statements.

Special Case of the Serre–Tate Theorem. We add the following to the conditions of the Serre–Tate theorem: $I^2 = 0$ and $pI = 0$.

Lemma on Strictness for p-Divisible Groups. Let B be a ring, $I \subset B$ an ideal, $I^2 = 0$, $pI = 0$, and let Γ_1 and Γ_2 be p-divisible groups over B. If the homomorphism φ: $\Gamma_1 \to \Gamma_2$ induces the zero homomorphism by the module I, then it is the zero homomorphism.

Starting here we shall suppose that $I^2 = 0$, $pI = 0$.

Proof of the Strictness Lemma. It is easily verified that $p\varphi = 0$.

Since $\Gamma_1 \xrightarrow{p} \Gamma_1$ is surjective, $\varphi = 0$.

It remains to prove the special case of the Serre–Tate theorem.

LEMMA 1. A_0 can be lifted to an Abelian scheme over B.

LEMMA 2. Let A and A' be lifts of A_0 to Abelian schemes over B. Then there exists a homomorphism $A \to A'$ which induces on A_0 multiplication by p.

LEMMA 3. Let Γ and Γ' be lifts of Γ_0 to p-divisible groups over B. Then there exists a homomorphism $\Gamma \to \Gamma'$ which induces on Γ_0 multiplication by p.

To Deduce the Special Case of the Serre–Tate Theorem from the Lemmas. a) Let Γ be a lift of Γ_0. We shall prove that Γ is induced by some lift of A_0. In fact, let A' be some lift of A_0, and let Γ' be the corresponding

lift of Γ_0. Then there exists a homomorphism ψ: $\Gamma' \to \Gamma$, which induces on Γ_0 multiplication by p. The kernel of ψ is a finite locally free group scheme H over B. Set A = A'/H. Then A is the required lift.

b) Let A and A' be lifts of A_0, Γ and Γ' the corresponding lifts of Γ_0. Let $\Gamma \simeq \Gamma'$: We shall prove that $A \simeq A'$. In fact, there exists a homomorphism ψ: $A \to A'$, which induces on A_0 multiplication by p. It is sufficient to prove that $\mathrm{Ker}(A \xrightarrow{\psi} A') = \mathrm{Ker}(A \xrightarrow{p} A)$. The homomorphism ψ: $A \to A'$ induces a homomorphism ψ_*: $\Gamma \to \Gamma'$ such that the corresponding homomorphism $\Gamma_0 \to \Gamma_0$ is multiplication by p. By the strictness lemma, ψ_* is a composition $\Gamma \xrightarrow{p} \Gamma \xrightarrow{\sim} \Gamma'$. Therefore, $\mathrm{Ker}(\Gamma \xrightarrow{\psi_*} \Gamma') = \mathrm{Ker}(\Gamma \xrightarrow{p} \Gamma)$, and hence $\mathrm{Ker}(A \xrightarrow{\psi} A') = \mathrm{Ker}(A \xrightarrow{p} A)$.

Proof of Lemma 1. Suppose a lift is prevented by the element $\alpha \in H^2(A_0, \Theta \otimes I) = H^2(A_0, O_{A_0}) \otimes \mathrm{Lie}(A_0) \otimes I$ (where Θ is the relative tangent sheaf to A_0). It is necessary to prove that $\alpha = 0$. This is easily done for $p \neq 2$. In fact, α must be mapped to itself under automorphisms of A_0, in particular, under multiplication by -1. Therefore, $\alpha = -\alpha$, hence, $\alpha = 0$. If now p = 2, we need to apply analogous arguments to $A_0 \times A_0$ and automorphism with the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proof of Lemma 2. Let A and A' be lifts of A_0. Cover A with affine charts. On each chart we may define a map into A', the identity for the module I. These maps are not defined uniquely, but after composition with the map $A' \xrightarrow{p} A'$ the arbitrariness is easily seen to disappear, and we obtain a morphism $A \to A'$.

Proof of Lemma 3. Let G and G' be finite locally free group schemes over B, G_0 and G_0' their reductions by the module I, and φ_0: $G_0 \to G_0'$ a homomorphism. We will construct a homomorphism ψ: $G \to G'$ inducing by the module I a homomorphism $p\varphi_0$, and this construction will be functorial (by G, G', B). Hence, the statement of the lemma will be proved.

To construct a homomorphism ψ: $G \to G'$, we need to construct for any B-algebra C a map $\psi_C: G(C) \to G'(C)$. The pair (C, IC) possesses the same properties as (B, I), and therefore, we may consider that C = B. Let $\hat{G}$, $\hat{G}'$ be the dual groups. We need to define a map $\mathrm{Hom}(\hat{G}, \mathbf{G}_m) \to \mathrm{Hom}(\hat{G}', \mathbf{G}_m)$. We define it as the composition of the obvious map $\mathrm{Hom}(\hat{G}, \mathbf{G}_m) \to \mathrm{Hom}(\hat{G}_0, \mathbf{G}_m)$, the map $(\varphi_0)_*$: $\mathrm{Hom}(\hat{G}_0, \mathbf{G}_m) \to \mathrm{Hom}(\hat{G}', \mathbf{G}_m)$, and the following map $\mathrm{Hom}(\hat{G}_0', \mathbf{G}_m) \to \mathrm{Hom}(\hat{G}', \mathbf{G}_m)$; the character χ of the group G_0' is associated with f^p, where f is any function on G' which induces χ.

LITERATURE CITED

1. V. G. Drinfel'd, "Elliptic modules," Matem. Sb., 94, 594-627 (1974).
2. J. Tate, "Endomorphisms of Abelian varieties over finite fields," Matematika, 4, No. 1, 31-40 (1960).
3. I. V. Cherednik, "Algebraic curves uniformized by discrete arithmetic subgroups," Usp. Matem. Nauk, 30, No. 3, 181-182 (1975).
4. M. Lazard, Commutative Formal Groups, Lecture Notes in Math., Vol. 443, Springer-Verlag, Berlin (1975).
5. W. Messing, The Crystals Associated to Barsotti–Tate Groups: with Applications to Abelian Schemes, Lecture Notes in Math., Vol. 264, Springer-Verlag, Berlin (1972).
6. D. Mumford, Geometric Invariant Theory, Springer-Verlag, Berlin (1965).