Synthetic Category Theory

(Previously: Formalization of Higher Categories)

Book project, in progress

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Abstract

This book aims at introducing higher category theory in an axiomatic way: instead of building higher category theory on top of set theory, as is done in the framework of quasi-categories, we will introduce higher category theory synthetically, with the aim of having access to its main features as quickly as possible: the Yoneda embedding, the straightening/unstraightening correspondence, the theory of Kan extensions, the theory of presentable categories, etcetera. Our axiomatic approach will not only provide tools to comprehend the important aspect of higher categories as they are used in practice (derived algebraic geometry, homotopical algebra, etc.) but also in more general contexts (e.g. higher category theory internally in any higher topos) and in logic (dependent type theory).

Preface

This book is under active construction: various parts of it are very much unfinished and will be refined and expanded over the course of 2025 and 2026. The most recent version of this book can be found here.

We would like to thank Ulrik Buchholtz, Daniel Gratzer and Jonathan Weinberger for useful discussions on (multimodal) type theory. We further thank the members of a reading group in Université Sorbonne Paris Nord for their useful constructive feedback that improved the exposition of this work.

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Change log

The following is an overview of the major updates in this book:

- **July 2026** New chapter on K-theory - Waldhausen's S-construction and its universal property. Localization theorem
- **June 2026:** New chapters on ind-objects and stable categories.
- **May 2026:** New chapter on left exact localizations.
- **March 2026:** Completed part on Peano arithmetic and filtered colimits of animae. Established equivalence between complete Segal animae and synthetic categories.
- **February 2026:** Completed the parts on straightening/unstraightening, Yoneda lemma, functoriality of the opposite category functor. Established fully faithfulness of the nerve functor by explicit computations of colimits.
- **November 2025:** Retitled book to *Synthetic Category Theory*; Rewrote Proposition 3.4.5 to emphasize the symmetry between the equivalent conditions from Proposition 3.4.4; Rewrote Section 4.2 on contexts and dependent sums; Changed conventions on underlining, see Axiom K.2; Fixed various typos; Removed lots of older material.
- **April 2025:** Redefined $[2]$ as $\mathrm{Fun}([1],[1])$, as suggested by Christian Sattler. Various reorganizations from Chapter 9 onwards. Changed terminology "fundamental groupoid" to "geometric realization". Changed convention regarding 'surjective' and 'essentially surjective', see Definition 6.3.1 and Definition 11.5.3;
- **November 2024:** Started a change log. Revisions and cleaner exposition in Chapters 7, 8 and 9. Reformulation of Axiom B.6 for functor categories;
- **October 2024:** Turned the 'lecture notes' into a 'book project'; Started Section 11.1 on naive set theory within synthetic category theory; Started Section 12.1 on the partially ordered set of natural numbers;
- **August 2024:** Removed axioms on diagram categories, replaced with simpler but more ad hoc approach merely using two synthetic categories $[1]$ and $[2]$;

- **July 2024:** Major rewrite of the introduction;
- **May 2024:** Added Chapter 4 on contexts;
- **April - July 2024:** Incorporated the material from Cisinski's lecture course *Formalization of Higher Categories II* during the summer term 2024 in Regensburg: the current Chapters 7, 8, 9 and 11 on universes, cofinality, the Yoneda embedding, peano arithmetic and the simplex category, plus some material which made its way into earlier chapters;
- **April 2024:** Major revision of the lecture notes into its current form: Introduced the language of 'naive category theory' in Chapter 1, rewrote the notes in terms of that language. Removed material on tribes, see Part II here;
- **December 2023:** Expanded Chapter 5 with more material on (co)cartesian fibrations;
- **October 2023 - February 2024:** Lecture notes for Cisinski's lecture course *Formalization of Higher Category* during the winter term 23/24 in Regensburg. Topics included: the theory of clans and tribes; a discussion of homotopy type theory within a tribe; disjoint unions and functor categories; a definition of simplices via the join construction; a discussion of diagram categories, the Segal axiom and the Rezk axiom; a discussion of groupoids; a (very brief) treatment of (co)cartesian fibrations; subcategories and localizations; the Fundamental Theorem of category theory; universes and directed univalence.

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Introduction

The goal of this book is to establish an axiomatic framework for higher category theory. This approach is best thought of as a form of *synthetic* category theory: instead of constructing higher categories from more primitive objects, the axioms impose the intended behavior of higher categories directly. This stands in contrast to *analytic* approaches, such as the theory of quasicategories, in which one first constructs higher categories inside a background foundation, typically set theory, and then proves that the resulting objects have the desired categorical properties.

Mathematical motivation

Category theory has long been a useful unifying language for abstract mathematics. Higher category theory[1] refines ordinary category theory by allowing all constructions and properties to be considered up to coherent homotopy. In recent decades, higher category theory and homotopy theory have become central tools in algebraic topology, algebraic geometry, representation theory, mathematical physics, and many other fields.

The traditional foundations of the field have rested on specific models, most notably the framework of quasicategories developed by Joyal [Joy08] and Lurie [Lur09; Lur17]. As the subject matured, however, it became clear that most of its results and constructions are *model-independent*: they never touch the combinatorial details of quasicategories, or of any other model. One way to say this is that there is a *language* of category theory, of which the classical set-based notion, a category as a set of objects and a set of morphisms, is one interpretation, and the theory of quasicategories another. The working content of higher category theory lives in the language, not in any single interpretation of it. Following this viewpoint, much recent work is written in a deliberately model-agnostic style that avoids the combinatorial scaffolding of quasicategories.

One primary goal of this book is to provide a detailed account of this model-independent language of category theory, and to identify a list of key categorical axioms one needs to assume to build up the rest of the theory. We believe this allows for a more direct and intuitive engagement with the core concepts of higher category theory, and that it leads to proofs carrying insights that may have been obscured by model-specific details in more traditional treatments of the theory.

Our framework has three further advantages:

- *Parametrized ∞-category theory*: Our formalism is designed to serve as the internal language of the ∞-category of sheaves of ∞-categories on any ∞-topos. Thus all results proved in this book are automatically valid in parametrized category theory, as in the work of Barwick et al.

[1] More specifically: $(\infty, 1)$-category theory.

[Bar+16] on genuine equivariant homotopy theory and the work of Martini and Wolf [MW21] on category theory internal to an ∞-topos.

- *Naive ∞-category theory*: The theory provides a semi-formal language in which one can teach and use ∞-category theory in the same way one uses naive set theory, making the subject more accessible.
- *Elementary ∞-category theory*: The axioms are constructive, and hence shed light on the scope of "elementary ∞-category theory": Even constructively it is possible to recover substantial parts of standard higher category theory.

An alternative formal approach to higher category theory was developed by Riehl and Verity [RV22]. The spirit of their approach is different from ours; for example, their formalism makes central use of the theory of quasicategories, something we wish to avoid. Their formalism shows that higher category theory can be developed in any choice of model satisfying some mild conditions, whereas our formalism shows that higher category theory can be developed without having any model at all.

Philosophical motivation

There is also a philosophical reason to formalize higher category theory: category theory can serve as a foundation for mathematics. This idea goes back at least to Lawvere, and it has been developed in many forms since then; see for instance Landry and Marquis [LM05] for historical context.

The point of view in this book is that categories, in fact $(\infty, 1)$-categories, should be treated as the building blocks of mathematics, in the role traditionally played by sets or types. One way to study mathematical objects within category theory is through the categories they form; this perspective is already a common practice in fields like algebraic geometry and algebraic topology. We wish to highlight a second way to recover ordinary mathematics as a fragment of the theory of categories: by first restricting the theory to *groupoids* (those categories in which any morphism is an isomorphism) and then further to *sets* (those groupoids that admit at most one isomorphism between every two objects); the resulting set theory can then be used to define other mathematical objects in the usual way.

The latter perspective also makes clear that the jump from set-based mathematics to category-based mathematics consists of two conceptual leaps, corresponding to the passage from sets to groupoids and the passage from groupoids to categories.

The first leap concerns the notion of *equality*. In set-based mathematics, equality is treated as a property: two elements of a set are either equal or they are not, and there is no further question of *how* they are equal. In groupoid-based mathematics, equality is instead treated as an intrinsic structure: think of how 'equality between two abelian groups' usually means that the two groups are isomorphic, and so different choices of isomorphism can be thought of as different 'equalities' between the groups. The structured perspective on equality is not new; for example:

- Makkai [Mak98] proposed the **Principle of Isomorphism**: All grammatically correct properties of objects of a category are to be invariant under isomorphism.

- Voevodsky's *homotopy type theory (HoTT)* [Uni13] builds in a flexible notion of equality which has the key feature that it is compatible with the **Univalence Axiom**, which roughly states that two isomorphic types (groups, topological spaces, etcetera) are equal to each other.

We take it as our main guiding principle throughout the book:

> **Fundamental Principle of Homotopy Theory:** When expressing that two entities are equal, we must always specify *how* they are equal by providing an isomorphism between them. All valid mathematical constructions should preserve these isomorphisms.

Taking this principle seriously, we are forced to replace sets as our basic notion of 'collections of mathematical objects': equality in sets is too rigid. The new fundamental object in this paradigm is called an *anima*, which may be described as a collection of objects such that for each two objects there is an anima of isomorphisms between them. This gives animae the structure of a groupoid. By considering equality of morphisms, we also have *isomorphisms between isomorphisms*, and repeating this process we see that animae naturally behave like ∞-*groupoids*. Other common names in the literature are *spaces* or *homotopy types*; HoTT for example uses the latter framing.

The second leap is to replace groupoids by categories as the primitive building blocks of the theory. In practice, a class of mathematical objects often naturally come with more general relations between objects than mere isomorphisms; think of homomorphisms between groups, continuous maps between topological spaces, functors between categories. The concept of a category encapsulates this by encoding morphisms between objects and behavior of their compositions.

The classical perspective on categories is analytic: a category consists of a *set* of objects and a *set* of morphisms, satisfying various axioms. This perspective clashes with Makkai's principle of isomorphism, as it allows one to distinguish two isomorphic objects: categories should have an *anima* of objects and a *anima* of morphisms. In light of this, it may be tempting to try to define categories directly within HoTT. However, we argue that it is better to make a second leap, and directly take categories to be the primitive objects of the theory, with sets and animae arising as special cases.

Just like the fundamental principle of homotopy theory forces animae to behave like ∞-groupoids, it also forces categories to behave like $(\infty, 1)$-*categories*: equality between morphisms must be encoded by some notion of 2-isomorphisms, equality between those by 3-isomorphisms, and so on. In this way, the language of category theory natively speaks about $(\infty, 1)$-categories, and strict $(1, 1)$-categories are merely a fragment of this theory.

For broader discussions of categorical and structural foundations of mathematics, we refer to Tsementzis [Tse17], Marquis [Mar13], Rodin [Rod14], and Awodey [Awo14].

Synthetic category theory eats itself

One of the distinctive features of synthetic category theory is that it can interpret itself. In fact, we will take this as a key design criterion: There should be a universe Cat whose objects are small synthetic categories, and all the axioms of the theory we develop should ultimately be satisfied in the internal language of Cat. In this sense, it is analogous to the role of Grothendieck universes in set theory: sets large enough to contain the usual set-theoretic constructions, which can therefore serve as a model of set theory inside set theory.

The axiom that makes this precise is *directed univalence*, which relates morphisms in Cat to functors between synthetic categories. The main consequence of this axiom is the familiar Grothendieck correspondence: for every synthetic category C, functors

$$C \to \mathrm{Cat}$$

correspond to small cocartesian fibrations over C. To illustrate why this relates the external theory of synthetic categories to the internal theory of Cat, it is illustrative to consider the terminal case $C = 1$. In this case, the Grothendieck correspondence says that objects of Cat correspond to small cocartesian fibrations over the terminal category, hence to small synthetic categories. Applying the principle to more general instances of C then provides the translation between the internal language of Cat and the external language of categories that we are setting up.

Restricting directed univalence to animae recovers Voevodsky's univalence axiom: isomorphisms between animae correspond to equalities in the subuniverse $\mathrm{An} \subseteq \mathrm{Cat}$ of animae. But the directed form is stronger: it also says that maps between animae correspond to morphisms in An. This also marks a crucial difference with homotopy type theory: HoTT has a universe of types, and univalence explains equality in that universe, but this does not yet give a self-interpretation of HoTT inside HoTT. The problem is to internalize the syntax of type theory together with its higher coherence in plain homotopy type theory. Synthetic category theory is designed so that the theory eats itself: the universe Cat is not only one object of the theory, but also a semantic universe in which the theory can be interpreted.

Relation to type theory

In this book, we deliberately present the theory of synthetic categories as a naive theory: it is written in semi-formal natural mathematical language. The main objective of this book is to provide a model-agnostic presentation of the language of naive category theory, and build up standard category theory from a sizable list of foundational axioms. No knowledge of type theory is required to read this book.

Nevertheless, our work is heavily inspired by type-theoretic considerations, and we expect it to be possible to write down rigorous type-theoretic syntax and derivation rules that allow the theory to be formalized in proof assistants. We are exploring several approaches:

- A presentation of the theory à la Martin-Löf is in preparation in collaboration with Michele Riva;
- A presentation of the theory within the type theory CaTT of ω-categories [FM17; BFM24] is in preparation in collaboration with Ivan Kobe.

For readers familiar with dependent type theory, we wish to flag two differences with synthetic category theory.

The first difference concerns the *Curry–Howard correspondence*: the idea that types may be interpreted as mathematical statements, and terms of a type as proofs of those statements. Constructions of types, like the disjoint union $X \sqcup Y$ or the product $X \times Y$, then admit logical interpretations, like the logical disjunction $P \vee Q$ and logical conjunction $P \wedge Q$. Under this correspondence, propositions appear as the special case of (-1)-truncated types: those types which admit at most one term.

In synthetic category theory, it is no longer the case that arbitrary types correspond to mathematical statements. General types are now interpreted as categories, which allow for non-invertible maps between objects. The types corresponding to statements are the *animae*: those categories in which every morphism is an isomorphism.

The second difference concerns *type dependency*. In ordinary dependent type theory, for each context Γ, the types in context Γ form a category, and any substitution

$$f\colon \Delta \to \Gamma$$

induces a pullback operation on types. This pullback operation preserves the logical structure of the theory: products, sums, identity types, and the other type-forming operations are stable under substitution. In this sense, substitution along an arbitrary map of contexts preserves the logic.

In synthetic category theory, this principle breaks in a controlled way. It continues to behave as expected when the contexts involved are animae. But if Γ is a general synthetic category, then a type in context Γ should be thought of as a category over Γ, that is, as a functor

$$p\colon X \to \Gamma.$$

The resulting theory of categories over Γ is not, in general, again a semantic interpretation of synthetic category theory.

However, there is a better category-theoretic replacement: among functors $X \to \Gamma$, one should restrict to cocartesian fibrations over Γ. The resulting category *does* form a semantic interpretation of synthetic category theory. This theory is not equivalent to the naive theory of all categories over Γ, except when Γ is an anima.

Still, even when we restrict to cocartesian fibrations, pullback along an arbitrary functor $f\colon \Delta \to \Gamma$ does not generally preserve all categorical structure: unlike in type theory, substitution does not fully preserve the logic.

We wish to regard this phenomenon as a feature rather than a bug: It is one of the ways in which synthetic category theory is more expressive than ordinary dependent type theory. The comparison between the naive theory of categories over Γ and the theory of cocartesian fibrations over Γ is itself structured: it is governed by adjunctions and by canonical comparison maps, many of which are not invertible. Such non-invertible comparison maps are not pathological in category theory: they often form an integral part of the mathematical discourse of the subject.

Consistency

The consistency of the theory presented in this book is controlled by the usual models of higher category theory. Cisinski's work on higher categories [Cis19], together with the work of Cisinski and Nguyen on the universal cocartesian cofibration [CN22], shows that Joyal's theory of quasicategories gives a semantic interpretation of synthetic category theory. Thus the theory is at least as consistent as the usual set-theoretic foundations used to construct quasicategories.

Let us note that the proof of directed univalence, namely the fact that there is a reasonable ∞-category of ∞-categories, subsumes Voevodsky's proof of the univalence axiom for Kan complexes [KL21].

As mentioned before, the work of Buchholtz, Gratzer and Weinberger [GWB26] on simplicial type theory is expected to provide a constructive semantic interpretation of synthetic category theory within the framework of Riehl and Shulman [RS17]. Conversely, the present book proves among other things that the synthetic category of synthetic categories is equivalent to the synthetic category of complete Segal objects in animae, which can be used as the basis for an internal version of simplicial homotopy type theory. This indicates that synthetic category theory and simplicial type theory are mutually consistent, constructively.

Organization of the book

This version of the book is preliminary and is currently being written and rewritten. The organization may fluctuate substantially between different versions; see here for the version of April 2024 and here for the version of November 2025. The present version focuses only on synthetic category theory and excludes older material such as the theory of tribes. The book should be expected to continue to change throughout 2025 and 2026.

The present version of the book is organized as follows:

(1) In Chapter 1, we introduce the language of *naive category theory*. The notions of synthetic categories, animae, functors and natural isomorphisms between functors are taken as primitive, and all our axioms are expressed in terms of them. We introduce products, coproducts, pullbacks and functor categories via universal properties. We further introduce morphisms and commutative diagrams in a synthetic category, and formulate the Segal axiom and the Rezk axiom to ensure that the objects we call synthetic categories do indeed behave like categories.

(2) Chapter 2 discusses the notion of *groupoids* and introduces the groupoid core of a synthetic category. Groupoids play a fundamental role in our approach: we define an object of a synthetic category C to be a functor $x\colon \Gamma \to C$ whose source is allowed to be an arbitrary groupoid Γ.

(3) In Chapter 3, we introduce various fundamental categorical constructions in the setting of synthetic category theory: full subcategories, localizations, geometric realizations, relative functor categories, joins and slices.

(4) Due to the way we set things up, all results proved about synthetic categories in this book also hold for families of synthetic categories indexed by some groupoid Γ. As a result, various object-level statements automatically imply statements that are fully coherent in the variable $\gamma \in \Gamma$. To formalize this idea, we introduce in Chapter 4 the notion of a *synthetic category in context* Γ.

(5) In Chapter 5, we introduce adjunctions, initial and terminal objects, limits and colimits, left and right fibrations, and cartesian and cocartesian fibrations, among other things. We also introduce the Functoriality of universals axiom, which states that in order to provide a left adjoint to a cartesian fibration $p\colon E \to C$ it is enough to show that all the fibers of p admit initial objects.

(6) One of the most fundamental theorems in category theory is the fact that a functor is an equivalence if and only if it is fully faithful and essentially surjective. We prove a synthetic analogue of this result in Chapter 6.

(7) The main goal of Chapter 7 is to axiomatize a synthetic category Cat of small synthetic categories. It comes equipped with a cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_{\bullet} \to \mathrm{Cat}$ satisfying the property that

for every synthetic category C there is an equivalence between functors from C to Cat and cocartesian fibrations over C with small fibers. The equivalence is given by pulling back the universal small cocartesian fibration π_{univ} along the given functor $C \to \text{Cat}$.

(8) The material in Chapter 8 and later chapters has been added only relatively recently and is likely to be reorganized in the near future. Topics discussed in these chapters include initial and terminal functors, proper and smooth functors, opposite categories, the Yoneda lemma, locally small categories, the simplex category and the epi-mono factorization of morphisms of groupoids.

Overview of the axioms

The axiomatic framework we propose for higher category theory is both comprehensive and intricate, reflecting the rich structure of higher categories themselves. Unlike the axioms of set theory, which can be stated succinctly, many of our axioms require the development of a substantial amount of basic theory before they can be fully articulated. To provide the reader with a roadmap and to illuminate the overall structure of our approach, we will now provide a high-level overview of our axiom system. While the full technical details will be developed in subsequent chapters, this overview aims to give the reader a conceptual understanding of the foundation we are building and how its various components fit together to create a coherent theory of higher categories.

Naive category theory

Our formalization of higher category theory rests upon a certain language of categories, functors and natural isomorphisms that we call *naive category theory*. We will now provide a rough overview of the language of naive category theory, leaving the details to Chapter 1.

Axiom A. We may speak of *categories* C, D, E, ..., of *functors* $f\colon C \to D$ between categories, of *natural isomorphisms* $\alpha\colon f \cong g$ between functors, of *isomorphisms*[2]

The precise version of the axiom above is that categories have *terms*. A term x in a category C should be seen as an object in C, possibly given *functorially*. We write

$$x\colon C$$

to say that x is a term of C. There is special class of categories called *animae*: those will be used to represent the concept of collection. Terms of animae are considered in a slightly more static way: the only way we can go from term a to a term b in some anima A is through an identification of a with b. However, the principle of our theory — taken from dependent type theory —, is that the collection of identifications of a with b form an anima denoted by $a \cong b$. In general, any well formed collection should correspond to some anima of some sort. In particular, given two categories C and D, the collection of functors from C to D form an anima denoted by $\mathrm{Map}(C, D)$. Identifications in $\mathrm{Map}(C, D)$ are called *natural isomorphisms* and are denoted as terms of the form $\alpha\colon f \cong g$. Of course, since natural isomorphisms between two functors $f, g\colon C \to D$ form an anima $f \cong g$, we can speak of *isomorphisms between isomorphisms* between natural isomorphisms, and so forth. One may compose functors, natural isomorphisms and 'higher' isomorphisms in the expected way.

[2]In the literature, the word 'homotopy' or 'equivalence' is often used in cases where we use 'isomorphism'.

The type theoretic notation for $\mathrm{Map}(C, D)$ is $C \to D$, so that $f : C \to D$ could mean that f is a term of type $\mathrm{Map}(C, D)$.

A functor $f : C \to D$ is said to be an *equivalence* if there exists another functor $g : D \to C$ and natural isomorphisms $\mathrm{id}_C \cong gf$ and $\mathrm{id}_D \cong fg$.

There are various ways to build new categories out of old ones. In each case, we will specify the behavior of these constructions by prescribing how to define functors into/out of them and how to define natural isomorphisms between functors into/out of them. To keep the theory simple, we omit the specification of all the expected higher coherences: we will see that we can develop the basic foundations without needing them.

Axiom B. We demand the existence of the following categories:

(B.1) A *terminal category* $*$: for every C there is a functor $p_C : C \to *$, and for any two functors $f, g : C \to *$ there is a natural isomorphism $f \cong g$.

(B.2) An *initial category* $\emptyset$: for every C there is a functor $\emptyset \to C$, and for any two functors $f, g : \emptyset \to C$ there is a natural isomorphism $f \cong g$.

(B.3) For every two categories C and D a *product* $C \times D$ with functors $\mathrm{pr}_C : C \times D \to C$ and $\mathrm{pr}_D : C \times D \to D$: functors $E \to C \times D$ are specified by giving the two components $E \to C$ and $E \to D$, and natural isomorphisms between functors $E \to C \times D$ are specified by giving natural isomorphisms between the components.

(B.4) Similarly there is a *coproduct* $C \sqcup D$ with functors $i_C : C \sqcup D$ and $i_D : D \to C \sqcup D$: functors $C \sqcup D \to E$ are specified by giving the two components $C \to E$ and $D \to E$, and natural isomorphisms between functors $C \sqcup D \to E$ are specified by giving natural isomorphisms between the components.

(B.5) For functors $f : C \to E$ and $g : D \to E$ there exists a *pullback* $C \times_E D$ with functors $\mathrm{pr}_C : C \times_E D \to C$ and $\mathrm{pr}_D : C \times_E D \to D$ and a natural isomorphism $f \circ \mathrm{pr}_C \cong g \circ \mathrm{pr}_D$. Functors $T \to C \times_E D$ are specified by giving two components $t : T \to C$ and $s : T \to D$ together with a natural isomorphism $f \circ t \cong g \circ s$. One may similarly specify natural isomorphisms between two functors $T \to C \times_E D$.

(B.6) For categories C and D there is a *functor category* $\mathrm{Fun}(C, D)$. Every functor $f : E \times C \to D$ gives rise to a *curried functor* $f_c : E \to \mathrm{Fun}(C, D)$ and conversely every functor $g : E \to \mathrm{Fun}(C, D)$ gives rise to an *uncurried functor* $g^u : E \times C \to D$, satisfying $f \cong (f_c)^u$ and $g \cong (g^u)_c$. We may similarly curry/uncurry natural isomorphisms between functors. Currying is demanded to be functorial in E.

We further demand that finite coproducts are *universal*, in the sense of Axiom B.2' and Axiom B.4'.

We define the collection of objects in a category C to be $C^{\simeq} = \mathrm{Map}(*, C)$. We define an *absolute object* of a category C to be a functor $x : * \to C$. To speak of *morphisms*, we need another axiom:

Axiom C. There is a category $[1]$, called the *walking morphism*, which comes equipped with objects 0 and 1. There is a category $[2]$, called the *walking commutative triangle*, which comes with objects 0, 1 and 2. There are functors $d_0, d_1, d_2 : [1] \to [2]$ and $s_0, s_1 : [2] \to [1]$ satisfying the simplicial identities.

We may then define a *morphism* in C to be a functor $f\colon [1] \to C$. We refer to the objects $f(0)$ and $f(1)$ of C as the *source* and *target* of f, respectively, and often write $f\colon x \to y$ to indicate that $f(0) = x$ and $f(1) = y$. Similarly, a *natural transformation* of functors $C \to D$ is defined as a functor $C \times [1] \to D$, or equivalently a functor $C \to \mathrm{Fun}([1], D)$.

A *commutative triangle* in C is a functor $[2] \to C$, and a *commutative square* is a functor $[1]\times[1] \to C$. Every commutative square determines two commutative triangles by precomposition with the maps $j_0 := (s_0, s_1)\colon [2] \to [1] \times [1]$ and $j_1 := (s_1, s_0)\colon [2] \to [1] \times [1]$, and we demand that the original square can be uniquely recovered from this data:

Axiom D (Commutative square axiom)**.** For every category C, the functor

$$(j_0^*, j_1^*)\colon \mathrm{Fun}([1] \times [1], C) \to \mathrm{Fun}([2], C) \times_{d_1,\mathrm{Fun}([1],C),d_1} \mathrm{Fun}([2], C)$$

is an equivalence.

Every object $x\colon * \to C$ gives rise to an *identity morphism* $\mathrm{id}_x\colon x \to x$, defined as the composite $[1] \xrightarrow{p_{[1]}} * \xrightarrow{x} C$. The definition of *composition* of morphisms in C requires another axiom:

Axiom E (Segal axiom)**.** For every category C, the restriction functor

$$(d_2^*, d_0^*)\colon \mathrm{Fun}([2], C) \to \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C)$$

is an equivalence of categories.

Given morphisms $f\colon x \to y$ and $g\colon y \to z$, the axiom provides an essentially unique commutative triangle $\sigma\colon [2] \to C$ satisfying $d_2^*(\sigma) \cong f$ and $d_0^*(\sigma) \cong g$. The resulting morphism $g \circ f := d_1^*(\sigma)\colon [1] \to C$ is called the *composite of f and g*. One can prove that composition of morphisms is automatically unital and associative.

Having both identity morphisms and composition, we may now introduce the notion of *isomorphism* in C: a morphism $f\colon x \to y$ in C is an isomorphism if it admits a left inverse $g\colon y \to x$ (satisfying $gf \sim \mathrm{id}_x$) as well as a right inverse $h\colon y \to x$ (satisfying $fh \sim \mathrm{id}_y$). Using diagram categories, we may in fact construct for every C a category $\mathrm{Iso}(C)$ whose objects are precisely the isomorphisms in C (equipped with the data of left and right inverses). The canonical example of an isomorphism in C is the identity morphism $\mathrm{id}_x\colon x \to x$, in which case the left and right inverses are simply id_x itself. This construction provides a functor $i\colon C \to \mathrm{Iso}(C)$, which we demand to be an equivalence:

Axiom F (Rezk axiom)**.** Every isomorphism in a category C is equivalent to an identity morphism. More precisely, the functor $i\colon C \to \mathrm{Iso}(C)$ is an equivalence.

We define groupoids as those categories in which all morphisms are isomorphisms. The fact that our theory will look like ordinary category theory in many ways is due to the following principle:

Axiom G. A category is an anima if and only it is a groupoids.

Axioms E and F are inspired by Rezk's notion of *complete Segal spaces*.

Constructions of categories

Equipped with a basic language of categories, functors and natural isomorphisms, we can start formulating our remaining axioms of category theory. As a first piece of structure, we require the existence of *groupoid cores* of categories. Using the above notion of isomorphisms in a category C, we say that C is a *groupoid* if all of its morphisms are isomorphisms.

For every category C, there is a groupoid $C^{\simeq}$ called the *groupoid core of* C, equipped with a functor $\gamma_C\colon C^{\simeq} \to C$ simply defined as

$$C^{\simeq} = \mathrm{Map}(*, C)\,.$$

Every functor $f\colon X \to C$ from a groupoid X factors through γ_C, and for functors $g, h\colon X \to C^{\simeq}$ every natural isomorphism $\gamma_C \circ g \cong \gamma_C \circ h$ may be lifted to a natural isomorphism $g \cong h$.

Furthermore, the groupoid core of $[1]$ is equivalent to $* \sqcup *$.

For categories C and D, we write $\mathrm{Map}(C, D)$ for the groupoid $\mathrm{Fun}(C, D)^{\simeq}$.

Groupoids will play a fundamental role in our development of category theory, and we devote Chapter 2 to their study. Among other things, they determine the correct notion of *objects* in a category. In ordinary category theory, one frequently encounters statements that can be tested 'on objects'; think for example of the fact that a natural transformation is a natural isomorphism if and only if it is an objectwise isomorphism. To get the correct analogue of this notion in category theory, we must define an *object of a category* C to be a functor $x\colon \Gamma \to C$ from an *arbitrary* groupoid Γ, or, equivalently, a term of the groupoid core $C^{\simeq}$. This is why we may think of $C^{\simeq}$ as the collection of objects in C.

We also demand the existence of subcategories, localizations, geometric realizations and joins, discussed in detail in Chapter 3:

Axiom H (Subcategory axiom)**.** Consider a category C equipped with a *collection of morphisms* $m\colon M \hookrightarrow \mathrm{Map}([1], C)$ of C which is *closed under composition* (see Definition 3.1.6 for a precise definition of these terms). Then there exists a category $\langle M\rangle_C$ equipped with a functor $i_M\colon \langle M\rangle_C \to C$ satisfying:

- The induced functor $(i_M)_*\colon\ \mathrm{Map}([1], \langle M\rangle_C) \to \mathrm{Map}([1], C)$ factors through M;
- The functor i_M is *universal* with respect to the previous property: every other functor $f\colon D \to C$ whose induced functor $f_*\colon\ \mathrm{Map}([1], D) \to \mathrm{Map}([1], C)$ factors through M admits an essentially unique factorization through $\langle M\rangle_C$.

Axiom I (Localization axiom)**.** For every category C and every collection of morphisms $W \hookrightarrow \mathrm{Map}([1], C)$, there exists a functor $l\colon C \to C[W^{-1}]$ exhibiting $C[W^{-1}]$ as a localization of C at the morphisms in W:

- The functor l sends the morphisms in W to isomorpisms $C[W^{-1}]$;
- The functor l is *universal* with respect to the previous property: every other functor $f\colon C \to D$ which sends morphisms in W to isomorphisms in D admits an essentially unique factorization through $C[W^{-1}]$.

In the case where W consists of *all* morphisms in C, we denote the localization by $\|C\|$ and call it the *geometric realization of* C.

Axiom J (Geometric realization axiom)**.** For every category C, the geometric realization $\|C\|$ is a groupoid.

Axiom K (Join axiom)**.** For every pair of categories C and D, there exists a category $C \star D$ and a pushout square

$$\begin{tikzcd} C \times D \sqcup C \times D \arrow[r] \arrow[d, "\langle \mathrm{pr}_C, \mathrm{pr}_D \rangle"'] & C \times [1] \times D \arrow[d] \\ C \sqcup D \arrow[r] & C \star D. \end{tikzcd}$$

If we write $[0]$ for the terminal category $*$, we have $[1] \simeq [0] \star [0]$, and we may inductively define $[n+1] := [n] \star [0]$.

Cartesian and cocartesian fibrations

An important concept in category theory is that of a cocartesian fibration: a functor $E \to C$ whose fibers E_c are in a certain 'covariantly functorial' in the object $c \in C$. In Chapter 5, we introduce a synthetic analogue of this definition: a functor $f\colon A \to B$ is called a cocartesian fibration if a certain 'directed evaluation map' $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1], A) \to A \vec{\times}_f B$ admits a left adjoint section $\mathrm{lift}_0^f\colon A \vec{\times}_f B \to \mathrm{Fun}([1], A)$. Informally, this left adjoint section provides the cocartesian lifts $a \to \beta_!(a)$ in A for morphisms $\beta\colon b \to b'$ in B with specified lift a of b. One may dually define *cartesian fibrations*.

Axiom M (Functoriality of universals axiom)**.** Let $p\colon E \to C$ be a cocartesian (resp. cartesian) fibration. If the fiber E_x of p admits a terminal (resp. initial) object for every object x of C, then these objects assemble into a section $s\colon C \to E$ of p. Furthermore, the natural transformation $\varepsilon\colon ps \xrightarrow{\sim} \mathrm{id}_C$ (resp. $\eta\colon \mathrm{id}_C \xrightarrow{\sim} ps$) exhibits s as a right (resp. left) adjoint of p.

Axiom N (Exponentiability axiom)**.** Cartesian and cocartesian fibrations are *exponentiable*, meaning that pullback along a (co)cartesian fibration admits a right adjoint, and these right adjoints satisfy a Beck-Chevalley condition.

Given a cocartesian fibration $p\colon U_\bullet \to U$, one can formulate what it means for p to satisfy *directed univalence*: roughly speaking, this means that for functors $f_0, f_1\colon C \to U$ with pullback fibrations $E_i := f_i^*(U_\bullet) \to C$, the groupoid of natural transformations $\alpha\colon f_0 \to f_1$ is equivalent to the groupoid of cocartesian functors $E_0 \to E_1$ over C. This axiom will in particular imply a form of *straightening/unstraightening*. A category U equipped with such a univalent cocartesian fibration p is called a *universe*. We say that U is a *universe of categories* if it is closed under all the categorical operations introduced so far; see Definition 7.4.5 for a complete list.

Axiom O (Directed univalence axiom)**.** For every cocartesian fibration $q\colon E \to B$, there exists a universe of categories U such that B is U-small and q has U-small fibers.

We will usually fix a universe of categories $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ and work with categories that are small with respect to this universe, i.e. those categories C that can be obtained from an object c of Cat via the following pullback square:

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "c"] & \mathrm{Cat}. \end{tikzcd}$$

A consequence of directed univalence is in fact that there is a canonical equivalence of the form

$$\mathrm{Fun}(C, \mathrm{Cat}) \xrightarrow{\sim} \mathrm{CoCart}(c)$$

where the right hand side denotes the subcategory of the lice category Cat/c whose objects are cocartesian functors $p\colon x \to c$ (corresponding through directed univalence to actual cocartesian functors of the form $p : X \to C$ with X small) and morphisms are cocartesian functors relatively to c. In fact the entirety of the axiom can be interpreted categories as terms of the universe Cat. Directed univalence says essentially that this interpretation is logically equivalent to the syntactic presentation of the theory in a precise way.

1 The language of naive category theory

At the heart of mathematics lies the concept of a set—a collection of mathematical objects. In modern mathematics, sets serve as the building blocks for defining everything from numbers and relations to more complex structures like groups, rings, and topological spaces. Sets are manipulated using basic operations like forming unions, intersections, and complements, and are compared using functions between sets. It is fair to say that the language of set theory forms an essential toolkit for mathematicians.

Although formal axiomatizations of set theory exist, such as those by Zermelo and Fraenkel, they often diverge from the working mathematician's intuitive understanding of sets. For instance, in formal set theory, every natural number needs to be defined as a set, and in particular one is allowed to speak of the elements of a natural number. Similarly, every function between sets needs to be defined as a set. While this approach is formally valid, it can seem disconnected from everyday mathematical discourse.

In practice, mathematicians often use a more pragmatic approach known as *naive set theory*. In naive set theory, one works with sets based on intuitive understanding, without invoking formal logical or axiomatic implementations. For example, the set of natural numbers is simply the set containing the numbers $0, 1, 2$, etcetera, and functions between sets X and Y are simply 'specifications' that assign to every element of X an element of Y. The simplicity and accessibility of naive set theory make it relatively easy for students to learn it, and indeed in many introductory mathematics textbooks sets are used in the naive way without being rigorously introduced.

A similar trend is occurring in the field of higher category theory. In recent decades, higher category theory has seen rapid growth, in large part thanks to contributions from people like Joyal, Lurie, and others. Their work provides solid foundations for the field, and the language of higher categories is now extending beyond its origins in algebraic topology to fields like algebraic geometry and representation theory. But while the original works of Joyal and Lurie are based on explicit set-theoretical models for higher categories called 'quasicategories', more and more research papers are written in a 'model-independent' way in which the usage of details about the specific choice of implementations is avoided. In other words, researchers are increasingly adopting a language of *naive category theory*, which enables a more intuitive approach to studying higher categorical structures.

We believe that naive category theory is easier to learn than the theory of quasicategories, just like naive set theory is easier to learn than formal set theory. With this in mind, the goal of this book is to provide a detailed treatment of the foundations of higher category theory written purely in the language of naive category theory, without reference to specific implementations. The goal of the present chapter is to give a careful introduction to this language: we introduce the relevant terms and explain the manipulations one is allowed to do. In particular, our approach is *synthetic* in nature rather than *analytic*: constructions of categories, like products or functor categories, are not *defined* in terms of set-level constructions, but are *axiomatized* to exist and satisfy the expected properties.

1.1 The basic vocabulary

We start in this section by introducing the basic vocabulary of naive category theory.

1.1.1 Categories, functors and natural isomorphisms

Axiom A.1. The following are the primitives in the vocabulary of naive category theory:

(1) There are *categories*, which we denote by symbols like C, D, E, etcetera. Among categories, there are *animae*; these are the categories that will encode the idea of collection.

(2) Categories have *terms*. We write $x\colon C$ to denote that x is a term of C. We should think of terms in C as objects in C given *functorially*; we refine this viewpoint in Section 1.1.3.

(3) Given two categories C and D, there is an anima $\mathrm{Map}(C, D)$, whose terms are called *functors* from C to D. We will frequently write $C \to D$ for the anima $\mathrm{Map}(C, D)$, so that we may write functors as $f\colon C \to D$.

(4) Every category C has an *identity functor* $\mathrm{id}_C\colon C \to C$. Given two functors $f\colon C \to D$ and $g\colon D \to E$, there is a *composite functor* $g \circ f\colon C \to E$.

(5) Given an anima Γ, as well as terms $x\colon \Gamma$ and $y\colon \Gamma$, there is the anima $x \cong_\Gamma y$, whose terms are called *identifications* of x with y in Γ (also written $x \cong y$ if this does not yield any confusion). The following principles hold for identifications of terms in an anima Γ:

- For any $x\colon \Gamma$, there is a canonical term $\mathrm{id}_x : x \cong_\Gamma x$.
- For any identification $u\colon x \cong_\Gamma y$, there is an identification $u^{-1}\colon y \cong_\Gamma x$.
- For any $u\colon x \cong_\Gamma y$ and $v\colon y \cong_\Gamma z$, there is an identification $vu\colon x \cong_\Gamma z$.

(6) Given a category Γ and, for each term $x\colon \Gamma$, a category $C(x)$, there is a category $\Sigma_{x:\Gamma} C(x)$ whose terms precisely are pairs of the form (x, c) with $x\colon \Gamma$ and $c\colon C(x)$. Furthermore, if both Γ and each $C(x)$ are animae, then so is $\Sigma_{x:\Gamma} C(x)$.

(7) For categories Γ and C, any rule assigning to each term $x\colon \Gamma$ a term $f(x)\colon C$ determines a map $(x \mapsto f(x))\colon \mathrm{Map}(\Gamma, C)$. If there is another rule assigning a term $g(x)\colon C$ to each term $x\colon \Gamma$ as well as a rule assigning an identification $u(x)\colon f(x) \cong_C g(x)$ to each term $x\colon \Gamma$, then there is an identification of the form

$$(x \mapsto u(x))\colon (x \mapsto f(x)) \cong_{\mathrm{Map}(\Gamma, C)} (x \mapsto g(x))\,.$$

Conversely, given any functor $f\colon \Gamma \to C$, there is a rule that assigns a term $f(x)\colon C$ to each term $x\colon \Gamma$ as well as an identification of the form

$$\lambda_f\colon (x \mapsto f(x)) \cong_{\mathrm{Map}(\Gamma,C)} f\,.$$

Any identification of the form $u\colon f \cong_{\mathrm{Map}(\Gamma,C)} g$ yields identifications of the form $u(x)\colon f(x) \cong_C g(x)$ for each $x\colon \Gamma$, and there is an identification of the form

$$\lambda_g(x \mapsto u(x))\lambda_f^{-1} \cong u$$

as terms of type $f \cong g$.

The notation $\cong$ for equalities is used to evoke the idea of isomorphisms in category theory. Recall the following fundamental principle guiding our theory:

Fundamental Principle of Homotopy Theory: Whenever we would like to express that two things *are equal*, we should always remember *how* the two things are equal by specifying an isomorphism between them.

Definition 1.1.1. Given two functors $f, f'\colon C \to D$, a *natural isomorphism* $\alpha\colon f \cong f'$ is an identification of f and f' in the anima $\mathrm{Map}(C, D)$.

Definition 1.1.2. A *commutative square of functors*, often displayed by means of a diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\ C' \arrow[r, "f'"'] & D', \end{tikzcd}$$

consists of four functors f, g, h and f' with sources and targets as displayed in the diagram, together with a specified natural isomorphism $\alpha\colon h \circ f \cong f' \circ g$. We may similarly define a *commutative square of natural isomorphisms*, etcetera.

We will often refer to the data of just the four functors f, g, h and f' as a *square*, and say that *the square commutes* whenever a natural isomorphism $\alpha\colon h \circ f \cong f' \circ g$ can be provided.

A *commutative triangle of functors*, displayed by means of a diagram

$$\begin{tikzcd} & D \arrow[dr, "g"] & \\ C \arrow[ur, "f"] \arrow[rr, "h"'] & & E, \end{tikzcd}$$

consists of three functors f, g and h with sources and targets as displayed in the diagram, together with a specified natural isomorphism $\alpha\colon h \cong g \circ f$. We may similarly define a *commutative triangle of natural isomorphisms*, etcetera.

We will similarly display more general diagrams of functors by drawing various directed planar graphs with vertices labeled by categories and with edges labeled by functors. We say the diagram *commutes* if every face comes equipped with a natural isomorphism between the two composites of functors that can be formed along the boundary of that face.

1.1.2 Coherences

In order for composition of functors to behave as expected, we require various 'coherence laws', like unitality and associativity.

Axiom A.2. The composition of functors is unital and associative up to specified natural isomorphisms: for functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$, there are natural isomorphisms

$$\lambda_f\colon \mathrm{id}_D \circ f \cong f, \qquad \rho_f\colon f \circ \mathrm{id}_C \cong f$$

and

$$\alpha_{f,g,h}\colon (h \circ g) \circ f \cong h \circ (g \circ f).$$

We demand analogous unitality and associativity laws for identifications. Moreover, for any identification $\alpha\colon x \cong y$ there are identifications $\alpha^{-1} \circ \alpha \cong \mathrm{id}_x$ and $\alpha \circ \alpha^{-1} \cong \mathrm{id}_y$ as terms of $x \cong x$ and $y \cong y$, respectively

The fundamental principle of higher category theory requires a bit of getting used to: since equality between two things has now become additional structure, in the form of an isomorphism, all definitions and constructions we do should keep track of this structure. In other words, any valid statement or construction in synthetic category theory must respect identifications: given identifications between the inputs of the construction, one obtains an identification between the outputs, compatibly with identity identifications and composition. In particular, we must demand:

Axiom A.3. Composition of functors respects natural isomorphisms: given a natural isomorphism $\alpha\colon f \cong f'$ of functors $C \to D$ and a natural isomorphism $\beta\colon g \cong g'$ of functors $D \to E$, we obtain a new natural isomorphism $\beta * \alpha\colon g \circ f \cong g' \circ f'$ of functors $C \to E$, called the *horizontal composite*.

Similarly, given an identification $\alpha_0 \cong \alpha_1$ of natural isomorphisms between f and f' and another identification $\beta_0 \cong \beta_1$ of natural isomorphisms between g and g', we obtain an identification $\beta_0 * \alpha_0 \cong \beta_1 * \alpha_1$ of natural isomorphisms between $g \circ f$ and $g' \circ f'$.

Horizontal composition further respects vertical composition:

- The horizontal composite $\mathrm{id}_g * \mathrm{id}_f\colon g \circ f \cong g \circ f$ is identified with $\mathrm{id}_{g \circ f}\colon g \circ f \cong g \circ f$;
- Given two more natural isomorphisms $\alpha'\colon f' \cong f''$ and $\beta'\colon g' \cong g''$, there is an identification
$$(\beta' \circ \beta) * (\alpha' \circ \alpha) \cong (\beta' * \alpha') \circ (\beta * \alpha)$$
of natural isomorphisms $g \circ f \cong g'' \circ f''$.

We similarly require the existence of horizontal composites for identifications between natural isomorphisms.

As a consequence of the subaxiom above, the two ways to compose the natural isomorphisms in the following diagram are identified:

$$\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow\cong"] \arrow[r, phantom, shift right=3, "\alpha'\Downarrow\cong"] & D \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3, "\beta\Downarrow\cong"] \arrow[r, phantom, shift right=3, "\beta'\Downarrow\cong"] & E. \end{tikzcd}$$

Axiom A.4. Horizontal composition of natural isomorphisms is compatible with the unitality and associativity isomorphisms from Axiom A.2:

- For a natural isomorphism $\alpha \colon f \cong f'$ of functors $C \to D$ there are commutative squares of natural isomorphisms

$$\begin{tikzcd} \mathrm{id}_D \circ f \arrow[r, "\mathrm{id}_{\mathrm{id}_D} * \alpha", "\cong"'] \arrow[d, "\lambda_f"'] & \mathrm{id}_D \circ f' \arrow[d, "\lambda_{f'}"] \\ f \arrow[r, "\alpha", "\cong"'] & f' \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} f \circ \mathrm{id}_C \arrow[r, "\alpha * \mathrm{id}_{\mathrm{id}_C}", "\cong"'] \arrow[d, "\rho_f"'] & f' \circ \mathrm{id}_C \arrow[d, "\rho_{f'}"] \\ f \arrow[r, "\alpha", "\cong"'] & f'. \end{tikzcd}$$

- For natural isomorphisms $\alpha \colon f \cong f'$, $\beta \colon g \cong g'$ and $\gamma \colon h \cong h'$, the following square of natural isomorphisms commutes:

$$\begin{tikzcd} (h \circ g) \circ f \arrow[r, "(\gamma * \beta) * \alpha", "\cong"'] \arrow[d, "\alpha_{f,g,h}"'] & (h' \circ g') \circ f' \arrow[d, "\alpha_{f',g',h'}"] \\ h \circ (g \circ f) \arrow[r, "\gamma * (\beta * \alpha)", "\cong"'] & h' \circ (g' \circ f'). \end{tikzcd}$$

The analogous conditions hold for horizontal composition of identifications between natural isomorphisms.

Remark 1.1.3. It will sometimes be convenient to abbreviate the conditions from Axiom A.4 as follows:

$$\mathrm{id} * \alpha \cong \alpha \cong \alpha * \mathrm{id} \qquad \text{and} \qquad (\gamma * \beta) * \alpha \cong \gamma * (\beta * \alpha).$$

More generally, if we are given natural isomorphisms $\alpha \colon f \cong g$ and $\alpha' \colon f' \cong g'$ of functors $C \to D$, and we happen to be given natural isomorphisms $f \cong f'$ and $g \cong g'$, we will write $\alpha \cong \alpha'$ to indicate the existence of a commutative square of natural isomorphisms as follows:

$$\begin{tikzcd} f \arrow[r, "\alpha", "\cong"'] \arrow[d, "\cong"'] & g \arrow[d, "\cong"] \\ f' \arrow[r, "\alpha'", "\cong"'] & g'. \end{tikzcd}$$

The unitors and associators from Axiom A.2 should satisfy various higher coherence identities. We will record the pentagon and triangle identities explicitly, but otherwise suppress higher coherence data in our naive language.

Axiom A.5 (Pentagon and triangle identities)**.** For functors $f: A \to B$, $g: B \to C$, $h: C \to D$ and $k: D \to E$, the following pentagon of natural isomorphisms commutes:

$$\begin{tikzcd} & (k \circ h) \circ (g \circ f) \arrow[dr, "\alpha_{k,h,g\circ f}"] & \\ ((k \circ h) \circ g) \circ f \arrow[ur, "\alpha_{k\circ h,g,f}"] \arrow[d, "\alpha_{k,h,g} * \mathrm{id}_f"'] & & k \circ (h \circ (g \circ f)) \\ (k \circ (h \circ g)) \circ f \arrow[rr, "\alpha_{k,h\circ g,f}"'] & & k \circ ((h \circ g) \circ f) \arrow[u, "\mathrm{id}_k * \alpha_{h,g,f}"'] \end{tikzcd}$$

Moreover, for every category C, the following triangle of natural isomorphisms commutes:

$$\begin{tikzcd} \mathrm{id}_C \arrow[rr, "\mathrm{id}_{\mathrm{id}_C}"] \arrow[dr, "\lambda^{-1}_{\mathrm{id}_C}"'] & & \mathrm{id}_C \\ & \mathrm{id}_C \circ \mathrm{id}_C \arrow[ur, "\rho_{\mathrm{id}_C}"'] & \end{tikzcd}$$

1.1.3 Contexts

Let us now get back to the interpretation of the notion of *terms* that appear in Axiom A.1. As mentioned before, we should think of constructions that depend on a term $x: C$ as being automatically fully functorial in C. We will now explain what we mean by this.

For this, we require the following refined language:

Axiom A.6 (Contexts of terms)**.** Let C be a category. A term x of C always lives in a *context*, which is some category Γ that is generally left implicit in the notation.

Definition 1.1.4. A term $x: C$ is called an *object* if its context Γ is an anima.

Convention 1.1.5. Let Γ and C be categories. We may sometimes refer to a term f of C in context Γ as a *rule* assigning a term $f(x) : C$ to each term $x: \Gamma$. Similarly, a category C in context Γ may be referred to as a *rule* assigning a category $C(x)$ to each term $x: \Gamma$. We may thus interpret items (6) and (7) in Axiom A.1 as follows:

(6) For every category C in context Γ, there is a category $\Sigma_\Gamma C$.

(7) Terms $f(x): C$ in context Γ correspond to functors $f: \Gamma \to C$.

Remark 1.1.6. Whenever several terms occur in the same construction, they are implicitly assumed to live in a common context. Consider functor composition, for example. Given terms $f: \mathrm{Map}(C, D)$ and $g: \mathrm{Map}(D, E)$, if f and g live in the same context Γ, we obtain their composite $g \circ f: \mathrm{Map}(C, E)$.

Thinking of f and g as rules

$$\Gamma \to \mathrm{Map}(C, D),\ x \mapsto f(x) \qquad\qquad \Gamma \to \mathrm{Map}(D, E),\ x \mapsto g(x),$$

we should think of $g \circ f$ as the rule

$$x \mapsto g(x) \circ f(x) \colon \Gamma \to \mathrm{Map}(C, E).$$

Axiom A.7. Any rule or axiom we state holds in context Γ for any anima Γ. Moreover, any rule or axiom that is underlined also holds in context Γ for an arbitrary category Γ.

Convention 1.1.7. Let Δ be a category, and let Γ and C be categories in context Δ. Per Axiom A.7, we must be able to speak of *terms of C in context Γ, in context Δ*. By convention, we understand this to mean *terms of C in context $\Sigma_\Delta\Gamma$*.

Remark 1.1.8. Recall that item (7) of Axiom A.1 requires that terms of C in context Γ correspond to functors $x \colon \Gamma \to C$. If we apply this rule in context Δ, we find that specifying a term $c(\delta, \gamma) \colon C$ in context $\Sigma_{\delta:\Delta}\Gamma(\delta)$ is the same as providing a functor

$$(\gamma \mapsto c(\delta, \gamma)) \colon \Gamma(\delta) \to C$$

in context Δ. The same applies when the target category C also varies with $\delta \colon \Delta$.

Although terms are represented by arbitrary functors into C, it is psychologically convenient to think of terms of C as individual objects of C, like we are used to from ordinary category theory. To reinforce this perspective, we sometimes use the following convention:

Convention 1.1.9. Given a category C and a term x of C, we write $\{x\}$ for the context Γ of x.

We emphasize that the category $\{x\}$ is allowed to be completely arbitrary, and that the notation is purely a psychological aid that lets us talk about terms as if they are individual objects.

One particular contextual term will come up a lot and deserves a special name:

Definition 1.1.10 (Universal term)**.** For a category C, we refer to the identity functor $\mathrm{id}_C \colon C \to C$ as the *universal term* of C. The corresponding rule is $x \mapsto x$.

The universal term is what allows us to make precise the idea that constructions that depend on a term $x \colon C$ are automatically functorial in x. Let us illustrate this with an example.

Construction 1.1.11. The mapping animae $\mathrm{Map}(C, D)$ are covariantly functorial in D and contravariantly functorial in C. For covariant functoriality, consider a functor $g \colon D \to E$. For any term $f \colon \mathrm{Map}(C, D)$, there is a term $g \circ f \colon \mathrm{Map}(C, E)$. Applying this to the universal term $\mathrm{id}_{\mathrm{Map}(C,D)} \colon \mathrm{Map}(C, D) \to \mathrm{Map}(C, D)$, this provides a term of $\mathrm{Map}(C, E)$ in context $\mathrm{Map}(C, D)$. This may be reinterpreted as a functor

$$g \circ - \colon \mathrm{Map}(C, D) \to \mathrm{Map}(C, E),$$

called *postcomposition with g*. We may describe this construction more succinctly by saying that $g \circ -$ is the functor associated to the rule $f' \mapsto g \circ f'$. Similarly, any functor $f \colon B \to C$ gives a functor

$$- \circ f \colon \mathrm{Map}(C, D) \to \mathrm{Map}(B, D)$$

associated to the rule $g' \mapsto g' \circ f$.

Proposition 1.1.12. *For any functors $f\colon A \to B$ and $g\colon B \to C$ and any category E, there is a canonical identification of the form*

$$g_* \circ f_* \;\cong\; (g \circ f)_*$$

in the anima $\mathrm{Map}(\mathrm{Map}(E, A), \mathrm{Map}(E, C))$.

Proof. This follows from the rule

$$h \mapsto \alpha_{g,f,h}\colon (g \circ (f \circ h)) \cong ((g \circ f) \circ h)$$

obtained from Axiom A.4. □

Similarly, we have:

Proposition 1.1.13. *For any functors $f\colon A \to B$ and $g\colon B \to C$ and any category E, there is a canonical identification of the form*

$$f^* \circ g^* \;\cong\; (g \circ f)^*$$

in the anima $\mathrm{Map}(\mathrm{Map}(C, E), \mathrm{Map}(A, E))$. □

1.1.4 Equivalence of categories

An important notion in category theory is that of an *equivalence of categories*:

Definition 1.1.14 (Equivalence of categories)**.** A functor $f\colon C \to D$ is said to be an *equivalence* if there exists another functor $g\colon D \to C$ equipped with natural isomorphisms $\mathrm{id}_C \cong gf$ and $\mathrm{id}_D \cong fg$. We refer to g as the *inverse* of f. Equivalences will often be introduced by writing $f\colon C \xrightarrow{\sim} D$. We may write $C \simeq D$ to indicate that C and D are equivalent via an unnamed equivalence.

Note that any functor $f'\colon C \to D$ isomorphic to f is automatically also an equivalence with the same inverse g. In practice, many equivalences will be obtained by applying the following two propositions altogether.

Axiom A.8. Any category equivalent to an anima is an anima.

The axiom above is redundant with axioms that will be introduced later on characterizing animae as groupoids (hence through a condition that is obviously invariant under equivalences). However, the notion of anima is solely introduced to represent the concept of collection and we would like to emphasize this in a synthetic way, as opposed to an analytic one.

Proposition 1.1.15. *Let $f\colon \Delta \to \Gamma$ be any functor (e.g. a map between animae). For f to be an equivalence, it suffices that the following two properties hold.*

- *Given any two terms $x, y\colon \Delta$, if there is an identification $\tau\colon f(x) \cong f(y)$ then there exists an identification $\sigma\colon x \cong y$.*
- *For any term $y\colon \Gamma$ there exists a term $x\colon \Delta$ as well as an identification $\tau\colon f(x) \cong y$.*

Proof. Applying the second condition to the universal term of Γ yields a map $g\colon \Gamma \to \Delta$ such that there is a term $\tau\colon f \circ g \cong \mathrm{id}_\Gamma$. It suffices to check that $g \circ f \cong \mathrm{id}_\Delta$ holds. Considering $g \circ f$ and id_Δ as terms of Δ, the first condition together with Axiom A.4 tell us that it suffices to prove that $(f \circ g) \circ f$ can be identified to f. Applying the operator $(-) \circ f$ to $\tau\colon f \circ g \cong \mathrm{id}_\Gamma$ thus provides the appropriate term. □

Proposition 1.1.16. *For a functor $f\colon C \to D$, the following are equivalent:*

(i) The functor f is an equivalence;

(ii) For any category E, the induced map

$$f_*\colon \mathrm{Map}(E, C) \to \mathrm{Map}(E, D)$$

is an equivalence.

(iii) For any category E, the induced map

$$f^*\colon \mathrm{Map}(D, E) \to \mathrm{Map}(C, E)$$

is an equivalence.

If furthermore C and D are animae then it suffices to check properties (i) or (ii) when E itself is an anima as well.

Proof. We will prove the equivalence between (i) and (ii); the proof of (i) $\iff$ (iii) is analogous. If f is an equivalence, it is immediate that f_* is an equivalence for every E. Conversely, assume that f_* is an equivalence for every E. By taking $E = D$, we obtain an equivalence $f_*\colon \mathrm{Map}(D, C) \xrightarrow{\sim} \mathrm{Map}(D, D)$, and hence there exists some functor $g\colon D \to C$ such that $f \circ g \cong \mathrm{id}_D$. It remains to show that also $g \circ f \cong \mathrm{id}_C$. To this end, we take $E = C$, so that we obtain an equivalence $f_*\colon \mathrm{Map}(C, C) \xrightarrow{\sim} \mathrm{Map}(C, D)$. To produce the desired isomorphism between $g \circ f$ and id_C, it will thus suffice to do so after postcomposing with f. But in that case we have the following string of isomorphisms:

$$f \circ (g \circ f) \cong (f \circ g) \circ f \cong \mathrm{id}_D \circ f \cong f \cong f \circ \mathrm{id}_C\,.$$

This finishes the proof. In the case where C and D are animae, it is clear that the proof above can be carried out with E ranging over animae only. □

Definition 1.1.17 (Section/retraction)**.** Consider a functor $f\colon C \to D$. A *section* of f is a functor $s\colon D \to C$ equipped with a natural isomorphism $fs \cong \mathrm{id}_D$. A *retraction* of f is a functor $r\colon D \to C$ equipped with a natural isomorphism $\mathrm{id}_C \cong rf$.

Lemma 1.1.18. *A functor $f\colon C \to D$ is an equivalence if and only if it admits both a section s and a retraction r. In this case there exists a natural isomorphism $s \cong r$ and s is an inverse of f.*

Proof. If f is an equivalence with inverse $g\colon D \to C$, then clearly g is both a section and a retraction of f. Conversely, given a section $s\colon D \to C$, $fs \cong \mathrm{id}_D$ and a retraction $r\colon D \to C$, $\mathrm{id}_C \cong rf$, we obtain a natural isomorphism $s \cong r$ by composing the following chain of natural isomorphisms:

$$s \cong \mathrm{id}_C \circ s \cong (r \circ f) \circ s \cong r \circ (f \circ s) \cong r \circ \mathrm{id}_D \cong r.$$

It follows that f is an equivalence with inverse s, since there are natural isomorphisms $fs \cong \mathrm{id}_D$ and $sf \cong rf \cong \mathrm{id}_C$. □

Corollary 1.1.19. *Any two inverses g and h of an equivalence f are isomorphic.* $\square$

Lemma 1.1.20. *If $f\colon C \to D$ and $f'\colon D \to E$ are equivalences with inverses $g\colon D \to C$ and $g'\colon E \to D$, then also the composite $f'f\colon C \to E$ is an equivalence with inverse $gg'\colon E \to C$.*

Proof. We have $gg'f'f \cong g\,\mathrm{id}_D\, f \cong gf \cong \mathrm{id}_C$ and $f'fgg' \cong f'\,\mathrm{id}_D\, g' \cong f'g' \cong \mathrm{id}_E$. $\square$

Lemma 1.1.21. *Equivalences of categories satisfy the 2-out-of-3 property: given functors $f\colon C \to D$ and $g\colon D \to E$, if two of the three functors f, g and gf are equivalences, then so is the third.*

Proof. If g and f are equivalences, then so is gf by Lemma 1.1.20. If f and gf are equivalences, then we may rewrite g as the composite $g \cong g \circ \mathrm{id}_D \cong g \circ f \circ f^{-1} = gf \circ f^{-1}$, so that g is the composite of two equivalences and hence itself an equivalence. A similar argument shows that f is an equivalence whenever g and gf are equivalences. $\square$

Lemma 1.1.22. *Equivalences of categories satisfy the 2-out-of-6 property: given functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$ such that gf and hg are equivalences, also the functors f, g, h and hgf are equivalences.*

Proof. Consider the functors $f' := f \circ (gf)^{-1}\colon E \to D$ and $h' := (hg)^{-1} \circ h\colon E \to D$. Then there are natural isomorphisms

$$g \circ f' = g \circ f \circ (gf)^{-1} = gf \circ (gf)^{-1} \cong \mathrm{id}_E$$

and

$$h' \circ g = (hg)^{-1} \circ h \circ g = (hg)^{-1} \circ hg \cong \mathrm{id}_D\,.$$

Hence it follows from Lemma 1.1.18 that g is an equivalence. In light of Lemma 1.1.21, we then get that also f and h are equivalences, and then so is hgf. $\square$

Definition 1.1.23. We say that a functor $f\colon C \to D$ is a *retract* of a functor $f'\colon C' \to D'$ if there exists a commutative diagram of the form

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & C' \arrow[r, "h"] \arrow[d, "f'"] & C \arrow[d, "f"] \\ D \arrow[r, "k"] & D' \arrow[r, "j"] & D, \end{tikzcd}$$

together with natural isomorphisms $\alpha\colon \mathrm{id}_C \cong h \circ g$ and $\beta\colon j \circ k \cong \mathrm{id}_D$ satisfying the property that the composite natural isomorphism

$$f \cong f \circ \mathrm{id}_C \overset{\alpha}{\cong} f \circ h \circ g \cong j \circ f' \circ g \cong j \circ k \circ f \overset{\beta}{\cong} \mathrm{id}_D \circ f \cong f$$

is isomorphic to $\mathrm{id}_f\colon f \cong f$.

Lemma 1.1.24. *Assume that $f\colon C \to D$ is a retract of $f'\colon C' \to D'$. If f' is an equivalence, then so is f.*

Proof. Let $(f')^{-1}\colon D' \to C'$ be an inverse of f'. We claim that the composite $g := h \circ (f')^{-1} \circ k\colon D \to C$ is inverse to f. Indeed, we have natural isomorphisms

$$f \circ g = f \circ h \circ (f')^{-1} \circ k \cong j \circ f' \circ (f')^{-1} \circ k \cong k \circ j \cong \mathrm{id}_D$$

and

$$g \circ f = h \circ (f')^{-1} \circ k \circ f \cong h \circ (f')^{-1} \circ f' \circ g \cong h \circ g \cong \mathrm{id}_C .$$

This finishes the proof. □

The mapping animae $\mathrm{Map}(-,-)$ will play an important role in the development of the theory: frequently, we may check conditions at the level of mapping animae. In the following, we will provide various examples of this phenomenon.

1.2 Products and coproducts of categories

On top of the language of categories, functors and natural isomorphisms, we need various basic categorical constructions, like products, pullbacks and functor categories. We will axiomatize these constructions via their corresponding universal properties, which specify the behavior of functors and natural isomorphisms into or out of these objects; we will not need any additional 'higher coherences' in our meta theory.

In this section, we introduce the terminal and initial categories, and discuss products and coproducts of categories.

The terminal category

Axiom B.1 (Terminal category)**.** There is an anima $*$, called the *terminal category*. For every category C, there is a functor $p_C\colon C \to *$. For any two functors $f, f'\colon C \to *$, there is a natural isomorphism $f \cong f'$.

Definition 1.2.1 (Absolute object)**.** An *absolute object* of a category C is a term of C in context $*$, i.e. a functor $x\colon * \to C$. We may sometimes introduce absolute objects of C by writing $x \in C$. Given two absolute objects x and x' of C, an *isomorphism* between x and x' is a natural isomorphism $x \cong x'$ of functors $* \to C$.

In Section 1.5, we will introduce a more general notion of objects in a category.

Definition 1.2.2 (Constant functor)**.** Let C and D be categories and let x be an absolute object of D. We define the *constant functor* $\mathrm{const}_x\colon C \to D$ as the composite $x \circ p_C\colon C \to D$:

$$\mathrm{const}_x : C \xrightarrow{p_C} * \xrightarrow{x} D.$$

Definition 1.2.3 (Contractible category)**.** A category C is called *contractible* if the functor $p_C\colon C \to *$ is an equivalence.

Remark 1.2.4. Observe that a category C is contractible if and only if there exists an absolute object $x\colon * \to C$ such that the composite $\mathrm{const}_x\colon C \to C$ is naturally isomorphic to the identity $\mathrm{id}_C\colon C \to C$. A natural isomorphism $H\colon\ \mathrm{const}_x \cong \mathrm{id}_C$ is called a *contraction of C onto x*.

The initial category

The initial category is introduced in a dual way:

Axiom B.2 (Initial category)**.** The *initial category* is a category denoted $\emptyset$. For every category C, there is a functor $\emptyset \to C$. For any two functors $f, f' : \emptyset \to C$, there is a natural isomorphism $f \cong f'$.

We further ask the initial category to be *strictly* initial, in the following sense:

Axiom B.2' (Strictness of the initial category)**.** Given a category C, any functor $C \to \emptyset$ is an equivalence.

Products of categories

Next we will introduce the *product* of two categories.

Axiom B.3 (Product of categories)**.** The *product* of two categories C and D is a category $C \times D$ equipped with two functors $\mathrm{pr}_C : C \times D \to C$ and $\mathrm{pr}_D : C \times D \to D$ called the *projection functors*.

Given functors $f : T \to C$ and $g : T \to D$, we obtain a functor $(f, g) : T \to C \times D$ equipped with natural isomorphisms $\mathrm{pr}_C \circ (f, g) \cong f$ and $\mathrm{pr}_D \circ (f, g) \cong g$.

Given two functors $h, k : T \to C \times D$ and two natural isomorphisms $\alpha : \mathrm{pr}_C \circ h \cong \mathrm{pr}_C \circ k$ and $\beta : \mathrm{pr}_D \circ h \cong \mathrm{pr}_D \circ k$, we obtain a natural isomorphism $(\alpha, \beta) : h \cong k$ satisfying $\mathrm{pr}_C \circ (\alpha, \beta) \cong \alpha$ and $\mathrm{pr}_D \circ (\alpha, \beta) \cong \beta$.

Remark 1.2.5. Given categories C and D, we define a *span* from C to D to be a triple (T, f, g) consisting of a category T together with functors $f : T \to C$ and $g : T \to D$. Axiom B.3 may be rephrased as saying that the triple $(C \times D, \mathrm{pr}_C, \mathrm{pr}_D)$ is the *terminal span* from C to D: for any other span (T, f, g) there is a map $(f, g) : T \to C \times D$ compatible with the spans, and any two maps $T \to C \times D$ compatible with the spans are naturally isomorphic.

From these simple rules, we may deduce that products are commutative, unital and associative. For commutativity of products, one argues as follows:

Lemma 1.2.6. *For categories C and D, the functor $(\mathrm{pr}_D, \mathrm{pr}_C) : C \times D \to D \times C$ is an equivalence, with inverse given by $(\mathrm{pr}_C, \mathrm{pr}_D) : D \times C \to C \times D$.*

Proof. We need to provide natural isomorphisms $(\mathrm{pr}_C, \mathrm{pr}_D) \circ (\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{id}_{C \times D}$ and $(\mathrm{pr}_D, \mathrm{pr}_C) \circ (\mathrm{pr}_C, \mathrm{pr}_D) \cong \mathrm{id}_{D \times C}$. By symmetry, it will suffice to produce the first natural isomorphism. For this, it will in turn suffice to produce natural isomorphisms after composing with the functors pr_C

and pr_D. For the composition with pr_C we compute that

$$\mathrm{pr}_C \circ ((\mathrm{pr}_C, \mathrm{pr}_D) \circ (\mathrm{pr}_D, \mathrm{pr}_C)) \cong (\mathrm{pr}_C \circ (\mathrm{pr}_C, \mathrm{pr}_D)) \circ (\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{pr}_C \circ (\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{pr}_C,$$

which is indeed isomorphic to $\mathrm{pr}_C \circ \mathrm{id}_{C\times D}$. The case for composition with pr_D is analogous. This finishes the proof. □

Unitality may be argued similarly:

Lemma 1.2.7. *For every category C, the projection functor $\mathrm{pr}_C \colon C \times * \to C$ is an equivalence, with inverse given by the functor $(\mathrm{id}_C, p_C) \colon C \to C \times *$.*

Proof. The composite $\mathrm{pr}_C \circ (\mathrm{id}_C, p_C) \colon C \to C$ is isomorphic to id_C by the defining property of (id_C, p_C). To show that the composite $(\mathrm{id}_C, p_C) \circ \mathrm{pr}_C \colon C \times * \to C \times *$ is isomorphic to $\mathrm{id}_{C\times *}$, it will suffice to do so after composing with pr_C and with pr_*. The case for pr_* is clear, since any two functors $C \times * \to *$ are isomorphic. In the case of pr_C we compute

$$\mathrm{pr}_C \circ ((\mathrm{id}_C, p_C) \circ \mathrm{pr}_C) \cong (\mathrm{pr}_C \circ (\mathrm{id}_C, p_C)) \circ \mathrm{pr}_C \cong \mathrm{id}_C \circ \mathrm{pr}_C \cong \mathrm{pr}_C \cong \mathrm{pr}_C \circ \mathrm{id}_{C\times *},$$

where the first isomorphism is associativity, the second is the relation proved in the first paragraph, and the third and fourth are unitality. This finishes the proof. □

Remark 1.2.8. In the previous proofs, we have been very explicit about the usage of associativity and naturality isomorphisms. In the remainder of this book, we will often leave the usage of such isomorphisms implicit in order to make the proofs more concise.

Exercise 1.2.9 (Exercise 1.9.1)**.** Formulate and prove the associativity of the product of categories.

Construction 1.2.10. The formation of products of categories is functorial: given two functors $f \colon C \to C'$ and $g \colon D \to D'$, we obtain a new functor

$$f \times g := (f \circ \mathrm{pr}_C, g \circ \mathrm{pr}_D) \colon C \times D \to C' \times D'.$$

Exercise 1.2.11 (Exercise 1.9.3)**.** Show that there are natural isomorphisms

$$\mathrm{id}_C \times \mathrm{id}_D \cong \mathrm{id}_{C\times D} \quad \text{and} \quad (f' \times g') \circ (f \times g) \cong (f' \circ f) \times (g' \circ g)$$

for functors $f \colon C \to C'$, $f' \colon C' \to C''$, $g \colon D \to D'$ and $g' \colon D' \to D''$.

Lemma 1.2.12. *For a category C, the projection map $C \times \emptyset \to \emptyset$ is an equivalence.*

Proof. This is immediate from Axiom B.2'. □

Proposition 1.2.13. *If C and D are animae, then $C \times D$ is an anima.*

Proof. Assigning for each term $x \colon C$ the anima D determined an anima $P = \Sigma_{x \colon C} D$ whose terms precisely are pairs of the form (x, y) with $x \colon C$ and $y \colon D$. There is a functor $p : P \to C$ and $q : P \to D$ associated to the rule $(x, y) \mapsto x$ and $(x, y) \mapsto y$, respectively. This determines a unique functor $f \colon P \to C \times D$. We see that this is an equivalence by definition of $C \times D$, using Proposition 1.1.16. □

Proposition 1.2.14. *Given two categories C and D, for any category E, the canonical projections induce an equivalence*

$$(\mathrm{pr}_{C,*}, \mathrm{pr}_{D,*}) \colon \operatorname{Map}(E, C \times D) \xrightarrow{\sim} \operatorname{Map}(E, C) \times \operatorname{Map}(E, D)\,.$$

The proof is left as an exercise.

Binary coproducts of categories

Binary coproducts of categories are introduced in the dual way to products.

> **Axiom B.4** (Coproduct of categories)**.** The *coproduct* of two categories C and D is a category $C \sqcup D$ equipped with two functors $i_C\colon C \to C \sqcup D$ and $i_D\colon D \to C \sqcup D$ called the *inclusion functors*. We will sometimes also refer to $C \sqcup D$ as the *disjoint union* of C and D.
>
> Given functors $f\colon C \to E$ and $g\colon D \to E$, we obtain a functor $\langle f, g\rangle\colon C \sqcup D \to E$ equipped with natural isomorphisms $\langle f, g\rangle \circ i_C \cong f$ and $\langle f, g\rangle \circ i_D \cong g$.
>
> Given two functors $h, k\colon C \sqcup D \to E$ and two natural isomorphisms $\alpha\colon h \circ i_C \cong k \circ i_C$ and $\beta\colon h \circ i_D \cong k \circ i_D$, we obtain a natural isomorphism $\langle \alpha, \beta\rangle\colon h \cong k$ satisfying $(\alpha, \beta) \circ i_C \cong \alpha$ and $(\alpha, \beta) \circ i_D \cong \beta$.

The following proposition is a direct consequence of the definition and of Proposition 1.1.16.

Proposition 1.2.15. *If C, D and E are categories, there is a canonical proof that the functor bellow is an equivalence.*

$$(i_C^*, i_D^*)\colon\ \mathrm{Map}(C \sqcup D, E) \xrightarrow{\sim} \mathrm{Map}(C, E) \times \mathrm{Map}(D, E)\,.$$

Exercise 1.2.16 (Exercise 1.9.2)**.** Show that the coproduct of categories is associative, commutative and unital:

$$(C \sqcup D) \sqcup E \xrightarrow{\sim} C \sqcup (D \sqcup E), \qquad C \sqcup D \xrightarrow{\sim} D \sqcup C, \qquad \emptyset \sqcup C \xrightarrow{\sim} C \xrightarrow{\sim} C \sqcup \emptyset.$$

Construction 1.2.17. The formation of coproducts of categories is functorial: given two functors $f\colon C \to C'$ and $g\colon D \to D'$, we obtain a new functor

$$f \sqcup g := \langle i_{C'} \circ f, i_{D'} \circ g\rangle\colon C \sqcup D \to C' \sqcup D'.$$

We leave it to the reader to verify that there are natural isomorphisms

$$\mathrm{id}_C \sqcup \mathrm{id}_D \cong \mathrm{id}_{C \sqcup D} \quad \text{and} \quad (f' \sqcup g') \circ (f \sqcup g) \cong (f' \circ f) \sqcup (g' \circ g).$$

1.3 Pullbacks of categories

We will now axiomatize the *pullback* of two functors $f\colon C \to E$ and $g\colon D \to E$ of categories. This will be similar to the axiom for the product $C \times D$, but it is more involved due to the fact that we need to record compatibilities with the structure maps to E.

Definition 1.3.1. Consider two functors $f\colon C \to E$ and $g\colon D \to E$. Given a category T, we define a *cone diagram on* T to be a triple $c = (t_C, t_D, \tau)$ consisting of functors $t_C\colon T \to C$ and $t_D\colon T \to D$ together with a natural isomorphism $\tau\colon f \circ t_C \cong g \circ t_D$. In other words, a cone diagram consists of a commutative square of the form

$$\begin{array}{ccc} T & \xrightarrow{t_C} & C \\ {\scriptstyle t_D}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ D & \xrightarrow{g} & E. \end{array}$$

We call T the *cone point* of the diagram. Given another cone diagram $c' = (t'_C, t'_D, \tau')$, an *isomorphism of cone diagrams* $c \cong c'$ is a triple $\Phi\colon (\varphi_C, \varphi_D, H)$, where $\varphi_C\colon t_C \cong t'_C$ and $\varphi_D\colon t_D \cong t'_D$ are natural isomorphisms, and H is an isomorphism $\tau \cong \tau'$ of natural isomorphisms.[1] If $\Psi\colon (\psi_C, \psi_D, G)$ is another isomorphism of cone diagrams $c \cong c'$, we say that Φ and Ψ are isomorphic if there are isomorphisms $\varphi_C \cong \psi_C$ and $\varphi_D \cong \psi_D$ under which H becomes isomorphic to G.

Given a functor $s\colon S \to T$ and a cone diagram $c = (t_C, t_D, \tau)$ on T we obtain another cone diagram $s^*(c) := (t_C \circ s, t_D \circ s, \tau * \mathrm{id}_s)$ on S. Similarly, every natural isomorphism $s_1 \cong s_2$ of functors $S \to T$ induces an isomorphism $s_1^*(c) \cong s_2^*(c)$ of cone diagrams.

Definition 1.3.2. Consider a commutative square of categories of the form

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E \end{tikzcd}$$

corresponding to a cone diagram $d = (p, q, \sigma)$. We say that such a square is *pullback square* (or is *cartesian*) if the following properties hold:

- For every category T equipped with a cone diagram $c = (t_C, t_D, \tau)$, there exists a functor $(t_C, t_D)\colon T \to P$ together with an isomorphism $c \cong (t_C, t_D)^*(d)$ of cone diagrams on T.
- Given two functors functors $s, t\colon T \to P$ equipped with an isomorphism of cone diagrams $s^*(d) \cong t^*(d)$, there exists a natural isomorphism $s \cong t$ that induces this isomorphism of cone diagrams.

If this property hold, we will write:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E \end{tikzcd}$$

Proposition 1.3.3. *Consider two functors $f\colon C \to E$ and $g\colon D \to E$ with C, D and E animae. Then there is a canonical pullback square*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E \end{tikzcd}$$

in which P is an anima.

Proof. Consider the assignment $(x, y) \mapsto f(x) \cong_{C\times D} g(y)$ for $(x, y)\colon C \times D = \Sigma_{x\colon C} D$. This defines an anima

$$P = \Sigma_{(x,y)\colon C\times D}\, f(x) \cong_{C\times D} g(y)\,.$$

The functors p and q are defined via the rules sending each $((x, y), \sigma)$ to x and y, respectively. The collection of all $\sigma\colon f(x) \cong g(y)$ determines an identification $\sigma\colon f \circ p \cong g \circ q$. That we obtain a pullback square in this way is a tautology. $\square$

[1] As in Remark 1.1.3 this should be interpreted as a commutative diagram of natural isomorphisms.

Proposition 1.3.4. *Consider two functors $f\colon C \to E$ and $g\colon D \to E$ as well as two cone diagrams $c = (p, q, \sigma)$ and $c' = (p', q', \sigma')$ yielding to two pullback squares*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} P' \arrow[r, "p'"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E \end{tikzcd}$$

respectively. Then any functor $u\colon P \to P'$ such that there is an isomorphism of cone diagrams $\tau\colon c \cong u^*(c')$ is an equivalence.

We also deduce the following proposition immediately from the definition.

Proposition 1.3.5. *A commutative square of categories*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E \end{tikzcd}$$

is a pullback square if and only if, for any category A, the induced commutative square

$$\begin{tikzcd} \mathrm{Map}(A, P) \arrow[r, "p_*"] \arrow[d, "q_*"'] & \mathrm{Map}(A, C) \arrow[d, "f_*"] \\ \mathrm{Map}(A, D) \arrow[r, "g_*"] & \mathrm{Map}(A, E) \end{tikzcd}$$

is a pullback square.

Axiom B.5 (Pullback of categories)**.** Consider two functors $f\colon C \to E$ and $g\colon D \to E$. The *pullback* of f and g, also called the *fiber product of C and D over E*, is a category $C \times_E D$ equipped with a cone diagram $c_{\mathrm{univ}} = (\mathrm{pr}_C, \mathrm{pr}_D, \tau_{\mathrm{univ}})$ making the commutative square

$$\begin{tikzcd} C \times_E D \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_D"'] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E. \end{tikzcd}$$

a pullback square.

The resulting map $(t_C, t_D)\colon T \to C \times_E D$ in the axiom may be pictorially displayed as follows:

$$\begin{tikzcd} T \arrow[dr, dashed, "(t_C, t_D)"] \arrow[drr, bend left, "t_C"] \arrow[ddr, bend right, "t_D"'] & & \\ & C \times_E D \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_D"'] & C \arrow[d, "f"] \\ & D \arrow[r, "g"] & E. \end{tikzcd}$$

Remark 1.3.6. While Axiom B.5 may look somewhat complicated, let us point out that it has the same form as that of Axioms B.1 and B.3:

- A category X is introduced, together with certain additional structure;
- Any other category T equipped with the same structure comes with a functor $T \to X$ that is compatible with this structure;
- If we have *two* functors $T \to X$ that are compatible with this structure, then they are naturally isomorphic.

The Axioms B.2 and B.4 have a similar form, except that they talk about functors *out of* X.

Exercise 1.3.7 (Exercise 1.9.4)**.** Show that the fiber product is commutative, associative and unital: for functors $C \to \Gamma$, $D \to \Gamma$ and $B \to \Gamma$, there are preferred equivalences

$$C \times_\Gamma D \xrightarrow{\sim} D \times_\Gamma C, \qquad B \times_\Gamma (C \times_\Gamma D) \xrightarrow{\sim} (B \times_\Gamma C) \times_\Gamma D, \qquad C \times_\Gamma \Gamma \xrightarrow{\sim} C.$$

Exercise 1.3.8 (Exercise 1.9.5)**.** Consider categories C and D, and let $p_C : C \to *$ and $p_D : D \to *$ be their functors to the terminal category. Show that the functor $(\mathrm{pr}_C, \mathrm{pr}_D): C \times_* D \to C \times D$ is an equivalence.

Construction 1.3.9. The construction of pullbacks of categories is functorial: given a commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\varphi"'] & \Gamma \arrow[d, "\chi"] & D \arrow[l, "g"'] \arrow[d, "\psi"] \\ C' \arrow[r, "f'"'] & \Gamma' & D' \arrow[l, "g'"] \end{tikzcd}$$

there is an induced functor

$$\varphi \times_\chi \psi := (\varphi \circ \mathrm{pr}_C, \psi \circ \mathrm{pr}_D): C \times_\Gamma D \to C' \times_{\Gamma'} D'.$$

Exercise 1.3.10. In the above situation, show that if each of the functors φ, ψ and χ is an equivalence, then so is $\varphi \times_\chi \psi$.

Corollary 1.3.11. *Equivalences are closed under pullback: if $g: D \to E$ is an equivalence and $f: C \to E$ is an arbitrary functor, then also $C \times_E D \to C$ is an equivalence.*

Proof. This is a special case of Exercise 1.3.10 by taking $C' = C$, $E' = D' = E$, $\varphi = \mathrm{id}_C$, $\chi = \mathrm{id}_E$ and $\psi = g$. □

1.3.1 Fibers of functors

Pullbacks of categories lead to the notion of a *fiber* of a functor:

Definition 1.3.12 (Fiber over a term)**.** Let $f: C \to D$ be a functor. For every term $x: \{x\} \to D$, we define the *fiber* of f over x as the pullback

$$\begin{tikzcd} C_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "f"] \\ \{x\} \arrow[r, "x"] & D. \end{tikzcd}$$

Lemma 1.3.13. *Let D be a category. Consider a functor over D of the form*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & C' \arrow[dl, "q"] \\ & D. & \end{tikzcd}$$

Then f is an equivalence over D if and only if for every term x of D the induced map $f_x\colon C_x \to C'_x$ on fibers is an equivalence.

Proof. The 'if'-statement is automatic by picking x to be the universal term $x = \mathrm{id}_D\colon D \to D$. For 'only if' we note that equivalences over D are preserved under pullback along an arbitrary functor $D' \to D$, hence for $d\colon \{d\} \to D$. □

Lemma 1.3.14. *A functor $f\colon C \to D$ is an equivalence if and only if for every term d of D the map $C_d \to \{d\}$ is an equivalence.*

Proof. This is a special case of the previous lemma by taking $C' = D$ and $q = \mathrm{id}_D$. □

Remark 1.3.15. Using the terminology of 'categories in contexts' from Chapter 4, we may phrase the last lemma as saying that $f\colon C \to D$ is an equivalence if the fibers over all terms of D are contractible.

1.3.2 Pullback squares

Lemma 1.3.16 (Pasting lemma for pullback squares)**.** *Consider a commutative diagram*

$$\begin{tikzcd} C_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & C_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\ D_1 \arrow[r, "h_1"'] & D_2 \arrow[r, "h_2"'] & D_3. \end{tikzcd}$$

(1) There is a preferred equivalence $D_1 \times_{D_3} C_3 \xrightarrow{\sim} D_1 \times_{D_2} (D_2 \times_{D_3} C_3)$.

(2) If the right-hand square is a pullback square, then the left-hand square is a pullback square if and only if the outer rectangle is a pullback square.

Proof. For part (1), we define the functor as $(\mathrm{pr}_{D_1}, (h_1 \circ \mathrm{pr}_{D_1}, \mathrm{pr}_{C_3}))$. We claim that an inverse is given by the functor

$$(\mathrm{pr}_{D_1}, \mathrm{pr}_{C_3} \circ \mathrm{pr}_{D_2 \times_{D_3} C_3})\colon D_1 \times_{D_2} (D_2 \times_{D_3} C_3) \to D_1 \times_{D_3} C_3.$$

To show that the two composites are isomorphic to the respective identities, it suffices to construct these isomorphisms after projecting to the two components and verifying that these isomorphisms agree after composing with the functors to D_3. But after unwinding the definitions, this holds tautologically by the very construction of the two functors in both directions.

Part (2) is an immediate consequence of the 2-out-of-3 property from Lemma 1.1.21 applied to the following commutative square:

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & D_1 \times_{D_2} C_2 \arrow[d, "\simeq"] \\ D_1 \times_{D_3} C_3 \arrow[r, "\simeq"] & D_1 \times_{D_2} (D_2 \times_{D_3} C_3), \end{tikzcd}$$

where the right vertical map is an equivalence by applying Exercise 1.3.10 to the equivalence $C_2 \xrightarrow{\sim} D_2 \times_{D_3} C_3$. □

Lemma 1.3.17. *Consider a commutative square*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\ E \arrow[r, "k"] & F \end{tikzcd}$$

and assume that h is an equivalence. Then g is an equivalence if and only if the square is a pullback square.

Proof. Since h is an equivalence, also the projection map $\mathrm{pr}_E \colon D \times_F E \to E$ is an equivalence by Corollary 1.3.11. Since the composite of the map $(f, g) \colon C \to D \times_F E$ with pr_E is isomorphic to g, the claim follows from the 2-out-of-3 property of equivalences, Lemma 1.1.21. □

Lemma 1.3.18. *Consider functors $f \colon C \to E$ and $g \colon D \to E$. Then the commutative square*

$$\begin{tikzcd} C \times_E D \arrow[r] \arrow[d, "(\mathrm{pr}_C, \mathrm{pr}_D)"'] & E \arrow[d, "\Delta"] \\ C \times D \arrow[r, "f \times g"] & E \times E \end{tikzcd}$$

is a pullback square.

Proof. This follows immediately from the pasting lemma 1.3.16 applied to the following commutative diagram:

$$\begin{tikzcd} C \times_E D \arrow[r, "\mathrm{pr}_D"] \arrow[d, "(\mathrm{pr}_C, \mathrm{pr}_D)"'] & D \arrow[r, "g"] \arrow[d, "(g, \mathrm{id}_D)"] & E \arrow[d, "(\mathrm{id}_E, \mathrm{id}_E)"] \\ C \times D \arrow[r, "f \times \mathrm{id}_D"] \arrow[d, "\mathrm{pr}_1"'] & E \times D \arrow[r, "\mathrm{id}_E \times g"] \arrow[d, "\mathrm{pr}_1"] & E \times E \\ C \arrow[r, "f"] & E. & \end{tikzcd}$$

□

Lemma 1.3.19. *For every functor $g \colon D \to E$, the commutative square*

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ D \arrow[r, "g"] & E \end{tikzcd}$$

is a pullback square.

Proof. We need to show that the induced map $\emptyset \to \emptyset \times_E D$ is an equivalence. But this follows directly from Axiom B.2' by applying 2-out-of-3 to the maps $\emptyset \to \emptyset \times_E D \xrightarrow{\mathrm{pr}_\emptyset} \emptyset$. □

1.3.3 Universality of coproducts

We now postulate an additional property on coproducts of categories:

Axiom B.4’ (Universality of coproducts)**.** Consider categories Γ_0 and Γ_1, define $\Gamma := \Gamma_0 \sqcup \Gamma_1$, and let $i_0 \colon \Gamma_0 \to \Gamma$ and $i_1 \colon \Gamma_1 \to \Gamma$ denote the two inclusions.

(1) For functors $f \colon C \to \Gamma_0$ and $g \colon D \to \Gamma_1$, the commutative squares

$$\begin{tikzcd} C \arrow[r, "i_C"] \arrow[d, "f"'] & C \sqcup D \arrow[d, "f \sqcup g"] \\ \Gamma_0 \arrow[r, "i_0"] & \Gamma_0 \sqcup \Gamma_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D \arrow[r, "i_D"] \arrow[d, "g"'] & C \sqcup D \arrow[d, "f \sqcup g"] \\ \Gamma_1 \arrow[r, "i_1"] & \Gamma_0 \sqcup \Gamma_1 \end{tikzcd}$$

are pullback squares.

(2) For a functor $h \colon E \to \Gamma$, the functor

$$\langle \mathrm{pr}_E, \mathrm{pr}_E \rangle \colon (E \times_\Gamma \Gamma_0) \sqcup (E \times_\Gamma \Gamma_1) \to E$$

is an equivalence.

Corollary 1.3.20 (Disjointness of coproducts)**.** *Coproducts are disjoint: for categories C and D, the commutative square*

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & C \arrow[d, "i_C"] \\ D \arrow[r, "i_D"] & C \sqcup D \end{tikzcd}$$

is a pullback square. □

Lemma 1.3.21. *For functors $f \colon C \to \Gamma$, $g \colon D \to \Gamma$ and $\varphi \colon \Gamma' \to \Gamma$, the commutative square*

$$\begin{tikzcd} (C \times_\Gamma \Gamma') \sqcup (D \times_\Gamma \Gamma') \arrow[r] \arrow[d] & C \sqcup D \arrow[d, "\langle f, g \rangle"] \\ \Gamma' \arrow[r, "\varphi"] & \Gamma \end{tikzcd}$$

is a pullback square.

Proof. We need to show that the induced functor

$$(C \times_\Gamma \Gamma') \sqcup (D \times_\Gamma \Gamma') \to (C \sqcup D) \times_\Gamma \Gamma'$$

is an equivalence. But this is a special case of Axiom B.4’ applied to the functor $f = \mathrm{pr}_1 \colon (C \sqcup D) \times_\Gamma \Gamma' \to C \sqcup D$: by the pasting law of pullback squares there are equivalences

$$C \times_{C \sqcup D} (C \sqcup D) \times_\Gamma \Gamma' \simeq C \times_\Gamma \Gamma' \qquad \text{and} \qquad D \times_{C \sqcup D} (C \sqcup D) \times_\Gamma \Gamma' \simeq D \times_\Gamma \Gamma'.$$

This finishes the proof. □

Corollary 1.3.22. *For categories C, D and E, the functor*

$$\langle E \times i_C, E \times i_D \rangle : E \times C \sqcup E \times D \to E \times (C \sqcup D)$$

is an equivalence.

Proof. This is an instance of the previous lemma by taking $\Gamma = *$, $\Gamma' = E$, $f = p_C : C \to *$, $g = p_D : D \to *$, and $\varphi = p_E : E \to *$. □

Lemma 1.3.23. *The following two conditions are equivalent:*

(1) The category $* \sqcup *$ is contractible;

(2) Every category is contractible.

Proof. It is clear that (2) implies (1). Conversely, assume that $* \sqcup *$ is contractible, i.e. that the map $* \sqcup * \to *$ is contractible. Since this map admits a section given by the left inclusion $l : * = * \sqcup \emptyset \hookrightarrow * \sqcup *$, it follows that l is an equivalence. In particular, its fiber over the right point must be contractible. Since this fiber is the empty category by Corollary 1.3.20, we see that the map $\emptyset \to *$ is an equivalence, and it follows from taking the product with any category C that also $\emptyset \to C$ is an equivalence. By 2-out-of-3 this implies that C is contractible. □

1.3.4 Embeddings of categories

We introduce the notion of *embeddings* of categories and prove basic properties about them. We will eventually see that embeddings precisely correspond to *subcategories*, see Proposition 6.4.1.

Definition 1.3.24 (Embedding)**.** A functor $f : C \to D$ is called an *embedding* if the diagonal $\Delta_f := (\mathrm{id}_C, \mathrm{id}_C) : C \to C \times_D C$ is an equivalence, or equivalently if the commutative diagram

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, equal] & C \arrow[d, "f"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

is a pullback square.

Proposition 1.3.25. *A functor $f : C \to D$ is an embedding if and only if for every category E the induced map $f_* :$ $\mathrm{Map}(E, C) \to \mathrm{Map}(E, D)$ *is an embedding.*

Proof. This follows immediately from Proposition 1.3.5 by considering the commutative diagram

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, equal] & C \arrow[d, "f"] \\ C \arrow[r, "f"] & D. \end{tikzcd}$$

□

Lemma 1.3.26. *A functor $f\colon C \to D$ is an embedding if and only if the commutative square*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "(\mathrm{id}_C,\mathrm{id}_C)"'] & D \arrow[d, "(\mathrm{id}_D,\mathrm{id}_D)"] \\ C \times C \arrow[r, "f\times f"] & D \times D \end{tikzcd}$$

is a pullback square.

Proof. By Lemma 1.3.18 the pullback of this diagram is $C \times_D C$. Since the induced map $C \to C \times_D C$ is precisely the diagonal $\Delta_f\colon C \to C \times_D C$ of f, the claim follows. □

Lemma 1.3.27. *Consider a pullback square*

$$\begin{tikzcd} C' \arrow[r, "g"] \arrow[d, "u"'] & C \arrow[d, "v"] \\ D' \arrow[r, "f"] & C \end{tikzcd}$$

of categories. If v is an embedding, then also u is an embedding.

Proof. This is left as an exercise for the reader, see Exercise 1.9.10 □

Lemma 1.3.28. *Let $f\colon C \to D$ be an embedding which admits a section $s\colon D \to C$. Then f is an equivalence.*

Proof. As $fs \cong \mathrm{id}_D$, it remains to show that $sf \cong \mathrm{id}_C$. Consider the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, dashed, "h"] \arrow[rr, bend left, equal] \arrow[d, "f"'] & C \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "f"] \\ D \arrow[r, "s"] & C \arrow[r, "f"] & D. \end{tikzcd}$$

Since the right square is a pullback square by assumption on f, there exists a map $h\colon C \to C$ making the diagram commute. It follows that $sf \cong h \cong \mathrm{id}_C$, finishing the proof. □

We will now prove some more intricate properties about embeddings, which may be skipped on first reading and referred back to whenever needed.

Lemma 1.3.29. *Let $f\colon C \to D$ and $g\colon D \to E$ be functors and assume that g is an embedding. Then the square*

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "f"'] & C \arrow[d, "gf"] \\ D \arrow[r, "g"] & E \end{tikzcd}$$

is a pullback square.

Proof. This is immediate from the pasting law of pullback squares applied to the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "f"'] & C \arrow[d, "f"] \\ D \arrow[r, equal] \arrow[d, equal] & D \arrow[d, "g"] \\ D \arrow[r, "g"] & E. \end{tikzcd}$$

Indeed, the bottom square is a pullback square by assumption on v, while the top square is clearly a pullback square. $\square$

Lemma 1.3.30. *Consider a commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "gf"'] & & D \arrow[dl, "g"] \\ & E & \end{tikzcd}$$

of functors. If g is an embedding, then f is an embedding if and only if gf is an embedding.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, equal] & C \arrow[r, equal] \arrow[d, "u"] & C \arrow[d, "vu"] \\ C \arrow[r, "u"] & D \arrow[r, "v"] & E. \end{tikzcd}$$

Since v is an embedding, the right square is a pullback square by Lemma 1.3.29. It follows from the pasting law of pullback squares, Lemma 1.3.16, that the left square is cartesian if and only if the outer rectangle is cartesian, proving the claim. $\square$

Corollary 1.3.31. *Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"] & D \end{tikzcd}$$

of functors such that v is an embedding. Then u is an embedding if and only if the induced map $(f,u)\colon A \to B \times_D C$ is an embedding.

Proof. The map $u\colon A \to C$ factors as the composite $A \xrightarrow{(f,u)} B \times_D C \xrightarrow{\mathrm{pr}_C} C$. The map pr_C is an embedding by Lemma 1.3.27, and hence the statement follows from Lemma 1.3.30. $\square$

Lemma 1.3.32. *Consider a commutative diagram*

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "v"] \arrow[d, "g'"'] & D \arrow[d, "g"] \\ E' \arrow[r, "w"] & E. \end{tikzcd}$$

If v and w are embeddings and the composite square is a pullback square, then also the top square is a pullback square.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
C' \arrow[r, equal] \arrow[d, "f'"'] & C' \arrow[r, "u"] \arrow[d, "(g'f', fu)"] & C \arrow[d, "f"] \\
D' \arrow[r, "(g', v)"] & E' \times_E D \arrow[r] \arrow[d] & D \arrow[d] \\
& E' \arrow[r, "w"] & E.
\end{tikzcd}$$

Since w is an embedding, the map $E' \times_E D \to D$ is an embedding by Lemma 1.3.27. Since $v\colon D' \to D$ is also an embedding, it thus follows from Lemma 1.3.30 that the map $(g', v)\colon D' \to E' \times_E D$ is an embedding, and thus by Lemma 1.3.29 the top left square is a pullback square. Since the right bottom square and right outer rectangle are pullback squares, so is the top right square by the pasting law Lemma 1.3.16. Another instance of Lemma 1.3.16 shows that the outer rectangle is a pullback square, as desired. □

1.4 Functor categories

Given two categories C and D, we are interested in forming the *functor category* $\mathrm{Fun}(C, D)$ whose objects are precisely the functors from C to D.

Given any functor

$$e\colon F \times C \to D$$

we can form, for any category T, a map

$$u_T\colon \mathrm{Map}(T, F) \to \mathrm{Map}(C \times T, D)$$

associated to the rule $g \mapsto e \circ (g \times \mathrm{id}_C)$.

$$\begin{tikzcd}
T \times C \arrow[r, "f_c \times \mathrm{id}_C"] \arrow[dr, "f"'] & F \times C \arrow[d, "e"] \\
& D.
\end{tikzcd}$$

Definition 1.4.1. A *functor category* from C to D is a category F equipped with a functor $e\colon F \times C \to D$ such that, for any category T, the induced map

$$u_T\colon \mathrm{Map}(T, F) \xrightarrow{\sim} \mathrm{Map}(C \times T, D)$$

is an equivalence. We then call F the category of functors from C to D and e the *evaluation functor*.

Proposition 1.4.2. *If F and F' both are categories of functors from C to D with evaluation functors e and e' respectively, then there is an equivalence $\varphi\colon F' \xrightarrow{\sim} F$ as well as $\sigma\colon e \circ (\varphi \times \mathrm{id}_C) \cong e'$.*

Proof. Using the equivalence u_T for $T = F'$, we see that there is a functor $\varphi\colon F' \to F$ as well as $\tau\colon u_T(\varphi) \cong e$. For $F' = F$ and $e = e'$, we see that φ must be (isomorphic to) the identity. Exchanging the roles of F and F' we thus get a functor $\psi\colon F \to F'$ that is inverse to φ. □

Axiom B.6 (Functor category)**.** For any two categories C and D, there is a category $\mathrm{Fun}(C, D)$ equipped with an *evaluation functor* $\mathrm{ev}\colon \mathrm{Fun}(C, D) \times C \to D$ that exhibits it as the category of functors from C to D.

Definition 1.4.3. Given another category T, we thus obtain for every functor $f\colon T\times C \to D$ a functor $f_c\colon T \to \mathrm{Fun}(C, D)$ that we refer to as the *currying* of f. It comes equipped with a commutative diagram as follows:

$$\begin{tikzcd} T \times C \arrow[r, "f_c\times\mathrm{id}_C"] \arrow[dr, "f"'] & \mathrm{Fun}(C, D) \times C \arrow[d, "\mathrm{ev}"] \\ & D. \end{tikzcd}$$

Conversely, given a functor $g\colon T \to \mathrm{Fun}(C, D)$, we refer to the composite

$$T \times C \xrightarrow{g\times\mathrm{id}_C} \mathrm{Fun}(C, D) \times C \xrightarrow{\mathrm{ev}} D$$

as the *uncurrying* of g and denote it by $g^u = u_T(g)$. Similarly, a natural isomorphism $\beta\colon g \cong h$ induces a natural isomorphism $\beta^u\colon g^u \cong h^u$. Uncurrying is thus an equivalence

$$(-)^u\colon \mathrm{Map}(T, \mathrm{Fun}(C, D)) \xrightarrow{\sim} \mathrm{Map}(T \times C, D)$$

with inverse defined by currying.

Remark 1.4.4. By considering the case of the terminal category $E = *$, we observe that terms $* \to \mathrm{Fun}(C, D)$ of the functor category correspond to functors $C \simeq C \times * \to D$ from C to D, justifying the notation and terminology of $\mathrm{Fun}(C, D)$.

Construction 1.4.5 (Evaluation functor)**.** Given categories C and D, and $x\colon * \to C$, the *evaluation functor at* x is defined as the composite

$$\mathrm{ev}_x\colon \mathrm{Fun}(C, D) \simeq \mathrm{Fun}(C, D) \times * \xrightarrow{\mathrm{id}\times x} \mathrm{Fun}(C, D) \times C \xrightarrow{\mathrm{ev}} D.$$

The functoriality of the product naturally provides functoriality for the functor categories:

Construction 1.4.6 (Functoriality of functor categories)**.** Given a functor $g\colon D \to D'$, we define the functor $g \circ -\colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, D')$ as the currying of the composite

$$\mathrm{Fun}(C, D) \times C \xrightarrow{\mathrm{ev}} D \xrightarrow{g} D'.$$

By assumption we obtain a commutative square as follows:

$$\begin{tikzcd} \mathrm{Fun}(C, D) \times C \arrow[r, "\mathrm{ev}"] \arrow[d, "(g\circ-)\times\mathrm{id}_C"'] & D \arrow[d, "g"] \\ \mathrm{Fun}(C, D') \times C \arrow[r, "\mathrm{ev}"] & D'. \end{tikzcd}$$

Similarly, given a functor $f\colon C' \to C$, we define the functor $-\circ f\colon \operatorname{Fun}(C', D) \to \operatorname{Fun}(C, D)$ as the currying of the composite

$$\operatorname{Fun}(C', D) \times C \xrightarrow{\mathrm{id}\times f} \operatorname{Fun}(C', D) \times C' \xrightarrow{\mathrm{ev}} D.$$

It satisfies the property that it makes the following square commute:

$$\begin{tikzcd}
\operatorname{Fun}(C', D) \times C \arrow[r, "\mathrm{id}\times f"] \arrow[d, "(-\circ f)\times \mathrm{id}_C"'] & \operatorname{Fun}(C', D) \times C' \arrow[d, "\mathrm{ev}"] \\
\operatorname{Fun}(C, D) \times C \arrow[r, "\mathrm{ev}"] & D.
\end{tikzcd}$$

In a completely analogous way, every natural isomorphism $\beta\colon g \cong g'$ of functors $D \to D'$ induces a natural isomorphism $(\beta\circ -)\colon (g\circ -) \cong (g'\circ -)$ of functors $\operatorname{Fun}(C, D) \to \operatorname{Fun}(C, D')$, and similarly for the construction $-\circ f$.

Exercise 1.4.7 (Exercise 1.9.6)**.** Formulate and prove that the assignments $g \mapsto (g\circ -)$ and $f \mapsto (-\circ f)$ are functorial, in the sense that they respect identity functors and composition of functors.

1.4.1 Properties of functor categories

The universal property of the functor category construction allows us to deduce various compatibilities of functor categories with the terminal category, the initial category, products, coproducts and pullbacks.

Lemma 1.4.8. *For every category C the functor category* $\operatorname{Fun}(C, *)$ *is contractible.*

Proof. We must show that the functor $p_{\operatorname{Fun}(C,*)}\colon \operatorname{Fun}(C, *) \to *$ is an equivalence. We will show that an inverse is given by the functor $(p_{*\times C})_c\colon * \to \operatorname{Fun}(C, *)$, obtained by uncurrying the functor $p_{*\times C}\colon *\times C \to *$. The composite $(p_{*\times C})_c \circ p_{\operatorname{Fun}(C,*)}\colon * \to *$ is automatically isomorphic to $\mathrm{id}_*\colon * \to *$, hence it remains to show that the composite $p_{\operatorname{Fun}(C,*)} \circ (p_{*\times C})_c\colon \operatorname{Fun}(C, *) \to \operatorname{Fun}(C, *)$ is isomorphic to $\mathrm{id}_{\operatorname{Fun}(C,*)}$. It will suffice to prove that the two uncurried functors $\operatorname{Fun}(C, *) \times C \to *$ are isomorphic. But this is again automatic. $\square$

Lemma 1.4.9. *Let C, D and E be categories. Then the functor*

$$(\mathrm{pr}_D \circ -, \mathrm{pr}_E \circ -)\colon \operatorname{Fun}(C, D \times E) \to \operatorname{Fun}(C, D) \times \operatorname{Fun}(C, E)$$

is an equivalence.

Proof. For the purpose of this proof, let us abbreviate $\operatorname{Fun}(C, T)$ by T^C for every category T. We define a functor $h\colon D^C \times E^C \to (D \times E)^C$ by currying the following composite:

$$D^C \times E^C \times C \xrightarrow{(\mathrm{pr}_{D^C}, \mathrm{pr}_C, \mathrm{pr}_{E^C}, \mathrm{pr}_C)} D^C \times C \times E^C \times C \xrightarrow{\mathrm{ev}\times\mathrm{ev}} D \times E.$$

We start by showing that the composite $(\mathrm{pr}_D \circ -, \mathrm{pr}_E \circ -) \circ h$ is the identity. It will suffice to show this after projecting to both D^C and E^C, and by symmetry we may just do this for D^C. In other words, we must show that the composite

$$D^C \times E^C \xrightarrow{h} (D \times E)^C \xrightarrow{\mathrm{pr}_D \circ -} D^C$$

is isomorphic to pr_{D^C}. It will in turn suffice to show this after uncurrying. But there it holds since the uncurrying of both functors can be seen to be isomorphic to the composite

$$D^C \times E^C \times C \xrightarrow{(\mathrm{pr}_{D^C}, \mathrm{pr}_C)} D^C \times C \xrightarrow{\mathrm{ev}} D.$$

We will now show that also the composite $h \circ (\mathrm{pr}_D \circ -, \mathrm{pr}_E \circ -)$ is isomorphic to the identity on $(D \times E)^C$. Since the target of these functors is $(D \times E)^C$, it will suffice to prove that the two uncurried functors $(D \times E)^C \times C \to D \times E$ are isomorphic. The target of these two functors is a product, hence it in turn suffices to show that the components of both functors are isomorphic. By symmetry, it again suffices to do this for the D-component. Using functoriality of uncurrying, and unwinding the constructions, we are reduced to showing that the following diagram commutes:

$$\begin{tikzcd}
(D \times E)^C \times C \arrow[r, "\mathrm{ev}"] \arrow[d, "(\mathrm{pr}_D \circ -) \times \mathrm{id}_C"'] & D \times E \arrow[d, "\mathrm{pr}_D"] \\
D^C \times C \arrow[r, "\mathrm{ev}"] & D.
\end{tikzcd}$$

But this is an instance of Lemma 1.4.6. □

Lemma 1.4.10. *Let T be a category and let $f\colon C \to E$ and $g\colon D \to E$ be functors. Then the functor*

$$(\mathrm{pr}_C \circ -, \mathrm{pr}_D \circ -)\colon \ \mathrm{Fun}(T, C \times_E D) \to \mathrm{Fun}(T, C) \times_{\mathrm{Fun}(T,E)} \mathrm{Fun}(T, D)$$

induced by the isomorphism $(f \circ -) \circ (\mathrm{pr}_C \circ -) \cong (g \circ -) \circ (\mathrm{pr}_D \circ -)$ is an equivalence.

Proof. We will again abbreviate $\mathrm{Fun}(T, -)$ by $(-)^T$. To construct an inverse to this functor, consider the two composites

$$(C^T \times_{E^T} D^T) \times T \xrightarrow{\mathrm{pr}_{C^T} \times \mathrm{id}_T} C^T \times T \xrightarrow{\mathrm{ev}} C$$

and

$$(C^T \times_{E^T} D^T) \times T \xrightarrow{\mathrm{pr}_{D^T} \times \mathrm{id}_T} D^T \times T \xrightarrow{\mathrm{ev}} D.$$

These two functors assemble into a functor to the pullback $C \times_E D$, due to the composite natural isomorphism exhibited by the following commutative diagram:

$$\begin{tikzcd}
 & (C^T \times_{E^T} D^T) \times T \arrow[dl, "\mathrm{pr}_{C^T} \times \mathrm{id}_T"'] \arrow[dr, "\mathrm{pr}_{D^T} \times \mathrm{id}_T"] & \\
\mathrm{Fun}(T, C) \times T \arrow[d, "\mathrm{ev}"'] \arrow[dr, "f \circ -"] & & \mathrm{Fun}(T, D) \times T \arrow[d, "\mathrm{ev}"] \arrow[dl, "g \circ -"'] \\
C \arrow[dr, "f"'] & T^E \times T \arrow[d, "\mathrm{ev}"] & D \arrow[dl, "g"] \\
 & E. &
\end{tikzcd}$$

By currying, we thus obtain a functor $h\colon C^T \times_{E^T} D^T \to (C \times_E D)^T$. We claim that this is the desired inverse. The proof is analogous to that of Lemma 1.4.9, with the added difficulty that one needs to make sure that the natural isomorphisms of functors into E are all compatible. We will leave the details as an exercise to the reader. □

Lemma 1.4.11. *For categories C, D and E, there is a preferred equivalence*

$$\mathrm{Fun}(C, \mathrm{Fun}(D, E)) \xrightarrow{\sim} \mathrm{Fun}(C \times D, E)\,.$$

Proof. We define the preferred functor as the curried functor of the following composite:

$$\mathrm{Fun}(C, \mathrm{Fun}(D, E)) \times C \times D \xrightarrow{\mathrm{ev}\times \mathrm{id}_D} \mathrm{Fun}(D, E) \times D \xrightarrow{\mathrm{ev}} E.$$

To construct an inverse, it will by currying suffice to construct a functor $\mathrm{Fun}(C \times D, E) \times C \to \mathrm{Fun}(D, E)$, which we may take to be the curried functor of the following composite

$$\mathrm{Fun}(C \times D, E) \times C \times D \xrightarrow{\mathrm{ev}} E.$$

By construction, these two functors are compatible with the chain of equivalences

$$\begin{aligned}
\mathrm{Map}(T, \mathrm{Fun}(C, \mathrm{Fun}(D, E))) &\xrightarrow{\sim} \mathrm{Map}(T \times C, \mathrm{Fun}(D, E)) \\
&\xrightarrow{\sim} \mathrm{Map}((T \times C) \times D, E) \\
&\xrightarrow{\sim} \mathrm{Map}(T \times (C \times D), E) \\
&\xleftarrow{\sim} \mathrm{Map}(T, \mathrm{Fun}(C \times D, E))\,.
\end{aligned}$$

This shows that applying $\mathrm{Map}(T, -)$ to the preferred functor is an equivalence for all T, hence that the preferred functor is an equivalence. □

Lemma 1.4.12. *For a category C, the functor* $\mathrm{ev}_*\colon\ \mathrm{Fun}(*, C) \to C$ *is an equivalence.*

Proof. We leave it to the reader to verify that an inverse is given by currying the equivalence $C \times * \xrightarrow{\sim} C$. □

Lemma 1.4.13. *For a category C, the functor category* $\mathrm{Fun}(\emptyset, C)$ *is contractible, i.e.* $\mathrm{Fun}(\emptyset, C) \xrightarrow{\sim} *$.

Proof. For the inverse we pick the preferred functor $\emptyset \to C$. The composite $* \to \mathrm{Fun}(\emptyset, C) \to *$ is automatically isomorphic to the identity. For the composite $\mathrm{Fun}(\emptyset, C) \to * \to \mathrm{Fun}(\emptyset, C)$, we may show this after uncurrying. But since $\mathrm{Fun}(\emptyset, C) \times \emptyset \xrightarrow{\sim} \emptyset$ by Lemma 1.2.12, this follows directly from the defining property of $\emptyset$. □

Lemma 1.4.14. *For categories C, D and E, resriction along the inclusions $i_C\colon C \hookrightarrow C \sqcup D$ and $i_D\colon D \hookrightarrow C \sqcup D$ induces an equivalence*

$$(i_C^*, i_D^*)\colon\ \mathrm{Fun}(C \sqcup D, E) \xrightarrow{\sim} \mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E).$$

Proof. Using the canonical equivalence

$$(C \sqcup D) \times (\mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E)) \cong C \times (\mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E)) \sqcup D \times (\mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E))\,,$$

the functors

$$C \times \mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E) \xrightarrow{\mathrm{pr}_{1,2}} C \times \mathrm{Fun}(C, E) \xrightarrow{\mathrm{ev}} E$$

and

$$D \times \mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E) \xrightarrow{\mathrm{pr}_{1,3}} D \times \mathrm{Fun}(D, E) \xrightarrow{\mathrm{ev}} E$$

assemble into a functor

$$(C \sqcup D) \times (\mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E)) \to E\,.$$

By currying this provides a preferred functor $\mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E) \to \mathrm{Fun}(C \sqcup D, E)$. On the other hand, for any category T, there are canonical equivalences

$$\begin{aligned}
\mathrm{Map}(T, \mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E)) &\xrightarrow{\sim} \mathrm{Map}(T, \mathrm{Fun}(C, E)) \times \mathrm{Map}(T, \mathrm{Fun}(D, E)) \\
&\xrightarrow{\sim} \mathrm{Map}(T \times C, E) \times \mathrm{Map}(T \times D, E) \\
&\xleftarrow{\sim} \mathrm{Map}((T \times C) \sqcup (T \times D), E) \\
&\xrightarrow{\sim} \mathrm{Map}(T \times (C \sqcup D), E)
\end{aligned}$$

where the last equivalence is induced by Corollary 1.3.22. We check that, by construction, the preferred functor is compatible with the chain of equivalences above, hence that it induces an equivalence on any mapping anima of the form $\mathrm{Map}(T, -)$. □

1.5 Objects in a category

Let X be an anima. We consider two categories C and D. Given a functor $u\colon X \to \mathrm{Map}(C, D)$, we obtain, for each term $x\colon X$ a functor $u(x)\colon C \to D$. Given a term $c\colon C$, this yields a term $u(x)(c)\colon D$. In other words, we obtain a rule that defines a functor of type $X \times C \to D$, hence a functor of the form $X \to \mathrm{Fun}(C, D)$. Applying this to the identity of $\mathrm{Map}(C, D)$ yields a canonical functor

$$\iota_{C,D}\colon\ \mathrm{Map}(C, D) \to \mathrm{Fun}(C, D)\,.$$

Proposition 1.5.1. *For any anima X, we get an induced equivalence*

$$(\iota_{C,D})_*\colon\ \mathrm{Map}(X, \mathrm{Map}(C, D)) \xrightarrow{\sim} \mathrm{Map}(X, \mathrm{Fun}(C, D))\,.$$

Proof. It suffices to check that we get an equivalence of the form

$$\mathrm{Map}(X, \mathrm{Map}(C, D)) \xrightarrow{\sim} \mathrm{Map}(X \times C, D)\,.$$

But the map is defined so that the image of $u\colon X \to \mathrm{Map}(C, D)$ is the map associated to the rule $(x, c) \mapsto u(x)(c)$. If two maps u and v have the same image, then, for each x, the rules $c \mapsto u(x)(c)$ and $c \mapsto v(x)(c)$ agree, hence there is a term of type $u(x) \cong v(x)$ for all x. This implies that $u \cong v$. On the other hand, given a map of the form $f\colon X \times C \to D$, for each term $x\colon X$, the rule $c \mapsto f(x, c)$ defines a term $u(x)\colon\ \mathrm{Map}(C, D)$. Therefore, the rule $x \mapsto u(x)$ defines a map $u\colon X \to \mathrm{Map}(C, D)$. It is clear that u is sent to f by construction. □

Definition 1.5.2. For any category C, we define the *animated core* of C to be

$$C^{\simeq} = \mathrm{Map}(*, C)\,.$$

It comes equipped with a canonical functor

$$\iota_C\colon C^{\simeq} \to C$$

defined as $\iota_C = \iota_{*,C}$.

Corollary 1.5.3. *For any anima X and any category D, we get an induced equivalence*

$$(\iota_D)_*\colon \operatorname{Map}(X, D^{\simeq}) \xrightarrow{\sim} \operatorname{Map}(X, D)\,.$$

Proof. This is a special case of the previous proposition in the case $C = *$, by virtue of Lemma 1.4.12. $\square$

Definition 1.5.4. Let C be a category. An *object* x of C is a term of $C^{\simeq}$.

Remark 1.5.5. By definition, understanding the objects of C is the same as understanding all functors of the form $x\colon \Gamma \to C$ where Γ is an anima. However, it also suffices to understand the universal such functor, i.e. the canonical functor $\iota_C\colon C^{\simeq} \to C$.

Definition 1.5.6. Let C be a category and let Γ be an anima. An *object of C in context* Γ is a functor $x\colon \Gamma \to C$.

Convention 1.5.7. From now onward, we will use the phrase *object of C* for an object x of C in an *arbitrary* context Γ. When the context of x is the terminal groupoid $*$, we will speak of an *absolute object*, cf. Definition 1.2.1.

Although the context Γ is allowed to be arbitrary, we may in practice always assume it is the terminal groupoid $*$ by replacing C with C_Γ and working with categories in context Γ instead. We will frequently reinforce this philosophy in our notation by denoting the context of an object x by $\{x\}$.

Proposition 1.5.8. *For any categories C and D there is a preferred equivalence*

$$\operatorname{Map}(C, D) \xrightarrow{\sim} \operatorname{Fun}(C, D)^{\simeq}\,.$$

Proof. The map $\iota_{C,D}$ induces a map $\bar{\iota}_{C,D}\colon \operatorname{Map}(C, D) \to \operatorname{Fun}(C, D)^{\simeq}$, by the preceding proposition. We also have the following equivalences, by definition:

$$\operatorname{Map}(X, \operatorname{Fun}(C, D)^{\simeq}) \xrightarrow{\sim} \operatorname{Map}(X, \operatorname{Fun}(C, D)) \xleftarrow{\sim} \operatorname{Map}(X, \operatorname{Map}(C, D))\,.$$

This implies that the map $\bar{\iota}_{C,D}$ induces an equivalence after applying $\operatorname{Map}(X, -)$ for any anima X. $\square$

Proposition 1.5.9. *A category C is an anima if and only if the canonical functor ι_C is an equivalence:*

$$\iota_C\colon C^{\simeq} \xrightarrow{\sim} C\,.$$

Proof. Since $C^{\simeq}$ is an anima, this is obviously a necessary condition. Conversely, if C is an anima, in order to prove that ι_C is an equivalence, it suffices to prove that, for any anima X, the induced map

$$(\iota_C)_*\colon \operatorname{Map}(X, C^{\simeq}) \to \operatorname{Map}(X, C)$$

is an equivalence, which is precisely Corollary 1.5.3. $\square$

1.5.1 Pushouts of categories

Using mapping animae, we may dualize the definition of a pullback square of categories:

Definition 1.5.10 (Pushout square)**.** A commutative square of categories

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "v"] & D \end{tikzcd}$$

is called a *pushout square* (or *cocartesian*) if, for every category E, the induced square

$$\begin{tikzcd} \mathrm{Map}(D,E) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Map}(D',C) \arrow[d, "(f')^*"] \\ \mathrm{Map}(C,E) \arrow[r, "u^*"] & \mathrm{Map}(C',E) \end{tikzcd}$$

is a pullback square.

Proposition 1.5.11. *A commutative square of categories*

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "v"] & D \end{tikzcd}$$

is called a pushout square (or cocartesian) if, for every category E, the induced square

$$\begin{tikzcd} \mathrm{Fun}(D,E) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Fun}(D',C) \arrow[d, "(f')^*"] \\ \mathrm{Fun}(C,E) \arrow[r, "u^*"] & \mathrm{Fun}(C',E) \end{tikzcd}$$

is a pullback square.

Proof. For any category T, the commutative square

$$\begin{tikzcd} \mathrm{Map}(T,\mathrm{Fun}(D,E)) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Map}(T,\mathrm{Fun}(D',C)) \arrow[d, "(f')^*"] \\ \mathrm{Map}(T,\mathrm{Fun}(C,E)) \arrow[r, "u^*"] & \mathrm{Map}(T,\mathrm{Fun}(C',E)) \end{tikzcd}$$

is equivalent to the commutative square

$$\begin{tikzcd} \mathrm{Map}(T\times D,E) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Map}(T\times D',C) \arrow[d, "(f')^*"] \\ \mathrm{Map}(T\times C,E) \arrow[r, "u^*"] & \mathrm{Map}(T\times C',E) \end{tikzcd}$$

which in turns is equivalent to the following one.

$$\begin{tikzcd} \mathrm{Map}(D, \mathrm{Fun}(T,E)) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Map}(D', \mathrm{Fun}(T,C)) \arrow[d, "(f')^*"] \\ \mathrm{Map}(C, \mathrm{Fun}(T,E)) \arrow[r, "u^*"] & \mathrm{Map}(C', \mathrm{Fun}(T,E)) \end{tikzcd}$$

The latter is a pullback square for all T and E if and only if it is so for all E in the special case where $T = *$ since there is a canonical equivalence $E \xrightarrow{\sim} \mathrm{Fun}(*, E)$. On the other hand, the first square is a pullback square for all T if and only if

$$\begin{tikzcd} \mathrm{Fun}(D, E) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Fun}(D', C) \arrow[d, "(f')^*"] \\ \mathrm{Fun}(C, E) \arrow[r, "u^*"] & \mathrm{Fun}(C', E) \end{tikzcd}$$

is a pullback square. This proves the proposition. □

Lemma 1.5.12 (Pasting lemma for pushout squares)**.** *Consider a commutative diagram*

$$\begin{tikzcd} C_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & C_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\ D_1 \arrow[r, "h_1"'] & D_2 \arrow[r, "h_2"'] & D_3. \end{tikzcd}$$

If the left-hand square is a pushout square, then the right-hand square is a pushout square if and only if the outer rectangle is a pushout square.

Proof. This follows immediately by applying Lemma 1.3.16 to the commutative diagrams

$$\begin{tikzcd} \mathrm{Map}(D_3, E) \arrow[r, "h_2^*"] \arrow[d, "f_3^*"'] & \mathrm{Map}(D_2, E) \arrow[r, "h_1^*"] \arrow[d, "f_2^*"] & \mathrm{Map}(D_1, E) \arrow[d, "f_1^*"] \\ \mathrm{Map}(C_3, E) \arrow[r, "g_2^*"'] & \mathrm{Map}(C_2, E) \arrow[r, "g_1^*"'] & \mathrm{Map}(C_1, E). \end{tikzcd}$$

for each E. □

1.6 Morphisms and commutative triangles

Our discussion of categories has so far completely ignored the *internal* structure that categories are expected to have. In this section we will start describing this internal structure by introducing the concepts of *morphisms* and commutative triangles in a category C.

Axiom C.1 (Walking morphism)**.** There is a category $[1]$ equipped with objects $0, 1 \colon * \to [1]$.

The category $[1]$ is called the *walking morphism* and should be thought of as the poset $\{0 \leq 1\}$.

Let us introduce some standard terminology regarding functors out of $[1]$.

Definition 1.6.1 (Morphisms)**.** An (absolute) *morphism* in a category C is defined as a functor $f\colon [1] \to C$, or equivalently as an absolute object of the arrow category $\mathrm{Fun}([1], C)$.

Every morphism $f\colon [1] \to C$ has a *source/domain* $f(0)\colon * \to C$ and a *target/codomain* $f(1)\colon * \to C$, obtained by precomposition with $0, 1\colon * \to [1]$. If f is a morphism in C, and x and y are objects of C, we will often write $f\colon x \to y$ to indicate that f comes equipped with isomorphisms $x \cong f(0)$ and $y \cong f(1)$, which we will mostly leave implicit in the notation. In this case we say that *f is a morphism from x to y*.

We write $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$ for the source and target functors, given by evaluation at 0 and 1, cf. Construction 1.4.5.

Definition 1.6.2 (Natural transformations)**.** If C and D are categories, we define a *natural transformation of functors* $C \to D$ to be a morphism in $\mathrm{Fun}(C, D)$. By (un)currying, this may equivalently be encoded as a functor $[1] \times C \to D$.

Definition 1.6.3 (Arrow category/ Hom groupoid)**.** Let C be a category. We refer to the functor category $\mathrm{Fun}([1], C)$ as the *arrow category* of C. Given terms x and y of C, we define the *hom groupoid* $C(x, y)$ via the following pullback square:

$$\begin{tikzcd} C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ \{(x,y)\} \arrow[r, "(x,y)"] & C \times C. \end{tikzcd}$$

Observe that the terms of $C(x, y)$ are triples (f, α, β), where $f\colon x' \to y'$ is a morphism in C and $\alpha\colon x \cong x'$ and $\beta\colon y \cong y'$ are isomorphisms in C. We will sometimes use the alternative notation $\mathrm{Hom}_C(x, y)$ for $C(x, y)$.

Construction 1.6.4. Let $f\colon C \to D$ be a functor and consider two terms x and y of C. Then the induced functor $f_*\colon \mathrm{Fun}([1], C) \to \mathrm{Fun}([1], D)$ induces a functor

$$f\colon C(x, y) \to D(f(x), f(y))$$

on fibers.

Example 1.6.5 (Identity morphism)**.** For an absolute object $x\colon * \to C$ we define the *identity morphism* $\mathrm{id}_x\colon x \to x$ in C as the composite $[1] \xrightarrow{p_{[1]}} * \xrightarrow{x} C$. Note that we have $\mathrm{id}_x \in C(x, x)$. For every functor $f\colon C \to D$ there is a preferred isomorphism $f(\mathrm{id}_x) \cong \mathrm{id}_{f(x)}$.

Definition 1.6.6 (Slice categories)**.** Let C be a category and let x be a term of C. We define the *slice categories* $C_{/x}$ and $C_{x/}$ via the following two pullback diagrams:

$$\begin{tikzcd} C_{/x} \arrow[r] \arrow[d, "p_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ C \arrow[r, "\mathrm{id}_C \times x"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \times C \arrow[d, "\mathrm{pr}_2"] \\ \{x\} \arrow[r, "x"] & C. \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_{x/} \arrow[r] \arrow[d, "q_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ C \arrow[r, "x \times \mathrm{id}_C"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \times C \arrow[d, "\mathrm{pr}_1"] \\ \{x\} \arrow[r, "x"] & C. \end{tikzcd}$$

In particular, an object of the category $C_{/x}$ consists of a pair (y, u) where y is an object of C and $u\colon y \to x$ is a morphism in C. Similarly the objects of $C_{x/}$ are pairs (y, v) where now v is a morphism in C of the form $v\colon x \to y$.

Definition 1.6.7 (Initial/terminal object)**.** A term x in a category C is called *terminal* if the functor $p_x \colon C_{/x} \to C$ is an equivalence. Dually, x is called *initial* if $q_x \colon C_{x/} \to C$ is an equivalence.

Remark 1.6.8. Observe that the fiber of p_x over some term y is the hom groupoid $C(x, y)$, using the pasting law for pullback squares. In particular, if x is terminal then $C(y, x)$ is contractible for every y. Dually, if x is initial then $C(x, y)$ is contractible for all y. The converse will be established later, once additional axioms have been introduced.

Axiom C.2. The object $0 : [1]$ is initial. The object $1 : [1]$ is terminal.

Construction 1.6.9. As a consequence of Axiom C.2, we obtain the following two pullback squares:

$$\begin{tikzcd}
{[1]} \arrow[r, "\overline{\min}"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],[1]) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
{[1]} \arrow[r, "(\mathrm{const}_0,\mathrm{id})"'] & {[1]\times[1]}
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
{[1]} \arrow[r, "\overline{\max}"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],[1]) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
{[1]} \arrow[r, "(\mathrm{id},\mathrm{const}_1)"'] & {[1]\times[1].}
\end{tikzcd}$$

The two top horizontal maps in these diagrams correspond under currying/uncurrying to functors

$$\min\colon [1]\times[1] \to [1] \qquad \text{and} \qquad \max\colon [1]\times[1] \to [1].$$

Lemma 1.6.10. *The maps* max *and* min *satisfy the equations*

$$\begin{aligned} \max(x,0) &= x = \max(0,x), & \max(x,1) &= 1 = \max(1,x),\\ \min(x,0) &= 0 = \min(0,x), & \min(x,1) &= x = \min(1,x), \end{aligned}$$

in the sense that the following diagrams commute:

$$\begin{tikzcd}
{[1]} \arrow[r, "(\mathrm{id},0)"] \arrow[dr, "\mathrm{const}_0"'] & {[1]\times[1]} \arrow[d, "\min"] & {[1]} \arrow[l, "(0,\mathrm{id})"'] \arrow[dl, "\mathrm{const}_0"] \\
& {[1]} &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[1]} \arrow[r, "(\mathrm{id},1)"] \arrow[dr, equal] & {[1]\times[1]} \arrow[d, "\min"] & {[1]} \arrow[l, "(1,\mathrm{id})"'] \arrow[dl, equal] \\
& {[1]} &
\end{tikzcd}$$

$$\begin{tikzcd}
{[1]} \arrow[r, "(\mathrm{id},0)"] \arrow[dr, equal] & {[1]\times[1]} \arrow[d, "\max"] & {[1]} \arrow[l, "(0,\mathrm{id})"'] \arrow[dl, equal] \\
& {[1]} &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[1]} \arrow[r, "(\mathrm{id},1)"] \arrow[dr, "\mathrm{const}_1"'] & {[1]\times[1]} \arrow[d, "\max"] & {[1]} \arrow[l, "(1,\mathrm{id})"'] \arrow[dl, "\mathrm{const}_1"] \\
& {[1].} &
\end{tikzcd}$$

Proof. The equations $\min(x,0) = 0$, $\min(x,1) = x$, $\max(x,0) = x$ and $\max(x,1) = 1$ are immediate from the two pullback squares stated before the lemma. For the other four relations we use that $q_0^{-1} \colon [1] \xrightarrow{\sim} [1]_{0/}$ sends 0 to $\mathrm{id}_0 \colon 0 \to 0$ and 1 to $\mathrm{can}\colon 0 \to 1$, while $p_1^{-1} \colon [1] \xrightarrow{\sim} [1]_{/1}$ sends 0 to $\mathrm{can}\colon 0 \to 1$ and 1 to $\mathrm{id}_1 \colon 1 \to 1$. $\square$

1.6.1 Commutative triangles

Apart from the poset $[1] = \{0 \leq 1\}$, which classifies morphisms, there should also be a poset $[2] = \{\overline{0} \leq \overline{1} \leq \overline{2}\}$ classifying *commutative triangles*.

Definition 1.6.11 (Walking commutative triangle)**.** We define the *walking commutative triangle* as

$$[2] \quad := \quad \mathrm{Fun}([1],[1]).$$

We define objects $\overline{0}, \overline{1}, \overline{2} : [2]$ by:

$$\overline{0} := \mathrm{id}_0, \qquad \overline{1} := \mathrm{can}, \qquad \overline{2} := \mathrm{id}_1,$$

where $\mathrm{can}\colon [1] \to [1]$ is the identity of $[1]$, regarded as a morphism $\mathrm{can}\colon 0 \to 1$ in $[1]$.

Construction 1.6.12. We construct three *face maps* $d_i\colon [1] \to [2]$ for $i = 0, 1, 2$:

- We set $d_0 := \overline{\max}\colon [1] \to \mathrm{Fun}([1],[1])$;
- We set $d_1 := p^*_{[1]}\colon [1] \to \mathrm{Fun}([1],[1])$;
- We set $d_2 := \overline{\min}\colon [1] \to \mathrm{Fun}([1],[1])$.

We also construct two *degeneracy maps* $s_i\colon [2] \to [1]$ for $i = 0, 1$:

- We set $s_0 := \mathrm{ev}_0\colon \mathrm{Fun}([1],[1]) \to [1]$;
- We set $s_1 := \mathrm{ev}_1\colon \mathrm{Fun}([1],[1]) \to [1]$.

Convention 1.6.13. It will sometimes be useful to denote the terminal category by $[0]$, and to denote the maps $0 : * \to [1]$ and $1 : * \to [1]$ by $d_1\colon [0] \to [1]$ and $d_0\colon [0] \to [1]$, respectively. We will visualize the three categories $[0]$, $[1]$ and $[2]$ as follows:

$$0, \qquad 0 \longrightarrow 1 \qquad \text{and} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2. \end{tikzcd}$$

The five face maps $d_0, d_1\colon [0] \to [1]$ and $d_0, d_1, d_2\colon [1] \to [2]$ and the relations between them may be displayed as follows:

$$\begin{tikzcd}
 & & & & 0 \arrow[ddl, dashed, "d_0"'] \arrow[ddr, dashed, "d_1"] & & & & \\
 & & & & & & & & \\
 & & & 1 & \cong & 0 \arrow[dr] & & & \\
 & & 0 \arrow[ur] \arrow[dr, dashed, "d_2"] & & 1 \arrow[dr] & & 1 \arrow[dl, dashed, "d_0"'] & & \\
 & & & 0 \arrow[ur] \arrow[rr] & & 2 & & & \\
 & & \cong & & & & \cong & & \\
0 \arrow[uuurr, dashed, "d_1"] \arrow[rrr, dashed, "d_1"] & & & 0 \arrow[uu, dashed, "d_1"] \arrow[rr] & & 1 & & & 0. \arrow[lll, dashed, "d_0"'] \arrow[uuull, dashed, "d_0"']
\end{tikzcd}$$

The two degeneracy maps $s_0, s_1 \colon [2] \to [1]$ may be interpreted the following 'projections':

$$\begin{tikzcd}
0 \arrow[r] \arrow[dr] & 1 \arrow[d] & & 0 \arrow[d] & & & 0 \arrow[dl] \arrow[d] \\
 & 2 \arrow[r, phantom, ""] & {} \arrow[r, dashed, "s_0"] & 1 & {} & 2 & 1. \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=tiny] {} \arrow[l, dashed, "s_1"'] \end{tikzcd}$$

Exercise 1.6.14. Show that the face and degeneracy maps satisfy the following simplicial identities:

$$d_0 \circ d_0 \cong d_1 \circ d_0, \qquad d_0 \circ d_1 \cong d_2 \circ d_0, \qquad d_1 \circ d_1 \cong d_2 \circ d_1 \quad \in \quad \mathrm{Fun}([0],[2])$$

$$s_0 \circ d_0 \cong \mathrm{id}_{[1]} \cong s_0 \circ d_1, \qquad s_0 \circ d_2 \cong \mathrm{const}_0 \quad \in \quad \mathrm{Fun}([1],[1])$$

$$s_1 \circ d_0 \cong \mathrm{const}_1, \qquad s_1 \circ d_1 \cong \mathrm{id}_{[1]} \cong s_1 \circ d_2 \quad \in \quad \mathrm{Fun}([1],[1]).$$

Definition 1.6.15 (Commutative triangle)**.** Given a category C, we define a *commutative triangle in C* to be a functor $[2] \to C$. The three functors $d_0, d_1, d_2 \colon [1] \to [2]$ induce three restriction functors

$$d_0^*, d_1^*, d_2^* : \mathrm{Fun}([2], C) \to \mathrm{Fun}([1], C).$$

We will frequently denote commutative triangles by

$$\sigma \quad = \quad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z, \end{tikzcd}$$

where $f = d_2^*(\sigma)$, $g = d_0^*(\sigma)$ and $h = d_1^*(\sigma)$. We will often leave commutative triangles unnamed, and only label their three edges. We may sometimes denote in the same manner triples of morphisms (f, g, h) that do *not* (or not necessarily) come from a commutative triangle, which will then be mentioned explicitly.

Definition 1.6.16. Given morphisms $f\colon x \to y$ and $g\colon y \to z$ in C, a *composite* of f and g is a morphism $h\colon x \to z$ that comes equipped with a commutative triangle σ of the above form:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "h"'] & & z. \end{tikzcd}$$

We define the *category of composites* $\mathrm{Comp}_{f,g}(C)$ as the following pullback:

$$\begin{tikzcd} \mathrm{Comp}_{f,g}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([2],C) \arrow[d, "(d_2^*, d_0^*)"] \\ * \arrow[r, "(f,g)"] & \mathrm{Fun}([1],C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_0} \mathrm{Fun}([1],C). \end{tikzcd}$$

Here the right vertical map is the map induced by the following commutative diagram of restriction maps:

$$\begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "d_0^*"] \arrow[d, "d_2^*"'] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_0"] \\ \mathrm{Fun}([1],C) \arrow[r, "\mathrm{ev}_1"'] & C. \end{tikzcd}$$

Example 1.6.17. The two functors $s_0, s_1\colon [2] \to [1]$ induce functors

$$s_0^*, s_1^*\colon \mathrm{Fun}([1],C) \to \mathrm{Fun}([2],C),$$

so that every morphism $f\colon x \to y$ in C gives rise to two commutative diagrams $s_0^*(f)$ and $s_1^*(f)$ in C. As a consequence of the simplicial identities from Exercise 1.6.14, these commutative triangles have the following form:

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y. \end{tikzcd}$$

In particular, these triangles exhibit f as a composite of id_x and f, as well as a composite of f and id_y.

1.6.2 Commutative squares

Throughout, C denotes a category.

Definition 1.6.18 (Commutative square)**.** We define a *commutative square in C* to be a functor $[1] \times [1] \to C$. We will often display commutative squares as follows:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & w. \end{tikzcd}$$

In ordinary category theory, a commutative square consists of morphisms f, g, h and k as above that satisfy the condition that $k \circ f = h \circ g$. We will now formulate a synthetic analogue of this.

Definition 1.6.19. We define two functors $j_0, j_1 \colon [2] \to [1] \times [1]$ as

$$j_0 := (s_0, s_1) \qquad \text{and} \qquad j_1 := (s_1, s_0).$$

These two commutative triangles in $[1] \times [1]$ may be visualized as follows:

$$j_0 \quad = \quad \begin{tikzcd} (0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] \\ & (1,1), \end{tikzcd} \qquad j_1 \quad = \quad \begin{tikzcd} (0,0) \arrow[d] \arrow[dr] & \\ (1,0) \arrow[r] & (1,1). \end{tikzcd}$$

The simplicial relations $s_0 \circ d_1 \cong \mathrm{id}_{[1]} \cong s_1 \circ d_1$ provide an isomorphism making the following square commute:

$$\begin{tikzcd} {[1]} \arrow[r, "d_1"] \arrow[d, "d_1"'] & {[2]} \arrow[d, "j_0"] \\ {[2]} \arrow[r, "j_1"'] & {[1] \times [1].} \end{tikzcd} \tag{1.1}$$

Axiom D (Commutative square axiom)**.** The commutative square (1.1) is a pushout square: For every category C, the commutative square

$$\begin{tikzcd} \mathrm{Fun}([1] \times [1], C) \arrow[r, "j_1^*"] \arrow[d, "j_0^*"'] & \mathrm{Fun}([2], C) \arrow[d, "d_1^*"] \\ \mathrm{Fun}([2], C) \arrow[r, "d_1^*"] & \mathrm{Fun}([1], C) \end{tikzcd}$$

is a pullback square.

Construction 1.6.20. Consider commutative triangles $\sigma \colon [2] \to C$ and $\tau \colon [2] \to C$ equipped with an isomorphism $\varphi \colon d_1^*(\sigma) \cong d_1^*(\tau)$ in $\mathrm{Fun}([1], C)$. By Axiom D, this data gives rise to a commutative square $\square(\sigma, \tau, \varphi) \colon [1] \times [1] \to C$ such that $\sigma \cong j_0^*(\square(\sigma, \tau, \varphi))$ and $\tau \cong j_1^*(\square(\sigma, \tau, \varphi))$:

$$\square(\sigma, \tau, \varphi) \quad = \quad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr] \arrow[dr, phantom, "\sigma", shift left=2ex] \arrow[dr, phantom, "\tau"', shift right=2ex] & y \arrow[d, "k"] \\ z \arrow[r, "h"'] & w \end{tikzcd}$$

We will sometimes just write $\square(\sigma, \tau)$ instead of $\square(\sigma, \tau, \varphi)$ when the isomorphism φ is left implicit.

Construction 1.6.21. We construct functors $p_0, p_2 \colon [1] \times [1] \to [2]$, pictorially represented by the following diagrams:

$$p_0 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] \arrow[dr] & 1 \arrow[d] \\ 0 \arrow[r] & 2, \end{tikzcd} \qquad p_2 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[d, equal] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

More precisely, $p_0 := \square(\mathrm{id}_{[2]}, d_1 s_0)$ is the commutative square determined by the two commutative triangles $\sigma = \mathrm{id}_{[2]}$ and $\tau = d_1 \circ s_0$, with isomorphism

$$d_1^*(\sigma) = \mathrm{id}_{[2]} \circ d_1 \cong d_1 \cong d_1 \circ \mathrm{id}_{[1]} \cong d_1 \circ s_0 \circ d_1 = d_1^*(\tau).$$

Similarly, $p_2 := \square(d_1 s_1, \mathrm{id}_{[2]})$ is the square determined by $\sigma = d_1 \circ s_1$ and $\tau = \mathrm{id}_{[2]}$, with analogous isomorphism between $d_1^*(\sigma)$ and $d_1^*(\tau)$.

By construction these two maps satisfy the following two relations:

$$p_0 \circ j_0 \cong \mathrm{id}_{[2]}, \qquad p_0 \circ j_1 \cong s_0 \circ d_1, \qquad p_2 \circ j_0 \cong s_1 \circ d_1, \qquad p_2 \circ j_1 \cong \mathrm{id}_{[2]}\,.$$

Lemma 1.6.22. *We have the relations*

$$\square(s_0, s_1) = \mathrm{pr}_1 \qquad \text{and} \qquad \square(s_1, s_0) = \mathrm{pr}_2$$

of functors $[1] \times [1] \to [1]$.

Proof. It suffices to observe that these two projection maps have the correct restrictions to the triangles j_0 and j_1. □

Lemma 1.6.23. *We have*

$$\max \cong \square(s_1, s_1) \qquad \text{and} \qquad \min \cong \square(s_0, s_0).$$

Proof. It suffices to check that max and min restrict correctly along j_0 and j_1, and this follows directly by unwinding the definitions of $\square(s_1, s_1)$ and $\square(s_0, s_0)$ from Construction 1.6.20. □

Lemma 1.6.24. *We have relations*

$$\begin{aligned} j_0 p_0(x, y) &= (\min(x, y), y), \qquad & j_1 p_0(x, y) &= (y, \min(x, y)), \\ j_0 p_2(x, y) &= (y, \max(x, y)), \qquad & j_1 p_2(x, y) &= (\max(x, y), y). \end{aligned}$$

Proof. Since $p_0 := \square(\mathrm{id}_{[2]}, d_1 s_0)$ and $p_2 := \square(d_1 s_1, \mathrm{id}_{[2]})$, we have

$$\begin{aligned} s_0 p_0 &\cong \square(s_0, s_0 d_1 s_0) \cong \min, \qquad & s_1 p_0 &\cong \square(s_1, s_1 d_1 s_0) \cong \square(s_1, s_0) \cong \mathrm{pr}_2 \\ s_0 p_1 &\cong \square(s_0 d_1 s_1, s_0) \cong \square(s_1, s_0) \cong \mathrm{pr}_2, \qquad & s_1 p_1 &\cong \square(s_1 d_1 s_1, s_1) = \max, \end{aligned}$$

where we used Lemma 1.6.22. Using $j_0 = (s_0, s_1)$ and $j_1 = (s_1, s_0)$ we get the desired four relations. □

1.7 The Segal axiom

In this section, we will introduce the *Segal axiom*, which will provide unique compositions of morphisms in categories.

Axiom E (Segal axiom)**.** For any category C, the functor

$$(d_0^*, d_2^*)\colon\ \mathrm{Fun}([2], C) \to \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C)$$

is an equivalence.

In particular, for every pair of composable morphisms $f: x \to y$ and $g: y \to z$ in C the category $\mathrm{Comp}_{f,g}(C)$ of composites of f and g is contractible, being defined as the fiber of the equivalence (d_0^*, d_2^*).

Notation 1.7.1. Given composable morphisms $f: x \to y$ and $g: y \to z$, we denote their unique composite by $g \circ f: x \to z$.

Observation 1.7.2 (Exercise 1.9.11)**.** Let C_0 and C_1 be categories and consider morphisms $f_0: x_0 \to y_0$ and $g_0: y_0 \to z_0$ in C_0 and morphisms $f_1: x_1 \to y_1$ and $g_1: y_1 \to z_1$ in C_1. Then the composite $(g_0, g_1) \circ (f_0, f_1): (x_0, x_1) \to (z_0, z_1)$ in $C_0 \times C_1$ is given by $(g_0 \circ f_0, g_1 \circ f_1)$.

Observation 1.7.3 (Exercise 1.9.12)**.** Show that a functor $F: C \to D$ preserves composition: for morphisms $f: x \to y$ and $g: y \to z$ in C there is a preferred isomorphism

$$F(g \circ f) \cong F(g) \circ F(f).$$

The definition of $g \circ f$ works for arbitrary terms f and g of the hom groupoids $C(x, y)$ and $C(y, z)$, showing that $g \circ f$ is automatically functorial in g and f. We may make the resulting functoriality more explicit as follows:

Construction 1.7.4 (Composition functors)**.** Consider the zig-zag

$$\mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C) \xleftarrow[\sim]{(d_2^*, d_0^*)} \mathrm{Fun}([2], C) \xrightarrow{(d_1^*)^*} \mathrm{Fun}([1], C)$$

of functors over C^3. The first map is an equivalence by the Segal axiom, and hence it admits an inverse over C^3, providing a composite functor

$$- \circ -: \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C) \xrightarrow{\sim} \mathrm{Fun}([2], C) \xrightarrow{(d_1^*)^*} \mathrm{Fun}([1], C).$$

Passing to fibers over $(x, y, z) \in C^3$ then produces a functor

$$- \circ -: C(x, y) \times C(y, z) \to C(x, z),$$

which we call the *composition functor*.

We will now show that composition in a category is unital and associative.

Lemma 1.7.5 (Unitality)**.** *For every morphism $f: x \to y$ in a category C, there are isomorphisms*

$$\mathrm{id}_y \circ f \cong f \cong f \circ \mathrm{id}_x$$

in $\mathrm{Fun}([1], C)$*. Moreover, these isomorphisms are natural in f, in the sense that they form natural isomorphisms of functors* $\mathrm{Fun}([1], C) \to \mathrm{Fun}([1], C)$.

Proof. For fixed f, these relations follow from Example 1.6.17. Observe that the arguments given there go through for morphisms $f: \Gamma \to \mathrm{Fun}([1], C)$ in an arbitrary context Γ. Taking $\Gamma = \mathrm{Fun}([1], C)$ then shows naturality in f.

In more detail: for the naturality of the relation $\mathrm{id}_y \circ f \cong f$ we have to show that the composite

$$\mathrm{Fun}([1], C) \xrightarrow{\sim} \mathrm{Fun}([1], C) \times_C C \xrightarrow{1 \times p_{[1]}^*} \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1]) \xrightarrow{- \circ -} \mathrm{Fun}([1], C)$$

is isomorphic to the identity functor. Unwinding the proof of Example 1.6.17, we see that this natural isomorphism is given by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \arrow[r, "s_1^*"] \arrow[d, "\sim"'] \arrow[rr, bend left, equal] & \mathrm{Fun}([2],C) \arrow[r, "d_1^*"] \arrow[d, "\simeq"] & \mathrm{Fun}([1],C) \arrow[d, equal] \\
\mathrm{Fun}([1],C) \times_C C \arrow[r, "1\times p_{[1]}^*"'] & \mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1]) \arrow[r, "-\circ-"'] & \mathrm{Fun}([1],C).
\end{tikzcd}$$

A similar discussion applies to the relation $f \cong \mathrm{id}_y \circ f$. $\square$

<u>Proposition 1.7.6</u> (Associativity, cf. [RS17, Proposition 5.9])**.** *For composable morphisms $f\colon x \to y$, $g\colon y \to z$ and $h\colon z \to w$ in a category C, there is an isomorphism*

$$h \circ (g \circ f) \cong (h \circ g) \circ f$$

in $\mathrm{Fun}([1],C)$. *Moreover, this isomorphism is natural in the triple (f,g,h), in the sense that it forms a natural isomorphism of functors*

$$- \circ (- \circ -) \cong (- \circ -) \circ -\colon \mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1],C) \to \mathrm{Fun}([1],C).$$

Proof. Consider the following two morphisms in $\mathrm{Fun}([1],C)$, regarded as objects of $\mathrm{Fun}([1],\mathrm{Fun}([1],C)) \simeq \mathrm{Fun}([1]\times[1],C)$:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, "g\circ f"] & y \arrow[d, "g"] \\
x \arrow[r, "g\circ f"'] & z
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
y \arrow[r, "h\circ g"] \arrow[d, "g"'] \arrow[dr, "h\circ g"] & w \arrow[d, "\mathrm{id}_w"] \\
z \arrow[r, "h"'] & w.
\end{tikzcd}$$

These morphisms are natural in f, g and h: we have functors

$$\mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1],C) \xrightarrow{\simeq} \mathrm{Fun}([2],C) \xrightarrow{p_0^*} \mathrm{Fun}([1]\times[1],C),$$

$$\begin{tikzcd}
x \arrow[r, "f"] & y \arrow[d, "g"] \\
 & z
\end{tikzcd}
\mapsto
\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "g\circ f"'] & y \arrow[d, "g"] \\
 & z
\end{tikzcd}
\mapsto
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, "g\circ f"] & y \arrow[d, "g"] \\
x \arrow[r, "g\circ f"'] & z,
\end{tikzcd}$$

and similarly for the other diagram. Forming the composite of these two morphisms in $\mathrm{Fun}([1],C)$, we thus obtain a commutative square in C of the form

$$\begin{tikzcd}
x \arrow[r, "(h\circ g)\circ f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr] & w \arrow[d, "\mathrm{id}_w"] \\
x \arrow[r, "h\circ(g\circ f)"'] & w,
\end{tikzcd}$$

natural in f, g and h. Using Lemma 1.7.5, the diagonal of this square is then naturally isomorphic to both $(h \circ g) \circ f$ and $h \circ (g \circ f)$, finishing the proof. $\square$

Observe that another way of formulating the Segal Axiom E is that the commutative square

$$\begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "d_2^*"] \arrow[d, "d_0^*"'] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Fun}([1],C) \arrow[r, "\mathrm{ev}_0"] & C \end{tikzcd}$$

is a pullback square. This has the following corollary:

Corollary 1.7.7. *For every category C, the commutative squares*

$$\begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "p_0^*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1]\times[1],C) \arrow[d, "\mathrm{res}"] \\ C \arrow[r, "(p_{[1]})^*"] & \mathrm{Fun}(\{0\}\times[1],C) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "p_2^*"] \arrow[d, "\mathrm{ev}_2"'] & \mathrm{Fun}([1]\times[1],C) \arrow[d, "\mathrm{res}"] \\ C \arrow[r, "(p_{[1]})^*"] & \mathrm{Fun}([1]\times\{1\},C) \end{tikzcd}$$

are pullback squares.

Proof. We will prove the claim for the left square and leave the analogous proof of the right square to the reader. Note that this square can be written as the outer square in the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "p_0^*"] \arrow[d, "d_1^*"'] & \mathrm{Fun}([1]\times[1],C) \arrow[d, "j_1^*"] \\ \mathrm{Fun}([1],C) \arrow[r, "s_1^*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([2],C) \arrow[d, "d_2^*"] \\ C \arrow[r, "(p_{[1]})^*"] & \mathrm{Fun}([1],C), \end{tikzcd}$$

and so it will suffice that each of these two squares is a pullback square. The first one follows from applying the pasting law of pullback squares to the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "p_0^*"] \arrow[d, "d_1^*"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1]\times[1],C) \arrow[r, "j_0^*"] \arrow[d, "j_1^*"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([2],C) \arrow[d, "d_1^*"] \\ \mathrm{Fun}([1],C) \arrow[r, "s_1^*"] & \mathrm{Fun}([2],C) \arrow[r, "d_1^*"] & \mathrm{Fun}([1],C). \end{tikzcd}$$

Here the outer square is evidently a pullback, and the right square is a pullback by the Commutative Square Axiom D. The second one similarly follows from applying the pasting law of pullback squares to the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fun}([1],C) \arrow[r, "s_1^*"] \arrow[d, "\mathrm{ev}_0"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([2],C) \arrow[r, "d_1^*"] \arrow[d, "d_2^*"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_0"] \\ C \arrow[r, "(p_{[1]})^*"] & \mathrm{Fun}([1],C) \arrow[r, "\mathrm{ev}_1"] & C. \end{tikzcd}$$

Here the outer rectangle is again evidently a pullback square, and the right square is a pullback square by the Segal Axiom E. $\square$

Remark 1.7.8. Informally, the previous corollary says that commutative triangles in C may equivalently be encoded in the following three ways:

(1) One might consider commutative triangles $[2] \to C$, which we will display as

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "gf"] & & z. \end{tikzcd}$$

(2) One might consider commutative squares $[1] \times [1] \to C$ whose restriction to $\{0\} \times [1]$ is constant, which we will display as

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] & y \arrow[d, "g"] \\ x \arrow[r, "gf"] & z. \end{tikzcd}$$

(3) One might consider commutative squares $[1] \times [1] \to C$ whose restriction to $[1] \times \{1\}$ is constant, which we will display as

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "gf"'] & y \arrow[d, "g"] \\ z \arrow[r, equal] & z. \end{tikzcd}$$

1.8 The Rezk axiom

The notions of identity morphisms and composition of morphisms in categories, provided by the Segal axiom, lead to the notion of an *isomorphism* in a category:

Definition 1.8.1 (Isomorphisms)**.** Consider a morphism $f : x \to y$ in a category C. We say that f is *invertible*, or that it is an *isomorphism*, if there are commutative triangles in C of the form

$$\begin{tikzcd} y \arrow[r, "g"] \arrow[dr, "\mathrm{id}_y"'] & x \arrow[d, "f"] \\ & y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "\mathrm{id}_x"'] & y \arrow[d, "h"] \\ & x. \end{tikzcd}$$

A priori, this use of the word ‘isomorphism’ clashes with that of Definition 1.2.1. For this reason, we will now introduce the *Rezk axiom*, which enforces that these two notions of isomorphisms agree with each other. We start by defining the category $\mathrm{Iso}(C)$ of isomorphisms in C.

Definition 1.8.2. Given a category C, we define the category $\mathrm{Iso}(C)$ via the following pullback diagram:

$$
\begin{tikzcd}
\mathrm{Iso}(C) \arrow[r] \arrow[dd] \arrow[rr, dashed, bend left, "\pi_{\mathrm{Iso}}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([2],C) \times_{d_0,\mathrm{Fun}([1],C),d_2} \mathrm{Fun}([2],C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\Delta"] \\
& \mathrm{Fun}([2],C) \times \mathrm{Fun}([2],C) \arrow[r, "d_0^* \times d_2^*"] \arrow[d, "d_1^* \times d_1^*"] & \mathrm{Fun}([1],C) \times \mathrm{Fun}([1],C) \\
C \times C \arrow[r, "(x,y) \mapsto (\mathrm{id}_y, \mathrm{id}_x)"] & \mathrm{Fun}([1],C) \times \mathrm{Fun}([1],C). &
\end{tikzcd}
$$

Explicitly, a term of $\mathrm{Iso}(C)$ is a commutative diagram in C of the form

$$
\begin{tikzcd}
& x \arrow[dr, "\mathrm{id}_x"] \arrow[dd, "f"] & \\
y \arrow[ur, "g"] \arrow[dr, "\mathrm{id}_y"'] & & x. \\
& y \arrow[ur, "h"'] &
\end{tikzcd}
$$

We denote the composite functor at the top of the diagram by

$$\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Fun}([1],C).$$

Remark 1.8.3. Notice that by definition a morphism $f \in \mathrm{Fun}([1],C)$ is invertible in C if and only if it lifts to a term of $\mathrm{Iso}(C)$. We may therefore think of the fiber of π_{Iso} over f as the category of 'proofs that f is invertible'. The following lemma shows that up to isomorphism there exists at most one such proof:

Lemma 1.8.4. *For a category C, the map $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Fun}([1],C)$ is an embedding.*

Proof. We need to show that the diagonal $\Delta \colon \mathrm{Iso}(C) \to \mathrm{Iso}(C) \times_{\mathrm{Fun}([1],C)} \mathrm{Iso}(C)$ is an equivalence, or equivalently that the first projection $\mathrm{pr}_1 \colon \mathrm{Iso}(C) \times_{\mathrm{Fun}([1],C)} \mathrm{Iso}(C) \to \mathrm{Iso}(C)$ is an equivalence. To this end, consider the two functors

$$
\begin{aligned}
\mathrm{Fun}([1],C) \times_{\mathrm{ev}_1,C,\mathrm{ev}_0} \mathrm{Iso}(C) &\to \mathrm{Fun}([1],C) \times_{\mathrm{ev}_1,C,\mathrm{ev}_1} \mathrm{Iso}(C), \qquad (g,f) \mapsto (g \circ f, f) \\
\mathrm{Iso}(C) \times_{\mathrm{ev}_1,C,\mathrm{ev}_0} \mathrm{Fun}([1],C) &\to \mathrm{Iso}(C) \times_{\mathrm{ev}_0,C,\mathrm{ev}_0} \mathrm{Fun}([1],C), \qquad (f,h) \mapsto (f, f \circ h).
\end{aligned}
$$

We claim that these functors are equivalences. For the first functor, note that for $f \in \mathrm{Iso}(C)$ we may functorially extract both a left inverse f_l with an isomorphism $f_l \circ f \cong \mathrm{id}_{s(f)}$ as well as a right inverse f_r with an isomorphism $f \circ f_r \cong \mathrm{id}_{t(f)}$. Using associativity of composition, a left inverse to the first functor is then given by $(g', f) \mapsto (g' \circ f_r, f)$, while a right inverse is given by $(g', f) \mapsto (g' \circ f_l, f)$. One argues similarly for the second functor.

The claim now follows from the observation that the projection pr_1 is the left vertical map in the following pullback square, where the right vertical map is the fiber product over $\mathrm{Iso}(C)$ of the two equivalences just constructed, hence an equivalence itself:

$$
\begin{tikzcd}
\mathrm{Iso}(C) \times_{\mathrm{Fun}([1],C)} \mathrm{Iso}(C) \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \times_{\mathrm{ev}_1,C,\mathrm{ev}_0} \mathrm{Iso}(C) \times_{\mathrm{ev}_1,C,\mathrm{ev}_0} \mathrm{Fun}([1],C) \arrow[d, "(g,f,h) \mapsto (g \circ f, f, f \circ h)"] \\
\mathrm{Iso}(C) \arrow[r, "f \mapsto (\mathrm{id}_y, f, \mathrm{id}_x)"] & \mathrm{Fun}([1],C) \times_{\mathrm{ev}_1,C,\mathrm{ev}_1} \mathrm{Iso}(C) \times_{\mathrm{ev}_0,C,\mathrm{ev}_0} \mathrm{Fun}([1],C).
\end{tikzcd}
$$

□

As one expects, the identity map $\mathrm{id}_x : x \to x$ is invertible in C for every term x of C:

Construction 1.8.5. We construct a functor $i : C \to \mathrm{Iso}(C)$ lifting the functor $1 : C \to \mathrm{Fun}([1], C) : x \mapsto \mathrm{id}_x$. Informally, i is given by sending a term x to the following commutative diagram in C:

$$\begin{tikzcd} & x \arrow[dd, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_x"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_x"'] & & x. \\ & x \arrow[ur, "\mathrm{id}_x"'] & \end{tikzcd}$$

More formally, restriction along the map $p_{[2]} : [2] \to [0]$ induces a functor $p^*_{[2]} : C \to \mathrm{Fun}([2], C)$, giving a functor $(p^*_{[2]}, p^*_{[2]}) : C \to \mathrm{Fun}([2], C) \times_{\mathrm{Fun}([1],C)} \mathrm{Fun}([2], C)$. Since the two relevant edges of this diagram are both the identity on x, this functor factors through $\mathrm{Iso}(C)$, providing the desired functor $i : C \to \mathrm{Iso}(C)$.

By construction the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[dr, "x \mapsto \mathrm{id}_x"'] & \mathrm{Iso}(C) \arrow[r, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \arrow[d, "\pi_{\mathrm{Iso}}"] & C \times C \\ & \mathrm{Fun}([1], C). \arrow[ur, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & \end{tikzcd}$$

> **Axiom F** (Rezk axiom)**.** For any category C, the functor $i : C \to \mathrm{Iso}(C)$ is an equivalence.

The Rezk axiom is important, as it tells us that two terms x and y of C are isomorphic in the sense of Definition 1.2.1 if and only if there exists an invertible morphism $f : x \to y$ between them:

Corollary 1.8.6. *Let x and y be terms of C and assume there exists an invertible morphism $f : x \to y$. Then there exists an isomorphism $x \cong y$ in C.*

Proof. Since the functor $i : C \to \mathrm{Iso}(C)$ is an equivalence, there exists a term z of C together with an isomorphism $i(z) \cong f$ in $\mathrm{Iso}(C)$. Using the source and target maps $s, t : \mathrm{Iso}(C) \to C$, this isomorphism produces isomorphisms $z \cong x$ and $z \cong y$ in C. By inversion we obtain $x \cong z$ and by composition we obtain $x \cong y$, as desired. □

Corollary 1.8.7. *For a category C, the functor $p^*_{[1]} : C \to \mathrm{Fun}([1], C)$ is an embedding.*

Proof. This functor factors as $C \xrightarrow{i} \mathrm{Iso}(C) \xrightarrow{\pi_{\mathrm{Iso}}} \mathrm{Fun}([1], C)$. The first functor is an equivalence by the Rezk axiom, while the second one is an embedding by Lemma 1.8.4. The claim follows. □

1.9 Exercises Chapter 1

Exercise 1.9.1. Show that the product of categories is associative: for categories C, D and E there is a preferred equivalence $C \times (D \times E) \xrightarrow{\sim} (C \times D) \times E$.

Exercise 1.9.2. Show that the coproduct of categories is associative, commutative and unital: for categories C, D and E there are preferred equivalences

$$(C \sqcup D) \sqcup E \xrightarrow{\sim} C \sqcup (D \sqcup E), \qquad C \sqcup D \xrightarrow{\sim} D \sqcup C, \qquad \emptyset \sqcup C \xrightarrow{\sim} C \xrightarrow{\sim} C \sqcup \emptyset.$$

Exercise 1.9.3. Given functors $f\colon C \to C'$, $f'\colon C' \to C''$, $g\colon D \to D'$ and $g'\colon D' \to D''$, show that there are natural isomorphisms

$$\mathrm{id}_C \times \mathrm{id}_D \cong \mathrm{id}_{C \times D} \quad \text{and} \quad (f' \times g') \circ (f \times g) \cong (f' \circ f) \times (g' \circ g).$$

Exercise 1.9.4. Show that the fiber product of categories is commutative, associative and unital: for functors $C \to E$, $D \to E$ and $B \to E$, there are preferred equivalences

$$C \times_E D \xrightarrow{\sim} D \times_E C, \qquad B \times_E (C \times_E D) \xrightarrow{\sim} (B \times_E C) \times_E D, \qquad C \times_E E \xrightarrow{\sim} C.$$

Exercise 1.9.5. Consider categories C and D, and let $p_C\colon C \to *$ and $p_D\colon D \to *$ be their functors to the terminal category. Show that the functor $(\mathrm{pr}_C, \mathrm{pr}_D)\colon C \times_* D \to C \times D$ is an equivalence.

Exercise 1.9.6. Prove that the functors

$$g \circ -\colon\ \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, D') \qquad \text{and} \qquad - \circ f\colon\ \mathrm{Fun}(C, D) \to \mathrm{Fun}(C', D)$$

constructed in Construction 1.4.6 respect identity functors and composition of functors:

$$\begin{aligned} (\mathrm{id}_D \circ -) &\cong \mathrm{id}_{\mathrm{Fun}(C,D)}, & g' \circ (g \circ -) &\cong (g' \circ g) \circ - \\ (- \circ \mathrm{id}_C) &\cong \mathrm{id}_{\mathrm{Fun}(C,D)}, & (- \circ f) \circ f' &\cong - \circ (f \circ f'). \end{aligned}$$

Exercise 1.9.7. Show that also the operation of currying is functorial in E: given functors $f\colon E \times C \to D$ and $h\colon E' \to E$, there is a preferred isomorphism between the currying of the functor $E' \times C \xrightarrow{h \times \mathrm{id}_C} E \times C \xrightarrow{f} D$ and the composite $E' \xrightarrow{h} E \xrightarrow{f_c} \mathrm{Fun}(C, D)$.

Exercise 1.9.8. Fill in the missing details at the end of the proof of Lemma 1.4.10.

Exercise 1.9.9. Let $f\colon C \to D$ and $g\colon D \to E$ be two embeddings of categories. Show that the composite $gf\colon C \to E$ is also an embedding.

Exercise 1.9.10. Consider a pullback square

$$\begin{tikzcd} C' \arrow[r, "g"] \arrow[d, "u"'] & C \arrow[d, "v"] \\ D' \arrow[r, "f"] & C \end{tikzcd}$$

of categories and assume that v is an embedding. Show that also u is an embedding.

Exercise 1.9.11. Let C_0 and C_1 be categories and consider morphisms $f_0\colon x_0 \to y_0$ and $g_0\colon y_0 \to z_0$ in C_0 and morphisms $f_1\colon x_1 \to y_1$ and $g_1\colon y_1 \to z_1$ in C_1. Show that the composite $(g_0, g_1) \circ (f_0, f_1)\colon (x_0, x_1) \to (z_0, z_1)$ in $C_0 \times C_1$ is given by $(g_0 \circ f_0, g_1 \circ f_1)$.

Exercise 1.9.12. Show that a functor $F\colon C \to D$ preserves composition: for morphisms $f\colon x \to y$ and $g\colon y \to z$ in C there is a preferred isomorphism

$$F(g \circ f) \cong F(g) \circ F(f).$$

<u>Exercise 1.9.13</u>. Let $f : x \to y$ be a morphism in a category C. Prove that the following assertions are equivalent:

(i) f is an isomorphism;

(ii) for any object c of C, f induces an equivalence of the form

$$f \circ - : C(c, x) \xrightarrow{\sim} C(c, y)\,;$$

(iii) for any object c of C, f induces an equivalence of the form

$$- \circ f : C(y, c) \xrightarrow{\sim} C(x, c)\,.$$

2 Groupoids

In this chapter, we discuss in detail the notion of a *groupoid*: a category in which every morphism is an isomorphism. Groupoids will play a fundamental role in our development of synthetic category theory: many statements about categories can ultimately be reduced to statements about groupoids. Every category C will come equipped with a maximal subgroupoid $C^{\simeq}$ called the *groupoid core* of C, which we may think of as remembering only the objects of C and forgetting its morphisms. The existence of the groupoid core is what will eventually allow us to check various properties about categories *objectwise*, in an appropriate sense.

2.1 Groupoids

In classical mathematics, a category is called a *groupoid* if all its morphisms are invertible. The correct way to translate this to the synthetic setting is as follows:

Definition 2.1.1 (Groupoid)**.** A category X is called a *groupoid* if for every functor $f\colon \Gamma \to \mathrm{Fun}([1], X)$ there exists a dashed arrow $\tilde{f}$ making the following diagram commute:

$$\begin{tikzcd} & \mathrm{Iso}(X) \arrow[d, "\pi_{\mathrm{Iso}}"] \\ \Gamma \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & \mathrm{Fun}([1], X). \end{tikzcd}$$

Note that when $\Gamma = *$, a functor $f\colon * \to \mathrm{Fun}([1], X)$ is precisely the data of a morphism in X, and a lift $\tilde{f}\colon * \to \mathrm{Iso}(X)$ is a proof that the morphism f is an isomorphism, cf. Remark 1.8.3. So the above definition in particular demands that every morphism in X is invertible. To get good behavior of this definition, we must not merely demand this condition for absolute objects of $\mathrm{Fun}([1], X)$ but more generally for arbitrary *terms* of $\mathrm{Fun}([1], X)$, cf. Axiom A.6.

The following result provides various alternative formulations for a category to be a groupoid:

Proposition 2.1.2. *Let X be a category. Then the following conditions are equivalent:*

(1) The category X is a groupoid;

(2) The functor $\pi_{\mathrm{Iso}}\colon \mathrm{Iso}(X) \to \mathrm{Fun}([1], X)$ admits a section, in the sense of Definition 1.1.17;

(3) The functor $p_{[1]}^*\colon X \to \mathrm{Fun}([1], X)$, induced by the map $p_{[1]}\colon [1] \to *$, admits a section;

(4) The functor $p_{[1]}^*\colon X \to \mathrm{Fun}([1], X)$ is an equivalence.

Proof. To see that (1) implies (2), take $\Gamma = \mathrm{Fun}([1], X)$ and let $f = \mathrm{id}_{\mathrm{Fun}([1],X)}$, so that $s = \tilde{f}$ provides a section. Conversely, if $s\colon \mathrm{Fun}([1], X) \to \mathrm{Iso}(X)$ is a section of π_{Iso}, a lift $\tilde{f}$ of f may be given as $\tilde{f} := s \circ f$.

The fact that (2) and (3) are equivalent is immediate from the following commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "i"', "\sim"] \arrow[dr, "(p_{[1]})^*"] & \\ \mathrm{Iso}(X) \arrow[rr, "\pi_{\mathrm{Iso}}"] & & \mathrm{Fun}([1], X), \end{tikzcd}$$

using that the map i is an equivalence by the Rezk axiom F. Finally, note that the functor $(p_{[1]})^*$ always admits a retraction given by $\mathrm{ev}_0\colon \mathrm{Fun}([1], X) \to X$, and thus the equivalence between (3) and (4) is an instance of Lemma 1.1.18. □

Convention 2.1.3. From now on we will mostly take (4) as our working definition of groupoids.

Example 2.1.4. The terminal category $*$ is a groupoid: there is a commutative triangle

$$\begin{tikzcd} & \mathrm{Fun}([1], *) \arrow[dr] & \\ * \arrow[ur, "(p_{[1]})^*"] \arrow[rr] & & *, \end{tikzcd}$$

and since the bottom and right diagonal maps are equivalences (see Lemma 1.4.8) the claim follows by 2-out-of-3.

Example 2.1.5. If X and Y are groupoids, then also their product $X \times Y$ is a groupoid: this is an immediate consequence of Lemma 1.4.9.

Lemma 2.1.6. *Let $C \to D$ be an equivalence of categories. Then C is a groupoid if and only if D is a groupoid.*

Proof. This follows from the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, "\sim"] \arrow[d, "(p_{[1]})^*"'] & D \arrow[d, "(p_{[1]})^*"] \\ \mathrm{Fun}([1], C) \arrow[r, "\sim"] & \mathrm{Fun}([1], D). \end{tikzcd}$$

Here the bottom functor is again an equivalence due to the functoriality of functor categories, Construction 1.4.6. □

Lemma 2.1.7. *Let X be a groupoid. Then for every category C, the functor category* $\mathrm{Fun}(C, X)$ *is a groupoid.*

Proof. This follows from the fact that the map $\mathrm{Fun}(C, X) \to \mathrm{Fun}([1], \mathrm{Fun}(C, X))$ is equivalent to the map $\mathrm{Fun}(C, X) \to \mathrm{Fun}(C, \mathrm{Fun}([1], X))$ induced by the equivalence $X \xrightarrow{\sim} \mathrm{Fun}([1], X)$. □

Lemma 2.1.8. *Groupoids are closed under retracts: if X is a groupoid and Y is a retract of X, then Y is also a groupoid.*

Proof. If Y is a retract of X, then the functor $Y \to \mathrm{Fun}([1], Y)$ is a retract of the functor $X \to \mathrm{Fun}([1], X)$. In particular, if the latter is an equivalence then so is the former by Lemma 1.1.24. □

Lemma 2.1.9. *Consider functors $f\colon X \to C$ and $g\colon Y \to C$ and assume that X and Y are groupoids. Then the pullback $X \times_C Y$ is a groupoid.*

Proof. The functor $X \times_C Y \to \mathrm{Fun}([1], X \times_C Y)$ factors as

$$X \times_C Y \xrightarrow{\sim} X \times_{\mathrm{Fun}([1],C)} Y \xrightarrow{\sim} \mathrm{Fun}([1], X) \times_{\mathrm{Fun}([1],C)} \mathrm{Fun}([1], Y) \simeq \mathrm{Fun}([1], X \times_C Y).$$

Here the second functor is an equivalence since X and Y are groupoids. The last equivalence is Lemma 1.4.10. Finally, we claim that the first functor is a pullback of the diagonal of the embedding $i\colon C \to \mathrm{Fun}([1], C)$ from Corollary 1.8.7 and hence is an equivalence. To see this, consider the following commutative diagram:

$$\begin{tikzcd}
X \times_C Y \arrow[r] \arrow[d] & X \times_{\mathrm{Fun}([1],C)} Y \arrow[rr] \arrow[d] & & X \times Y \arrow[d] \\
C \arrow[r, "\Delta_i", "\sim"'] & C \times_{\mathrm{Fun}([1],C)} C \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & & C \times C \arrow[d, "i \times i"] \\
& \mathrm{Fun}([1], C) \arrow[rr, "\Delta"] & & \mathrm{Fun}([1], C) \times \mathrm{Fun}([1], C).
\end{tikzcd}$$

The right bottom square, the right outer rectangle and the top outer rectangle are pullback squares by Lemma 1.3.18. It thus follows from the pasting lemma 1.3.16 that also the top left square is a pullback square. $\square$

2.2 The homotopy hypothesis

Axiom G (Homotopy Hypothesis)**.** A category is an anima if and only if it is a groupoid.

Remark. Axiom G plays a similar role in our theory as Axiom B.6: it may only be stated in a *groupoidal* context. We will explain what this means in Axiom K.2.

Definition 1.5.2 tells us that, for every category C, there exists thus a groupoid $C^{\simeq} = \mathrm{Map}(*, C)$ called the *groupoid core of* C, which comes equipped with a functor $\gamma_C\colon C^{\simeq} \to C$. For every other groupoid X and every functor $f\colon X \to C$, there exists a functor $\tilde{f}\colon X \to C^{\simeq}$ and a natural isomorphism $f \cong \gamma_C \circ \tilde{f}$. Given two functors $g, h\colon X \to C^{\simeq}$, there is for every natural isomorphism $\alpha\colon \gamma_C \circ g \cong \gamma_C \circ h$ of functors $X \to C$ a natural isomorphism $\tilde{\alpha}\colon f \cong g$ of functors $X \to C^{\simeq}$, together with an isomorphism of natural isomorphisms $\gamma_D \circ \tilde{\alpha} \cong \alpha$.

We will see in Corollary 2.2.12 that the functor $\gamma_C\colon C^{\simeq} \to C$ is always an embedding of categories.

The groupoid core $C^{\simeq}$ is automatically functorial in C:

Construction 2.2.1. Let $f\colon C \to D$ be a functor. Then the composite $f \circ \gamma_C\colon C^{\simeq} \to D$ is a functor from a groupoid into D, and thus we obtain a functor

$$f^{\simeq} := (\widetilde{f \circ \gamma_C})\colon C^{\simeq} \to D^{\simeq}$$

making the following diagram commute:

$$\begin{tikzcd} C^{\simeq} \arrow[r, "f^{\simeq}"] \arrow[d, "\gamma_C"'] & D^{\simeq} \arrow[d, "\gamma_D"] \\ C \arrow[r, "f"'] & D. \end{tikzcd}$$

We next prove various basic properties of the groupoid core construction $C \mapsto C^{\simeq}$.

Lemma 2.2.2. *Let X be a groupoid. Then the functor $\gamma_X \colon X^{\simeq} \to X$ is an equivalence.*

Proof. This is part of Proposition 1.5.9. □

Proposition 2.2.3. *For any groupoids X and any category A, there is a canonical equivalence of the form*

$$\iota_{A,X} \colon \operatorname{Map}(A, X) \xrightarrow{\sim} \operatorname{Fun}(A, X)\,.$$

Proof. By virtue of the preceding lemma, this follows right away from Proposition 1.5.8 and Lemma 2.1.7. □

Lemma 2.2.4. *For categories C and D, the functor $(\mathrm{pr}_C^{\simeq}, \mathrm{pr}_D^{\simeq}) \colon (C \times D)^{\simeq} \to C^{\simeq} \times D^{\simeq}$ is an equivalence.*

Proof. This is a special case of Proposition 1.2.14. □

Lemma 2.2.5. *For functors $C \to E$ and $D \to E$, the functor $(\mathrm{pr}_C^{\simeq}, \mathrm{pr}_D^{\simeq}) \colon (C \times_E D)^{\simeq} \to C^{\simeq} \times_{E^{\simeq}} D^{\simeq}$ is an equivalence.*

Proof. This is a special case of Proposition 1.3.5. □

Corollary 2.2.6. *Given an embedding $f \colon C \to D$, the functor $f^{\simeq} \colon C^{\simeq} \to D^{\simeq}$ is also an embedding.*

Proof. In light of the definition of embeddings, this is immediate from Lemma 2.2.5. □

The walking morphism $[1]$ comes with two objects 0 and 1. The following axiom expresses that these are the only objects of $[1]$:

Axiom G.1. The map $\langle 0, 1\rangle \colon * \sqcup * \to [1]^{\simeq}$ is an equivalence.

Corollary 2.2.7. *The category $* \sqcup *$ is a groupoid.*

Proof. By assumption it is equivalent to the groupoid $[1]^{\simeq}$. □

Corollary 2.2.8. *The map $\langle 0, 1, 2\rangle \colon * \sqcup * \sqcup * \to [2]^{\simeq}$ is an equivalence.*

Proof. Since equivalences are closed under retracts (Lemma 1.1.24) this follows from the observation that this map is a retract of the equivlaence $\langle 0, 1\rangle^2 \colon (* \sqcup *)^2 \to ([1]^{\simeq})^2$. □

Lemma 2.2.9. *For categories C and D, the maps $C^{\simeq} \to (C \sqcup D)^{\simeq}$ and $D^{\simeq} \to (C \sqcup D)^{\simeq}$ induce an equivalence $C^{\simeq} \sqcup D^{\simeq} \xrightarrow{\sim} (C \sqcup D)^{\simeq}$.*

Proof. Let $i_0 \colon * \to * \sqcup *$ and $i_1 \colon * \to * \sqcup *$ denote the inclusions of the first and second point, respectively. By combining Lemma 2.2.5 and Axiom B.4', there are pullback squares

$$\begin{tikzcd} C^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (C \sqcup D)^{\simeq} \arrow[d] \\ (*)^{\simeq} \arrow[r, "i_0^{\simeq}"] & (* \sqcup *)^{\simeq} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (C \sqcup D)^{\simeq} \arrow[d] \\ (*)^{\simeq} \arrow[r, "i_1^{\simeq}"] & (* \sqcup *)^{\simeq}. \end{tikzcd}$$

By virtue of Lemma 1.3.16 and of Corollary 1.3.22 it will thus suffice to show that the two maps $i_0^{\simeq}$ and $i_1^{\simeq}$ induce an equivalence $(*)^{\simeq} \sqcup (*)^{\simeq} \xrightarrow{\sim} (* \sqcup *)^{\simeq}$. But this follows immediately from Axiom G.1: both sides are equivalent to $* \sqcup *$. □

Remark 2.2.10. By Corollary 2.2.8, we see that $\mathrm{Map}([1],[1]) = \mathrm{Fun}([1],[1])^{\simeq} = [2]^{\simeq}$ is a disjoint union of three points. We may think of these three points picking out the three morphisms $\mathrm{id}_0 \colon 0 \to 0$, $\mathrm{id}_1 \colon 1 \to 1$ and $\mathrm{can} \colon 0 \to 1$ of $[1]$. In particular, we see that $[1]$ only has three morphisms.

Proposition 2.2.11. *Let X be a groupoid and let C be a category. Then the functor*

$$\gamma_C \circ - \colon \mathrm{Map}(X, C^{\simeq}) \to \mathrm{Map}(X, C)$$

is an equivalence.

Proof. Since the notions of groupoids and of animae coincide, this is Corollary 1.5.3. □

The mapping spaces $\mathrm{Map}(-,-)$ will play an important role in the development of the theory: frequently, we may check conditions at the level of mapping spaces. In the following, we will provide various examples of this phenomenon.

Corollary 2.2.12. *Let C be a category. Then the functor $\gamma_C \colon C^{\simeq} \to C$ is an embedding.*

Proof. We need to show that the diagonal $C^{\simeq} \to C^{\simeq} \times_C C^{\simeq}$ is an equivalence. Since $C^{\simeq} \times_C C^{\simeq}$ is a groupoid by Lemma 2.1.9, it will suffice to show that for every groupoid Z the induced map

$$\mathrm{Map}(Z, C^{\simeq}) \to \mathrm{Map}(Z, C^{\simeq} \times_C C^{\simeq}) \simeq \mathrm{Map}(Z, C^{\simeq}) \times_{\mathrm{Map}(Z,C)} \mathrm{Map}(Z, C^{\simeq})$$

is an equivalence, i.e. we need to show that the map $\mathrm{Map}(Z, C^{\simeq}) \to \mathrm{Map}(Z, C)$ is an embedding. But this map is even an equivalence by Proposition 2.2.11. □

2.3 The synthetic theory of groupoids

In this section, we show that the collection of groupoids satisfy all the postulates and axioms of synthetic category theory introduced so far.

Lemma 2.3.1. *Consider a pullback square*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

of categories. If X, Y and Y' are groupoids, then so is X'.

Proof. This is a special case of Lemma 2.1.9. □

Lemma 2.3.2. *The empty category $\emptyset$ is a groupoid.*

Proof. We need to show that the functor $\emptyset \to \mathrm{Fun}([1], \emptyset)$ is an equivalence, or equivalently that the map $\mathrm{ev}_0 \colon \mathrm{Fun}([1], \emptyset) \to \emptyset$ is an equivalence. But this is immediate from Axiom B.2'. □

Lemma 2.3.3. *Let C be a category and let X be a groupoid. Then also the functor category $\mathrm{Fun}(C, X)$ is a groupoid.*

Proof. We need to show that the functor $\mathrm{Fun}(C, X) \to \mathrm{Fun}([1], \mathrm{Fun}(C, X))$ is an equivalence. But under the equivalences

$$\mathrm{Fun}([1], \mathrm{Fun}(C, X)) \simeq \mathrm{Fun}([1] \times C, X) \simeq \mathrm{Fun}(C, \mathrm{Fun}([1], X)),$$

from Lemma 1.4.11, this functor corresponds to the one induced by the equivalence $X \xrightarrow{\sim} \mathrm{Fun}([1], X)$, hence it is an equivalence. □

Lemma 2.3.4. *For every two categories C and D, the canonical map*

$$\mathrm{Fun}([1], C) \sqcup \mathrm{Fun}([1], D) \to \mathrm{Fun}([1], C \sqcup D)$$

is an equivalence.

Proof. When $C = * = D$, we have $\mathrm{Fun}([1], *) \cong *$ and the statement says that the map $* \sqcup * \to \mathrm{Fun}([1], * \sqcup *)$ is an equivalence. This is precisely the content of Corollary 2.2.7. For arbitrary C and D, it follows from Lemma 1.4.10 that there are pullback squares

$$\begin{tikzcd} \mathrm{Fun}([1], C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C \sqcup D) \arrow[d] & \mathrm{Fun}([1], D) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ \mathrm{Fun}([1], *) \arrow[r] & \mathrm{Fun}([1], * \sqcup *) & \mathrm{Fun}([1], *). \arrow[l] \end{tikzcd}$$

The claim thus follows from Lemma 1.3.21. □

Corollary 2.3.5. *Let X and Y be groupoids. Then their disjoint union $X \sqcup Y$ is a groupoid.*

Proof. The map $X \sqcup Y \to \mathrm{Fun}([1], X \sqcup Y)$ factors as

$$X \sqcup Y \to \mathrm{Fun}([1], X) \sqcup \mathrm{Fun}([1], Y) \to \mathrm{Fun}([1], X \sqcup Y),$$

where the first map is the disjoint union of the two equivalences $X \xrightarrow{\sim} \mathrm{Fun}([1], X)$ and $Y \xrightarrow{\sim} \mathrm{Fun}([1], Y)$, while the second is the equivalence from Corollary 2.3.4. □

Definition 2.3.6 (Hom groupoid)**.** Let C be any category. Given objects x and y of C, we define the *hom groupoid* $C(x, y)$ via the following pullback square:

$$\begin{tikzcd} C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ \{(x,y)\} \arrow[r, "(x,y)"] & C \times C. \end{tikzcd}$$

We will often treat $C(x, y)$ as a category in context $\{(x, y)\}$.

Warning 2.3.7. We have not yet shown that $C(x, y)$ is actually a groupoid! We will do so in Corollary 5.3.5.

2.4 Exercises Chapter 2

Exercise 2.4.1. Show that the category $*$ is a groupoid.

Exercise 2.4.2. Show that the product of two groupoids X and Y is again a groupoid.

3 Constructions of categories

In this chapter, we introduce various fundamental categorical constructions in the setting of synthetic category theory: (full) subcategories, localizations, geometric realizations, joins and slices.

Remark. As will be explained in Chapter 4, all these axioms will be assumed in an arbitrary *groupoidal* context, i.e. they are assumed to hold in context Γ for an arbitrary groupoid Γ.

3.1 Subcategories

In ordinary category theory, if one is given a collection M of morphisms in a category C which is closed under composition, then one may form the subcategory of C whose morphisms are precisely the morphisms in M (and the objects in the subcategory are precisely the objects whose identity morphism is in M). In this section, we discuss the analogous situation in the context of synthetic category theory.

We start by definition the notion of a subcategory. For this, we need the following auxiliary construction:

Construction 3.1.1. For categories C, D and E, we construct a map

$$\mathrm{Map}(C, D) \to \mathrm{Map}(\mathrm{Map}(E, C), \mathrm{Map}(E, D))$$

which informally is given by sending a functor $f \colon C \to D$ to the functor $f_* \colon \mathrm{Map}(E, C) \to \mathrm{Map}(E, D)$ given by postcomposition with f. More precisely, consider the composition functor

$$- \circ - \colon \mathrm{Fun}(E, C) \times \mathrm{Fun}(C, D) \to \mathrm{Fun}(E, D),$$

defined as the functor adjoint to the composite

$$E \times \mathrm{Fun}(E, C) \times \mathrm{Fun}(C, D) \xrightarrow{\mathrm{ev} \times 1} C \times \mathrm{Fun}(C, D) \xrightarrow{\mathrm{ev}} D.$$

Applying the groupoid core construction $(-)^{\simeq}$, we thus obtain a map

$$\mathrm{Map}(E, C) \times \mathrm{Map}(C, D) \to \mathrm{Map}(E, D),$$

which by currying defines a map

$$\mathrm{Map}(C, D) \to \mathrm{Fun}(\mathrm{Map}(E, C), \mathrm{Map}(E, D)).$$

Since $\mathrm{Map}(C, D)$ is a groupoid, this factors canonically through the groupoid core of the target, producing the desired map

$$\mathrm{Map}(C, D) \to \mathrm{Map}(\mathrm{Map}(E, C), \mathrm{Map}(E, D)).$$

Definition 3.1.2 (Subcategory)**.** A functor $f: A \to C$ is called a *subcategory* if the following two conditions are satisfied:

(1) The map $f_*: \mathrm{Map}([1], A) \to \mathrm{Map}([1], C)$ is an embedding;

(2) For every category D, the induced square

$$\begin{tikzcd}
\mathrm{Map}(D, A) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}(D, C) \arrow[d] \\
\mathrm{Map}(\mathrm{Map}([1], D), \mathrm{Map}([1], A)) \arrow[r, "(f_*)_*"] & \mathrm{Map}(\mathrm{Map}([1], D), \mathrm{Map}([1], C))
\end{tikzcd}$$

is a pullback square. Here the two vertical maps are given by Construction 3.1.1.

In other words, if $f: A \to C$ is a subcategory, then a given functor $D \to C$ factors through A if and only if the associated map on morphisms $\mathrm{Map}([1], D) \to \mathrm{Map}([1], C)$ factors through $\mathrm{Map}([1], A)$.

Remark 3.1.3. The above notion of subcategory corresponds to what is known as a *replete subcategory* in the higher category theory literature.

Example 3.1.4. The identity functor $\mathrm{id}_C: C \to C$ and the empty category $\emptyset \to C$ are both seen to be subcategories.

Lemma 3.1.5. *Every subcategory $f: A \to C$ is an embedding.*

Proof. By Proposition 1.3.25, it will suffice to show that the induced map $\mathrm{Map}(D, A) \to \mathrm{Map}(D, C)$ is an embedding for every category D. But this follows from its defining property, as it is the base change of the map $(f_*)_*$ induced by the map $f_*: \mathrm{Map}([1], A) \to \mathrm{Map}([1], C)$ which was assumed to be an embedding. □

We would now like to construct subcategories of C out of the data of a collection of morphisms in C closed under composition. Let us start by defining this notion.

Definition 3.1.6 (Collection of morphisms)**.** Let C be a category. A *collection of morphisms in C* is a groupoid M equipped with an embedding

$$m: M \hookrightarrow \mathrm{Map}([1], C).$$

We say that M *is closed under composition* if the following two conditions are satisfied:

- If a morphism $f: x \to y$ in C is in M, then so are id_x and id_y. More concretely, there are commutative diagrams of the following form:

$$\begin{tikzcd}
M \arrow[d, hook, "m"'] & M \arrow[l, dashed] \arrow[r, dashed] \arrow[d, hook, "m"] & M \arrow[d, hook, "m"] \\
\mathrm{Map}([1], C) & \mathrm{Map}([1], C) \arrow[l, "f \mapsto \mathrm{id}_x"'] \arrow[r, "f \mapsto \mathrm{id}_y"] & \mathrm{Map}([1], C).
\end{tikzcd}$$

- For maps $f: x \to y$ and $g: y \to z$ in M, also the composition of f and g is in M. More concretely, there exists a dotted arrow in the following diagram:

$$\begin{tikzcd}
M \times_C M \arrow[rr, dashed] \arrow[d, hook] & & M \arrow[d, hook, "m"] \\
\mathrm{Map}([1],C) \times_C \mathrm{Map}([1],C) & \mathrm{Map}([2],C) \arrow[l, "\sim"'] \arrow[r, "d_1^*"] & \mathrm{Map}([1],C).
\end{tikzcd}$$

Example 3.1.7. Taking $M = \mathrm{Map}([1], C)$, it is clear that the identity map $M \to \mathrm{Map}([1], C)$ is a collection of morphisms of C closed under composition.

Example 3.1.8. Let $A \to C$ be a subcategory. Then the map $M_A := \mathrm{Map}([1], A) \to \mathrm{Map}([1], C)$ is a collection of morphisms in C closed under composition. This is immediate from the fact that the functor $A \to C$ preserves compositions and identities.

Also the collection of isomorphisms in C determines a collection of morphisms closed under composition:

Lemma 3.1.9. *For a category C, the map $\pi_{\mathrm{Iso}}^{\simeq}\colon \mathrm{Iso}(C)^{\simeq} \to \mathrm{Map}([1], C)$ defines a collection of morphisms in C closed under composition.*

Proof. We saw in Lemma 1.8.4 that the functor π_{Iso} is an embedding, hence so is $\pi_{\mathrm{Iso}}^{\simeq}$ by Lemma 2.2.6. Since the Rezk axiom F gives an equivalence $C^{\simeq} \xrightarrow{\sim} \mathrm{Iso}(C)^{\simeq}$, it will thus suffice to show that the map $C^{\simeq} \to \mathrm{Map}([1], C)$ defines a collection of morphisms in C closed under composition. But for this map the first diagram becomes a tautology and the second diagram holds by the relation $\mathrm{id}_x \circ \mathrm{id}_x = \mathrm{id}_x$. $\square$

We may now state the subcategory axiom:

Axiom H (Subcategory axiom)**.** Let C be a category and let $M \hookrightarrow \mathrm{Map}([1], C)$ be a collection of morphisms in C closed under composition. Then there exists a category $\langle M \rangle_C$ equipped with a functor $i_M\colon \langle M \rangle_C \to C$ satisfying the following two conditions:

(1) The functor i_M sends morphisms in $\langle M \rangle_C$ into M: the first projection pr_1 in the following pullback square is an equivalence:

$$\begin{tikzcd}
\mathrm{Map}([1], \langle M \rangle_C) \times_{\mathrm{Map}([1],C)} M \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"', "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, hook, "m"] \\
\mathrm{Map}([1], \langle M \rangle_C) \arrow[r, "(i_M)_*"] & \mathrm{Map}([1], C).
\end{tikzcd}$$

Equivalently: the map $(i_M)_*\colon \mathrm{Map}([1], \langle M \rangle_C) \to \mathrm{Map}([1], C)$ factors through the map $m\colon M \to \mathrm{Map}([1], C)$.

(2) The functor i_M is universal with this property: for any other category D, the following commutative square is a pullback square:

$$\begin{tikzcd}
\mathrm{Map}(D, \langle M\rangle_C) \arrow[r, "(i_M)_*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(D, C) \arrow[d] \\
\mathrm{Map}(\mathrm{Map}([1], D), M) \arrow[r, hook, "m_*"] & \mathrm{Map}(\mathrm{Map}([1], D), \mathrm{Map}([1], C)).
\end{tikzcd}$$

Here the vertical maps use Construction 3.1.1 and the map from (1).

Corollary 3.1.10. *In the situation of Axiom H, the functor* $\mathrm{Map}([1], \langle M\rangle_C) \to M$ *is an equivalence.*

Proof. Applying the universal property of $\langle M\rangle_C$ to $D = [1]$, and using the equivalence $\mathrm{Map}([1], [1]) \simeq \{\mathrm{id}_0, \mathrm{id}_1, \mathrm{can}\}$ from Remark 2.2.10, we obtain a pullback square

$$\begin{tikzcd}
\mathrm{Map}([1], \langle M\rangle_C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1], C) \arrow[d, "{(f\colon x\to y) \mapsto (\mathrm{id}_x, \mathrm{id}_y, f)}"] \\
M^3 \arrow[r, hook] & \mathrm{Map}([1], C)^3.
\end{tikzcd}$$

The claim now follows from the fact that the map $M \hookrightarrow \mathrm{Map}([1], C)$ is an embedding and that M is closed under composition, so that f lies in M if and only if each of $\mathrm{id}_x, \mathrm{id}_y, f$ lie in M. ◻

Corollary 3.1.11. *In the situation of Axiom H, the functor $i_M\colon \langle M\rangle_C \to C$ exhibits $\langle M\rangle_C$ as a subcategory of C. In particular, i_M is an embedding.*

Proof. By Corollary 3.1.10, the map $\mathrm{Map}([1], \langle M\rangle_C) \to \mathrm{Map}([1], C)$ agrees with the embedding $M \hookrightarrow \mathrm{Map}([1], C)$, giving condition (1) of Definition 3.1.2. Condition (2) then follows immediately from the universal property of $\langle M\rangle_C$. ◻

Lemma 3.1.12. *Let $f\colon A \to C$ be a subcategory and let $M_A := \mathrm{Map}([1], A) \hookrightarrow \mathrm{Map}([1], C)$ denote the induced collection of morphisms in C. Then the map $i_{M_A}\colon \langle M_A\rangle_C \to C$ factors through an equivalence $\langle M_A\rangle_C \xrightarrow{\sim} A$.*

Proof. Since the functor $\langle M_A\rangle_C \to C$ induces the inclusion $\mathrm{Map}([1], A) \hookrightarrow \mathrm{Map}([1], C)$, the fact that A is a subcategory of C implies that this functor factors through A. It further follows that the induced map $\mathrm{Map}(D, \langle M_A\rangle_C) \to \mathrm{Map}(D, A)$ is an equivalence for every category D, since both sides sit in the same pullback square. The claim thus follows from Proposition 1.1.16. ◻

In particular, this shows:

Corollary 3.1.13. *When $M = \mathrm{Map}([1], C)$ is the collection of all morphisms, the map $i_M\colon \langle M\rangle_C \to C$ is an equivalence.* ◻

We will now explain the functoriality of the construction of the subcategory $\langle M\rangle_C$ in the pair (C, M).

Definition 3.1.14. Let C and D be categories, and assume that they come equipped with collections of morphisms $m\colon M \hookrightarrow \mathrm{Map}([1], C)$ and $n\colon N \hookrightarrow \mathrm{Map}([1], D)$. A functor $f\colon C \to D$ is said to *send morphisms in M to morphisms in N* if there exists a map $\tilde{f}\colon M \to N$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[r, dashed, "\tilde{f}"] \arrow[d, hook, "m"'] & N \arrow[d, hook, "n"] \\ \mathrm{Map}([1], C) \arrow[r, "f_*"] & \mathrm{Map}([1], D). \end{tikzcd}$$

Since the map n is an embedding, the map $\tilde{f}\colon M \to N$ is unique up to natural isomorphism if it exists.

Construction 3.1.15. Let C and D be categories equipped with collections of morphisms M and N closed under composition, and let $f\colon C \to D$ be a functor which sends morphisms in M to morphisms in N. We construct a functor $f|\colon \langle M\rangle_C \to \langle N\rangle_D$ making the following diagram commute:

$$\begin{tikzcd} \langle M\rangle_C \arrow[r, dashed, "f|"] \arrow[d, "i_M"'] & \langle N\rangle_D \arrow[d, "i_N"] \\ C \arrow[r, "f"] & D. \end{tikzcd}$$

By the defining property of $\langle N\rangle_D$, it suffices to show that the the map $f \circ i_M\colon \langle M\rangle_C \to D$ sends all morphisms to morphisms in N. But this follows from the following commutative diagram:

$$\begin{tikzcd} \mathrm{Map}([1], \langle M\rangle_C) \arrow[r, "\simeq"] \arrow[dr, "(i_M)_*"'] & M \arrow[r, "\tilde{f}"] \arrow[d, hook, "m"] & N \arrow[d, hook, "n"] \\ & \mathrm{Map}([1], C) \arrow[r, "f_*"] & \mathrm{Map}([1], D). \end{tikzcd}$$

Remark 3.1.16 (Exercise 3.8.4)**.** The construction $f \mapsto f|$ satisfies the following properties:

(1) The map $f|$ is (up to natural isomorphism) uniquely determined by f.

(2) Given a natural isomorphism $f \cong f'$, we obtain an induced isomorphism $f| \cong f'|$.

(3) The construction is compatible with composition: $(\mathrm{id}_C)| = \mathrm{id}_{\langle M\rangle_C}$ and $(g \circ f)| = g| \circ f|$.

(4) If the maps f and $\tilde{f}$ are equivalences, then also $f|\colon \langle M\rangle_C \to \langle N\rangle_D$ is an equivalence.

3.2 Full subcategories

As a consequence of the subcategory axiom, we may in particular form *full* subcategories.

Construction 3.2.1. Given categories C and D, we construct a map $\mathrm{Map}(C, D) \to \mathrm{Map}(C^{\simeq}, D^{\simeq})$ by currying the following composite:

$$\mathrm{Map}(C, D) \times C^{\simeq} \simeq (\mathrm{Fun}(C, D) \times C)^{\simeq} \xrightarrow{\mathrm{ev}^{\simeq}} D^{\simeq}.$$

Definition 3.2.2 (Full subcategory)**.** A functor $f\colon A \to C$ is called a *full subcategory* if the induced map $f^{\simeq}\colon A^{\simeq} \to C^{\simeq}$ is an embedding and for every category D, the commutative square

$$\begin{tikzcd}
\mathrm{Map}(D, A) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}(D, C) \arrow[d] \\
\mathrm{Map}(D^{\simeq}, A^{\simeq}) \arrow[r, "(f^{\simeq})_*"] & \mathrm{Map}(D^{\simeq}, C^{\simeq})
\end{tikzcd}$$

is a pullback square.

More informally, this says that a functor $D \to C$ factors through A if and only if the map $D^{\simeq} \to C^{\simeq}$ on objects factors through the embedding $A^{\simeq} \to C^{\simeq}$.

Corollary 3.2.3. *Every full subcategory is an embedding.*

Proof. The proof is similar to Lemma 3.1.5 and will be omitted. ◻

Definition 3.2.4 (Collection of objects)**.** Consider a category C. A *collection of objects* is a pair (Γ, j) consisting of a groupoid Γ equipped with an embedding $j\colon \Gamma \hookrightarrow C^{\simeq}$ which is an embedding. In this case, we define a groupoid M_Γ via the following pullback:

$$\begin{tikzcd}
M_\Gamma \arrow[r, hook, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
\Gamma \times \Gamma \arrow[r, hook, "j \times j"] & C^{\simeq} \times C^{\simeq}.
\end{tikzcd}$$

Note that the map m is an embedding, as it is a base change of the embedding $j \times j$. It is also easy to verify that the resulting collection of morphisms M_Γ is closed under composition in C. This allows us to make the following definition:

Definition 3.2.5. Given an embedding $j\colon \Gamma \hookrightarrow C^{\simeq}$, we define

$$\langle \Gamma \subseteq C \rangle := \langle M_\Gamma \rangle_C,$$

and refer to it as the *full subcategory of C spanned by* Γ.

Lemma 3.2.6. *There is a preferred equivalence $\langle \Gamma \subseteq C \rangle^{\simeq} \xrightarrow{\sim} \Gamma$ fitting in a commutative triangle*

$$\begin{tikzcd}
\langle \Gamma \subseteq C \rangle^{\simeq} \arrow[rr, "\simeq"] \arrow[dr] & & \Gamma \arrow[dl, "j"] \\
& C^{\simeq}. &
\end{tikzcd}$$

Proof. Given a category D, consider the following two commutative squares:

$$\begin{tikzcd}
\mathrm{Map}(D, \langle \Gamma \subseteq C \rangle) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(D, C) \arrow[d] \\
\mathrm{Map}(\mathrm{Map}([1], D), M_\Gamma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\mathrm{Map}([1], D), \mathrm{Map}([1], C)) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)_*"] \\
\mathrm{Map}(\mathrm{Map}([1], D), \Gamma \times \Gamma) \arrow[r] & \mathrm{Map}(\mathrm{Map}([1], D), C^{\simeq} \times C^{\simeq}).
\end{tikzcd}$$

The top square is a pullback by definition of $\langle \Gamma \subseteq C \rangle$, and the bottom square is a pullback by definition of M. Taking $D = *$ we thus obtain a pullback square of the form

$$\begin{tikzcd} \langle \Gamma \subseteq C \rangle^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\simeq} \arrow[d, "\Delta"] \\ \Gamma \times \Gamma \arrow[r, "j \times j"] & C^{\simeq} \times C^{\simeq}. \end{tikzcd}$$

Since j is an embedding, this implies that $\langle \Gamma \subseteq C \rangle^{\simeq} \simeq \Gamma$, cf. Lemma 1.3.26. It is immediate from the construction of this equivalence that it lives over $C^{\simeq}$. □

Proposition 3.2.7. *The inclusion $\tilde{j} \colon \langle \Gamma \subseteq C \rangle \to C$ is a full subcategory, in the sense of Definition 3.2.2.*

Proof. We showed in Lemma 3.2.6 that the induced functor $\tilde{j}^{\simeq} \colon \langle \Gamma \subseteq C \rangle \to C^{\simeq}$ agrees with $j \colon \Gamma \to C^{\simeq}$, hence is an embedding by assumption. We thus need to show that for every category D, the top square in the following commutative diagram is a pullback square:

$$\begin{tikzcd} \mathrm{Map}(D, \langle \Gamma \subseteq C \rangle) \arrow[r] \arrow[d] & \mathrm{Map}(D, C) \arrow[d] \\ \mathrm{Map}(D^{\simeq}, \langle \Gamma \subseteq C \rangle^{\simeq}) \arrow[r] \arrow[d] & \mathrm{Map}(D^{\simeq}, C^{\simeq}) \arrow[d] \\ \mathrm{Map}(D^{\simeq} \times D^{\simeq}, \langle \Gamma \subseteq C \rangle^{\simeq} \times \langle \Gamma \subseteq C \rangle^{\simeq}) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)^*"] & \mathrm{Map}(D^{\simeq} \times D^{\simeq}, C^{\simeq} \times C^{\simeq}) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)^*"] \\ \mathrm{Map}(\mathrm{Map}([1], D), \langle \Gamma \subseteq C \rangle^{\simeq} \times \langle \Gamma \subseteq C \rangle^{\simeq}) \arrow[r] & \mathrm{Map}(\mathrm{Map}([1], D), C^{\simeq} \times C^{\simeq}). \end{tikzcd}$$

The bottom two horizontal maps are embeddings since they are induced by the embedding $\langle \Gamma \subseteq C \rangle^{\simeq} \to C^{\simeq}$. The outer rectangle agrees with the outer pullback square in the proof of Lemma 3.2.6, hence is a pullback square itself. It then follows from Lemma 1.3.32 that the top square is also a pullback square. □

Corollary 3.2.8. *For any category D, there is a preferred pullback*

$$\begin{tikzcd} \mathrm{Map}(D, \langle \Gamma \subseteq C \rangle) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(D, C) \arrow[d] \\ \mathrm{Map}(D^{\simeq}, \Gamma) \arrow[r, "j_*"] & \mathrm{Map}(D^{\simeq}, C^{\simeq}). \end{tikzcd}$$

Proof. This is immediate from Proposition 3.2.7 and Lemma 3.2.6. □

Lemma 3.2.9. *The inclusion $\langle C^{\simeq} \subseteq C \rangle \hookrightarrow C$ is an equivalence.*

Proof. Since the map $j \colon C^{\simeq} \to C^{\simeq}$ is the identity, it follows from Proposition 3.2.8 that for every category D the induced map

$$\mathrm{Map}(D, \langle C^{\simeq} \subseteq C \rangle) \to \mathrm{Map}(D, C)$$

is an equivalence. The claim thus follows from Proposition 1.1.16. □

Lemma 3.2.10. *Let C and D be categories, and let $j\colon \Gamma \hookrightarrow C^{\simeq}$ and $k\colon \Lambda \hookrightarrow D^{\simeq}$ be embeddings. Then there is a preferred equivalence*

$$\langle \Gamma \times \Lambda \subseteq C \times D \rangle \xrightarrow{\sim} \langle \Gamma \subseteq C \rangle \times \langle \Lambda \subseteq D \rangle.$$

Proof. This is an exercise in unwinding definitions and is left to the reader: see Exercise 3.8.3. □

Lemma 3.2.11. *Consider a functor $f\colon B \to C$. Given an embedding $\Gamma \hookrightarrow C^{\simeq}$, we define $\Lambda \hookrightarrow B^{\simeq}$ via the pullback*

$$\begin{tikzcd} \Lambda \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\simeq} \arrow[d, "f^{\simeq}"] \\ \Gamma \arrow[r, hook] & C^{\simeq}. \end{tikzcd}$$

Then the square

$$\begin{tikzcd} \langle \Lambda \subseteq B \rangle \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ \langle \Gamma \subseteq C \rangle \arrow[r] & C \end{tikzcd}$$

is a pullback square.

Proof. By Proposition 1.3.5, it will suffice to check that this square becomes a pullback square after applying the functor $\mathrm{Map}(D, -)$ for every category D. To this end, consider the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Map}(D, B) \arrow[rr] \arrow[dd] & & \mathrm{Map}(D^{\simeq}, B^{\simeq}) \arrow[dd] \\ \mathrm{Map}(D, \langle \Lambda \subseteq B \rangle) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Map}(D^{\simeq}, \Lambda) \arrow[ur] \arrow[dd] & \\ & \mathrm{Map}(D, C) \arrow[rr] & & \mathrm{Map}(D^{\simeq}, C^{\simeq}) \\ \mathrm{Map}(D, \langle \Gamma \subseteq C \rangle) \arrow[ur] \arrow[rr] & & \mathrm{Map}(D^{\simeq}, \Gamma). \arrow[ur] & \end{tikzcd}$$

Here the top and bottom face of the cube are pullback squares by Proposition 3.2.8, while the right face is a pullback square since $\mathrm{Map}(D^{\simeq}, -)$ preserves pullback squares by Proposition 1.3.5. It follows from the pasting law of pullback squares that also the left face of the cube is a pullback square, finishing the proof. □

Lemma 3.2.12. *Let $f\colon B \to C$ be a functor exhibiting B as a full subcategory of C. Then f is an equivalence if and only if the induced map $f^{\simeq}\colon B^{\simeq} \to C^{\simeq}$ is an equivalence.*

Proof. This is clear from the definitions. □

3.3 Localizations

In ordinary category theory, if one is given a category C equipped with a collection of morphisms W of C, one may form the *localization* of C at W: the initial functor $C \to C[W^{-1}]$ out of C which sends all morphisms in W to isomorphisms. Our goal in this section is to define localizations in the context of synthetic category theory.

Construction 3.3.1. Let C be a category equipped with a collection of morphisms $w\colon W \hookrightarrow \mathrm{Map}([1], C)$, in the sense of Definition 3.1.6. We will construct for every category D a full subcategory

$$\mathrm{Fun}^W(C, D) \subseteq \mathrm{Fun}(C, D)$$

In order to do this, we first define the groupoid $\mathrm{Map}^W(C, D)$ via the following pullback square:

$$\begin{tikzcd}
\mathrm{Map}^W(C,D) \arrow[r, hook, "\lambda^W_{C,D}"] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathrm{Map}(C,D) \arrow[d] \\
& \mathrm{Map}(\mathrm{Map}([1],C), \mathrm{Map}([1],D)) \arrow[d, "w^*"] \\
\mathrm{Map}(W, \mathrm{Iso}(D)^{\simeq}) \arrow[r, hook, "(\pi_{\mathrm{Iso}})_*"] & \mathrm{Map}(W, \mathrm{Map}([1], D)),
\end{tikzcd}$$

where the top right vertical map is the map from Construction 3.1.1. The bottom map is induced by the embedding $\pi^{\simeq}_{\mathrm{Iso}}$ from Lemma 1.8.4 and hence is an embedding. It follows that also $\lambda^W_{C,D}$ is an embedding. We can now define

$$\mathrm{Fun}^W(C, D) := \langle \mathrm{Map}^W(C, D) \subseteq \mathrm{Fun}(C, D)\rangle.$$

In particular, the map $\lambda^W_{C,D}$ extends to a functor $\tilde{\lambda}^W_{C,D}\colon \mathrm{Fun}^W(C, D) \to \mathrm{Fun}(C, D)$. We may think of the category $\mathrm{Fun}^W(C, D)$ as the full subcategory of $\mathrm{Fun}(C, D)$ spanned by those functors sending the morphisms in W to isomorphisms in D.

Remark 3.3.2. The construction of $\mathrm{Fun}^W(C, D)$ is functorial in the pair (C, W): if C' is another category equipped with a collection of morphisms $w'\colon W' \hookrightarrow \mathrm{Map}([1], C')$ and $f\colon C \to C'$ is a functor which sends the morphisms of W to the morphisms of W' (in the sense of Definition 3.1.14), then for every category D there is an induced square

$$\begin{tikzcd}
\mathrm{Map}^{W'}(C', D) \arrow[r, "f^*"] \arrow[d, hook, "\lambda^{W'}_{C',D}"'] & \mathrm{Map}^W(C, D) \arrow[d, hook, "\lambda^W_{C,D}"] \\
\mathrm{Map}(C', D) \arrow[r, "f^*"] & \mathrm{Map}(C, D),
\end{tikzcd}$$

which in turn induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{W'}(C', D) \arrow[r, "f^*"] \arrow[d, "\tilde{\lambda}^{W'}_{C',D}"'] & \mathrm{Fun}^W(C, D) \arrow[d, "\tilde{\lambda}^W_{C,D}"] \\
\mathrm{Fun}(C', D) \arrow[r, "f^*"] & \mathrm{Fun}(C, D)
\end{tikzcd}$$

via Construction 3.1.15.

Before moving on to the localization axiom, we will establish some basic results that express that the categories $\mathrm{Fun}^W(C, D)$ behave as we expect. As a first result, we show that every functor $C \to D$ lies in $\mathrm{Fun}^W(C, D)$ whenever D is a groupoid:

Lemma 3.3.3. *Let C be a category equipped with a collection of morphisms $W \hookrightarrow \mathrm{Map}([1], C)$, and let Γ be a groupoid. Then the functor*

$$\mathrm{Fun}^W(C, \Gamma) \to \mathrm{Fun}(C, \Gamma)$$

is an equivalence.

Proof. The functor $\mathrm{Iso}(\Gamma)^{\simeq} \to \mathrm{Map}([1], \Gamma)$ is an equivalence, since Γ is a groupoid. It follows from its defining property that also the map $\mathrm{Map}^W(C, \Gamma) \to \mathrm{Map}(C, \Gamma)$ is an equivalence, and hence both sides define the same full subcategory of $\mathrm{Fun}(C, \Gamma)$. Since the full subcategory determined by $\mathrm{Map}(C, \Gamma)$ is just $\mathrm{Fun}(C, \Gamma)$ by Lemma 3.2.9, this finishes the proof. □

The following lemma expresses the fact that any functor $C \to D$ takes isomorphisms in C to isomorphisms in D:

Proposition 3.3.4. *In the case of $W := \mathrm{Iso}(C)^{\simeq} \hookrightarrow \mathrm{Map}([1], C)$, the forgetful functor*

$$\mathrm{Fun}^W(C, D) \to \mathrm{Fun}(C, D)$$

is an equivalence.

Proof. Since the comparison map is the embedding of a full subcategory by construction, it suffices to prove that it induces an equivalence on objects. In other words, it suffices to check that the induced map

$$\mathrm{Map}^W(C, D) \to \mathrm{Map}(C, D)$$

is an equivalence. Since this is an embedding of animae, it suffices to check that any functor of the form $f : C \to D$ has the property that the induced map

$$f_* : \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$$

sends isomorphisms of C to isomorphisms of D. For this, since there is a canonical equivalence $C^{\simeq} \simeq \mathrm{Iso}(C)^{\simeq}$ given by the rule $x \mapsto \mathrm{id}_x$, it suffices to check that $f(\mathrm{id}_x) \cong \mathrm{id}_{f(x)}$ holds for any object x of C, which is obviously true. □

Construction 3.3.5. Let $l : C \to L$ be a functor which sends all objects of W to isomorphism of L, in the sense of Definition 3.1.14. Then we see that for a category the induced map $l^* : \mathrm{Fun}(L, D) \to \mathrm{Fun}(C, D)$ factors through $\mathrm{Fun}^W(C, D)$, using the following composite:

$$\mathrm{Fun}(L, D) \simeq \mathrm{Fun}^{\mathrm{Iso}(L)^{\simeq}}(L, D) \xrightarrow{l^*} \mathrm{Fun}^W(C, D).$$

Here the map l^* is obtained via Remark 3.3.2.

Definition 3.3.6 (Localization)**.** Given a collection of morphisms W of C, we say that a functor $l\colon C \to L$ is a *localization of C at W* if it sends W to isomorphisms of L and is universal with respect to this property: for every category D, the functor

$$l^*\colon \operatorname{Fun}(L, D) \to \operatorname{Fun}^W(C, D)$$

is an equivalence. A functor $l\colon C \to L$ is called a *localization functor* if it is a localization of C at *some* class of morphisms W.

If a localization exists, it is unique up to preferred equivalence; see Exercise 3.8.5. We will frequently denote it by $C[W^{-1}]$, and refer to the functor $l\colon C \to C[W^{-1}]$ as the *localization functor*.

Axiom I (Localization axiom)**.** For every collection of morphisms $w\colon W \to \operatorname{Map}([1], C)$ in C there exists a localization $l\colon C \to C[W^{-1}]$ of C at W.

Proposition 3.3.7. *Given a collection of morphisms W in a category C as well as a functor $\lambda\colon C \to L$ sending all maps in W to isomorphisms in L, if, for any synthetic category D, the induced map*

$$\lambda^*\colon \operatorname{Map}^W(L, D) \to \operatorname{Map}(C, D)$$

is an equivalence, then l exhibits L as a localization of C by W.

Proof. We observe that, in particular, the identity of L is the unique functor $i\colon L \to L$ with the property that $i\lambda$ is isomorphic to λ. Let $l\colon C \to C[W^{-1}]$ be a localization of C by W. Then the induced map

$$l^*\colon \operatorname{Map}^W(C[W^{-1}], D) \to \operatorname{Map}(C, D)$$

is an equivalence because it is equivalent to the equivalence

$$l^*\colon \operatorname{Map}(*, \operatorname{Fun}^W(C, D)) \to \operatorname{Map}(*, \operatorname{Fun}(C[W^{-1}], D))\,.$$

Therefore, for $D = L$, we see that there is a unique functor $u\colon C[W^-1] \to L$ such that $ul \cong \lambda$. Similarly, there is a unique functor $v\colon L \to C[W^{-1}]$ such that $v\lambda \cong l$. Since $uv\lambda$ is isomorphic to λ, uv must be isomorphic to the identity of L. Similarly, vu is isomorphic to the identity of $C[W^{-1}]$. □

Remark 3.3.8. The previous proposition and the localization axiom literally tell us that there exists a pushout square

$$\begin{tikzcd}
{[1] \times W} \arrow[r, "\tilde{w}"] \arrow[d, "\mathrm{pr}_2"'] & C \arrow[d] \\
W \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C[W^{-1}],
\end{tikzcd}$$

where the map $\tilde{w}$ is obtained by uncurrying the composite $W \xrightarrow{w} \operatorname{Map}([1], C) \to \operatorname{Fun}([1], C)$; see Exercise 3.8.6.

Construction 3.3.9. The construction $C \mapsto C[W^{-1}]$ is functorial in the pair (C, W): if D is another category equipped with a collection of morphisms W' and $f\colon C \to D$ is a functor which sends W to W', the composite $C \to D \to D[W'^{-1}]$ sends all morphisms in W to isomorphisms, and hence induces a canonical dashed functor making the square commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\iota"'] & D \arrow[d, "\iota"] \\ C[W^{-1}] \arrow[r, dashed] & D[W'^{-1}]. \end{tikzcd}$$

3.4 Geometric realizations

The localization of a category C at *all* its morphisms deserves special notation and terminology:

Definition 3.4.1 (Geometric realization)**.** In the case where $W := \mathrm{Map}([1], C)$ consists of all morphisms in C, we write

$$\|C\| := C[W^{-1}],$$

and we denote the localization functor by $\lambda_C\colon C \to \|C\|$. We refer to $\|C\|$ as the *geometric realization* of C.

In other words, the geometric realization is characterized by the property that for every other category C, the functor

$$\lambda_C^*\colon \operatorname{Fun}(\|C\|, D) \to \operatorname{Fun}^{\mathrm{Map}([1],C)}(C, D)$$

is an equivalence.

It is immediate from Construction 3.3.9 that the assignment $C \mapsto \|C\|$ is functorial in C: every functor $f\colon C \to D$ induces a functor

$$\|f\|\colon \|C\| \to \|D\|.$$

Lemma 3.4.2. *Assume that X is a groupoid. Then the localization functor $\lambda_X\colon X \to \|X\|$ is an equivalence.*

Proof. Using that $\mathrm{Iso}(X) \simeq \operatorname{Fun}([1], X)$, we obtain by Proposition 3.3.4 equivalences

$$\operatorname{Fun}^{\mathrm{Map}([1],X)}(X, D) \simeq \operatorname{Fun}^{\mathrm{Iso}(X)^{\simeq}}(X, D) \simeq \operatorname{Fun}(X, D),$$

which shows that the identity map $X \to X$ exhibits X as a localization of X at all its morphisms. This gives the claim. □

Lemma 3.4.3. *If C is a category and Γ is a groupoid, then precomposition with the localization functor $\lambda_C\colon C \to \|C\|$ defines an equivalence*

$$\lambda_C^*\colon \operatorname{Fun}(\|C\|, \Gamma) \xrightarrow{\sim} \operatorname{Fun}(C, \Gamma)$$

Proof. We may simply compose the equivalence $\operatorname{Fun}(\|C\|, \Gamma) \xrightarrow{\sim} \operatorname{Fun}^{\mathrm{Map}([1],C)}(C, \Gamma)$ with the equivalence $\operatorname{Fun}^{\mathrm{Map}([1],C)}(C, \Gamma) \xrightarrow{\sim} \operatorname{Fun}(C, \Gamma)$ from Lemma 3.3.3. □

Since we have obtained $\|C\|$ from C by inverting all its morphisms, one may expect it to be a groupoid. While this is indeed the case, this question is somewhat more subtle than it may seem at first, since a priori the localization may create new morphisms that do not lift to C itself. This problem is closely related to a variety of other statements we may expect to hold. Let us first formulate these and prove that they are equivalent; we will see afterwards that they are in fact satisfied.

Proposition 3.4.4. *The following conditions are equivalent:*

(1) *For every category C, the category $\|C\|$ is a groupoid;*

(2) *The geometric realization $\|[1]\|$ of the walking morphism $[1]$ is contractible;*

(3) *For every category C, the functor $p_C^*\colon C \to \mathrm{Fun}([1], C)$ exhibits C as a full subcategory of $\mathrm{Fun}([1], C)$, in the sense of Definition 3.2.2;*

(4) *For every category C, the functor $\pi_{\mathrm{Iso}}\colon \mathrm{Iso}(C) \to \mathrm{Fun}([1], C)$ is a full subcategory.*

(5) *Every category C for which the map $C^{\simeq} \simeq \mathrm{Map}(*, C) \xrightarrow{p_{[1]}^*} \mathrm{Map}([1], C)$ is an equivalence is a groupoid;*

(6) *For every category C, the subcategory $\langle \mathrm{Iso}(C)^{\simeq} \rangle_C \subseteq C$ spanned by all the isomorphisms in C is a groupoid;*

(7) *For every category C, the inclusion $C^{\simeq} \hookrightarrow C$ induces an equivalence $C^{\simeq} \xrightarrow{\sim} \langle \mathrm{Iso}(C)^{\simeq} \rangle_C$ between the groupoid core of C and the subcategory of C spanned by all its isomorphisms.*

Proof. We start by showing that (1) implies (2). Since $\|[1]\|$ is a groupoid by Proposition 3.4.5 it suffices by Proposition 1.1.16 to show that for every groupoid Γ the induced map

$$\mathrm{Map}(*, \Gamma) \to \mathrm{Map}(\|[1]\|, \Gamma)$$

is an equivalence. By Lemma 3.4.3, we may identify this map (up to equivalence) with the map

$$p_{[1]}^*\colon \Gamma \to \mathrm{Map}([1], \Gamma).$$

But this map is an equivalence since Γ is a groupoid, finishing the proof.

For (2) $\implies$ (3), we need to show that there exists an embedding $j\colon \Gamma \to \mathrm{Fun}([1], C)^{\simeq}$ such that C is equivalent to $\langle \Gamma \subseteq \mathrm{Fun}([1], C) \rangle$. But by (2), the map $p_{[1]}\colon [1] \to *$ induces an equivalence

$$C \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Map}([1],[1])}([1], C),$$

which is a full subcategory of $\mathrm{Fun}([1], C)$ as desired.

The equivalence between (3) and (4) is clear from the Rezk Axiom F.

For (3) $\implies$ (5), recall that C is a groupoid if and only if the map $C \to \mathrm{Fun}([1], C)$ is an equivalence. Since this functor is a full subcategory by (3), it follows from Lemma 3.2.12 that we may check this on objects, giving the criterion from (5).

We will now deduce (6) from (5). To show $\langle \mathrm{Iso}(C)^{\simeq} \rangle_C$ is a groupoid, it suffices by (5) to show that the map $(\langle \mathrm{Iso}(C)^{\simeq} \rangle_C)^{\simeq} \to \mathrm{Map}([1], \langle \mathrm{Iso}(C)^{\simeq} \rangle_C)$ is an equivalence. To this end, consider the following commutative diagram, where the two horizontal maps are induced by the inclusion $i\colon \langle \mathrm{Iso}(C)^{\simeq}\rangle_C \hookrightarrow C$:

$$\begin{tikzcd}
(\langle \mathrm{Iso}(C)^{\simeq}\rangle_C)^{\simeq} \arrow[rr, "i^{\simeq}"] \arrow[d, "p_{[1]}^*"'] & & C^{\simeq} \arrow[dl, "\simeq"'] \arrow[d, "p_{[1]}^*"] \\
\mathrm{Map}([1], \langle \mathrm{Iso}(C)^{\simeq}\rangle_C) \arrow[r, "\simeq"] & \mathrm{Iso}(C)^{\simeq} \arrow[r, hook, "\pi_{\mathrm{Iso}}^{\simeq}"'] & \mathrm{Map}([1], C).
\end{tikzcd}$$

By the 2-out-of-3-property it remains to show that the top map is an equivalence. To construct an inverse, we consider the inclusion $\gamma_C\colon C^{\simeq} \to C$. It sends all morphisms to isomorphisms, so induces a functor $C^{\simeq} \to \langle \mathrm{Iso}(C)^{\simeq}\rangle_C$. Since the source is a groupoid, it factors through a map $\varphi\colon C^{\simeq} \to (\langle \mathrm{Iso}(C)^{\simeq}\rangle_C)^{\simeq}$. To check the relations $\varphi \circ i^{\simeq} \cong \mathrm{id}$ and $i^{\simeq} \circ \varphi \cong \mathrm{id}$, we may do so after composing with the embedding $i_C\colon C^{\simeq} \hookrightarrow C$ from Corollary 2.2.12 and the embeddings $(\langle \mathrm{Iso}(C)^{\simeq}\rangle_C)^{\simeq} \hookrightarrow \langle \mathrm{Iso}(C)^{\simeq}\rangle_C$ and $\langle \mathrm{Iso}(C)^{\simeq}\rangle_C \hookrightarrow C$ (Lemma 3.1.5) after which the claim becomes a tautology. This establishes (6).

Assuming (6), we see that the map $C^{\simeq} \to \langle \mathrm{Iso}(C)^{\simeq}\rangle_C$ is a map between groupoids, hence it is an equivalence if and only if it is an equivalence on groupoid cores. This follows using the same argument as before (note that that step did not use (5)). Thus (6) implies (7).

Finally, we show that (7) implies (1). The localization functor $\lambda_C\colon C \to \|C\|$ sends all morphisms of C to isomorphisms, hence it factors through the subcategory of $\|C\|$ spanned by its isomorphisms. By (7) this agrees with the groupoid core of $\|C\|$. The resulting functor $C \to \|C\|^{\simeq}$ still inverts all morphisms in C, hence induces a functor $\|C\| \to \|C\|^{\simeq}$. By uniqueness, the composite $\|C\| \to \|C\|^{\simeq} \hookrightarrow \|C\|$ is the identity map. But this means that we have constructed a section of the embedding $\|C\|^{\simeq} \hookrightarrow \|C\|$ and hence this embedding is an equivalence by Lemma 1.3.28. It follows that $\|C\|$ is a groupoid, as desired. □

Proposition 3.4.5. *All of the equivalent conditions from Proposition 3.4.4 are satisfied.*

Proof. It will suffice to show that the map $\|[1]\| \to *$ is an equivalence. By Proposition 1.1.16, it will suffice to show that for any category D, the induced functor $D^{\simeq} \simeq \mathrm{Map}(*, D) \to \mathrm{Map}(\|[1]\|, D) \simeq \mathrm{Map}^{\mathrm{Map}([1],[1])}([1], D)$ is an equivalence. Recall from Remark 2.2.10 that the groupoid $\mathrm{Map}([1], [1])$ is a disjoint union of three points, corrsponding to the identity on 0, the identity on 1, and the canonical map from 0 to 1 in $[1]$. Spelling out the definition of $\mathrm{Map}^{\mathrm{Map}([1],[1])}([1], D)$, we obtain a pullback square

$$\begin{tikzcd}
\mathrm{Map}^{\mathrm{Map}([1],[1])}([1], D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}([1], D) \arrow[d, "{(f\colon x\to y) \mapsto (\mathrm{id}_x, \mathrm{id}_y, f)}"] \\
D^{\simeq 3} \arrow[r] & \mathrm{Map}([1], D)^3.
\end{tikzcd}$$

Since the map $D^{\simeq} \to \mathrm{Map}([1], D)$ is an embedding (see Corollary 1.8.7) and its image contains id_x and id_y for all $x, y \in D^{\simeq}$, we see that the pullback of this square is $D^{\simeq}$, and the claim follows by 2-out-of-3. □

Corollary 3.4.6. *Consider a pushout square of categories*

$$\begin{tikzcd}
C' \arrow[r] \arrow[d] & C \arrow[d] \\
D' \arrow[r] & D.
\end{tikzcd}$$

Then the induced square

$$\begin{array}{ccc} \|C'\| & \longrightarrow & \|C\| \\ \big\downarrow & & \big\downarrow \\ \|D'\| & \longrightarrow & \|D\| \end{array}$$

is again a pushout square.

Proof. By Definition 1.5.10, it will suffice to show that the square becomes a pullback square after applying $\mathrm{Map}(-, E)$ for every category E. But using the equivalences

$$\mathrm{Map}(\|C\|, E) \overset{2.2.11}{\simeq} \mathrm{Map}(\|C\|, E^{\simeq}) \overset{3.4.3}{\simeq} \mathrm{Map}(C, E^{\simeq}),$$

and similarly for C replaced with C', D and D', this directly follows from the assumption on the original square. □

3.5 Exponentiable functors

For a category T, we may consider categories C equipped with a functor $f: C \to T$, which we will refer to as *categories over* T (to be discussed in more detail in Chapter 4). Given a functor $p: S \to T$, we obtain a notion of *base change along* p:

$$\begin{array}{ccc} \{\text{ Categories over } T \text{ }\} & \to & \{\text{ Categories over } S \text{ }\} \\ f: C \to T & \mapsto & p^*(f): C \times_T S \to S. \end{array}$$

This assignment admits a left adjoint, given by postcomposition with p:

$$\begin{array}{ccc} \{\text{ Categories over } S \text{ }\} & \to & \{\text{ Categories over } T \text{ }\} \\ g: D \to S & \mapsto & p \circ f: D \to T. \end{array}$$

However, in general this assignment will not admit a *right* adjoint. Let us illustrate this with the following simple counterexample:

Example 3.5.1. Consider the functor $p = d_1: [1] \to [2]$ given by $d_1(0) = 0$ and $d_1(1) = 2$. We will argue that pullback along p does not preserve pushouts of categories, and hence cannot admit a right adjoint. To see this, consider the following commutative square:

$$\begin{tikzcd} {[0]} \arrow[r, "d_2"] \arrow[d, "d_0"'] & {[1]} \arrow[d, "d_0"] \\ {[1]} \arrow[r, "d_2"'] & {[2].} \end{tikzcd} \tag{3.1}$$

We see as a consequence of the Segal Axiom E that this is a pushout square of categories. If we pull back this square along the functor $d_1: [1] \to [2]$, we obtain the commutative square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & {[0]} \arrow[d, "d_0"] \\ {[0]} \arrow[r, "d_1"'] & {[1].} \end{tikzcd} \tag{3.2}$$

But this is no longer a pushout square of categories: if it were, then it would remain a pushout square after passing to geometric realization by Corollary 3.4.6. But since $\|[1]\| \xrightarrow{\sim} *$ by part (2) of Proposition 3.4.4, this square takes the form

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & * \arrow[d, equal] \\ * \arrow[r, equal] & *, \end{tikzcd}$$

and this is only a pushout square if $* \sqcup * \xrightarrow{\sim} *$. This is false in essentially all interesting models for our axioms.

As the previous example indicates, we cannot expect the existence of a right adjoint to p^* in full generality. That being said, there is a fairly large class of functors, the so-called *exponentiable functors*, for which this right adjoint *does* exist. This class for example includes the classes of cartesian fibrations and cocartesian fibrations, to be introduced in Chapter 5.

In this section, we will give a precise definition of this class of functors. We start with some auxiliary definitions and constructions.

Definition 3.5.2. Let S be a category and let $f\colon C \to S$ and $g\colon D \to S$ be categories over S. We define the category $\mathrm{Fun}_S(C, D)$ of *functors $C \to D$ over S* as the following pullback:

$$\begin{tikzcd} \mathrm{Fun}_S(C,D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(C,D) \arrow[d, "g_*"] \\ * \arrow[r, "f"] & \mathrm{Fun}(C,S). \end{tikzcd}$$

Note that objects of $\mathrm{Fun}_S(C, D)$ are pairs (h, α) where $h\colon C \to D$ is a functor and α is a natural isomorphism $f \cong g \circ h$. In other words, objects are commutative triangles of the form

$$\begin{tikzcd} C \arrow[rr, "h"] \arrow[dr, "f"'] & & D \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

We write $\mathrm{Map}_S(C, D)$ for the groupoid core of $\mathrm{Fun}_S(C, D)$, which fits into a pullback square of the form

$$\begin{tikzcd} \mathrm{Map}_S(C,D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(C,D) \arrow[d, "g_*"] \\ * \arrow[r, "f"] & \mathrm{Map}(C,S). \end{tikzcd}$$

Construction 3.5.3. Let $p\colon S \to T$ be a functor. For categories C and D over T, we obtain an induced functor

$$p^*\colon \mathrm{Fun}_T(C, D) \to \mathrm{Fun}_S(C \times_T S, D \times_T S)$$

by noting that the composite $\mathrm{Fun}_T(C, D) \to \mathrm{Fun}(C, D) \xrightarrow{p^*} \mathrm{Fun}(C \times_T S, D \times_T S)$ lands in the correct fiber.

Definition 3.5.4 (Dependent product)**.** Consider a functor $p\colon S \to T$ and let $f\colon C \to S$ by a category over S. A *dependent product of C along p* is a category $g\colon D \to T$ over T equipped with a functor $\varepsilon\colon D \times_T S \to C$ over S which satisfies the property that for any other category $E \to T$ over T the composite functor

$$\mathrm{Map}_T(E, D) \xrightarrow{p^*} \mathrm{Map}_T(E \times_T S, D \times_T S) \xrightarrow{\varepsilon\circ -} \mathrm{Map}_S(E \times_T S, C)$$

is an equivalence.

If a dependent product of C along p exists, it is unique up to equivalence, see Exercise 3.8.7, and we will denote it by $p_*(f)\colon p_*(C) \to T$.

Terminology 3.5.5. Assume that C admits a dependent product $p_*(C)$ along f exists. Given a functor $\varphi\colon D \to p_*(C)$ over T, we obtain a functor

$$\varphi^u := \varepsilon \circ p^*(\varphi)\colon D \times_T S \to C$$

over S that we will refer to as the *relative uncurrying of φ*. Conversely, there is for every functor $\psi\colon D \times_T S \to C$ over S a *relatively curried* functor $\psi_c\colon D \to p_*(C)$, satisfying

$$(\varphi^u)_c \cong \varphi \qquad \text{and} \qquad (\psi_c)^u \cong \psi.$$

In this way, we may think of the equivalence in Definition 3.5.4 as a *relative* version of currying/uncurrying.

Construction 3.5.6. Consider a pullback square

$$\begin{tikzcd} S' \arrow[r, "h"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "p"] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

of categories, and let $f\colon C \to S$ be a category over S. Assume that:

- There exists a dependent product $p_*(C) \to T$ of C along p;
- There exists a dependent product $p'_*(C \times_S S') \to T'$ of $C \times_S S'$ along p'.

We will construct a functor

$$\mathrm{BC}_*\colon p_*(C) \times_T T' \to p'_*(C \times_S S')$$

over T' called the *Beck-Chevalley map*. By relative uncurrying, we may equivalently define BC_* by specifying a map $(p_*(C) \times_T T') \times_{T'} S' \to C \times_S S'$, which we take to be the map

$$(p_*(C) \times_T T') \times_{T'} S' \simeq (p_*(C) \times_T S) \times_S S' \xrightarrow{\varepsilon \times_S S'} C \times_S S'.$$

We are now ready to introduce the class of exponentiable functors.

Definition 3.5.7. Let $p\colon S \to T$ be a functor. We say that p is *exponentiable* if the following two conditions are satisfied:

(1) For every pullback square

$$\begin{tikzcd} S' \arrow[r, "h"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "p"] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

and every category C' over S', there exists a dependent product $p'_*(C') \to T'$ of C' along p'.

(2) For every pullback diagram

$$\begin{tikzcd} S'' \arrow[r, "h'"] \arrow[d, "p''"'] \arrow[dr, phantom, "\lrcorner", very near start] & S' \arrow[r, "h"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "p"] \\ T'' \arrow[r, "g'"] & T' \arrow[r, "g"] & T \end{tikzcd}$$

and every $f' \colon C' \to S'$, the Beck-Chevalley map $\mathrm{BC}_* \colon p'_*(C') \times_{T'} T'' \to p''_*(C' \times_{S'} S'')$ is an equivalence.

Remark 3.5.8. Note that exponentiable functors are by definition closed under base change: if $p \colon S \to T$ is exponentiable, then so is any base change $p' \colon S' \to T'$ of p.

Construction 3.5.9. Let $p \colon S \to T$ be an exponentiable functor. Then every functor $\varphi \colon C \to D$ over S induces a map $p_*(\varphi) \colon p_*(C) \to p_*(D)$ over T. The construction of this map is completely analogous to the construction of the functoriality of functor categories in Construction 1.4.6 and the details are left to the reader.

Lemma 3.5.10. *Let $p \colon S \to T$ be an exponentiable functor. Then the assignment $f \mapsto p_*(f)$ preserves pullbacks: given functors $C_1 \to C_3 \leftarrow C_2$ over S, there exists a preferred equivalence*

$$p_*(C_1) \times_{p_*(C_2)} p_*(C_3) \xrightarrow{\sim} p_*(C_1 \times_{C_3} C_2)$$

over T.

Proof. The preferred functor is constructed by adjunction, using that $p^*(-) = - \times_T S$ preserves pullbacks. The inverse is constructed using the functoriality of $p_*(-)$. It follows directly from universal properties that these two maps are mutual inverses; the details are left to the reader. □

Corollary 3.5.11. *Let $p \colon S \to T$ be an exponentiable functor. If $f \colon C \to S$ is an embedding, then also $p_*(f) \colon p_*(C) \to T$ is an embedding.* □

The terminology 'exponentiable' comes from the fact that it allows us to construct certain 'exponential objects' relative to a base category S:

Definition 3.5.12. Let $p \colon C \to S$ and $q \colon D \to S$ be two functors and assume that p is an exponentiable functor. We define the *relative functor category* $\underline{\mathrm{Fun}}_S(C, D) \to S$ as

$$\begin{array}{c} \underline{\mathrm{Fun}}_S(C,D) \\ \big\downarrow \\ S \end{array} \quad := \quad p_* \left(\begin{array}{c} C \times_S D \\ \big\downarrow {\scriptstyle p^*(q)} \\ C \end{array} \right).$$

It comes equipped with an evaluation map

$$\mathrm{ev}\colon \underline{\mathrm{Fun}}_S(C,D)\times_S C \to D$$

over S satisfying the property that for every other functor $E\to S$ the composite functor

$$\mathrm{Map}_S(E,\underline{\mathrm{Fun}}_S(C,D)) \xrightarrow{-\times_S C} \mathrm{Map}_S(E\times_S C,\underline{\mathrm{Fun}}_S(C,D)\times_S C) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Map}_S(E\times_S C,D).$$

is an equivalence. In other words, $\underline{\mathrm{Fun}}_S(C,D)$ is an 'internal hom object' or 'exponential object' for categories over S.

Remark 3.5.13. Note that, for a category T, the data of a functor $T\to\underline{\mathrm{Fun}}_S(C,D)$ is equivalent to the data of a pair (t,f), where $t\colon T\to S$ is a functor and $f\colon T\times_S C\to T\times_S D$ is a functor over T.

Observation 3.5.14. For any functor $t\colon T\to S$ and categories C and D over S such that $C\to S$ is exponentiable, there is a pullback square

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_T(T\times_S C, T\times_S D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Fun}}_S(C,D) \arrow[d] \\
T \arrow[r, "t"] & S.
\end{tikzcd}$$

In particular, the fiber of $\underline{\mathrm{Fun}}_S(C,D)\to S$ over an object s of S is the functor category $\mathrm{Fun}(C_s,D_s)$, where C_s and D_s are the fibers of C and D over s, respectively.

Proposition 3.5.15. *Let Γ be an anima and $f\colon S\to\Gamma$ a category in context Γ. We consider two functors $p\colon C\to\mathbb{S}$ and $q\colon D\to S$ and we assume that p is exponentiable. Then there is a canonical equivalence*

$$\mathrm{Fun}_S(C,D)\xrightarrow{\sim} f_*\,\underline{\mathrm{Fun}}_S(C,D)$$

in context Γ.

Proof. Since we work in context Γ, we may as well assume that Γ is the terminal category. The evaluation functor

$$\mathrm{ev}\colon C\times_S \mathrm{Fun}_S(C,D)\to D$$

defines a functor $(\mathrm{ev},\mathrm{pr}_2)\colon C\times_S\mathrm{Fun}_S(C,D)\to D\times_S\mathrm{Fun}_S(C,D)$ over $\mathrm{Fun}_S(C,D)$ (with pr_2 the projection of $C\times_S\mathrm{Fun}_S(C,D)$ to $\mathrm{Fun}_S(C,D)$), hence a section of the structural projection of $\underline{\mathrm{Fun}}_S(C,D)$ to S over $\mathrm{Fun}_S(C,D)$, in particular a functor of the form

$$\varphi\colon\ \mathrm{Fun}_S(C,D)\to f_*\,\underline{\mathrm{Fun}}_S(C,D)$$

To prove that the latter is an equivalence, it suffice to prove that, for any category E, the induced map

$$\varphi_*\colon\ \mathrm{Map}(E,\mathrm{Fun}_S(C,D))\to\mathrm{Map}(E,f_*\,\underline{\mathrm{Fun}}_S(C,D))$$

is an equivalence. The left hand side fits in a pullbacks square of the form

$$\begin{tikzcd}
\mathrm{Map}(E,\mathrm{Fun}_S(C,D)) \arrow[r] \arrow[d] & \mathrm{Map}(E\times C,D) \arrow[d] \\
* \arrow[r, "\mathrm{pr}_2"] & \mathrm{Map}(E\times S,S)
\end{tikzcd}$$

In other words, a term of $\mathrm{Map}(E, \mathrm{Fun}_S(C, D))$ amounts to a commutative square of the following form.

$$\begin{tikzcd} E \times C \arrow[r, "u"] \arrow[d, "\mathrm{id}_E \times p"'] & D \arrow[d, "q"] \\ E \times S \arrow[r, "\mathrm{pr}_2"] & S \end{tikzcd}$$

Taking into account the canonical equivalence of the form $(E \times S) \times_S C \xrightarrow{\sim} C$, those correspond to commutative triangles of the form

$$\begin{tikzcd} E \times C \arrow[rr, "h"] \arrow[dr, "\mathrm{id}_E \times p"'] & & E \times D \arrow[dl, "\mathrm{id}_E \times q"] \\ & E \times S & \end{tikzcd}$$

through the rules $u \mapsto (\mathrm{pr}_1, u)$ and $h \mapsto \mathrm{pr}_2 \circ h$ that are obviously inverse to each other. The map φ_* simply is another way to name this correspondence. □

Corollary 3.5.16. *Let* Γ *be an anima and* $f \colon S \to \Gamma$ *a category in context* Γ. *We consider two two functors* $p \colon C \to \mathbb{S}$ *and* $q \colon D \to S$ *and we assume that* p *is exponentiable. Then there is a canonical equivalence*

$$\mathrm{Map}_S(C, D) \xrightarrow{\sim} (f_* \underline{\mathrm{Fun}}_S(C, D))^{\simeq}$$

in context Γ.

Proof. Since $\mathrm{Map}(X, Y) \xrightarrow{\sim} \mathrm{Fun}(X, Y)^{\simeq}$ for all X and Y and since $(-)^{\simeq}$ preserves pullback squares, we see that there is a canonical equivalence of the form

$$\mathrm{Map}_S(C, D) \xrightarrow{\sim} \mathrm{Fun}_S(X, Y)^{\simeq}.$$

This corollary follows thus directly from the previous proposition. □

3.6 Joins

In this section, we introduce the *join* $C \star D$ of two categories C and D. We will see that it has two different universal properties: one for mapping *out of* $C \star D$ and one for mapping *into* $C \star D$. Using the join construction we define $C^{\triangleright} := C \star *$ and $C^{\triangleleft} := * \star C$. We may then inductively define the categories $[n]$ by $[0] = *$ and $[n+1] := [n]^{\triangleright}$.

Two universal properties

We start with the universal property of mapping out of $C \star D$:

Axiom J.1 (Join axiom)**.** For categories C and D, there is a category $C \star D$ and a pushout square

$$\begin{tikzcd} C \times (\{0\} \sqcup \{1\}) \times D \arrow[r] \arrow[d] & C \times [1] \times D \arrow[d] \\ C \sqcup D \arrow[r, "i_{C,D}"] \arrow[ur, phantom, "\ulcorner", very near end] & C \star D, \end{tikzcd}$$

where the top map is induced by the inclusion $\{0\} \sqcup \{1\} \to [1]$ and the left vertical map is given by

$$C \times (\{0\} \sqcup \{1\}) \times D \xrightarrow{\sim} C \times \{0\} \times D \sqcup C \times \{1\} \times D \xrightarrow{\mathrm{pr}_C \sqcup \mathrm{pr}_D} C \sqcup D.$$

Remark 3.6.1. In other words, for every third category E, the commutative square

$$\begin{tikzcd} \mathrm{Fun}(C \star D, E) \arrow[r] \arrow[d] & \mathrm{Fun}(C \times [1] \times D, E) \arrow[d] \\ \mathrm{Fun}(C \sqcup D, E) \arrow[r] & \mathrm{Fun}(C \times \partial[1] \times D, E) \end{tikzcd}$$

is a pullback square.

Example 3.6.2. For $C = D = [0]$, we see that $[0] \star [0] \xrightarrow{\sim} [1]$.

Example 3.6.3. By Corollary 1.7.7, the commutative squares

$$\begin{tikzcd} \{0\} \times [1] \arrow[r] \arrow[d] & {[1] \times [1]} \arrow[d, "p_0"] \\ * \arrow[r, "0"] & {[2]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {[1] \times \{0\}} \arrow[r] \arrow[d] & {[1] \times [1]} \arrow[d, "p_2"] \\ * \arrow[r, "2"] & {[2]} \end{tikzcd}$$

are pushout squares. It follows that there are equivalences

$$[0] \star [1] \simeq [2] \simeq [1] \star [0].$$

Notation 3.6.4. For any functor $g \colon C \star D \to E$, we denote by

$$g_C \colon C \to E \qquad \text{and} \qquad g_D \colon D \to E$$

the composites of g with the inclusions $C \to C \star D$ and $D \to C \star D$, respectively.

Construction 3.6.5 (Functoriality of joins)**.** Consider functors $f \colon C \to C'$ and $g \colon D \to D'$. Then the functors $f \times \mathrm{id}_{[1]} \times g \colon C \times [1] \times D \to C' \times [1] \times D'$ and $f \sqcup g \colon C \sqcup D \to C' \sqcup D'$ induce a functor $f \star g \colon C \star D \to C' \star D'$. This construction respects identities and composition.

Example 3.6.6. Taking $f = p_C \colon C \to [0]$ and $g = p_D \colon D \to [0]$, we obtain a map $p_C \star p_D \colon C \star D \to [0] \star [0] = [1]$ fitting in a commutative diagram as follows:

$$\begin{tikzcd} C \sqcup D \arrow[r, "i_{C,D}"] \arrow[d, "f \sqcup g"] & C \star D \arrow[d, "p_C \star p_D"] \\ \partial[1] = [0] \sqcup [0] \arrow[r, hook] & {[1].} \end{tikzcd}$$

We now come to the second universal property of $C \star D$:

Axiom J.2 (Join as dependent product)**.** Let C and D be categories. Then the map $C \sqcup D \to (C \star D) \times_{[1]} \partial[1]$ is an equivalence, and its inverse exhibits $C \star D$ as a dependent product of $C \sqcup D$ along the inclusion $\partial[1] \hookrightarrow [1]$, in the sense of Definition 3.5.4.

Remark 3.6.7. Let us spell out the resulting universal property of $C \star D$ more explicitly. Consider a category E over $[1]$ and denote by E_0 and E_1 its fibers over 0 and 1:

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ {[0]} \arrow[r, "0"] & {[1]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ {[0]} \arrow[r, "1"] & {[1]}. \end{tikzcd}$$

Then Axiom J.2 demands that for all E there is an equivalence

$$\mathrm{Map}_{[1]}(E, C \star D) \xrightarrow{\sim} \mathrm{Map}_{\partial[1]}(E \times_{[1]} \partial[1], C \sqcup D) \simeq \mathrm{Map}(E_0, C) \times \mathrm{Map}(E_1, D).$$

In other words, giving a functor $\varphi\colon E \to C \star D$ over $[1]$ amounts to giving a pair of functors $\varphi_0\colon E_0 \to C$ and $\varphi_1\colon E_1 \to D$.

Proposition 3.6.8. *Let A and B be two categories. The canonical embedding $A \sqcup B \hookrightarrow A \star B$ induces an equivalence*

$$A^{\simeq} \sqcup B^{\simeq} \xrightarrow{\sim} (A \star B)^{\simeq}.$$

Proof. We have a canonical commutative square of the form

$$\begin{tikzcd} (A \star B)^{\simeq} \arrow[r, hook] \arrow[d] & A \star B \arrow[d] \\ {[1]}^{\simeq} \arrow[r, hook] & {[1]} \end{tikzcd}$$

in which the horizontal maps are embeddings as well as a pullback square of the form

$$\begin{tikzcd} A \sqcup B \arrow[r, hook] \arrow[d] & A \star B \arrow[d] \\ {[1]}^{\simeq} \arrow[r, hook] & {[1]} \end{tikzcd}$$

thus inducing a canonical embedding

$$(A \star B)^{\simeq} \hookrightarrow A \sqcup B.$$

Since $A^{\simeq} \sqcup B^{\simeq} = (A \sqcup B)^{\simeq}$, this defines an embedding of the form

$$(A \star B)^{\simeq} \hookrightarrow A^{\simeq} \sqcup B^{\simeq}$$

that has an obvious section given by the embedding

$$A^{\simeq} \sqcup B^{\simeq} \hookrightarrow (A \star B)^{\simeq}$$

obtained by applying $(-)^{\simeq}$ to $A \sqcup B \hookrightarrow A \star B$. Since embeddings with sections are equivalences, this achieves the proof. $\square$

Unitality and associativity

We will now show that the join operation $(C, D) \mapsto C \star D$ is unital and associative.

Lemma 3.6.9 (Unitality of joins)**.** *For every category C, the maps $C \to \emptyset \star C$ and $C \to C \star \emptyset$ are equivalences.*

Proof. By definition the category $\emptyset \star C$ sits in a pushout square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ C \arrow[r] & \emptyset \star C \end{tikzcd}$$

and hence the bottom map is an equivalence. A similar argument applies to $C \star \emptyset$. □

Construction 3.6.10 (Associativity of joins)**.** Given categories C, D and E, we construct an equivalence

$$a_{C,D,E} \colon C \star (D \star E) \xrightarrow{\cong} (C \star D) \star E.$$

Consider the following diagram:

$$\begin{tikzcd} C \sqcup (D \sqcup E) \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_C \sqcup i_{D,E}"'] & (C \sqcup D) \sqcup E \arrow[d, "i_{C,D} \sqcup \mathrm{id}_E"] \\ C \sqcup (D \star E) \arrow[d, "i_{C,D\star E}"'] & (C \star D) \sqcup E \arrow[d, "i_{C\star D,E}"] \\ C \star (D \star E) \arrow[r, dashed, "a_{C,D,E}"] \arrow[d] & (C \star D) \star E \arrow[d] \\ {[0] \star [1]} \arrow[r, "\simeq"] & {[1] \star [0],} \end{tikzcd}$$

where the bottom equivalence is the composite $[0] \star [1] \simeq [2] \simeq [1] \star [0]$ from Example 3.6.3. One may easily check that the outer rectangle commutes by reducing to the case $C = D = E = [0]$. It then follows from a double use of the universal property of the join that there exists an essentially unique dashed arrow $a_{C,D,E}$ making the diagram commute. One may construct a map $a^{-1}_{C,D,E} \colon (C \star D) \star E \to C \star (D \star E)$ in an analogous fashion, and it follows by iterated use of the universal property of the join constrution that these maps are inverses to each other, showing that $a_{C,D,E}$ is an equivalence.

Proposition 3.6.11 (Monoidal coherence)**.** *For all categories C, D, E and F, the following two diagrams commute:*

$$\begin{tikzcd} & C \star D \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ C \star (\emptyset \star D) \arrow[rr, "a_{C,\emptyset,D}"'] & & (C \star \emptyset) \star D \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
 & (C\star D)\star(E\star F) \arrow[dl, "a_{C\star D,E,F}"'] & \\
((C\star D)\star E)\star F & & C\star(D\star(E\star F)) \arrow[ul, "a_{C,D,E\star F}"'] \arrow[d, "\mathrm{id}_C\star a_{D,E,F}"] \\
(C\star(D\star E))\star F \arrow[u, "a_{C,D,E}\star\mathrm{id}_F"] & & C\star((D\star E)\star F) \arrow[ll, "a_{C,D\star E,F}"]
\end{tikzcd}$$

Proof. TO DO.

Proof strategy: first prove it in the case where all the categories are $[0]$, and then deduce the general case using the universal property of the join. For the second diagram, the special case will need the assumption that the category $[1]$ is 0-truncated, which we may or may not have already established/assumed at this point. □

Relative joins

We will at some point also need to talk about *relative* joins:

Definition 3.6.12. Let $p\colon C \to S$ and $q\colon D \to S$ be two exponentiable functors of categories. A *relative join* of C and D over S is an exponentiable fibration $C \star_S D \to S$ together with a pushout square

$$\begin{tikzcd}
(\{0\}\sqcup\{1\})\times(C\times_S D) \arrow[r] \arrow[d] & {[1]}\times(C\times_S D) \arrow[d] \\
C\sqcup D \arrow[r, "i_{C,D}"] & C\star_S D \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

over S.

Definition 3.6.13. Let C be a category. We define its *left cone* $C^{\triangleleft}$ and its *right cone* $C^{\triangleright}$ as follows:

$$C^{\triangleleft} := [0]\star C, \qquad \text{and} \qquad C^{\triangleright} := C\star[0].$$

3.7 Slice categories

The join operation can be used to describe functors into slice categories.

Construction 3.7.1 (Slice category)**.** Let $\psi\colon Y \to C$ be a functor. We construct the *slice* $C_{/\psi}$ via the following pullback square:

$$\begin{tikzcd}
C_{/\psi} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1]\times Y, C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
C = C\times * \arrow[r, "(p_Y)^*\times\psi"] & \mathrm{Fun}(Y,C)\times\mathrm{Fun}(Y,C).
\end{tikzcd}$$

Objects of $C_{/\psi}$ are pairs (x,α) consisting of an object x of C together with a natural transformation $\alpha\colon \mathrm{const}_x \to \psi$, where $\mathrm{const}_x\colon Y \xrightarrow{p_Y} * \xrightarrow{x} C$ is the constant function with values x.

Remark 3.7.2. We will mostly be interested in the case $Y = *$. In this case, we may identify functors $x\colon * \to C$ with (absolute) objects of C, and write $C_{/x}$ for the slice of C over x.

Lemma 3.7.3. *For every category X and every functor $\psi\colon Y \to C$, there is a pullback square of the form*

$$\begin{tikzcd} \mathrm{Fun}(X, C_{/\psi}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(X \star Y, C) \arrow[d] \\ \mathrm{Fun}(X, C) \arrow[r, "1\times\psi"'] & \mathrm{Fun}(X, C) \times \mathrm{Fun}(Y, C). \end{tikzcd}$$

Proof. Since the assignment $C \mapsto \mathrm{Fun}(X, C)$ preserves pullbacks, applying it to the defining pullback square of $C_{/\psi}$ gives

$$\begin{tikzcd} \mathrm{Fun}(X, C_{/\psi}) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}(X \times [1] \times Y, C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ \mathrm{Fun}(X, C) \arrow[r, "1\times\psi"] & \mathrm{Fun}(X, C) \times \mathrm{Fun}(Y, C) \arrow[r, "\mathrm{pr}_X^* \times \mathrm{pr}_Y^*"] & \mathrm{Fun}(X \times Y, C) \times \mathrm{Fun}(X \times Y, C). \end{tikzcd}$$

Because of the factorization of the bottom map, we may compute this pullback by first pulling back along the map $\mathrm{pr}_X^* \times \mathrm{pr}_Y^*$, giving a pullback square

$$\begin{tikzcd} \mathrm{Fun}(X, C_{/\psi}) \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(X \sqcup Y, C) \times_{\mathrm{Fun}(X\times\partial[1]\times Y, C)} \mathrm{Fun}(X \times [1] \times Y, C) \arrow[d] \\ & \mathrm{Fun}(X \sqcup Y, C) \arrow[d, "\cong"] \\ \mathrm{Fun}(X, C) = \mathrm{Fun}(X, C) \times * \arrow[r, "1\times\psi"] & \mathrm{Fun}(X, C) \times \mathrm{Fun}(Y, C). \end{tikzcd}$$

Since the upper right corner is equivalent to $\mathrm{Fun}(X \star Y, C)$ by Remark 3.6.1, this finishes the proof. ◻

Remark 3.7.4. Informally speaking, the previous lemma says that a functor $X \to C_{/\psi}$ into the slice consists of a map $g\colon X \star Y \to C$ fitting in a commutative diagram as follows:

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr, "\psi"] & \\ X \star Y \arrow[rr, "g"'] & & C. \end{tikzcd}$$

Completely analogously we obtain a dual slice construction:

Construction 3.7.5. For a functor $\varphi\colon X \to C$, we define the slice category $C_{\varphi/}$ via the following pullback square:

$$\begin{tikzcd} C_{\varphi/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1] \times X, C) \arrow[d] \\ C = * \times C \arrow[r, "\varphi\times p_X^*"] & \mathrm{Fun}(X, C) \times \mathrm{Fun}(X, C). \end{tikzcd}$$

3.8 Exercises Chapter 3

Exercise 3.8.1. Consider a category Γ, a category D and a category $(E, q\colon E \to \Gamma)$ over Γ. Define another category $\mathrm{Fun}_\Gamma(D_\Gamma, E)$ over Γ via the following pullback square:

$$\begin{tikzcd}
\mathrm{Fun}_\Gamma(D_\Gamma, E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(D, E) \arrow[d, "q_*"] \\
\Gamma \arrow[r, "p_D^*"] & \mathrm{Fun}(D, \Gamma).
\end{tikzcd}$$

Show that $\mathrm{Fun}_\Gamma(D_\Gamma, E)$ is indeed equivalent to the functor category in context Γ from D_Γ to E.

Exercise 3.8.2. Consider a groupoid Γ. Show that a commutative square

$$\begin{tikzcd}
C' \arrow[r, "h"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\
D' \arrow[r, "g"] & D
\end{tikzcd}$$

in context Γ is a pushout square if and only if the induced square

$$\begin{tikzcd}
\sum_{\gamma:\Gamma} C' \arrow[r, "\sum_{\gamma:\Gamma} h"] \arrow[d, "\sum_{\gamma:\Gamma} f'"'] & \sum_{\gamma:\Gamma} C \arrow[d, "\sum_{\gamma:\Gamma} f"] \\
\sum_{\gamma:\Gamma} D' \arrow[r, "\sum_{\gamma:\Gamma} g"] & \sum_{\gamma:\Gamma} D
\end{tikzcd}$$

is a pushout square (in the absolute context).

Exercise 3.8.3. Prove Lemma 3.2.10: show that for embeddings $\Gamma \hookrightarrow C$ and $\Lambda \hookrightarrow D$ the projections $C \times D \to C$ and $C \times D \to D$ induce an equivalence

$$\langle \Gamma \times \Lambda \subseteq C \times D \rangle \xrightarrow{\sim} \langle \Gamma \subseteq C \rangle \times \langle \Lambda \subseteq D \rangle.$$

Exercise 3.8.4. Prove the assertions from Remark 3.1.16: given a functor $f\colon C \to D$ sending a collection of morphisms M in C to a collection of morphisms N in D, we have:

(1) The map $f|\colon \langle M \rangle_C \to \langle N \rangle_D$ is uniquely determined by f (up to natural isomorphism).

(2) Given a natural isomorphism $f \cong f'$, we obtain an induced isomorphism $f| \cong f'|$.

(3) The construction is compatible with composition: we have $(\mathrm{id}_C)| = \mathrm{id}_{\langle M \rangle_C}$ and $(g \circ f)| = g| \circ f|$ if $g\colon D \to E$ is another functor preserving a collection of morphisms.

(4) If the functors $f\colon C \to D$ and $\tilde{f}\colon M \to N$ are equivalences, then also $f|\colon \langle M \rangle_C \to \langle N \rangle_D$ is an equivalence.

Exercise 3.8.5. Let W be a collection of morphisms of a category C, and let $l\colon C \to L$ and $l'\colon C \to L'$ be two localizations of C at W. Construct an equivalence $L \xrightarrow{\sim} L'$.

Exercise 3.8.6. Let W be a collection of morphisms of a category C, and let $C \to C[W^{-1}]$ be a localization of C at W. Construct a pushout square

$$\begin{tikzcd}
{[1] \times W} \arrow[r, "\tilde{w}"] \arrow[d, "\mathrm{pr}_2"'] & C \arrow[d] \\
W \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & C[W^{-1}],
\end{tikzcd}$$

where the map $\tilde{w}$ is adjoint to the composite $W \xrightarrow{w} \mathrm{Map}([1], C) \to \mathrm{Fun}([1], C)$.

Exercise 3.8.7. Let $p\colon S \to T$ be a functor, and let $f\colon C \to S$ be a category over S. Let $g\colon D \to T$ and $g'\colon D' \to T$ be categories over T that are both a dependent product of C along p. Show that there is an equivalence $D \simeq D'$ of categories over T (in the sense of Definition 4.1.3).

4 Synthetic categories and contexts

We have already explained that we consider terms of categories in various contexts. We also would like to consider synthetic category theory in any given context as well. In other words, for any category Γ, we would like to speak of categories defined in context Γ.

Definition 4.0.1. Let Γ be a context (i.e. a category). A category in context $Gamma$ is a pair $C = (X, p)$ where C is a category and $p\colon C \to \Gamma$ a functor.

A context in context Γ is thus any functor of the form $\gamma\colon \Gamma' \to \Gamma$. for any category $C = (X, p)$ in context Γ there is a category

$$C(\gamma) = (X', p')$$

where $X' = \Gamma' \times_\Gamma X$ and $p' = \mathrm{pr}_{\Gamma'}$ is the canonical projection. We can thus think of C as a family of categories $C(\gamma)$ indexed by the terms of Γ. Forgetting the context yields a category in the absolute context

$$\Sigma_{\gamma\colon \Gamma} C(\gamma) = X\,.$$

The caveat is the following: the formation of each $C(\gamma)$ is functorial with respect to morphisms of contexts over Γ in the external sense but not with respect to morphisms between terms of C. The fact that the latter functoriality exists is the prerogative of cocartesian fibrations. However, for Γ an anima, the external functoriality and the functoriality with respect to morphisms of terms in Γ will coincide. This will make the theory very transparent for animae: for each anima Γ, synthetic category theory in context Γ is a semantic interpretation of synthetic category theory, and for each map of animae $f\colon \Delta \to \Gamma$, pulling back along f is a map of logics: it will preserve absolutely every construction that is the translation of a formal statement made in the language of synthetic category theory.

In Section 4.1, we will formalize the idea of a Γ-indexed family of categories in terms of functors $C \to \Gamma$. In order to ensure that families of categories satisfy the same axioms as individual categories, we will introduce in Section 4.2 a notion of *categories in context* Γ and axiomatize a one-to-one correspondence with Γ-indexed families of categories. We may then demand that the axioms hold for categories in context Γ for either an arbitrary category Γ or merely for an arbitrary *groupoid* Γ.

4.1 Families of categories

Definition 4.1.1. Let Γ be a fixed category. A *category over* Γ is a functor $p\colon C \to \Gamma$. Given another such functor $q\colon D \to \Gamma$, a *functor from C to D over* Γ is a commutative triangle

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & \Gamma. & \end{tikzcd}$$

We will often abuse notation and leave the choice of isomorphism $q \circ f \cong p$ implicit in the notation, referring to such a triangle with just the symbol f. If g (read: a pair $(g, q \circ g \cong p)$) is another functor from C to D over Γ, a *natural isomorphism over* Γ *between f and g* is a natural isomorphism $\alpha\colon f \cong g$ of functors $C \to D$ equipped with an isomorphism of natural isomorphisms between id_p and the composite

$$p \cong q \circ f \overset{q\circ\alpha}{\cong} q \circ g \cong p.$$

This approach can be iterated to define even higher iterations of isomorphisms over Γ, but these first three layers will suffice for our purposes.

Construction 4.1.2 (Identities and compositions)**.** In fact contexts themselves will always come in the context of an anima Δ. In other words, we give first an anima Δ and then a functor $\pi\colon \Gamma \to \Delta$ (in context Δ, we can always see Δ as the terminal object). Given any functors $p\colon C \to \Gamma$ and $q\colon D \to \Gamma$ we can form the following pullback square.

$$\begin{tikzcd} \mathrm{Map}_\Gamma(C, D) \arrow[r] \arrow[d] & \mathrm{Map}_\Delta(C, D) \arrow[d, "q_*"] \\ \Delta \arrow[r, "p"] & \mathrm{Map}_\Delta(C, \Gamma) \end{tikzcd}$$

In other words, terms of $\mathrm{Map}_\Gamma(C, D)$ literally are commutative triangles of the form

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & \Gamma. & \end{tikzcd}$$

over the context Δ. For any other functor $r\colon E \to \Gamma$, the composition law

$$- \circ -\colon\ \mathrm{Map}_\Delta(C, D) \times \mathrm{Map}_\Delta(D, E) \to \mathrm{Map}_\Delta(C, E)$$

induces a composition law

$$- \circ -\colon\ \mathrm{Map}_\Gamma(C, D) \times \mathrm{Map}_\Gamma(D, E) \to \mathrm{Map}_\Gamma(C, E)$$

and the identity of C canonically yields a term

$$\mathrm{id}_C\colon\ \mathrm{Map}_\Gamma(C, C)$$

corresponding to the commutative diagram

$$\begin{tikzcd} C \arrow[rr, "\mathrm{id}_C"] \arrow[dr, "p"'] & & C \arrow[dl, "p"] \\ & \Gamma, & \end{tikzcd}$$

where the commutativity is exhibited by the unitality isomorphism $p \cong p \circ \mathrm{id}_C$.

For any commutative triangles

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & \Gamma. & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[rr, "g"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & \Gamma. & \end{tikzcd}$$

we can make sense of $f \cong g$ in context Γ simply by considering the concept of identification in the anima $\mathrm{Map}_\Gamma(C, D)$.

Definition 4.1.3. We say that a functor $f \colon C \to D$ over Γ is an *equivalence over* Γ if there exists another functor $g \colon D \to C$ over Γ such that the composites $g \circ f$ and $f \circ g$ are naturally isomorphic over Γ to the respective identities.

4.2 Synthetic categories in context

For future purposes, we would like to be able to manipulate categories over Γ as if they are individual categories. In order to do this, we will expand our language of naive categories by introducing the concept of categories in *contexts*:

Axiom K.1 (Categories in context)**.** Let Γ be a category. We may speak of:

(0) *Synthetic categories in context* Γ;

(1) *Functors in context* Γ;

(2) *Natural isomorphisms in context* Γ;

(3+) Etcetera...

All the rules from Axioms A.1, A.3, A.2 and A.4 are still valid in context Γ. In particular, we may talk about commutative squares, commutative triangles and equivalences in context Γ.

We will use the phrase *absolute context* if we wish to emphasize that we are talking about categories in the sense of Axiom A.1 as opposed to categories in some context Γ. If Γ is a groupoid, we will speak of a *groupoidal context*.

Recall that we think of our naive category theory as capturing the internal language of the universe of small categories Cat, to be officially introduced in Chapter 7 below. From this perspective, we may think of synthetic category theory in context Γ as capturing the internal language of the slice category $\mathrm{Cat}_{/\Gamma}$.

Notation 4.2.1. Synthetic categories in context Γ may be denoted by letters like C, D, E, etcetera, just like for categories in the absolute context. If we wish to emphasize we are working in context Γ, we will sometimes also use the notation $C(\gamma)$, $D(\gamma)$, etcetera to denote a category in context Γ. This notation is meant to evoke the idea that a category in context Γ contains the data of a synthetic $C(\gamma)$ for every object γ of Γ. In a similar way, we may denote functors in context Γ by $f(\gamma)\colon C(\gamma) \to D(\gamma)$, natural isomorphisms in context Γ by $\alpha(\gamma)\colon f(\gamma) \cong g(\gamma)$, etcetera.

It may sometimes be useful to employ type-theoretic notation to incidate contexts:

$$\Gamma \vdash C \text{ category}, \qquad \Gamma \vdash F\colon C \to D, \qquad \Gamma \vdash \alpha\colon F \cong G, \qquad \ldots.$$

We will now impose that all previous axioms hold in arbitrary groupoidal context, and some even in arbitrary contexts.

<u>Axiom K.2</u> (Axioms in context)**.** Let Γ be a category.

(1) If Γ is a groupoid, then every axiom about categories also holds for categories in context Γ.

(2) Every axiom which is *underlined* also holds for categories in an *arbitrary* context Γ.

So for example, we may form products, coproducts and pullbacks of categories in an arbitrary context Γ, but we may form functor categories, groupoid cores and subcategories only in a groupoidal context. Note that all of the content of Chapter 3 only holds in groupoidal content.

Note that Axioms K.1 and K.2 are themselves underlined, which allows us to form *iterated contexts*: if Γ_2 is a category in context Γ_1, we may work with categories in context Γ_2 in context Γ_1:

$$\Gamma_1, \Gamma_2 \vdash C \text{ category}, \qquad \Gamma_1, \Gamma_2 \vdash F\colon C \to D, \qquad \ldots.$$

We will usually leave contexts implicit, so while iterated contexts are allowed they will make no explicit appearances in the book.

Next, we will make precise the one-to-one correspondence between categories in context Γ and categories over Γ.

<u>Axiom K.3</u> (Weakening)**.** Let Γ be a category. For every category C in the absolute context, there is a category $w_\Gamma C$ in context Γ, called the *weakening of C to* Γ. For every functor $F\colon C \to D$ in the absolute context, there is a functor $w_\Gamma F\colon w_\Gamma C \to w_\Gamma D$, called the *weakening of F to* Γ. This process continues: for every n-isomorphism in the absolute context, there is an n-isomorphism in context Γ. These assignments preserve identities, composition, and inverses, and are compatible with the associativity, unitality and invertibility isomorphisms.

<u>Remark 4.2.2</u>. In the semantic picture of the universe of categories Cat, the weakening construction $C \mapsto w_\Gamma C$ encodes the pullback functor $\mathrm{Cat} \to \mathrm{Cat}_{/\Gamma}$. In particular, the fiber of $w_\Gamma C$ over $\gamma : \Gamma$ is supposed to be C for every γ.

Convention 4.2.3. We will usually want to think of $w_\Gamma C$ as being simply an instantiation of C itself. To avoid cluttering of notation, we will frequently write C instead of $w_\Gamma C$.

Axiom K.4 (Dependent sum)**.** Let Γ be a category. Given a category $C(\gamma)$ in context Γ, there is a category $\Sigma_\Gamma C$ in the absolute context, called the *dependent sum of* $C(\gamma)$ *over* Γ. It comes equipped with a functor

$$(\gamma, -)\colon C(\gamma) \to w_\Gamma(\Sigma_\Gamma C)$$

in context Γ, called the *pair functor*. This pair functor is universal among functors from $C(\gamma)$ into a weakening of an absolute category:

- Given a category D in the absolute context and a functor $f(\gamma)\colon C(\gamma) \to w_\Gamma D$, there exists a functor $\overline{f}\colon \sum_\Gamma C \to D$ in the absolute context, equipped with a natural isomorphism in context Γ making the following triangle commute:

$$\begin{tikzcd} C(\gamma) \arrow[r, "(\gamma,-)"] \arrow[dr, "f(\gamma)"'] & w_\Gamma(\sum_\Gamma C) \arrow[d, "w_\Gamma \overline{f}"] \\ & w_\Gamma D. \end{tikzcd}$$

- Given two functors $g, h\colon \sum_\Gamma C \to D$ in the absolute context and a natural isomorphism $\alpha(\gamma)\colon w_\Gamma g \circ (\gamma, -) \cong w_\Gamma h \circ (\gamma, -)$ in context Γ, there exists a natural isomorphism $\overline{\alpha}\colon g \cong h$ such that $\alpha(\gamma) \cong (\gamma, -) \star w_\Gamma \overline{\alpha}$.

In particular, there is a one-to-one correspondence between functors $C(\gamma) \to w_\Gamma D$ in context Γ and functors $\Sigma_\Gamma C \to D$ in the absolute context.

Remark 4.2.4. In the semantic picture of the universe of categories Cat, the dependent sum construction $C(\gamma) \mapsto \Sigma_\Gamma C$ is encoding a left adjoint $\Sigma_\Gamma\colon \mathrm{Cat}_{/\Gamma} \to \mathrm{Cat}$ to the weakening functor. It corresponds to the forgetful functor that forgets the reference map to Γ.

Construction 4.2.5 (Functoriality of direct sum)**.** Using the axiom we see that every functor $F(\gamma)\colon C(\gamma) \to C'(\gamma)$ induces a functor $\sum_\Gamma F\colon \sum_\Gamma C \to \sum_\Gamma C'$ in the absolute context which makes the following diagram commute in context Γ:

$$\begin{tikzcd} C(\gamma) \arrow[r, "(\gamma,-)"] \arrow[d, "F(\gamma)"'] & w_\Gamma(\Sigma_\Gamma C) \arrow[d, "w_\Gamma(\Sigma_\Gamma F)"] \\ C'(\gamma) \arrow[r, "(\gamma,-)"] & w_\Gamma(\Sigma_\Gamma C'). \end{tikzcd} \tag{4.1}$$

Construction 4.2.6 (Second projection functor)**.** Given a category D in the absolute context, we may apply Axiom K.4 to the identity functor $\mathrm{id}_{w_\Gamma D}\colon w_\Gamma D \to w_\Gamma D$ in context Γ to obtain a functor

$$\mathrm{pr}_2\colon \Sigma_\Gamma w_\Gamma D \to D$$

in the absolute context. We refer to pr_2 as the *second projection functor*. Observe that for every functor $F\colon D \to E$ we obtain a commutative square as follows:

$$\begin{tikzcd} \Sigma_\Gamma w_\Gamma D \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\Sigma_\Gamma w_\Gamma F"'] & D \arrow[d, "F"] \\ \Sigma_\Gamma w_\Gamma E \arrow[r, "\mathrm{pr}_2"] & E. \end{tikzcd} \tag{4.2}$$

Note that so far the notion of categories in context Γ did not actually talk about Γ yet (even though we are heavily using Γ in our suggestively chosen notation). The following final two axioms about contexts will make sure categories in context Γ really behave like categories over Γ:

Axiom K.5. There is an equivalence $\mathrm{pr}_1\colon \Sigma_\Gamma * \xrightarrow{\sim} \Gamma$.

Definition 4.2.7. Under the equivalence from the axiom, the pair functor $(\gamma, -)\colon * \to w_\Gamma \Sigma_\Gamma *$ corresponds to a functor

$$\gamma_{\mathrm{univ}}\colon * \to w_\Gamma \Gamma$$

that we refer to as the *universal object* of Γ.

Axiom K.6. (1) For every functor $F(\gamma)\colon C(\gamma) \to C'(\gamma)$ in context Γ the square (4.1) is a pullback square;

(2) For every functor $F\colon D \to E$ in the absolute context the square (4.2) is a pullback square;

(3) The assignment $C(\gamma) \mapsto \Sigma_\Gamma C$ preserves pullbacks: For every pullback square

$$\begin{tikzcd} C'(\gamma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C(\gamma) \arrow[d] \\ D'(\gamma) \arrow[r] & D(\gamma), \end{tikzcd}$$

the induced commutative square

$$\begin{tikzcd} \Sigma_\Gamma C' \arrow[r] \arrow[d] & \Sigma_\Gamma C \arrow[d] \\ \Sigma_\Gamma D' \arrow[r] & \Sigma_\Gamma D \end{tikzcd}$$

is a pullback square.

Notation 4.2.8. Note that the map $\Sigma_\Gamma * \to \Gamma$ corresponds to a functor $* \to w_\Gamma \Gamma$ in context Γ. We will denote this object by γ_{univ}.

Observation 4.2.9. There is a one-to-one correspondence between categories in context Γ and categories over Γ:

- To a category C in context Γ we assign its dependent sum $\Sigma_\Gamma C$, equipped with the composite

$$\mathrm{pr}_1 \colon \Sigma_\Gamma C \xrightarrow{\Sigma_\Gamma p_C} \Sigma_\Gamma * \xrightarrow{\sim} \Gamma,$$

called the *first projection functor*.

- To a category $p\colon D \to \Gamma$ over Γ, we assign the category D_γ in context Γ defined by the following pullback square:

$$\begin{tikzcd} D_\gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & w_\Gamma D \arrow[d, "w_\Gamma p"] \\ * \arrow[r, "\gamma_{\mathrm{univ}}"] & w_\Gamma \Gamma. \end{tikzcd}$$

- We have $C(\gamma) \simeq (\Sigma_\Gamma C)_\gamma$ as a special case of the pullback square (4.1):

$$\begin{tikzcd} C(\gamma) \arrow[r, "{(\gamma,-)}"] \arrow[d, "p_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & w_\Gamma(\Sigma_\Gamma C) \arrow[d, "w_\Gamma \Sigma_\Gamma p_C"] \\ * \arrow[r, "{(\gamma,-)}"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & w_\Gamma(\Sigma_\Gamma *) \arrow[d, "\sim"] \\ * \arrow[r, "\gamma_{\mathrm{univ}}"] & w_\Gamma \Gamma. \end{tikzcd}$$

- We have $\Sigma_\Gamma D_\gamma \xrightarrow{\sim} D$ by virtue of the outer pullback square in the following diagram:

$$\begin{tikzcd} \Sigma_\Gamma D_\gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_\Gamma w_\Gamma D \arrow[r, "pr_D"] \arrow[d, "\Sigma_\Gamma w_\Gamma p"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "p"] \\ \Sigma_\Gamma * \arrow[r, "\Sigma_\Gamma \gamma_{\mathrm{univ}}"] \arrow[rr, bend right, "\sim"'] & \Sigma_\Gamma w_\Gamma \Gamma \arrow[r, "\mathrm{pr}_\Gamma"] & \Gamma. \end{tikzcd}$$

  Here the left-hand square is a pullback by (3) of Axiom K.6 while the right-hand square is a pullback by (2).

Example 4.2.10. Given a functor $f\colon C \to D$ and a term $x\colon \{x\} \to D$, we defined in Definition 1.3.12 the *fiber* C_x of f over x as the pullback

$$\begin{tikzcd} C_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ \{x\} \arrow[r, "x"] & D. \end{tikzcd}$$

In particular, C_x is naturally a category over $\{x\}$, and Theorem 4.2.9 allows us to regard C_x as a category in context $\{x\}$. We will frequently treat C_x in this way: it will allow us to assume without loss of generality that x is an absolute term.

Remark 4.2.11. For every category C in the absolute context, there is a functor $p_C\colon C \to *$, allowing us to regard C as a category in context $*$. Conversely, every category D in context $*$ determines a category $\sum_* D$. The axioms on $*$ imply that these two assignments determine a one-to-one correspondence between categories in the absolute context and categories in context $*$.

Lemma 4.2.12. *Every category in context $\emptyset$ is contractible.*

Proof. Under the correspondence from Theorem 4.2.9 between categories in context $\emptyset$ and categories over $\emptyset$, we see that a category C in context $\emptyset$ is contractible if and only if the induced functor $\sum_\emptyset C \to \sum_\emptyset * \simeq \emptyset$ is an equivalence. This is an instance of Axiom B.2'. □

Lemma 4.2.13. *For every category Γ, the functor $\emptyset \to \sum_{\gamma:\Gamma} \emptyset$ is an equivalence.*

Proof. By the universal property of $\emptyset$, the given functor is a functor over Γ. We will construct an explicit inverse. Under the corresponcende from Theorem 4.2.9, the functor $\emptyset \to \Gamma$ corresponds to a category $\emptyset_\gamma$ in context Γ. By the universal property of the initial object in context Γ there is a functor $\emptyset \to \emptyset_\gamma$, and translating this back this gives rise to a functor $\sum_{\gamma:\Gamma} \emptyset \to \emptyset$. Using the respective universal properties of the initial categories in the absolute context and in context Γ it follows that this functor is an inverse to the given functor. (Alternatively one may conclude using Axiom B.2'.) □

Lemma 4.2.14. *Let Γ be a groupoid and let $p\colon X \to \Gamma$ be a category over Γ. Then X is a groupoid in context Γ if and only if X is a groupoid in the absolute context.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathrm{Fun}_\Gamma([1]_\Gamma, X) \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], X) \arrow[d] \\ & \Gamma \arrow[r, "\simeq"] & \mathrm{Fun}([1], \Gamma). \end{tikzcd}$$

The bottom map is an equivalence since Γ is a groupoid, and thus it follows that also the top horizontal map is an equivalence. It then follows by 2-out-of-3 that the map $X \to \mathrm{Fun}([1], X)$ is an equivalence if and only if the map $X \to \mathrm{Fun}_\Gamma([1]_\Gamma, X)$ is an equivalence, finishing the proof. □

Corollary 4.2.15. *The data of a term of C in context Γ is equivalent to the data of an absolute object of C_Γ in context Γ.*

Proof. By Axiom K.5, there is an equivalence $\mathrm{pr}_1\colon \sum_{\gamma:\Gamma} * \xrightarrow{\sim} \Gamma$ is an equivalence. In particular, the data of a term $x\colon \Gamma \to C$ of C in context Γ is the same as the data of a functor $\sum_{\gamma:\Gamma} * \to C$. By Axiom K.4, this is in turn equivalent to the data of a functor $* \to C_\Gamma$ in context Γ, i.e. the data of an absolute object of C_Γ in context Γ. □

We will now show that the assignment $C \mapsto C_\Gamma$ preserves all categorical structures we have introduced so far.

Proposition 4.2.16. *Let Γ be a category.*

(1) For a category C, there is a natural isomorphism $(\mathrm{id}_C)_\Gamma \cong \mathrm{id}_{C_\Gamma}$ of functors $C_\Gamma \to C_\Gamma$ in context Γ.

(2) *For functors $f\colon C \to D$ and $g\colon D \to E$, there is a natural isomorphism $(g \circ f)_\Gamma \cong g_\Gamma \circ f_\Gamma$ of functors $C_\Gamma \to E_\Gamma$ in context Γ.*

(3) *The category $*_\Gamma$ in context Γ is contractible;*

(4) *For categories C and D, the functor*

$$((\mathrm{pr}_C)_\Gamma, (\mathrm{pr}_D)_\Gamma)\colon (C \times D)_\Gamma \to C_\Gamma \times D_\Gamma$$

is an equivalence;

(5) *For functors $f\colon C \to E$ and $g\colon D \to E$, the functor*

$$((\mathrm{pr}_C)_\Gamma, (\mathrm{pr}_D)_\Gamma)\colon (C \times_E D)_\Gamma \to C_\Gamma \times_{E_\Gamma} D_\Gamma$$

is an equivalence;

(6) *The category $\emptyset_\Gamma$ is an initial category in context Γ;*

(7) *For categories C and D, the functor*

$$\langle (i_C)_\Gamma, (i_D)_\Gamma \rangle\colon C_\Gamma \sqcup D_\Gamma \to (C \sqcup D)_\Gamma$$

is an equivalence.

Proof. For parts (1) and (2) we may prove the analogous statements for categories over Γ, which are special cases of Exercise 1.2.11.

For (3), let us temporarily write $\star$ for the terminal category in context Γ, in order to distinguish it from the terminal category $*$ in the absolute context. We need to show that the functor $*_\Gamma \to \star$ is an equivalence in context Γ. Under Theorem 4.2.9, this amounts to showing that the induced functor

$$\begin{tikzcd} \Gamma \times * \arrow[rr] \arrow[dr, "\mathrm{pr}_1"'] & & \sum_{\gamma:\Gamma} \star \arrow[dl, "\mathrm{pr}_1"] \\ & \Gamma & \end{tikzcd}$$

is an equivalence over Γ. But this is an immediate consequence of 2-out-of-3, since the two diagonal functors are equivalences by Lemma 1.2.7 and Axiom K.5, respectively.

The proofs of (4) and (5) are similar. For (4), it will suffice to show that the induced functor $\Gamma \times (C \times D) \to (\Gamma \times C) \times_\Gamma (\Gamma \times D)$ is an equivalence. For (5), we may equivalently show that the induced functor

$$\Gamma \times (C \times_E D) \to \sum_{\gamma:\Gamma} (C_\Gamma \times_{E_\Gamma} D_\Gamma) \xrightarrow[\sim]{K.6} (\Gamma \times C) \times_{\Gamma \times E} (\Gamma \times D)$$

is an equivalence. Both of these statements are relatively straightforward to prove and will be left to the reader.

For part (6), let us temporarily denote the initial category in context Γ by $\varnothing$, to distinguish it from the initial category in the absolute context. It will suffice to produce a functor $\emptyset_\Gamma \to \varnothing$ in context Γ, since this is then automatically an equivalence by Axiom B.2'. Via Theorem 4.2.9, such functor corresponds to a functor $\emptyset \times \Gamma \to \sum_{\gamma:\Gamma} \varnothing$ over Γ. But since $\emptyset \times \Gamma \xrightarrow{\sim} \emptyset$ by Lemma 1.2.12, there is a preferred such functor, finishing the proof of (6).

For part (7), we start by constructing a functor $(C \sqcup D)_\Gamma \to C_\Gamma \sqcup D_\Gamma$ in context Γ. Via Theorem 4.2.9, this corresponds to a functor $\Gamma \times (C \sqcup D) \to \sum_{\gamma:\Gamma}(C_\Gamma \sqcup D_\Gamma)$ over Γ. The source of this map is equivalent to $\Gamma \times C \sqcup \Gamma \times D$ by Corollary 1.3.22, and hence such a functor may be given by the functors

$$\Gamma \times C \simeq \sum_{\gamma:\Gamma} C_\Gamma \xrightarrow{\sum_{\gamma:\Gamma} i_{C_\Gamma}} \sum_{\gamma:\Gamma}(C_\Gamma \sqcup D_\Gamma) \qquad \text{and} \qquad \Gamma \times D \simeq \sum_{\gamma:\Gamma} D_\Gamma \xrightarrow{\sum_{\gamma:\Gamma} i_{D_\Gamma}} \sum_{\gamma:\Gamma}(C_\Gamma \sqcup D_\Gamma)$$

over Γ. We will leave it to the reader to verity that the resulting functor $(C \sqcup D)_\Gamma \to C_\Gamma \sqcup D_\Gamma$ is an inverse of the functor $\langle (i_C)_\Gamma, (i_D)_\Gamma \rangle \colon C_\Gamma \sqcup D_\Gamma \to (C \sqcup D)_\Gamma$, finishing the proof. $\square$

4.3 Exercises Chapter 4

Exercise 4.3.1. Consider a category Γ. Show that a commutative square

$$\begin{array}{ccc} C' & \xrightarrow{h} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow{g} & D \end{array}$$

in context Γ is a pushout square if and only if the induced square

$$\begin{array}{ccc} \sum_{\gamma:\Gamma} C' & \xrightarrow{\sum_{\gamma:\Gamma} h} & \sum_{\gamma:\Gamma} C \\ {\scriptstyle \sum_{\gamma:\Gamma} f'}\downarrow & & \downarrow{\scriptstyle \sum_{\gamma:\Gamma} f} \\ \sum_{\gamma:\Gamma} D' & \xrightarrow{\sum_{\gamma:\Gamma} g} & \sum_{\gamma:\Gamma} D \end{array}$$

is a pushout square.

5 Cartesian and cocartesian fibrations

An important concept in category theory is that of a cocartesian fibration: a functor $E \to C$ whose fibers E_c are in a certain sense 'covariantly functorial' in the object $c \in C$. Similarly there is the dual concept of a cartesian fibration, for which the fibers are 'contravariantly functorial' in c. In this chapter, we will introduce synthetic analogues of these notions.

The material in this chapter is similar in spirit to the article [BW21] by Buchholtz and Weinberger, who give an extensive treatment of cartesian and cocartesian fibrations in the context of simplicial type theory.

We emphasize that most results in this chapter will hold true in an *arbitrary* context Γ and not merely in groupoidal context: most of the content of this chapter is completely independent of all the axioms introduced in Chapters 2 and Chapter 3. All definitions and results that work in an arbitrary context are underlined.

5.1 Directed pullbacks

We begin by introducing the notion of the *directed pullback* of two functors.

Construction 5.1.1 (Directed pullback)**.** Consider two functors $f\colon A \to C$ and $g\colon B \to C$. We define the *directed pullback* $A \vec{\times}_{f,g} B$ *of* A *and* B *over* C (sometimes called 'lax pullback') via the pullback square

$$\begin{tikzcd} A \vec{\times}_{f,g} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ A \times B \arrow[r, "f \times g"] & C \times C. \end{tikzcd}$$

Note that a term of $A \vec{\times}_{f,g} B$ consists of a triple (a, b, φ) where a is a term of A, b is a term of B, and $\varphi\colon f(a) \to g(b)$ is a morphism in C.

We often abuse notation and write $A \vec{\times}_g B$ in the case where $C = A$ and $f = \mathrm{id}_A\colon A \to A$ and dually write $A \vec{\times}_f B$ when $C = B$ and $g = \mathrm{id}_B\colon B \to B$.

Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & C \arrow[d, "\gamma"] & B \arrow[l, "g"'] \arrow[d, "\beta"] \\ A' \arrow[r, "f'"'] & C' & B', \arrow[l, "g'"] \end{tikzcd}$$

we obtain an induced functor

$$\alpha \vec{\times}_\gamma \beta \colon A \vec{\times}_{f,g} B \to A' \vec{\times}_{f',g'} B',$$
$$(a,b,\varphi) \mapsto (\alpha(a), \beta(b), \gamma(\varphi) \colon f'(\alpha(a)) \cong \gamma(f(a)) \to \gamma(g(b)) \cong g'(\beta(b))).$$

Remark 5.1.2. Combining Lemma 1.3.18 with the Rezk axiom (Axiom F), we obtain a pullback square

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ A \times B \arrow[r, "f \times g"] & C \times C. \end{tikzcd}$$

We will see in Corollary 6.1.9 below that the inclusion $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \hookrightarrow \mathrm{Fun}([1], C)$ is a full subcategory, and hence it induces a full subcategory $A \times_C B \hookrightarrow A \vec{\times}_C B$. Therefore, we may think of the pullback $A \times_C B$ as the full subcategory of $A \vec{\times}_C B$ spanned by those triples (a, b, γ) such that the map $\gamma \colon f(a) \xrightarrow{\sim} g(b)$ is an isomorphism in C.

Construction 5.1.3 (Directed evaluation maps)**.** Consider a functor $f \colon A \to B$. We define the *directed evaluation maps*

$$\begin{aligned} \vec{\mathrm{ev}}_0^f \colon \mathrm{Fun}([1], A) &\to A \vec{\times}_f B \\ (\varphi \colon a \to a') &\mapsto (a, f(a'), f(\varphi)) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \vec{\mathrm{ev}}_1^f \colon \mathrm{Fun}([1], A) &\to B \vec{\times}_f A \\ (\varphi \colon a \to a') &\mapsto (f(a), a', f(\varphi)) \end{aligned}$$

via the following two commutative diagrams:

$$\begin{tikzcd} \mathrm{Fun}([1], A) \arrow[r, dashed, "\vec{\mathrm{ev}}_0^f"] \arrow[rr, bend left, "f_*"] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & A \vec{\times}_f B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], B) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ A \times A \arrow[r, "\mathrm{id}_A \times f"] & A \times B \arrow[r, "f \times \mathrm{id}_B"] & B \times B \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Fun}([1], A) \arrow[r, dashed, "\vec{\mathrm{ev}}_1^f"] \arrow[rr, bend left, "f_*"] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & B \vec{\times}_f A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], B) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ A \times A \arrow[r, "f \times \mathrm{id}_A"] & B \times A \arrow[r, "\mathrm{id}_B \times f"] & B \times B. \end{tikzcd}$$

Remark 5.1.4. In case $B = *$ there are equivalences $A \vec{\times}_f B \simeq A \simeq B \vec{\times}_f A$ and the directed evaluation maps $\vec{\mathrm{ev}}_0^f$ and $\vec{\mathrm{ev}}_1^f$ simplify to the evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}([1], A) \to A$.

For later use, we record various properties of the directed evaluation maps.

Lemma 5.1.5. *Assume that $f \colon A \to B$ is an equivalence. Then also $\vec{\mathrm{ev}}_0^f$ and $\vec{\mathrm{ev}}_1^f$ are equivalences.*

Proof. For $\vec{\mathrm{ev}}_0^f$, this follows from the 2-out-of-3 property applied to the following two maps:

$$\mathrm{Fun}([1], A) \xrightarrow{\vec{\mathrm{ev}}_0^f} A \vec{\times}_f B \to \mathrm{Fun}([1], B).$$

Indeed, the composite is induced by f and is hence an equivalence, and the second map is a base change of f and thus also an equivalence. The proof for $\vec{\mathrm{ev}}_1^f$ is similar. □

<u>Lemma 5.1.6</u> (Base change)**.** *Consider a pullback square of the form*

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B.
\end{tikzcd}$$

Then the commutative square

$$\begin{tikzcd}
\mathrm{Fun}([1], A') \arrow[r, "u_*"] \arrow[d, "\vec{\mathrm{ev}}_0^{f'}"'] & \mathrm{Fun}([1], A) \arrow[d, "\vec{\mathrm{ev}}_0^{f}"] \\
A' \vec{\times}_{f'} B' \arrow[r, "v \vec{\times}_f u"] & A \vec{\times}_f B
\end{tikzcd}$$

is a pullback square. The dual result for $\vec{\mathrm{ev}}_1^f$ *and* $\vec{\mathrm{ev}}_1^{f'}$ *also holds.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
& \mathrm{Fun}([1], B') \arrow[rr] \arrow[dd, equal] & & \mathrm{Fun}([1], B) \arrow[dd, equal] \\
\mathrm{Fun}([1], A') \arrow[ur] \arrow[rr] \arrow[dd, "\vec{\mathrm{ev}}_0^{f'}"'] & & \mathrm{Fun}([1], A) \arrow[ur] \arrow[dd, "\vec{\mathrm{ev}}_0^{f}"] & \\
& \mathrm{Fun}([1], B') \arrow[rr] \arrow[dd, "(\mathrm{ev}_0, \mathrm{ev}_1)"] & & \mathrm{Fun}([1], B) \arrow[dd, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
A' \vec{\times}_{f'} B' \arrow[ur] \arrow[rr] \arrow[dd] & & A \vec{\times}_f B \arrow[ur] \arrow[dd] & \\
& B' \times B' \arrow[rr] & & B \times B \\
A' \times B' \arrow[ur] \arrow[rr] & & A \times B. \arrow[ur] &
\end{tikzcd}$$

Observe that the bottom square is a pullback square, while the lower vertical squares on the left and right sides are pullback squares by definition of $A' \vec{\times}_{f'} B'$ and $A \vec{\times}_f B$. It follows by the pasting law of pullback squares that also the middle horizontal square is a pullback square. Since also the upper square at the back and the top square are pullback squares, it thus follows that the upper front square is a pullback square, as desired. □

<u>Lemma 5.1.7</u> (Composition)**.** *For functors* $f \colon A \to B$ *and* $g \colon B \to C$*. Then there is a commutative diagram*

$$\begin{tikzcd}
& \mathrm{Fun}([1], A) \arrow[dl, "\vec{\mathrm{ev}}_0^{f}"'] \arrow[dr, "\vec{\mathrm{ev}}_0^{gf}"] & \\
A \vec{\times}_f B \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & A \vec{\times}_{gf} C \arrow[d] \\
\mathrm{Fun}([1], B) \arrow[rr, "\vec{\mathrm{ev}}_0^{g}"] & & B \vec{\times}_g C,
\end{tikzcd}$$

where the bottom square is a pullback square.

Proof. It is easy to verify that the diagram commutes, hence it remains to show that the bottom square is a pullback square. To this end, consider the following commutative diagram:

$$
\begin{tikzcd}
A \vec{\times}_f B \arrow[rr] \arrow[dd] \arrow[dr] & & A \vec{\times}_{gf} C \arrow[dd] \arrow[dr] & \\
& \mathrm{Fun}([1], B) \arrow[rr, "\vec{\mathrm{ev}}_0^g"] \arrow[dd] & & B \vec{\times}_g C \arrow[dd] \\
A \times B \arrow[rr, "1\times g"] \arrow[dr, "f\times 1"'] & & A \times C \arrow[dr, "f\times 1"] & \\
& B \times B \arrow[rr, "1\times g"] & & B \times C.
\end{tikzcd}
$$

We want to show that the top square is a pullback square. The left square is a pullback square by definition and the bottom square is easily seen to be a pullback square, hence by the pasting law of pullback squares it will suffice to show that the right square is a pullback square. This follows from another instance of the pasting law, applied to the following commutative diagram:

$$
\begin{tikzcd}
A \vec{\times}_{gf} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \vec{\times}_g C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
A \times C \arrow[r, "f\times 1"] & B \times C \arrow[r, "g\times 1"] & C \times C.
\end{tikzcd}
$$

Indeed, by the definitions of $A \vec{\times}_{gf} C$ and $B \vec{\times}_g C$ the right and outer squares are pullback squares and hence so is the left square. □

Lemma 5.1.8 (Retracts)**.** *Consider a retract diagram of functors:*

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d, "f'"'] & A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\
B' \arrow[r] & B \arrow[r] & B'.
\end{tikzcd}
$$

Then the map $\vec{\mathrm{ev}}_0^{f'}$ *is a retract of* $\vec{\mathrm{ev}}_0^{f}$*:*

$$
\begin{tikzcd}
\mathrm{Fun}([1], A') \arrow[r] \arrow[d, "\vec{\mathrm{ev}}_0^{f'}"'] & \mathrm{Fun}([1], A) \arrow[r] \arrow[d, "\vec{\mathrm{ev}}_0^{f}"] & \mathrm{Fun}([1], A') \arrow[d, "\vec{\mathrm{ev}}_0^{f'}"] \\
A' \vec{\times}_{f'} B' \arrow[r] & A \vec{\times}_f B \arrow[r] & A' \vec{\times}_{f'} B'.
\end{tikzcd}
$$

Proof. This is an immediate consequence of the definitions. □

5.2 Left and right fibrations

We define the notions of *left* and *right fibrations*, and show that they admit a notion of *covariant/contravariant transport*.

Definition 5.2.1 (Left/right fibration)**.** A functor $f\colon A \to B$ is called a *left fibration* if the directed evaluation map $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1], A) \to A \vec{\times}_f B$ is an equivalence. Dually, we say that f is a *right fibration* if $\vec{\mathrm{ev}}_1^f\colon \mathrm{Fun}([1], A) \to B \vec{\times}_f A$ is an equivalence.

Lemma 5.2.2. *A functor $f\colon A \to B$ is a left fibration if and only if the commutative square*

$$\begin{tikzcd} \mathrm{Fun}([1], A) \arrow[r, "f_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1], B) \arrow[d, "\mathrm{ev}_0"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback. A dual characterization for right fibrations is obtained by replacing $\vec{\mathrm{ev}}_0^f$ by $\vec{\mathrm{ev}}_1^f$ and ev_0 *by* ev_1.

Proof. This is immediate from the fact that the directed evaluation map $\vec{\mathrm{ev}}_0^f$ is defined as the unique dashed map filling the following diagram:

$$\begin{tikzcd} \mathrm{Fun}([1], A) \arrow[dr, dashed, "\vec{\mathrm{ev}}_0^f"] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "\mathrm{ev}_0"'] & & \\ & A \vec{\times}_f B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], B) \arrow[d, "\mathrm{ev}_0"] \\ & A \arrow[r, "f"] & B. \end{tikzcd}$$

The proof for right fibrations is dual. ◻

Corollary 5.2.3. *Let $f\colon A \to B$ be a left (resp. right) fibration. Then the commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "(p_{[1]})^*"'] & B \arrow[d, "(p_{[1]})^*"] \\ \mathrm{Fun}([1], A) \arrow[r, "f_*"] & \mathrm{Fun}([1], B) \end{tikzcd}$$

is a pullback square.

Proof. We prove the claim for left fibrations; the claim for right fibrations is dual. Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "(p_{[1]})^*"'] & B \arrow[d, "(p_{[1]})^*"] \\ \mathrm{Fun}([1], A) \arrow[r, "f_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1], B) \arrow[d, "\mathrm{ev}_0"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

The outer square is clearly cartesian, and the bottom square is cartesian by Lemma 5.2.2. It thus follows from the pasting law of pullback squares that also the top square is cartesian. ◻

Proposition 5.2.4. *Let $p\colon X \to Y$ be a functor and assume that Y is a groupoid. Then the following conditions are equivalent:*

(1) The category X is a groupoid;

(2) The map p is a left fibration;

(3) The map p is a right fibration.

Proof. We prove the equivalence between (1) and (2); the equivalence between (1) and (3) is dual. Consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fun}([1],X) \arrow[r, "p_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1],Y) \arrow[d, "\mathrm{ev}_0", "\sim"'] \\ X \arrow[r, "p"] & Y. \end{tikzcd}$$

Since Y is a groupoid, the right vertical map is an equivalence. It follows that this square is a pullback square if and only if the left vertical map is an equivalence. But the former is by Lemma 5.2.2 equivalent to (2), while the latter is equivalent to (1). $\square$

Corollary 5.2.5. *If $f\colon X \to Y$ is a left (resp. right) fibration and Y is a groupoid, then also X is a groupoid.* $\square$

Corollary 5.2.6. *Let $f\colon A \to B$ be a left (resp. right) fibration. Then every fiber of f is a groupoid.*

Proof. For every term b of B, the map $A_b \to \{b\}$ is again a left (resp. right) fibration, hence the claim follows from the previous corollary. $\square$

Covariant transport

For a left fibration $f\colon A \to B$, the fibers A_b are suitably *functorial* in the object $b \in B$: every morphism $\beta\colon b \to b'$ in B induces a functor $\beta_!\colon A_b \to A_{b'}$ between fibers, called the *covariant transport functor*. Dually, right fibrations admit *contravariant transport functors*. We will now construct these functors and prove various of their basic properties.

Definition 5.2.7 (Category of lifts)**.** Let $f\colon A \to B$ be a functor. Consider a term (a,β) of $A \vec{\times}_f B$ in context $\{(a,\beta)\}$, given by a term a of A and a morphism $\beta\colon f(a) \to b'$ in B. We define the category $\mathrm{Lift}_0^f(a,\beta)$ of *covariant lifts* as the fiber of $\vec{\mathrm{ev}}_0^f$ over (a,β):

$$\begin{tikzcd} \mathrm{Lift}_0^f(a,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],A) \arrow[d, "\vec{\mathrm{ev}}_0^f"] \\ \{(a,\beta)\} \arrow[r, "(a,\beta)"] & A \vec{\times}_f B. \end{tikzcd}$$

As discussed in Example 4.2.10, we may regard $\mathrm{Lift}_0^f(a,\beta)$ as a category in context $\{(a,\beta)\}$. A term of $\mathrm{Lift}_0^f(a,\beta)$ is called a *covariant lift of β starting in a* and consists of a morphism $\alpha\colon a \to a'$ in A whose source is a and whose image under f is β.

Dually, we define for every term (a', β) of $B \vec{\times}_f A$ the category $\mathrm{Lift}_1^f(a', \beta)$ of *contravariant lifts* as the fiber of $\vec{\mathrm{ev}}_1^f$ over (a', β):

$$\begin{tikzcd} \mathrm{Lift}_1^f(a',\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], A) \arrow[d, "\vec{\mathrm{ev}}_1^f"] \\ \{(a',\beta)\} \arrow[r, "(a',\beta)"] & B \vec{\times}_f A. \end{tikzcd}$$

A term of $\mathrm{Lift}_1^f(a', \beta)$ is called a *contravariant lift of β ending in a'* and consists of a morphism $\alpha \colon a \to a'$ in A whose target is a and whose image under f is β.

Corollary 5.2.8. *A functor $f \colon A \to B$ is a left fibration if and only if the category $\mathrm{Lift}_0^f(a, \beta)$ is contractible for every term (a, β) of $A \vec{\times}_f B$. Dually, f is a right fibration if and only if $\mathrm{Lift}_1^f(a', \beta)$ is contractible for every term (a', β) of $B \vec{\times}_f A$.*

Proof. This is an instance of Lemma 1.3.14, applied to the functor $\vec{\mathrm{ev}}_0^f \colon \mathrm{Fun}([1], A) \to A \vec{\times}_f B$ in context $\{(\alpha, \beta)\}$. □

Remark 5.2.9. After imposing some further axioms, we will show in Lemma 6.4.4 below that it in fact suffices to check the contractibility of $\mathrm{Lift}_0^f(a, \beta)$ only for *objects* (a, β).

In what follows, we will spell out various constructions in the case of left fibrations, and be more terse about the dual constructions for right fibrations.

Construction 5.2.10 (Covariant transport)**.** Let $f \colon A \to B$ be a left fibration. We denote by

$$\mathrm{lift}_0^f \colon A \vec{\times}_f B \to \mathrm{Fun}([1], A)$$

the inverse of the functor $\vec{\mathrm{ev}}_0^f \colon \mathrm{Fun}([1], A) \to A \vec{\times}_f B$. For a term (a, β) of $A \vec{\times}_f B$, we denote the morphism $\mathrm{lift}_0^f(a, \beta)$ in A by

$$\mathrm{lift}_0^f(a, \beta) \colon a \to \beta_!(a),$$

and refer to its target $\beta_!(a)$ as the *covariant transport of a along β*. Observe that $\mathrm{lift}_0^f(a, \beta)$ is a covariant lift of β starting in a, and in particular we see that $\beta_!(a)$ is a term of $A_{b'}$. Fixing a morphism $\beta \colon b \to b'$ of B, the assignment $a \mapsto \beta_!(a)$ thus provides a functor

$$\beta_! \colon A_b \to A_{b'}.$$

Dually, if f is instead a *right* fibration, we denote the inverse of $\vec{\mathrm{ev}}_1^f$ by lift_1^f. Its value at a term (a', β) of $B \vec{\times}_f A$ is denoted by $\mathrm{lift}_1^f(a', \beta) \colon \beta^*(a') \to a$ and we refer to $\beta^*(a')$ as the *contravariant transport of a' along β*. If we again fix a morphism $\beta \colon b \to b'$ of B, this construction assembles into a functor $\beta^* \colon A_{b'} \to A_b$.

Remark 5.2.11. We remind the reader that we may construct functors like $\beta_! \colon A_b \to A_{b'}$ simply by providing for every term of A_b a term of $A_{b'}$: a 'term' is defined as a functor $a \colon \Gamma \to A_b$ from an arbitrary category Γ, and Construction 5.2.10 provides for us another functor $\beta_!(a) \colon \Gamma \to A_{b'}$. Applying this to the identity functor $a = \mathrm{id}_{A_b} \colon A_b \to A_b$ thus provides the desired functor $\beta_! \colon A_b \to A_{b'}$.

Since this is the first time we use this method for constructing a functor, let us give an alternative definition of $\beta_! \colon A_b \to A_{b'}$ that does not use the language of terms. As a first step, we observe that the assignment $a \mapsto (a, \beta)$ extends to a functor $A_b \to A \vec{\times}_f B$, defined as the unique dashed arrow making the following diagram commute:

$$\begin{tikzcd}
A_b \arrow[r, dashed, "a \mapsto (a,\beta)"] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\lrcorner", very near start] & A \vec{\times}_f B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
\{\beta\} \arrow[r, "\beta"] & \mathrm{Fun}([1], B) \arrow[r, "\mathrm{ev}_0"] & B.
\end{tikzcd}$$

Here the outer rectangle is the commutative square defining the fiber A_b. We may now consider the composite

$$A_b \xrightarrow{a \mapsto (a,\beta)} A \vec{\times}_f B \xrightarrow{\mathrm{lift}_0^f} \mathrm{Fun}([1], A) \xrightarrow{\mathrm{ev}_1} A.$$

Observe that the composite of ev_1 with f is isomorphic to the composite $\mathrm{Fun}([1], A) \xrightarrow{\vec{\mathrm{ev}}_0^f} A \vec{\times}_f B \to \mathrm{Fun}([1], B) \xrightarrow{\mathrm{ev}_1} B$, and since lift_0^f is a section of $\vec{\mathrm{ev}}_1$ we see that this composite factors uniquely through the fiber $A_{b'}$. All in all we obtain a functor $\beta_! \colon A_b \to A_{b'}$.

The next proposition shows that the covariant transport functors $\beta_! \colon A_b \to A_{b'}$ are suitably 'functorial':

Proposition 5.2.12. *Let $f \colon A \to B$ be a left fibration.*

(1) For $\beta = \mathrm{id}_b \colon b \to b$, the functor $(\mathrm{id}_b)_! \colon A_b \to A_b$ is naturally isomorphic to the identity functor;

(2) For morphisms $\beta \colon b \to b'$ and $\beta' \colon b' \to b''$ in B, the composite functor

$$\beta'_! \circ \beta_! \colon A_b \to A_{b''}$$

is naturally isomorphic to $(\beta' \circ \beta)_! \colon A_b \to A_{b''}$.

The dual result for the contravariant transport functors of a right fibration is also valid.

Proof. For (1), observe that for any term $a : A_b$ the map $\mathrm{id}_a \colon a \to a$ is a covariant lift of $\mathrm{id}_b \colon b \to b$ starting in a. By uniqueness of covariant lifts, it follows that $\mathrm{id}_a \cong \mathrm{lift}_0^f(a, \mathrm{id}_b)$, and in particular we have $a \cong (\mathrm{id}_b)_! a$. Applying this to the universal term of A_b provides the desired natural isomorphism $(\mathrm{id}_b)_! \cong \mathrm{id}_{A_b}$.

For (2), we have for term $a : A_b$ a covariant lift

$$\mathrm{lift}_0^f(a, \beta) \colon a \to \beta_!(a)$$

of β starting in a, which in turn leads to a covariant lift

$$\mathrm{lift}_0^f(\beta_!(a), \beta') \colon \beta_!(a) \to \beta'_!(\beta_!(a)).$$

Since the composite of these two morphisms is a covariant lift of $\beta' \circ \beta$ starting in a, it follows that there is an isomorphism

$$\mathrm{lift}_0^f(\beta_!(a), \beta') \circ \mathrm{lift}_0^f(a, \beta) \cong \mathrm{lift}_0^f(a, \beta' \circ \beta),$$

and in particular we get $(\beta' \circ \beta)_!(a) \cong \beta'_!(\beta_!(a))$. Taking a to be the universal term of A_b then provides the desired natural isomorphism. $\square$

The covariant transport functors are compatible with maps of left fibrations:

Lemma 5.2.13. *Consider a morphism of left fibrations over B:*

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B. & \end{tikzcd}$$

Then there is for every morphism $\beta\colon b \to b'$ in B and every term a of A_b a natural isomorphism

$$\beta_!(\varphi_b(a)) \cong \varphi_{b'}(\beta_!(a))$$

of functors $A_b \to A'_{b'}$.

Proof. This is clear from the observation that the morphism

$$\varphi(\mathrm{lift}^f_0(a, \beta))\colon \varphi_b(a) \to \varphi_{b'}(\beta_!(a))$$

in A' is a covariant lift of β starting in $\varphi_b(a)$, so that it must be isomorphic to the morphism $\mathrm{lift}^f_0(\varphi_b(a), \beta)\colon \varphi_b(a) \to \beta_!(\varphi_b(a))$. □

Closure properties left/right fibrations

We now prove various preservation properties of left and right fibrations. While the results below are only stated for left fibrations, all of them admit dual versions for right fibrations whose formulations we leave to the reader.

Lemma 5.2.14. *Every equivalence $f\colon A \xrightarrow{\sim} B$ is a left fibration.*

Proof. This is immediate from Lemma 5.1.5. □

Proposition 5.2.15 (Base change)**.** *Consider a pullback square of categories*

$$\begin{tikzcd} A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B' \arrow[r, "v"] & B. \end{tikzcd}$$

(1) If f is a left fibration, then also f' is a left fibration;

(2) For a morphism $\beta\colon b \to b'$ in B', the following square commutes:

$$\begin{tikzcd} A'_b \arrow[r, "\beta_!"] \arrow[d, "u_b"', "\simeq"] & A'_{b'} \arrow[d, "\simeq"', "u_{b'}"] \\ A_{v(b)} \arrow[r, "v(\beta)_!"] & A_{v(b')}. \end{tikzcd}$$

Proof. Part (1) is immediate from Lemma 5.1.6, which says that the map $\vec{\mathrm{ev}}^{f'}_0$ is a base change of $\vec{\mathrm{ev}}^f_0$. This implies that the inverse map $\mathrm{lift}^{f'}_0$ is a base change of lift^f_0: given a morphism $\beta\colon b \to b'$ in B' and a term a of A'_b, the covariant lift $\mathrm{lift}\, f'_0\colon a \to \beta_!(a)$ of β starting in a is sent under u to the covariant lift $\mathrm{lift}^f_0\colon u(a) \to v(\beta)_!(u(a))$ of $v(\beta)\colon v(b) \to v(b')$ starting in $u(a)$. This proves part (2). □

Proposition 5.2.16 (Composition)**.** *Consider a commutative diagram*

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "gf"'] & & C \end{tikzcd}$$

of functors, and assume that g is a left fibration. Then f is a left fibration if and only if gf is a left fibration.

Proof. By Lemma 5.1.7, there is a commutative triangle

$$\begin{tikzcd} & \mathrm{Fun}([1], A) \arrow[dl, "\vec{\mathrm{ev}}_0^f"'] \arrow[dr, "\vec{\mathrm{ev}}_0^{gf}"] & \\ B \vec{\times}_f A \arrow[rr] & & C \vec{\times}_{gf} A, \end{tikzcd}$$

where the bottom map is a base change of the map $\vec{\mathrm{ev}}_0^g$ and hence is an equivalence as g is a left fibration. It follows by 2-out-of-3 that the map $\vec{\mathrm{ev}}_0^f$ is an equivalence if and only if $\vec{\mathrm{ev}}_0^{gf}$ is an equivalence. This finishes the proof. □

Lemma 5.2.17. *The class of left fibrations is stable under retracts.*

Proof. This is immediate from Lemma 5.1.8. □

Lemma 5.2.18. *Let $f\colon C \to D$ be a left fibration. Then for every category E also the induced functor $f_*\colon \mathrm{Fun}(E, C) \to \mathrm{Fun}(E, D)$ is a left fibration.*

Proof. Observe that the directed evaluation map for f_* is obtained by applying $\mathrm{Fun}(E, -)$ to the directed evaluation map of f, and hence is again an equivalence. □

5.3 Slice categories

In this section we provide two fundamental examples of left and right fibrations: the forgetful functors $C_{/x} \to C$ and $C_{x/} \to C$ for a category C and a term x of C. We further show that resulting functoriality on the fibers $C(x, y)$ of these two fibrations is given by precomposition and postcomposition in C, respectively.

Recall first from Definition 1.6.6 the definitions of the slice categories $C_{/x}$ and $C_{x/}$:

$$\begin{tikzcd} C_{/x} \arrow[r] \arrow[d, "p_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ C \arrow[r, "\mathrm{id}_C \times x"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \times C \arrow[d, "\mathrm{pr}_2"] \\ \{x\} \arrow[r, "x"] & C. \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_{x/} \arrow[r] \arrow[d, "q_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ C \arrow[r, "x \times \mathrm{id}_C"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \times C \arrow[d, "\mathrm{pr}_1"] \\ \{x\} \arrow[r, "x"] & C. \end{tikzcd}$$

Remark 5.3.1. The slice category $C_{/x}$ is a special case of the more general slice category construction $C_{/\psi}$ from Construction 3.7.1 by taking $Y = *$. It is also a special case of the directed pullback construction 5.1.1 by taking $f = \mathrm{id}_C \colon C \to C$ and $g = x \colon * \to C$.

Lemma 5.3.2. *Consider categories C and D, let x be a term of C, and let $\tilde{x} \in \mathrm{Fun}(D,C)$ denote the constant functor on x. Then there are equivalences*

$$\mathrm{Fun}(D, C_{/x}) \xrightarrow{\sim} \mathrm{Fun}(D,C)_{/\tilde{x}} \qquad \text{and} \qquad \mathrm{Fun}(D, C_{x/}) \xrightarrow{\sim} \mathrm{Fun}(D,C)_{\tilde{x}/}.$$

Proof. Since $\mathrm{Fun}(D,-)$ preserves pullbacks, we have a pullback square

$$\begin{tikzcd}
\mathrm{Fun}(D, C_{/x}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], \mathrm{Fun}(D,C)) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
\mathrm{Fun}(D,C) \times \{\tilde{x}\} \arrow[r, "\mathrm{id}_{\mathrm{Fun}(D,C)} \times \tilde{x}"] & \mathrm{Fun}(D,C) \times \mathrm{Fun}(D,C),
\end{tikzcd}$$

showing that $\mathrm{Fun}(D, C_{/x}) \xrightarrow{\sim} \mathrm{Fun}(D,C)_{/\tilde{x}}$. The equivalence $\mathrm{Fun}(D, C_{x/}) \xrightarrow{\sim} \mathrm{Fun}(D,C)_{\tilde{x}/}$ is analogous. □

The following is the main result of this section:

Theorem 5.3.3. *Let C be a category and let x be a term of C. Then the forgetful functor $p_x \colon C_{/x} \to C$ is a right fibration. Dually, the map $q_x \colon C_{x/} \to C$ is a left fibration.*

Proof. By symmetry it will suffice to prove the claim for $C_{/x} \to C$. We have to show that the functor

$$\vec{\mathrm{ev}}_1^{p_x} \colon \mathrm{Fun}([1], C_{/x}) \to C \vec{\times}_{p_x} C_{/x}$$

is an equivalence. By Lemma 5.3.2, the source of this isofibration sits in a pullback square of the form

$$\begin{tikzcd}
\mathrm{Fun}([1], C_{/x}) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & & \mathrm{Fun}([1], \mathrm{Fun}([1], C)) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
\mathrm{Fun}([1], C) \times \{x\} \arrow[r, "1 \times x"] & \mathrm{Fun}([1], C) \times C \arrow[r, "1 \times p_{[1]}^*"] & \mathrm{Fun}([1], C) \times \mathrm{Fun}([1], C),
\end{tikzcd}$$

and as a consequence of Corollary 1.7.7 we thus obtain a pullback square

$$\begin{tikzcd}
\mathrm{Fun}([1], C_{/x}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, C) \arrow[d, "\mathrm{ev}_2"] \\
\{x\} \arrow[r, "x"] & C.
\end{tikzcd}$$

On the other hand, the target $C \vec{\times}_{p_x} C_x$ is defined via the pullback square

$$\begin{tikzcd}
C \vec{\times}_{p_C} C_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], C) \times \{x\} \times \mathrm{id} \arrow[d, "\mathrm{ev}_1"] \\
C_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \times \{x\} \arrow[d, "1 \times x"] \\
\mathrm{Fun}([1], C) \arrow[r, "(\mathrm{ev}_0, \mathrm{ev}_1)"] & C \times C,
\end{tikzcd}$$

and consequently we obtain a pullback square

$$\begin{tikzcd}
C \vec{\times}_{pC} C_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Lambda^2_1, C) \arrow[d, "\mathrm{ev}_2"] \\
\{x\} \arrow[r, "x"] & C.
\end{tikzcd}$$

Under these two identifications, the map $\vec{\mathrm{ev}}_1^{p_x} \colon \mathrm{Fun}([1], C_{/x}) \to C \vec{\times}_{p_x} C_{/x}$ is obtained by pulling back the map $\mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Lambda^2_1, C)$ along $x \colon \{x\} \to C$, and thus it is an equivalence by the Segal axiom E. □

Remark 5.3.4. In more down-to-earth terms, the above proof is capturing the equivalence between three ways of encoding maps in the slice $C_{/x}$:

$$\begin{tikzcd}
y \arrow[r] \arrow[d] & z \arrow[d] \\
x \arrow[r, equal] & x
\end{tikzcd}
\overset{\text{Corollary 1.7.7}}{\Longleftrightarrow}
\begin{tikzcd}
 & z \arrow[dr] & \\
y \arrow[ur] \arrow[rr] & & x
\end{tikzcd}
\overset{\text{Axiom E}}{\Longleftrightarrow}
\begin{tikzcd}
 & z \arrow[dr] & \\
y \arrow[ur] & & x.
\end{tikzcd}$$

Corollary 5.3.5. *For two terms x and y in a category C, the hom groupoid $C(x, y)$ is a groupoid.*

Proof. Since $C(x, y)$ is the fiber over x of the right fibration $C_{/y} \to C$ from Theorem 5.3.3, this follows from Corollary 5.2.6. □

Since the fiber of the left fibration $C_{x/} \to C$ over a term y of C is the hom groupoid $C(x, y)$ from Definition 1.6.3, Construction 5.2.10 produces for every morphism $g \colon y \to z$ a covariant transport functor

$$g_! \colon C(x, y) \to C(x, z).$$

This functor may be identified with a functor we have already encountered:

Proposition 5.3.6. *For every morphism $g \colon y \to z$, the functor $g_! \colon C(x, y) \to C(x, z)$ agrees with the composition functor*

$$g \circ - \colon C(x, y) \to C(x, z)$$

from Construction 1.7.4. Dually, the contravariant transport functor $g^* \colon C(z, x) \to C(y, x)$ associated to the right fibration $C_{/x} \to C$ agrees with the composition functor $- \circ g \colon C(z, x) \to C(y, x)$.

Proof. As we have seen in the proof of Theorem 5.3.3, the directed evaluation map $\vec{\mathrm{ev}}_0^{q_x} \colon \mathrm{Fun}([1], C_{x/}) \to C \vec{\times}_{q_x} C_{x/}$ is a base change of the map $\mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Lambda^2_1, C)$, and hence the map $\mathrm{lift}_0^{q_x}$ is given by the inverse of this map. But this inverse precisely defines composition in C. In more detail, fixing the morphism $g \colon y \to z$, the covariant transport functor $g_!$ is given as the composite

$$C(x, y) \xrightarrow{f \mapsto (f, g)} C \vec{\times}_{q_x} C_{x/} \xrightarrow{\mathrm{lift}_0^{q_x}} \mathrm{Fun}([1], C_{x/}) \xrightarrow{\mathrm{ev}_1} C_{x/},$$

where one checks that this lands in the fiber $C(x, z)$ over z. Due to the above identification of the middle map, we may rewrite this as

$$\begin{aligned}
C(x, y) \to \mathrm{Fun}([1], C) &\xrightarrow{f \mapsto (f, g)} \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C) \\
&\simeq \mathrm{Diag}(\Lambda^2_1, C) \xrightarrow{\sim} \mathrm{Diag}(\Delta^2, C) \xrightarrow{\mathrm{ev}_{\{1,2\}}} \mathrm{Fun}([1], C),
\end{aligned}$$

where again one checks that this lands in $C(x,z)$. But this is precisely the map $g \circ - \colon C(x,y) \to C(x,z)$. □

The previous results directly imply versions for the *relative slice categories* $C_{/d}$ for a functor $f\colon C \to D$ and a term d of D:

Definition 5.3.7 (Relative slice)**.** Given a functor $f\colon C \to D$ and a term d of D we define the *relative slices* $C_{/d}$ and $C_{d/}$ via the following pullback squares:

$$\begin{tikzcd} C_{/d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D_{/d} \arrow[d] \\ C \arrow[r, "f"] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_{d/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D_{d/} \arrow[d] \\ C \arrow[r, "f"] & D. \end{tikzcd}$$

Corollary 5.3.8. *For any functor $f\colon C \to D$ and any term d of D, the functor $C_{/d} \to C$ is a right fibration and the functor $C_{d/} \to C$ is a left fibration.*

Proof. This is immediate from Theorem 5.3.3 and Proposition 5.2.15. □

Corollary 5.3.9. *For a functor $\psi\colon X \to C$, the forgetful functor $C_{/\psi} \to C$ from Construction 3.7.1 is a right fibration. Dually, $C_{\psi/} \to C$ is a left fibration.*

Proof. We prove the first statement; the second statement is dual. Unwinding definitions, we see that the category $C_{/\psi}$ sits in a pullback square of the form

$$\begin{tikzcd} C_{/\psi} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(Y,C)_{/\psi} \arrow[d] \\ C \arrow[r, "p_Y^*"] & \mathrm{Fun}(Y,C), \end{tikzcd}$$

hence the result is a special case of the previous corollary. □

Proposition 5.3.10. *Let C be a category. Then the commutative diagram*

$$\begin{tikzcd} \mathrm{Fun}([1],C) \arrow[rr, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \arrow[dr, "\mathrm{ev}_1"'] & & C \times C \arrow[dl, "\mathrm{pr}_2"] \\ & C & \end{tikzcd}$$

determines a right fibration in context C. Dually, the diagram

$$\begin{tikzcd} \mathrm{Fun}([1],C) \arrow[rr, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \arrow[dr, "\mathrm{ev}_0"'] & & C \times C \arrow[dl, "\mathrm{pr}_1"] \\ & C & \end{tikzcd}$$

determines a left fibration in context C.

Proof. We prove the first statement, the second statement being dual. For the purpose of this proof, consider the category $D := C_C$ in context C corresponding to the category $(C \times C \xrightarrow{\mathrm{pr}_2} C)$ over C. The diagonal map $\Delta\colon C \to C \times C$ determines an absolute object x of D in context C. We claim that the map $D_{/x} \to D$ in context C corresponds to the functor $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}([1], C) \to C \times C$ over C from the proposition, so that the statement is a special case of Theorem 5.3.3 applied in context C. To see this, notice that the map $D_{/x} \to D$ is defined by the following pullback square in context C:

$$\begin{tikzcd} D_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}_C([1]_C, D) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ D \arrow[r, "(\mathrm{id}_D, x)"] & D \times D. \end{tikzcd}$$

When translating this to categories over C, the arrow category $\mathrm{Fun}_C([1]_C, D)$ of D in context C corresponds to the functor $\mathrm{pr}_2\colon \mathrm{Fun}([1], C) \times C \to C$, and the above pullback square turns into the following pullback square over C:

$$\begin{tikzcd} \mathrm{Fun}([1], C) \arrow[r] \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C) \times C \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)\times \mathrm{id}_C"] \\ C \times C \arrow[r, "\mathrm{id}_C \times \Delta"] & C \times C \times C, \end{tikzcd}$$

where the fact that the pullback is $\mathrm{Fun}([1], C)$ follows from an easy instance of the pasting lemma of pullback squares. We conclude that the right fibration $D_{/x} \to D$ in context C corresponds to the functor $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}([1], C) \to C \times C$ over C, finishing the proof. $\square$

5.4 Adjunctions

The concept of an adjunction is fundamental in category theory, hence we will need an analogue in our setup of synthetic category theory. Given two functors $f\colon C \to D$ and $g\colon D \to C$ between ordinary categories, there are two equivalent ways to encode the data of an adjunction $f \dashv g$ between f and g:

(1) One may provide the data of a unit $\eta\colon \mathrm{id}_C \to gf$ and a counit $\varepsilon\colon fg \to \mathrm{id}_D$ satisfying the *triangle identities*;

(2) One may provide a natural isomorphism $D(fx, y) \xrightarrow{\sim} C(x, gy)$ for all x in C and y in D.

In this section, we will introduce synthetic analogues of these two definitions.

5.4.1 Units and counits

Our first definition of adjunctions will use units and counits.

Definition 5.4.1 (Adjunction)**.** Consider two functors $l\colon C \to D$ and $r\colon D \to C$. An *adjunction* $l \dashv r$ consists of natural transformations $\eta\colon \mathrm{id}_C \to rl$ and $\varepsilon\colon lr \to \mathrm{id}_D$ satisfying the *triangle identities*:

$$\begin{tikzcd} & lrl \arrow[dr, "\varepsilon l"] & \\ l \arrow[ur, "l\eta"] \arrow[rr, equal] & & l \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & rlr \arrow[dr, "r\varepsilon"] & \\ r \arrow[ur, "\eta r"] \arrow[rr, equal] & & r. \end{tikzcd}$$

We call η the *unit* of the adjunction and ε the *counit*. We often write $(l \dashv r)\colon C \rightleftarrows D$ to indicate the data $(l, r, \eta, \varepsilon)$ of an adjunction.

Warning 5.4.2. To obtain the correct *category of adjunctions*, Definition 5.4.1 needs to be modified as follows: an adjunction $l \vdash r$ consists of a natural transformation $\eta\colon \mathrm{id}_C \to rl$ and two natural transformations $\varepsilon, \varepsilon'\colon lr \to \mathrm{id}_D$ satisfying the triangle identities:

$$\begin{tikzcd} & lrl \arrow[dr, "\varepsilon l"] & \\ l \arrow[ur, "l\eta"] \arrow[rr, equal] & & l \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & rlr \arrow[dr, "r\varepsilon'"] & \\ r \arrow[ur, "\eta r"] \arrow[rr, equal] & & r. \end{tikzcd}$$

One may then deduce a posteriori that the counits ε and ε' are equivalent to each other, much like the left and right inverses of an equivalence are canonically equivalent.

Since we will not have much interest in the category of adjunctions in this course, we will keep working with the somewhat simpler and more familiar definition of an adjunction given in Definition 5.4.1.

Proposition 5.4.3. *Let $(l \vdash r)\colon C \rightleftarrows D$ and $(l' \vdash r')\colon D \rightleftarrows E$ be adjunctions. Then the composite functors $l'l\colon C \to E$ and $rr'\colon E \to C$ constitute an adjunction $(l'l \vdash rr')\colon C \rightleftarrows E$.*

Proof. The corresponding unit and counit are given as the composites

$$\mathrm{id}_C \xrightarrow{\eta} rl \xrightarrow{r\eta' l} rr'l'l \qquad \text{and} \qquad l'lrr' \xrightarrow{l'\varepsilon r'} l'r' \xrightarrow{\varepsilon'} \mathrm{id}_E\,.$$

We leave the verification of the triangle identity to the reader, see Exercise 5.13.1 □

Proposition 5.4.4. *Let $(l \vdash r)\colon C \leftrightarrows D$ be an adjunction. Then for every category E we obtain adjunctions*

$$l_*\colon \mathrm{Fun}(E, C) \rightleftarrows \mathrm{Fun}(E, D) : r_* \qquad \textit{and} \qquad r^*\colon \mathrm{Fun}(D, E) \rightleftarrows \mathrm{Fun}(C, D) : l^*.$$

Proof. We construct the first adjunction and leave the second adjunction to the reader. To construct the unit $\eta_*\colon \mathrm{id}_{\mathrm{Fun}(E,C)} \to r_* l_*$, we may by currying equivalently produce a transformation between the two curried functors $E \times \mathrm{Fun}(E, C) \to C$, which we do by whiskering the transformation $\eta\colon \mathrm{id}_C \to rl$ with the evaluation functor $\mathrm{ev}\colon E \times \mathrm{Fun}(E, C) \to C$. The counit $\varepsilon_*\colon l_* r_* \to \mathrm{id}_{\mathrm{Fun}(E,D)}$ is constructed similarly. The triangle identities for η_* and ε_* follow directly from the triangle identities of η and ε after whiskering. □

Proposition 5.4.5. *Adjoints are unique: if $(l \dashv r)\colon C \rightleftarrows D$ and $(l \dashv r')\colon C \rightleftarrows D$ are adjunctions, there is a preferred natural isomorphism $r \cong r'$.*

Proof. Consider the following two composites:

$$r \xrightarrow{\eta' r} r'lr \xrightarrow{r'\varepsilon} r' \qquad \text{and} \qquad r' \xrightarrow{\eta r'} rlr' \xrightarrow{r\varepsilon'} r.$$

These two maps can be shown to be inverse to each other, finishing the proof. We leave the details to the reader, see Exercise 5.13.2 □

Proposition 5.4.6 (cf. [Uni13, Theorem 4.2.3])**.** *Let $f\colon C \to D$ be an equivalence with inverse $g\colon D \to C$. Then there exists an adjunction $f \dashv g$ between f and g.*

Proof. By assumption, there are natural isomorphisms $\eta\colon \mathrm{id}_C \xrightarrow{\sim} gf$ and $\varepsilon\colon fg \xrightarrow{\sim} \mathrm{id}_D$. We define a new transformation ε' as the composite

$$fg \xrightarrow{\varepsilon^{-1}fg} fgfg \xrightarrow{f\eta^{-1}g} fg \xrightarrow{\varepsilon} \mathrm{id}_D\,.$$

We claim that the transformations η and ε' satisfy the two triangle identities. To show that the composite $f \xrightarrow{f\eta} fgf \xrightarrow{\varepsilon' f} f$ is isomorphic to the identity on f, we consider the following commutative diagram:

$$\begin{tikzcd}
 & fgf \arrow[r, "\varepsilon^{-1}fgf"] \arrow[rrrd, bend left, "\varepsilon' f"] & fgfgf \arrow[dr, "f\eta^{-1}gf"] & & \\
f \arrow[ur, "f\eta"] \arrow[r, "\varepsilon^{-1}f"] \arrow[rrrr, bend right, equal] & fgf \arrow[ur, "fgf\eta"] \arrow[r, "f\eta^{-1}"] \arrow[rr, bend right, equal] & f \arrow[r, "f\eta"] & f \arrow[r, "\varepsilon f"] & f.
\end{tikzcd}$$

The top composite is $(\varepsilon' f) \circ (\varepsilon' f)$. The two faces of the diagram commute by naturality.

To show that the composite $g \xrightarrow{\eta g} gfg \xrightarrow{g\varepsilon'} g$ is isomorphic to the identity on g, consider the following commutative diagram:

$$\begin{tikzcd}
 & gfg \arrow[r, "g\varepsilon'"] \arrow[dr, "\eta gfg"] & g \arrow[r, "\eta g"] & gfg \arrow[dr, "g\varepsilon'"] & \\
g \arrow[ur, "\eta g"] \arrow[dr, "\eta g"'] & & gfgfg \arrow[ur, "gfg\varepsilon'"] \arrow[dr, "g\varepsilon' fg"] & & g \\
 & gfg \arrow[ur, "gf\eta g"] \arrow[rr, equal] & & gfg. \arrow[ur, "g\varepsilon"] &
\end{tikzcd}$$

The three squares each commute by naturality and the bottom triangle commutes by the triangle identity established above. Therefore, the diagram expresses an isomorphism

$$(g\varepsilon') \circ (\eta g) \circ (g\varepsilon') \circ (\eta g) \cong (g\varepsilon') \circ (\eta g).$$

Since the transformations ηg and $g\varepsilon'$ are invertible, we may cancel them to obtain the desired isomorphism $(g\varepsilon') \circ (\eta g) \cong \mathrm{id}_g$. This finishes the proof. □

5.4.2 Adjunctions via hom groupoids

As explained in the introduction of this section, there is a second common way to define adjunctions $(l \dashv r)\colon C \rightleftarrows D$ between categories: rather than providing unit and counit transformations, one may also provide a natural equivalence

$$D(lx, y) \xrightarrow{\sim} C(x, ry)$$

between hom groupoids for all x in C and y in D.

Proposition 5.4.7. *Let $l\colon C \to D$ and $r\colon D \to C$ be functors. Then the following three conditions are equivalent:*

(1) There exist a unit $\eta\colon \mathrm{id}_C \to rl$ and counit $\varepsilon\colon lr \to \mathrm{id}_D$ which exhibit r as a right adjoint of l;

(2) For every term $(x, y)\colon C \times D$ there is an equivalence $\alpha_{x,y}\colon C(lx, y) \xrightarrow{\sim} D(x, ry)$.

(3) There exists an equivalence $\alpha\colon C \vec{\times}_l D \xrightarrow{\sim} C \vec{\times}_r D$ over $C \times D$.

Proof. It is clear that (2) and (3) are equivalent: (3) follows from (2) by taking (x, y) to be the universal term of $C \times D$, while (2) follows from (3) by pulling back the equivalence along $(x, y)\colon \{(x, y)\} \to C \times D$.

To show that (1) implies (2), assume that the data (η, ε) of an adjunction has been given, and consider terms $x : C$ and $y : D$. We construct the map $\alpha_{x,y}\colon D(lx, y) \to C(x, ry)$ as the following composite:

$$\alpha_{x,y}\colon D(lx, y) \xrightarrow{r_*} C(rlx, ry) \xrightarrow{\eta_x^*} C(x, ry).$$

Here the first map is the map induced by r from Construction 1.6.4, while the second map is defined as the contravariant transport functor along the unit map $\eta_x\colon x \to rlx$ applied to the right fibration $C_{/ry} \to C$ from Theorem 5.3.3. Dually, we may construct a map $\beta_{x,y}\colon C(x, ry) \to D(lx, y)$ as the following composite:

$$\beta_{x,y}\colon C(x, ry) \xrightarrow{l_*} D(lx, lry) \xrightarrow{(\varepsilon_y)_!} D(lx, y).$$

Here the first map is induced by l and the second map is defined as the covariant transport functor along the counit map $\varepsilon_y\colon lry \to y$. We will now show that $\beta_{x,y}$ is an inverse to $\alpha_{x,y}$. We will show that $\beta_{x,y} \circ \alpha_{x,y} \sim \mathrm{id}_{D(lx,y)}$, leaving the other case to the reader. To this end, consider the following diagram:

$$\begin{tikzcd}
D(lx, y) \arrow[r, "r_*"] \arrow[rr, bend left, "\alpha_{x,y}"] \arrow[dr, "(lr)_*"'] \arrow[ddr, bend right, "\varepsilon_{lx}^*"'] & C(rlx, ry) \arrow[r, "\eta_x^*"] \arrow[d, "l_*"'] \arrow[dd, bend left=60, "\beta_{rlx,y}"] & C(x, ry) \arrow[dd, "\beta_{x,y}"] \\
& D(lrlx, lry) \arrow[d, "(\varepsilon_y)_!"'] & \\
& D(lrlx, y) \arrow[r, "(l\eta_x)^*"'] & D(lx, y).
\end{tikzcd}$$

The composite along the top right is $\beta \circ \alpha$. We claim that the bottom composite is the identity functor. To see this, notice that by Proposition 5.2.12 the bottom composite is isomorphic to the functor $(\varepsilon_l x \circ l\eta_x)^*$. By the triangle identity, we may identify this with the functor $(\mathrm{id}_{lx})^*$. It thus follows from another application of Proposition 5.2.12 that this is indeed the identity on $D(lx, y)$.

We will now show that the diagram commutes. The square on the right commutes by applying (the dual of) Lemma 5.2.13 to the morphism $\beta\colon C_{/ry} \to C \vec{\times}_l D$ of right fibrations over C. Here we have used Proposition 5.2.15 to identify the bottom horizontal map. The top left triangle is easily seen to commute. It thus remains to see that the triangle on the bottom left commutes. Recall from Proposition 5.3.6 that the functors $(\varepsilon_y)_!$ and ε_{lx}^* are given by postcomposition with $\varepsilon_y\colon lrx \to y$

and precomposition with $\varepsilon_{lx}\colon lrlx \to lx$, respectively. The claim is thus equivalent to the claim that for every morphism $\varphi\colon lx \to y$, the following square commutes in D:

$$\begin{tikzcd} lrlx \arrow[r, "lr(\varphi)"] \arrow[d, "\varepsilon_{lx}"'] & lry \arrow[d, "\varepsilon_y"] \\ lx \arrow[r, "\varphi"] & y. \end{tikzcd}$$

But this is an instance of naturality of the transformation ε. We conclude that $\beta \circ \alpha \sim \mathrm{id}_{D(lx,y)}$. The proof that $\alpha \circ \beta \sim \mathrm{id}_{C(x,ry)}$ is entirely analogous and is left to the reader.

We now show that (3) implies (1). Consider an equivalence $\alpha\colon C \vec{\times}_l D \xrightarrow{\sim} C \vec{\times}_r D$ over $C \times D$. We define the unit $\eta\colon \mathrm{id}_C \to rl$ as the composite

$$C \to C \vec{\times}_l D \simeq C \vec{\times}_r D \to \mathrm{Fun}([1], C).$$

The fact that the source and target of this natural transformation are indeed id_C and rl, respectively, follows from the following commutative diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\Delta"'] & C \vec{\times}_l D \arrow[r, "\alpha"] \arrow[d] & C \vec{\times}_r D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ C \times C \arrow[r, "\mathrm{id}_C \times l"] & C \times D \arrow[r, equal] & C \times D \arrow[r, "\mathrm{id}_C \times r"] & C \times C. \end{tikzcd}$$

Similarly, we define the counit $\varepsilon\colon lr \to \mathrm{id}_D$ as the composite

$$D \to C \vec{\times}_r D \simeq C \vec{\times}_l D \to \mathrm{Fun}([1], C).$$

We now need to verify the two triangle identities. By symmetry, it will suffice to show one of the triangle identities: we will show that the composite

$$r \xrightarrow{\eta r} rlr \xrightarrow{r\varepsilon} r$$

is isomorphic to the identity on r. Picking an arbitrary term y of D, we must show that the composite $ry \xrightarrow{\eta_{ry}} rlr \xrightarrow{r\varepsilon_y} ry$ is the identity on ry. As a consequence of Lemma 5.2.13 we see that the following diagram commutes:

$$\begin{tikzcd} D(lry, lry) \arrow[r, "\alpha_{lry,lry}"] \arrow[d, "(\varepsilon_y)_!"'] & C(ry, rlry) \arrow[d, "(r\varepsilon_y)_!"] \\ D(lry, y) \arrow[r, "\alpha_{lry,y}"] & D(ry, ry). \end{tikzcd}$$

But by Proposition 5.3.6, the two covariant transport functors $(\varepsilon_y)_!$ and $(r\varepsilon_y)_!$ are given by post-composition with ε_y and $r\varepsilon_y$ respectively. Since $\alpha_{lry,lry}(\mathrm{id}_{rly}) = \eta_{ry}$ and $\alpha_{lry,y}(\varepsilon_y) = \mathrm{id}_{ry}$ by definition of η and ε, plugging in the map $\mathrm{id}_{rly}\colon rly \to rly$ in the top left corner thus gives the desired isomorphism

$$r\varepsilon_y \circ \eta_{ry} = (r\varepsilon_y)_!(\alpha_{lry,lry}(\mathrm{id}_{rly})) \cong \alpha_{lry,y}((\varepsilon_y)_!(\mathrm{id}_{rly})) = \alpha_{lry,y}(\varepsilon_y) = \mathrm{id}_{ry}\,.$$

This finishes the proof. $\square$

5.5 Initial/terminal objects and left/right adjoint sections

In this section, we introduce the notion of an *initial object* of a category. More generally, we consider the notion of a *left adjoint section* of a functor, which we may think of as a fiberwise analogue of initial objects. Dually we obtain notions of *terminal objects* and *right adjoint sections*.

5.5.1 Left and right adjoint sections

Definition 5.5.1 (Left/right adjoint section)**.** Let $f\colon C \to D$ be a functor. A *left adjoint section* of f is a left adjoint $s\colon D \to C$ such that the unit transformation $\eta\colon \mathrm{id}_D \to fs$ is a natural isomorphism. Dually, a *right adjoint section* of f is a right adjoint $s\colon D \to C$ such that the counit transformation $\varepsilon\colon fs \to \mathrm{id}_D$ is a natural isomorphism.

In other words, f admits a left adjoint section if and only if it admits a section $s\colon D \to C$ (i.e. a functor satisfying $fs \cong \mathrm{id}_D$) which comes equipped with a natural transformation $\varepsilon\colon sp \to \mathrm{id}_C$ of functors $C \to C$ satisfying the following two conditions:

- The map ε is a transformation living over D: the induced map $f\varepsilon\colon f \cong fsf \to f$ is naturally isomorphic to the identity transformation.
- The map ε restricts to the identity on D: the map $\varepsilon s\colon s \cong sps \to s$ is naturally isomorphic to the identity transformation.

Definition 5.5.2 (Left/right Bousfield localization)**.** A functor $f\colon C \to D$ is called a *left Bousfield localization* if it admits a right adjoint section. We call f a *right Bousfield localization* if it admits a left adjoint section.

We will see in Proposition 6.1.7 that a left/right adjoint section $s\colon D \to C$ of a functor $f\colon C \to D$ is automatically a full subcategory. We may then think of f as reflecting the category C into this subcategory D.

Lemma 5.5.3. *Every equivalence $C \xrightarrow{\sim} D$ is both a left and a right Bousfield localization.*

Proof. This is immediate from Proposition 5.4.6. □

Lemma 5.5.4. *For a category C, the target functor $\mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$ is a left Bousfield localization with right adjoint section given by*

$$s = p_{[1]}^*\colon C \to \mathrm{Fun}([1], C).$$

Dually, the source functor $\mathrm{ev}_0\colon \mathrm{Fun}([1], C) \to C$ is a right Bousfield localization.

Proof. We prove the claim for ev_1; the claim for ev_0 is dual. Consider the following functor:

$$\mathrm{Fun}([1], C) \times [1] \to \mathrm{Fun}([1], C); \qquad (f, t) \mapsto (s \mapsto f(\max(t, s))).$$

Using Lemma 1.6.10, we see that $f(\max(0, y)) = f(y)$ and $f(\max(1, y)) = f(1)$, so that this assignment defines a natural transformation

$$\eta\colon \mathrm{id}_{\mathrm{Fun}([1],C)} \to s\mathrm{ev}_1$$

of functors $\mathrm{Fun}([1], C) \to \mathrm{Fun}([1], C)$. It also follows from Lemma 1.6.10 that:

- The evaluation of η at 1 gives the identity of ev_1;
- Applying η to $s(c)$ gives the identity of $s(c)$.

This shows that η exhibits s as a right adjoint section of ev_1, finishing the proof. □

We can relax the condition on the transformation ε as follows:

Proposition 5.5.5. *Let $f: C \to D$ be a functor. Assume that there exists a section $s: D \to C$ of f and a natural transformation $\varepsilon: sf \to \mathrm{id}_C$ over D such that the transformation $\varepsilon s: s \cong sfs \to s$ is a natural isomorphism. Then s is a left adjoint section of f.*

The analogous statement for right adjoint sections also holds.

Proof. Let $(\varepsilon s)^{-1}: s \to s$ be an inverse of $\varepsilon s: s \to s$. We define a new transformation $\varepsilon': sf \to \mathrm{id}_C$ as the following composite:

$$sf \xrightarrow{(\varepsilon s)^{-1} f} sf \xrightarrow{\varepsilon} \mathrm{id}_C .$$

This is again a transformation over D, and it satsifies

$$\varepsilon' s = (\varepsilon s) \circ ((\varepsilon s)^{-1} fs) \cong (\varepsilon s) \circ (\varepsilon s)^{-1} \cong \mathrm{id}_C .$$

This shows that the map $\varepsilon': sf \to \mathrm{id}_C$ exhibits s as a left adjoint section of f. □

Left and right Bousfield localizations are closed under a variety of operations.

Proposition 5.5.6 (Composition)**.** *Left (resp. right) Bousfield localizations are closed under composition.*

Proof. By Proposition 5.4.3 the composite functors form again an adjunction, and it is immediate from the descriptions of the unit and counit in the proof that the composite is again a left/right Bousfield localization. □

Proposition 5.5.7 (Functor categories)**.** *Let $f: C \to D$ be a left (resp. right) Bousfield localization. Then also the functor $f_*: \mathrm{Fun}(A, C) \to \mathrm{Fun}(A, D)$ is a left (resp. right) Bousfield localization for any category A.*

Proof. This is immediate from Proposition 5.4.4. □

Proposition 5.5.8 (Base change)**.** *Left (resp. right) Bousfield localizations are stable under base change.*

Proof. We prove the claim for right Bousfield localizations; the claim for left Bousfield localizations is dual. Consider a pullback square

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "f"] \\ D' \arrow[r, "v"] & D \end{tikzcd}$$

where f is a right Bousfield localization. Let (s, ε) be a left adjoint section of f. By the universal property of the pullback, we obtain a section $s'\colon D' \to C'$ of f' satisfying $us' = sv$:

$$\begin{tikzcd} D' \arrow[r, dashed, "s'"] \arrow[rr, bend left, "sv"] \arrow[dr, equal] & C' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ & D' \arrow[r, "v"] & D. \end{tikzcd}$$

Similarly, we define the map $\varepsilon'\colon C' \times [1] \to C'$ over D' via the following commutative diagram:

$$\begin{tikzcd} C' \times [1] \arrow[r, dashed, "\varepsilon'"] \arrow[rr, bend left, "\varepsilon\circ(u\times 1)"] \arrow[dr, "f'\circ\mathrm{pr}_1"'] & C' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ & D' \arrow[r, "v"] & D. \end{tikzcd}$$

Note that ε' is indeed a transformation from $s'f'$ to $\mathrm{id}_{C'}$ over D'. Finally, we construct the desired commutative diagram $\sigma'\colon D' \times [2] \to C'$ via the following commutative diagram:

$$\begin{tikzcd} D' \times [2] \arrow[r, dashed, "\sigma'"] \arrow[rr, bend left, "\sigma\circ(v\times 1)"] \arrow[dr, "\mathrm{pr}_1"'] & C' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ & D' \arrow[r, "v"] & D. \end{tikzcd}$$

□

Proposition 5.5.9 (Retracts)**.** *Left and right Bousfield localizations are stable under retracts: given a diagram*

$$\begin{tikzcd} C \arrow[r] \arrow[rr, bend left, "\mathrm{id}_C"] \arrow[d, "f"'] & C' \arrow[r] \arrow[d, "f'"] & C \arrow[d, "f"] \\ D \arrow[r] \arrow[rr, bend right, "\mathrm{id}_D"'] & D' \arrow[r] & D, \end{tikzcd}$$

if f' is a left/right Bousfield localization then so is f.

Proof. The proof is similar to the previous result, and we leave the proof as an exercise to the reader, see Exercise 5.13.3. □

Proposition 5.5.10. *Let $f\colon C \to D$ be a left (resp. right) Bousfield localization. Then f is a localization, in the sense of Definition 3.3.6.*

Proof. Let $s\colon D \to C$ be a left adjoint section of f, with natural isomorphism $\eta\colon \mathrm{id}_D \cong fs$ and counit $\varepsilon\colon sf \to \mathrm{id}_C$. For any category E, the induced functor

$$f^*\colon \mathrm{Fun}(D, E) \to \mathrm{Fun}(C, E)$$

is a left adjoint section of $s^*\colon \mathrm{Fun}(C, E) \to \mathrm{Fun}(D, E)$, see Proposition 5.4.4. Let $W \hookrightarrow \mathrm{Map}([1], C)$ denote the collection of morphisms in C that are sent to isomorphisms in D under f, i.e.

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1], C) \arrow[d, "f_*"] \\ \mathrm{Iso}(D) \arrow[r, hook] & \mathrm{Map}([1], D). \end{tikzcd}$$

Since the map $f(\varepsilon_x)\colon f(s(f(x))) \to f(x)$ is an isomorphism by the triangle identity, it follows that the map $\varepsilon_x\colon s(f(x)) \to x$ lies in W for any object x of C. In particular, for any functor $F\colon C \to E$ in $\mathrm{Fun}^W(C, E)$, the counit of the adjunction $f^* \dashv s^*$ is a natural isomorphism: $f^*s^*F \xrightarrow{\cong} F$. It follows that this adjunction restricts to an adjunction

$$f^*\colon \mathrm{Fun}(D, E) \rightleftarrows \mathrm{Fun}^W(C, E) : s^*$$

in which both the unit and counit are an equivalence. We conclude that the induced functor

$$f^*\colon \mathrm{Fun}(D, E) \to \mathrm{Fun}^W(C, E)$$

is an equivalence, showing that f exhibits D as a localization of C at W as desired. □

5.5.2 Initial and terminal objects

The notion of leftr/right adjoint sections specializes to the notion of initial/terminal objects from Definition 1.6.7:

Lemma 5.5.11. *Let C be a category and let x be an absolute object of C. Then the following conditions are equivalent:*

(1) The object x is initial in the sense of Definition 1.6.7: the forgetful functor $C_{x/} \to C$ is an equivalence;

(2) The object x defines a left adjoint section $x\colon * \to C$ of the functor $p_C\colon C \to *$;

(3) There exists a transformation $\varepsilon\colon \mathrm{const}_x \to \mathrm{id}_C$ and a natural isomorphism $\varepsilon_x \cong \mathrm{id}_x$ of functors $* \to C$. Here const_x is defined as the composite $C \xrightarrow{p_C} * \xrightarrow{x} C$.

(4) There exists a transformation $\varepsilon\colon \mathrm{const}_x \to \mathrm{id}_C$ such that the map $\varepsilon_x\colon x \to x$ is an isomorphism in C.

The dual statement for terminal objects is also valid.

Proof. The equivalence between (1) and (2) is a special case of Proposition 5.4.7, applied to $l = x\colon * \to C$ and $r = p_C\colon C \to *$. To see that (2) is equivalent to (3), note that there is only a single functor $C \times [1] \to *$, and hence the unit transformation $\eta\colon \mathrm{id}_* \to p_C \circ x$ is unique. One of the two triangle identities is automatic, and the remaining triangle identity amounts to condition (3). The equivalence between (3) and (4) is a special case of Proposition 5.5.5. □

Remark 5.5.12. Let 0 be an initial object of C. Point (2) says that the functor $C_{0/} \to C$ is an equivalence, and in particular for every $y \in C$ the fiber $C(0, y)$ of this map over y is contractible. Informally, this says that for every y there exists a unique morphism $0 \to y$ in C.

We will show in Lemma 5.9.8 below that the converse also holds true, but proving this requires an additional axiom which we introduce in Section 5.9.

Lemma 5.5.13. *Let $f\colon C \to D$ be a right Bousfield localization with left adjoint section $s\colon D \to C$. For every object d of D, the object $s(d) \in C_d$ is an initial object of the fiber $C_d := \{d\} \times_C D$.*

The dual statement for left Bousfield localizations and terminal objects also holds.

Proof. It follows from Proposition 5.5.8 that the base change $f|_{C_d}\colon C_d \to \{d\}$ of f along $\{d\} \to D$ is again a right Bousfield localization with left adjoint section given by the restriction $s|_{\{d\}}\colon \{d\} \to C_d$. By definition this means that $s(d)$ is an initial object of C_d. □

We will now provide some basic examples of initial and terminal objects. The most important example will be the walking morphism:

Lemma 5.5.14. *The category* $[1]$ *has both an initial object* $0 : [1]$ *and a terminal object* $1 : [1]$.

Proof. This is true as Axiom C.2. □

This lets us compute all the Hom groupoids of $[1]$:

Corollary 5.5.15. *There are equivalences*

$$\mathrm{Hom}_{[1]}(0,0) \simeq *, \qquad \mathrm{Hom}_{[1]}(0,1) \simeq * \qquad \textit{and} \qquad \mathrm{Hom}_{[1]}(1,1) \simeq *.$$

Proof. This is immediate from Lemma 5.5.14. □

Corollary 5.5.16. *There is an equivalence* $\mathrm{Hom}_{[1]}(1,0) \simeq \emptyset$.

Proof. We first show that every morphism $f\colon 1 \to 0$ in $[1]$ is an isomorphism with inverse the canonical morphism $\mathrm{can}\colon 0 \to 1$. Indeed, by Corollary 5.5.15 there are unique homotopies $f \circ \mathrm{can} \sim \mathrm{id}_0$ and $\mathrm{can} \circ f \sim \mathrm{id}_1$, so that can provides both a left and a right inverse to f. Since $[1] \simeq \mathrm{Iso}([1]) \hookrightarrow \mathrm{Fun}([1],[1])$ is an embedding by Lemma 1.8.4, we deduce that $\mathrm{Hom}_{[1]}(1,0)$ sits in a pullback square as follows:

$$\begin{tikzcd}
\mathrm{Hom}_{[1]}(1,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[1]} \arrow[d, "\Delta"] \\
* \arrow[r, "{(1,0)}"] & {[1] \times [1].}
\end{tikzcd}$$

Furthermore, since $[1]^{\simeq} \hookrightarrow [1]$ is an embedding Corollary 2.2.12, the right-hand square in the following commutative diagram is a pullback square, and hence by the pasting law so must be the left-hand square:

$$\begin{tikzcd}
\mathrm{Hom}_{[1]}(1,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[1]^{\simeq}} \arrow[r] \arrow[d, "\Delta"] & {[1]} \arrow[d, "\Delta"] \\
* \arrow[r, "{(1,0)}"] & {[1]^{\simeq} \times [1]^{\simeq}} \arrow[r] & {[1] \times [1].}
\end{tikzcd}$$

But since $[1]^{\simeq} \simeq \{0,1\}$ is a disjoint union of two points, we see that

$$[1]^{\simeq} \times [1]^{\simeq} \simeq \{(0,0),(0,1),(1,0),(1,1)\}$$

is a disjoint union of four points, and the diagonal map corresponds to the inclusion

$$\{(0,0),(1,1)\} \hookrightarrow \{(0,0),(0,1),(1,0),(1,1)\}\,.$$

It thus follows from the universality of disjoint unions, Axiom B.4', that the pullback over the point $(1,0)$ is empty. □

Lemma 5.5.17. *For a term x of a category C, the slice category $C_{/x}$ admits a terminal object $(x, \mathrm{id}_x \colon x \to x)$. Similarly, $C_{x/}$ admits an initial object $(x, \mathrm{id}_x \colon x \to x)$.*

Proof. We showed in Lemma 5.5.4 that the evaluation map $\mathrm{ev}_1\colon \mathrm{Fun}([1],C) \to C$ admits a right adjoint section given by $p_{[1]}^*\colon C \to \mathrm{Fun}([1],C)$. Since $C_{/x}$ is the fiber over x of this map, the claim thus follows from Lemma 5.5.13. □

If one already knows the existence of an initial object x, then other initial objects are particularly easy to recognize:

Exercise 5.5.18. Let x and y be objects in a category C and assume that x is an initial object. Then y is an initial object if and only if the canonical map $x \to y$ from (the dual of) Remark 5.5.12 is an isomorphism.

The dual statement for terminal objects also holds.

Corollary 5.5.19. *All terminal objects of a category C are canonically isomorphic. Similarly, all initial objects of C are canonically isomorphic.* □

5.6 Cartesian and cocartesian fibrations

In Section 5.2 we introduced the notion of a *left fibration*: a functor $f\colon A \to B$ such that every morphism $\beta\colon b \to b'$ admits a unique covariant lift starting in a given lift a of b. The goal of this section is to introduce a variant of this notion, called a *cocartesian fibration*, in which the covariant lift is not necessarily unique but where there exists always an *initial* covariant lift with given source. Dually, we obtain a notion of *cartesian fibration*, in which there exists a *terminal* contravariant lift with given target.

Definition 5.6.1 ((Co)cartesian fibration)**.** An isofibration $f\colon A \to B$ is called a *cocartesian fibration* the directed evaluation map $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1],A) \to A \vec{\times}_f B$ admits a left adjoint section $\mathrm{lift}_0^f\colon A \vec{\times}_f B \to \mathrm{Fun}([1],A)$.

Dually, f is called a *cartesian fibration* if the map $\vec{\mathrm{ev}}_1^f\colon \mathrm{Fun}([1],A) \to B \vec{\times}_f A$ admits a right adjoint section $\mathrm{lift}_1^f\colon B \vec{\times}_f A \to \mathrm{Fun}([1],A)$.

Throughout this section, we will frequently state results only for cocartesian fibrations and leave their dual formulation for cartesian fibrations to the reader.

Example 5.6.2. The following are the first few basic examples of (co)cartesian fibrations:

- Every left fibration is a cocartesian fibration: the map $\vec{\mathrm{ev}}_0^f$ is an equivalence and thus in particular admits a left adjoint section by Lemma 5.5.3. Dually, every right fibration is a cartesian fibration.

- In particular, every equivalence is both a cartesian fibration as well as a cocartesian fibration.

- For a category C, the map $p_C\colon C \to *$ is both a cartesian fibration and a cocartesian fibration. Indeed, since the map $\vec{\mathrm{ev}}_0^{p_C}\colon \mathrm{Fun}([1], C) \to C \vec{\times}_{p_C} *$ is isomorphic to $\mathrm{ev}_0\colon \mathrm{Fun}([1], C) \to C$ by Remark 5.1.4, it admits left adjoint section by Lemma 5.5.4, showing that p_C is a cocartesian fibration. The cartesian case is dual.

- For categories C and D, the projection maps $\mathrm{pr}_1\colon C \times D \to C$ and $\mathrm{pr}_2\colon C \times D \to D$ are both a cartesian fibration and a cocartesian fibration. This can again be argued directly, but also follows from the previous example and Proposition 5.6.14 below.

5.6.1 Covariant transport

Cocartesian fibrations allow for a notion of *covariant transport* generalizing the situation for left fibration:

Notation 5.6.3 (Covariant transport)**.** For a term (a, β) of $A \vec{\times}_f B$, we denote the morphism $\mathrm{lift}_0^f(a, \beta)$ in A as

$$\mathrm{lift}_0^f(a, \beta)\colon a \to \beta_!(a)$$

and refer to its target $\beta_!(a)$ as the *covariant transport of a along* β. Observe that this map is a term of the category $\mathrm{Lift}_0^f(a, \beta)$ of covariant lifts of β starting in a, defined in Definition 5.2.7. In fact, $\mathrm{lift}_0^f(a, \beta)$ is even an *initial object* of this category, see Lemma 5.5.13.

Just like in Construction 5.2.10, the assignment $a \mapsto \beta_!(a)$ determines a functor

$$\beta_!\colon A_b \to A_{b'}.$$

called *covariant transport along* β.

Remark 5.6.4. When we write $\mathrm{lift}_0^f(a, \beta)$ we allow a and β to be a term of A and a morphism of B, respectively, defined not just in a global or groupoidal context, but in an arbitrary context Γ. Concretely, this means that for every category Γ and any solid commutative square of the form

$$\begin{tikzcd} \Gamma \times \{0\} \arrow[r, "a"] \arrow[d] & A \arrow[d, "f"] \\ \Gamma \times [1] \arrow[r, "\beta"] \arrow[ur, dashed] & B \end{tikzcd}$$

we have the dashed lift defined as the uncurrying of the composite

$$\Gamma \xrightarrow{(a,\beta)} A \vec{\times}_f B \xrightarrow{\mathrm{lift}_0^f} \mathrm{Fun}([1], A)$$

(where we are implicitly currying β). We call the dashed arrow a *cocartesian lift in context* Γ and simply denote it by $\mathrm{lift}_0^f(a, \beta)$. We can view it as a transformation $a \to \beta_!(a)$, whose codomain we call the *cocartesian transport in context* Γ of a along β.

Note that by construction, cocartesian lifts are compatible with change of context. Explicitly, for every $g\colon \Gamma' \to \Gamma$ we have $\mathrm{lift}_0^f(a,\beta)|_g = \mathrm{lift}_0^f(a|_g, \beta|_g)$, and analogously for cocartesian transport.

Finally, note that the definition of left adjoint section means precisely that for each term of $A \vec{\times}_f B$, i.e. each pair (a,β) in arbitrary context Γ as above, the object $\mathrm{lift}_0^f(a,\beta)$ is characterized as an initial object of the category $\mathrm{Lift}_0^f(a,\beta)$ in context Γ.

We would like to show that the covariant transport functors satisfy the relations $(\mathrm{id}_b)_! = \mathrm{id}$ and $(\beta' \circ \beta)_! = \beta'_! \circ \beta_!$ as in Proposition 5.2.12. While the statement for left fibrations was immediate due to uniqueness of covariant lifts, the statement for cocartesian fibrations is slightly more involved. We will need the following auxiliary notion of *cocartesian morphisms*:

Construction 5.6.5. Let $f\colon A \to B$ be a cocartesian fibration. Since the functor lift_0^f is a left adjoint section of $\vec{\mathrm{ev}}_0^f$, it comes equipped with a natural transformation $\varepsilon\colon \mathrm{lift}_0^f \circ \vec{\mathrm{ev}}_0^f \to \mathrm{id}_{\mathrm{Fun}([1],A)}$. In particular, for a morphism $\alpha\colon a \to a'$ in A with underlying morphism $\beta := f(\alpha)\colon f(a) \to f(a')$ in B, we obtain a morphism $\varepsilon_\alpha\colon \mathrm{lift}_0^f(a,\beta) \to \alpha$ in $\mathrm{Fun}([1],A)$, corresponding to a commutative square in A of the form

$$\begin{tikzcd} a \arrow[r, "\mathrm{lift}_0^f(a,\beta)"] \arrow[d, equal] & \beta_!(a) \arrow[d, dashed, "\varepsilon_\alpha"] \\ a \arrow[r, "\alpha"] & a'. \end{tikzcd}$$

By Corollary 1.7.7, we may think of this square in A as a commutative triangle in A of the form

$$\begin{tikzcd} & \beta_!(a) \arrow[dr, dashed, "\varepsilon_\alpha"] & \\ a \arrow[ur, "\mathrm{lift}_0^f(a,\beta)"] \arrow[rr, "\alpha"'] & & a'. \end{tikzcd}$$

Definition 5.6.6 (Cocartesian morphism)**.** In the setting of Construction 5.6.5 we say that the morphism α is *f-cocartesian* (or simply *cocartesian* if f is clear from context) if the morphism $\varepsilon_\alpha\colon \beta_!(a) \to a'$ is an isomorphism in A.

Remark 5.6.7. Note that by abuse of notation, we used the symbol ε_α both for the transformation $\mathrm{lift}_0^f(a,\beta) \to \alpha$ and for its image under the projection $\mathrm{ev}_1\colon \mathrm{Fun}([1],A) \to A$. In the definition of cocartesian morphisms, we are only asking for the latter to be an isomorphism. Nevertheless, we will prove in Corollary 6.2.5 below that isomorphisms in $\mathrm{Fun}([1],A)$ can be detected after evaluation at 0 and 1, and since the component of ε_α at 0 is always an isomorphism, this will imply that cocartesian morphisms are equivalently characterized by the whole transformation $\mathrm{lift}_0^f(a,\beta) \to \alpha$ being invertible.

Lemma 5.6.8. *Let $f\colon A \to B$ be a cocartesian fibration. Fix a term a of A and composable morphisms $\beta\colon b := f(a) \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$, both in some context Γ. Then for every $\gamma\colon a \to a''$ over $\beta'\beta$ there exists a dashed filling*

$$\begin{tikzcd} a \arrow[r, "\mathrm{lift}_0^f(a,\beta)"] \arrow[d, "\gamma"'] & \beta_!(a) \arrow[d, dashed, "\bar{\gamma}"] \\ a'' \arrow[r, equal] & a'' \end{tikzcd} \quad \textit{over} \quad \begin{tikzcd} b \arrow[r, "\beta"] \arrow[d, "\beta'\beta"'] & b' \arrow[d, "\beta'"] \\ b'' \arrow[r, equal] & b'' \end{tikzcd}$$

which is unique up to isomorphism.

Proof. This follows from the fact that lift^f_0 is a left adjoint section of $\vec{\mathrm{ev}}^f_0$. Indeed, consider the objects (a,β) and $(\mathrm{id}_{a''}\colon a'' \to a'')$ of $A \vec{\times}_f B$ and $\mathrm{Fun}([1],A)$, respectively. Then the solid data describes an arrow $(a,\beta) \to \vec{\mathrm{ev}}^f_0(\mathrm{id}_{a''})$ while the dashed extension describes an extension to an arrow $\mathrm{lift}^f_0(a,\beta) \to \mathrm{id}_{a''}$. By the adjunction equivalence

$$\mathrm{Hom}_{\mathrm{Fun}([1],A)}(\mathrm{lift}^f_0(a,\beta), \mathrm{id}_{a''}) \xrightarrow{\sim} \mathrm{Hom}_{A \vec{\times}_f B}((a,\beta), \vec{\mathrm{ev}}^f_0(\mathrm{id}_{a''}))$$

there is a unique such extension up to isomorphism. □

We will often use Corollary 1.7.7 to rewrite such lifting problems in the following form:

$$\begin{tikzcd} & \beta_!(a) \arrow[dr, dashed, "\tilde{\gamma}"] & \\ a \arrow[ur, "\mathrm{lift}^f_0(a,\beta)"] \arrow[rr, "\gamma"] & & a'' \end{tikzcd} \quad \overset{f}{\rightsquigarrow} \quad \begin{tikzcd} & b' \arrow[dr, dashed, "\beta'"] & \\ b \arrow[ur, "\beta"] \arrow[rr, "\beta'\beta"] & & b''. \end{tikzcd}$$

Proposition 5.6.9. *Let $f\colon A \to B$ be a cocartesian fibration.*

(1) For each term a of A with $b \coloneqq f(a)$, the cocartesian lift $a \to (\mathrm{id}_b)_! a$ of the identity can be chosen to be the identity map $\mathrm{id}_a\colon a \to a$.

(2) Given a term a of A and composable morphisms $b \coloneqq f(a) \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$, the canonical map $c\colon (\beta'\beta)_!(a) \to \beta'_!\beta_!(a)$ over b'' obtained by filling

$$\begin{tikzcd} a \arrow[r, "\mathrm{lift}^f_0(a,\beta'\beta)"] \arrow[d, "\mathrm{lift}^f_0(a,\beta)"'] & (\beta'\beta)_!(a) \arrow[d, dashed, "c"] \\ \beta_!(a) \arrow[r, "\mathrm{lift}^f_0(\beta_!(a),\beta')"'] & \beta'_!\beta_!(a) \end{tikzcd} \quad \textit{over} \quad \begin{tikzcd} b \arrow[r, "\beta'\beta"] \arrow[d, "\beta"'] & b'' \arrow[d, equal] \\ b' \arrow[r, "\beta'"'] & b'' \end{tikzcd}$$

is an ismorphism.

Proof. (1) Recall that $(\beta_! a, \mathrm{lift}^f_0)$ is an initial object of the category $\mathrm{Lift}^f_0(a,\beta)$. In the case where $\beta = \mathrm{id}_b$ is the identity, this category is exactly the slice $(A_b)_{a/}$ which already has an initial object given by (a, id_a). The claim follows because initial objects are unique up to isomorphism, see Corollary 5.5.19.

(2) We construct an explicit inverse $c'\colon \beta'_!\beta_!(a) \to (\beta'\beta)_!(a)$ as follows. Consider the solid diagram

$$\begin{tikzcd} a \arrow[r, "\mathrm{lift}^f_0(a,\beta)"] \arrow[d, "\mathrm{lift}^f_0(a,\beta'\beta)"'] & \beta_!(a) \arrow[r, "\mathrm{lift}^f_0(\beta_! a,\beta')"] \arrow[d, dashed] & \beta'_!\beta_!(a) \arrow[d, dashed, "c'"] \\ (\beta'\beta)_!(a) \arrow[r, equal] & (\beta'\beta)_!(a) \arrow[r, equal] & (\beta'\beta)_!(a) \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} b \arrow[r, "\beta"] \arrow[d, "\beta'\beta"'] & b' \arrow[r, "\beta'"] \arrow[d, "\beta'"] & b'' \arrow[d, equal] \\ b'' \arrow[r, equal] & b'' \arrow[r, equal] & b'' \end{tikzcd}$$

By applying Lemma 5.6.8 twice, we obtain the indicated dashed extensions, yielding the desired map c'. The fact that this is an inverse follows by the uniqueness part of Lemma 5.6.8 applied to the diagrams

$$\begin{tikzcd}
a \arrow[r, "\mathrm{lift}_0^f(a,\beta)"] \arrow[d, "\mathrm{lift}_0^f(a,\beta'\beta)"] \arrow[dd, bend right, "\mathrm{lift}_0^f(\beta_!a,\beta')\,\mathrm{lift}_0^f(a,\beta)"'] & \beta_!(a) \arrow[r, "\mathrm{lift}_0^f(\beta_!a,\beta')"] \arrow[d, dashed, "u"] \arrow[dd, bend right, "\mathrm{lift}_0^f(\beta_!a,\beta')"'] & \beta'_!\beta_!(a) \arrow[d, dashed, "c'"] \arrow[dd, dashed, bend right, "\mathrm{id}_{\beta'_!\beta_!(a)}"'] \\
(\beta'\beta)_!(a) \arrow[r, equal] \arrow[d, "c"] & (\beta'\beta)_!(a) \arrow[r, equal] \arrow[d, "c"] & (\beta'\beta)_!(a) \arrow[d, "c"] \\
\beta'_!\beta_!(a) \arrow[r, equal] & \beta'_!\beta_!(a) \arrow[r, equal] & \beta'_!\beta_!(a)
\end{tikzcd}$$

(twice, yielding $\mathrm{lift}_0^f(\beta_!a,\beta') \simeq cu$ and then $\mathrm{id} \simeq cc'$) and

$$\begin{tikzcd}
a \arrow[r, "\mathrm{lift}_0^f(a,\beta'\beta)"] \arrow[d, equal] & (\beta'\beta)_!(a) \arrow[d, "c'"] \arrow[dd, dashed, bend left, "\mathrm{id}_{(\beta'\beta)_!(a)}"] \\
a \arrow[r, "\mathrm{lift}_0^f(\beta_!a,\beta')\,\mathrm{lift}_0^f(a,\beta)"] \arrow[d, equal] & \beta'_!\beta_!(a) \arrow[d, "c"] \\
a \arrow[r, "\mathrm{lift}_0^f(a,\beta'\beta)"] & (\beta'\beta)_!(a)
\end{tikzcd}$$

(yielding $\mathrm{id} \simeq c'c$). □

Corollary 5.6.10. *Let $f\colon A \to B$ be a cocartesian fibration. Covariant transport is functorial in the following sense:*

(1) For every object b of B, covariant transport $(\mathrm{id}_b)_!\colon A_b \to A_b$ along the identity is isomorphic to the identity functor.

(2) Given composable morphisms $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$, there is a canonical natural isomorphism $(\beta'\beta)_! \cong \beta'_!\beta_!\colon A_b \to A_{b''}$.

Proof. This follows directly from Proposition 5.6.9 applied to the tautological term $a\colon A_b \to A$ of A in context A_b (which is just the fiber inclusion). □

Lemma 5.6.11. *Let $f\colon A \to B$ be a cocartesian fibration.*

(1) Every isomorphism of A is f-cocartesian.

(2) Let $a \xrightarrow{\alpha} a' \xrightarrow{\alpha'} a''$ be composable arrows of A over $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$. Assume that α is f-cocartesian. Then α' is f-cocartesian if and only if $\alpha'\alpha$ is f-cocartesian.

Proof. (1) Without loss of generality we may treat the case of an identity $\mathrm{id}_a\colon a \to a$. In this case, we can choose $\mathrm{lift}_0^f(a,\mathrm{id}_{f(a)})$ to be the identity, so that ε_α must be the identity too.

(2) Consider the following diagram

$$
\begin{tikzcd}
a \arrow[rr, "\mathrm{lift}_0^f(a,\beta'\beta)"] \arrow[d, equal] & & (\beta'\beta)_!(a) \arrow[d, "\cong"'] \arrow[ddd, dashed, bend left=60, "\varepsilon_{\alpha'\alpha}"] \\
a \arrow[r, "\mathrm{lift}_0^f(a,\beta)"] \arrow[d, "\alpha"] & \beta_!a \arrow[r, "\mathrm{lift}_0^f(\beta_!a,\beta')"] \arrow[d, dashed, "\varepsilon_\alpha"] & \beta'_!\beta_!a \arrow[d, dashed, "\beta'_!\varepsilon_\alpha"] \\
a' \arrow[r, equal] \arrow[d, "\alpha'"] & a' \arrow[r, "\beta'_!a'"] \arrow[d, "\alpha'"] & \beta'_!a' \arrow[d, dashed, "\varepsilon_{\alpha'}"] \\
a'' \arrow[r, equal] & a'' \arrow[r, equal] & a''
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
b \arrow[rr, "\beta'\beta"] \arrow[d, equal] & & b'' \arrow[d, equal] \\
b \arrow[r, "\beta"] \arrow[d, "\beta"] & b' \arrow[r, "\beta'"] \arrow[d, equal] & b'' \arrow[d, equal] \\
b' \arrow[r, equal] \arrow[d, "\beta'"] & b' \arrow[r, "\beta'"] \arrow[d, "\beta'"] & b'' \arrow[d, equal] \\
b'' \arrow[r, equal] & b'' \arrow[r, equal] & b''
\end{tikzcd}
$$

where the the rightmost semicircle commutes by the uniqueness statement of Lemma 5.6.8. Since α is cocartesian by assumption, ε_α is an isomorphism, hence also its cocartesian transport along β'. We conclude that $\varepsilon_{\alpha'\alpha}$ is an isomorphism if and only if $\varepsilon_{\alpha'}$ is an isomorphism, which is what we wanted to show. □

5.6.2 Closure properties of cartesian and cocartesian fibrations

We will now show that the classes of cartesian and cocartesian fibrations satisfy various closure properties.

Proposition 5.6.12. *Let $f\colon A \to B$ be a cocartesian fibration. Then for every category E, the functor $f_*\colon \mathrm{Fun}(E, A) \to \mathrm{Fun}(E, B)$ is a cocartesian fibration.*

Proof. Since $\mathrm{Fun}(E, -)$ preserves pullbacks, one observes that the directed evaluation map of f_* is given by applying $\mathrm{Fun}(E, -)$ to $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1], A) \to A \vec{\times}_f B$. The claim is now an immediate consequence from the fact that $\mathrm{Fun}(E, -)$ preserves adjunctions, see Proposition 5.4.4. □

Proposition 5.6.13. *The classes of cartesian and cocartesian fibrations are stable under composition.*

Proof. We will only prove the claim about cocartesian fibrations, leaving the dual claim for cartesian fibrations to the reader. If $f\colon A \to B$ and $g\colon B \to C$ are two cocartesian fibrations, we have by Lemma 5.1.7 a commutative diagram

$$
\begin{tikzcd}
& \mathrm{Fun}([1], A) \arrow[dl, "\vec{\mathrm{ev}}_0^f"'] \arrow[dr, "\vec{\mathrm{ev}}_0^{gf}"] & \\
A \vec{\times}_f B \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & A \vec{\times}_{gf} C \arrow[d] \\
\mathrm{Fun}([1], B) \arrow[rr, "\vec{\mathrm{ev}}_0^g"] & & B \vec{\times}_g C,
\end{tikzcd}
$$

where the bottom square is a pullback square. As f and g are cocartesian fibrations, the maps $\vec{\mathrm{ev}}_0^g$ and $\vec{\mathrm{ev}}_0^f$ admit left adjoint sections, and it then follows from Proposition 5.5.6 and Proposition 5.5.8 that also $\vec{\mathrm{ev}}_0^{gf}$ admits a left adjoint section. □

Proposition 5.6.14. *The classes of cartesian and cocartesian fibrations are stable under base change.*

Proof. Again we only treat the case of cocartesian fibrations, leaving the dual statement for cartesian fibrations to the reader. Consider a pullback square of the form

$$\begin{tikzcd} A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B' \arrow[r, "v"] & B, \end{tikzcd}$$

and assume that f is a cocartesian fibration. By Lemma 5.1.6, there is a pullback square of the form

$$\begin{tikzcd} \mathrm{Fun}([1], A') \arrow[r] \arrow[d, "\vec{\mathrm{ev}}_0^{f'}"'] & \mathrm{Fun}([1], A) \arrow[d, "\vec{\mathrm{ev}}_0^{f}"] \\ A' \vec{\times}_{f'} B' \arrow[r] & A \vec{\times}_f B, \end{tikzcd}$$

and thus it follows from Proposition 5.5.8 that also $\vec{\mathrm{ev}}_0^{f'}$ admits a left adjoint section. □

Proposition 5.6.15. *The classes of cartesian and cocartesian fibrations are stable under retracts.*

Proof. This follows from a combination of Lemma 5.1.8 and Proposition 5.5.9. □

Proposition 5.6.16. *Let $f\colon A \to B$ be a cocartesian fibration. If for every morphism $u\colon a \to b$ the covariant transport functor $u_!\colon A_a \to A_b$ admits a right adjoint $u^*\colon A_b \to A_a$, then f is a cartesian fibration as well.*

Dually, every cartesian fibration whose contravariant transport functors admit left adjoints is also a cocartesian fibration.

Proof. Missing. □

5.7 Cocartesian fibrations in context S

Let S be a category. As indicated at the start of this chapter, all the axioms that were needed to make sense of the definition of a (co)cartesian fibration are also valid in context S, and so we may speak of (co)cartesian fibrations in context S. In this subsection, we will discuss the relationship between (co)cartesian fibrations in context S and (co)cartesian fibrations in the absolute context.

Lemma 5.7.1. *Let S be a category and consider a functor*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

over S. Then f admits a left (resp. right) adjoint section in context S if and only if it admit a left (resp. right) adjoint section in the absolute context.

Proof. If f admits a left (resp. right) adjoint section $s\colon D \to C$ in context S, then by forgetting that it lives over S it also provides a left (resp. right) adjoint section to f in the absolute context. Conversely, assume that $s\colon D \to C$ is a left adjoint section to f in the absolute context, and let $\varepsilon\colon sf \to \mathrm{id}_C$ be a counit transformation exhibiting this adjunction, i.e. the two composites

$$f\varepsilon\colon f \cong fsf \to f \qquad \text{and} \qquad \varepsilon s\colon s \cong sps \to s$$

are naturally isomorphic to the respective identity transformations. The fact that s is a section of f means that we may regard s as a functor over D, hence it is in particular a functor over S. The first condition on the transformation ε expresses the fact that ε is a transformation over D, and hence it is in particular a transformation over S. The second condition on ε expresses the fact that it restricts to the identity on D, and this will then automatically hold true in context S as well. All in all, we conclude that ε also exhibits s as a left adjoint section of f in context S, as desired. $\square$

Lemma 5.7.2. *Let S be a category and consider a functor*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

over S. Then there exists a pullback square

$$\begin{tikzcd} \mathrm{Fun}_S([1]_S, C) \arrow[r] \arrow[d, "\vec{\mathrm{ev}}^f_{0,S}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C) \arrow[d, "\vec{\mathrm{ev}}^f_0"] \\ C \vec{\times}_{f,S} D \arrow[r] & C \vec{\times}_f D, \end{tikzcd}$$

where we denote by $\vec{\mathrm{ev}}^f_{0,S}$ the underlying functor of the directed evaluation map of f in context S

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fun}_S([1]_S, C) \arrow[r] \arrow[d, "\vec{\mathrm{ev}}^f_{0,S}"'] & \mathrm{Fun}([1], C) \arrow[d, "\vec{\mathrm{ev}}^f_0"] \\ C \vec{\times}_{f,S} D \arrow[r] \arrow[d] & C \vec{\times}_f D \arrow[d] \\ S \arrow[r] & \mathrm{Fun}([1], S). \end{tikzcd}$$

By definition, the outer rectangle is a pullback square, hence by the pasting law of pullback squares it will suffice to show that also the bottom square is a pullback square. This follows from an iterated use of the pasting law of pullback squares applied to the following large commutative diagram:

$$\begin{tikzcd}
& C \vec{\times}_f D \arrow[rrr] \arrow[ddd, two heads] \arrow[ddr] & & & \mathrm{Fun}([1], D) \arrow[ddd, two heads] \arrow[ddr] & \\
C \vec{\times}_{f,S} D \arrow[ur] \arrow[rrr] \arrow[ddd, two heads] & & & \mathrm{Fun}_S([1]_S, D) \arrow[ur] \arrow[ddd, two heads] & & \\
& & C \times D \arrow[rrr] \arrow[ddd, two heads] & & & D \times D \arrow[ddd, two heads] \\
& C \vec{\times}_S D \arrow[ur] \arrow[rrr] \arrow[ddd, two heads] & & & D \vec{\times}_S D \arrow[ur] \arrow[ddd, two heads] & \\
C \times_S D \arrow[ur] \arrow[rrr] \arrow[ddd, two heads] & & & D \times_S D \arrow[ur] \arrow[ddd, two heads] & & \\
& & S \times S \arrow[rrr, equal] & & & S \times S \\
& \mathrm{Fun}([1], S) \arrow[ur] \arrow[rrr, equal] & & & \mathrm{Fun}([1], S) \arrow[ur] & \\
S \arrow[ur] \arrow[rrr, equal] & & & S. \arrow[ur] & &
\end{tikzcd}$$

Indeed, in the bottom half of the diagram, all left and right faces are pullback squares, and since the bottom faces are trivially pullback squares it follows that the middle two faces are pullback squares. On the top half, the slanted back face is a pullback square by definition, hence so is the back face of the top front cube. As the front face is a pullback by definition, it thus remains to show that its right face is a pullback square. But this also follows from the pasting lemma as the composite vertical face on the right is a pullback square. □

Proposition 5.7.3. *Let S be a category and consider a functor*

$$\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

over S. If f is a left (resp. right) fibration in the absolute context, then it is a left (resp. right) fibration in context S.

Proof. This is immediate from Lemma 5.7.2. □

Proposition 5.7.4. *Let S be a category and consider a functor*

$$\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

over S. If f is a cocartesian (resp. cartesian) fibration in the absolute context, then it is a cocartesian (resp. cartesian) fibration when regarded as a functor in context S.

Proof. By symmetry it will suffice to prove the claim for cocartesian fibrations. Let us denote by

$$\vec{\mathrm{ev}}^f_{0,S} \colon \mathrm{Fun}_S([1]_S, A) \to A \vec{\times}_{f,S} B$$

the directed evaluation map of f in context S. To show that this map admits a left adjoint section in context S, it suffices by Lemma 5.7.1 to show that the underlying map the absolute admits a left adjoint section. But by Lemma 5.7.2 this map is a pullback of $\vec{\mathrm{ev}}^f_0 \colon \mathrm{Fun}([1], A) \to A \vec{\times}_f B$, and hence the claim is a consequence of Proposition 5.5.8. □

Proposition 5.7.5. *Let Γ be a groupoid and consider a functor over Γ of the form*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & \Gamma. & \end{tikzcd}$$

Then f is a cocartesian (resp. cartesian) fibration when regarded as a functor in context Γ if and only if it is a cocartesian (resp. cartesian) fibration in the absolute context.

Proof. By duality it suffices to prove the claim for cocartesian fibrations. Consider the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Fun}([1], C) \arrow[rr, two heads, "\vec{\mathrm{ev}}^f_0"] \arrow[dd, two heads] & & C \vec{\times}_f D \arrow[dd, two heads] \\ \mathrm{Fun}_\Gamma([1]_\Gamma, C) \arrow[ur] \arrow[rr, two heads, "\vec{\mathrm{ev}}^f_{0,\Gamma}"] \arrow[dd, two heads] & & C \vec{\times}_{f,\Gamma} D \arrow[ur] \arrow[dd, two heads] & \\ & \mathrm{Fun}([1], \Gamma) \arrow[rr, equal] & & \mathrm{Fun}([1], \Gamma) \\ \Gamma \arrow[ur] \arrow[rr, equal] & & \Gamma \arrow[ur] & \end{tikzcd}$$

The left face is a pullback square by construfction. The right face was shown to be a pullback square in the proof of Proposition 5.7.4. Since Γ is a groupoid, the map $p^*_{[1]} \colon \Gamma \to \mathrm{Fun}([1], \Gamma)$ is an equivalence, and it follows that the two slanted maps in the top square of the cube are equivalences. We conclude that the map $\vec{\mathrm{ev}}^f_0$ admits a left adjoint section if and only if the map $\vec{\mathrm{ev}}^f_{0,\Gamma}$ admits a left adjoint section, which by Lemma 5.7.1 implies that this map even admits a left adjoint section in context Γ. This finishes the proof. □

Corollary 5.7.6. *Let Γ be a groupoid and let $p \colon C \to \Gamma$ be a functor. Then p is both a cartesian fibration as well as a cocartesian fibration.*

Proof. It follows from Lemma 5.6.2 that the map

$$\begin{tikzcd} C \arrow[rr, "p"] \arrow[dr, "p"'] & & \Gamma \arrow[dl, equal] \\ & \Gamma & \end{tikzcd}$$

is both a cartesian and a cocartesian fibration in context Γ. The claim thus immediately follows from Proposition 5.7.5. □

5.8 The source and target fibrations

In this section, we will provide two fundamental examples of (co)cartesian fibrations: the source and target maps $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}([1], C) \to C$ for a category C. The fibers of these two functors are the slice categories $C_{/x}$ and $C_{x/}$ constructed in Section 5.3 below, and the fact that ev_0 and ev_1 are cartesian and a cocartesian fibration, respectively, encodes the functoriality of these slices in x.

We start with an auxiliary result, for which we need to recall the maps $j_0, j_1 \colon \Delta^2 \to [1] \times [1]$ and $p_0, p_2 \colon [1] \times [1] \to \Delta^2$ from Constructions 1.6.19 and 1.6.21, pictorially represented by the following diagrams:

$$j_0 \quad = \quad \begin{tikzcd} (0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] \\ & (1,1), \end{tikzcd} \qquad\qquad j_1 \quad = \quad \begin{tikzcd} (0,0) \arrow[d] \arrow[dr] & \\ (1,0) \arrow[r] & (1,1), \end{tikzcd}$$

$$p_0 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & 1 \arrow[d] \\ 0 \arrow[r] & 2, \end{tikzcd} \qquad\qquad p_2 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 2 \arrow[d, equal] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

Proposition 5.8.1. *The map j_0 is a left adjoint section of the map p_0. The map j_1 is right adjoint section of the map p_2.*

Proof. We prove the claim for j_1 and p_2 and leave the analogous case for j_0 and p_0 to the reader. For simplicity we write $j := j_1$ and $p := p_2$ for the remainder of this proof. By construction j is a section of p. We will construct a natural transformation $\eta \colon \mathrm{id}_{[1]\times[1]} \to jp$, or equivalently a map

$$h \colon ([1] \times [1]) \times [1] \to ([1] \times [1])$$

such that $h(x, y, 0) = (x, y)$ and $h(x, y, 1) = jp(x, y) = (\max(x, y), y)$, where the last equality holds by Lemma 1.6.24. Formally, we may define h by

$$h(x, y, t) = (\max(\min(t, y), x), y),$$

using the maps max and min from Construction 1.6.9. Indeed, when $t = 0$, we have $\min(t, x) = \min(0, x) = 0$, so that the second component is $\max(0, y) = y$. When $t = 1$, we have $\min(t, x) = \min(1, x) = x$ so that the second component is $\max(x, y)$.

Pictorially, we may display h as the following commutative cube in $[1] \times [1]$ (read from front to back):

$$\begin{tikzcd} & (0,0) \arrow[rr] \arrow[dd] & & (1,1) \arrow[dd, equal] \\ (0,0) \arrow[ur, equal] \arrow[rr, crossing over] \arrow[dd] & & (0,1) \arrow[ur] & \\ & (1,0) \arrow[rr] & & (1,1) \\ (1,0) \arrow[ur, equal] \arrow[rr] & & (1,1). \arrow[ur, equal] \arrow[from=uu, crossing over] & \end{tikzcd}$$

An explicit computation using the relations from Lemma 1.6.10 shows that the composite ph is the identity transformation $\mathrm{id}_p \colon p \to p$. Similarly, one can show that the composite $h(j \times 1)$ is the identity transformation $\mathrm{id}_j \colon j \to j$. This finishes the proof. $\square$

Construction 5.8.2. Given a natural transformation $H \colon [1] \times C \to D$ between two functors $f, g \colon C \to D$, and a category A, we construct a natural transformation

$$H^* \colon [1] \times \mathrm{Fun}(D, A) \to \mathrm{Fun}(C, A)$$

between the precomposition functors $f^*, g^* \colon \mathrm{Fun}(D, A) \to \mathrm{Fun}(C, A)$ as follows:

$$\mathrm{Fun}(D, A) \xrightarrow{H^*} \mathrm{Fun}([1] \times C, A) \cong \mathrm{Fun}([1], \mathrm{Fun}(C, A)).$$

It is clear that this construction sends identity transformations to identity transformations and compositions of transformations to compositions of transformations.

Theorem 5.8.3. *For any category C, the functor $\mathrm{ev}_0 \colon \mathrm{Fun}([1], C) \to C$ is a cartesian fibration. Dually, the functor $\mathrm{ev}_1 \colon \mathrm{Fun}([1], C) \to C$ is a cocartesian fibration.*

Proof. We show the statement for ev_0 and leave the dual case of ev_1 to the reader. We need to show that the map

$$\vec{\mathrm{ev}}_1^{\mathrm{ev}_0} \colon \mathrm{Fun}([1], \mathrm{Fun}([1], C)) \to C \vec{\times}_{\mathrm{ev}_0} \mathrm{Fun}([1], C)$$

admits a right adjoint section. We may identify the source of this map with $\mathrm{Fun}([1] \times [1], C)$. The target sits by definition in a pullback square

$$\begin{tikzcd}
C \vec{\times}_{\mathrm{ev}_0} \mathrm{Fun}([1], C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], C) \arrow[d, "\mathrm{ev}_1"] \\
\mathrm{Fun}([1], C) \arrow[r, "\mathrm{ev}_0"] & C,
\end{tikzcd}$$

and thus we may identify it with $\mathrm{Fun}(\Lambda^2_1, C)$. Unwinding definitions, we see that the map $\vec{\mathrm{ev}}_1^{\mathrm{ev}_0}$ factors as

$$\mathrm{Fun}([1] \times [1], C) \xrightarrow{j_1^*} \mathrm{Fun}(\Delta^2, C) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Sp}_2, C),$$

where the second map is a trivial fibration by the Segal axiom. It will thus suffice to show that j_1^* admits a right adjoint section.

We claim that p_2^* is a right adjoint section of j_1^*. Indeed, we have $j_1^* p_2^* = (p_2 j_1)^* = (\mathrm{id})^* = \mathrm{id}$. The transformation $h \colon 1 \to j_1 p_2$ gives rise to a transformation $h^* \colon 1 \to (j_1 p_2)^* = p_2^* j_1^*$ via Construction 5.8.2. Furthermore, we have

$$j_1^* h^* = (h j_1)^* = \mathrm{id}_{j_1^*} \qquad \text{and} \qquad h^* p_2^* = (p_2 h)^* = \mathrm{id}_{p_2^*}.$$

This finishes the proof. $\square$

Remark 5.8.4. Pictorially, the functor $\vec{\mathrm{ev}}_1^{\mathrm{ev}_0}$ may be thought of as the following assignment:

$$\begin{tikzcd}
x \arrow[r, "u"] \arrow[d, "\alpha"'] & y \arrow[d, "\beta"] \\
x' \arrow[r, "u'"'] & y'
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
x \arrow[d, "\alpha"'] \arrow[dr, "u'\alpha"] & \\
x' \arrow[r, "u'"'] & y'
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
x \arrow[d, "\alpha"'] & \\
x' \arrow[r, "u'"'] & y'.
\end{tikzcd}$$

Its right adjoint section is given informally by the following assignment:

$$
\begin{tikzcd}
x \arrow[d, "\alpha"'] & \\
x' \arrow[r, "u'"'] & y'
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
x \arrow[d, "\alpha"'] \arrow[dr, "u'\alpha"] & \\
x' \arrow[r, "u'"'] & y'
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
x \arrow[d, "\alpha"'] \arrow[r, "u'\alpha"] & y' \arrow[d, equal] \\
x' \arrow[r, "u'"'] & y'.
\end{tikzcd}
$$

We will in fact need a slight generalization of Theorem 5.8.3. Given a functor $f\colon C \to D$, we may consider the directed pullbacks $C \vec{\times}_f D$ and $D \vec{\times}_f C$. They both come equipped with canonical maps to D, which we claim to be (co)cartesian fibrations:

Proposition 5.8.5. *For a functor $f\colon C \to D$, the projection $\mathrm{ev}_1\colon C \vec{\times}_f D \to D$ is a cocartesian fibration, while the projection $\mathrm{ev}_0\colon D \vec{\times}_f C \to D$ is a cartesian fibration.*

Proof. By symmetry, it suffices to prove the claim for $\mathrm{ev}_1\colon C \vec{\times}_f D \to D$. We thus need to show that the map

$$\vec{\mathrm{ev}}^0_{\mathrm{ev}_1}\colon \mathrm{Fun}([1], C \vec{\times}_f D) \to (C \vec{\times}_f D) \vec{\times}_\pi D$$

admits a left adjoint section. The proof is analogous to the proof of Theorem 5.8.3; to make the proof more readable we will explain what happens on objects and leave it to the reader to write a formal proof.

First observe that terms of the category $\mathrm{Fun}([1], C \vec{\times}_f D)$ are pairs (α, τ) where $\alpha\colon x \to x'$ is a morphism in C, and σ is a commutative square in D of the form

$$
\begin{tikzcd}
f(x) \arrow[r, "u"] \arrow[d, "f(\alpha)"'] & y \arrow[d, "\beta"] \\
f(x') \arrow[r, "u'"] & y'.
\end{tikzcd}
$$

On the other hand, terms of $(C \vec{\times}_f D) \vec{\times}_\pi D$ are of the form $(x, f(x) \xrightarrow{u} y \xrightarrow{v} z)$, which we might alternatively display as

$$
\left(x,\
\begin{tikzcd}
f(x) \arrow[r, "u"] & y \arrow[d, "v"] \\
& z
\end{tikzcd}
\right).
$$

The map $\vec{\mathrm{ev}}^0_{\mathrm{ev}_1}$ is then given by the assignment

$$
\left(
\begin{tikzcd}
x \arrow[d, "\alpha"'] \\
x'
\end{tikzcd},\
\begin{tikzcd}
f(x) \arrow[r, "u"] \arrow[d, "f(\alpha)"'] & y \arrow[d, "\beta"] \\
f(x') \arrow[r, "u'"] & y'
\end{tikzcd}
\right)
\quad \mapsto \quad
\left(x,\
\begin{tikzcd}
f(x) \arrow[r, "u"] & y \arrow[d, "\beta"] \\
& y'
\end{tikzcd}
\right)
$$

A left adjoint section is then given by the assignment

$$
\left(x,\
\begin{tikzcd}
f(x) \arrow[r, "u"] & y \arrow[d, "v"] \\
& z
\end{tikzcd}
\right)
\quad \mapsto \quad
\left(
\begin{tikzcd}
x \arrow[d, equal] \\
x
\end{tikzcd},\
\begin{tikzcd}
f(x) \arrow[r, "u"] \arrow[d, equal] & y \arrow[d, "v"] \\
f(x) \arrow[r, "vu"] & z
\end{tikzcd}
\right).
$$

The precise definition of this functor and the fact that it is a left adjoint section proceed just like the (dual version of the) proof of Theorem 5.8.3, using the functors $j_0\colon \Delta^2 \to [1] \times [1]$ and $p_0\colon [1] \times [1] \to \Delta^2$. $\square$

5.9 Functoriality of universals axiom

Let $f\colon C \to D$ be a functor. If f admits a left adjoint section $s\colon D \to C$, we saw in Lemma 5.5.13 that then every fiber $C(d)$ of f admits an initial object given by $s(d) \in C(d)$. If f is a *cartesian fibration*, we expect to be able to also go the other way: if every fiber $C(d)$ admits an initial object s_x, then these objects should assemble into a left adjoint section $s\colon D \to C$ of f. Informally, the functoriality of s is given as follows: given a morphism $\beta\colon d \to d'$ in D, its image $s_d \to s_{d'}$ under s corresponds to a morphism $s_d \to \beta^*(s_{d'})$ in the fiber $C(d)$, which is uniquely determined by the fact that s_d is an initial object of $C(d)$. We make this heuristic precise by introducing the following axiom:

> **Axiom L** (Functoriality of universals axiom)**.** Let $p\colon E \to C$ be a cartesian (resp. cocartesian) fibration. Assume that for every object x in C, the fiber $E(x)$ of p at x admits an initial (resp. terminal) object s_x. Then there is a left adjoint (resp. right adjoint) section $s\colon C \to E$ of p equipped with an isomorphism $s_x \xrightarrow{\sim} s(x)$ in $E(x)$ for every object x of C.

Remark 5.9.1. Recall from Convention 1.5.7 that by 'object', we really mean 'an object $x\colon \{x\} \to C$ in an arbitrary groupoidal context $\{x\}$'. One easily observes that it suffices to check the condition for the *universal* object, i.e. the map $\gamma_C\colon C^{\simeq} \to C$. In this case, the fiber $E(x)$ is given by the left vertical map $\tilde{p}$ in the following pullback square:

$$\begin{tikzcd}
C^{\simeq} \times_C E \arrow[r] \arrow[d, "\tilde{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
C^{\simeq} \arrow[r, "\gamma_C"] & C.
\end{tikzcd}$$

Hence the following is an equivalent way of phrasing the axiom: If $\tilde{p}$ admits a left (resp. right) adjoint section $\tilde{s}\colon C^{\simeq} \to C^{\simeq} \times_C E$, then also p admits a left (resp. right) adjoint section $s\colon C \to E$ which the following diagram commute:

$$\begin{tikzcd}
C^{\simeq} \times_C E \arrow[r] & E \\
C^{\simeq} \arrow[r, "\gamma_C"] \arrow[u, "\tilde{s}"] & C. \arrow[u, "s"']
\end{tikzcd}$$

Axiom L will be of fundamental importance throughout our development of synthetic category theory: it allows us to reduce functorial statements to objectwise statements. In the remainder of this section, we will prove a wide range of examples of such phenomena.

5.9.1 Fiberwise criteria

We will start by showing that various proprties of (maps between) left/right fibrations may be tested at the level of fibers.

Proposition 5.9.2. *Let $f\colon E \to B$ be a left (resp. right) fibration. Assume that the fiber E_b of f at b is contractible for every object b of B. Then f is an equivalence.*

Proof. We will prove the claim for left fibrations; the claim for right fibrations is dual. For any object b of B, contractibility of E_b implies that there is an object s_b in E_b such that s_b is terminal. By Axiom L, it follows that the assignment $b \mapsto s_b$ is the restriction of a right adjoint section $s\colon B \to E$ of f. In particular, we have $fs = \mathrm{id}_B$ and a given transformation $\eta\colon \mathrm{id}_E \to sf$ such that $f\eta = \mathrm{id}_f$. In particular, the transformation $f\eta\colon f \to f$ is a natural isomorphism. But by combining Corollary 5.2.3 with the Rezk Axiom, we see that the commutative square

$$\begin{tikzcd}
\mathrm{Iso}(E) \arrow[r, "\pi_{\mathrm{Iso}}"] \arrow[d, "f_*"'] & \mathrm{Fun}([1], E) \arrow[d, "f_*"] \\
\mathrm{Iso}(B) \arrow[r, "\pi_{\mathrm{Iso}}"] & \mathrm{Fun}([1], C)
\end{tikzcd}$$

is a pullback square, and hence it follows that already the transformation $\eta\colon \mathrm{id}_E \to sf$ is a natural isomorphism. This shows that s is in fact an inverse to f, thus finishing the proof that f is an equivalence. □

Lemma 5.9.3. *Consider a commutative triangle*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& C, &
\end{tikzcd}$$

where p and q are left (resp. right) fibrations. Assume that the induced map $A_c \to B_c$ on fibers is an equivalence for every object c of C. Then f is an equivalence.

Proof. The functor f is also a left (resp. right) fibration by Proposition 5.2.16. By the previous proposition, it will thus suffice to show that the fibers of f are contractible. To this end, consider an object $b\colon \{b\} \to B$, and define $c := q \circ b\colon \{b\} \to C$. We then get a pullback diagram of the form

$$\begin{tikzcd}
A_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\
\{b\} \arrow[r] & B_c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "q"] \\
& \{b\} \arrow[r, "c"] & C.
\end{tikzcd}$$

Since the induced functor $A_c \to B_c$ is an equivalence by assumption , it follows that also its base change $A_b \to \{b\}$ is an equivalence, showing that A_b is a contractible category in context $\{b\}$ as desired. □

Corollary 5.9.4. *Consider a commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

and assume that p and q are left (resp. right) fibrations. Then the square is a pullback square if and only if for every object x of C the induced map $f_x\colon A_x \to B_{f(x)}$ on fibers is an equivalence.

Proof. Since left (resp. right) fibrations are closed under base change by Proposition 5.2.15, the functor $B \times_D C \to C$ is a right fibration. The square is a pullback square if and only if the map $(f, p)\colon A \to B \times_D C$ is an equivalence over C. By Lemma 5.9.3, we may check this condition fiberwise, where it precisely becomes the claim that the induced map $A(x) \to B(f(x))$ is an equivalence. □

Corollary 5.9.5. *Let $f\colon E \to C$ be a left (resp. right) fibration. Then the commutative square*

$$\begin{tikzcd} E^{\simeq} \arrow[r, "\gamma_E"] \arrow[d, "f^{\simeq}"'] & E \arrow[d, "f"] \\ C^{\simeq} \arrow[r, "\gamma_C"] & C \end{tikzcd}$$

is a pullback square.

Proof. Note that the functor $f^{\simeq}$ is again a left (resp. right) fibration by Proposition 5.2.4, since both its source and target are groupoids. By Corollary 5.9.4 it will thus suffice to show that γ_E induces equivalences on all fibers over $C^{\simeq}$. But given an object x of C, the induced map on fibers is $\gamma_{E_x}\colon (E_x)^{\simeq} \to E_x$, which is an equivalence since E_x is a groupoid by Corollary 5.2.6. □

Lemma 5.9.6. *For every category C, the commutative square*

$$\begin{tikzcd} \mathrm{Map}([1], C) \arrow[r, "\gamma_{\mathrm{Fun}([1],C)}"] \arrow[d, "(ev_0, ev_1)"'] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ C^{\simeq} \times C^{\simeq} \arrow[r, "\gamma_C \times \gamma_C"] & C \times C \end{tikzcd}$$

is a pullback square.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Map}([1], C) \arrow[r] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & C^{\simeq} \vec{\times}_{\gamma_C} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ C^{\simeq} \times C^{\simeq} \arrow[r, "\mathrm{id}_{C^{\simeq}} \times \gamma_C"] & C^{\simeq} \times C \arrow[r, "\gamma_C \times \mathrm{id}_C"] & C \times C. \end{tikzcd}$$

By the pasting law of pullback squares, it will suffice to show that the left square is a pullback square. Using the first projection map $\mathrm{pr}_1\colon C^{\simeq} \times C \to C^{\simeq}$, we may regard this square as a diagram of categories in context $C^{\simeq}$, and it will suffice it is a pullback square in context $C^{\simeq}$. Since $C^{\simeq}$ is a groupoid, we may prove this claim after pulling back along an arbitrary map $x\colon \{x\} \to C^{\simeq}$ from some groupoid $\{x\}$. (This step is redundant since we may take $\{x\} = C^{\simeq}$, but the change in notation makes the proof psychologically easier to follow.) Observe that the resulting diagram of categories in context $\{x\}$ is given as follows:

$$\begin{tikzcd} (C_{x/})^{\simeq} \arrow[r, "\gamma_{C_{x/}}"] \arrow[d] & C_{x/} \arrow[d] \\ C^{\simeq} \arrow[r, "\gamma_C"] & C. \end{tikzcd}$$

But since the functor $C_{x/} \to C$ is a left fibration by Theorem 5.3.3, this square is a pullback square by Corollary 5.9.5. This finishes the proof. $\square$

Recall that we defined in Definition 1.6.3 for any pair (x, y) of absolute objects in a category C the *hom groupoid* $C(x, y)$, which was generalized in Definition 2.3.6 to objects in arbitrary context. We now obtain the following alternative description of the hom groupoid:

Corollary 5.9.7. *For a category C and objects x and y of C, there is a pullback square*

$$\begin{tikzcd} C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ \{(x,y)\} \arrow[r, "(x,y)"] & C^{\simeq} \times C^{\simeq}. \end{tikzcd}$$

Proof. The objects $x, y \colon \Gamma \to C$ uniquely factor through the groupoid core $C^{\simeq}$. Since $C(x, y)$ is defined as the fiber over Γ of the map $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}([1], C) \to C \times C$, it follows from Lemma 5.9.6 that it is equivalently the fiber of the map $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Map}([1], C) \to C^{\simeq} \times C^{\simeq}$. $\square$

Note that this gives an alternative proof of the claim in Corollary 5.3.5 that the hom groupoids $C(x, y)$ are indeed groupoids.

5.9.2 Objectwise criteria

Using the fiberwise criteria discussed above, we may reformulate various categorical notions in terms of objectwise properties.

Lemma 5.9.8. *Let x be an object of a category C. Then x is initial if and only if for every other object y of C the hom groupoid $C(x, y)$ is contractible. Dually, x is terminal if and only if $C(y, x)$ is contractible for every object y of C.*

Proof. By Lemma 5.5.11, the object x is initial if and only if the forgetful functor $C_{x/} \to C$ is an equivalence. Since this functor is a left fibration by Theorem 5.3.3, it follows from Proposition 5.9.2 that it is an equivalence if and only if all of its fibers $C(x, y)$ are contractible. $\square$

Lemma 5.9.9. *Let $f\colon A \to B$ be a functor and let $\alpha\colon a \to a'$ be a morphism in A. Then α is f-cocartesian if and only if for every object x of A the commutative square*

$$\begin{tikzcd} A(a', x) \arrow[r, "-\circ\alpha"] \arrow[d, "f"'] & A(a, x) \arrow[d, "f"] \\ B(f(a'), f(x)) \arrow[r, "-\circ f(\alpha)"] & B(f(a), f(x)) \end{tikzcd}$$

is a pullback square.

Proof. By definition, α is f-cocartesian if and only if the square

$$\begin{tikzcd} A_{a'/} \arrow[r, "\alpha^*"] \arrow[d, "f"'] & A_{a/} \arrow[d, "f"] \\ B_{f(a')/} \arrow[r, "f(\alpha)^*"] & B_{f(a)/} \end{tikzcd}$$

is a pullback square. Equivalently, the canonical map $A_{a'/} \to A_{a/} \times_{B_{f(a)/}} B_{f(a')/}$ is an equivalence. This map fits into a commutative triangle

$$\begin{tikzcd} A_{a'/} \arrow[rr] \arrow[dr] & & A_{a/} \times_{B_{f(a)/}} B_{f(a')/} \arrow[dl] \\ & A & \end{tikzcd}$$

of left fibrations over A, and thus by Lemma 5.9.3 it suffices to show that this map is an equivalence on fibers over x for every object x of A. But the induced map on fibers is the map $A(a', x) \to A(a, x) \times_{B(f(a'), f(x))} B(f(a), f(x))$ induced by the commutative square from the statement of the lemma, and hence it is an equivalence if and only if this square is a pullback square. □

Remark 5.9.10. In more informal terms, the previous result says that a morphism $\alpha \colon a \to a'$ is f-cocartesian if and only if for any morphism $v \colon a \to x$ in A and morphism $w \colon f(a') \to f(x)$ in B such that the triangle

$$\begin{tikzcd} & f(a') \arrow[dr, "w"] & \\ f(a) \arrow[ur, "f(\alpha)"] \arrow[rr, "f(v)"] & & f(x) \end{tikzcd}$$

commutes, there exists a unique morphism $\tilde{w} \colon a' \to x$ such that the triangle

$$\begin{tikzcd} & a' \arrow[dr, "\tilde{w}"] & \\ a \arrow[ur, "\alpha"] \arrow[rr, "v"] & & x \end{tikzcd}$$

commutes and is sent to the previous triangle under f.

Lemma 5.9.11. *Let $f \colon A \to B$ be a cocartesian fibration. Then f is a left fibration if and only if the fibers of f are groupoids.*

Proof. We have seen the 'only if'-direction in Corollary 5.2.6. For the 'if'-direction, assume that all fibers of f are groupoids. It follows from Proposition 5.2.4 that for every functor $\Gamma \to B$ from a groupoid Γ the base change functor $A \times_B \Gamma \to \Gamma$ is a left fibration. To show that f itself is a left fibration, we must show that all morphisms in A are cocartesian morphisms. Consider a morphism $u \colon a \to a'$ in A and let $v \colon b \to b'$ be its image in B. Since f is cocartesian, u factors uniquely as a composite of a cocartesian morphism $a \to u_! a$ and some map $u_! a \to a'$. It thus remains to show that the map $u_! a \to a'$ is an equivalence. But this map lives in the fiber A_b of A over $\{b\}$, which by assumption is a groupoid. □

5.9.3 The pointwise criterion for adjunctions

We will now prove a useful characterization for adjunctions.

Theorem 5.9.12. *Let $f\colon C \to D$ be a functor. Then the following conditions are equivalent:*

(1) The functor f admits a right adjoint $g\colon D \to C$;

(2) The projection functor $p\colon C \vec{\times}_f D \to D$ admits a right adjoint section $s\colon D \to C \vec{\times}_f D$;

(3) For every object d in D, the relative slice $C_{/d} = C \times_D D_{/d}$ admits a terminal object s_d.

Dually, f admits a left adjoint if and only if the projection functor $D \vec{\times}_f C \to D \to D$ admits a left adjoint section, or equivalently if and only if the relative coslice $C_{d/}$ admits an initial object for every object d of D.

Remark 5.9.13. We will in fact prove the following somewhat more refined statement: assume that for every object d of D, the relative slice $C_{/d}$ admits a terminal object $s_d = (g_d, \varepsilon_d)$, where g_d is an object of C and $\varepsilon_d\colon f(g_d) \to d$ is a morphism in D. Then the right adjoint $g\colon D \to C$ of f may be chosen in such a way that for every object d of D the following two conditions are satisfied:

- There is an isomorphism $g_d \cong g(d)$;
- Under this isomorphism, the counit map $\varepsilon\colon f(g(d)) \to d$ of the adjunction corresponds to the given map $\varepsilon_d\colon f(g_d) \to d$.

Proof of Theorem 5.9.12. We will prove the equivalence between (1), (2) and (3), and leave the dual claim to the reader. Recall from Construction 5.1.1 the directed pullback $C \vec{\times}_f D$, defined via the following pullback diagram:

$$\begin{tikzcd}
C \vec{\times}_f D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], D) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
C \times D \arrow[r, "f\times \mathrm{id}_D"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \times D \arrow[d, "\mathrm{pr}_1"] \\
C \arrow[r, "f"] & D.
\end{tikzcd}$$

Objects of $C \vec{\times}_f D$ are pairs (x, u) where x is an object in C and u is an object in $\mathrm{Fun}([1], D)$ of the form $u\colon f(x) \to y$. There are two canonical projection maps

$$p\colon C \vec{\times}_f D \to D \qquad \text{and} \qquad q\colon C \vec{\times}_f D \to C$$

which are given on terms by $p(x, u\colon f(x) \to y) = y$ and $q(x, u) = x$. We have:

- The functor p is a cocartesian fibration by Proposition 5.8.5;
- The functor q is a base change of the evaluation map $\mathrm{ev}_0\colon \mathrm{Fun}([1], D) \to D$. Since ev_0 admits a left adjoint section given by $y \mapsto \mathrm{id}_y$ by Lemma 5.5.4, we get that also q admits a left adjoint section $i\colon C \to C \vec{\times}_f D$ given by $i(x) = (x, f(x) \xrightarrow{=} f(x))$.

- The composite
$$C \xrightarrow{i} C \vec{\times}_f D \xrightarrow{p} D$$
is equal to f.

For (1) $\implies$ (2), recall from Proposition 5.4.7 that the data of an adjunction $f \dashv g$ is equivalent to the data of an equivalence $C \vec{\times}_f D \simeq C \vec{\times}_g D$ over $C \times D$. We may therefore equivalently show that the projection $C \vec{\times}_g D \to D$ admits a right adjoint section. But this follows from Lemma 5.5.4, as this map is a base change of the evaluation functor $\mathrm{ev}_1 \colon \mathrm{Fun}([1], C) \to C$.

For (2) $\implies$ (1), assume that p admits a right adjoint section $s \colon D \to C \vec{\times}_f D$. Since adjunctions compose by Proposition 5.4.3, we conclude that the composite $g := qs \colon D \to C$ is the desired right adjoint of $f = pi$:

$$\begin{tikzcd} C \arrow[r, shift left, "i"] \arrow[rr, bend left, "f"] \arrow[r, phantom, "\perp"] & C \vec{\times}_f D \arrow[l, shift left, "q"] \arrow[r, shift left, "p"] \arrow[r, phantom, "\perp"] & D. \arrow[l, shift left, "s"] \arrow[ll, bend left, "g"] \end{tikzcd}$$

The claim that (2) and (3) are equivalent is a direct consequence of the Functoriality of Universals Axiom L: p is a cocartesian fibration, and the fiber of p over an object d of D is precisely the relative slice $C_{/d}$.

For the the refined statement from Remark 5.9.13, we use that the terminal objects $s_d = (g_d, \varepsilon_d)$ of the relative slices $C_{/d}$ assemble into the right adjoint section $s \colon D \to C \vec{\times}_f D$, with counit $ps \cong \mathrm{id}_D$. The right adjoint g of f is then given by qs, with counit $fg \to \mathrm{id}_D$ given by the composite

$$fg(d) = piqs(d) \xrightarrow{p\varepsilon'_{s(d)}} ps(d) \cong d,$$

where the transformation $\varepsilon' \colon iq \to \mathrm{id}_{C \vec{\times}_f D}$ is the counit of the adjunction $i \dashv q$. After unwinding the definitions, one sees that this map is precisely the given map $\varepsilon_d \colon f(g_d) \to d$, as claimed. □

Proposition 5.9.14 (Pointwise criterion adjunctions)**.** *Let $f \colon C \to D$ be a functor. Assume that, for every object $d \in D$, there exists an object $g(d)$ in C together with a morphism $\varepsilon_d \colon f(g(d)) \to d$ (resp. $\eta_d \colon d \to f(g(b))$) such that for any object c in C the map*

$$C(c, g(d)) \xrightarrow{f} D(f(c), f(g(d))) \xrightarrow{\varepsilon_d \circ -} D(f(c), d)$$

is an equivalence (resp. for every c in C the map

$$C(g(d), c) \xrightarrow{f} D(f(g(d)), f(c)) \xrightarrow{- \circ \eta_d} D(d, f(c))$$

is an equivalence.) Then there is a right adjoint (resp. left adjoint) $g \colon D \to C$ of f such that $g(d)$ coincides on objects with the data above, and the counit (resp. unit) is on objects given by the maps ε_d (resp. η_d).

Proof. We will only prove the first case; the other case is dual. By Theorem 5.9.12 it suffices to check that, for each object d of D, the pair $(g(d), \varepsilon_d)$ is a final object of the relative slice $C_{/d} = C \times_D D_{/d}$. By Lemma 5.9.8, it suffices to show that the hom groupoid $\mathrm{Hom}_{C_{/d}}((c, \alpha), (g(d), \varepsilon_d))$ is contractible for any object $(c, \alpha\colon f(c) \to d)$ of $C_{/d}$. By definition of $C_{/d}$ as a pullback, we may compute this hom groupoid via the following pullback square:

$$\begin{tikzcd}
\operatorname{Hom}_{C_{/d}}((c,\alpha),(g(d),\varepsilon_d)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Hom}_{D_{/d}}((f(c),\alpha),(f(g(d)),\varepsilon_d)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
\operatorname{Hom}_C(c,g(d)) \arrow[r, "f"] & \operatorname{Hom}_D(f(c),f(g(d))) \arrow[r, "(\varepsilon_d)_*"] & \operatorname{Hom}_D(f(c),d).
\end{tikzcd}$$

But since the bottom composite is assumed to be an equivalence, we see that the top composite is also an equivalence, and so the left-upper corner is contractible as desired. This finishes the proof. □

Proposition 5.9.15. *Consider a pullback square*

$$\begin{tikzcd}
C' \arrow[r, "u"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\
D' \arrow[r, "v"] & D
\end{tikzcd}$$

such that p is a cocartesian fibration. If v admits a left adjoint, then also u admits a left adjoint.

Proof. By Theorem 5.9.12, the projection $D' \vec{\times}_v D \to D$ admits a left adjoint section. By that same theorem, it will suffice to show that the projection $C' \vec{\times}_u C \to C$ admits a left adjoint section. To this end, we consider the following commutative diagram:

$$\begin{tikzcd}
 & C' \vec{\times}_u C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Fun}([1],C) \arrow[d, "\vec{\mathrm{ev}}_0^p"] \\
D \vec{\times}_v D' \arrow[d] & C \vec{\times}_{p,v} D' \arrow[l] \arrow[r, "\mathrm{id} \vec{\times} v"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & C \vec{\times}_p D \\
D & C. \arrow[l, "p"] &
\end{tikzcd}$$

Since p is a cocartesian fibration, the functor $\vec{\mathrm{ev}}_0^p$ admits a left adjoint section. Since functors admitting left adjoint sections are closed under base change and composition, it remains to show that the two squares in the diagram are pullback squares. For the left bottom square, this follows by applying the pasting lemma of pullback squares to the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Fun}([1],D) \arrow[d] & D \vec{\times}_v D' \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & C \vec{\times}_{p,v} D' \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
D \times D & D \times D' \arrow[l] \arrow[d] & C \times D' \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
 & D & C \arrow[l, "p"]
\end{tikzcd}$$

For the top right square, this is a consequence of the pasting law of pullback squares applied to the following commutative diagram:

$$\begin{tikzcd}
C \vec{\times}_u C' \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Fun}([1],C) \arrow[dr, "\mathrm{id}_C \times p'"] \arrow[dd] & & \\
& C \vec{\times}_{p,v} D' \arrow[rr] \arrow[dd] & & C \vec{\times}_p D \arrow[r] \arrow[dd] & \mathrm{Fun}([1],D) \arrow[dd] \\
C \times C' \arrow[rr] \arrow[dr] & & C \times C \arrow[dr, "\mathrm{id}_C \times p"] & & \\
& C \times D' \arrow[rr, "\mathrm{id}_C \times v"] & & C \times D \arrow[r] & D \times D
\end{tikzcd}$$

This finishes the proof. □

Let us finish the section with a recognition principle for (co)cartesian fibrations:

Proposition 5.9.16. *Let $f\colon C \to D$ be a functor. Assume that, for any object x of C and any morphism $v\colon y \to f(x)$ in D, there exists an f-cartesian lift $u\colon \tilde{y} \to x$ of v, meaning that $f(u) = v$. Then f is a cartesian fibration.*

Proof. Consider the directed evaluation map $\vec{\mathrm{ev}}_1^f\colon \mathrm{Fun}([1],C) \to D \vec{\times}_f C$. We have to show that it admits a right adjoint section. We will apply Theorem 5.9.12, so we have to prove that for every object (v,y) in $D \vec{\times}_f C$ the slice $\mathrm{Fun}([1],C)_{/(v,y)}$ admits a terminal object. We proved in Lemma 5.5.17 that the slice $\mathrm{Fun}([1],C)_{/u}$ admits a terminal object. Hence it will suffice to show that the functor

$$\begin{tikzcd}
\mathrm{Fun}([1],C)_{/u} \arrow[rr] \arrow[dr] & & \mathrm{Fun}([1],C)_{/(v,y)} \arrow[dl] \\
& \mathrm{Fun}([1],C) &
\end{tikzcd}$$

is an equivalence over $\mathrm{Fun}([1],C)$. Since both functors are right fibrations, also the horizontal functor is a right fibration, and thus it will suffice to show that the fibers over any object in $\mathrm{Fun}([1],C)_{/(v,y)}$ are contractible. Such an object is given by a pair (α,τ) where $\alpha\colon y' \to y$ is a map fitting in a commutative square as follows:

$$\begin{tikzcd}
f(x') \arrow[r, "v'=f(u')"] \arrow[d, "w"'] & f(y') \arrow[d, "f(\alpha)"] \\
f(x) \arrow[r, "v"] & f(y).
\end{tikzcd}$$

But since u is f-cartesian, there exists a unique morphism $\tilde{w}$ making the following diagram commute:

$$\begin{tikzcd}
x' \arrow[r] \arrow[d, "\tilde{w}"'] & y' \arrow[r] & y \\
x. \arrow[urr, "u"'] & &
\end{tikzcd}$$

This proves that $\vec{\text{ev}}_1^f$ admits a right adjoint $\sigma\colon D \vec{\times}_f C \to \text{Fun}([1], C)$, with unit $\eta\colon 1 \to \sigma\text{ev}_f$ and counit $\vec{\text{ev}}_1^f\sigma \to 1$ such that $\sigma(v, y)$ is the chosen f-cartesian lift u of v and such that ε restricts to the identity on objects. It follows that ε is a natural isomorphism, and which means that σ is a right adjoint section of ε. This shows that f is a cartesian fibration. □

5.10 Limits and colimits

In this section, we start to discuss the concept of *limits* and *colimits* in categories.

Definition 5.10.1. Let I and C be categories. We refer to objects of the functor category $\text{Fun}(I, C)$ as *I-indexed diagrams in C*. Given such a diagram $F \in \text{Fun}(I, C)$, we define the category $C_{/F}$ of *cones over F* as the following relative slice category:

$$\begin{tikzcd} C_{/F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \text{Fun}(I, C)_{/F} \arrow[d] \\ C \arrow[r, "x \mapsto \text{const}_x"'] & \text{Fun}(I, C). \end{tikzcd}$$

In other words, a cone over F consists of a pair (x, ε) where $x \in C$ and $\varepsilon\colon \text{const}_x \to F$ is a natural transformation. A cone (x, ε) over F is called a *limit cone* if it is a terminal object of the cone category $C_{/F}$. If it exists, a limit cone is unique up to isomorphism. The underlying object x will then be called the *limit of F* and denoted by $\lim(F)$, of $\lim_I F$ or $\lim_{i\in I} F(i)$.

Dually, we define a *cocone under F* to be an object of the dually defined category $C_{F/}$. A *colimit cocone* is an initial object of $C_{F/}$, and its underlying object, if it exists, will be denoted by $\text{colim}(F)$, or $\text{colim}_I F$ or $\text{colim}_{i\in I} F(i)$.

The condition that C admits *all* I-indexed (co)limits admits

Definition 5.10.2. Let I and C be categories. We say that C *admits I-indexed limits (resp. colimits)* if the functor $p_I^*\colon C \to \text{Fun}(I, C)$ admits a right (resp. left) adjoint

$$\lim_I\colon \text{Fun}(I, C) \to C \qquad (\text{ resp. } \text{colim}_I\colon \text{Fun}(I, C) \to C).$$

Proposition 5.10.3. *A category C admits I-indexed colimit if and only if for every functor $F\colon I \to C$ (in arbitrary groupoidal context Γ) there exists a colimit $\text{colim}_I(F)$ of F.*

Proof. This is a particular case of Proposition 5.9.14. □

Definition 5.10.4. Let I be a category and let $f\colon C \to D$ be a functor. If C and D admit I-indexed colimits, we say that *f preserves I-indexed colimits* if the commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\text{const}_D"'] & D \arrow[d, "\text{const}_C"] \\ \text{Fun}(I, C) \arrow[r, "f_*"] & \text{Fun}(I, D) \end{tikzcd}$$

is left-adjointable, in the sense that the Beck-Chevalley map

$$\text{BC}\colon \text{colim}_I \circ f_* \to f \circ \text{colim}_I$$

is a natural isomorphism of functors $\text{Fun}(I, C) \to D$.

A straightforward reformulation of the previous definition is:

Proposition 5.10.5. *A functor $f: C \to D$ preserves I-indexed colimits if and only if for every I-indexed diagram $F: I \to C$ (in an arbitrary context Γ) and an arbitrary colimit cocone* $(\operatorname{colim}_I F, \varepsilon)$ *for F, the induced cocone $(f(\operatorname{colim}_I F), f(\varepsilon))$ is a colimit cocone for $f \circ F$.*

Lemma 5.10.6 (Adjoints preserve (co)limits)**.** *Let $f: C \to D$ be a functor and let I be a category such that both C and D admit I-indexed colimits. If f admits a right adjoint $g: D \to C$, then f commutes with I-indexed colimits. The dual statement that right adjoints preserve limits also holds true.*

Proof. We have to show that the Beck-Chevalley transformation in the diagram

$$\begin{tikzcd} C \arrow[r, "f"] & D \\ \operatorname{Fun}(I, C) \arrow[u, "\operatorname{colim}_I"] \arrow[r, "f_*"] & \operatorname{Fun}(I, D) \arrow[u, "\operatorname{colim}_I"'] \arrow[ul, Rightarrow] \end{tikzcd}$$

is a natural isomorphism. By passing to right adjoints, this is equivalent to showing that the Beck-Chevalley transformation in the diagram

$$\begin{tikzcd} C \arrow[d, "\operatorname{const}_D"'] \arrow[dr, Rightarrow] & D \arrow[l, "g"'] \arrow[d, "\operatorname{const}_C"] \\ \operatorname{Fun}(I, C) & \operatorname{Fun}(I, D) \arrow[l, "g_*"] \end{tikzcd}$$

is a natural isomorphism. But this is clear since the adjunction $f_*: \operatorname{Fun}(I, C) \rightleftarrows \operatorname{Fun}(I, D) : g_*$ is natural in I. □

Lemma 5.10.7. *Let D and I be small categories and assume that D admits I-indexed colimits. Then for every category C also* $\operatorname{Fun}(C, D)$ *admits I-indexed colimits and for every functor $f: C' \to C$ the restriction functor $f^*:$* $\operatorname{Fun}(C, D) \to \operatorname{Fun}(C', D)$ *preserves I-indexed colimits.*

The dual statement for I-indexed limits also holds true.

Proof. If we have an adjunction of the form

$$\operatorname{colim}_I: \operatorname{Fun}(I, D) \rightleftarrows D: \operatorname{const}_I \ .$$

Therefore, for any category C, there is an induced adjunction of the form

$$(\operatorname{colim}_I)_*: \operatorname{Fun}(C, \operatorname{Fun}(I, D)) \rightleftarrows \operatorname{Fun}(C, D): (\operatorname{const}_I)_* \, .$$

By virtue of Proposition 5.4.4, this yields an adjunction of the form

$$\operatorname{colim}_I: \operatorname{Fun}(I, \operatorname{Fun}(C, D)) \rightleftarrows \operatorname{Fun}(C, D): \operatorname{const}_I \ .$$

By construction, the unit and co-unit of the latter adjunction are those in D, termwise with respect to C. This implies right away that, for any functor $f: C' \to C$ the functor

$$f^*: \operatorname{Fun}(C, D) \to \operatorname{Fun}(C', D)$$

preserves I-indexed colimits by Proposition 5.10.5. □

Examples of limits and colimits

Throughout, we let C denote a category.

Observation 5.10.8. Consider the empty diagram $I = \emptyset$. Then the category $\mathrm{Fun}(\emptyset, C)$ is contractible by Lemma 1.4.13, providing a unique functor $F\colon \emptyset \to C$. Observe that $C_{/F} \simeq C \simeq C_{F/}$. So cones and cocones on F are simply given by objects $x \in C$, and as a consequence we see:

- Terminal objects in C are the same as $\emptyset$-indexed limits;
- Initial objects in C are the same as $\emptyset$-indexed colimits.

Definition 5.10.9. Consider the two-point set $I = \{0,1\} := * \sqcup *$. Then the two evaluation maps define an equivalence $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}(\{0,1\}, C) \xrightarrow{\sim} C \times C$, so that a functor $\{0,1\} \to C$ is the same as a pair (x, y) of objects of C.

- A *product of x and y*, denoted $x \times y$, is a limit of the $\{0,1\}$-indexed diagram (x, y).
- A *coproduct of x and y*, denoted $x \sqcup y$, is a colimit of the $\{0,1\}$-indexed diagram (x, y).

Unwinding the definitions, we see that $x \times y$ is terminal among objects equipped with maps $\mathrm{pr}_x\colon x \times y \to x$ and $\mathrm{pr}_y\colon x \times y \to y$. Dually, $x \sqcup y$ is initial among objects equipped with maps $x \to x \sqcup y$ and $y \to x \sqcup y$.

We say that C *admits binary (co)products* if C admits $\{0,1\}$-indexed (co)limits.

Definition 5.10.10. We define categories $\lrcorner$ and $\ulcorner$ as follows:

$$\lrcorner := \{0,1\} \star * \qquad \text{and} \qquad \ulcorner := * \star \{0,1\},$$

using the join operation from Axiom J.1.

The notations for these categories reflect the shape of the diagrams they classify: a functor $\lrcorner \to C$ corresponds to a diagram in C of the form

$$\begin{tikzcd} & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z, \end{tikzcd}$$

while a functor $\ulcorner \to C$ corresponds to a diagram of the form

$$\begin{tikzcd} z \arrow[r, "f"] \arrow[d, "g"'] & x \\ y. & \end{tikzcd}$$

More precisely, there are pullback squares

$$\begin{tikzcd} \mathrm{Fun}(\lrcorner, C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], C) \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Fun}([1], C) \arrow[r, "\mathrm{ev}_1"] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Fun}(\ulcorner, C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], C) \arrow[d, "\mathrm{ev}_0"] \\ \mathrm{Fun}([1], C) \arrow[r, "\mathrm{ev}_0"] & C. \end{tikzcd}$$

Definition 5.10.11. Consider a $\lrcorner$-indexed diagram $x \xrightarrow{f} z \xleftarrow{g} y$ in C. We refer to its limit as the *pullback of f and g*, and denote it by $x \times_z y$. We say that C *admits pullbacks* if it admits $\lrcorner$-indexed limits.

Dually, given a $\ulcorner$-indexed diagram $x \xleftarrow{f} z \xrightarrow{g} y$, we refer to its colimit in C as the *pushout of f and g*, and denote it by $x \sqcup_z y$. We say that C *admits pushouts* if it admits $\ulcorner$-indexed colimits.

5.11 Locally cocartesian fibrations

In this section, we introduce the notion of a locally (co)cartesian fibration. Our exposition closely follows that of [Lur, Section 5.1.3, 01TW].

Definition 5.11.1. Let $p\colon C \to S$ be a functor, and consider a morphism $u\colon x \to y$ in C, say in context Γ. This morphism uniquely lifts to a morphism u' in the category C' defined by the following pullback square in context Γ:

$$\begin{tikzcd}
{[1]} \arrow[dr, dashed, "u'"] \arrow[drr, bend left, "u"] \arrow[ddr, equal, bend right] & & \\
& C' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\
& {[1]} \arrow[r, "p(u)"] & S.
\end{tikzcd}$$

We say that u is *locally p-cocartesian* if the morphism u' is p'-cocartesian. We say that p is a *locally cocartesian fibration* if for every morphism $w\colon s \to t$ in S and any $x\colon C_s$ there exists a locally p-cocartesian lift $u\colon x \to y$ of w starting in x.

One may dually define the notion of a *locally p-cartesian* morphism and the associated notion of locally cartesian fibrations.

Remark 5.11.2. Notice that the morphism $u\colon x \to y$ is locally p-cocartesian fibration if for every object $z \in C_{p(y)}$ the functor $v \mapsto vu$ given by composing with u induces an equivalence

$$C_{p(y)}(y, z) \to C(x, z)_{p(u)},$$

where $C_{p(y)}$ is the fiber of p over $p(y)$ and $C(x, z)_{p(u)}$ is defined as the fiber

$$\begin{tikzcd}
C(x', y)_{p(u)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C(x', y) \arrow[d, "p"] \\
* \arrow[r, "p(u)"] & S(p(x), p(y)).
\end{tikzcd}$$

More informally, this says that for any morphism $u'\colon x \to z$ such that $p(u') = p(u)$, there is a unique morphism $v\colon y \to z$ in $C_{p(y)}$ fitting in a commutative triangle

$$\begin{tikzcd}
& y \arrow[dr, dashed, "\exists! v"] & \\
x \arrow[ur, "u"] \arrow[rr, "u'"'] & & z
\end{tikzcd}$$

in C which is sent under p to the degenerate commutative triangle on $p(u)$ in S.

Observation 5.11.3. Note that a functor $p\colon C \to S$ is a locally cocartesian fibration if and only if for every functor $\Gamma \times [1] \to S$ the base change $p'\colon C' \to \Gamma \times [1]$ of p is a cocartesian fibration.

Example 5.11.4. Any p-cartesian morphism is locally p-cartesian.

Example 5.11.5. Any isomorphism is p-cartesian, hence in particular locally p-cartesian.

Example 5.11.6. Let A and B be two categories. A morphism (u, v), consisting of a pair of morphisms $u\colon x_0 \to x_1$ and $v\colon y_0 \to y_1$ in A and B, respectively, is locally pr_A-cocartesian if and only if v is an isomorphism. To see this, we observe that $(u, \mathrm{id}_b)\colon (x_0, b) \to (x_1, b)$ is pr_A-cocartesian for any object b by direct inspection of the universal properties. On the other hand, for any morphism $u\colon x_0 \to x_1$ in A any object of the form (x_0, b) in $A \times B$, the morphism (u, id_b) is a locally pr_A-cocartesian lift of u. This implies that a morphism of $A \times B$ is pr_A-cocartesian if and only if it is isomorphic to a morphism of the form (u, id_b) with v an isomorphism in B. It is straightforward to see that the latter conditions characterizes the morphisms of $A \times B$ whose second component is an isomorphism.

Observation 5.11.7. Consider a pullback square

$$\begin{tikzcd}
C' \arrow[r, "f"] \arrow[d, "p'"'] & C \arrow[d, "p"] \\
S' \arrow[r] & S
\end{tikzcd}$$

and let $u'\colon x' \to y'$ be a morphism in C' with image $u := f(u')\colon x \to y$ in C. Then u' is locally p'-cocartesian if and only if u is locally p-cocartesian.

Proposition 5.11.8. *Let $p\colon C \to S$ be a functor and let $u\colon x \to y$ be a morphism in C. Assume that $p(u)\colon p(x) \to p(y)$ is an isomorphism in C. Then the following conditions are equivalent:*

(1) u is locally p-cocartesian;

(2) u is locally p-cartesian;

(3) u is an isomorphism in C.

Proof. By Observation 5.11.7 each of these three properties are stable under base change. Since the morphism $p(u)\colon [1] \to S$ factors through $[0]$ (since it is an isomorphism), we may thus assume that S is the terminal category $[0]$. In particular p is a cocartesian fibration by Proposition 5.7.6. But in this case, a morphism u in C living over an isomorphism in S is an isomorphism itself if and only if it is p-cocartesian, showing the equivalence between (1) and (3). Similarly one shows (1) $\Longleftrightarrow$ (2). $\square$

Lemma 5.11.9. *Assume that $S = T^{\triangleright} = T \star [0]$ for some category T. Let $p\colon C \to T^{\triangleright}$ be a functor and let $u\colon x \to y$ be a morphism in C such that $p(y)$ is the cocone point of $T^{\triangleright}$. Then u is locally p-cocartesian if and only if it is p-cocartesian.*

Proof. This is immediate from the characterization from Remark 5.11.2, since the image of any morphism $y \to z$ in C under $p\colon C \to T^{\triangleright}$ is the identity on $p(y)$: Since there are no morphimsms from the cocone point of $T^{\triangleright}$ to any term´ coming from T (because there is no morphism from 1 to 0 in $[1]$), the morphism u, seen as a functor of the form $[1] = [0] \star [0] \to T \star [0]$ must send all of $[1]$ to the full subcategory $\emptyset \star [0] \simeq [0]$ of $T^{\triangleright}$, hence must be the identity of $p(y)$. $\square$

Corollary 5.11.10. *Let $p\colon C \to S$ be a functor and consider a commutative triangle $\sigma\colon [2] \to C$ of the form*

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ f \arrow[ur, "f"] \arrow[rr, "h"'] & & z. \end{tikzcd}$$

Assume that f is p-cocartesian. Then g is locally p-cocartesian if and only if h is locally p-cocartesian.

Proof. By pulling back along the map $p(\sigma)\colon [2] \to C$ it will suffice to prove the claim when $S = [2]$ and $p(\sigma) = \mathrm{id}_{[2]}$. In this case Lemma 5.11.9 tells us that the morphisms g and h are locally p-cocartesian if and only if they are p-cocartesian. The claim thus follows from Lemma 5.6.11. □

Proposition 5.11.11. *Let $p\colon C \to S$ be a locally cocartesian fibration. The following conditions are equivalent:*

(1) The functor p is a cocartesian fibration;

(2) Every locally p-cocartesian morphism of C is p-cocartesian;

(3) Locally p-cocartesian morphisms are closed under composition: if $f\colon x \to y$ and $g\colon y \to z$ are locally p-cocartesian morphisms in C then so is $gf\colon x \to z$.

Proof. If p is a cocartesian fibration, then condition (2) holds: indeed, if $f\colon x \to y$ is a locally p-cocartesian morphism, then there is a unique p-cocartesian lift $g\colon x \to y'$ of $p(f)$. Since both f and g are locally p-cocartesian lifts of $p(f)$ with the same source, there is a unique isomorphism $u\colon y \to y'$ in the fiber of $p(y)$ such that $gu = f$. In particular, f is the composition of two p-cocartesian morphisms, hence is p-cocartesian as well. Since p-cocartesian morphisms are stable under composition, it is clear that condition (3) follows from condition (2). It remains to prove that condition (1) follows from condition (3). Let us assume that condition (3) holds and that $f\colon x \to y$ is a locally p-cocartesian morphism. Lemma 5.9.9 tells us that, order to prove that f is p-cocartesian, it suffices to prove that, for any object z in C,

$$\begin{tikzcd} C(y,z) \arrow[r, "-\circ f"] \arrow[d, "p"'] & A(x,z) \arrow[d, "p"] \\ S(p(y),p(z)) \arrow[r, "-\circ p(f)"] & S(p(x),p(z)) \end{tikzcd}$$

is a pullback square of animae. This means that we have to check that the map induced by this pullback square

$$C(y,z) \to S(p(y),p(z)) \times_{S(p(x),p(z))} A(x,z)$$

is an equivalence. In other words, by virtue of Proposition 1.1.15, if we consider any morphism $g\colon x \to z$ in C as well as a morphism $v\colon p(y) \to p(z)$ in S such that $p(g)$ is the composition of v and $p(f)$, we have to prove that there is a unique morphism $u\colon y \to z$ in X such that $p(u) = v$ and such that g is the composition of u and f. We define $u'\colon y \to z'$ as the unique locally p-cocartesian lift of v with source y. We define $u = hu'$. Since $u'f$ is a locally p-cocartesian lift of $vp(f)$ there is a unique morphism $h\colon z' \to z$ in the fiber over $p(z)$ such that $uf = hu'f$ can be identified with g.

If we have another morphism of the form $w\colon y \to z$ in C such that $wf = g$, and such that $p(w) = v$, then we have a unique morphism $k\colon z' \to z$ in the fiber over $p(z)$ such that $ku' = w$. In particular, we must have

$$h\,u'\,f = g \quad \text{and} \quad k\,u'\,f = g$$

and the universal property of $u'\,f$ as locally p-cocartesian lift of $v\,p(f)$ implies that k and h must agree as terms of $C(z', z)$. Therefore, $u = w$ must hold true. $\square$

Remark 5.11.12. Let $p\colon C \to S$ bs a cocartesian fibration. Given any morphism $v\colon y_0 \to y_1$ in S and any object x_0 in C such that $p(x_0) = y_0$, there is a (locally) p-cocartesian morphism $u\colon x_0 \to x_1$ in C such that $p(u) \cong v$ in $S(y_0, y_1)$. On the other hand, there is a functor

$$v_!\colon C_{y_0} \to C_{y_1}$$

defined by covariant transport. Each object x_0 of C_{y_0} comes with a canonical map

$$\varepsilon_{x_0} := \mathrm{lift}^p_0(x_0, v)\colon x_0 \to v_!(x_0)$$

and there is a canonical map $c_u\colon v_!(x_0) \to x_1$ in C_{x_1} such that $c_u \varepsilon_{x_0} = u$ holds in C. By uniqueness of locally p-cocartesian lifts, the property that u is a locally p-cocartesian lift of v is equivalent to the property that it is a p-cocartesian morphism in C, which in turns means that c_u is an isomorphism in C. In other words, up to isomorphism in the category of morphism in C, all (locally) p-cocartesian morphisms are of the form ε_{x_0} for some object x_0 of C and some morphism v in S as above.

Proposition 5.11.13. *Let $p\colon C \to A \times B$ be a functor. We denote by $\mathrm{pr}_A\colon A \times B \to A$ and $\mathrm{pr}_B\colon A \times B \to B$ the obvious projections, and we assume that $\mathrm{pr}_B\, p\colon C \to B$ is a cocartesian fibration. The property that p is a cocartesian fibration is equivalent to the following list of properties to be fulfilled altogether:*

(i) the functor $p\colon C \to A \times B$ is cocartesian over B.

(ii) for any object b of B, the induced functor $p_b\colon C_b \to A$ is a cocartesian fibration;

(iii) for any morphism $v\colon b_0 \to b_1$ in B, the covariant transport functor

$$v_!\colon C_{b_0} \to C_{b_1}$$

sends p_{b_0}-cocartesian morphisms to p_{b_1}-cocartesian morphisms (i.e. preserves the property of being a locally p-cocartesian morphism in X if we see each fiber X_b as a subcategory of X).

Proof. By virtue of Example 5.11.6 and of Proposition 5.11.11, pr_B-cocartesian morphisms are pairs of the form (u, v), with u any morphism in A and v any isomorphism in B. It is clear that all properties (i), (ii) and (iii) are verified whenever p is a cocartesian fibration. Conversely, let us assume all three of them, and let us verify that p is a cocartesian fibration.

Given any object b of B, we can identify A with the fiber of pr_B at b (in context $\{b\}$) and the induced covariant transport functor associated to any morphism $v\colon b_0 \to b_1$ in B with respect to the projection pr_B simply is the identity of A. Condition (i) thus ensures that we have canonical commutative trangles of the form

$$\begin{tikzcd} C_{b_0} \arrow[rr, "v_!"] \arrow[dr, "p_{b_0}"'] & & C_{b_1} \arrow[dl, "p_{b_1}"] \\ & A & \end{tikzcd}$$

for each morphism $v : b_0 \to b_1$ in B. Condition (ii) ensures that each projection p_b is a cocartesian fibration and condition (iii) means that each of these functor $v_!$ in the triangles above is cocartesian over A.

Let us consider a morphism $f : x_0 \to x_1$ in C. Its image by p is of the form $p(f) = (u, v)$ with $u : a_0 \to a_1$ and $v : b_0 \to b_1$ morphisms in A and in B, respectively. This yields a commutative diagram of the following form in $A \times B$:

$$\begin{array}{ccc} (a_0, b_0) & \xrightarrow{(u,\mathrm{id}_{b_0})} & (a_1, b_0) \\ {\scriptstyle (\mathrm{id}_{a_0},v)}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id}_{a_1},v)} \\ (a_0, b_1) & \xrightarrow{(u,\mathrm{id}_{b_1})} & (a_1, b_1) \end{array}$$

We consider the morphism $\varepsilon_{x_0} : x_0 \to u_!(x_0)$ obtained as the image in C of the p_{b_0}-cocartesian lift of u. We may assume that $\mathrm{pr}_B(p(\varepsilon_{x_0}))$ is the identity of b_0. Seeing $u_!(x_0)$ as an object of the fiber C_{b_0} we may now apply the functor $v_!$ and get an object $v_!u_!(x_0)$ in the fiber C_{b_1}. Alternatively, we may as well apply covariant transport along v to x_0, yielding an object $v_!(x_0)$ in the fiber C_{b_1}, observing that we may assume that $\mathrm{pr}_A\, p(v_!(x_0)) = a_0$ (by condition (i)) and then apply covariant along $u_!$. The fact that $v_!(\varepsilon_{x_0})$ is a p_{b_1}-cocartesian morphism ensures that we have a canonical identication of the form

$$v_!u_!(x_0) \xrightarrow{\sim} u_!v_!(x_0)$$

inducing a commutative square of the form

$$\begin{array}{ccc} x_0 & \xrightarrow{\varepsilon_{x_0}} & u_!(x_0) \\ {\scriptstyle \eta_{x_0}}\big\downarrow & & \big\downarrow{\scriptstyle \eta_{u_!(x_0)}} \\ v_!(x_0) & \xrightarrow{\varepsilon_{v_!(x_0)}} & v_!u_!(x_0) \end{array}$$

in which the vertical maps are $\mathrm{pr}_B\, p$-cocartesian morphisms (the commutativity of the square comes from the fact that the formation of $v_!(x_0)$ is functorial in x_0 and from the identification $\varepsilon_{v_!(x_0)} = v_!(\varepsilon_{x_0})$ due to condition (iii)). There is a unique morphism $g : v_!(x_0) \to x_1$ such that $g\eta_{x_0} = f$. We may assume furthermore that $\mathrm{pr}_B\, p(g)$ is the identity of b_1. In other words, we can see g as a map in C_{b_1}. There is thus a unique morphism of the form $h : v_!u_!(x_0) \to x_1$ in C such that $h\varepsilon_{v_!(x_0)} = g$ and such that $\mathrm{pr}_B\, p(h) = \mathrm{id}_{b_1}$. In other words, the morphism $\varepsilon_{v_!(x_0)}\eta_{x_0}$ is a locally p-cocartesian lift of (u, v). This proves that p is a local cocartesian fibration.

By virtue of Proposition 5.11.11, it remains to prove that locally p-cocartesian morphisms are stable under composition in C. Since locally p-cocartesian morphisms are those isomorphic to those of the form $\varepsilon_{v_!(x_0)}\eta_{x_0}$ constructed above it suffices to check that those are stable under composition. This boils down to the fact that (locally) $p_B p$-cocartesian morphisms are stable under composition and that p_b-cocartesian morphisms are stable under composition for each object b of B, which yields the property that commutative squares as above are stable under horizontal as well as under vertical composition. □

5.12 Cocartesian functors

In Lemma 5.2.13, we saw that any functor between left fibrations is compatible with the covariant transport functors. The situation is different for cocartesian fibration: a functor between cocartesian fibrations is compatible with the covariant transport functors if and only if it preserves cocartesian morphisms. Functors satisfying this property are called *cocartesian functors* and will be made precise in this section.

Construction 5.12.1 (Beck-Chevalley transformation)**.** Let $p\colon C \to S$ and $q\colon D \to S$ be two cocartesian fibrations, and consider a commutative triangle

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & S. & \end{tikzcd}$$

By definition of cocartesian fibrations, the map $\vec{\mathrm{ev}}_0^p\colon \mathrm{Fun}([1], C) \to C \vec{\times}_p S$ admits a left adjoint section lift_0^p, and similarly $\vec{\mathrm{ev}}_0^q$ admits a left adjoint section lift_0^q. In particular, we have natural isomorphisms $\vec{\mathrm{ev}}_0^p \,\mathrm{lift}_0^p \cong \mathrm{id}$ and $\vec{\mathrm{ev}}_0^q \,\mathrm{lift}_0^q\, \mathrm{id}\,\mathrm{id}$, and transformations $\varepsilon_p\colon \mathrm{lift}_0^p \,\vec{\mathrm{ev}}_0^p \to \mathrm{id}$ and $\varepsilon_q\colon \mathrm{lift}_0^q \,\vec{\mathrm{ev}}_0^q \to \mathrm{id}$. Observe that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Fun}([1], C) \arrow[r, "f_*"] \arrow[d, "\vec{\mathrm{ev}}_0^p"'] & \mathrm{Fun}([1], D) \arrow[d, "\vec{\mathrm{ev}}_0^q"] \\ C \vec{\times}_p S \arrow[r, "\vec{f}"] & D \vec{\times}_q S. \end{tikzcd}$$

We now define the *Beck-Chevalley transformation* $\mathrm{BC}_!\colon \mathrm{lift}_0^q \,\vec{f} \to f_* \,\mathrm{lift}_0^p$ as the following composite:

$$\mathrm{lift}_0^q \,\vec{f} \cong \mathrm{lift}_0^q \,\vec{f}\vec{\mathrm{ev}}_0^p \,\mathrm{lift}_0^p \cong \mathrm{lift}_0^q \,\vec{\mathrm{ev}}_0^q f_* \,\mathrm{lift}_0^p \xrightarrow{\varepsilon_q} f_* \,\mathrm{lift}_0^p .$$

Definition 5.12.2 (Cocartesian functor)**.** The functor f over S as above is said to be a *cocartesian functor over S* if the natural transformation $\mathrm{BC}_!\colon \mathrm{lift}_0^q \,\vec{f} \to f_* \,\mathrm{lift}_0^p$ is invertible.

There is a dual notion of a *cartesian functor* between cartesian fibrations over S.

We can make the Beck-Chevalley transformation a bit more concrete as follows. For every morphism $v\colon x \to y$ in S and every term a of C_x, the map $\mathrm{BC}_!(v)\colon \mathrm{lift}_0^q \,\vec{f}(a, v) \to f_* \,\mathrm{lift}_0^p(a, v)$ takes the form of a commutative square

$$\begin{tikzcd} f(a) \arrow[r, equal] \arrow[d, "\mathrm{lift}_0^q(f(a), v)"'] & f(a) \arrow[d, "f(\mathrm{lift}_0^p(a, v))"] \\ v_!(f(a)) \arrow[r, "\overline{v}"] & f(v_!(a)). \end{tikzcd}$$

In this sense, we see that the Beck-Chevalley transformation induces a canonical morphism of functors

$$\overline{v}\colon v_! f_x \to f_y v_!$$

where $v_!\colon C_x \to C_y$ and $v_!\colon D_x \to D_y$ are the covariant transport functors and $f_x\colon C_x \to D_x$ is the functor induced by f on each fiber.

Lemma 5.12.3. *Given two cocartesian fibrations $p\colon C \to S$ and $q\colon D \to S$, and a functor $f\colon C \to D$ over S, the following assertions are equivalent:*

(i) the functor f is cocartesian over S;

(ii) for any morphism $v\colon x \to y$ in S, the Beck-Chevalley transformation induces a commutative square of the form

$$\begin{tikzcd} C_x \arrow[r, "f_x"] \arrow[d, "v_!"'] & D_x \arrow[d, "v_!"] \\ C_y \arrow[r, "f_y"] & D_y \end{tikzcd} \quad ;$$

(iii) the functor f sends locally p-cocartesian morphisms to locally q-cocartesian morphisms;

(iv) the functor f sends p-cocartesian morphisms to q-cocartesian morphisms.

Proof. Proposition 5.11.11 tells us that conditions (iii) and (iv) obviously are equivalent. The previous remark explains why conditions (i) and (ii) are equivalent. Finally, the equivalence between conditions (ii) and (i) is a straightforward consequence of the characterization of p-cocartesian morphisms given in Remark 5.11.12. □

Example 5.12.4. For $S = *$, every functor $f\colon C \to D$ is a cocartesian functor over $*$. Indeed, we have $C \vec{\times}_p * \simeq C$, with the directed evaluation map corresponding to $\mathrm{ev}_0\colon \mathrm{Fun}([1], C) \to C$, which admits a left adjoint section $p^*_{[1]}\colon C \to \mathrm{Fun}([1], C)$. The same holds for D, and the claim follows from naturality of $p^*_{[1]}$.

Corollary 5.12.5. *Every equivalence $C \xrightarrow{\sim} D$ over S is a cocartesian functor.* □

Example 5.12.6. If p and q are left fibrations, then any functor f over S from C to D is a cocartesian functor over S: this is a direct consequence of the fact that the two adjunctions $\mathrm{lift}_0^p \dashv \vec{\mathrm{ev}}_0^p$ and $\mathrm{lift}_0^q \dashv \vec{\mathrm{ev}}_0^q$ are adjoint equivalences.

Lemma 5.12.7. *Consider cocartesian fibrations $f\colon C \to D$ and $q\colon D \to S$, and let $p := qf\colon C \to S$ denote the composite cocartesian fibration from Proposition 5.6.13. Then f is a cocartesian functor over S.*

Proof. By construction of the Beck-Chevalley map, it will suffice to show that the map

$$\mathrm{lift}_0^q \, \vec{\mathrm{ev}}_0^q f_* \, \mathrm{lift}_0^p \xrightarrow{\varepsilon_q} f_* \, \mathrm{lift}_0^p$$

is an equivalence. Spelling out the left adjoint section lift_0^p obtained from Proposition 5.6.13, we see that it sits in the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fun}([1], C) \arrow[r, shift right, "\vec{\mathrm{ev}}_0^f"'] \arrow[dr, "f_*"'] & C \vec{\times}_f D \arrow[l, shift right, "\mathrm{lift}_0^f"'] \arrow[r, shift right] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \vec{\times}_{qf} S \arrow[l, shift right] \arrow[d] \arrow[ll, bend right, "\mathrm{lift}_0^{qf}"'] \\ & \mathrm{Fun}([1], D) \arrow[r, shift right, "\vec{\mathrm{ev}}_0^q"'] & D \vec{\times}_q S. \arrow[l, shift right, "\mathrm{lift}_0^q"'] \end{tikzcd}$$

In particular, we see that $f_* \operatorname{lift}_0^p$ factors through lift_0^q, and thus it suffices to show that the map

$$\operatorname{lift}_0^q \vec{\operatorname{ev}}_0^q \operatorname{lift}_0^q \xrightarrow{\varepsilon_q \operatorname{lift}_0^q} \operatorname{lift}_0^q$$

is an equivalence. But this is immediate from the triangle identities and the fact that the unit $\mathrm{id} \to \vec{\operatorname{ev}}_0^q \operatorname{lift}_0^q$ of the adjunction $\operatorname{lift}_0^q \dashv \vec{\operatorname{ev}}_0^q$ is a natural isomorphism. □

Proposition 5.12.8. *For any category C the diagram*

$$\begin{tikzcd} \operatorname{Fun}([1], C) \arrow[rr, "(\operatorname{ev}_0, \operatorname{ev}_1)"] \arrow[dr, "\operatorname{ev}_1"'] & & C \times C \arrow[dl, "\operatorname{pr}_2"] \\ & C & \end{tikzcd}$$

is a cocartesian functor over C.

Since locally ev_1-cocartesian morphisms in $\operatorname{Fun}([1], C)$ precisely are those commutative squares in C of the form

$$\begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "u"'] & y' \arrow[d, "v"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in which both u and f' are isomorphisms, this is a direct consequence of:

Proposition 5.12.9. *Let $p\colon X \to B$ be any cocartesian fibration and $u\colon C \to C$ any functor. Then the diagram*

$$\begin{tikzcd} X \arrow[rr, "(p,u)"] \arrow[dr, "p"'] & & B \times C \arrow[dl, "\operatorname{pr}_B"] \\ & B & \end{tikzcd}$$

is a cocartesian functor over B if and only if the functor u sends locally p-cocartesian morphisms to isomorphisms.

Proof. We know from Example 5.11.6 that the (locally) pr_B-cocartesian morphisms in $B \times C$ are nothing else than pairs (u, v) with $u\colon b_0 \to b_1$ any morphism in B and $v\colon c_0 \to c_1$ any isomorphism in C. □

Definition 5.12.10. Let us consider a commutative square of categories of the form bellow.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "p"'] & D \arrow[d, "q"] \\ S \arrow[r, "u"] & T \end{tikzcd}$$

We assume that both p and q are cocartesian (cartesian) fibrations. We say that f is a *cocartesian functor over u* (a *cartesian functor over u*) if the induced functor $C \to S \times_T D$ is a cocartesian functor over S (a cartesian functor over S, respectively).

Remark 5.12.11. For a commutative square as in the definition above, the property that f is a (co)cartesian functor over T is equivalent to the property that it sends p-(co)cartesian morphism in C to q-(co)cartesian morphisms in D.

Proposition 5.12.12. *For any functor $f: C \to D$, the diagram*

$$\begin{tikzcd} \mathrm{Fun}([1],C) \arrow[r, "\vec{\mathrm{ev}}_1^f"] \arrow[d, "\mathrm{ev}_1"'] & D \vec{\times}_f C \arrow[d, "\mathrm{ev}_1"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

determines $\vec{\mathrm{ev}}_1^f$ as a cocartesian functor over f.

Proof. Recall that terms of $D \vec{\times}_f C$ are pairs (x,u) where x is a term of C and $u: f(x) \to y$ a term of $\mathrm{Fun}([1],D)$ whose source agrees with $f(x)$. In particular, morphisms in $D \vec{\times}_f C$ are essentially characterized as morphisms $\alpha: x \to x'$ in C together with commutative squares of the form

$$\begin{tikzcd} f(x) \arrow[r, "u"] \arrow[d, "f(\alpha)"'] & y \arrow[d, "\beta"] \\ f(x') \arrow[r, "u'"] & y' \end{tikzcd}$$

in D. Such data yield an ev_1-cocartesian morphism if and only if α is an isomorphism in C (more precisely, what the proof of Proposition 5.8.5 tells us is that morphisms that are isomorphic to morphisms obtained from data as above in which α is an identity are ev_1-cocartesian). On the other hand ev_1-cocartesian morphisms in $\mathrm{Fun}([1],C)$ correspond to commutative squares in which the maps going out of the upper left corner are isomorphisms. It is thus clear that the functor $\vec{\mathrm{ev}}_1^f$ preserves ev_1-cocartesian morphisms, so that Lemma 5.12.3 applies to this situation. □

Lemma 5.12.13. *For a functor $f: C \times [1] \to D$, the induced functor $(f, \mathrm{pr}_{[1]}): C \times [1] \to D \times [1]$ over $[1]$ is a cocartesian functor over $[1]$ if and only if f is a natural isomorphism of functors $C \to D$.*

Proof. Recall from Example 5.11.6 that the the C-component of a cocartesian morphisms in $C \times [1]$ is necessarily invertible, and similarly for cocartesian morphisms in $D \times [1]$. It follows that $(f, \mathrm{pr}_{[1]})$ is a cocartesian functor over $[1]$ if and only if for every term x of C the morphism $f_x: \{x\} \times [1] \to D$ is invertible in D. But this is equivalent to the condition that f is a natural isomorphism. □

Proposition 5.12.14. *Let $p: C \to S$ and $q: D \to S$ be cocartesian fibrations and let $f: C \to D$ be a functor over S. If f is a cocartesian functor over S, then for every functor $T \to S$ the induced functor $f \times_S T: C \times_S T \to D \times_S T$ is a cocartesian functor over T.*

Proof. Let us abbreviate $C_T := C \times_S T$, $D_T := C \times_S T$ and $f_T := f \times_S T$. In light of (the proof of) Proposition 5.6.14, the functors $p_T: C_T \to T$ and $q_T: D_T \to T$ are again cocartesian fibrations, which left adjoint sections of the directed evaluation maps given via pullback:

$$\begin{tikzcd} \mathrm{Fun}([1],C_T) \arrow[r] \arrow[d, shift left, "\vec{\mathrm{ev}}_0^{p_T}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}([1],C) \arrow[d, shift left, "\vec{\mathrm{ev}}_0^{p}"] \\ C_T \vec{\times}_{p_T} T \arrow[r] \arrow[u, dashed, bend left, "\mathrm{lift}_0^{p_T}"] & C \vec{\times}_p S \arrow[u, bend left, "\mathrm{lift}_0^{p}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Fun}([1],D_T) \arrow[r] \arrow[d, shift left, "\vec{\mathrm{ev}}_0^{q_T}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}([1],D) \arrow[d, shift left, "\vec{\mathrm{ev}}_0^{q}"] \\ D_T \vec{\times}_{q_T} T \arrow[r] \arrow[u, dashed, bend left, "\mathrm{lift}_0^{q_T}"] & D \vec{\times}_q S. \arrow[u, bend left, "\mathrm{lift}_0^{q}"] \end{tikzcd}$$

We need to show that the Beck-Chevalley transformation $\mathrm{BC}_!: \mathrm{lift}_0^{q_T} \vec{f}_T \to (f_T)_* \mathrm{lift}_0^{p_T}$ is a natural isomorphism. Since this is a natural transformation of functors $T \vec{\times}_{p_T} C_T \to \mathrm{Fun}([1], D_T)$, it suffices to show this after application of $\vec{\mathrm{ev}}_0^{q_T}$ and $\mathrm{Fun}([1], D_T) \to \mathrm{Fun}([1], D)$. For the first, this is a consequence of the fact that the lifts form sections of the directed evaluation maps. For the second, this is a consequence of the assumption that f is a cocartesian functor. □

Proposition 5.12.15. *Consider a commutative diagram*

$$\begin{tikzcd} C \times_E D \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "h"] \arrow[ddr, bend left, "q"] & \\ C \arrow[r, "g"] \arrow[drr, bend right, "p"'] & E \arrow[dr, "r"] & \\ & & S \end{tikzcd}$$

of categories, and assume that the functors p, q, r are cocartesian fibrations and f and g are cocartesian functors over S.

(1) The induced functor $C \times_E D \to S$ is a cocartesian fibration;

(2) The projection functors pr_1 and pr_2 are cocartesian functors over S;

(3) For every other cocartesian fibration $T \to S$ and cocartesian functors $t_C : T \to C$ and $t_D : T \to D$ over S such that $f \circ t_C \cong g \circ t_D$ over S, the induced functor $(t_C, t_D) : T \to C \times_E D$ is a cocartesian functor over S.

Proof. We will prove that $\pi : C \times_E D \to S$ is a locally cocartesian fibration with locally π-cocartesian morphisms those pairs (u, v) with u a p-cocartesian morphism in C and v a q-cocartesian morphism in D such that $g(u) = h(v)$ holds. Since such pairs are stable under composition, this will show that π is a cocartesian fibration. Given any morphism $w : s_0 \to s_1$ in S and any object $z_0 = (x_0, y_0)$ in $C \times_E D$, i.e. x_0 is an object of C, y_0 an object of D, and $g(x_0) = h(y_0)$ holds true, we can consider a p-cocartesian lift $u : x_0 \to x_1$ as well as q-cocartesian lift $v : y_0 \to y_1$. For any morphism $f : z_0 \to z_2$ in $C \times_E D$ with $\pi(f) = w$, we have $z_2 = (x_2, y_2)$ with x_2 an object of C and y_2 an object of D such that $g(x_2) = h(y_2)$ holds. In particular, there is a unique morphism $a : x_1 \to x_2$ such that $au = \mathrm{pr}_1(f)$ as well as a unique morphism $b : y_1 \to y_2$ such that $bv = \mathrm{pr}_2(f)$. The pair (a, b) is the unique morphism from z_1 to z_2 factorizing f through (u, v). This shows that (u, v) is a locally cartesian lift of w. This achieves the proof that π is a cocartesian fibration. The explicit characterization of π-cocartesian morphisms in $C \times_E D$ turns the other assertions into obvious statements. □

5.13 Exercises Chapter 5

Exercise 5.13.1. Spell out the missing details in the proof of Proposition 5.4.3.

Exercise 5.13.2. Spell out the missing details in the proof of Proposition 5.4.5.

Exercise 5.13.3. Show that left and right Bousfield localizations are stable under retracts: given a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"] \arrow[rr, bend left, "\mathrm{id}_X"] & X' \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Y \arrow[r] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Y' \arrow[r] & Y, \end{tikzcd}$$

if p' is a left/right Bousfield localization then so is p.

Exercise 5.13.4. Let $p\colon A \to S$ and $q\colon B \to S$ be left fibrations over S. Show that any functor $f\colon A \to B$ over S is cocartesian over S.

6 The Fundamental Theorem of category theory

In ordinary category theory, it is a well-known fact that a functor $f : C \to D$ is an equivalence if and only if it is both fully faithful and essentially surjective. The goal of this chapter is to discuss an analogue of this result in synthetic category theory, and draw various important conclusions from it. All results in this section hold in context Γ for an arbitrary groupoid Γ.

6.1 Fully faithful functors

In this section, we introduce the notion of *fully faithful* functors in the context of synthetic category theory.

Definition 6.1.1 (Fully faithful)**.** A functor $f : C \to D$ is said to be *fully faithful* if for all objects x and y of C the induced functor

$$f : C(x, y) \to D(f(x), f(y))$$

on hom groupoids is an equivalence.

Remark 6.1.2. The map $C(x, y) \to D(f(x), f(y))$ is obtained by passing to fibers over (x, y) in the following commutative diagram:

$$\begin{tikzcd} \mathrm{Map}([1], C) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}([1], D) \arrow[d] \\ C^{\simeq} \times C^{\simeq} \arrow[r, "f^{\simeq} \times f^{\simeq}"] & D^{\simeq} \times D^{\simeq}, \end{tikzcd} \tag{6.1}$$

where we use Corollary 5.9.7 to identify the fibers of the two vertical maps with $C(x, y)$ and $D(f(x), f(y))$, respectively. It follows that fully faithfulness of f is equivalent to the condition that the square (6.1) is a pullback square.

Corollary 6.1.3. *Let $f : C \to D$ and $g : D \to E$ be two functors such that g is fully faithful. Then f is fully faithful if and only if $gf : C \to E$ is fully faithful.*

Proof. This follows from applying the pasting lemma for pullbacks to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}([1],D) \arrow[r, "g_*"] \arrow[d] & \mathrm{Map}([1],E) \arrow[d] \\
C^{\simeq} \times C^{\simeq} \arrow[r, "f^{\simeq}\times f^{\simeq}"] & D^{\simeq} \times D^{\simeq} \arrow[r, "g^{\simeq}\times g^{\simeq}"] & E^{\simeq} \times E^{\simeq}.
\end{tikzcd}$$ ◻

Example 6.1.4. Every equivalence $u\colon C \xrightarrow{\sim} D$ is fully faithful, see Exercise 6.5.1.

Lemma 6.1.5. *For a functor $f\colon C \to D$, the following conditions are equivalent:*

(1) The functor f is fully faithful;

(2) For every object x in C, the following commutative square is a pullback square:

$$\begin{tikzcd}
C_{/x} \arrow[r, "f_*"] \arrow[d] & D_{/f(x)} \arrow[d] \\
C \arrow[r, "f"] & D.
\end{tikzcd}$$

(3) For every object x in C, the following commutative square is a pullback square:

$$\begin{tikzcd}
C_{x/} \arrow[r, "f_*"] \arrow[d] & D_{f(x)/} \arrow[d] \\
C \arrow[r, "f"] & D.
\end{tikzcd}$$

Proof. We will prove the equivalence between (1) and (2); the equivalence between (1) and (3) is dual. The vertical two maps in (2) are both right fibrations by Theorem 5.3.3, hence by Corollary 5.9.4 the square is a pullback square if and only if it induces an equivalence on all fibers. But for an object y of C, the induced map on slices is the map $C(y,x) \to D(f(y), f(x))$, so this condition is satisfied if and only if f is fully faithful. ◻

Lemma 6.1.6. *Let X and Y be groupoids. Then a map $u\colon X \to Y$ is fully faithful if and only if it is an embedding.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\sim"'] & Y \arrow[d, "\sim"] \\
\mathrm{Map}([1],X) \arrow[r, "u_*"] \arrow[d] & \mathrm{Map}([1],Y) \arrow[d] \\
X \times X \arrow[r, "u\times u"] & Y \times Y.
\end{tikzcd}$$

Since the top two vertical maps are equivalences, we see that the outer rectangle is a pullback square if and only if the bottom square is a pullback square. Since the former means that u is an embedding and the latter that u is fully faithful, this finishes the proof. ◻

Proposition 6.1.7. *Let $f\colon C \to D$ be a functor with a left (resp. right) adjoint $s\colon D \to C$. The following conditions are equivalent.*

(i) The functor s is a left (resp. right) adjoint section of f.

(ii) The functor s is fully faithful.

(iii) For any object $y\colon D^{\simeq}$, the co-unit map (resp. the unit map) is an isomorphism between $f(s(y))$ and y.

Proof. We will prove the case for right adjoints; the other case is similar.
It is clear that (iii) follows from (i) by definition. Since a natural transformation is invertible if only if it is objectwise invertible, this means that condition (i) and (iii) are equivalent.

Given objects x and y of D, and let us consider the co-unit map $\varepsilon_x\colon fsx \xrightarrow{\sim} x$. By the triangle identity, the composite

$$sx \xrightarrow{\eta_{sx}} sfsx \xrightarrow{s\varepsilon_x} sx$$

is the identity and we may now consider the following commutative diagram:

$$\begin{tikzcd}
D(x,y) \arrow[r, "s_*"] \arrow[d, "(\varepsilon_x)^*"] & C(sx,sy) \arrow[d, "(s\varepsilon_x)^*"] \\
D(fsx,y) \arrow[r, "s_*"] \arrow[dr, "\simeq"'] & C(sfsx,sy) \arrow[d, "(\eta_{sx})^*"] \\
 & C(sx,sy).
\end{tikzcd}$$

As discussed in the proof of Proposition 5.4.7, the diagonal composite

$$D(fsx,y) \xrightarrow{s_*} C(sfsx,sy) \xrightarrow{\eta^*_{sx}} C(sx,sy)$$

is precisely the adjunction equivalence. This shows that the map $(\varepsilon_x)^*$ is an equivalence for all x and y if and only if s_* is an equivalence for all x and y. The latter property expresses precisely the property for s to be fully faithful. The first property expresses the invertibility of ε_x for all x (Exercise 1.9.13). This shows that assertions (ii) and (iii) are equivalent. □

By virtue of Lemma 5.5.4, Proposition 6.1.7 immediately implies:

Corollary 6.1.8. *The functor $p^*_{[1]}\colon C \to \mathrm{Fun}([1],C)$ is fully faithful.* □

Corollary 6.1.9. *The inclusion functor $\pi_{\mathrm{Iso}}\colon \mathrm{Iso}(C) \to \mathrm{Fun}([1],C)$ is fully faithful.* □

6.2 Conservative functors

We introduce the notion of a *conservative* functor and prove some basic properties about them.

Definition 6.2.1 (Conservative functor)**.** A functor $f\colon C \to D$ is said to be *conservative* if the following condition is satisfied: for every pair of objects x and y of C and every morphism $u\colon x \to y$ in C, if the image $f(u)\colon f(x) \to f(y)$ of f is invertible in D then f is already invertible in C.

Conservative functors may be characterized in a variety of different ways:

Proposition 6.2.2. *Let $f\colon C \to D$ be any functor. The following conditions are equivalent:*

(i) *The functor f is conservative.*

(ii) *The commutative square*

$$\begin{tikzcd}
\mathrm{Iso}(C)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] \arrow[d, "f_*"'] & \mathrm{Map}([1], C) \arrow[d, "f_*"] \\
\mathrm{Iso}(D)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] & \mathrm{Map}([1], D)
\end{tikzcd}$$

is a pullback square.

(iii) *The commutative square*

$$\begin{tikzcd}
C^{\simeq} = \mathrm{Map}(*, C) \arrow[r, "(p_{[1]})^*"] \arrow[d] & \mathrm{Map}([1], C) \arrow[d, "f_*"] \\
D^{\simeq} = \mathrm{Map}(*, D) \arrow[r, "(p_{[1]})^*"] & \mathrm{Map}([1], D)
\end{tikzcd}$$

is a pullback square.

(iv) *For any object d of D, the fiber of f at d is a groupoid in context $\{d\}$;*

(v) *For any object d of D, the fiber of f at d is a groupoid in the absolute context;*

(vi) *A given commutative square*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & C \arrow[d, "f"] \\
Y \arrow[r] & D
\end{tikzcd}$$

with X and Y groupoids is a pullback square whenever the associated square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & C^{\simeq} \arrow[d, "f^{\simeq}"] \\
Y \arrow[r] & D^{\simeq}
\end{tikzcd}$$

is a pullback square.

(vii) *The square*

$$\begin{tikzcd}
C^{\simeq} \arrow[r] \arrow[d, "f^{\simeq}"'] & C \arrow[d, "f"] \\
D^{\simeq} \arrow[r] & D
\end{tikzcd}$$

is a pullback square.

Proof. We first show the equivalence between (i) and (ii). Recall from Lemma 1.8.4 that the map

$$(\pi_{\mathrm{Iso}})^{\simeq}\colon \mathrm{Iso}(C)^{\simeq} \to \mathrm{Map}([1], C)$$

is an embedding for every category C. It thus follows from Lemma 1.3.30 that also the comparison map

$$\mathrm{Iso}(C)^{\simeq} \to \mathrm{Iso}(D)^{\simeq} \times_{\mathrm{Map}([1],D)} \mathrm{Map}([1], C)$$

is an embedding. It thus follows from Lemma 1.3.28 that it is an equivalence if and only if it admits a section. Since the target is a groupoid, the latter is equivalent to the essential surjectivity of this map. But since the objects in the target of this map are precisely those morphism $u\colon x \to y$ in C whose image in D is invertible, the essential surjectivity of this map is precisely the condition on f that every such morphism u in fact comes from an isomorphism in C, which is the definition of conservativity.

The equivalence between (ii) and (iii) is immediate from the equivalence $C^{\simeq} \xrightarrow{\sim} \mathrm{Iso}(C)^{\simeq}$ from the Rezk Axiom F. The equivalence between (iv) and (v) is immediate from Lemma 4.2.14. To show that (v) implies (vi), consider the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \times_D C \arrow[dl] \\ & Y & \end{tikzcd}$$

of groupoids. By the hypothesis of (v), the category $Y \times_D C$ is a groupoid, hence $Y \times_D C \simeq (Y \times_D C)^{\simeq} \simeq Y \times_{D^{\simeq}} C^{\simeq}$ and the claim follows. The implication (vi) $\implies$ (vii) is clear. To see that (vii) implies (iii), we use that the functor $\mathrm{Map}([1], -)$ preserves pullback squares, and apply the equivalence $C^{\simeq} \xrightarrow{\sim} \mathrm{Map}([1], C^{\simeq})$ coming from the fact that $C^{\simeq}$ is a groupoid.

It thus remains to show that (iii) implies (v). Consider a pullback square

$$\begin{tikzcd} X = f^{-1}(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ \Gamma \arrow[r, "b"] & D \end{tikzcd}$$

with Γ a groupoid. We must show that X is a groupoid as well. To this end, consider the following commutative diagram:

$$\begin{tikzcd} X^{\simeq} \arrow[rr] \arrow[dr] \arrow[dd] & & C^{\simeq} \arrow[dr] \arrow[dd] & \\ & \mathrm{Map}([1], X) \arrow[rr] \arrow[dd] & & \mathrm{Map}([1], C) \arrow[dd] \\ \Gamma^{\simeq} \arrow[rr] \arrow[dr] & & D^{\simeq} \arrow[dr] & \\ & \mathrm{Map}([1], \Gamma) \arrow[rr] & & \mathrm{Map}([1], D). \end{tikzcd}$$

The back and front faces are pullback squares, and the right face is a pullback square by the hypothesis from (ii). We conclude from the pasting lemma of pullback squares that the left face

$$\begin{tikzcd} X^{\simeq} \arrow[r] \arrow[d] & \mathrm{Map}([1], X) \arrow[d] \\ \Gamma^{\simeq} \arrow[r, "\simeq"] & \mathrm{Map}([1], \Gamma) \end{tikzcd}$$

is a pullback square. Since the bottom map $\Gamma^{\simeq} \to \mathrm{Map}([1], \Gamma)$ is an equivalence, it follows that also the top map $X^{\simeq} \to \mathrm{Map}([1], X)$ is an equivalence, and thus we conclude from part (5) of Proposition 3.4.4 that X is a groupoid as desired. □

Lemma 6.2.3. *Compositions of conservative functors are conservative.*

Proof. This is clear from the definition. □

Lemma 6.2.4. *Let $f : C \to D$ be a functor admitting a retraction $r : D \to C$, i.e. we have $rf \cong \mathrm{id}_C$. Then f is conservative.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Iso}(C)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] \arrow[d, "f_*"'] & \mathrm{Map}([1], C) \arrow[d, "f_*"] \\ \mathrm{Iso}(D)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] \arrow[d, "r_*"'] & \mathrm{Map}([1], D) \arrow[d, "r_*"] \\ \mathrm{Iso}(C)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] & \mathrm{Map}([1], C). \end{tikzcd}$$

Since the horizontal maps are embeddings and the outer square is clearly a pullback square, it follows from Lemma 1.3.32 that the top square is a pullback square. The claim thus follows from Proposition 6.2.2. □

Lemma 6.2.5. *For every category C, the functor $(\mathrm{ev}_0, \mathrm{ev}_1) : \mathrm{Fun}([1], C) \to C \times C$ is conservative.*

Proof. We will check condition (iii) from Proposition 6.2.2. Under the currying-uncurrying equivalence from Lemma 1.4.11, this translates to showing that the square

$$\begin{tikzcd} \mathrm{Map}([1], C) \arrow[r] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] & \mathrm{Map}([1] \times [1], C) \arrow[d] \\ C^{\simeq} \times C^{\simeq} \arrow[r] & \mathrm{Map}([1] \sqcup [1], C) \end{tikzcd}$$

is a pullback square. For this, it will in turn suffice to express the category $[1]$ as a pushout of $[0] \sqcup [0]$ and $[1] \times [1]$ along $[1] \sqcup [1]$. For this, we consider the following commutative square:

$$\begin{tikzcd} {[0] \sqcup [0]} \arrow[r] \arrow[d] & {[1] \sqcup [1]} \arrow[r] \arrow[d] & {[1]} \arrow[d, "\Delta"] \\ {[1] \sqcup [1]} \arrow[r] \arrow[d] & {[2] \sqcup [2]} \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & {[1] \times [1]} \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] \\ {[0] \sqcup [0]} \arrow[rr] & & {[1]}. \end{tikzcd}$$

In this diagram, both of the top two squares are pushouts as a consequence of the Commutative Square Axiom D, where for the top right square we use the reformulation of pullbacks from Lemma 1.3.18. Since the outer square is vertically degenerate and hence also a pushout square by Lemma 1.3.17, it follows from the pasting law of pushout squares that the bottom rectangle is a pushout square, as desired. ◻

Lemma 6.2.6. *Every left (resp. right) fibration is conservative.*

Proof. In light of Corollary 5.2.3, this follows immediately from characterization (iii) in the previous proposition. ◻

Corollary 6.2.7. *Let $f\colon C \to S$ and $g\colon D \to S$ be functors and assume that g is a left (resp. right) fibration. Then the canonical inclusion map*

$$\mathrm{Map}_S(C, D) \to \mathrm{Fun}_S(C, D)$$

from Definition 3.5.2 is an equivalence.

Proof. Since g is a left (resp. right) fibration, also the induced functor $g_*\colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, S)$ is a left (resp. right) fibration by Lemma 5.2.18, and in particular g_* is conservative by Lemma 6.2.6. It thus follows from Proposition 6.2.2 that we have a pullback square of the form

$$\begin{tikzcd} \mathrm{Map}(C, D) \arrow[r, hook] \arrow[d, "g_*"'] & \mathrm{Fun}(C, D) \arrow[d, "g_*"] \\ \mathrm{Map}(C, S) \arrow[r, hook] & \mathrm{Fun}(C, S). \end{tikzcd}$$

Passing to vertical fibers, it follows that canonical map $\mathrm{Map}_S(C, D) \to \mathrm{Fun}_S(C, D)$ is an equivalence, as claimed. ◻

Proposition 6.2.8. *If a functor $f\colon C \to D$ is fully faithful, then it is conservative.*

Proof. Consider a morphism $u\colon x \to y$ in C whose image $f(u)$ in D is invertible. We will show that u admits a left inverse; the argument for the existence of a right inverse will be analogous. Since $f(u)$ is invertible in D, it admits a left inverse $v\colon f(y) \to f(x)$, i.e. we have $vf(u) = \mathrm{id}_{f(x)}$. Since f is fully faithful, there exists a morphism $\tilde{v}\colon y \to x$ in C satisfying $f(\tilde{v}) = v$. We claim that the relation $\tilde{v}u = \mathrm{id}_x$ is satisfied. Since f is fully faithful, it suffices to show that $f(\tilde{v}u) = f(\mathrm{id}_x)$. But since f preserves identities and compositions, this is equivalent to the relation $f(\tilde{v})f(u) = vf(u) = \mathrm{id}_{f(x)}$, which holds by assumption. ◻

Corollary 6.2.9. *Let $f\colon C \to D$ be a fully faithful functor. Then the induced map $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ is an embedding.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} C^{\simeq} \arrow[r, "f^{\simeq}"] \arrow[d] & D^{\simeq} \arrow[d] \\ \mathrm{Map}([1], C) \arrow[r] \arrow[d] & \mathrm{Map}([1], D) \arrow[d] \\ C^{\simeq} \times C^{\simeq} \arrow[r, "f^{\simeq} \times f^{\simeq}"] & D^{\simeq} \times D^{\simeq}. \end{tikzcd}$$

The bottom square is a pullback square by the assumption that f is fully faithful. The top square is a pullback square since the map $f: C \to D$ is conservative by the previous proposition. It follows that the outer rectangle is a pullback square, which is precisely the condition that the functor $f^{\simeq}$ is an embedding. □

Corollary 6.2.10. *Let $f: C \to D$ be a fully faithful functor such that $f^{\simeq}: C^{\simeq} \to D^{\simeq}$ admits a section. Then the functor $f^{\simeq}: C^{\simeq} \to D^{\simeq}$ is an equivalence.*

Proof. We have seen in Corollary 6.2.9 that the functor $f^{\simeq}$ is an embedding. Since it also admits a section, it follows that $f^{\simeq}$ is an equivalence by Lemma 1.3.28. □

We now come to the main result of this section: the fact that a natural transformation is a natural isomorphism if and only if it is so objectwise.

Theorem 6.2.11 (Objectwise criterion natural isomorphisms)**.** *For categories C and D, the canonical embedding $\gamma_C: C^{\simeq} \to C$ induces a conservative functor $\gamma_C^*: \operatorname{Fun}(C, D) \to \operatorname{Fun}(C^{\simeq}, D)$.*

In other words: given a natural transformation $\alpha: f \to g$ of functors $C \to D$, if the induced map $\alpha(x): f(x) \to g(x)$ is an isomorphism in D for every object x of C, then α is a natural isomorphism.

Proof. By Proposition 6.2.2, we may show that the square

$$\begin{tikzcd}
\operatorname{Map}(C, D) \arrow[r] \arrow[d, "\gamma_C^*"'] & \operatorname{Map}([1], \operatorname{Fun}(C, D)) \arrow[d, "\gamma_C^*"] \\
\operatorname{Map}(C^{\simeq}, D) \arrow[r] & \operatorname{Map}([1], \operatorname{Fun}(C^{\simeq}, D))
\end{tikzcd}$$

is a pullback square. Note that this square is isomorphic to the square

$$\begin{tikzcd}
\operatorname{Map}(C, D) \arrow[r] \arrow[d] & \operatorname{Map}(C, \operatorname{Fun}([1], D)) \arrow[d] \\
\operatorname{Map}(C^{\simeq}, D^{\simeq}) \arrow[r] & \operatorname{Map}(C^{\simeq}, \operatorname{Map}([1], D)).
\end{tikzcd}$$

But this square is a pullback square since the functor $D \to \operatorname{Fun}([1], D)$ is a full subcategory by part (3) of Proposition 3.4.4. □

We may use this theorem to give a more concrete description of when a functor between cocartesian fibrations is a cocartesian functor.

Corollary 6.2.12. *Let $p: A \to S$ and $q: B \to S$ be two cocartesian fibrations, and consider a commutative triangle*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& S. &
\end{tikzcd}$$

Then f is a cocartesian functor over S if and only if for every morphism $v: x \to y$ in S and every object a of $A(x)$, the map $\overline{v}: v_!(f(a)) \to f(v_!(a))$ is an isomorphism.

Proof. By definition, f is a cocartesian functor if and only if the natural transformation $\mathrm{BC}_!\colon \mathrm{lift}_0^q \vec{f} \to f_* \mathrm{lift}_0^p$ constructed in Construction 5.12.1 is a natural isomorphism. By Theorem 6.2.11, this may be tested pointwise: it is equivalent to asking that for every morphism $v\colon x \to y$ in S and every object a of $A(x)$, the morphism $\mathrm{BC}_!(a, v)\colon \mathrm{lift}_0^q \vec{f}(a, v) \to f_* \mathrm{lift}_0^p(a, v)$ in $\mathrm{Fun}([1], B)$ is an isomorphism. This morphism takes the form of a commutative square

$$\begin{tikzcd} f(a) \arrow[r, equal] \arrow[d, "\mathrm{lift}_0^q(f(a),v)"'] & f(a) \arrow[d, "f(\mathrm{lift}_0^p(a,v))"] \\ v_!(f(a)) \arrow[r, "\overline{v}"] & f(v_!(a)), \end{tikzcd}$$

and by Corollary 6.2.5 this is in turn equivalent to asking both horizontal maps to be isomorphisms. Since the top vertical morphism is an identity map and thus always an isomorphism, this finishes the proof. □

Proposition 6.2.13. *Let p and q as above and let $f\colon C \to D$ be a functor over S. Then the following conditions are equivalent:*

(1) The functor f is a (co)cartesian functor over S;

(2) For any functor $T \to S$, the induced functor $f_T\colon T \times_S C \to T \times_S D$ is a (co)cartesian functor over T;

(3) Condition (2) holds whenever T is of the form $T = [1] \times \Gamma$ for some groupoid Γ;

(4) Condition (2) holds for $T = [1] \times \mathrm{Map}([1], S)$ equipped with the evaluation map $[1] \times \mathrm{Map}([1], S) \to S$.

Proof. We showed that (1) implies (2) in Proposition 5.12.14. The implications (2) $\implies$ (3) $\implies$ (4) are clear. To see that (4) implies (3), consider a map $t\colon [1] \times \Gamma \to S$. We then obtain by currying a map $\tilde{t}\colon \Gamma \to \mathrm{Fun}([1], S)^{\simeq} = \mathrm{Map}([1], S)$, and the map t is given as the composite

$$[1] \times \Gamma \xrightarrow{1 \times \tilde{t}} [1] \times \mathrm{Map}([1], S) \xrightarrow{\mathrm{ev}} S.$$

In particular, if the pullback of f to $[1] \times \mathrm{Map}([1], S)$ is a (co)cartesian functor, then its pullback to $[1] \times \Gamma$ will also be a (co)cartesian functor by Proposition 5.12.14, as desired.

It thus remains to show that (3) implies (1). By the previous corollary, it suffices to show that for every morphism $v\colon x \to y$ in S and every object c of $C(x)$, the map $\overline{v}\colon v_!(f(c)) \to f(v_!(c))$ is an isomorphism. But if x and y are objects in context Γ for some groupoid Γ, then v is a map of the form $[1] \times \Gamma \to S$. By assumption, the pullback f_T of f to $T = [1] \times \Gamma$ is a cocartesian functor of cocartesian fibrations over T. But the map $\overline{v}$ is precisely the Beck-Chevalley map induced by the canonical morphism in T (regarded as a category in context Γ) and hence it is an isomorphism. □

6.3 The Fundamental Theorem of category theory

The goal of this section is to prove the synthetic analogue of the fundamental theorem of category theory: a functor is an equivalence if and only if it is both fully faithful and essentially surjective. In the synthetic setting the notion of 'surjectivity' is somewhat more subtle than in the classical setting. While we will eventually introduce a notion of essential surjectivity in Definition 11.5.3, we will work for now with the following stronger notion:

Definition 6.3.1 (Strongly surjective functor)**.** We say that a functor $f\colon C \to D$ is *strongly surjective* if it comes equipped with a section $s\colon D^{\simeq} \to C^{\simeq}$ of the underlying map $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ on groupoid cores.

Remark 6.3.2. Using the convention from Definition 1.5.6, note that f is strongly surjective if and only if for every object d of D there exists an object c in C together with an isomorphism $f(c) \xrightarrow{\sim} d$ in D. For this reason, this notion was previously called 'essentially surjectivity' in this text.

Theorem 6.3.3 (Fundamental Theorem of Category Theory)**.** *A functor $f\colon C \to D$ is an equivalence if and only if it is fully faithful and strongly surjective.*

Proof. It is clear that every equivalence is fully faithful and strongly surjective. Conversely, let $f\colon C \to D$ be a functor which is fully faithful and strongly surjective. Our goal is to show that f is an equivalence.

We start by showing that f admits a right adjoint $g\colon D \to C$. By Theorem 5.9.12, it will suffice to show that the relative slice category $C_{/d} = C \times_D D_{/d}$ admits a terminal object for every object d of D. Since f is strongly surjective, we may assume that d is of the form $f(c)$ for some object c in C, and thus it suffices to show that $C_{/f(c)}$ admits a terminal object for every object c of C. But since f is fully faithful, it follows from Lemma 6.1.5 that the canonical map $C_{/c} \to C_{/f(c)} = C \times_D D_{/f(c)}$ is an equivalence, and hence it will suffice to show that the slice category $C_{/c}$ admits a terminal object for every object c of C. This was proved in Lemma 5.5.17.

Recall that the adjoint g we obtain is given by a composite of adjoints, as displayed as follows:

$$\begin{tikzcd} C \arrow[r, shift left, "i"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left, "f"] & C \vec{\times}_f D \arrow[l, shift left, "q"] \arrow[r, shift left, "p"] \arrow[r, phantom, "\perp"] & D. \arrow[l, shift left, "s"] \arrow[ll, bend left, "g"] \end{tikzcd}$$

If we denote by $\eta\colon \mathrm{id}_{C \vec{\times}_f D} \to sp$ the unit of the adjunction $p \dashv s$ and by $\varepsilon\colon iq \to \mathrm{id}_{C \vec{\times}_f D}$ the counit of the adjunction $i \dashv q$, it will thus remain to show that the to natural transformations

$$fg = piqs \xrightarrow{p\varepsilon s} ps = \mathrm{id}_D$$

and

$$\mathrm{id}_C = qi \xrightarrow{q\eta i} qspi = gf$$

are natural isomorphisms.

To show that $p\varepsilon s$ is a natural isomorphism, consider an arbitrary type X, a term $x\colon X \to C$ of C and a morphism $u\colon f(x) \to y$ of D. Then the evaluation of the transformation $\varepsilon\colon iq \to \mathrm{id}_{C \vec{\times}_f D}$ at the object $(x, u\colon f(x) \to y)$ of $C \vec{\times}_f D$ is by the following morphism in $C \vec{\times}_f D$ (to be read from top to bottom):

$$\left(\begin{tikzcd} x \arrow[d, equal] \\ x \end{tikzcd} , \begin{tikzcd} f(x) \arrow[r, equal] \arrow[d, equal] & f(x) \arrow[d, "u"] \\ f(x) \arrow[r, "u"] & y \end{tikzcd} \right).$$

In particular, we see that the induced map $p\varepsilon_{(x,u)}\colon x \to x$ is the identity in D. This shows that the transformation $p\varepsilon s\colon fg \to \mathrm{id}_D$ is a natural isomorphism.

To show that $q\eta i$ is a natural isomorphism, it will by Theorem 6.2.11 suffice to do so on objects. To this end, we will start by giving an explicit description of the transformation $\eta\colon \mathrm{id}_{C \vec{\times}_f D} \to sp$ at an object $(x, u\colon f(x) \to y)$. By essential surjectivity of f, there exists an object x' in C with $f(x') \cong y$. By fully faithfulness, there is in turn a morphism $v\colon x \to x'$ in C such that $f(v)\colon f(x) \to f(x') \cong y$ is equal to u. Unwinding definitions, the section s is given on an object y of D by the pair $(x', f(x') \xrightarrow{\cong} y)$. The map $\eta_{(x,u)}\colon (x,u) \to (x', f(x') \xrightarrow{\cong} y)$ is given by the following morphism (to be read from top to bottom):

$$\left(\begin{tikzcd} x \arrow[d, "v"'] \\ x' \end{tikzcd}, \quad \begin{tikzcd} f(x) \arrow[r, "u"] \arrow[d, "f(v)"] & y \arrow[d, equal] \\ f(x') \arrow[r, "\cong"] & y \end{tikzcd}\right).$$

In particular, the induced map $q\eta_{(x,u)}\colon y \to y$ is the identity as well. We conclude that $q\eta_i$ is an (objectwise) natural isomorphism, finishing the proof. ◻

Corollary 6.3.4. *A functor $f\colon C \to D$ is an equivalence if and only if the induced map*

$$f_*\colon \mathrm{Map}([1], C) \xrightarrow{\sim} \mathrm{Map}([1], D)$$

is an equivalence.

Proof. If f is an equivalence, it is clear that f_* is an equivalence. Conversely, assume that f_* is an equivalence. By the Fundamental Theorem, it will suffice to show that f is strongly surjective and fully faithful. Consider the following commutative diagram:

$$\begin{tikzcd} C^{\simeq} \arrow[r, "p_{[1]}^*"] \arrow[d, "f^{\simeq}"'] & \mathrm{Map}([1], C) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "f_*"] & C^{\simeq} \arrow[d, "f^{\simeq}"] \\ D^{\simeq} \arrow[r, "p_{[1]}^*"] & \mathrm{Map}([1], D) \arrow[r, "\mathrm{ev}_0"] & D^{\simeq}. \end{tikzcd}$$

It follows that $f^{\simeq}$ is a retract of f_*, and thus in particular it is an equivalence. In particular, f is strongly surjective. Now consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Map}([1], C) \arrow[r, "f_*"] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & \mathrm{Map}([1], D) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] \\ C^{\simeq} \times C^{\simeq} \arrow[r, "f^{\simeq} \times f^{\simeq}"] & D^{\simeq} \times D^{\simeq}. \end{tikzcd}$$

Since the top and bottom maps are equivalences, it follows that the square is a pullback square, and thus f is fully faithful by Remark 6.1.2. ◻

Exercise 6.3.5. Given types C and D, construct a canonical pullback square of the form

$$\begin{tikzcd} \mathrm{Equiv}(C, D) \arrow[r] \arrow[d] & \mathrm{Equiv}(\mathrm{Map}([1], C), \mathrm{Map}([1], D)) \arrow[d] \\ \mathrm{Map}(C, D) \arrow[r] & \mathrm{Map}(\mathrm{Map}([1], C), \mathrm{Map}([1], D)) \end{tikzcd}$$

6.4 Applications

We will now discuss a variety of applications of the Fundamental Theorem. We start by showing that the various notions of 'subcategories' and 'full subcategories' we have previously introduced are compatible with each other.

Proposition 6.4.1. *Consider a functor $f\colon C \to D$. Then the following conditions are equivalent:*

(1) The functor f is an embedding, in the sense of Definition 1.3.24;

(2) The induced map $f_*\colon M_C := \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$ is an embedding and the induced map $C \to \langle M_C \rangle_D$ is an equivalence;

(3) The functor f exhibits C as a subcategory of D, in the sense of Definition 3.1.2.

Proof. It is clear that (2) implies (3), and we showed in Lemma 3.1.5 that (3) implies (1). To see that (1) implies (2), assume that f is an embedding. Then also the induced map $f_*\colon \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$ is an embedding. To show that the functor $C \to \langle M_C \rangle_D$ is an equivalence, it suffices by Corollary 6.3.4 to show that the induced functor $\mathrm{Map}([1], C) \to \mathrm{Map}([1], \langle M_C \rangle_D)$ is an equivalence. But this is clear since under the fixed identification $\mathrm{Map}([1], \langle M_C \rangle_D) \xrightarrow{\sim} M_C = \mathrm{Map}([1], C)$ this map corresponds to the identity map on $\mathrm{Map}([1], C)$. □

Proposition 6.4.2. *Consider a functor $f\colon C \to D$. Then the following conditions are equivalent:*

(1) The functor f is fully faithful;

(2) The functor $f^\simeq\colon C^\simeq \to D^\simeq$ is an embedding and the canonical functor $C \to \langle C^\simeq \subseteq D \rangle$ is an equivalence;

(3) The functor f exhibits C as a full subcategory of D, in the sense of Definition 3.2.2.

Proof. First note that in all three cases the induced map $f^\simeq\colon C^\simeq \to D^\simeq$ is an embedding, where in case (1) we use Corollary 6.2.9.

To prove that (1) implies (2), assume that f is fully faithful. We already argued that $f^\simeq\colon C^\simeq \to D^\simeq$ is an embedding. To show that $C \to \langle C^\simeq \subseteq D \rangle$ is an equivalence, it suffices by Corollary 6.3.4 to show that the induced functor $\mathrm{Map}([1], C) \to \mathrm{Map}([1], \langle \Gamma \subseteq C \rangle)$ is an equivalence. By Proposition 3.2.8, this is equivalent to the condition that the top square in the following commutative diagram is a pullback square:

$$\begin{tikzcd}
\mathrm{Map}([1], C) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}([1], D) \arrow[d] \\
\mathrm{Map}(([1])^\simeq, C^\simeq) \arrow[r, "(f^\simeq)_*"] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"', "\sim"] & \mathrm{Map}(([1])^\simeq, D^\simeq) \arrow[d, "\sim"', "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
C^\simeq \times C^\simeq \arrow[r, "f^\simeq \times f^\simeq"] & D^\simeq \times D^\simeq.
\end{tikzcd}$$

Since the bottom two vertical maps are equivalences by Axiom G.1, the bottom square is always a pullback square, hence by the pasting law for pullback squares it will suffice to show that the outer square is a pullback square. But this precisely fully faithfulness of f.

The fact that (2) implies (3) follows from Proposition 3.2.7, which shows that the functor $\langle C^{\simeq} \subseteq D\rangle \to D$ satisfies the conditions of (3).

Finally, condition (3) demands that for every category B the commutative square

$$\begin{tikzcd}
\mathrm{Map}(B,C) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}(B,D) \arrow[d] \\
\mathrm{Map}(B^{\simeq},C^{\simeq}) \arrow[r, "(f^{\simeq})_*"] & \mathrm{Map}(B^{\simeq},D^{\simeq})
\end{tikzcd}$$

is a pullback square, and as explained above this square for $B = [1]$ precisely says that f is fully faithful, giving (1). □

Corollary 6.4.3. *For every category C the inclusion $\mathrm{Iso}(C) \hookrightarrow \mathrm{Fun}([1],C)$ is fully faithful.*

Proof. Under Proposition 6.4.2 this is a reformulation of part (4) of Proposition 3.4.4. □

Next, we show that various properties we have introduced can be checked at the level of objects. To start with, recall that a functor $f\colon C \to D$ is a left fibration if and only if the commutative square

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \arrow[r, "f_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1],D) \arrow[d, "\mathrm{ev}_0"] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

is a pullback square, and dually for right fibrations. We may interpret this as saying that every morphism in B admits a unique lift to A with prespecified source, in a way which is fully functorial in the morphism in B. The next lemma says that it suffices to find these lifts objectwise:

Lemma 6.4.4. *Let $f\colon C \to D$ be a functor. Then f is a left (resp. right) fibration if and only if the square*

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r, "f_*"] \arrow[d, "\mathrm{ev}_i"'] & \mathrm{Map}([1],D) \arrow[d, "\mathrm{ev}_i"] \\
C^{\simeq} \arrow[r, "f^{\simeq}"] & D^{\simeq}
\end{tikzcd}$$

is a pullback for $i = 0$ (resp. $i = 1$).

Proof. By symmetry, it suffices to treat the case for left fibrations. One direction is clear by Lemma 5.2.2, so assume conversely that the above square is a pullback square. We need to show that the map

$$\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1],C) \to C \vec{\times}_f D$$

is an equivalence. By assumption, we know that the underlying map on groupoids

$$(\vec{\mathrm{ev}}_0^f)^{\simeq}\colon \mathrm{Map}([1],C) = \mathrm{Fun}([1],C)^{\simeq} \to C^{\simeq} \times_{D^{\simeq}} \mathrm{Fun}([1],D)^{\simeq} = (C \vec{\times}_f D)^{\simeq}$$

is an equivalence, and in particular $\vec{\mathrm{ev}}_0^f$ is strongly surjective. By the Fundamental Theorem 6.3.3, it will thus suffice to show that it is also fully faithful. To this end, consider objects $u\colon x_0 \to x_1$ and $v\colon y_0 \to y_1$ of $\mathrm{Fun}([1],C)$. We need to prove that the induced map

$$\mathrm{Hom}_{\mathrm{Fun}([1],C)}(u,v) \to \mathrm{Hom}_{C \vec{\times}_f D}((x_0,f(u)),(y_0,f(v)))$$

is an equivalence. Since it is a functor between groupoids, we may equivalently show that all its fibers are contractible. A map $(x_0, f(u)) \to (y_0, f(v))$ in $C \vec{\times}_f D$ is given by a morphism $\alpha\colon x_0 \to y_0$ in C and a morphism $\beta\colon f(x_1) \to f(y_1)$ in D together with a commutative diagram in D of the form

$$\begin{tikzcd} f(x_0) \arrow[r, "f(u)"] \arrow[d, "f(\alpha)"'] & f(x_1) \arrow[d, "\beta"] \\ f(y_0) \arrow[r, "f(v)"] & f(y_1). \end{tikzcd}$$

A lift of this data to a morphism $u \to v$ in $\mathrm{Fun}([1], C)$ consists of a morphism $\overline{\beta}\colon x_1 \to y_1$ and a commutative square in C of the form

$$\begin{tikzcd} x_0 \arrow[r, "u"] \arrow[d, "\alpha"'] & x_1 \arrow[d, dashed, "\overline{\beta}"] \\ y_0 \arrow[r, "\alpha_1"] & y_1 \end{tikzcd}$$

such that f sends this commutative square to the above square in D. By invoking the Segal Axiom E, we thus have to show the existence of a unique dashed arrow in the following triangle on the left which gets sent to the triangle on the right:

$$\begin{tikzcd} x_0 \arrow[rr, "u"] \arrow[dr, "\beta u"'] & & x_1 \arrow[dl, dashed, "\overline{\beta}"] \\ & y_1 & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} f(x_0) \arrow[rr, "f(u)"] \arrow[dr, "f(\beta u)"'] & & x_1 \arrow[dl, "\beta"] \\ & y_1. & \end{tikzcd}$$

By assumption on f there exists an essentially unique lift $\beta'\colon x_1 \to y'$ of β. But then the composite $\beta' u\colon x_0 \to y'$ is a lift of $f(\beta u)$, so by uniqueness it agrees with $\beta u\colon x_0 \to y_1$. In particular, $y' = y_1$ and β' is the desired unique lift $\overline{\beta}$. □

Proposition 6.4.5. *Let $f\colon C \to D$ be a right fibration, $c\colon C^{\simeq}$ an object of C, and $d := f(c)$. Then the induced functor*

$$q\colon C_{c/} \to D_{d/}\,, (x, u\colon c \to x) \mapsto (f(x), f(u)\colon d \to f(x))$$

is a right fibration as well.

Proof. We apply the criterion of Lemma 6.4.4. We consider an object of $C_{c/}$, that is a morphism $u\colon c \to x$ in C. We define $v = f(u)$ and $y = f(x)$, so that $v\colon d \to y$. A morphism in $D_{d/}$ with codomain $q(x, u) = (y, v)$, is a diagram of the form

$$d \xrightarrow{v'} y' \xrightarrow{k} y$$

in D with an identification $t\colon k \circ v' \simeq v$. Since $y = f(x)$ there is a unique morphism $h\colon x' \to x$ with $f(h) = k$. Similarly, there is a unique morphism $u_0\colon c_0 \to x'$ with $f(u_0) = v'$. But then $h \circ u_0$ is sent to v and there is thus a unique isomorphism $i\colon c \xrightarrow{\sim} c_0$ such that $f(i) = \mathrm{id}_d$ and $h \circ u_0 \circ i = u$. Defining $u' = h \circ u_0$ we thus get a diagram of the form

$$c \xrightarrow{u'} x' \xrightarrow{u} x$$

with an identification $s\colon h \circ u' = u$ such that $f(s) = t$. This list is unique up to isomorphism in any context, and this implies that the criterion of Lemma 6.4.4 is fulfilled for q. □

Our next goal is to provide a fiberwise criterion for a map between (co)cartesian fibrations to be an equivalence. In contrast to the statement for left (resp. right) fibrations from Lemma 5.9.3, the statement for (co)cartesian fibrations requires that we restrict attention to *(co)cartesian functors*, a notion that was defined in Section 5.12.

Lemma 6.4.6. *If a (co)cartesian functor between (co)cartesian fibrations is fully faithful fiber-wise then it is fully faithful. In other words, given a commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

such that p and q are cartesian (resp. cocartesian) fibrations and that f is a cartesian (resp. cocartesian) functor over S, if moreover, for each object $s\colon S^{\simeq}$, the induced functor $f_s\colon C_s \to D_s$ is fully faithful, then the functor $f\colon C \to D$ is fully faithful.

Proof. We will consider the cartesian case, the cocartesian case being deduced by duality (i.e through the semantic interpretation of synthetic category theory obtained by exchanging the roles of 0 and 1 in the interval $[1]$). Let x_0 and x_1 be two objects of C, and put $s_i = p(x_i)$ for $i = 0$ and $i = 1$. We consider a morphism $v\colon f(x_0) \to f(x_1)$. We want to show that there is a unique morphism $u\colon x_0 \to x_1$ such that $f(u) = v$. Let $w = q(v)$ be the image of v in S.

Existence. We choose a p-cartesian lift of w

$$\tilde{u}\colon w^*(x_1) \to x_1$$

as well as a q-cartesian lift of w

$$\tilde{v}\colon w^*(f(x_1)) \to f(x_1)\,.$$

Since the functor f is cartesian over S, we have $f(\tilde{u}) = \tilde{v}$. On the other hand, there is a unique morphism in the fiber D_s of the form

$$v_s\colon f(x_0) \to w^*(f(x_1))$$

such that $\tilde{v} \circ v_s = v$. Since $f_s\colon C_s \to D_s$ is fully faithful, there is a unique morphism in C_s of the form

$$u_s\colon x_0 \to w^*(x_1)$$

such that $f(u_s) = v_s$. Finally, defining $u = \tilde{u} \circ u_s$ we get $f(u) = v$.

Unicity. Let $u\colon x_0 \to x_1$ be a morphism in C such that $f(u) = v$. We can factorise uniquely $u = \tilde{u} \circ u_s$ with u_s a morphism in the fiber C_s and $\tilde{u}$ the p-cartesian lift of w determined by x_1. Since $f(\tilde{u}) = \tilde{v}$, the universal property of q-cartesian lifts tells us that we must have $f(u_s) = v_s$. □

Theorem 6.4.7. *A (co)cartesian functor between (co)cartesian fibrations is an equivalence if and only if it is an equivalence on fibers. More precisely, consider commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

such that p and q are cartesian (resp. cocartesian) fibrations and that f is a cartesian (resp. cocartesian) functor over S. If for every object s of S the induced functor $f_s\colon C_s \to D_s$ is an equivalence, then f is an equivalence.

Proof. It is clear that, if f is an equivalence, so are the functors induced by f on fibers. Let us assume that $f_s\colon C_s \to D_s$ is an equivalence for each object s of S. First observe that the fibers of f are contractible. Indeed, for an object y of D in context $\{y\}$, let $s := q(y) \in S$. Then there is a pullback square of the form:

$$\begin{tikzcd}
C_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{y\} \arrow[d, "y"] & \\
C_s \arrow[r, "f_s"] \arrow[d] & D_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{y\} \arrow[d, "s"] \\
C \arrow[r, "f"] & D \arrow[r, "q"] & S,
\end{tikzcd}$$

and thus the assumption that $f_s\colon C_s \to D_s$ is an equivalence implies that also $C_y \to \{y\}$ is an equivalence. This implies that f is strongly essentially surjective: for each object y of D with image s in S, there is an object x in C_s such that $f(x) = y$. Therefore, by virtue of the fundamental theorem of category theory, it suffices to get a proof that f is fully faithful, which is provided by Lemma 6.4.6. $\square$

Corollary 6.4.8. *Consider a commutative square of categories over S, i.e. a commutative diagram*

$$\begin{tikzcd}
& C' \arrow[rr, "g"] \arrow[dl, "f"'] \arrow[ddr, "p'"'] & & D' \arrow[dr, "k"] \arrow[ddl, "q'"] & \\
C \arrow[rrrr, "h"] \arrow[drr, "p"'] & & & & D \arrow[dll, "q"] \\
& & S. & &
\end{tikzcd}$$

Assume that all four functors p, p', q, q' are cocartesian fibrations and that the functors f, g, h, k are cocartesian functors over S. Further assume that for every object s of S the induced square

$$\begin{tikzcd}
C'_s \arrow[r, "g_s"] \arrow[d, "f_s"'] & D'_s \arrow[d, "k_s"] \\
C_s \arrow[r, "h_s"] & D_s
\end{tikzcd}$$

is a pullback square. Then also the top square of the original diagram is a pullback square.

Proof. Define the category E as the pullback $C \times_D D'$. By Proposition 5.12.15, the resulting functor $C \times_D D' \to S$ is a cocartesian fibration and the induced functor $C' \to E$ is a cocartesian functor. Our task is to show that it is an equivalence. By Theorem 6.4.7 it will suffice to show that the induced map $C'_s \to E_s$ on fibers is an equivalence for every object s of S. But since taking the fiber commutes with pullbacks, this map is the canonical comparison map $C'_s \to C_s \times_{D_s} D'_s$ coming from the induced square on fibers. Since this square was assumed to be a pullback square, this map is indeed an equivalence, finishing the proof. $\square$

Theorem 6.4.9. *Let $f\colon C \to D$ be any functor. Then the following conditions are equivalent:*

(1) The functor f is fully faithful;

(2) The functor f induces a pullback square

$$\begin{tikzcd}
\mathrm{Fun}([1], C) \arrow[r, "f_*"] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & \mathrm{Fun}([1], D) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
C \times C \arrow[r, "f \times f"] & D \times D.
\end{tikzcd}$$

(3) For every category E, the induced functor $f_*\colon \mathrm{Fun}(E, C) \to \mathrm{Fun}(E, D)$ is fully faithful.

Proof. We first show that (2) and (3) are equivalent. By Proposition 1.3.5, we see that condition (2) is equivalent to the condition that for every category E the commutative square

$$\begin{tikzcd}
\mathrm{Map}(E, \mathrm{Fun}([1], C)) \arrow[r] \arrow[d] & \mathrm{Map}(E, \mathrm{Fun}([1], D)) \arrow[d] \\
\mathrm{Map}(E, C \times C) \arrow[r] & \mathrm{Map}(E, D \times D)
\end{tikzcd}$$

is a pullback square. Notice that this square is isomorphic to the following square:

$$\begin{tikzcd}
\mathrm{Map}([1], \mathrm{Fun}(E, C)) \arrow[r] \arrow[d] & \mathrm{Map}([1], \mathrm{Fun}(E, D)) \arrow[d] \\
\mathrm{Fun}(E, C)^{\simeq} \times \mathrm{Fun}(E, C)^{\simeq} \arrow[r] & \mathrm{Fun}(E, D)^{\simeq} \times \mathrm{Fun}(E, D)^{\simeq}.
\end{tikzcd}$$

But this square is a pullback square if and only if the functor $f_*\colon \mathrm{Fun}(E, C) \to \mathrm{Fun}(E, D)$ is fully faithful, which is condition (3).

Now we show that (1) and (2) are equivalent. It is clear that (2) implies (1) by passing to groupoid cores. It thus remains to show that (1) implies (2). To this end, recall from Lemma 6.1.5 that for every object x of C the commutative square

$$\begin{tikzcd}
C_{/x} \arrow[r] \arrow[d] & D_{/f(x)} \arrow[d] \\
C \arrow[r, "f"] & D
\end{tikzcd} \tag{6.2}$$

is a pullback square. Now consider the following diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1], C) \arrow[r] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & D \vec{\times}_f C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], D) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
C \times C \arrow[r, "f \times \mathrm{id}"] & D \times C \arrow[r, "\mathrm{id} \times f"] & D \times D.
\end{tikzcd}$$

We need to show that the outer rectangle is a pullback square. Since the right square is a pullback square, it suffices to show that the left square is a pullback square.

To show this, we will make use of Corollary 6.4.8. Note that we may regard this square as a square over C as follows:

$$\begin{tikzcd}
 & \mathrm{Fun}([1], C) \arrow[rr, "\vec{\mathrm{ev}}_1^f"] \arrow[dl, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"'] \arrow[ddr, "\mathrm{ev}_1"] & & D \vec{\times}_f C \arrow[dr, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \arrow[ddl, "\mathrm{pr}_2"'] & \\
C \times C \arrow[rrrr, "f \times \mathrm{id}"] \arrow[drr, "\mathrm{pr}_2"'] & & & & D \times C \arrow[dll, "\mathrm{pr}_2"] \\
 & & C. & &
\end{tikzcd}$$

All four functors to C are cocartesian fibrations: for the maps $\mathrm{pr}_2 \colon C \times C \to C$ and $\mathrm{pr}_2 \colon D \times C \to C$ this holds by Lemma 5.6.2, for $\mathrm{ev}_1 \colon \mathrm{Fun}([1], C) \to C$ this holds by Theorem 5.8.3, and for $D \vec{\times}_f C \to C$ this follows from Proposition 5.12.14 since it is the base change of the cocartesian fibration $\mathrm{ev}_1 \colon \mathrm{Fun}([1], D) \to D$.

Furthermore, all four functors in the top square are cocartesian functors over C. For $f \times \mathrm{id} \colon C \times C \to D \times C$ this follows from [ref] since it is a pullback along $C \to *$ of the functor $f \colon C \to D$ over $*$. For $\vec{\mathrm{ev}}_1^f$ this was argued in Proposition 5.12.12. For the functor $(\mathrm{ev}_0, \mathrm{ev}_1) \colon \mathrm{Fun}([1], C) \to C \times C$ this was argued in Proposition 5.12.8. For the functor $(\mathrm{ev}_0, \mathrm{ev}_1) \colon D \vec{\times}_f C \to D \times C$ this then follows from Proposition 5.12.14 as it is a base change along $C \to D$ of the cocartesian functor $(\mathrm{ev}_0, \mathrm{ev}_1) \colon \mathrm{Fun}([1], D) \to D \times D$ over D.

By Corollary 6.4.8, it now remains to show that for every object x of C the induced square on fibers over C is a pullback square. But this induced square is precisely the square (6.2), which we have already argued is a pullback square. □

Corollary 6.4.10. *Let $u \colon A \hookrightarrow C$ be a full subcategory, in the sense of Definition 3.2.2. Then for every category E, the induced functor $\mathrm{Fun}(E, u) \colon \mathrm{Fun}(E, A) \hookrightarrow \mathrm{Fun}(E, C)$ is again a full subcategory, with essential image those functors $F \colon E \to C$ satisfying the property that, for any object x of E, the object $F(x)$ of C is contained in A.*

Proof. The functor u is fully faithful by Proposition 6.4.2, hence so is $\mathrm{Fun}(E, u)$ by Theorem 6.4.9. Another application of Proposition 6.4.2 thus shows that $\mathrm{Fun}(E, A)$ is a full subcategory of $\mathrm{Fun}(E, C)$. The characterization of its essential image in $\mathrm{Fun}(E, C)$ is then a consequence of the following pullback square from Proposition 3.2.8:

$$\begin{tikzcd}
\mathrm{Map}(E, A) \arrow[r] \arrow[d] & \mathrm{Map}(E, C) \arrow[d] \\
\mathrm{Map}(E^{\simeq}, A^{\simeq}) \arrow[r] & \mathrm{Map}(E^{\simeq}, C^{\simeq}).
\end{tikzcd}$$

□

Remark 6.4.11. For every category E, we have a pullback square

$$\begin{tikzcd}
\mathrm{Fun}(E, \langle \Gamma \subseteq C \rangle) \arrow[r] \arrow[d] & \mathrm{Fun}(E, C) \arrow[d] \\
\mathrm{Fun}(E^{\simeq}, \langle \Gamma \subseteq C \rangle) \arrow[r] & \mathrm{Fun}(E^{\simeq}, C).
\end{tikzcd}$$

Indeed, we may show that the map from the left upper corner to the pullback is both strongly surjective and fully faithful. Essential surjectivity follows from the usual claim on mapping spaces.

For fully faithfulness, notice that both the top map as well as the bottom map are fully faithful by Theorem 6.4.9. By the cancellation property of fully faithful functors from Corollary 6.1.3, it follows that also the map from $\mathrm{Fun}(E, \langle \Gamma \subseteq C \rangle)$ to the pullback of this square is fully faithful. In light of the Fundamental Theorem of Category Theory, Theorem 6.3.3, this finishes the proof.

6.5 Exercises Chapter 6

Exercise 6.5.1. Show that every equivalence $u\colon C \xrightarrow{\sim} D$ is fully faithful.

Exercise 6.5.2. Show that a composite of fully faithful functors is fully faithful.

7 Universes and directed univalence

In this chapter, we will introduce a synthetic category Cat called the *universe of small categories*. It comes equipped with a cocartesian fibration $\pi_{\mathrm{univ}} \colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ called the *universal small cocartesian fibration*. The conditions on Cat and π_{univ} will imply that all the constructions of synthetic category theory can be performed internally in the universe Cat. We will also construct the universe of groupoids Grpd (for historic reasons often denoted $\mathcal{S}$ and called the '∞*-category of spaces*' in the literature), together with its natural functors to and from Cat:

$$\begin{tikzcd} \mathrm{Grpd} \arrow[r, hook] & \mathrm{Cat}. \arrow[l, bend right, "\|-\|"'] \arrow[l, bend left, "(-)^{\simeq}"] \end{tikzcd}$$

A fundamental property π_{univ} is required to satisfy is *directed univalence*. To state what that says, note that to every object c of Cat we may assign a synthetic category C by forming the following pullback:

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "c"] & \mathrm{Cat}. \end{tikzcd}$$

We will call a synthetic category C *small* if it is obtained in this way for a given object c of Cat. Directed univalence then demands that for two small categories C and D, corresponding to objects c and d of Cat, morphisms from c to d in Cat are in one-to-one correspondence with functors $C \to D$; more precisely, we require a certain map

$$\mathrm{Hom}_{\mathrm{Cat}}(c, d) \to \mathrm{Map}(C, D)$$

to be an equivalence. Informally, this means that working with small categories and functors between them is the same as working with objects and morphisms of Cat, justifying the terminology 'universe of small categories'.

7.1 The subcategory of cocartesian functors

In order to phrase directed univalence, we need to have a category of cocartesian functors between two cocartesian fibrations $p \colon C \to S$ and $q \colon D \to S$. Recall from Definition 5.12.2 that a functor $f \colon C \to D$ over S is a cocartesian functor if it sends p-cocartesian morphisms in C to q-cocartesian morphisms in D, see Lemma 5.12.3. The goal of this section is to construct a category $\underline{\mathrm{CoCart}}_S(C, D)$

over S whose global sections $S \to \underline{\mathrm{CoCart}}_S(C, D)$ correspond to cocartesian functors from C to D over S.

We will construct this category as a subcategory of the relative functor category $\underline{\mathrm{Fun}}_S(C, D)$ of functors $C \to D$ over S, defined in Definition 3.5.12. For the latter to make sense, recall that we need the functor p to be an *exponentiable functor*, in the sense of Definition 3.5.7. In light of this, we will impose the following axiom:

> **Axiom M** (Exponentiability Axiom)**.** Let $p\colon S \to T$ be a cartesian or cocartesian fibration. Then p is an exponentiable functor.

Example 7.1.1. Assume that $S = \Gamma$ is a groupoid. By Proposition 5.7.6 any functor $p\colon C \to \Gamma$ is a cocartesian fibration, hence in particular exponentiable. It follows that the relative functor category $\underline{\mathrm{Fun}}_\Gamma(C, D)$ exists for *any* categories C and D over Γ.

Construction 7.1.2. Let $p\colon C \to S$ and $q\colon D \to S$ be two cocartesian fibrations, and let $f\colon C \to D$ be a functor over S. We will construct a category $\texttt{is\_Cocart}(f)$ whose objects are proofs that f is a cocartesian functor.

Recall from Corollary 6.2.12 that a functor $f\colon C \to D$ over S is a cocartesian functor over S if and only if for every morphism $v\colon x \to y$ in S and for every object $c \in C(x)$ a certain canonical map $\mathrm{BC}_!(v, c)\colon v_! f(c) \to f(v_! c)$ is an isomorphism in D. This map is natural in the pair (v, x), in the sense that it is part of a natural transformation of functors $C \vec{\times}_p S \to D$. Precomposing with the directed evaluation functor $\vec{\mathrm{ev}}\colon \mathrm{Fun}([1], C) \to C \vec{\times}_p S$, we thus obtain a functor

$$\mathrm{BC}_!\colon \mathrm{Fun}([1], C) \to \mathrm{Fun}([1], D)$$

and f is a cocartesian functor over S if and only if $\mathrm{BC}_!$ factors through the subcategory $D \simeq \mathrm{Iso}(C) \hookrightarrow \mathrm{Fun}([1], D)$ of isomorphisms.

Consider now the following pullback square:

$$\begin{tikzcd} P \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(D)^{\simeq} \arrow[d, hook, "\pi_{\mathrm{Iso}}"] \\ \mathrm{Map}([1], C) \arrow[r, "\mathrm{BC}_!"] & \mathrm{Map}([1], D). \end{tikzcd}$$

Consider also the map $p = p_{\mathrm{Map}([1],C)}\colon \mathrm{Map}([1], C) \to *$. We then define

$$\texttt{is\_Cocart}(f) := p_*(P),$$

where $p_*(P)$ denotes the dependent product of P along p as defined in Definition 3.5.4.

Lemma 7.1.3. *For every f as above, the functor* $\texttt{is\_Cocart}(f) \to *$ *is an embedding. In particular, if* $\texttt{is\_Cocart}(f)$ *admits an (absolute) object, then it is contractible.*

Proof. Note that $P \to \mathrm{Map}([1], C)$ is an embedding, since it is the base change of the embedding π_{Iso} from Lemma 1.8.4. It follows from Corollary 3.5.11 that also the map $\texttt{is\_Cocart}(f) \to *$ is an embedding. An absolute object of $\texttt{is\_Cocart}(f)$ determines a section of this embedding, which then by Lemma 1.3.28 implies that this embedding is an equivalence. $\square$

Observation 7.1.4. Note that $\mathtt{is\_Cocart}(f)$ admits an absolute object if and only if f is a cocartesian functor over S. Indeed, an object $* \to \mathtt{is\_Cocart}(f)$ corresponds by adjunction to a section of the map $P \to \mathrm{Map}([1], C)$, which in turn is equivalent to a lift of $\mathrm{BC}_!$ along π_{Iso}:

$$\begin{tikzcd} & \mathrm{Iso}(D)^{\simeq} \arrow[d, "\pi_{\mathrm{Iso}}"] \\ \mathrm{Map}([1], C) \arrow[r, "\mathrm{BC}_!"'] \arrow[ur, dashed] & \mathrm{Map}([1], D). \end{tikzcd}$$

Such a section precisely says that the map $\mathrm{BC}_!(\varphi)$ is an equivalence for every morphism φ in C, which is the condition for f to be a cocartesian functor.

Remark 7.1.5. The above constructions go through in the context of an arbitrary groupoid Γ: for a category S in context Γ, cocartesian fibration $C \to S$ and $D \to S$ in context Γ, and a functor $f: C \to D$ over S in context Γ, we obtain a category $\mathtt{is\_Cocart}_\Gamma(f)$ in context Γ.

We continue to fix two cocartesian fibrations $p: C \to S$ and $q: D \to S$ over S. Our next goal is to define a subcategory $\underline{\mathrm{CoCart}}_S(C, D) \subseteq \underline{\mathrm{Fun}}_S(C, D)$ spanned by the cocartesian functors $C \to D$ over S. More precisely, we would like that for any other functor $T \to S$, a functor $T \to \underline{\mathrm{Fun}}_S(C, D)$ over S factors through $\underline{\mathrm{CoCart}}_S(C, D)$ if and only if the associated functor $T \times_S C \to T \times_S D$ over T from Remark 3.5.13 is a cocartesian functor over T.

Definition 7.1.6. Consider a morphism in $\underline{\mathrm{Fun}}_S(C, D)$, i.e. an object of $\mathrm{Fun}([1], \underline{\mathrm{Fun}}_S(C, D))$. Such an object is given by a groupoid Γ together with a functor $[1] \times \Gamma \to S$ and a functor over $[1] \times \Gamma$ of the form

$$f: ([1] \times \Gamma) \times_S C \to ([1] \times \Gamma) \times_S D.$$

We say that the morphism is *cocartesian* if the associated map f is a cocartesian functor over $[1] \times \Gamma$. Equivalently, we may regard f as a functor over Γ and ask that it is a cocartesian functor over $[1]_\Gamma$ in context Γ. By Remark 7.1.5, this is classified by a category $\mathtt{is\_Cocart}_\Gamma(f)$ in context Γ.

We will apply this to the *universal* morphism of $\underline{\mathrm{Fun}}_S(C, D)$:

Construction 7.1.7. Define $T := [1] \times \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(C, D))$. Then evaluation determines a map

$$\mathrm{ev}: T \to \underline{\mathrm{Fun}}_S(C, D),$$

which corresponds to a functor $T \to S$ together with a functor

$$f_{\mathrm{univ}}: C_T \to D_T.$$

Applying Definition 7.1.6 to the case $\Gamma = \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(C, D))$, we thus obtain a category $\mathtt{is\_Cocart}_\Gamma(f_{\mathrm{univ}})$ in context Γ, i.e. a functor

$$\mathtt{is\_Cocart}_\Gamma(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(C, D)).$$

Lemma 7.1.8. *The above functor is an embedding.*

Proof. This is a special case of Lemma 7.1.3, applied in context Γ. $\square$

Note that a morphism f in $\underline{\mathrm{Fun}}_S(C, D)$ lies in the image of this embedding if and only if the corresponding functor $([1] \times \Gamma) \times_S C \to ([1] \times \Gamma) \times_S D$ over $[1] \times \Gamma$ is a cocartesian functor.

Lemma 7.1.9. *The embedding $M := \mathtt{is\_Cocart}_\Gamma(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(C, D))$ forms a collection of morphisms in $\underline{\mathrm{Fun}}_S(C, D)$ which is closed under composition, in the sense of Definition 3.1.6.*

Proof. We first show that M is closed under identity maps. But this is clear: given a morphism f in $\underline{\mathrm{Fun}}_S(C, D)$, the corresponding functors over $[1] \times \Gamma$ corresponding to $\mathrm{id}_{f(0)}$ and $\mathrm{id}_{f(1)}$ are given by pulling back the corresponding functor to f along the two maps

$$[1] \times \Gamma \to \{0\} \times \Gamma \hookrightarrow [1] \times \Gamma \qquad \text{and} \qquad [1] \times \Gamma \to \{0\} \times \Gamma \hookrightarrow [1] \times \Gamma,$$

and cocartesian functors are closed under base change, see Lemma 6.2.13.

We now show that M is closed under composition. Consider two morphisms f and g in $\underline{\mathrm{Fun}}_S(C, D)$. By the Segal Axiom, we may just as well assume they arise as the restrictions of some map $\sigma\colon [2] \times \Gamma \to \underline{\mathrm{Fun}}_S(C, D)$ along the two inclusions $[1]_{01} \hookrightarrow [2]$ and $[1]_{12} \hookrightarrow [2]$, where for $0 \le i < j \le 2$ we write $[1]_{ij} \subseteq [2]$ for the subcategory spanned by i and j. This map σ corresponds to a functor

$$\begin{tikzcd} C_{[2]\times\Gamma} \arrow[rr, "\sigma"] \arrow[dr, "p"'] & & D_{[2]\times\Gamma} \arrow[dl] \\ & {[2] \times \Gamma} & \end{tikzcd}$$

over $[2] \times \Gamma$. By assumption on f and g, if we form the pullbacks of this map along the inclusions $[1]_{01} \times \Gamma \hookrightarrow [2] \times \Gamma$ and $[1]_{12} \times \Gamma \hookrightarrow [2] \times \Gamma$ we obtain cocartesian functors over $[1] \times \Gamma$. Our task is to show that also the pullback of this map along the inclusion $[1]_{02} \times \Gamma \hookrightarrow [2] \times \Gamma$ is a cocartesian functor over $[1] \times \Gamma$. To this end, consider a cocartesian morphism $x_0 \to x_2$ in $C|_{[1]_{02}\times\Gamma}$ living over the edge $0 \to 2$. By cocartesian lifting, we may factor this morphism as a composite of two morphisms $x_0 \to x_1$ and $x_1 \to x_2$ living over $0 \to 1$ and $1 \to 2$, respectively. By the assumption on f and g, these morphisms are then sent to cocartesian morphisms in $D_{[1]_{01}\times\Gamma}$ and $D_{[1]_{12}\times\Gamma}$, respectively. But then their composite must be a cocartesian morphism in $D_{[1]_{02}\times\Gamma}$. This finishes the proof. □

Definition 7.1.10. We define the (non-full) subcategory $\underline{\mathrm{CoCart}}_S(C, D) \subseteq \underline{\mathrm{Fun}}_S(C, D)$ as the subcategory spanned by M:

$$\underline{\mathrm{CoCart}}_S(C, D) := \langle M \rangle_{\underline{\mathrm{Fun}}_S(C,D)} \subseteq \underline{\mathrm{Fun}}_S(C, D).$$

Lemma 7.1.11. *Let C and D be categories, regarded as cocartesian fibrations over the terminal category $S = *$. Then we have an equivalence*

$$\underline{\mathrm{CoCart}}_*(C, D) \simeq \mathrm{Map}(C, D) \subseteq \mathrm{Fun}(C, D).$$

Proof. We need to show that a morphism $f\colon [1] \to \mathrm{Fun}(C, D)$ lies in the image of $M \hookrightarrow \mathrm{Map}([1], \mathrm{Fun}(C, D))$ if and only if it is a natural isomorphism of functors $C \to D$. By definition, the former is the case if and only if the corresponding functor $[1] \times C \to [1] \times D$ is a cocartesian functor of cocartesian fibrations over $[1]$. The claim then follows from Lemma 5.12.13. □

Lemma 7.1.12. *Let $p\colon X \to S$ and $q\colon Y \to S$ be two left fibrations. Then the inclusion*

$$\underline{\mathrm{CoCart}}_S(X,Y) \hookrightarrow \underline{\mathrm{Fun}}_S(X,Y)$$

is an equivalence.

Proof. Since every functor between two left fibrations is a cocartesian functor, it follows that the embedding $\mathtt{is\_Cocart}_\Gamma(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(X,Y))$ is an equivalence. The claim thus follows from Lemma 3.1.13. □

Proposition 7.1.13. *Consider a functor $T \to S$ and let $\varphi\colon T \to \underline{\mathrm{Fun}}_S(C,D)$ be a functor over S, corresponding to a functor $f\colon T \times_S C \to T \times_S D$ over T. Then f is cocartesian over T if and only if φ factors through the subcategory $\underline{\mathrm{CoCart}}_S(C,D)$.*

Proof. By definition of $\underline{\mathrm{CoCart}}_S(C,D)$ as a subcategory, φ factors through $\underline{\mathrm{CoCart}}_S(C,D)$ if and only if the induced map $\mathrm{Map}([1],T) \to \mathrm{Map}([1],\mathrm{Fun}_S(C,D)) = \mathrm{Map}_S([1]\times C, D)$ factors through M. This is in turn equivalent to the condition that for every Γ and every map $\alpha\colon \Gamma \to \mathrm{Map}_S([1],T)$, the object

$$\varphi(\alpha)\colon \Gamma \to \mathrm{Map}_S([1],\mathrm{Fun}_S(C,D))$$

factors through M. The map α corresponds to a map $\tilde{\alpha}\colon [1]\times\Gamma \to T$, and by definition of M this happens if and only if the induced functor

$$f_{\Gamma\times[1]}\colon (\Gamma\times[1])\times_S C \to (\Gamma\times[1])\times_S D$$

is a cocartesian functor over $\Gamma\times[1]$. Now, observe that this map is obtained from $f_T\colon T\times_S C \to T\times_S D$ by pulling back along $\tilde{\alpha}\colon \Gamma\times[1] \to T$, and thus it follows from Lemma 6.2.13 that the condition is equivalent to the condition that f_T is a cocartesian functor over T. This finishes the proof. □

Corollary 7.1.14. *For every functor $T \to S$, there is a pullback square of the form*

$$\begin{tikzcd}
\underline{\mathrm{CoCart}}_T(T\times_S C, T\times_S D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{CoCart}}_S(C,D) \arrow[d] \\
T \arrow[r, "t"] & S.
\end{tikzcd}$$

Proof. The pullback square from Observation 3.5.14 restricts a pullback square on subcategories of cocartesian fibrations due to Proposition 7.1.13. □

7.2 Straightening over the walking morphism

Recall that for a cocartesian fibration $W \to [1]$, the canonical morphism $v\colon 0 \to 1$ in $[1]$ determines a covariant transport functor $v_!\colon W_0 \to W_1$. We will now construct a version of covariant transport that works over an arbitrary base category.

Construction 7.2.1. Let S be a category, let $p\colon W \to [1]\times S$ be a cocartesian fibration, and denote by $p_i\colon W_i \to S$ for $i = 0, 1$ the base change of $p\colon W \to S$ along the inclusion $(i, \mathrm{id})\colon S \to [1]\times S$. We will construct a cocartesian functor over $[1]\times S$ of the form

$$\begin{tikzcd} {[1]\times W_0} \arrow[rr, "l"] \arrow[dr, "\mathrm{id}_{[1]}\times p_0"'] & & W \arrow[dl, "p"] \\ & {[1]\times S.} & \end{tikzcd}$$

The functor l will be constructed as a dashed filler fitting in the following commutative diagram:

$$\begin{tikzcd} W_0 \arrow[r, "\iota_0"] \arrow[d, "{(0,\mathrm{id}_{W_0})}"'] & W \arrow[d, "p"] \\ {[1]\times W_0} \arrow[r, "\mathrm{id}_{[1]}\times p_0"'] \arrow[ur, dashed, "l"] & {[1]\times S.} \end{tikzcd}$$

Observe that such a choice of lift is equivalent to finding a filler for the following commutative diagram:

$$\begin{tikzcd} \{0\} \arrow[r, "\iota_0"] \arrow[d] & \mathrm{Fun}(W_0, W) \arrow[d, "p_*"] \\ {[1]} \arrow[r] \arrow[ur, dashed, "k"] & \mathrm{Fun}(W_0, [1]\times S), \end{tikzcd}$$

where the right vertical functor is given by postcomposition with p and the bottom map is adjoint to $[1]\times p_0\colon [1]\times W_0 \to [1]\times S$. In other words: we need to find a morphism in $\mathrm{Fun}(W_0, W)$ whose source is the inclusion $\iota_0\colon W_0 \to W$ and which lifts the morphism in $\mathrm{Fun}(W_0, [1]\times S)$ adjoint to $\mathrm{id}_{[1]}\times p_0\colon [1]\times W_0 \to [1]\times S$. Since p_* is a cocartesian fibration by Proposition 5.6.12, we may simply choose k to be a cocartesian lift which starts in ι_0.

Unwinding the proof of Proposition 5.6.12, we obtain an explicit description of l at the level of objects: given an object $(\varepsilon, w) \in [1]\times W_0$, write $s := \pi_0(w) \in S$ and let $u\colon 0 \to \varepsilon$ denote the morphism in $[1]$ obtained from the fact that $0 \in [1]$ is an initial object, see Lemma 5.5.14. Then we have

$$l(\varepsilon, w) = (u, \mathrm{id}_s)_!(w),$$

where the functor $(u, \mathrm{id}_s)_!\colon W_{(0,s)} \to W_{(\varepsilon,s)}$ denotes cocartesian transport along the morphism $(u, \mathrm{id}_s)\colon (0, s) \to (\varepsilon, s)$ in $[1]\times S$.

Lemma 7.2.2. *The functor $l\colon [1]\times W_0 \to W$ constructed in Construction 7.2.1 is a cocartesian functor over $[1]\times S$.*

Proof. Consider a morphism $\psi\colon (\varepsilon, s) \to (\varepsilon', s')$ in $[1]\times S$ and an object $(\varepsilon, w) \in \{\varepsilon\}\times W_s = ([1]\times W)_{(\varepsilon,s)}$. By Corollary 6.2.12, it will suffice to show that the canonical map

$$\overline{\psi}\colon \psi_!(l(\varepsilon, w)) \to l(\psi_!(\varepsilon, w))$$

is an isomorphism in W. Observe that we may factor ψ as a composite

$$(\varepsilon, s) \to (\varepsilon', s) \to (\varepsilon', s'),$$

and hence since the covariant transport functors compose, Corollary 5.6.10, it will suffice to show the claim in the two cases where either $\varepsilon = \varepsilon'$ or $s = s'$:

- Assume that $s = s'$ and $\psi = (v, \mathrm{id}_s)\colon (\varepsilon, s) \to (\varepsilon', s)$ for some morphism $v\colon \varepsilon \to \varepsilon'$ in $[1]$. Letting $u_\varepsilon\colon 0 \to \varepsilon$ and $u_{\varepsilon'}\colon 0 \to \varepsilon'$ denote the canonical maps in $[1]$, the map $\overline{\psi}$ then simplifies to the map
$$(v, \mathrm{id}_s)_!(u_\varepsilon, \mathrm{id}_s)_!(w) \to (u_{\varepsilon'}, \mathrm{id}_s)(w).$$
But since $u_{\varepsilon'}\colon 0 \to \varepsilon'$ is the composite of $u_\varepsilon\colon 0 \to \varepsilon$ and $v\colon \varepsilon \to \varepsilon'$, this map is an isomorphism by Corollary 5.6.10.

- Assume that $\varepsilon = \varepsilon'$ and that $\psi = (\mathrm{id}_\varepsilon, v)$ for some morphism $v\colon s \to s'$ in S. Since $(\varepsilon, w) = (u_\varepsilon, \mathrm{id}_s)_!(0, w)$, we may by the previous case reduce to $\varepsilon = 0$. The map $\overline{\psi}$ then simplifies to the map
$$(\mathrm{id}_0, v)_!(u_0, \mathrm{id}_s)_!(0, w) \to (u_0, \mathrm{id}_s)_!(\mathrm{id}_0, v)_!(0, w),$$
which is clearly the identity since $u_0\colon 0 \to 0$ is the identity and hence so is $(u_0, \mathrm{id}_s)_!$.

This finishes the proof. □

Definition 7.2.3 (Straightening)**.** Let $p\colon W \to [1] \times S$ be a cocartesian fibration, and denote by $p_i\colon W_i \to S$ for $i = 0, 1$ the base change of $p\colon W \to S$ along the inclusion $(i, \mathrm{id})\colon S \to [1] \times S$. We define the *straightening* of p over $[1]$ to be the cocartesian functor over S of the form
$$\begin{tikzcd} W_0 \arrow[rr, "\mathrm{St}(p)"] \arrow[dr, "p_0"'] & & W_1 \arrow[dl, "p_1"] \\ & S, & \end{tikzcd}$$
obtained from the cocartesian functor $l\colon [1] \times W_0 \to W$ over $[1] \times S$ by forming the base change along $(1, \mathrm{id})\colon S \to [1] \times S$. Note that $\mathrm{St}(p)$ corresponds to a section
$$\sigma_p\colon S \to \underline{\mathrm{CoCart}}_S(W_0, W_1)$$
of the structural functor $\underline{\mathrm{CoCart}}_S(W_0, W_1) \to S$.

Remark 7.2.4. If $S' \to S$ is any functor and if $p\colon W \to [1] \times S$ is a cocartesian fibration, we may form the pullback
$$\begin{tikzcd} W' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "p"] \\ {[1] \times S'} \arrow[r] & {[1] \times S.} \end{tikzcd}$$
We obtain a pullback square
$$\begin{tikzcd} \underline{\mathrm{CoCart}}_S(W_0', W_1') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{CoCart}}_S(W_0, W_1) \arrow[d] \\ {[1] \times S'} \arrow[r] & {[1] \times S.} \end{tikzcd}$$
There are now a priori two ways of constructing a cocartesian functor $W_0' \to W_1'$ over S': we can either apply Definition 7.2.3 directly to p', or we first apply Definition 7.2.3 to p to get a cocartesian

functor $W_0 \to W_1$ over S and then pull back along $S' \to S$. One can show that these two functors agree. In other words, the diagram

$$\begin{tikzcd} \underline{\mathrm{CoCart}}_S(W_0', W_1') \arrow[r] & \underline{\mathrm{CoCart}}_S(W_0, W_1) \\ {[1]} \times S' \arrow[r] \arrow[u, "\sigma_{p'}"] & {[1]} \times S \arrow[u, "\sigma_p"'] \end{tikzcd}$$

is commutative.

Weak functoriality of straightening over $[1]$

Given a cocartesian fibration $W \to [2]$, we obtain functors $W_0 \to W_1$, $W_1 \to W_2$ and $W_0 \to W_2$ by performing covariant transport along the three maps $0 \to 1$, $1 \to 2$ and $0 \to 2$ in $[2]$. We have seen in Corollary 5.6.10 that these three functors fit into a commutative triangle

$$\begin{tikzcd} & W_1 \arrow[dr] & \\ W_0 \arrow[ur] \arrow[rr] & & W_2. \end{tikzcd}$$

Similarly, if the cocartesian fibration $W \to [1]$ were pulled back along the map $[1] \to [0]$ from some functor $W' \to [0]$, then the corresponding covariant transport functor $W' \to W'$ is naturally isomorphic to the identity functor. This expresses the functoriality of covariant transport.

As we shall now see, the analogous statements still hold true over an arbitrary base S.

Lemma 7.2.5. *Consider a cocartesian fibration $p\colon W \to [1] \times S$ obtained from a cocartesian fibration $p'\colon W \to S$ via the following pullback square:*

$$\begin{tikzcd} W \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & W' \arrow[d, "p'"] \\ {[1]} \times S \arrow[r, "\mathrm{pr}_S"] & S. \end{tikzcd}$$

Then the straightening functor $\mathrm{St}(p)\colon W' \to W'$ *of p is isomorphic to the identity of W' over S.*

Proof. This is clear from the fact that the dashed diagonal map in the diagram in Construction 7.2.1 may be taken to be the identity map on $W = [1] \times W'$. ◻

Proposition 7.2.6. *Consider a cocartesian fibration $p\colon W \to [2]$. For $0 \le i < j \le 2$, define the cocartesian fibration $p_{ij}\colon W_{ij} \to [1]_{ij} \times S$ via the following pullback square:*

$$\begin{tikzcd} W_{ij} \arrow[r] \arrow[d, "p_{ij}"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "p"] \\ {[1]}_{ij} \times S \arrow[r, hook] & {[2]} \times S, \end{tikzcd}$$

where $[1]_{ij}$ is the subcategory of $[2]$ *spanned by i and j. Then there is a natural isomorphism*

$$\mathrm{St}(p_{02}) \cong \mathrm{St}(p_{12}) \circ \mathrm{St}(p_{01})$$

of cocartesian functors $W_0 \to W_2$ over S.

Proof. Via Construction 7.2.1, we obtain cocartesian functors

$$\begin{tikzcd}
{[1]_{01} \times W_0} \arrow[rr, "l_{01}"] \arrow[dr, "\mathrm{id} \times p_0"'] & & W_{01} \arrow[dl, "p_{01}"] \\
& {[1]_{01} \times S,} &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[1]_{12} \times W_1} \arrow[rr, "l_{12}"] \arrow[dr, "\mathrm{id} \times p_1"'] & & W_{12} \arrow[dl, "p_{12}"] \\
& {[1]_{12} \times S} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[1]_{02} \times W_0} \arrow[rr, "l_{02}"] \arrow[dr, "\mathrm{id} \times p_0"'] & & W_{02} \arrow[dl, "p_{02}"] \\
& {[1]_{02} \times S.} &
\end{tikzcd}$$

Restricting these three functors to $\{j\} \times S \hookrightarrow [1]_{ij} \times S$ produces the three functors $\mathrm{St}(p_{01})$, $\mathrm{St}(p_{12})$ and $\mathrm{St}(p_{02})$. We claim that these three diagrams can all be extended to cocartesian functors over $[2] \times S$ of the form

$$\begin{tikzcd}
{[2] \times W_0} \arrow[rr, "\bar{l}_{01}"] \arrow[dr, "\mathrm{id} \times p_0"'] & & {W_{01} \times_{[1]_{01}} [2]} \arrow[dl] \\
& {[2] \times S,} &
\end{tikzcd}
\qquad
\begin{tikzcd}
{W_{01} \times_{[1]_{01}} [2]} \arrow[rr, "\bar{l}_{12}"] \arrow[dr, "\mathrm{id} \times p_1"'] & & W \arrow[dl, "p"] \\
& {[2] \times S} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[2] \times W_0} \arrow[rr, "\bar{l}_{02}"] \arrow[dr, "\mathrm{id} \times p_0"'] & & W \arrow[dl, "p"] \\
& {[2] \times S.} &
\end{tikzcd}$$

The map $\bar{l}_{01}$ is obtained from l_{01} by pulling back along the map $s_1 \colon [2] \to [1]$ (given by $0 \mapsto 0$, $1 \mapsto 1$ and $2 \mapsto 1$). To obtain the map $\bar{l}_{12}$, we write $[2]$ as a pushout of the span $[1]_{01} \leftarrow \{1\} \to [1]_{12}$, giving a pushout square

$$\begin{tikzcd}
W_1 \arrow[r, hook] \arrow[d, "{(1,\mathrm{id})}"'] & W_{01} \arrow[d] \\
{[1]_{12} \times W_1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & {W_{01} \times_{[1]_{01}} [2].}
\end{tikzcd}$$

The map $\bar{l}_{12}$ is then given on W_{01} by the inclusion $W_{01} \hookrightarrow W$ and on $[1]_{12} \times W_1$ by the composite

$$[1]_{12} \times W_1 \xrightarrow{l_{12}} W_{12} \hookrightarrow W;$$

this is well-defined since the restriction of both functors to W_1 is the inclusion $W_1 \hookrightarrow W$. Since both l_{12} and the inclusion preserve cocartesian morphisms, it follows that $\bar{l}_{12}$ preserves cocartesian morphisms lying over either $[1]_{01}$ or $[1]_{12}$, from which it follows that $\bar{l}_{12}$ is indeed a cocartesian functor over $[2] \times S$.

Finally, the map $\bar{l}_{02}$ is defined by finding a lift in the following commutative diagram:

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d] & {[1]_{01}} \arrow[r, "k_{01}"] \arrow[d] & \mathrm{Fun}(W_0, W) \arrow[d, "p_*"] \\
{[1]_{12}} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & {[2]} \arrow[r] \arrow[ur, dashed, "\overline{k_{02}}"] & {\mathrm{Fun}(W_0, [2] \times S).}
\end{tikzcd}$$

Since the left square is a pushout square, it suffices to construct a diagonal map $[1] \to \mathrm{Fun}(W_0, W)$ making the top and bottom triangle commute, but such a map exists uniquely due to the fact that p_* is a cocartesian fibration. Observe that the pullback of the map $\bar{l}_{02}$ along the inclusion $[1]_{02} \times S \hookrightarrow [2] \times S$ is indeed the map l_{02}: the restriction of its curried functor $\overline{k_{02}}$ to $[1]_{02} \hookrightarrow [2]$ is a solution of the lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathrm{Fun}(W_0, W_{02}) \arrow[d, "p_*"] \\
{[1]_{02}} \arrow[r] \arrow[ur, dashed, "\overline{k_{02}}"] & \mathrm{Fun}(W_0, [1]_{02} \times S)
\end{tikzcd}$$

and must hence agree with the (unique) solution k_{02}.

To finish the proof, observe that the composite $\bar{l}_{12} \circ \bar{l}_{01} \colon [2] \times W_0 \to W$ is another cocartesian functor over $[2] \times S$ that satisfies the same lifting criterion as $\bar{l}_{02}$. By uniqueness, we thus obtain a natural isomorphism

$$\bar{l}_{02} \cong \bar{l}_{12} \circ \bar{l}_{01}.$$

Pulling back along the inclusion $\{2\} \times S \hookrightarrow [2] \times S$ then provides the desired natural isomorphism

$$\mathrm{St}(p_{02}) \cong \mathrm{St}(p_{12}) \circ \mathrm{St}(p_{01}). \qquad \square$$

7.3 Directed univalence

In this section, we introduce a class of cocartesian fibrations called *directed univalent* cocartesian fibrations. Let us start by giving an informal overview of the definition.

Consider a cocartesian fibration $\pi \colon U_\bullet \to U$, taken to be fixed for now. We will say that another cocartesian fibration $E \to S$ is *U-small* if it is a base change of π along some map $S \to U$. Given a U-small cocartesian fibration $p \colon W \to S \times [1]$, we may apply the straightening procedure from the previous section to obtain a cocartesian functor $\mathrm{St}(p) \colon W_0 \to W_1$ of U-small cocartesian fibrations over S. We then say that $\pi \colon U_\bullet \to U$ is *directed univalent* if this procedure determines, for every S, a one-to-one correspondence between U-small cocartesian fibrations over $S \times [1]$ and cocartesian functors over S.

7.3.1 Small cocartesian fibrations

We start by defining what it means for a cocartesian fibration to be 'small', and by introducing various categories that classify small cocartesian fibrations. Throughout this section, we fix a cocartesian fibration $\pi \colon U_\bullet \to U$.

Definition 7.3.1. We say that a cocartesian fibration $p \colon E \to S$ is *U-small* if it comes equipped with a pullback square of the form

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\
S \arrow[r, "\ulcorner p \urcorner"] & U.
\end{tikzcd}$$

We say that the map $\ulcorner p \urcorner$ *classifies the fibration p*, or that *p is classified by the map $\ulcorner p \urcorner$*. We say that a category C is *U-small* if the functor $p_C \colon C \to *$ is U-small.

Terminology 7.3.2. We say that the category U *classifies* U-small cocartesian fibrations, and that the fibration $\pi\colon U_\bullet \to U$ is the *universal U-small cocartesian fibration*.

Observe that the category $U \times U$ classifies *pairs* of U-small cocartesian fibrations:

Observation 7.3.3. Let S be a category equipped with two U-small cocartesian fibrations $p_1\colon E_1 \to S$ and $p_2\colon E_2 \to S$, classified by functors $\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner\colon S \to U$. Then p_1 and p_2 may be obtained via pullback from the functors $\pi \times \mathrm{id}\colon U_\bullet \times U \to U \times U$ and $\mathrm{id} \times \pi\colon U \times U_\bullet \to U \times U$:

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \times U \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\pi\times\mathrm{id}"] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\
S \arrow[r, "(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)"'] & U \times U \arrow[r, "\mathrm{pr}_1"] & U
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
E_2 \arrow[r] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \times U_\bullet \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{id}\times\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\
S \arrow[r, "(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)"'] & U \times U \arrow[r, "\mathrm{pr}_2"] & U.
\end{tikzcd}$$

We may think of the pair of cocartesian fibrations $\pi\times\mathrm{id}\colon U_\bullet \times U \to U \times U$ and $\mathrm{id} \times \pi\colon U \times U_\bullet \to U \times U$ as the *universal pair of U-small cocartesian fibrations*.

Observation 7.3.4. Given the pair of cocartesian fibrations $\pi\times\mathrm{id}\colon U_\bullet \times U \to U \times U$ and $\mathrm{id} \times \pi\colon U \times U_\bullet \to U \times U$, Definition 3.5.12 provides a category $\underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ equipped with a functor

$$\underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \to U \times U.$$

Given a category S, it follows from Remark 3.5.13 that the data of a functor $S \to \underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ is the same as the data of a functor $(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)\colon S \to U \times U$ together with a functor $E_1 \to E_2$ over S between the cocartesian fibrations $p_1\colon E_1 \to S$ and $p_2\colon E_2 \to S$ classified by $\ulcorner p_1\urcorner$ and $\ulcorner p_2\urcorner$. In other words: the category $\underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ classifies *functors between two U-small cocartesian fibrations*.

Observation 7.3.5. Consider the subcategory

$$\underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \subseteq \underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$$

of cocartesian functors over $U\times U$, defined in Definition 7.1.10. By Proposition 7.1.13, a functor $S \to \underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ factors through this subcategory if and only if the functor $E_1 \to E_2$ over S from Observation 7.3.4 is a cocartesian functor over S. In other words: the category $\underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ classifies *cocartesian* functors between two U-small cocartesian fibrations.

Remark 7.3.6. Recall from Observation 3.5.14 and Corollary 7.1.14 that the constructions $\underline{\mathrm{Fun}}$ and $\underline{\mathrm{CoCart}}$ are compatible with pullbacks. It thus follows that for any pair (p_1, p_2) of U-small cocartesian fibrations over S, classified by functors $\ulcorner p_1\urcorner, \ulcorner p_2\urcorner\colon S \to U$, there are pullback squares of the form

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_S(E_1, E_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \arrow[d] \\
S \arrow[r, "(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)"] & U \times U
\end{tikzcd}$$

and

$$\begin{tikzcd}
\underline{\mathrm{CoCart}}_S(E_1, E_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \arrow[d] \\
S \arrow[r, "(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)"] & U \times U.
\end{tikzcd}$$

Using this remark, we may be more precise about the way that the category $\underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ 'classifies' cocartesian functors between two cocartesian fibrations.

Definition 7.3.7. Write $T := \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ for short. Define cocartesian fibrations $\overline{\pi}_1 : \overline{U}_1 \to T$ and $\overline{\pi}_2 : \overline{U}_2 \to T$ via the following pullback squares:

$$\begin{tikzcd} \overline{U}_1 \arrow[r] \arrow[d, "\overline{\pi}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \times U \arrow[d, "\pi\times\mathrm{id}"] \\ T \arrow[r] & U \times U \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \overline{U}_2 \arrow[r] \arrow[d, "\overline{\pi}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \times U_\bullet \arrow[d, "\mathrm{id}\times\pi"] \\ T \arrow[r] & U \times U. \end{tikzcd}$$

Since the map $T \to U\times U$ lifts to a map $T \to \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U\times U_\bullet)$ (namely the identity map on T), it follows from Observation 7.3.5 that the cocartesian fibrations $\overline{\pi}_1$ and $\overline{\pi}_2$ come equipped with a cocartesian functor

$$\begin{tikzcd} \overline{U}_1 \arrow[rr, "f_{\mathrm{univ}}"] \arrow[dr, "\overline{\pi}_1"'] & & \overline{U}_2 \arrow[dl, "\overline{\pi}_2"] \\ & T. & \end{tikzcd}$$

Lemma 7.3.8. *Let $p_1 : E_1 \to S$ and $p_2 : E_2 \to S$ be two U-small cocartesian fibrations, and let $f : E_1 \to E_2$ be a cocartesian functor over S. Then there exists a functor $\ulcorner f \urcorner : S \to T := \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ such that f is the base change of f_{univ} along $\ulcorner f \urcorner$:*

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & S & \end{tikzcd} \simeq \begin{tikzcd} \ulcorner f \urcorner^*(\overline{U}_1) \arrow[rr, "\ulcorner f \urcorner^*(f_{\mathrm{univ}})"] \arrow[dr, "\ulcorner f \urcorner^*(\overline{\pi}_1)"'] & & S\ulcorner f \urcorner^*(\overline{U}_2) \arrow[dl, "\ulcorner f \urcorner^*(\overline{\pi}_2)"] \\ & S. & \end{tikzcd}$$

Proof. We saw in Observation 7.3.5 that the cocartesian functors between U-small cocartesian fibrations over a category S are classified by functors $S \to T = \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$. This process is functorial in S: given functors $\varphi : S \to S'$ and $\psi : S' \to T$, the cocartesian functor over S classified by the composite $\psi \circ \varphi : S \to T$ is obtained from the one over S' classified by ψ by pulling back along φ. Taking $S' = T$ now gives the claim. □

7.3.2 Directed univalence

We now come to the definition of directed univalence, still fixing a cocartesian fibration $\pi : U_\bullet \to U$. As indicated in the introduction of this section, this should mean that for every S there is a one-to-one correspondence between U-small cocartesian fibrations over $[1] \times S$ and cocartesian functors between U-small cocartesian fibrations over S.

By definition, U-small cocartesian fibrations over $[1] \times S$ are classified by functors $[1] \times S \to U$, or equivalently by functors $S \to \mathrm{Fun}([1], U)$. Similarly, we have seen in Lemma 7.3.8 that cocartesian functors between U-small cocartesian fibrations over S are classified by functors $S \to \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$. Hence directed univalence essentially boils down to the existence of an equivalence

$$\mathrm{St}_U : \mathrm{Fun}([1], U) \xrightarrow{\sim} \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet).$$

Let us explain how to construct such a comparison functor:

Construction 7.3.9 (Universal straightening)**.** Let $\pi\colon U_\bullet \to U$ be a cocartesian fibration. We will construct a *straightening functor*

$$\begin{tikzcd} \mathrm{Fun}([1],U) \arrow[rr, "\mathrm{St}_U"] \arrow[dr, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U\times U_\bullet) \arrow[dl] \\ & U\times U. & \end{tikzcd}$$

To this end, consider the cocartesian fibration $p\colon W \to [1]\times \mathrm{Fun}([1],U)$ defined as the base change of π along the evaluation functor:

$$\begin{tikzcd} W \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ {[1]}\times \mathrm{Fun}([1],U) \arrow[r, "\mathrm{ev}"] & U. \end{tikzcd}$$

(Observe that p is the universal U-small cocartesian fibration of the form $W \to [1]\times S$ for some S.) Applying the straightening procedure from Section 7.2, we thus obtain a cocartesian functor

$$\begin{tikzcd} W_0 \arrow[rr, "\mathrm{St}(p)"] \arrow[dr, "p_0"'] & & W_1 \arrow[dl, "p_1"] \\ & \mathrm{Fun}([1],U) & \end{tikzcd}$$

of cocartesian fibrations over $\mathrm{Fun}([1],U)$, where W_0 and W_1 are defined as the following base changes of π:

$$\begin{tikzcd} W_i \arrow[r] \arrow[d, "p_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ \mathrm{Fun}([1],U) \arrow[r, "\mathrm{ev}_i"] & U. \end{tikzcd}$$

Applying Observation 7.3.5 to the case $S = \mathrm{Fun}([1],U)$, we see that the cocartesian functor $\mathrm{St}(p)$ over S is precisely classifying a lift St_U in the following commutative diagram:

$$\begin{tikzcd} & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U\times U_\bullet) \arrow[d] \\ \mathrm{Fun}([1],U) \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] \arrow[ur, dashed, "\mathrm{St}_U"] & U\times U. \end{tikzcd}$$

This finishes the construction of St_U.

Definition 7.3.10 (Directed univalence)**.** A cocartesian fibration $\pi\colon U_\bullet \to U$ is called *directed univalent* if the functor $\mathrm{St}_U\colon \mathrm{Fun}([1],U) \to \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U\times U_\bullet)$ is an equivalence.

Perhaps the most important consequence of directed univalence is that it provides a translation between morphisms in U and functors of U-small categories. To see this, consider two objects c and d of U, and consider their corresponding U-small categories C and D:

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ * \arrow[r, "c"] & U \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ * \arrow[r, "d"] & U. \end{tikzcd}$$

Since the map St_U is a functor over $U \times U$, it induces a functor on fibers over (c, d). This functor takes the form

$$\mathrm{Hom}_U(c, d) \to \underline{\mathrm{CoCart}}_{\{(c,d)\}}(C, D) \simeq \mathrm{Map}(C, D),$$

where the equivalence comes from Lemma 7.1.11.

Corollary 7.3.11. *Let $\pi\colon U_\bullet \to U$ be a directed univalent cocartesian fibration. For every pair of objects c and d of U with corresponding U-small categories C and D, the induced map*

$$\mathrm{Hom}_U(c, d) \to \mathrm{Map}(C, D)$$

is an equivalence of groupoids. □

In other words, morphisms in U from c to d are nothing but functors from C to D. It then follows from Proposition 7.2.6 and Lemma 7.2.5 that under this correspondence identities and compositions in U correspond to identity functors and composition of functors. All in all we get the following dictionary:

Objects c of U	$\Longleftrightarrow$	U-small categories C
Morphisms $c \to d$ in U	$\Longleftrightarrow$	Functors $C \to D$
Identity id_c in U	$\Longleftrightarrow$	The identity functor $\mathrm{id}_C\colon C \to C$
Composition in U	$\Longleftrightarrow$	Composition of functors
Commutative triangles $[2] \to U$	$\Longleftrightarrow$	Commutative triangles $\begin{tikzcd}[sep=small] & D \arrow[dr] & \\ C \arrow[ur] \arrow[rr] & & E \end{tikzcd}$
Commutative squares $[1] \times [1] \to U$	$\Longleftrightarrow$	Commutative squares $\begin{tikzcd}[sep=small] C \arrow[r] \arrow[d] & D \arrow[d] \\ C' \arrow[r] & D' \end{tikzcd}$

Remark 7.3.12. Using the same reasoning as above, it follows that for U-small cocartesian fibrations $C_1 \to S$ and $C_2 \to S$, classified by functors $F_1, F_2\colon S \to U$, there is a one-to-one correspondence between cocartesian functors $C_1 \to C_2$ over S and natural transformations $F_1 \to F_2$. Under this correspondence, the cocartesian functor $C_1 \to C_2$ is an equivalence if and only if the natural transformation $F_1 \to F_2$ is a natural isomorphism.

As a special case, assume that a cocartesian fibration $p\colon C \to S$ is classified by two a priori different functors $F, F'\colon S \to U$:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U_\bullet \arrow[d, "\pi"] \\ S \arrow[r, "F"] & U \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U_\bullet \arrow[d, "\pi"] \\ S \arrow[r, "F'"] & U. \end{tikzcd}$$

Then there is a natural isomorphism $F \cong F'$ corresponding to the identity $C = C$. We will see this below in Remark 7.3.17. We conclude that the notion of smallness is really just a *property* of the underlying cocartesian fibration.

7.3.3 Consequences of directed univalence

We continue to fix a directed univalent cocartesian fibration $\pi\colon U_\bullet \to U$.

Definition 7.3.13. Consider two cocartesian fibrations $p_0\colon E_0 \to S$ and $p_1\colon E_1 \to S$. We define the *cocartesian mapping space over S* as

$$\mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1) := (p_S)_* \left(\begin{array}{c} \underline{\mathrm{CoCart}}_S(E_0, E_1) \\ \downarrow \\ S \end{array} \right)^{\simeq} .$$

We observe that by virtue of Corollary 3.5.16, there is a canonical equivalence of the form

$$\mathrm{Map}_S(E_0, E_1) \xrightarrow{\sim} (p_S)_* \left(\begin{array}{c} \underline{\mathrm{Fun}}_S(E_0, E_1) \\ \downarrow \\ S \end{array} \right)^{\simeq} .$$

There is thus an embedding $\mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1) \hookrightarrow \mathrm{Map}_S(E_0, E_1)$.

Lemma 7.3.14. *Given two left fibrations $p_0\colon X_0 \to S$ and $p_1\colon X_1 \to S$, the inclusion*

$$\mathrm{Map}_S^{\mathrm{CoCart}}(X_0, X_1) \hookrightarrow \mathrm{Map}_S(X_0, X_1)$$

is an equivalence.

Proof. This is an immediate consequence of Lemma 7.1.12. □

Now assume that the cocartesian fibrations $p_i\colon E_i \to S$ are U-small: there are functors $F_i\colon S \to U$ for $i = 0, 1$ and pullback squares

$$\begin{tikzcd} E_i \arrow[r, "\widetilde{F_i}"] \arrow[d, "p_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ S \arrow[r, "F_i"] & U. \end{tikzcd}$$

Lemma 7.3.15. *Given functors $F_0, F_1\colon S \to U$, the fiber of the functor*

$$\mathrm{Fun}(S, \mathrm{CoCart}_{U\times U}(U_\bullet \times U, U \times U_\bullet)) \to \mathrm{Fun}(S, U) \times \mathrm{Fun}(S, U)$$

at the pair (F_0, F_1) is $\mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1)$.

Proof. The fibers of this functor are groupoids: indeed, we are talking of the fibers of a functor equivalent to the functor

$$\mathrm{Fun}(S, \mathrm{Fun}([1], U)) \cong \mathrm{Fun}([1], \mathrm{Fun}(S, U)) \to \mathrm{Fun}(S, U) \times \mathrm{Fun}(S, U)\,,$$

and since the latter is known to be conservative, this proves our claim. Consider a groupoid Γ. Then a map $\Gamma \to \mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1)$ corresponds by definition to a cocartesian functor $f\colon \Gamma \times E_0 \to \Gamma \times E_1$ over $\Gamma \times S$.

On the other hand, a map from Γ to the fiber is given as a map $Y_0 \to Y_1$ over $\Gamma \times S$ together with a proof that it is a cocartesian functor over $\Gamma \times S$, where Y_0 and Y_1 are defined via the pullback squares

$$\begin{tikzcd} Y_i \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & U_\bullet \arrow[d, "\pi"] \\ \Gamma \times S \arrow[r, "\mathrm{pr}"] & S \arrow[r, "F_i"] & U \end{tikzcd}$$

But it is easy to see that Y_0 and Y_1 are equivalent to $\Gamma \times E_0$ and $\Gamma \times E_1$, respectively. □

Theorem 7.3.16. *For U-small cocartesian fibrations $p_i\colon E_i \to S$ classified by functors $F_i\colon S \to U$, there is an equivalence of groupoids*

$$\mathrm{Fun}(S,U)(F_0,F_1) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart}}(E_0,E_1).$$

Proof. By definition, the hom groupoid $\mathrm{Hom}_{\mathrm{Fun}(S,U)}(F_0,F_1)$ is defined via the following pullback square:

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Fun}(S,U)}(F_0,F_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],\mathrm{Fun}(S,U)) \arrow[d] \\ * \arrow[r, "(F_0,F_1)"] & \mathrm{Fun}(S,U) \times \mathrm{Fun}(S,U). \end{tikzcd}$$

By the previous lemma, it will thus suffice to show that the right vertical map is equivalent to the map in the lemma. But this follows directly from the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fun}([1],\mathrm{Fun}(S,U)) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Fun}(S,\mathrm{Fun}([1],U)) \arrow[r, "\mathrm{St}_*"] & \mathrm{Fun}(S,\mathrm{CoCart}_{U\times U}(U_\bullet \times U, U \times U_\bullet)) \arrow[d] \\ \mathrm{Fun}(S,U) \times \mathrm{Fun}(S,U) \arrow[rr, equal] & & \mathrm{Fun}(S,U) \times \mathrm{Fun}(S,U), \end{tikzcd}$$

since the top two functors are equivalences. □

Remark 7.3.17. Given two cocartesian fibrations $p_i\colon X_i \to S$, every equivalence $X_0 \xrightarrow{\sim} X_1$ over S is automatically a cocartesian functor over S, Corollary 5.12.5, so the inclusion $\mathrm{Equiv}_S(X_0,X_1) \to \mathrm{Map}_S(X_0,X_1)$ factors through $\mathrm{Map}_S^{\mathrm{CoCart}}(X_0,X_1)$. If each cocartesian fibration p_i is classified by a functor $F_i\colon S \to \mathrm{Cat}$, Theorem 7.3.16 implies right away that the equivalence of the previous theorem restricts to a map of the form

$$\mathrm{Fun}(S,\mathrm{Cat})^{\simeq}(F_0,F_1) \xrightarrow{\sim} \mathrm{Equiv}_S(X_0,X_1)$$

that is an equivalence. Indeed the induced comparison map is an embedding since its composition with the embedding into $\mathrm{Map}_S^{\mathrm{CoCart}}(X_0,X_1)$ is the composition of an embedding with an equivalence, hence an embedding. The compatibility of the straightening equivalence with compositions by Proposition 7.2.6 implies the surjectivity as well: given an equivalence $u\colon X_0 \xrightarrow{\sim} X_1$ over S, with inverse $v\colon X_1 \to X_0$, we can find $\varphi\colon F_0 \to F_1$ and $\psi\colon F_1 \to F_0$ such that u is the straightening of φ and v the straightening of ψ. This means that $\varphi \circ \psi$ and $\psi \circ \varphi$ are straightenings of identities. Theorem 7.3.16 thus implies that they must be equivalent to identities themselves.

Consider now the following diagram:

$$\begin{tikzcd}
\mathrm{Iso}(U) \arrow[r, dashed] \arrow[d] & \mathrm{Equiv}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \arrow[d] \\
\mathrm{Fun}([1], U) \arrow[r, "\mathrm{St}"] & \mathrm{CoCart}_{U\times U}(U_\bullet \times U, U \times U_\bullet).
\end{tikzcd}$$

We claim that this is a pullback square and thus that the dashed map is an equivalence. Indeed, given any category S, we obtain a commutative square of the form

$$\begin{tikzcd}
\mathrm{Map}(S, \mathrm{Iso}(U)) \arrow[r, dashed] \arrow[d] & \mathrm{Map}(S, \mathrm{Equiv}_{U\times U}(U_\bullet \times U, U \times U_\bullet)) \arrow[d] \\
\mathrm{Map}(S, \mathrm{Fun}([1], U)) \arrow[r, "\mathrm{St}_*"] & \mathrm{Map}(S, \mathrm{CoCart}_{U\times U}(U_\bullet \times U, U \times U_\bullet)).
\end{tikzcd}$$

This is a pullback square: since this is a diagram of animae over $\mathrm{Map}(S, U \times U)$, it suffices to check this property fiber-wise over each object of $\mathrm{Map}(S, U \times U)$, which we already know.

The equivalence

$$\mathrm{Iso}(U) \xrightarrow{\sim} \mathrm{Equiv}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$$

precisely means that the cocartesian fibration $\pi\colon U_\bullet \to U$ is univalent in Voevodsky's sense. In practical terms, this is the precise way to say that there is a correspondence between cocartesian fibrations with U-small fibers of the form $p\colon X \to S$ and functors of the form $F\colon S \to U$: given a pullback square of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] & U_\bullet \arrow[d, "\pi"] \\
S \arrow[r, "F"] & U
\end{tikzcd}$$

p and F determine each other uniquely up to equivalence.

7.4 The universe of categories

Our next goal is to define the *universe of all (small) categories*, i.e. a category Cat whose objects classify (small) categories. We require that Cat comes equipped with a directed univalent cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$, so that we may identify the objects and morphisms of Cat with (small) categories and functors between them. To ensure that Cat does indeed behave like a universe of categories, we will have to enforce that all the categorical constructions of categories we have introduced so far admit analogues internal to Cat.

7.4.1 Universes

Let us start by introducing some general terminology regarding universes.

Definition 7.4.1 ((Regular) Universe)**.** A *universe* is a category U equipped with a directed univalent cocartesian fibration $\pi\colon U_\bullet \to U$.

A universe U is called *regular* if U-small cocartesian fibrations are closed under composition.

Remark 7.4.2. In light of Remark 7.3.12, we will frequently be somewhat sloppy with our use of the adjective 'U-small' and treat it as a property. A more constructive approach will be given in Section 7.8.

While there can a priori be many different universes, the collection of universes behaves like a partially ordered set:

Theorem 7.4.3. *Let U be a universe with directed univalent cocartesian fibration $\pi\colon U_\bullet \to U$, and let $i\colon V \to U$ be a functor. Then the following two conditions are equivalent:*

(1) The base change $V_\bullet := V \times_U U_\bullet \to V$ exhibits V as a universe;

(2) The functor i is fully faithful.

Proof. Consider the map $\pi'\colon V_\bullet \to V$ defined as the base change of π along i:

$$\begin{tikzcd} V_\bullet \arrow[r, "j"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ V \arrow[r, "i"] & U. \end{tikzcd}$$

By taking the cartesian product of this square with a degenerate pullback square, we obtain two pullback squares

$$\begin{tikzcd} V_\bullet \times V \arrow[r, "i\times j"] \arrow[d, "\pi'\times\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \times U \arrow[d, "\pi\times\mathrm{id}"] \\ V \times V \arrow[r, "i\times i"] & U \times U \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} V \times V_\bullet \arrow[r, "j\times i"] \arrow[d, "\mathrm{id}\times\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \times U_\bullet \arrow[d, "\mathrm{id}\times\pi"] \\ V \times V \arrow[r, "i\times i"] & U \times U. \end{tikzcd}$$

As a consequence, we obtain from Corollary 7.1.14 a pullback square

$$\begin{tikzcd} \underline{\mathrm{CoCart}}_{V\times V}(V_\bullet \times V, V \times V_\bullet) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \arrow[d] \\ V \times V \arrow[r, "i\times i"] & U \times U. \end{tikzcd}$$

Since the square

$$\begin{tikzcd} {[1] \times \mathrm{Fun}([1], V)} \arrow[r, "\mathrm{ev}"] \arrow[d, "1\times i_*"'] & V \arrow[d, "i"] \\ {[1] \times \mathrm{Fun}([1], U)} \arrow[r, "\mathrm{ev}"] & U \end{tikzcd}$$

commutes, we obtain pullback squares

$$\begin{tikzcd} W_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ {[1] \times \mathrm{Fun}([1], U)} \arrow[r, "1\times i_*"] & {[1] \times \mathrm{Fun}([1], U)} \arrow[r, "\mathrm{ev}"] & U. \end{tikzcd}$$

Consider now the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],V) \arrow[r, "i_*"] \arrow[d, "\mathrm{St}_V"] & \mathrm{Fun}([1],U) \arrow[d, "\mathrm{St}_U"] \\
\underline{\mathrm{CoCart}}_{V\times V}(V_\bullet \times V, V \times V_\bullet) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \arrow[d] \\
V \times V \arrow[r, "i\times i"] & U \times U.
\end{tikzcd}$$

Since the bottom square is a pullback square, it follows from the pasting rule for pullback squares that i is fully faithful if and only if the top square is cartesian. Since the upper right vertical map St_U is an equivalence, this happens if and only if also the left vertical map St_V being an equivalence, i.e. if the cocartesian fibration $V_\bullet \to V$ is directed univalent. This finishes the proof. $\square$

Lemma 7.4.4. *Let Γ be a groupoid and let U be a universe. Then the category U_Γ in context Γ is a universe for categories in context Γ.*

Proof. If is clear that U_Γ is a cocartesian fibration. Since Γ is a groupoid, a family of functors $f(x)\colon A(x) \to B(x)$ parametrized by terms $x\colon \Gamma$ is a cocartesian fibration in conext Γ if and only if its associated functor

$$\Sigma_{x\colon\Gamma} f(x)\colon \Sigma_{x\colon\Gamma} A(x) \to \Sigma_{x\colon\Gamma} B(x)$$

is cocartesian in the absolute context. Similarly, if $S(x)$ is a family of categories indexed by terms $x\colon \Gamma$ with associated category in the absolute context $S = \Sigma_{x\colon\Gamma} S(x)$, a functor between cocartesian fibrations over $S(x)$ in context Γ is cocartesian over $S(x)$ in context Γ if and only if it is cocartesian in context S in the absolute context. This implies that

$$\underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)_\Gamma \to (U\times U)_\Gamma = U_\Gamma \times_\Gamma U_\Gamma$$

classifies cocartesian functors between cocartesian fibrations that are pullbacks of the cocartesian fibration $U_\bullet \to U$ in context Γ. Since we also have

$$\mathrm{Fun}([1],U)_\Gamma = \mathrm{Fun}_\Gamma([1],U_\Gamma)$$

this implies right away that U_Γ is a directed univalent universe in context Γ simply because the operation $(-)_\Gamma$ preserves equivalences. $\square$

7.4.2 The universe of categories

We will now introduce the universe Cat.

Definition 7.4.5 (Universe of categories)**.** We say that a universe U is a *universe of categories* if the following conditions are satisfied:

(Cat1) (Regularity) The composite of two U-small cocartesian fibrations is U-small;

(Cat2) (Terminal category) The category $*$ is U-small;

(Cat3) (Pullbacks) Given U-small cocartesian fibrations $C \to S$, $D \to S$ and $E \to S$, and cocartesian functors $C \to E$ and $D \to E$ over S, the cocartesian fibration $C \times_D E \to S$ from Proposition 5.12.15 is U-small;

(Cat4) (Subcategories) Let $p\colon C \to S$ be a U-small cocartesian firbration and let $i\colon C' \hookrightarrow C$ be an embedding such that $ip\colon C' \to S$ is still a cocartesian fibration. Then ip is a U-small cocartesian fibration;

(Cat5) (Joins) If $C \to S$ and $D \to S$ are U-small cocartesian fibrations, then their relative join $C \star_S D \to S$ (in the sense of Definition 3.6.12) exists and is again a U-small cocartesian fibration;

(Cat6) (Dependent products) For any U-small category S, any U-small cocartesian fibration $f\colon C \to D$ and any functor $D \to S$, the functor $(p_S)_*(f)\colon (p_S)_*(C) \to (p_S)_*(D)$ is a U-small cocartesian fibration.

(Cat7) (Localizations) If C is U-small and $W \to \mathrm{Map}([1], C)$ is an embedding, then there exists a localization of C at W which is U-small;

(Cat8) (Regular subuniverses) For any U-small subcategory V of U there exists a U-small subcategory $V \subseteq V' \subseteq U$ such that V' is a regular universe.

We may or may not need the following additional conditions:

(Cat9) (Retracts) Any retract of a U-small category is U-small;

(Cat10) A cocartesian fibration which has U-small fibers is a U-small cocartesian fibration;

(Cat11) (Dependent products 2) For any U-small cocartesian fibration $p\colon C \to D$ and a functor $E \to C$ with U-small fibers, also the functor $p_*(E) \to D$ has U-small fibers.

(Cat12) A category C is U-small whenever the groupoid $\mathrm{Map}([1], C)$ is U-small.

Here in (Cat10) and (Cat11) we used the following:

Definition 7.4.6 (U-small fibers)**.** Let U be a universe in the sense of Definition 7.4.1, and let $f\colon C \to D$ be a functor. We say that f *has U-small fibers* if for any U-small category T and any functor $T \to D$ the pullback $C \times_D T$ is U-small.

Axiom N (Directed Univalence Axiom)**.** For any cocartesian fibration $q\colon E \to B$, there exists a universe of categories U such that B is U-small and q has U-small fibers.

In practice, we will mostly just need a single universe of categories; we will frequently fix a choice and denote it by Cat. We denote the associated universal cocartesian fibration by

$$\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}.$$

We will say that a category C is *small* if it is Cat-small in the sense of Definition 7.3.1, i.e. if there exists an object $c\colon * \to \mathrm{Cat}$ and a pullback square

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "c"] & \mathrm{Cat}. \end{tikzcd}$$

We will similarly speak of *small cocartesian fibrations*, etcetera. The reader should keep in mind that all results stated for the fixed universe Cat will hold for any universe of categories U.

We will now deduce various further closure properties of Cat.

Lemma 7.4.7 (Identity functors)**.** *For every category S, the identity functor $\mathrm{id}_S\colon S \to S$ is a small cocartesian fibration.*

Proof. The identity functor on S is a base change along $p_S\colon S \to *$ of the functor $\mathrm{id}_*\colon * \to *$, and hence this follows (Cat2). □

Lemma 7.4.8 (Initial category)**.** *For every category S, the functor $\emptyset \to S$ is a small cocartesian fibration.*

Proof. This follows from applying (Cat4) to the embedding $\emptyset \hookrightarrow S$, combined with the fact that $\mathrm{id}_S\colon S \to S$ is small by the previous lemma. □

Lemma 7.4.9 (Products)**.** *Let $p\colon C \to S$ and $q\colon D \to S$ be small cocartesian fibrations. Then also $C \times_S D \to S$ is a small cocartesian fibration.*

Proof. This functor may be written as a composite $C \times_S D \xrightarrow{\mathrm{pr}_C} C \xrightarrow{p} S$ of two small cocartesian fibrations, hence the claim follows from (Cat1). □

Lemma 7.4.10 (Groupoid cores)**.** *Let C be a small category. Then also its groupoid core $C^{\simeq}$ is small.*

Proof. This follows by applying (Cat4) to the embedding $C^{\simeq} \hookrightarrow C$ from Corollary 2.2.12. □

Lemma 7.4.11 (Universality of coproducts)**.** *Let S and T be categories. A cocartesian fibration $p\colon C \to S \sqcup T$ is small if and only if its pullbacks $p_S\colon C_S \to S$ and $p_T\colon C_T \to T$ are small.*

Proof. The 'only if'-direction is clear since U-small cocartesian fibrations are closed under base change. Conversely, assume that the cocartesian fibrations $p_S\colon C_S \to S$ and $p_T\colon C_T \to T$ are represented by functors $F_S\colon S \to \mathrm{Cat}$ and $F_T\colon T \to \mathrm{Cat}$, respectively. We claim that p is represented by the functor $F := \langle F_S, F_T\rangle\colon S \sqcup T \to \mathrm{Cat}$. Indeed, if we let $D \to S \sqcup T$ denote the small cocartesian fibration classified by F, then we see that D pulls back to the functors $C_S \to S$ and $C_T \to T$, hence it follows from Axiom B.4' that $D \simeq C_S \sqcup C_D = C$ as desired. □

Lemma 7.4.12 (Coproducts)**.** *Given small cocartesian fibrations $p\colon C \to S$ and $q\colon D \to S$, their coproduct $\langle p, q\rangle\colon C \sqcup D \to S$ is a small cocartesian fibration.*

Proof. We may factor this functor as $C \sqcup D \xrightarrow{p \sqcup q} S \sqcup S \xrightarrow{\langle \mathrm{id}_S, \mathrm{id}_S \rangle} S$. The first functor is a small cartesian fibration by Lemma 7.4.11. The second functor is a base change of the functor $* \sqcup * \to *$, hence it remains to show that $* \sqcup *$ is small. But since $* \sqcup * = [1]^{\simeq}$, this follows from applying Lemma 7.4.10 to the small category $[1]$. □

Lemma 7.4.13. *The product $p \times q \colon E \times F \to C \times D$ of two small cocartesian fibrations $p \colon E \to C$ and $q \colon F \to D$ is again small.*

Proof. We may factor $p \times q$ as a composite

$$E \times F \xrightarrow{p \times \mathrm{id}_F} C \times F \xrightarrow{\mathrm{id}_C \times q} C \times D.$$

The maps $p \times \mathrm{id}_F$ and $\mathrm{id}_C \times q$ are small, and hence so is their composite by (Cat1). □

Lemma 7.4.14. *Let $p \colon C \to S$ be a small cocartesian fibration. Then p has small fibers.*

Proof. We may assume without loss of generality that S is small. But then also the composite $C \to S \to *$ is small by (Cat1). □

Lemma 7.4.15. *Let C be a small category and $p \colon D \to S$ a small cocartesian fibration. Then also $p_* \colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, S)$ is a small cocartesian fibration.*

Proof. The functor $p \times \mathrm{id}_C \colon D \times C \to S \times C$ is a small cocartesian fibration, being a base change of p. The projection maps to C exhibit it as a cocartesian fibration over C. By (Cat6), the functor $(p_C)_*(D \times C) \to (p_C)_*(S \times C)$ is again a small cocartesian fibration. But this functor is precisely $p_* \colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, S)$, hence we are done. □

Lemma 7.4.16 (Functor categories)**.** *If C and D are small categories, then also* $\mathrm{Fun}(C, D)$ *is small.*

Proof. This is a special case of Lemma 7.4.15 by taking $S = *$. □

7.5 Category theory internal to the universe

In the previous chapters of this book, we have introduced various axioms for constructing and manipulating categories, and we used these axioms to build up some basic category theory. We will sometimes refer to this theory as the 'external' category theory: the notion of 'category' is taken for granted, and it is up to the user of the theory to interpret it in different contexts.

In this section, we will study 'internal' category theory. Recall that we introduced the *universe of categories* Cat in the previous section, whose objects we may think of as small categories. The goal of this section is to show that all aspects of the external theory admit reflections for small categories, purely formulated in terms of the category Cat. We may loosely refer to this as 'internal category theory in Cat'.

7.5.1 Constructions of categories

We start by discussing internal versions of initial/terminal categories, products/coproducts, pullbacks and functor categories.

Products and coproducts

Notation 7.5.1 (Initial/terminal categories)**.** By condition (Cat2) and Lemma 7.4.8, the terminal category $*$ and the initial category $\emptyset$ are both small, hence are classified by objects $\lfloor * \rfloor$ and $\lfloor \emptyset \rfloor$ of Cat:

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "\lfloor * \rfloor"] & \mathrm{Cat} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "\lfloor \emptyset \rfloor"] & \mathrm{Cat}. \end{tikzcd}$$

Lemma 7.5.2. *The object* $* \in \mathrm{Cat}$ *is a terminal object. The object* $\emptyset \in \mathrm{Cat}$ *is an initial object.*

Proof. We start with the statement about $*$. By Lemma 5.9.8, it suffices to show that for any other object $c \in \mathrm{Cat}$ the Hom groupoid $\mathrm{Hom}_{\mathrm{Cat}}(c, *)$ is contractible. By directed univalence, this groupoid is equivalent to $\mathrm{Map}(C, *)$, where C is the small category classified by c. But we have $\mathrm{Fun}(C, *) \xrightarrow{\sim} *$ by Lemma 1.4.8 and $*^{\simeq} \xrightarrow{\sim} *$ by Example 2.1.4.

The argument for $\emptyset$ is analogous, except that we use the equivalence $\mathrm{Fun}(\emptyset, C) \xrightarrow{\sim} *$ from Lemma 1.4.13. □

Notation 7.5.3 (Products)**.** By Lemma 7.4.13, the product functor $\pi_{\mathrm{univ}} \times \pi_{\mathrm{univ}} \colon \mathrm{Cat}_\bullet \times \mathrm{Cat}_\bullet \to \mathrm{Cat} \times \mathrm{Cat}$ is a small cocartesian fibration, and hence is classified by some functor $- \times - \colon \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$:

$$\begin{tikzcd} \mathrm{Cat}_\bullet \times \mathrm{Cat}_\bullet \arrow[r] \arrow[d, "\pi_{\mathrm{univ}} \times \pi_{\mathrm{univ}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Cat} \times \mathrm{Cat} \arrow[r, dashed, "\times"] & \mathrm{Cat}. \end{tikzcd}$$

Given two objects c and d of Cat corresponding to small categories C and D, note that the fiber of $\pi_{\mathrm{univ}} \times \pi_{\mathrm{univ}}$ over (c, d) is the product category $C \times D$, so that the functor $- \times - \colon \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$ does indeed encode the product of categories.

Notation 7.5.4 (Coproducts)**.** By Lemma 7.4.12, the coproduct functor

$$\langle \pi_{\mathrm{univ}} \times \mathrm{id}, \mathrm{id} \times \pi_{\mathrm{univ}} \rangle \colon (\mathrm{Cat}_\bullet \times \mathrm{Cat}) \sqcup (\mathrm{Cat} \times \mathrm{Cat}_\bullet) \to \mathrm{Cat} \times \mathrm{Cat}$$

is a small cocartesian fibration, and hence is classified by some functor $- \sqcup - \colon \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$:

$$\begin{tikzcd} (\mathrm{Cat}_\bullet \times \mathrm{Cat}) \sqcup (\mathrm{Cat} \times \mathrm{Cat}_\bullet) \arrow[r] \arrow[d, "\langle \pi_{\mathrm{univ}} \times \mathrm{id}, \mathrm{id} \times \pi_{\mathrm{univ}} \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Cat} \times \mathrm{Cat} \arrow[r, dashed, "\sqcup"] & \mathrm{Cat}. \end{tikzcd}$$

Given two objects c and d of Cat corresponding to small categories C and D, note that the fiber of the left vertical map over (c, d) is the coproduct category $C \sqcup D$, so that the functor $- \sqcup - \colon \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$ does indeed encode the coproduct of categories.

Lemma 7.5.5. *For objects* $c, d \in \mathrm{Cat}$*, the object* $c \times d$ *is a product in* Cat*, and the object* $c \sqcup d$ *is a coproduct in* Cat*.*

Proof. The argument is similar to Lemma 7.5.2. For the product, we need to show that for any other $e \in \mathrm{Cat}$ the two projection maps induce an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}}(e, c \times d) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}(e, c) \times \mathrm{Hom}_{\mathrm{Cat}}(e, d) \simeq \mathrm{Hom}_{\mathrm{Cat}\times\mathrm{Cat}}((e, e), (c, d)).$$

Under directed univalence this follows from the equivalence $\mathrm{Map}(E, C \times D) \xrightarrow{\sim} \mathrm{Map}(E, C) \times \mathrm{Map}(E, D)$ from Lemma 1.4.9. The argument for the coproduct is dual. □

Pullbacks

Recall from Definition 5.10.10 the category $\lrcorner$, which for the purposes of this section we will display as follows:

$$\lrcorner \quad = \quad \begin{tikzcd} & 0 \arrow[d] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

In particular we denote its three objects by 0, 1 and 2. We would like to construct a pullback functor $\lim\colon \mathrm{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$.

Construction 7.5.6. Let us write $\mathrm{Cat}^{\lrcorner} := \mathrm{Fun}(\lrcorner, \mathrm{Cat})$. For $i = 0, 1, 2$, we denote by $\mathrm{Cat}^{\lrcorner}_{\bullet,i} \to \mathrm{Cat}^{\lrcorner}$ the small cocartesian fibration defined via the following pullback diagram:

$$\begin{tikzcd} \mathrm{Cat}^{\lrcorner}_{\bullet,i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Cat}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Cat}^{\lrcorner} \arrow[r, "\mathrm{ev}_i"] & \mathrm{Cat}. \end{tikzcd}$$

Using the unstraightening procedure from Definition 7.2.3, the morphisms $0 \to 2$ and $1 \to 2$ in $\lrcorner$ give rise to cocartesian functors

$$\begin{tikzcd} \mathrm{Cat}^{\lrcorner}_{\bullet,0} \arrow[rr] \arrow[dr] & & \mathrm{Cat}^{\lrcorner}_{\bullet,2} \arrow[dl] \\ & \mathrm{Cat}^{\lrcorner} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Cat}^{\lrcorner}_{\bullet,1} \arrow[rr] \arrow[dr] & & \mathrm{Cat}^{\lrcorner}_{\bullet,2} \arrow[dl] \\ & \mathrm{Cat}^{\lrcorner}. & \end{tikzcd}$$

It then follows from condition (Cat3) that the resulting cocartesian fibration

$$\mathrm{Cat}^{\lrcorner}_{\bullet,0} \times_{\mathrm{Cat}^{\lrcorner}_{\bullet,2}} \mathrm{Cat}^{\lrcorner}_{\bullet,1} \to \mathrm{Cat}^{\lrcorner}$$

is again a small cocartesian fibration, and hence is classified by some functor $\lim\colon \mathrm{Cat}^{\lrcorner} \to \mathrm{Cat}$:

$$\begin{tikzcd} \mathrm{Cat}^{\lrcorner}_{\bullet,0} \times_{\mathrm{Cat}^{\lrcorner}_{\bullet,2}} \mathrm{Cat}^{\lrcorner}_{\bullet,1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Cat}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Cat}^{\lrcorner} \arrow[r, dashed, "\lim"] & \mathrm{Cat}. \end{tikzcd}$$

Observation 7.5.7. Note that an object of $\mathrm{Cat}^{\lrcorner}$ corresponds to a pair of morphisms $c \to e$ and $d \to e$ in Cat, which in turn corresponds under directed univalence to a pair of functors $C \to E$ and $D \to E$. A computation shows that the fiber of the left vertical map over this object is the pullback $C \times_E D$, showing that the functor $\lim\colon \mathrm{Cat}^{\lrcorner} \to \mathrm{Cat}$ does indeed classify pullbacks of categories.

Lemma 7.5.8. *The functor* $\lim\colon \operatorname{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$ *is right adjoint to the diagonal map* $p_{\lrcorner}^{*}\colon \mathrm{Cat} \to \operatorname{Fun}(\lrcorner, \mathrm{Cat})$.

Proof. Consider a cospan $c \to e \leftarrow d$ in Cat and an object x of Cat. We need to show that the map

$$\operatorname{Hom}_{\mathrm{Cat}^{\lrcorner}}(x \to x \leftarrow x, c \to e \leftarrow d) \to \operatorname{Hom}_{\mathrm{Cat}}(x, c \times_e d)$$

induced by the cone diagram of $c \times_e d$ is an equivalence. A computation shows that the left-hand side is equivalent to

$$\operatorname{Hom}_{\mathrm{Cat}}(x, c) \times_{\operatorname{Hom}_{\mathrm{Cat}}(x,e)} \operatorname{Hom}_{\mathrm{Cat}}(x, d).$$

By directed univalence, the map is then equivalent to the map

$$\operatorname{Map}(X, C) \times_{\operatorname{Map}(X,E)} \operatorname{Map}(X, D) \to \operatorname{Map}(X, C \times_E D),$$

which is an equivalence by Lemma 1.4.10 and Lemma 2.2.5. □

Corollary 7.5.9. *The category* Cat *admits pullbacks.* □

Functor categories

We will now discuss functor categories internal to Cat.

Construction 7.5.10. Let c be an object of Cat and let C be the associated small category. By Lemma 7.4.15, the functor

$$(\pi_{\mathrm{univ}})_*\colon \operatorname{Fun}(C, \mathrm{Cat}_\bullet) \to \operatorname{Fun}(C, \mathrm{Cat})$$

is a small cocartesian fibration, hence it is classified by some functor $\Gamma_C\colon \operatorname{Fun}(C, \mathrm{Cat}) \to \mathrm{Cat}$:

$$\begin{tikzcd}
\operatorname{Fun}(C, \mathrm{Cat}_\bullet) \arrow[r] \arrow[d, "(\pi_{\mathrm{univ}})_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\operatorname{Fun}(C, \mathrm{Cat}) \arrow[r, dashed, "\Gamma_C"] & \mathrm{Cat}.
\end{tikzcd}$$

We define the functor $(-)^c\colon \mathrm{Cat} \to \mathrm{Cat}$ as the following composite:

$$\mathrm{Cat} \xrightarrow{p_C^*} \operatorname{Fun}(C, \mathrm{Cat}) \xrightarrow{\Gamma_C} \mathrm{Cat}.$$

Note that for an object $d \in \mathrm{Cat}$ corresponding to a small category D, the image $d^c \in \mathrm{Cat}$ classifies the small category $\operatorname{Fun}(C, D)$.

Lemma 7.5.11. *The functor* $(-)^c\colon \mathrm{Cat} \to \mathrm{Cat}$ *is right adjoint to* $- \times c\colon \mathrm{Cat} \to \mathrm{Cat}$*.*

Proof. Using directed univalence this follows from Lemma 1.4.11, similar to the proofs of Lemmas 7.5.2, 7.5.5 and 7.5.8. □

Construction 7.5.12. Let $f\colon c \to d$ be a morphism in Cat corresponding under directed univalence to a functor $f\colon C \to D$ between small categories. We will construct a natural transformation

$$f^*\colon (-)^d \to (-)^c.$$

We denote by $\mathrm{Cat}_\bullet^c \to \mathrm{Cat}$ the small cocartesian fibration classified by $(-)^c\colon \mathrm{Cat} \to \mathrm{Cat}$:

$$\begin{tikzcd}
\mathrm{Cat}_\bullet^c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(C, \mathrm{Cat}_\bullet) \arrow[r] \arrow[d, "(\pi_{\mathrm{univ}})_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Cat} \arrow[r, "p_C^*"] \arrow[rr, bend right, "(-)^c"'] & \mathrm{Fun}(C, \mathrm{Cat}) \arrow[r, "\Gamma_C"] & \mathrm{Cat}.
\end{tikzcd}$$

Precomposition with f gives rise to a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(D, \mathrm{Cat}_\bullet) \arrow[r] \arrow[d] & \mathrm{Fun}(C, \mathrm{Cat}_\bullet) \arrow[d] \\
\mathrm{Fun}(D, \mathrm{Cat}) \arrow[r] & \mathrm{Fun}(C, \mathrm{Cat}),
\end{tikzcd}$$

which induces a cocartesian functor $\mathrm{Cat}_\bullet^d \to \mathrm{Cat}_\bullet^c$ over Cat. Under directed univalence this corresponds to a natural transformation $f^*\colon (-)^d \to (-)^c$ as desired.

Lemma 7.5.13. *There is a natural isomorphism* $(-)^* \cong \mathrm{id}_{\mathrm{Cat}}$ *of functors* $\mathrm{Cat} \to \mathrm{Cat}$.

Proof. This is immediate from the equivalence $\mathrm{Cat}_\bullet^* \simeq \mathrm{Cat}_\bullet$ over Cat, which follows from the equivalences $\mathrm{Fun}(*, \mathrm{Cat}) \simeq \mathrm{Cat}$ and $\mathrm{Fun}(*, \mathrm{Cat}_\bullet) \simeq \mathrm{Cat}_\bullet$ from Lemma 1.4.12. □

Lemma 7.5.14. *For small categories C and D there is a natural isomorphism* $((-)^C)^D \cong (-)^{C\times D}$ *of functors* $\mathrm{Cat} \to \mathrm{Cat}$.

Proof. The small cocartesian fibration classified by $((-)^C)^D$ sits in a pullback square as follows:

$$\begin{tikzcd}
(\mathrm{Cat}_\bullet^c)^d \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}(D, \mathrm{Fun}(C, \mathrm{Cat}_\bullet)) \arrow[d, "(\pi_{\mathrm{univ}})_*"] \\
\mathrm{Cat} \arrow[r, "p_D^*"] & \mathrm{Fun}(D, \mathrm{Cat}) \arrow[r, "(p_C^*)_*"] & \mathrm{Fun}(D, \mathrm{Fun}(C, \mathrm{Cat})).
\end{tikzcd}$$

Under the equivalences $\mathrm{Fun}(D, \mathrm{Fun}(C, \mathrm{Cat})) \simeq \mathrm{Fun}(C \times D, \mathrm{Cat})$ and $\mathrm{Fun}(D, \mathrm{Fun}(C, \mathrm{Cat}_\bullet)) \simeq \mathrm{Fun}(C \times D, \mathrm{Cat}_\bullet)$ from Lemma 1.4.11, the bottom functor is $p_{C\times D}^*$ and we see that the left vertical map agrees with the small cocartesian fibration $\mathrm{Cat}_\bullet^{c\times d} \to \mathrm{Cat}$ classifying $(-)^{c\times d}$. The claim thus follows from directed univalence. □

Remark 7.5.15. Proceeding in this way, one may prove each of the following natural isomorphisms:

- For $c \in \mathrm{Cat}$ there is a natural isomorphism $(- \times -)^c \cong (-)^c \times (-)^c$ of functors $\mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$, cf. Lemma 1.4.9;
- For $t \in \mathrm{Cat}$ there is a natural isomorphism $(\lim(-))^t \cong \lim((-)^t)$ of functors $\mathrm{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$, cf. Lemma 1.4.10;
- There is a natural isomorphism $(-)^\emptyset \cong \mathrm{const}_\emptyset$, cf. Lemma 1.4.13;
- There is a natural isomorphism $(-)^{c \sqcup d} \cong (-)^c \times (-)^d$, cf. Lemma 1.4.14.

7.6 The universe of groupoids

In this section we construct the *universe of groupoids* Grpd and equip it with the *universal small left fibration* $\pi_{\mathrm{univ}}\colon \mathrm{Grpd}_\bullet \to \mathrm{Grpd}$. We further show that the inclusion $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$ admits both a right adjoint $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{Grpd}$ and a left adjoint $\|-\|\colon \mathrm{Cat} \to \mathrm{Grpd}$.

Recall that a category X is a groupoid if all its morphisms are invertible, or equivalently if the functor $p^*_{[1]}\colon X \to \mathrm{Fun}([1], X)$ is an equivalence. This immediately leads to an internal definition of groupoids:

Definition 7.6.1. Applying Construction 7.5.12 to the projection map $[1] \to [0]$, we obtain a natural transformation

$$p^*_{[1]}\colon \ \mathrm{id}_{\mathrm{Cat}} \cong (-)^{[0]} \to (-)^{[1]}$$

of functors $\mathrm{Cat} \to \mathrm{Cat}$, which corresponds to a functor $\mathrm{Cat} \to \mathrm{Fun}([1], \mathrm{Cat})$ sending x to $p^*_{[1]}\colon x \to x^{[1]}$. We say that $x \in \mathrm{Cat}$ is a *groupoid* if this morphism is an equivalence in Cat. We define the category Grpd of groupoids via the pullback square

$$\begin{tikzcd} \mathrm{Grpd} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(\mathrm{Cat}) \arrow[d, hook] \\ \mathrm{Cat} \arrow[r, "x \mapsto (x \to x^{[1]})"'] & \mathrm{Fun}([1], \mathrm{Cat}). \end{tikzcd}$$

Since the right vertical map is fully faithful by Corollary 6.4.3, we see that the resulting functor $\mathrm{Grpd} \to \mathrm{Cat}$ is fully faithful. We define the cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Grpd}_\bullet \to \mathrm{Grpd}$ via the pullback square

$$\begin{tikzcd} \mathrm{Grpd}_\bullet \arrow[r] \arrow[d, "\pi_{\mathrm{univ}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Grpd} \arrow[r] & \mathrm{Cat}. \end{tikzcd}$$

Note that the cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Grpd}_\bullet \to \mathrm{Grpd}$ is directed univalent by Theorem 7.4.3, because it is classified by a fully faithful functor $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$.

Definition 7.6.2. A small cocartesian fibration $p\colon X \to S$ which is a left fibration is called a *small left fibration*.

Theorem 7.6.3. *The functor* $\pi_{\mathrm{univ}}\colon \mathrm{Grpd}_\bullet \to \mathrm{Grpd}$ *is the universal small left fibration: given a small cocartesian fibration* $p\colon X \to S$ *classified by a functor* $F\colon S \to \mathrm{Cat}$*, we have that* p *is a left fibration if and only if* F *factors through the full subcategory* Grpd *of* Cat.

Proof. Let $p\colon X \to S$ be a small cocartesian fibration classified by a functor F:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ S \arrow[r, "F"] & \mathrm{Cat}. \end{tikzcd}$$

We know from Lemma 5.9.11 that p is a left fibration if and only if the fibers of p are groupoids. Note that the fiber of p at an object x of S is the category classified by the object $F(x)$ of Cat, so we see that p is a left fibration if and only if $F(x) \in \mathrm{Grpd}$ for every object x of S. Since the inclusion $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$ is fully faithful, the latter condition is equivalent to the condition that the functor F factors through the full subcategory Grpd of Cat. $\square$

Proposition 7.6.4. *The category* Grpd *admits pullbacks, and the inclusion* $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$ *preserves pullbacks.*

Proof. Recall from Corollary 7.5.9 that Cat admits pullbacks: given morphisms $c \to e$ and $d \to e$ in Cat corresponding to functors $C \to E$ and $D \to E$, the pullback $c \times_e d$ in Cat corresponds to the pullback $C \times_E D$. We have seen in Lemma 2.1.9 that if C, D and E are groupoids then so is $C \times_E D$. This shows that the functor $\lim\colon \mathrm{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$ restricts to a functor $\lim\colon \mathrm{Fun}(\lrcorner, \mathrm{Grpd}) \to \mathrm{Grpd}$ right adjoint to the diagonal functor. This means that Grpd admits pullbacks and that they are preserved by the inclusion to Cat. $\square$

7.6.1 Groupoid cores

We would like to show that the inclusion $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$ admits a right adjoint

$$(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{Grpd}$$

which assigns to every small category C its groupoid core $C^{\simeq}$. For this, we start by constructing the *fiberwise maximal groupoid* $C^{\simeq/S} \to S$ of a cocartesian fibration $p\colon C \to S$.

Construction 7.6.5. Let $p\colon C \to S$ be a cocartesian fibration. Consider a morphism $u\colon x_0 \to x_1$ in C, and write $v = p(u)$ for the resulting morphism in S. We then get a unique map $\psi(u)\colon v_!(x_0) \to x_1$ in C_{s_1} fitting in the following commutative diagram:

$$\begin{tikzcd} & v_!(x_0) \arrow[dr, "\psi(u)"] & \\ x_0 \arrow[ur] \arrow[rr] & & x_1. \end{tikzcd}$$

This process defines a functor

$$\psi\colon \mathrm{Map}([1], C) \to \mathrm{Map}([1], C)\colon u \mapsto \psi(u).$$

Observe that we have $p(\psi(u)) = \mathrm{id}_{p(x_0)}$. We now define a collection of maps in C via the following pullback square:

$$\begin{tikzcd} M \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(C)^{\simeq} \arrow[d, hook] \\ \mathrm{Map}([1], C) \arrow[r, "\psi"] & \mathrm{Map}([1], C). \end{tikzcd}$$

Observe that a morphism is in M if and only if it is p-cocartesian. Since the p-cocartesian morphisms in C are closed under composition by **??**, it follows that the collection of maps M is closed under composition, and thus we may make the following definition:

Definition 7.6.6. Given a cocartesian fibration $C \to S$, we define the *fiberwise maximal groupoid* of C over S as the subcategory

$$C^{\simeq/S} := \langle M \rangle_C \subseteq C.$$

Proposition 7.6.7. *The resulting map $p'\colon C^{\simeq/S} \to S$ is a left fibration and the functor $C^{\simeq/S} \hookrightarrow C$ is a cocartesian functor over S.*

Proof. A direct inspection of universal properties shows that any morphism in $C^{\simeq/S}$ whose image in C is p-cocartesian is in fact p'-cocartesian. Since all p-cocrtesian morphisms of C belong to $C^{\simeq/S}$, it is clear that any morphism of S equipped with a lift of its source in $C^{\simeq/S}$ has a unique p'-cocartesian lift with the prescribed source: it suffices to provide a p-cocartesian lift in C. This proves that p' is a cocartesian fibration and that the embedding of $C^{\simeq/S}$ into C is a cocartesian functor over S. □

Lemma 7.6.8. *Let $p\colon C \to S$ be a cocartesian fibration and let $f\colon T \to S$ be any functor. For any left fibration $q\colon Y \to T$, there is an equivalence*

$$\mathrm{Map}_T(Y, S \times_T C^{\simeq/S}) \xrightarrow{\simeq} \mathrm{Map}_T^{\mathrm{CoCart}}(Y, T \times_S C).$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} (S \times_T C)^{\simeq/T} \arrow[r, "\simeq"] & S \times_T C^{\simeq/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\simeq/S} \arrow[d] \\ & T \arrow[r] & S. \end{tikzcd}$$

Now use the universal property of the subcategory $\langle M \rangle_C$. □

We are now ready to construct the groupoid core functor $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{Cat}$.

Construction 7.6.9 (Groupoid core)**.** Applying Proposition 7.6.7 to the universal small cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$, we obtain a left fibration

$$\begin{tikzcd} \mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} \arrow[rr, hook] \arrow[dr] & & \mathrm{Cat}_\bullet \arrow[dl, "\pi_{\mathrm{univ}}"] \\ & \mathrm{Cat}. & \end{tikzcd}$$

Since the top functor is an embedding, it follows from the closure condition (Cat4) that the left fibration $\mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} \to \mathrm{Cat}$ is a small cocartesian fibration, hence is classified by a functor $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{Cat}$:

$$\begin{tikzcd} \mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Cat} \arrow[r, dashed, "(-)^{\simeq}"] & \mathrm{Cat}. \end{tikzcd}$$

The inclusion map $\mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} \to \mathrm{Cat}_\bullet$ is a cocartesian functor over Cat, and hence corresponds under directed univalence to a natural transformation

$$\gamma\colon (-)^{\simeq} \to \mathrm{id}_{\mathrm{Cat}}$$

of functors $\mathrm{Cat} \to \mathrm{Cat}$.

Lemma 7.6.10. *The functor* $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{Cat}$ *factors through the full subcategory* $\mathrm{Grpd} \subseteq \mathrm{Cat}$.

Proof. By definition of full subcategories, it suffices to show that for every object c of Cat the object $c^{\simeq}$ of Cat is a groupoid. If C is the small category classified by $c \in \mathrm{Cat}$, then a simple computation shows that its groupoid core $C^{\simeq}$ is the small category classified by $c^{\simeq} \in \mathrm{Cat}$ (here one uses the equivalence $C^{\simeq} \simeq \langle \mathrm{Iso}(C) \rangle_C$.) Since $C^{\simeq}$ is a groupoid, the claim follows. □

Proposition 7.6.11. *The functor* $(-)^{\simeq} \colon \mathrm{Cat} \to \mathrm{Grpd}$ *is right adjoint to the inclusion* $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$*, with counit given by the natural transformation* $\gamma \colon (-)^{\simeq} \to \mathrm{id}_{\mathrm{Cat}}$.

Proof. Consider objects $c, x \in \mathrm{Cat}$ and assume that $x \in \mathrm{Grpd}$. We have to show that postcomposition with the map $c^{\simeq} \to c$ induces an equivalence of groupoids

$$\mathrm{Hom}_{\mathrm{Grpd}}(x, c^{\simeq}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}(x, c).$$

Under directed univalence, this map agrees with the functor

$$\mathrm{Map}(X, C^{\simeq}) \to \mathrm{Map}(X, C)$$

induced by the inclusion $\gamma_C \colon C^{\simeq} \to C$. The fact that this is an equivalence is Proposition 2.2.11. □

7.6.2 Geometric realizations and localizations

Our next goal is to construct a left adjoint

$$\| - \| \colon \mathrm{Cat} \to \mathrm{Grpd}$$

to the inclusion $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$, which is given on objects by sending a small category C to its fundamental groupoid $\|C\|$.

The main ingredient will be to understand how to localize a cocartesian fibration fiberwise. We continue to fix a universe Cat of small categories, equipped with a universal cocartesian fibration $\pi_{\mathrm{univ}} \colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ with small fibers. Recall that we then defined the universal *left* fibration with small fibers as the pullback

$$\begin{tikzcd} \mathrm{Grpd}_\bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Grpd} \arrow[r, hook, "i"] & \mathrm{Cat}. \end{tikzcd}$$

Last time, we investigated the right adjoint to the inclusion i. This time, we would like to understand the *left* adjoint of i.

Proposition 7.6.12. *The inclusion functor* $i \colon \mathrm{Grpd} \hookrightarrow \mathrm{Cat}$ *admits a left adjoint* $\| - \| \colon \mathrm{Cat} \to \mathrm{Grpd}$.

Proof. We will verify the conditions of Proposition 5.9.14. Let c be an object of Cat and let C denote its associated small category. By condition 7 there exists a small category $\|C\|$ together with a functor $\eta_C \colon C \to \|C\|$ that exhibits it as a localization of C at all its morphisms. This implies that $\|C\|$ is the fundamental groupoid of C and hence is a groupoid by Proposition 3.4.5. It is thus classified by some object $\|C\| \in \mathrm{Grpd}$, and under directed univalence the localization map $C \to \|C\|$ is classified by a morphism $c \to \|C\|$ in Cat.

We claim the assignment $c \mapsto \|C\|$ enhances to a functor $\| - \|\colon \mathrm{Cat} \to \mathrm{Grpd}$ which is left adjoint to the inclusion functor. By Proposition 5.9.14, it will suffice to show that for any object $x \in \mathrm{Grpd}$ the map

$$\mathrm{Hom}_{\mathrm{Grpd}}(\|C\|, x) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}(i\|C\|, ix) \xrightarrow{-\circ\eta_C} \mathrm{Hom}_{\mathrm{Cat}}(c, ix)$$

is an equivalence. Under directed univalence, this map agrees with the map

$$- \circ \eta_C\colon\ \mathrm{Map}(\|C\|, X) \to \mathrm{Map}(C, X).$$

The fact that this map is an equivalence then follows from Lemma 3.4.3. □

The geometric realization functor is a special case of the process of localizing by a family of maps.

Construction 7.6.13. Let C be a category with pullbacks. There is a functor

$$\delta\colon\ \mathrm{Fun}([1], C) \to \mathrm{Fun}([1], C)$$

sending each morphism $f\colon x \to y$ in C to the diagonal map $\delta_f\colon x \to x \times_y x$. induced by the canonical commutative square

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\mathrm{id}_x"'] & x \arrow[d, "f"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in C. We define the category of embeddings of C as the full subcategory of $\mathrm{Fun}([1], C)$ spanned by embeddings in C. In other words, this is the category obtained by forming the following pullback square.

$$\begin{tikzcd} \mathrm{Emb}(C) \arrow[r, hook] \arrow[d] & \mathrm{Fun}([1], C) \arrow[d, "\delta"] \\ \mathrm{Iso}(C) \arrow[r, hook] & \mathrm{Fun}([1], C) \end{tikzcd}$$

For $C = \mathrm{Cat}$, we define the categories of pairs of small categories as

$$\mathrm{Pair} = \mathrm{Emb}(\mathrm{Cat})$$

By a slight abuse of notation, we will often think of objects in Pair as a pair (C, W) where C is a small category and W a subcategory of C (which makes sense, using Proposition 6.4.1). There is a functor

$$\mathrm{ev}_1 \mathrm{Pair} \to C$$

sending (C, W) to C. The latter has a unique section

$$(-)^\flat\colon \mathrm{Cat} \to \mathrm{Pair}$$

with the property of sending each small category C to the pair $C^\flat = (C, C^\simeq)$. The functor $(-)^\flat$ is easily seen to be fully faithful (this boils down to the fact that any functor sends isomorphisms to isomorphisms).

Proposition 7.6.14. *The functor $C \mapsto (C, C^\simeq)$ has a left adjoint*

$$L\colon \mathrm{Pair} \to \mathrm{Cat}\,.$$

More precise, for a pair (C, W), the category $L(C, W)$ corresponds to the localization of C by the family of maps $\mathrm{Map}([1], W)$.

Proof. A morphism of pair of the form $(C, W) \to (D, D^{\simeq})$ corresponds to a commutative square in Cat of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & D^{\simeq} \arrow[d] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

in which both vertical maps are embeddings. But by virtue of Proposition 6.4.1, such a commutative square is totally determined by the functor $f : C \to D$ and the fact that the induced map

$$f_* \colon \operatorname{Map}([1], C) \to \operatorname{Map}([1], D)$$

sends $\operatorname{Map}([1], W)$ to $\operatorname{Iso}(D)^{\simeq}$. In other words, this is the same thing (in any groupoidal context) as a functor from $L(C, W)$ to D, where $L(C, W)$ denotes the localization of C by $\operatorname{Map}([1], W)$. □

Remark 7.6.15. For any category, $i(\|C\|) = L(C, C)$, where $i \colon \mathrm{Grpd} \to \mathrm{Cat}$ denotes the canonical embedding.

7.6.3 Fiberwise localizations

As a consequence of the adjunction between $(-)^\flat$ and L from the previous paragraph, we get for every category S an adjunction between the induced functors

$$L_S \colon \operatorname{Fun}(S, \mathrm{Pair}) \to \operatorname{Fun}(S, \mathrm{Cat}) \qquad \text{and} \qquad (-)^\flat_S \colon \operatorname{Fun}(S, \mathrm{Pair}) \to \operatorname{Fun}(S, \mathrm{Cat}).$$

The existence of the functor L_S allows us to form *fiberwise localizations* of (small) cocartesian fibrations:

Proposition 7.6.16. *Let $p \colon C \to S$ be a small cocartesian fibration. We consider, for each object $x \colon S^{\simeq}$ a subcategory $W_x \hookrightarrow C_x$ and assume that, for each morphism $f : x \to y$ in S, the induced functor $f_! \colon C_x \to C_y$ restricts to a functor from W_x to W_y. Then there exists a commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "\eta_C"] \arrow[dr, "p"'] & & L_S(C, W) \arrow[dl, "L_S(p)"] \\ & S & \end{tikzcd}$$

satisfying the following properties:

(1) The functor $L_S(p)$ is a cocartesian fibration;

(2) For any other small cocartesian fibration $D \to S$, precomposition with η_C induces an equivalence

$$\eta_C^* \colon \operatorname{Map}_S^{\mathrm{CoCart}}(L_S(C, W), D) \xrightarrow{\sim} \operatorname{Map}_S^{\mathrm{CoCart}, W}(C, D)$$

where $\operatorname{Map}_S^{\mathrm{CoCart}, W}(C, D)$ is the collection of cocartesian functor $\varphi \colon C \to D$ over S with the property that, for any object $x \colon S^{\simeq}$, the induced functor on fibers $\varphi_x \colon C_x \to D_x$ sends all morphisms of W_x to isomorphisms of D_x.

(3) The formation of η_C and $L_S(C, W)$ is stable under base change along any functor $S' \to S$. In particular, for any object $s \colon S^{\simeq}$, the functor $(\eta_C)_s \colon C_s \to L_S(C, W)_s$ induces an equivalence

$$L(C_s, W_s) \xrightarrow{\sim} L_S(C, W)_s\,.$$

Proof. Each morphism u of C factors uniquely into a p-cocartesian morphism $v\colon x \to y$ followed by a morphism $w\colon y \to y'$ that is sent to the identity of $p(y)$. We define W s the subcategory of C whose morphisms are characterized by the property that, in the factorization above, w belongs to $W_{p(y)}$. It is obvious that the restriction of p to W remains a cocartesian fibration and that the embedding $W \hookrightarrow C$ is a cocartesian functor over S. Such data correspond by directed univalence to a functor $F\colon S \to \mathrm{Pair}$. Then the composite

$$L \circ F\colon S \to \mathrm{Cat}$$

classifies a cocartesian fibration ${}_S(p)\colon L_S(C,W) \to S$:

$$\begin{tikzcd} L_S(C,W) \arrow[rr] \arrow[d, "L_S(p)"'] \arrow[drr, phantom, "\lrcorner"] & & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ S \arrow[r, "F"] & \mathrm{Pair} \arrow[r, "L"] & \mathrm{Cat}. \end{tikzcd}$$

The natural transformation $\mathrm{id}_{\mathrm{Pair}} \to (L(-))^\flat$ provides a natural transformation $F \to (L(F))^\flat$, which, by directed univalence, yields a cocartesian functor from C to $L_S(C,W)$ over S:

$$\begin{tikzcd} C \arrow[rr, "\eta_C"] \arrow[dr, "p"'] & & L_S(C,W) \arrow[dl, "L_S(p)"] \\ & S. & \end{tikzcd}$$

This produces the desired commutative triangle and proves property (1). For (2), we note that postcomposition with the functors L and $(-)^\flat$ extends to an adjunction

$$L_S = (L)_*\colon \mathrm{Fun}(S,\mathrm{Pair}) \rightleftarrows \mathrm{Fun}(S,\mathrm{Cat}) : ((-)^\flat)_*$$

whose unit is induced by the unit map of the adjoint pair $(L, (-)^\flat)$, by Proposition 5.4.4. In particular, for functors $F\colon S \to \mathrm{Pair}$ and $G\colon S \to \mathrm{Cat}$ the map

$$\mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{Cat})}(L\circ F, G) \xrightarrow{(-)^\flat \circ -} \mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{Pair})}((L\circ F)^\flat, G^\flat) \xrightarrow{-\circ \eta} \mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{Pair})}(F, G^\flat)$$

is an equivalence. The functor $F\colon S \to \mathrm{Pair}$ corresponds to a small cocartesian fibration $C \to S$, equipped with a subcategory $W \hookrightarrow C$ such that the restriction of p to W remains a cocartesian fibration and the embedding of W into C a cocartesian functor over S, while the functor $G\colon S \to \mathrm{Cat}$ corresponds to a small cocartesian fibration $D \to S$. Under directed univalence, the previous functor then translates into the functor

$$\eta_C^*\colon \mathrm{Map}_S^{\mathrm{CoCart}}(\|C\|_S, D) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart},W}(C,D),$$

from part (2), showing that it is an equivalence.

For part (3), we observe that the map η_C can in fact be recovered as a pullback from the universal case $S = \mathrm{Pair}$ and $F = \mathrm{id}_{\mathrm{Pair}}$. More precisely, The embedding

$$\mathrm{Pair} \hookrightarrow \mathrm{Fun}([1],\mathrm{Cat}) \xrightarrow{\sim} \underline{\mathrm{CoCart}}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}_\bullet \times \mathrm{Cat}, \mathrm{Cat}\times\mathrm{Cat}_\bullet)$$

yields a commutative triangle of the form

$$\begin{tikzcd} W_{\mathrm{univ}} \arrow[rr, hook, "i_{\mathrm{univ}}"] \arrow[dr] & & C_{\mathrm{univ}} \arrow[dl, "p_{\mathrm{univ}}"] \\ & \mathrm{Pair} & \end{tikzcd}$$

in which p_{univ} as well as its restriction to W_{univ} are cocartesian fibrations and i_{univ} is a cocartesian functor over Pair and an embedding (fiberwise, hence globally as well). Pulling back the diagram above along $F\colon S \to \mathrm{Pair}$ recovers $W \hookrightarrow C$ over S and we have canonical pullback squares of the form:

$$\begin{tikzcd} L_S(C,W) \arrow[r] \arrow[d, "L_S(p)"'] & L_{\mathrm{Pair}}(C_{\mathrm{univ}}, W_{\mathrm{univ}}) \arrow[r] \arrow[d, "L_{\mathrm{Pair}}(p_{\mathrm{univ}})"'] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ S \arrow[r, "F"] & \mathrm{Pair} \arrow[r, "L"] & \mathrm{Cat}\,. \end{tikzcd}$$

This readily implies the final statement in part (3). □

Corollary 7.6.17. *Let $p\colon C \to S$ be a small cocartesian fibration. Then there exists a commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "\eta_C"] \arrow[dr, "p"'] & & \|C\|_S \arrow[dl, "\|p\|_S"] \\ & S & \end{tikzcd}$$

satisfying the following properties:

(1) The functor $\|p\|_S$ is a small left fibration;

(2) For any other small left fibration $X \to S$, precomposition with η_C induces an equivalence

$$\eta_C^*\colon\ \mathrm{Map}_S^{\mathrm{CoCart}}(\|C\|_S, X) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart}}(C, X).$$

(3) The formation of η_C and $\|C\|_S$ is stable under base change along any functor $S' \to S$. In particular, for any object s of S the functor $(\eta_C)_s\colon C_s \to (\|C\|_S)_s$ induces an equivalence

$$\|C_s\| \xrightarrow{\sim} (\|C\|_S)_s.$$

Proof. Under the assumptions of the preceding proposition, if we suppose furthermore that $W_x = C_x$ for all $x\colon S^{\simeq}$, we define

$$\|C\|_S = L_S(C,C)\,.$$

We see that each fiber of $L_S(p)$ at $x\colon S^{\simeq}$ is the geometric realization of the fiber of p:

$$\|C_x\| = L_S(C,C)_x\,.$$

This means that the cocartesian fibration $L_S(p)$ is in fact a left fibration. This corollary is an immediate consequence of Proposition 7.6.16. □

7.6.4 Fiberwise localization as a localization

In the previous subsection we have constructed the fiberwise localization $\eta_C : C \to \|C\|_S$ over S for every cocartesian fibration $C \to S$ and showed that it has a universal property among left fibrations over S receiving a cocartesian functor from C. It turns out that the functor $\eta_C : C \to \|C\|_S$ also admits a non-relative universal property: it is the functor obtained those maps in C that are inverted by $p : C \to S$, i.e. those maps which are contained in a single fiber of p. The goal of this subsection is to prove this claim.

The main ingredient for the proof will be a certain result about relative functor categories. Let $p : C \to S$ be a cocartesian fibration. For any other category D, we may consider the projection map $S \times D \to S$, which defines another category over S. Since p is exponentiable by Axiom M, we may form the relative functor category over S:

$$\pi : \underline{\mathrm{Fun}}_S(C, S \times D) \to S.$$

Given another functor $q : E \to S$, a map $f : D \to \underline{\mathrm{Fun}}_S(C, S \times D)$ over S corresponds to a map $\varphi : D \times_S C \to S \times D$ over D:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "q"'] & \underline{\mathrm{Fun}}_S(C, S \times D) \arrow[d, "\pi"] \\ & S \end{tikzcd} \iff \begin{tikzcd} E \times_S C \arrow[r, "\varphi"] \arrow[dr] & D \times D \arrow[d, "\mathrm{pr}_1"] \\ & S. \end{tikzcd}$$

Proposition 7.6.18. *Let $p : C \to S$ be a cocartesian (cartesian) fibration and let D be a category. Then the functor*

$$\pi : T := \underline{\mathrm{Fun}}_S(C, D \times S) \to S$$

is a cartesian (cocartesian) fibration whose fiber over an object s of S is given by $\mathrm{Fun}(C_s, D)$.

Similarly, for any cocartesian (cartesian) functor $C \to C'$ over S, the induced functor $\underline{\mathrm{Fun}}_S(C', D \times S) \to \underline{\mathrm{Fun}}_S(C, D \times S)$ is a cartesian functor over S.

Proof. It suffices to prove the case where p is a cocartesian fibration: the case of cartesian fibrations follows from the semantic interpretation of synthetic category theory obtained by exchanging the roles of 0 and 1 in the interval $[1]$.

Recall that for a map $\alpha : S' \to S$, we have a pullback square

$$\begin{tikzcd} \underline{\mathrm{Fun}}_{S'}(C', S' \times D) \arrow[r] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Fun}}_S(C, S \times D) \arrow[d, "\pi"] \\ S' \arrow[r, "\alpha"] & S, \end{tikzcd}$$

where we write $C' := S' \times_S C \to S'$. In particular, it follows directly that the fibers of π are given by $\mathrm{Fun}(C_s, D)$, and it remains to show that π is a cartesian fibration.

We will first show that π is a locally cocartesian fibration and then show that locally cartesian morphisms compose. By (the dual of) Proposition 5.11.11 this will imply the claim.

Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],T) \arrow[r, "\vec{\mathrm{ev}}_1^\pi"] \arrow[d, "\pi_*"'] & S \vec{\times}_\pi T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & T \arrow[d, "\pi"] \\
\mathrm{Fun}([1],S) \arrow[r, equal] & \mathrm{Fun}([1],S) \arrow[r, "\mathrm{ev}_1"] & S.
\end{tikzcd}$$

Given a morphism $u\colon a \to b$ in S, we may consider the fiber of the map $\vec{\mathrm{ev}}_1^\pi$ over $u \in \mathrm{Fun}([1],S)$ to obtain a functor

$$l := (\vec{\mathrm{ev}}_1^\pi)_u\colon\ \mathrm{Fun}([1],T)_u \to (S \vec{\times}_\pi T)_u \simeq T_b.$$

We claim that π is a locally cartesian fibration if and only if this functor l admits a right adjoint section $r\colon T_b \hookrightarrow \mathrm{Fun}([1],T)_u$. Indeed, the existence of r means that for every object $t \in T_b$ there exists a morphism $r(t)\colon t' \to t$ over u such that for every diagram

$$\begin{tikzcd}
s' \arrow[r] \arrow[d, dashed] & s \arrow[d] \\
t' \arrow[r, "r(t)"] & t
\end{tikzcd}$$

in T living over $u\colon a \to b$, there exists a dashed arrow filling the square. But this is equivalent to the condition that $r(t)\colon t' \to t$ is a locally cartesian lift of u.

This characterization of locally cocartesian fibrations works for arbitrary functors $T \to S$. For our specific T we must now compute what the map $\mathrm{Fun}([1],T)_u \to T_b$ is. In the source we get $\mathrm{Fun}(C_u, D)$, where C_u is defined via the following pullback diagram:

$$\begin{tikzcd}
C_u \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & C \arrow[d, "p"] \\
{[1]} \arrow[r, "u"] & S.
\end{tikzcd}$$

We also have $T_b \xrightarrow{\sim} \mathrm{Fun}(C_b, D)$. The map $\mathrm{Fun}(C_u, D) \to \mathrm{Fun}(C_b, D)$ is given by restriction along the inclusion $C_b \hookrightarrow C_u$. Our goal now is to find a right adjoint section of this functor.

Recall from Proposition 5.4.4 that for an arbitrary right adjoint section $l \dashv s\colon C \rightleftarrows D$, the induced maps

$$s^*\colon\ \mathrm{Fun}(C,D) \rightleftarrows \mathrm{Fun}(D,D)\ :l^*$$

still form a right adjoint section (where now l^* is the section). So it will suffice to show that the inclusion $C_b \hookrightarrow C_u$ is the right adjoint section of some functor $C_u \to C_b$. This is a general fact about cocartesian fibrations: we have a pullback square

$$\begin{tikzcd}
C_b \arrow[r, hook] \arrow[d, "p_b"'] \arrow[dr, phantom, "\lrcorner" very near start] & C_u \arrow[d, "p_u"] \\
\{1\} \arrow[r, hook] & {[1]},
\end{tikzcd}$$

and since the functor p_u is a cocartesian fibration and $\{1\} \hookrightarrow [1]$ admits a left adjoint $[1] \to \{1\}$ it follows from Proposition 5.9.15 that also $C_b \hookrightarrow C_u$ admits a right adjoint $C_u \to C_b$ as desired. This finishes the proof that $\pi\colon T \to S$ is a locally cartesian fibration.

To see that locally π-cartesian morphisms are closed under composition, we may equivalently show that the locally cartesian transport functors compose. In other words, given two morphisms $u\colon a \to b$ and $v\colon b \to c$ in S, we need to show that the composite

$$T_c = \mathrm{Fun}(C_c, D) \xrightarrow{v^*} \mathrm{Fun}(C_b, D) \xrightarrow{u^*} \mathrm{Fun}(C_a, D) = T_a$$

agrees with $(vu)^*\colon T_c \to T_a$. To see this, we give an explicit description of v^* and u^*.

We start by giving an explicit description of the left adjoint $t\colon C_u \to C_b$ to the inclusion functor $C_b \to C_u$: on objects, it sends a pair (t, x) consisting of an object t of $[1]$ and an object x of $C_{u(t)}$ to the object $u(t \to 1)_!(x)$, given by the cocartesian transport of x along the morphism $u(t \to 1)$, where $t \to 1$ is the unique map in $[1]$ from t to 1. The locally cartesian transport $u^*\colon T_b \to T_a$ along $u\colon a \to b$ is then given as the following composite:

$$u^*\colon\ \mathrm{Fun}(C_b, D) \xrightarrow{t^*} \mathrm{Fun}(C_u, D) \xrightarrow{\mathrm{res}} \mathrm{Fun}(C_a, D).$$

The first functor sends $F\colon C_b \to D$ to the assignment

$$(t, x) \mapsto F(u(t \to 1)_!(x)).$$

Restricting to $t = 0$ thus shows that the composite sends F to the functor $x \mapsto F(u_!(x))$, which is the precomposition of F with $u_!\colon C_a \to C_b$.

Now, given two morphisms $u\colon a \to b$ and $v\colon b \to c$ in S, the composite

$$\mathrm{Fun}(C_c, D) \xrightarrow{v^*} \mathrm{Fun}(C_b, D) \xrightarrow{u^*} \mathrm{Fun}(C_a, D)$$

is given by precomposition with $v_! \circ u_!$. Since p is a cocartesian fibration, this functor is $(v \circ u)_!\colon C_a \to C_c$, hence precomposition with it agrees with $(vu)^*$. This finishes the proof. $\square$

Remark 7.6.19. More generally, if $p\colon C \to S$ is merely *locally* cocartesian, $\pi\colon T \to S$ will be locally cartesian. Indeed, for a functor $S' \to S$ we have a pullback square

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_{S'}(C \times_S S', D \times S') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Fun}}_S(C, D \times S) \arrow[d] \\
S' \arrow[r] & S.
\end{tikzcd}$$

Taking $S' = [1]$ then shows that π being locally cartesian only depends on the pullbacks of $C \to S$ to $[1]$.

Theorem 7.6.20. *Let $p\colon C \to S$ be a cocartesian fibration. We assume given a subcategory $i\colon W \hookrightarrow C$ wuch that $p \circ i$ is a cocartesian fibration and i is a cocartesian functor over S. Then the functor $\eta_C\colon C \to L_S(C, W)$ exhibits $L_S(C, W)$ as the localization of C by* $\mathrm{Map}([1], W)$.

Proof. We must show that precomposition with η_C induces for every category D a fully faithful functor

$$\eta_C^*\colon\ \mathrm{Fun}(L_S(C, W), D) \hookrightarrow \mathrm{Fun}(C, D)$$

whose image agrees with the full subcategory $\mathrm{Fun}^W(C, D)$. To this end, consider the following functor over S:

$$\begin{tikzcd} \underline{\mathrm{Fun}}_S(L_S(C,W), S \times D) \arrow[rr, "\eta_C^*"] \arrow[dr] & & \underline{\mathrm{Fun}}_S(C, S \times D) \arrow[dl] \\ & S. & \end{tikzcd}$$

By Proposition 7.6.18 this functor is a cocartesian functor between cartesian fibrations over S, which is fiberwise given by the fully faithful functors

$$\mathrm{Fun}(L(C_x, W_x), D) \hookrightarrow \mathrm{Fun}(C_x, D)\,.$$

Recall from Lemma 6.4.6 that a cartesian functor $T' \to T$ over S is fully faithful if the induced functors $T'_x \to T_x$ on fibers is fully faithful for all $x\colon S^{\simeq}$. Not that the essential image is obviously detected fiberwise. We conclude that $\eta_C^*\colon \underline{\mathrm{Fun}}_S(\|C\|_S, S \times D) \to \underline{\mathrm{Fun}}_S(C, S \times D)$ is fully faithful, and that a functor $f \in \mathrm{Fun}(C_x, D) = \underline{\mathrm{Fun}}_S(C, S \times D)_x$ is in the essential image if and only if it inverts all morphisms in W_x. Now, applying $(p_S)_*(-)$ with $P_S\colon S \to [0]$, we obtain the functor

$$\eta_C^*\colon\ \mathrm{Fun}(L_S(C,W), D) \hookrightarrow \mathrm{Fun}(C, D),$$

which we conclude to be also fully faithful. Its essential image consists of those functors $f\colon C \to D$ whose associated cocartesian functor $(p, f)\colon C \to S \times D$ over S sends all morphisms in C_a to isomorphisms for all $a \in S$. But this is precisely the definition of the full subcategory $\mathrm{Fun}^W(C, D)$, finishing the proof. □

7.7 Straightening/unstraightening

Given a category S, functors $S \to \mathrm{Cat}$ correspond by their very definition to small cocartesian fibrations over S. Furthermore, directed univalence of π implies that natural transformations of functors $S \to \mathrm{Cat}$ correspond to cocartesian functors between small cocartesian fibrations over S, and we have seen in Lemma 7.2.5 and Proposition 7.2.6 that this assignment is compatible with identity functors and with composition of functors.

The goal of this section is to enhance the correspondence between functors $S \to \mathrm{Cat}$ and small cocartesian fibrations over S to a fully fledged equivalence of categories: if S is a small category corresponding to an object s of Cat, we will define a (non-full) subcategory $\mathrm{CoCart}(S) \subseteq \mathrm{Cat}_{/s}$ of *(small) cocartesian fibrations over S* and construct an equivalence

$$\mathrm{Fun}(S, \mathrm{Cat}) \xrightarrow{\simeq} \mathrm{CoCart}(S)$$

called the *straightening/unstraightening equivalence*.

7.7.1 Directed univalence in arbitrary contexts

Let S be a small category with associated object $s\colon \mathrm{Cat}^{\simeq}$.

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "s"] & \mathrm{Cat} \end{tikzcd}$$

We will write $\mathrm{Cat}_{/S} := \mathrm{Cat}_{/s}$. We consider similarly two small categories X and Y corresponding to

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "x"] & \mathrm{Cat} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "y"] & \mathrm{Cat} \end{tikzcd}$$

as well as functors $p\colon X \to S$ and $q\colon Y \to S$. We will also denote by $p\colon x \to s$ and $q\colon y \to s$ the associated morphisms in Cat. Since straightening is compatible with composition of functors (by Proposition 7.2.6) the canonical embedding

$$\mathrm{Cat}_{/S}((x,p),(y,q)) \hookrightarrow \mathrm{Map}([1],\mathrm{Cat}) \xrightarrow{\sim} \underline{\mathrm{CoCart}}(\mathrm{Cat}_\bullet \times \mathrm{Cat}, \mathrm{Cat} \times \mathrm{Cat}_\bullet)^{\simeq}$$

corresponds to a commutative triangle of the form bellow.

$$\begin{tikzcd} X \times \mathrm{Cat}_{/S}((x,p),(y,q)) \arrow[rr, "f_{\mathrm{univ}}"] \arrow[dr, "p\times \mathrm{id}"'] & & Y \times \mathrm{Cat}_{/S}((x,p),(y,q)) \arrow[dl, "q\times \mathrm{id}"] \\ & S \times \mathrm{Cat}_{/S}((x,p),(y,q)) & \end{tikzcd}$$

Therefore, in context $\mathrm{Cat}_{/S}((x,p),(y,q))$, there is a unique term of $\mathrm{Map}_S(X,Y)$ corresponding to this diagram. In other words, this defines a canonical map

$$\mathrm{Cat}_{/S}((x,p),(y,q)) \to \mathrm{Map}_S(X,S)\,.$$

Lemma 7.7.1. *There is a canonical equivalence* $\mathrm{Cat}_{/S}((x,p),(y,q)) \xrightarrow{\sim} \mathrm{Map}_S(X,S)$.

Proof. Using the fact that π_{univ} is directed univalent as well as Proposition 7.2.6, we see that the map

$$\mathrm{Cat}_{/S}((x,p),(y,q)) \to \mathrm{Map}_S(X,S)$$

constructed above is an equivalence because it is obiously bijective (i.e. by applying Proposition 1.1.15). □

7.7.2 The subcategory of cocartesian fibrations

We start by construction the subcategory $\mathrm{CoCart}(S)$ of $\mathrm{Cat}_{/s}$. For this, we will 'internalize' the definition of cocartesian fibrations given in Definition 5.6.1, so that it makes sense for a morphism in Cat to be a 'cocartesian fibration'.

Remark 7.7.2. In what follows, we will use the following previously established facts about the category Cat:

- The walking morphism $[1]$ is small and thus defines an object $[1] \in \mathrm{Cat}$;
- There is a pullback functor $\mathrm{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$ right adjoint to the constant diagram functor, see Corollary 7.5.9;
- There is a product functor $-\times-\colon \mathrm{Cat}\times\mathrm{Cat} \to \mathrm{Cat}$ right adjoint to the diagonal $\mathrm{Cat} \to \mathrm{Cat}\times\mathrm{Cat}$;

- For every $d \in \mathrm{Cat}$, the functor $- \times d\colon \mathrm{Cat} \to \mathrm{Cat}$ admits a right adjoint $(-)^d\colon \mathrm{Cat}' \to \mathrm{Cat}'$.
- For any object c in Cat and any object x of c (i.e. any morphism $x\colon [0] \to c$), one can define the slice $c_{/x}$ by forming the following pullback square in Cat.

$$\begin{tikzcd} c_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & c^{[1]} \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ c \times [0] \arrow[r, "\mathrm{id}_c \times x"] & c \times c. \end{tikzcd}$$

- An object x of an object c of Cat' is terminal if the projection $c_{/x} \to c$ is an isomorphism;
- all of the above holds for Cat and the embedding $\iota\colon \mathrm{Cat} \to \mathrm{Cat}$ commutes with these operations.

Construction 7.7.3. For two morphisms $f\colon a \to c$ and $g\colon b \to c$ in Cat, we define their *directed pullback* $a \vec{\times}_{f,g} b$ via the following pullback square in Cat:

$$\begin{tikzcd} a \vec{\times}_{f,g} b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & c^{[1]} \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ a \times b \arrow[r, "f \times g"] & c \times c. \end{tikzcd}$$

When $b = c$ and $g = \mathrm{id}_b$, we write $a \vec{\times}_f b := a \vec{\times}_{f,g} b$. For a morphism $f\colon a \to b$ in Cat, we define the *directed evaluation map* $\vec{\mathrm{ev}}_0^f\colon a^{[1]} \to a \vec{\times}_f b$ as the map whose two components are $\mathrm{ev}_0\colon a^{[1]} \to a$ and $f \circ \mathrm{ev}_1\colon a^{[1]} \to a \to b$.

We can now define notions of (co)cartesian fibrations internally to Cat'. We can do this in two equivalent ways: we can interpret any map in Cat' as a functor between large categories and then ask that this functor is a (co)cartesian fibration. Equivalently, we can reproduce the definition using the internal language of Cat'. Are we are about to see, the external language remains useful to define the collection of all (co)cartesian fibrations over a given base as well as the collection of all cocartesian functors between them.

Construction 7.7.4. Consider an object s of Cat', corresponding to some small category S:

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "s"] & \mathrm{Cat}. \end{tikzcd}$$

Recall that we write $\mathrm{Cat}_{/S} = \mathrm{Cat}_{/s}$. We define In order to define $\mathrm{CoCart}(S)$ as a subcategory of $\mathrm{Cat}_{/S}$, we need to define the collection $\mathrm{CoCart}(S)^{\simeq}$ of all cocartesian fibrations between small categories. For this we recall the following fact. Given a functor $p\colon X \to B$ in a given groupoidal context Γ, one can define type-theoretically the proposition asserting that f is a p-coCartesian functor as follows. For each morphism $f\colon x \to y$ of X in context Γ, one gets a commutative square of the form

$$\begin{tikzcd} X_{y/} \arrow[r, "f^*"] \arrow[d, "p"'] & X_{x/} \arrow[d, "p"] \\ A_{p(y)/} \arrow[r, "p(f)^*"] & A_{p(x)/} \end{tikzcd}$$

hence a canonical functor

$$c_{f/p}\colon X_{y/} \to A_{p(y)/} \times_{A_{p(x)/}} X_{x/}$$

and we have the proposition asserting that the latter is an equivalence in context Γ:

$$\texttt{is\_Equiv}_\Gamma(c_{f/p}) \hookrightarrow \Gamma\,.$$

The proposition $\texttt{is\_Equiv}_\Gamma(c_{f/p})$ is also the assertion, in context Γ, that f is a p-coCartesian morphism in X. Let us consider the universal morphism in X, in context $\mathrm{Map}([1], X)$, defined through the evaluation functor;

$$\mathit{univ}_X\colon\ [1] \times \mathrm{Map}_\Gamma([1], X) \hookrightarrow [1] \times \mathrm{Fun}_\Gamma([1], X) \to X\,.$$

If we still denote by p the pullback of p along the projection of $\mathrm{Map}_\Gamma([1], X)$ to Γ, the proposition asserting that univ_X is p-cocartesian in context $\mathrm{Map}_\Gamma([1], X)$ defines the collection of p-coCartesian morphisms in X:

$$\mathrm{Map}_\Gamma([1], X)_{p\text{-coCart}} := \texttt{is\_Equiv}_{\mathrm{Map}_\Gamma([1],X)}(c_{\mathit{univ}_X/p}) \hookrightarrow \mathrm{Map}_\Gamma([1], X)\,.$$

The property that p is a coCartesian fibration can be reformulated by asserting, for any object x_0 in X and any morphism $g\colon\ p(x_0) \to y_1$ in A, the existence of a unique p-coCartesian morphism in X of the form $f\colon x_0 \to x_1$ such that $p(f) = g$. In other words, evaluation at 0 induces a commutative square of the form

$$\begin{tikzcd}
\mathrm{Map}_\Gamma([1], X)_{p\text{-coCart}} \arrow[r, "\mathrm{ev}_0"] \arrow[d, "p_*"'] & X^{\simeq} \arrow[d, "p"] \\
\mathrm{Map}_\Gamma([1], A) \arrow[r, "\mathrm{ev}_0"] & A^{\simeq}
\end{tikzcd}$$

hence a canonical functor

$$\varphi_p\colon\ \mathrm{Map}_\Gamma([1], X)_{p\text{-coCart}} \to \mathrm{Map}_\Gamma([1], A) \times_{A^{\simeq}} X^{\simeq}\,,$$

and the property for the latter to be an equivalence is equivalent to the property that the functor p is a coCartesian fibration in context Γ. We thus define

$$\texttt{is\_coCartFib}(p) := \texttt{is\_Equiv}_\Gamma(\varphi_p) \hookrightarrow \Gamma\,.$$

We consider now the universal functor from a small category to S. This is the functor of the form

$$p_{\mathrm{univ}/S}\colon X_{\mathrm{univ}/S} \to (\mathrm{Cat}_{/S})^{\simeq} \times S$$

corresponding through directed univalence to the morphism

$$\mathrm{univ}_{/S}\colon\ [1] \times (\mathrm{Cat}_{/S})^{\simeq} \hookrightarrow [1] \times \mathrm{Map}([1], \mathrm{Cat}) \hookrightarrow [1] \times \mathrm{Fun}([1], \mathrm{Cat}) \to \mathrm{Cat}$$

induced by the obvious evaluation functor. The collection of coCartesian fibrations of the form $p\colon X \to S$ with X small is defined to be the proposition asserting that $p_{\mathrm{univ}_{/S}}$ is a coCartesian fibration in context $(\mathrm{Cat}_{/S})^{\simeq}$:

$$\mathrm{CoCart}(S)^{\simeq} := \texttt{is\_coCartFib}(p_{\mathrm{univ}_{/S}}) \hookrightarrow (\mathrm{Cat}_{/S})^{\simeq}\,.$$

We define as an intermediate step the full subcategory $\mathrm{CoCart}(S)'$ of $\mathrm{Cat}_{/S}$ spanned by coCartesian fibrations of codomain S. By directed univalence, in context $\mathrm{Map}([1], \mathrm{CoCart}(S)')$, there is a universal functor between cocartesian fibrations over S:

$$\begin{tikzcd} X_0 \arrow[rr, "f_{\mathrm{univ}/S}"] \arrow[dr, "p_0"'] & & X_1 \arrow[dl, "p_1"] \\ & \mathrm{Map}([1], \mathrm{CoCart}(S)') \times S & \end{tikzcd}$$

Asserting that f_{univ} is a cocartesian functor over S in context $\mathrm{Map}([1], \mathrm{CoCart}(S)')$ thus defines a collection of morphisms in $\mathrm{CoCart}(S)'$:

$$\mathrm{Map}([1], \mathrm{CoCart}(S)')_{\mathrm{CoCart}} := \mathtt{is\_Cocart}(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1], \mathrm{CoCart}(S)') \hookrightarrow \mathrm{Map}([1], \mathrm{Cat}_{/S})\,.$$

We finally define $\mathrm{CoCart}(S)$ as the subcategory of $\mathrm{Cat}_{/S}$ generated by $\mathrm{Map}([1], \mathrm{CoCart}(S)')_{\mathrm{CoCart}}$:

$$\mathrm{CoCart}(S) = \langle \mathrm{Map}([1], \mathrm{CoCart}(S)')_{\mathrm{CoCart}} \rangle_{\mathrm{Cat}_{/S}}\,.$$

By definition, a functor $f\colon B \to \mathrm{Cat}_{/S}$ factors through $\mathrm{CoCart}(S)$ if and only it sends each object of B to cocartesian fibrations over S and each morphism of B to cocartesian functors between cocartesian fibrations over S.

Construction 7.7.5. Given S as above, we now construct a functor

$$\mathrm{Un}\colon\ \mathrm{Fun}(S, \mathrm{Cat}) \to \mathrm{Cat}_{/S}.$$

We observe that, given any category B and any functor $\Phi\colon B \to \mathrm{Fun}(S, \mathrm{Cat})$, we may uncurry Φ into a functor of the form

$$F\colon B \times S \to \mathrm{Cat}$$

and then consider the small cocartesian fibration classified by F:

$$p\colon E \to B \times S\,.$$

Observe that the projection map $\mathrm{pr}_B\colon B \times S \to B$ sits in a pullback square of the form

$$\begin{tikzcd} B \times S \arrow[r, "\mathrm{pr}_S"] \arrow[d, "\mathrm{pr}_B"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ B \arrow[r] & * \arrow[r, "s"] & \mathrm{Cat}, \end{tikzcd}$$

and hence pr_B is a small cocartesian fibration classified by the constant functor $\mathrm{const}_s\colon B \to \mathrm{Cat}$. Since Cat is a regular universe, the composite $q := \mathrm{pr}_B \circ p\colon E \to B$ is a small cocartesian fibration as well, thus classified by some functor $e\colon B \to \mathrm{Cat}$.

We may regard the functor $p\colon E \to B \times S$ as a functor over B in this way, and it follows from Lemma 5.12.7 that p is a cocartesian functor over B. By directed univalence of Cat, more specifically by Theorem 7.3.16, it follows that p corresponds to a natural transformation $e \to \mathrm{const}_s$ of functors $B \to \mathrm{Cat}$. But by Lemma 5.3.2 such a natural transformation precisely corresponds to a functor

$$\mathrm{Un}(\Phi)\colon B \to \mathrm{Cat}_{/S}$$

of the desired form. Consider now $B = \mathrm{Fun}(S, \mathrm{Cat})$ and the identity of B. The previous construction applied to the identity of $\mathrm{Fun}(S, \mathrm{Cat})$ defines a functor of the desired type:

$$\mathrm{Un}\colon\ \mathrm{Fun}(S, \mathrm{Cat}) \to \mathrm{Cat}_{/S}\,.$$

Lemma 7.7.6. *The functor* $\mathrm{Un}\colon \mathrm{Fun}(S, \mathrm{Cat}) \to \mathrm{Cat}_{/S}$ *lands in the subcategory* $\mathrm{CoCart}(S)$.

Proof. By definition of a subcategory it suffices to check this on individual morphisms in $\mathrm{Fun}(S, \mathrm{Cat})$. Let us check what this functor does on objects. Note that the above construction is functorial in B, in the sense that if we precompose $F\colon B \times S \to \mathrm{Cat}$ by the functor $g \times \mathrm{id}_S\colon B' \times S \to B \times S$ induced by some functor $g\colon B' \to B$, we will end up with the composite $B' \xrightarrow{g} B \xrightarrow{\mathrm{Un}} \mathrm{Cat}_{/S}$. So to compute $\mathrm{Un}(f)$ for some functor $f\colon S \to \mathrm{Cat}$, we may directly do the above construction where we replace B by $*$. But then the natural transformation $e \to \mathrm{const}_a$ is just the morphism in Cat corresponding under directed univalence to the cocartesian fibration $p : E \to S$ classified by the given functor $f : S \to \mathrm{Cat}$. So on objects we see that it does indeed factor through $\mathrm{CoCart}(S)$.

On morphisms, the proof is similar. It suffices to interpret how the functor Un interprets cocartesian fibrations with small fibers of the form

$$p\colon E \to [1] \times S$$

as a morphism in $\mathrm{Cat}_{/S}$. Let l be a p-cocartesian lift of the term $\mathrm{id}_{[1]} \times p_0\colon E_0 \times [1] \to [1] \times S$ extending the term given by the canonical embedding ι_0 of the fiber E_0 into E obtained from Lemma 7.2.2.

$$\begin{tikzcd}
E_0 \arrow[rr, "\iota_0"] \arrow[d, "{(0,\mathrm{id}_{E_0})}"'] & & E \arrow[r, "\mathrm{id}_E"] \arrow[d, "p"] & E \arrow[d, "\mathrm{pr}_{[1]}\, p"] \\
{[1] \times E_0} \arrow[rr, "\mathrm{id}_{[1]} \times p_0"'] \arrow[urr, dashed, "l"] & & {[1] \times S} \arrow[r, "\mathrm{pr}_{[1]}"] & {[1]}
\end{tikzcd}$$

Taking fibers over 1 turns l into a functor $l_1\colon E_0 \to E_1$ over S that is a cocartesian functor over S. The extra commutative square on the right hand side of the diagram above is there to make obvious to the eye of the reader that forgetting the base S is a transparent operation: the induced functor l_1, interpreted in Cat through directed univalence, is indeed the underlying morphism associated to p. In other words, the induced morphism in $\mathrm{Cat}_{/S}$ simply is the the morphism in Cat associated to l_1 together with the information that the latter is compatible with the projections to S. In particular, this morphism is cocartesian over S. □

Lemma 7.7.7. *The functor* $\mathrm{Un}\colon \mathrm{Fun}(S, \mathrm{Cat}) \to \mathrm{CoCart}(S)$ *is fully faithful.*

Proof. Let us consider for $i = 0$ and $i = 1$ a pullbacks square of the following form.

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d] & \mathrm{Cat}_\bullet \arrow[d] \\
S \arrow[r, "F_i"] & \mathrm{Cat}
\end{tikzcd}$$

If we denote by x_i the object of $\mathrm{CoCart}(S)$ corresponding to the cocartesian fibration p_i, we see that, by definition of $\mathrm{CoCart}(S)$, the equivalence

$$\mathrm{Cat}_{/S}((x_0, x_1)) \xrightarrow{\sim} \mathrm{Map}_S(X_0, X_1)$$

from Lemma 7.7.1 induces a pullback square of the form

$$\begin{tikzcd}
\mathrm{CoCart}(S)(x_0, x_1) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{Map}_S^{\mathrm{CoCart}}(X_0, X_1) \arrow[d, hook] \\
\mathrm{Cat}_{/S}((x_0, x_1)) \arrow[r, "\sim"] & \mathrm{Map}_S(X_0, X_1)
\end{tikzcd}$$

We observe that the map induced by Un

$$\mathrm{Un}\colon \mathrm{Fun}(S,\mathrm{Cat})(F_0,F_1) \to \mathrm{CoCart}(S)(x_0,x_1)$$

composed with the equivalence

$$\mathrm{CoCart}(S)(x_0,x_1) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart}}(X_0,X_1)$$

coincides termwise (in any groupoidal context) with the canonical equivalence

$$\mathrm{Fun}(S,\mathrm{Cat})(F_0,F_1) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart}}(X_0,X_1)\,.$$

provided by Theorem 7.3.16. This shows that this map is an equivalence by applying Proposition 1.1.15. □

Theorem 7.7.8 (Straightening/unstraightening)**.** *For every small category S, the functor*

$$\mathrm{Un}\colon \mathrm{Fun}(S,\mathrm{Cat}) \xrightarrow{\sim} \mathrm{CoCart}(S)$$

is an equivalence.

Proof. By virtue of the preceding lemma and of the fundamental theorem of category theory (Theorem 6.3.3), it suffices to prove that the functor Un has a section. We observe that a functor Φ from B to $\mathrm{Cat}_{/S}$ factors through $\mathrm{Fun}(S,\mathrm{Cat})$ if and only if it factors through $\mathrm{CoCart}(S)$: this is precisely what Proposition 5.11.13 means. Considering $B = \mathrm{CoCart}(S)$, and Φ the canonical embedding of $\mathrm{CoCart}(S)$ into $\mathrm{Cat}_{/S}$, this proves that the functor Un has a section. □

Corollary 7.7.9. *For every small groupoid Γ, there is a canonical equivalence of the form*

$$\mathrm{Un}\colon \mathrm{Fun}(\Gamma,\mathrm{Cat}) \xrightarrow{\sim} \mathrm{Cat}_{/\Gamma}\,.$$

Proof. For any functor $p\colon X \to \Gamma$, we know that p is a cocartesian fibration and that any morphism in X is p-cocartesian. This readily implies that

$$\mathrm{CoCart}(\Gamma) = \mathrm{Cat}_{/\Gamma}$$

for any small groupoid Γ. The corollary is thus a special case of the previous theorem. □

7.7.3 Straightening/unstraightening for left fibrations

Construction 7.7.10. Let A be a small category, with universal object $\iota_A\colon A^{\simeq} \hookrightarrow A$. We define the category $\mathrm{LFib}(A)$ of *left fibrations over A* as the full subcategory

$$\mathrm{LFib}(A) \subseteq \mathrm{CoCart}(A) \subseteq \mathrm{Cat}_{/A}$$

spanned by those cocartesian fibrations $B \to A$ which are in fact left fibrations (i.e. have groupoidal fibers). In other words, it is defined by forming the pullback square bellow.

$$\begin{tikzcd}
\mathrm{LFib}(A) \arrow[rr] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Grpd}_{/A^{\simeq}} \arrow[d, hook] \\
\mathrm{CoCart}(A) \arrow[r, hook] & \mathrm{Cat}_{/A} \arrow[r, "\iota_A^*"] & \mathrm{Cat}_{/A^{\simeq}}
\end{tikzcd}$$

Since the vertical functor at the right hand side is fully faithful, so is its counter part at the left hand side. We observe that, since Grpd is a full subcategory of Cat, there is a canonical pullback square of the form bellow.

$$\begin{tikzcd}
\mathrm{Fun}(A, \mathrm{Grpd}) \arrow[r, "\iota_A^*"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(A^{\simeq}, \mathrm{Grpd}) \arrow[d, hook] \\
\mathrm{Fun}(A, \mathrm{Cat}) \arrow[r, "\iota_A^*"] & \mathrm{Fun}(A^{\simeq}, \mathrm{Cat})
\end{tikzcd}$$

Proposition 7.7.11 ((Un)Straightening for left fibrations)**.** *For every small category A, the straightening/unstraightening equivalence from Theorem 7.7.8 restricts to an equivalence*

$$\mathrm{Un}\colon \mathrm{Fun}(A, \mathrm{Grpd}) \xrightarrow{\sim} \mathrm{LFib}(A).$$

Proof. For a cocartesian fibration of the form $p\colon X \to A$, the property to be a left fibration is equivalent to the property of being conservative, or. equivalently, that all fibers of p are groupoids. In other words, p is a left fibration if and only if the pullback $A^{\simeq} \times_A X$ is a groupoid. In view of the two pullback squares above, in order to prove this proposition, by virtue of Theorem 7.7.8 and of Corollary 7.7.9, it suffices to prove the special case where A is a groupoid. In other words, if suffices to prove the next corollary. □

Corollary 7.7.12. *For every small groupoid* Γ*, there is a canonical equivalence of the form*

$$\mathrm{Un}\colon \mathrm{Fun}(\Gamma, \mathrm{Grpd}) \xrightarrow{\sim} \mathrm{Grpd}_{/\Gamma}\,.$$

Proof of Corollary 7.7.12. Since left fibrations precisely are conservative cocartesian fibrations, and since any functor with codomain a groupoid is a cocartesian fibration, left fibrations with codomain a groupoid are the same as functors between groupoids. This implies right away that Corollary 7.7.9 induces the claimed equivalence. □

7.8 A constructive approach to universes

In the previous sections, we have been somewhat sloppy with our use of the phrase 'U-small cocartesian fibration', treating the U-smallness as if it was a *property* rather than additional data. While we have somewhat justified this approach in Remark 7.3.12, it is also possible to provide a fully constructive treatment in which one keeps track of all this additional data. The goal of this section is to provide such a treatment, which is based on Awodey's work [Awo18].[1] A reader not interested in constructive mathematics may safely ignore the content of this section.

Throughout, we shall fix a universe U and a directed univalent cocartesian fibration $\pi\colon U_\bullet \to U$. The overall strategy of constructively reformulating certain closure properties for U-small cocartesian categories is to show that the question reduces to a certain 'universal' instance of the question.

7.8.1 Regular universes constructively

We will start by making precise the definition of a 'regular universe' from Definition 7.4.1, which is about composites of U-small cocartesian fibrations again being U-small.

[1] We apologize if there is earlier work on this that deserves to be cited as well.

Construction 7.8.1. We construct a category U^Σ equipped with two U-small cocartesian fibrations $\pi_1^\Sigma \colon U_\bullet^\Sigma \to U^\Sigma$ and $\pi_2^\Sigma \colon U_{\bullet,\bullet}^\Sigma \to U_\bullet^\Sigma$:

- We let $U^\Sigma := \underline{\mathrm{Fun}}_U(U_\bullet, U \times U)$. It comes equipped with a canonical map $p_\Sigma \colon U^\Sigma \to U$.
- We define π_1^Σ via the following pullback square:

$$\begin{tikzcd} U_\bullet^\Sigma \arrow[r] \arrow[d, "\pi_1^\Sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ U^\Sigma \arrow[r, "p^\Sigma"] & U. \end{tikzcd}$$

  Note that it comes equipped with an evaluation map

$$\mathrm{ev} \colon U_\bullet^\Sigma = \underline{\mathrm{Fun}}_U(U_\bullet, U \times U) \times_U U_\bullet \to U.$$

- We define π_2^Σ via the following pullback square:

$$\begin{tikzcd} U_{\bullet,\bullet}^\Sigma \arrow[r] \arrow[d, "\pi_2^\Sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ U_\bullet^\Sigma \arrow[r, "\mathrm{ev}"] & U. \end{tikzcd}$$

Lemma 7.8.2. *The category U^Σ classifies composable pairs of U-small cocartesian fibrations: the data of a functor $S \to U^\Sigma$ is the same as the data of a pair of U-small cocartesian fibrations $C' \xrightarrow{p_2} C \xrightarrow{p_1} S$:*

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_{\bullet,\bullet}^\Sigma \arrow[d, "\pi_2^\Sigma"] \\ C \arrow[r] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet^\Sigma \arrow[d, "\pi_1^\Sigma"] \\ S \arrow[r] & U^\Sigma. \end{tikzcd}$$

Proof. The data of a functor $S \to U^\Sigma = \underline{\mathrm{Fun}}_U(U_\bullet, U \times U)$ is the same as the data of a functor $F \colon S \to U$ together with a lift

$$\begin{tikzcd} & \underline{\mathrm{Fun}}_U(U_\bullet, U \times U) \arrow[d] \\ S \arrow[ur, dashed] \arrow[r, "F"'] & U. \end{tikzcd}$$

Let $p_1 \colon C \to S$ denote the cocartesian fibration classified by F:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ S \arrow[r, "F"] & U. \end{tikzcd}$$

Since the pullback of the first projection $\mathrm{pr}_1 : U \times U \to U$ along F is the first projection $\mathrm{pr}_1 : S \times U \to S$, it follows from Observation 7.3.4 that such a lift contains the same data as that of a functor $C \to U \times S$ over S. But this in turn contains the same data as a functor $C \to U$, or equivalently a U-small cocartesian fibration $C' \to C$:

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ C \arrow[r] & U. \end{tikzcd}$$

This shows the desired one-to-one correspondence. Observe that in the case $S = U^\Sigma$, the cocartesian fibrations p_1 and p_2 classified by the identity functor $\mathrm{id} : U^\Sigma \to U^\Sigma$ are precisely the fibrations π_1^Σ and π_2^Σ from Construction 7.8.1. By unwinding of definitions we see that in general p_1 and p_2 are obtained by pulling back π_1^Σ and π_2^Σ along the classifying functor $S \to U^\Sigma$. □

In other words, the lemma says that the U-small cocartesian fibrations π_1^Σ and π_2^Σ from Construction 7.8.1 form the *universal* pair of two composable U-small cocartesian fibrations. We can now use this observation to give a constructive definition of regular universes:

Definition 7.8.3 (cf. [Awo18, Section 3.1])**.** A universe U is called *constructively regular* if it comes equipped with functors

$$\Sigma : U^\Sigma \to U \qquad \text{and} \qquad U^\Sigma_{\bullet,\bullet} \to U_\bullet$$

which fit into a pullback square as follows:

$$\begin{tikzcd} U^\Sigma_{\bullet,\bullet} \arrow[r, "\mathrm{pair}"] \arrow[d, "\pi_1^\Sigma \circ \pi_2^\Sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ U^\Sigma \arrow[r, "\Sigma"] & U. \end{tikzcd}$$

Proposition 7.8.4. *A universe is regular if and only if it is constructively regular.*

Proof. Let U be a consturctively regular universe. Consider a pair of U-small cocartesian fibrations $C' \xrightarrow{p_2} C \xrightarrow{p_1} S$. By Lemma 7.8.2 we have a pullback diagram as follows:

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "p_2"] \arrow[dd, bend right=60, "p_1 \circ p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & U^\Sigma_{\bullet,\bullet} \arrow[r, "\mathrm{pair}"] \arrow[d, "\pi_2^\Sigma"'] \arrow[ddr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[dd, "\pi"] \\ C \arrow[r] \arrow[d, "p_1"] \arrow[dr, phantom, "\lrcorner", very near start] & U^\Sigma_\bullet \arrow[d, "\pi_1^\Sigma"'] & \\ S \arrow[r] & U^\Sigma \arrow[r, "\Sigma"] & U. \end{tikzcd}$$

The outer pullback square then expresses the composite $p_1 \circ p_2$ as a U-small cocartesian fibration. Conversely, if U is regular, then the composition $\pi_1^\Sigma \circ \pi_2^\Sigma$ is a cocartesian fibration with small fibers and there is thus a unique cartesian square is in the previous definition. □

7.8.2 The closure properties of Cat

TO DO: Write down analogous constructive versions of each of the other closure properties of U-small cocartesian fibrations from Definition 7.4.5.

8 Cofinality

In this chapter, we will introduce the notions of *initial* and *terminal* functors,[1] and show that every functor admits a preferred factorization into an initial functor followed by a left fibration, as well as a dual factorization into a terminal functor followed by a right fibration. We will deduce from this the existence of left Kan extensions in the synthetic category Grpd of groupoids, see Theorem 8.1.19. In Section 8.2 we introduce proper and smooth functors and in Section 8.3 we investigate weak homotopy equivalences of synthetic categories. In Section 8.4 we prove smooth and proper base change, see Theorem 8.4.1 and Theorem 8.4.6, respectively.

8.1 Initial and terminal functors

Let S be a category and consider two categories over S given by functors $u\colon C \to S$ and $p\colon X \to S$. Recall from Definition 3.5.2 the groupoid $\mathrm{Map}_S(C, X)$ of functors $C \to X$ over S, defined via the pullback square

$$\begin{tikzcd} \mathrm{Map}_S(C,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(C,X) \arrow[d, "p_*"] \\ * \arrow[r, "u"] & \mathrm{Map}(C,S). \end{tikzcd}$$

Definition 8.1.1. A *covariant equivalence* (resp. a *contravariant equivalence*) over S is a commutative triangle

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "u"'] & & D \arrow[dl, "v"] \\ & S & \end{tikzcd}$$

such that for any left (resp. right) fibration $p\colon X \to S$ the induced functor

$$- \circ f\colon \mathrm{Map}_S(D, X) \to \mathrm{Map}_S(C, X)$$

is an equivalence.

Definition 8.1.2 (Initial/terminal functors)**.** A functor $f\colon C \to D$ is called *initial* (or *limit-cofinal*) if for any category S and any functor $v\colon D \to S$, f is a covariant equivalence over S (with respect to

[1] In the literature these are usually called *final* and *cofinal*, but it depends on the author which word is used for which notion. To avoid confusion, we opt for using *initial* and *terminal* instead, for which we think the meaning is unambiguous and easy to remember, cf. Proposition 8.1.13 below.

the composite $C \xrightarrow{f} D \xrightarrow{v} S$). Dually, f is called *terminal* (or *colimit-cofinal*) if for any $v\colon D \to S$, f is a contravariant equivalence over S.

Lemma 8.1.3. *Consider a covariant (resp. contravariant) equivalence*

$$\begin{tikzcd} C \arrow[rr] \arrow[dr, "u"'] & & D \arrow[dl, "v"] \\ & S & \end{tikzcd}$$

over S. Then for every functor $\varphi\colon S \to T$, the functor f is also a covariant (resp. contravariant) equivalence over T.

Proof. Let $X \to T$ be a left (resp. right) fibration. We obtain a commutative square as follows:

$$\begin{tikzcd} \mathrm{Map}_T(D, X) \arrow[r, "-\circ f"] \arrow[d, "\simeq"'] & \mathrm{Map}_T(C, X) \arrow[d, "\simeq"] \\ \mathrm{Map}_S(D, S \times_T X) \arrow[r, "-\circ f"] & \mathrm{Map}_S(C, S \times_T X). \end{tikzcd}$$

We need to show that the top map is an equivalence. But this follows from the fact that the bottom map is an equivalence, since the functor $S \times_T X \to S$ is again a left (resp. right) fibration. $\square$

Proposition 8.1.4. *Let $f\colon C \to D$ be a functor. Then the following conditions are equivalent:*

(1) The functor f is initial (resp. terminal);

(2) For every left (resp. right) fibration $X \to D$ the induced map

$$-\circ f\colon \mathrm{Map}_D(D, X) \to \mathrm{Map}_D(C, X)$$

is an equivalence.

(3) For every left (resp. right) fibration $p\colon X \to S$ the commutative square

$$\begin{tikzcd} \mathrm{Fun}(D, X) \arrow[r, "f^*"] \arrow[d, "p_*"'] & \mathrm{Fun}(C, X) \arrow[d, "p_*"] \\ \mathrm{Fun}(D, S) \arrow[r, "f^*"'] & \mathrm{Fun}(C, S) \end{tikzcd}$$

is a pullback square.

Proof. The equivalence between (1) and (2) is immediate from Lemma 8.1.3. For the equivalence between (1) and (3), note that the two vertical maps labeled p_* are still left fibrations by Lemma 5.2.18, so that the square is a pullback square if and only if it induces equivalences on fibers by Corollary 5.9.4. But by Corollary 6.2.7 the map on fibers is precisely the map

$$-\circ f\colon \mathrm{Map}_S(D, X) \to \mathrm{Map}_S(C, X)$$

appearing in the definition of initial (resp. terminal) functors. $\square$

Remark 8.1.5. We may understand part (3) as saying that for every solid commutative square

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"'] & X \arrow[d, "p"] \\ D \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

there exists a unique diagonal filler making both of the triangles commute. One sometimes also says that the initial functors are *left orthogonal* to the left fibrations, and similarly that the terminal functors are left orthogonal to the right fibrations.

We will now prove various basic properties of initial and terminal functors.

Lemma 8.1.6. *Consider a commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & S, & \end{tikzcd}$$

where p and q are left (resp. right) fibrations. Then the following three conditions are equivalent:

(1) f is an equivalence;

(2) f is initial (resp. terminal);

(3) f is a covariant (resp. contravariant) equivalence over S.

Proof. The implications (1) $\implies$ (2) and (2) $\implies$ (3) are obvious. The proof of (3) $\implies$ (1) is entirely analogous to Proposition 1.1.16 and is left to the reader. □

Lemma 8.1.7. *Let $f\colon C \to D$ be a left (resp. right) fibration. Then f is initial (resp. terminal) if and only if it is an equivalence.*

Proof. This follows from Lemma 8.1.6 by taking $q = \mathrm{id}_D$ and $p = f$. □

Proposition 8.1.8. *(1) Let $u\colon C \to D$ be an initial (resp. terminal) functor. Then for every category E, also the functor $u \times \mathrm{id}_E\colon C \times E \to D \times E$ is initial (resp. terminal).*

(2) Consider two functors $u\colon C \to D$ and $v\colon D \to E$ and assume that u is initial (resp. terminal). Then v is initial (resp. terminal) if and only if vu is initial (resp. terminal).

Proof. For (1), consider a functor $f\colon D \times E \to S$ and a left (resp. right) fibration $X \to S$. We want to show that the induced functor

$$- \circ (u \times \mathrm{id}_E)\colon \mathrm{Map}_S(D \times E, X) \to \mathrm{Map}_S(C \times E, X)$$

is an equivalence. By uncurrying, the functor f corresponds to a functor $\tilde{f}\colon D \to \mathrm{Fun}(E, S)$, and the above functor is equivalent to the functor

$$\mathrm{Map}_{\mathrm{Fun}(E,S)}(D, \mathrm{Fun}(E, X)) \xrightarrow{- \circ u} \mathrm{Map}_{\mathrm{Fun}(E,S)}(C, \mathrm{Fun}(E, X)).$$

Since the functor $\mathrm{Fun}(E, X) \to \mathrm{Fun}(E, S)$ is still a left (resp. right) fibration by Lemma 5.2.18, it follows from the assumption on u that this functor is an equivalence.

For (2), let $X \to E$ be a left (resp. right) fibration, and consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Map}_E(E,X) \arrow[rr, "v\circ-"] \arrow[dr, "vu\circ-"'] & & \mathrm{Map}_E(D,X) \arrow[dl, "u\circ-"] \\ & \mathrm{Map}_E(A,X). & \end{tikzcd}$$

By assumption, the right diagonal map is an equivalence. It follows by 2-out-of-3 that the left diagonal map is an equivalence if and only if the top horizontal map is an equivalence. □

Proposition 8.1.9. *Let $f\colon C \to D$ be a localization, i.e. it factors as $C \to C[W^{-1}] \simeq D$ for some collection of morphisms $W \hookrightarrow \mathrm{Map}([1], C)$. Then f is both initial and terminal.*

Proof. Recall that we denote by $\mathrm{Map}^W(C, T)$ the full subgroupoid of $\mathrm{Map}(C, T)$ on those maps that send all morphisms in W to equivalences in T. Consider a left (resp. right) fibration $p\colon X \to D$. By Lemma 6.2.6, p is conservative, and hence for a functor $F\colon C \to X$ the functor F sends morphisms in W to isomorphisms in X if and only if the composite $p \circ F\colon C \to D$ sends morphisms in W to isomorphisms in D. In particular, we see that the right square in the following commutative diagram is a pullback square:

$$\begin{tikzcd} \mathrm{Map}(D,X) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Map}^W(C,X) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(C,X) \arrow[d] \\ \mathrm{Map}(D,D) \arrow[r, "\simeq"] & \mathrm{Map}^W(C,D) \arrow[r, hook] & \mathrm{Map}(C,D). \end{tikzcd}$$

The left square is also a pullback square since both horizontal functors are equivalences. By passing to fibers over the identity $\mathrm{id}_D\colon D \to D$, this shows that the induced functor

$$\mathrm{Map}_D(D, X) \to \mathrm{Map}_D(C, X)$$

is an equivalence, which is what we needed to show. □

The following example of initial/terminal functors will generate a large class of other examples:

Example 8.1.10. The inclusion $\{1\} \hookrightarrow [1]$ is terminal. The inclusion $\{0\} \hookrightarrow [1]$ is initial.

Proof. Let us treat the case of $\{0\} \hookrightarrow [1]$, the other case being dual. Consider a left fibration $p\colon X \to S$. By Proposition 8.1.4, it will suffice to show that the commutative square

$$\begin{tikzcd} \mathrm{Fun}([1],X) \arrow[r, "p_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1],S) \arrow[d, "\mathrm{ev}_0"] \\ \mathrm{Fun}(\{0\},X) \arrow[r, "p_*"] & \mathrm{Fun}(\{0\},S) \end{tikzcd}$$

is a pullback square. But since the bottom functor agrees with $p\colon X \to S$, this holds true by Lemma 5.2.2. □

Our next goal is to show that right adjoints are terminal and that left adjoints are initial. We start with the case for fully faithful functors.

Proposition 8.1.11. *Let $v\colon D \to C$ be a left Bousfield localization with right adjoint section $u\colon C \hookrightarrow D$. Then u is a terminal functor. Dually, a left adjoint section of v is an initial functor.*

Proof. We prove the first case, and leave the second dual case to the reader. Consider a right fibration $p\colon X \to D$. The unit transformation $\eta\colon \mathrm{id} \to uv$ corresponds to a functor $\eta\colon [1]\times D \to D$, allowing us to regard $[1] \times D$ as a category over D. Consider now the following square:

$$\begin{tikzcd}
\mathrm{Map}_D([1]\times D, X) \arrow[r] \arrow[d] & \mathrm{Map}_D([1]\times C, X) \arrow[d] \\
\mathrm{Map}_D(\{1\}\times D, X) \arrow[r] & \mathrm{Map}_D(\{1\}\times C, X).
\end{tikzcd}$$

Since the inclusion $\{1\} \hookrightarrow [1]$ is terminal by Example 8.1.10, so is $\{1\} \times C \to [1] \times C$ by Proposition 8.1.8, and hence both vertical functors are equivalences. It follows that the square is a pullback square, i.e. we have an equivalence

$$\mathrm{Map}_D([1]\times D, X) \xrightarrow{\sim} \mathrm{Map}_D([1]\times C, X) \times_{\mathrm{Map}_D(\{1\}\times C, X)} \mathrm{Map}_D(\{1\}\times D, X).$$

We will finish the proof by exhibiting the map $-\circ u\colon \mathrm{Map}_D(D, X) \to \mathrm{Map}_D(C, X)$ as a retract of this equivalence, implying that it is itself an equivalence as desired. To this end, we consider the following diagram:

$$\begin{tikzcd}
\mathrm{Map}_D(D, X) \arrow[r, "\mathrm{pr}_2^*"] \arrow[d, "-\circ u"'] & \mathrm{Map}_D([1]\times D, X) \arrow[r, "i_0^*"] \arrow[d, "\simeq"] & \mathrm{Map}_D(D, X) \arrow[d, "-\circ u"] \\
\mathrm{Map}_D(C, X) \arrow[r, "(\mathrm{pr}_2^*, v^*)"] & \mathrm{Map}_D([1]\times C, X) \times_{\mathrm{Map}_D(\{1\}\times C, X)} \mathrm{Map}_D(\{1\}\times D, X) \arrow[r, "i_0^*\circ \mathrm{pr}_1"] & \mathrm{Map}_D(C, X).
\end{tikzcd}$$

The fact that all these maps are well-defined and that the two squares commute is a consequence of the following commutative diagram over D:

$$\begin{tikzcd}
 & C \arrow[r, "u"] \arrow[d, "i_0"] & D \arrow[d, "i_0"'] \arrow[dd, bend left, equal] \\
\{1\}\times C \arrow[r, "i_1"] \arrow[d, "u"'] & {[1]}\times C \arrow[r, "{[1]}\times u"] \arrow[d, "\mathrm{pr}_2"] & {[1]}\times D \arrow[d, "\eta"'] \\
\{1\}\times D \arrow[r, "v"] & C \arrow[r, "u"] & D.
\end{tikzcd}$$

This finishes the proof. $\square$

Proposition 8.1.12. *Any right adjoint is terminal. Any left adjoint is initial.*

Proof. By symmetry it suffices to do the right adjoint case, so let $f\colon C \to D$ be a right adjoint. As in the proof of Theorem 5.9.12, we may factorize f as a composite

$$C \xhookrightarrow{i} C \vec{\times}_f D \xrightarrow{\pi} D,$$

where i is a fully faithful right adjoint and where π admits a left adjoint section, i.e. is a right Bousfield localization. The functor i is terminal by Proposition 8.1.11. The functor π is a localization by Lemma 5.5.10, and hence also terminal by Proposition 8.1.9. It follows from Proposition 8.1.8 that also their composite f is terminal. $\square$

Proposition 8.1.13. *Let x be an object of a category C. Then x is an initial (resp. terminal) object of x if and only if the functor $x\colon * \to C$ is initial (resp. terminal).*

Proof. If x is an initial object, then the functor $x\colon * \to C$ is a left adjoint and the claim follows from Proposition 8.1.12. Conversely, assume that $x\colon * \to C$ is initial. Since the functor $(x, \mathrm{id}_x)\colon * \to C_{x/}$ is a left adjoint it is also initial, and hence it follows from Proposition 8.1.8 that also the forgetful functor $C_{x/} \to C$ is initial. Since this functor is a morphism between left fibrations over C, it follows from Lemma 8.1.6 that it is an equivalence, and hence x is an initial object. $\square$

Corollary 8.1.14. *Let xbe an object of a category C. For any left (right) fibration $p\colon A \to C$ with fiber A_x over x, there is a canonical equivalence*

$$\mathrm{Map}_C(C_{x/}), A) \xrightarrow{\sim} A_x \qquad (\mathrm{Map}_C(C_{/x}, A) \xrightarrow{\sim} A_x\,, \quad \textit{(respectively)}).$$

The following theorem will be of crucial importance throughout the remainder of this chapter.

Theorem 8.1.15. *Any functor $f\colon C \to D$ admits a factorization of the form*

$$f\colon C \xrightarrow{i} C' \xrightarrow{f'} E \xrightarrow{p} D,$$

where

- *i is a fully faithful left (resp. right) adjoint;*
- *f' is a localization, and remains a localization after base change along an arbitrary functor $D' \to D$;*
- *p is a left (resp. right) fibration.*

Proof. We start by factoring f as

$$C \xrightarrow{i} C \vec{\times}_f D \xrightarrow{\pi} D,$$

where i is a fully faithful left (resp. right) adjoint and π is a cocartesian (resp. cartesian) fibration. We may now factor π as

$$C \vec{\times}_f D \xrightarrow{f'} \Pi_\infty(C \vec{\times}_f D/D) \xrightarrow{\Pi_\infty(\pi/D)} D.$$

We showed in Proposition 7.6.16 that $\Pi_\infty(\pi/D)$ is a left fibration, and we showed in Theorem 7.6.20 that f' is a localization. Since this construction is compatible with base change along $D' \to D$ (part (3) of Proposition 7.6.16) we see that in fact any base change of f' along $D' \to D$ is still a localization. $\square$

Corollary 8.1.16. *Any functor $f\colon C \to D$ admits a factorization of the form*

$$C \xrightarrow{j} X \xrightarrow{p} D,$$

where $j\colon C \to X$ is an initial (resp. terminal) functor, and $p\colon X \to D$ is a left (resp. right) fibration.

Remark 8.1.17. If C and D are small and f has small fibers, then all the functors in the above diagrams are again small. (For the factorization using $\Pi_\infty(-/D)$ we in fact only need small fibers, we don't even need source and target to be small.)

Proposition 8.1.18. *Consider a commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & S, & \end{tikzcd}$$

where q is a left (resp. right) fibration. Then f is initial (resp. terminal) if and only if it is a covariant (resp. contravariant) equivalence over S.

Proof. It is clear that if f is initial, then it is a covariant equivalence over S. Conversely, assume that f is a covariant equivalence over S. We may factorize f as pi where i is initial and p is a left fibration. By Lemma 8.1.6 we know that i is also a covariant equivalence over S, hence so is p by 2-out-of-3. It then follows from the previous lemma that p is an equivalence, hence $f \cong pi$ is initial. □

The main consequence of the above factorization is the following explicit formula for computing colimits of groupoids:

Theorem 8.1.19. *Let $u\colon C \to D$ be a functor which has small fibers.*

(1) Then the pullback functor

$$u^*\colon \mathrm{Fun}(D, \mathrm{Grpd}) \to \mathrm{Fun}(C, \mathrm{Grpd})$$

admits a left adjoint $u_!\colon \mathrm{Fun}(C, \mathrm{Grpd}) \to \mathrm{Fun}(D, \mathrm{Grpd})$.

(2) This left adjoint is given on objects as follows: for a functor $F\colon C \to \mathrm{Grpd}$, consider the pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi"] \\ C \arrow[r, "F"] & \mathrm{Grpd}, \end{tikzcd}$$

and apply Corollary 8.1.16 to factor the composite $u \circ p\colon E \to D$ as a composite

$$E \xrightarrow{j} Y \xrightarrow{q} D,$$

where j is initial and q is a left fibration. Since u has small fibers, also q has small fibers, and thus there is a pullback square of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi"] \\ D \arrow[r, dashed, "u_!(F)"] & \mathrm{Grpd}. \end{tikzcd}$$

Under directed univalence, the functor $(p, j)\colon E \to C \times_D Y$ over C then corresponds to a natural transformation $\eta_F\colon F \to u^* u_!(F)$. Then η_F is a unit map exhibiting $u_!(F)$ as a left adjoint object to F.

Proof. Consider a functor $G\colon D \to \mathrm{Grpd}$. We have to show that the canonical map

$$\mathrm{Hom}_{\mathrm{Fun}(D,\mathrm{Grpd})}(u_!(F), G) \to \mathrm{Hom}_{\mathrm{Fun}(C,\mathrm{Grpd})}(F, u^*(G))$$

is an equivalence. Define the left fibration $r\colon Z \to D$ via the following pullback square:

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi"] \\ D \arrow[r, "G"] & \mathrm{Grpd}. \end{tikzcd}$$

Under directed univalence, in the form of Theorem 7.3.16, and using the equivalence from Lemma 7.3.14, the above map then becomes equivalent to the map

$$\mathrm{Map}_D(Y, Z) \to \mathrm{Map}_C(E, C \times_D Z) \simeq \mathrm{Map}_D(E, Z)$$

induced by precomposition with $j\colon E \to Y$. But since j is initial and $r\colon Z \to D$ is a left fibration, this map is an equivalence, finishing the proof. □

8.2 Proper and smooth functors

We will now introduce the classes of smooth and proper functors.

Definition 8.2.1 (Proper/smooth functors, Grothendieck)**.** A functor $p\colon E \to S$ is called *proper* (resp. *smooth*) if for any pullback diagram

$$\begin{tikzcd} E'' \arrow[r, "u'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ S'' \arrow[r, "v'"] & S' \arrow[r, "v"] & S \end{tikzcd}$$

in which v' is terminal (resp. initial), also u' is terminal (resp. initial).

Remark 8.2.2. Some remarks about this definition:

- We will see that a functor $u\colon C \to D$ is initial if and only if for every object x of D, the relative slice $C_{/x}$ is weakly contractible (meaning that its fundamental groupoid $\Pi_\infty(C_{/x})$ is contractible).
- We have also seen that a functor $u\colon C \to D$ is a left adjoint if and only if for any object x of D, the slice $C_{/x}$ admits a terminal object.
- More generally, every predicate on categories, we obtain a notion of “initiality”, which has a corresponding notion of “smoothness”.
- For the choice of left adjoints, we claim that this notion of smoothness corresponds to locally cocartesian fibrations.

Eventually, we will prove that for a category E with small limits and any functor $u\colon C \to D$ with C and D small, the restriction functor

$$u^*\colon \operatorname{Fun}(D, E) \to \operatorname{Fun}(A, E)$$

admits a right adjoint $u_*\colon \operatorname{Fun}(C, E) \to \operatorname{Fun}(D, E)$. Then u is proper if and only if for any E with small limits and any pullback square

$$\begin{tikzcd} C' \arrow[r, "v"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "u"] \\ D' \arrow[r, "w"] & D \end{tikzcd}$$

the canonical map $w^*u_* \to u'_*v^*$ is a natural isomorphism of functors $\operatorname{Fun}(C, E) \to \operatorname{Fun}(D', E)$, and in fact the same property holds for any base change of u.

Lemma 8.2.3. *The collection of proper (resp. smooth) functors is closed under composition and base change.*

Proof. It is clear that it is closed under base change. For compositions, let $p\colon X \to B$ and $q\colon B \to C$ be proper functor. For functors $C'' \to C' \to C$ with $C'' \to C'$ terminal, we consider

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ B'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "q"] \\ C'' \arrow[r] & C' \arrow[r] & C. \end{tikzcd}$$

Since q is proper, also $B'' \to B'$ is terminal, but then by properness of p also $X'' \to X'$ is terminal. This finishes the proof. ◻

Theorem 8.2.4. *Let $p\colon E \to S$ be a functor, and assume that p satisfies the property that for any pullback diagram of the form*

$$\begin{tikzcd} E'' \arrow[r, "u'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ S'' = \{\varepsilon\} \times S'' \arrow[r, hook, "v'"] & {[1] \times S''} \arrow[r, "v"] & S \end{tikzcd}$$

for $\varepsilon = 1$ (resp. $\varepsilon = 0$) the base change u' is terminal (resp. initial). Then p is proper (resp. smooth).

Proof. We prove the statement for proper functors; the statement for smooth functors is dual. Consider an arbitrary pullback diagram

$$\begin{tikzcd} E'' \arrow[r, "u'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ S'' \arrow[r, "v'"] & S' \arrow[r, "v"] & S \end{tikzcd}$$

in which v' is terminal. We need to show that also u' is terminal. To this end, we factor v' as

$$S'' = S''_0 \xrightarrow{j} S''_1 \xrightarrow{\lambda} S''_2 = \Pi_\infty(S''_1/S') \xrightarrow{q} S',$$

where j is a fully faithful right adjoint, λ is a localization and remains so after pullback, and q is a right fibration. Consider now the following commutative triangle:

$$\begin{tikzcd} S'' \arrow[rr] \arrow[dr, "v'"'] & & S''_2 \arrow[dl, "q"] \\ & S'. & \end{tikzcd}$$

The top map is a right adjoint and thus terminal by Proposition 8.1.12. It follows that also q is terminal, and since it is also a right fibration it is an equivalence, see Lemma 8.1.7. Now consider the following pullback diagram:

$$\begin{tikzcd} E'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E''_1 \arrow[r, "\lambda'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E''_2 \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ S'' \arrow[r, "j"] & S''_1 \arrow[r, "\lambda"] & S''_2 \arrow[r, "q", "\simeq"'] & S' \arrow[r, "v"] & S. \end{tikzcd}$$

Since λ' is a localization, it is terminal by Proposition 8.1.9. It thus remains to show that the pullback of j along p is terminal. In other words: without loss of generality we may assume that v' is a fully faithful right adjoint. Let $w\colon S' \to S''$ be a left adjoint of v', so that we have maps $\varepsilon\colon wv' \cong \mathbb{1}_{S''}$ and $\eta\colon \mathbb{1}_{S'} \to v'w$. Now form the following pullback:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "p'"] \\ {[1] \times S'} \arrow[r, "\eta"] & S'. \end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd} \{1\} \times S'' \arrow[r, "v'"] \arrow[d, hook] & \{1\} \times S' \cong S' \arrow[d, hook] \arrow[dr, "w"] & \\ {[1] \times S''} \arrow[r, "\mathrm{id}_{[1]} \times v'"] \arrow[dr, "\mathrm{pr}_2"'] & {[1] \times S'} \arrow[dr, "\eta"] & S'' \arrow[d, "v'"] \\ & S'' \arrow[r, "v'"] & S'. \end{tikzcd}$$

If we pull back this diagram along p', we obtain

$$\begin{tikzcd} \{1\} \times X'' \arrow[r] \arrow[d, hook, "i"] & S' \times_{S''} X'' \arrow[d, "j"] & \\ {[1] \times X''} \arrow[r] & \tilde{X} \arrow[dr, "\tilde{\eta}"] & \\ & & X'. \end{tikzcd}$$

The map i is clearly terminal, and j are terminal by the assumption since it is a pullback of the map $\{1\} \times S' \hookrightarrow [1] \times S'$:

$$\begin{tikzcd} S' \times_{S''} X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "p'"] \\ \{1\} \times S' \arrow[r, hook] & {[1] \times S'} \arrow[r, "\eta"] & S'. \end{tikzcd}$$

Now, to show that $u' \colon X'' \to X'$ is terminal, we have to show that or every right fibration $q \colon Z \to X'$ the induced map

$$\mathrm{Map}_{X'}(X', Z) \xrightarrow{(u')^*} \mathrm{Map}_{X'}(X'', Z)$$

is an equivalence. Observe that the above commutative square over X' induces a commutative square

$$\begin{tikzcd} \mathrm{Map}_{X'}(\tilde{X}, Z) \arrow[r] \arrow[d, "j^*"'] & \mathrm{Map}_{X'}([1] \times X'', Z) \arrow[d, "i^*"] \\ \mathrm{Map}_{X'}(S' \times_{S''} X'') \arrow[r] & \mathrm{Map}_{X'}(\{1\} \times X'', Z) \end{tikzcd}$$

where the two vertical maps are equivalences since i and j are terminal. It follows that we have an equivalence

$$\mathrm{Map}_{X'}(\tilde{X}, Z) \xrightarrow{\sim} \mathrm{Map}_{X'}(S' \times_{S''} X'', Z) \times_{\mathrm{Map}_{X'}(X'', Z)} \mathrm{Map}_{X'}([1] \times X'', Z).$$

Now we are done, since the map we wanted to show is an equivalence is a retract of this map. [Work this out!] □

Corollary 8.2.5. *Any cocartesian fibration is proper. Any cartesian fibration is smooth.*

Proof. Let $p \colon E \to S$ be a cocartesian fibration and consider a pullback square of the following form:

$$\begin{tikzcd} E'' \arrow[r, "u'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ S'' = \{\varepsilon\} \times S'' \arrow[r, hook, "v'"] & {[1] \times S''} \arrow[r, "v"] & S \end{tikzcd}$$

Since p' is a cocartesian fibration and v' is a right adjoint, it follows from Proposition 5.9.15 that also u' is a right adjoint, and hence terminal by Proposition 8.1.12. The claim now follows from Theorem 8.2.4. □

Remark 8.2.6. The statement of the corollary in facts this holds true for *locally* (co)cartesian fibrations.

Corollary 8.2.7. *Any left fibration is proper. Any right fibration is smooth.* □

Corollary 8.2.8. *Any projection of the form $C \times D \to C$ is both smooth and proper.* □

8.3 Weak homotopy equivalences and Quillen's Theorem A

We will now introduce the notion of a weak homotopy equivalence of categories and prove a synthetic version of Quillen's Theorem A.

Definition 8.3.1. A functor $f \colon C \to B$ is called a *weak homotopy equivalence* if it is a covariant equivalence over $*$. A category C is called *weakly contractible* if the functor $p_C \colon C \to *$ is a weak homotopy equivalence.

Lemma 8.3.2. *For a functor $f \colon C \to B$, the following are equivalent:*

(1) *The functor f is a weak homotopy equivalence;*

(2) *For every groupoid X, the induced map*

$$\mathrm{Map}(B, X) \to \mathrm{Map}(C, X)$$

is an equivalence;

(3) *The induced map $\Pi_\infty(f)\colon \Pi_\infty(C) \to \Pi_\infty(B)$ is an equivalence.*

Proof. The equivalence between (1) and (2) is clear, since by Proposition 5.2.4 a functor $X \to *$ is a left fibration if and only if X is a groupoid. By Lemma 3.4.3, (2) is equivalent to the condition that f induces an equivalence

$$\mathrm{Map}(\Pi_\infty(B), X) \xrightarrow{\sim} \mathrm{Map}(\Pi_\infty(C), X)$$

for every groupoid X. This is equivalent to (3) by Proposition 1.1.16. □

Corollary 8.3.3. *A category C is weakly contractible if and only if its fundamental groupoid $\Pi_\infty(C)$ is contractible.* □

Example 8.3.4. Any initial (or terminal) functor is a weak homotopy equivalence.

Lemma 8.3.5. *Consider a commutative triangle of the form*

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "u"'] & & X \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

where i is initial and where p is a left fibration. Then, for any object s of S, the following two functors are weak homotopy equivalences:

$$A_{/s} \to X_{/s} \leftarrow X_s.$$

In fact, $A_{/s} \to X_{/s}$ is initial and $X_s \to X_{/s}$ is terminal.

Proof. Consider the following pullback diagram:

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_{/s} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "p"] \\ * \arrow[r, "(s,\mathrm{id}_s)"] & S_{/s} \arrow[r] & S. \end{tikzcd}$$

Since p is a left fibration, it is proper, so the terminal functor $* \to S_{/s}$ pulls back to a terminal functor $X_s \to X_{/s}$. Consider now the following commutative diagram:

$$\begin{tikzcd} A_{/s} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_{/s} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S_{/s} \arrow[d] \\ A \arrow[r, "i"] & X \arrow[r, "p"] & S. \end{tikzcd}$$

Since the functor $S_{/s} \to S$ is a right fibration, it is in particular smooth, so the initial functor $i\colon A \to X$ pulls back to an initital functor $A_{/s} \to X_{/s}$. □

Proposition 8.3.6. *Consider a commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "u"'] & & D \arrow[dl, "v"] \\ & S. & \end{tikzcd}$$

Then f is a covariant equivalence if and only if for every object x of S the induced map

$$C_{/x} \to D_{/x}$$

is a weak homotopy equivalence.

Proof. The first step is to construct a commutative diagram of the form

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[d] & & D \arrow[d] \\ C' \arrow[rr, "f'"] \arrow[dr, "u'"'] & & D' \arrow[dl, "v'"] \\ & S, & \end{tikzcd}$$

where the two vertical maps are initial and the two diagonal maps u' and v' are left fibrations. In particular, the maps $C \to C'$ and $D \to D'$ are covariant equivalences, and we see that f is a covariant equivalence if and only if f' is a covariant equivalence. But since u' and v' are left fibrations, this is equivalent to saying that f is an equivalence. By Lemma 5.9.3 this is in turn equivalent to the statement that the induced map $C'_x \to D'_x$ is an equivalence on all fibers. Again using the lemma, this translates into the statement that the induced map $C_{/x} \to D_{/x}$ is a weak equivalence. □

Corollary 8.3.7. *A functor $u\colon C \to D$ is initial if and only if for any object d of D the relative slice $C_{/d}$ is weakly contractible.*

Proof. The functor u is initial if and only if the diagram

$$\begin{tikzcd} C \arrow[rr, "u"] \arrow[dr, "u"'] & & D \arrow[dl, "\mathrm{id}_D"] \\ & D & \end{tikzcd}$$

is a covariant equivalence. The statement now follows as the relative slice of the identity functor id_D at any object of D has a terminal object and is thus weakly contractible. □

Corollary 8.3.8 (Quillen's Theorem A)**.** *Let $f\colon C \to D$ be a functor such that the relative slice $C_{/d}$ is weakly contractible for any object d of D. Then f is a weak homotopy equivalence.*

Proof. This is immediate from the previous corollary. □

8.4 Proper and smooth base change

Let $f\colon A \to B$ be a functor between two small categories A and B. Recall from Theorem 8.1.19 that the restriction functor

$$f^*\colon \operatorname{Fun}(B, \mathrm{Grpd}) \to \operatorname{Fun}(A, \mathrm{Grpd})$$

admits a left adjoint $f_!$. In particular, taking $B = *$ shows that the category Grpd of small groupoids admits all small colimits.

Now consider a commutative square of small categories

$$\begin{tikzcd} A' \arrow[r, "u"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B' \arrow[r, "v"] & B \end{tikzcd}$$

and any functor $F\colon A \to \mathrm{Grpd}$. The goal of this section is to show that under suitable assumptions, the canonical Beck-Chevalley map

$$\mathrm{BC}_!\colon f'_! u^*(F) \to v^* f_!(F)$$

in $\operatorname{Fun}(B', \mathrm{Grpd})$ is an equivalence. This Beck-Chevalley map is defined as the composite

$$f'_! u^*(F) \xrightarrow{\eta} f'_! u^* f^* f_!(F) \simeq f'_! f'^* v^* f_!(F) \xrightarrow{\varepsilon} v^* f_!(F),$$

where $\eta\colon \mathrm{id}_A \to f^* f_!$ is the unit of the adjunction $f_! \dashv f^*$ and where $\varepsilon\colon f'_! f'^* \to \mathrm{id}_{B'}$ is the counit of the adjunction $f'_! \dashv f'^*$. Alternatively, it is the map adjoint to the map

$$u^*(F) \xrightarrow{\eta} u^* f^* f_! F \simeq f'^* v^* f_!(F).$$

Before continuing, it is useful to first give a more concrete description of the map $\mathrm{BC}_!$. Consider the following pullback square:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d] \\ A \arrow[r, "F"] & \mathrm{Grpd}, \end{tikzcd}$$

where the right vertical map is the universal left fibration with small fibers. Choose a factorization of $f \circ p\colon X \to B$ as

$$X \xrightarrow{j} Y \xrightarrow{q} B,$$

where j is initial and q is a left fibration. In particular, q is given via a pullback square

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & Y \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r, dashed, "f_!(F)"] & \mathrm{Grpd} \end{tikzcd}$$

for some functor $f_!(F)$. The commutative square on the left exhibits the unit $F \to f^* f_! F$. Pulling back along v and u, we obtain the commutative diagram

$$\begin{tikzcd} A' \times_A X \arrow[r, "f' \times_f j"] \arrow[d, "u^* p"'] & B' \times_B Y \arrow[d, "v^* q"] \\ A' \arrow[r, "f'"] & B'. \end{tikzcd}$$

We may factorize $f' \times_f j$ into

$$A' \times_A X \xrightarrow{j'} Y' \xrightarrow{r} B' \times_B Y,$$

where j' is initial and r is a left fibration. Then the composite

$$q' := v^* q \circ r \colon Y' \to B$$

is again a left fibration. All in all, we have constructed a commutative diagram as follows:

$$\begin{tikzcd}
X' \arrow[rr, "j'"] \arrow[dd, "u^*p"'] \arrow[dr] & & Y' \arrow[dd, "q'" near start] \arrow[dr] & \\
& X \arrow[rr, "j" near start, crossing over] & & Y \arrow[dd, "q"] \\
A' \arrow[rr, "f'" near start] \arrow[dr, "u"'] & & B' \arrow[dr, "v"] & \\
& A \arrow[rr, "f"] \arrow[from=uu, "p" near start, crossing over] & & B.
\end{tikzcd}$$

We have pullback squares

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Grpd}_\bullet \arrow[d] \\
B' \arrow[r, "f'_! u^*(F)"'] & \mathrm{Grpd}
\end{tikzcd}$$

and

$$\begin{tikzcd}
B' \times_B Y \arrow[r] \arrow[d, "v^*q"'] \arrow[dr, phantom, "\lrcorner" very near start] & Y \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Grpd}_\bullet \arrow[d] \\
B' \arrow[r, "v"] \arrow[rr, bend right, "v^* f_!(F)"'] & B \arrow[r, "f_!(F)"] & \mathrm{Grpd}.
\end{tikzcd}$$

All in all, we see that the Beck-Chevalley map $\mathrm{BC}_! \colon f'_! u^*(F) \to v^* f_!(F)$ corresponds under directed univalence to the map $Y' \to B' \times_B Y$ over B'. We may now state the main theorem of this section:

Theorem 8.4.1 (Smooth base change)**.** *Consider a pullback square of small categories*

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B,
\end{tikzcd}$$

where v is smooth. Then for any functor $F \colon A \to \mathrm{Grpd}$, the Beck-Chevalley map

$$\mathrm{BC}_! \colon f'_! u^*(F) \to v^* f_!(F)$$

is an equivalence.

Proof. Denote by $p \colon X \to A$ the left fibration associated to F:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
A \arrow[r, "F"] & \mathrm{Grpd}.
\end{tikzcd}$$

We may factor the composite $f \circ p \colon X \to B$ into an initial functor $j \colon X \to Y$ and a left fibration $q \colon Y \to B$:

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

The left fibration q corresponds to a functor $f_!(F) \colon B \to \mathrm{Grpd}$. Similarly, we may factor the induced functor $A' \times_A X \to B' \times_B Y$ into

$$A' \times_A X \xrightarrow{j'} Y' \xrightarrow{r} B' \times_B Y,$$

where j is initial and r is a left fibration. Then we get a composite left fibration $q' := v^*(q)r \colon Y' \to B'$. All of these functors fit into the following commutative square:

$$\begin{tikzcd}
A' \times_A X \arrow[rr, "j'"] \arrow[dr] \arrow[dd, "u^* p"'] & & Y' \arrow[dr] \arrow[dd] & \\
& X \arrow[rr, "j"] \arrow[dd] & & Y \arrow[dd] \\
A' \arrow[rr, "f'"] \arrow[dr, "u"] & & B' \arrow[dr, "v"] & \\
& A \arrow[rr, "f"] & & B.
\end{tikzcd}$$

The commutative diagram

$$\begin{tikzcd}
Y' \arrow[rr, "r"] \arrow[dr, "q'"'] & & B' \times_B Y \arrow[dl, "v^*(q)"] \\
& B' &
\end{tikzcd}$$

corresponds to a natural transformation $f'_! u^*(F) \to v^* f_!(F)$. We will prove that the map $r' \colon Y' \to B' \times_B Y$ is initial.

To this end, consider the following commutative square:

$$\begin{tikzcd}
A' \times_A X \arrow[rr] \arrow[dr] \arrow[dd, "u^* p"] & & B' \times_B Y \arrow[dr] \arrow[dd] & \\
& X \arrow[rr, "j"] \arrow[dd] & & Y \arrow[dd] \\
A' \arrow[rr, "f'"] \arrow[dr, "u"] & & B' \arrow[dr, "v"] & \\
& A \arrow[rr, "f"] & & B.
\end{tikzcd}$$

Then the top square is a pullback square, hence the functor $A' \times_A X \to B' \times_B Y$ is a base change of $j \colon X \to Y$ along v. Since v is smooth and j is initial, it follows that also $A' \times_A X \to B' \times_B Y$ is initial. Since j' is also initial, it follows by Proposition 8.1.8 that also r' is initial, as claimed.

This finishes the proof: since r' is both initial and a left fibration, it is an equivalence by Lemma 8.1.7, as desired. $\square$

Recall: if C is a category that admits a terminal object ω, the functor

$$\mathrm{ev}_\omega\colon \operatorname{Fun}(C, \mathrm{Grpd}) \to \mathrm{Grpd}$$

admits a left adjoint given by

$$p^*\colon \mathrm{Grpd} \to \operatorname{Fun}(C, \mathrm{Grpd}),$$

where $p = p_C\colon C \to *$. (This is an instance of Proposition 5.4.4.)

Notation 8.4.2. For a general (small) category C with $p\colon C \to *$, we define

$$\operatorname{colim}_C := p_!\colon \operatorname{Fun}(C, \mathrm{Grpd}) \to \mathrm{Grpd}.$$

Sometimes this is also denoted by $\varinjlim$.

Remark 8.4.3. Consider a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ C \arrow[r, "F"'] & \mathrm{Grpd}. \end{tikzcd}$$

Then the groupoid $\operatorname{colim}_C F$ corresponds to the weak homotopy type of X: it is obtained by factoring the functor $X \to *$ into an initial functor $X \to \Pi_\infty(X)$ followed by a left fibration $\Pi_\infty(X) \to *$. In other words, there is a pullback square:

$$\begin{tikzcd} \Pi_\infty(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "\operatorname{colim}_C F"'] & \mathrm{Grpd}. \end{tikzcd}$$

Proposition 8.4.4. *Let A and B be small categories. A functor $i\colon A \to B$ is terminal if and only if, for any functor $F\colon B \to \mathrm{Grpd}$, the Beck-Chevalley map*

$$\mathrm{BC}_!\colon \operatorname{colim}_A i^*F \to \operatorname{colim}_B F$$

from Construction 5.12.1 is an equivalence.

Proof. As usual we will start by rewriting the Beck-Chevalley map in terms of a morphism of left fibrations. Consider the following pullback diagram:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ A \arrow[r, "i"] & B \arrow[r, "F"] & \mathrm{Grpd}. \end{tikzcd}$$

Since q is a left fibration, it is in particular proper. Therefore, if i is terminal we get that j is terminal as well. If we denote by $p\colon C \to *$ the terminal functor, the Beck-Chevalley map is given by

$$\operatorname{colim}_A i^*F = p_! i_! i^* F \xrightarrow{p_!(\text{counit})} p_! F = \operatorname{colim}_B F,$$

hence this corresponds to the map induced by j on fundamental groupoids:

$$\Pi_\infty(j)\colon \Pi_\infty(X) \to \Pi_\infty(Y).$$

But this is an equivalence since j is terminal.

For the converse, we can consider the special case where q is the projection $B_{b/} \to B$ for some object b of B. In this case, we have $X = A_{b/}$ and $\Pi_\infty(j)$ is the map from $\Pi_\infty(A_{b/})$ to the terminal groupoid. In other words, of the Beck-Chevalley map $\mathrm{BC}_!$ is an equivalence for all F, the co-slice $A_{b/}$ is weakly contractible for any object b of B. We conclude with Corollary 8.3.8 that the functor i must be terminal. □

Lemma 8.4.5. *Let $f\colon A \to B$ be a proper functor between small categories. Consider an absolute object b of B and let A_b denote the fiber of f over b. Then for any functor $F\colon A \to \mathrm{Grpd}$, there is a canonical isomorphism*

$$\operatorname{colim}_{A_b} i^*F \xrightarrow{\sim} b^* f_!(F) = f_!(F)(b)$$

of groupoids, where $i\colon A_b \to A$ denotes the inclusion of the fiber.

Proof. We will again translate this to a statement about left fibrations. Consider the left fibration $p\colon X \to A$ associated to F:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ A \arrow[r, "F"] & \mathrm{Grpd}. \end{tikzcd}$$

Consider now the following pullback diagram:

$$\begin{tikzcd} X_b \arrow[r, "k_b"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{/b} \arrow[r, "u_b"] \arrow[d] & X \arrow[d, "p"] \\ A_b \arrow[r, "i_b"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_{/b} \arrow[r, "v_b"] \arrow[d, "f_{b/}"] & A \arrow[d, "f"] \\ * \arrow[r, "{(b,\mathrm{id}_b)}"] \arrow[rr, bend right, "b"'] & B_{/b} \arrow[r, "w_b"] & B. \end{tikzcd}$$

Since w_b is a right fibration, it is in particular smooth, hence by smooth base change (Theorem 8.4.1) we see that the canonical map

$$\mathrm{BC}_!\colon (f_{/b})_! v_b^*(F) \xrightarrow{\sim} w_b^* f_!(F)$$

is an isomorphism. Passing to colimits over $B_{/b}$, we obtain an equivalence

$$\operatorname{colim}_{A_{/b}} v_b^*(F) = \operatorname{colim}_{B/b} (f_{/b})_! v_b^*(F) \xrightarrow{\sim} \operatorname{colim}_{B_{/b}} w_b^* f_!(F).$$

Now consider the following commutative diagram:

$$\begin{tikzcd} \operatorname{colim}_{A_b} i^*F \arrow[r] \arrow[d] & b^* f_!(F) \arrow[d] \\ \operatorname{colim}_{A_{/b}} v_b^*F \arrow[r] & \operatorname{colim}_{B_{/b}} w_b^* f_!(F). \end{tikzcd}$$

Since the inclusion $(b, \mathrm{id}_b)\colon * \to B_{/b}$ is terminal, the right vertical map is an equivalence by Proposition 8.4.4. But also i_b is initial, since it is a base change of $(b, \mathrm{id}_b)\colon * \to B_{/b}$, hence also the left vertical map is an equivalence. By 2-out-of-3 it follows that also the top map is an equivalence, finishing the proof. $\square$

Theorem 8.4.6 (Proper base change)**.** *Consider a pullback square*

$$\begin{tikzcd} A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B' \arrow[r, "v"] & B, \end{tikzcd}$$

where f is proper. Then for every functor $F\colon A \to \mathrm{Grpd}$, the Beck-Chevalley map

$$f'_! u^*(F) \to v^* f_!(F)$$

is a natural isomorphism.

Proof. It suffices to check that for any object b' in B', the map

$$f'_! u^*(F)(b') \to v^* f_!(F)(b') = f_!(F)(v(b'))$$

is an isomorphism in Grpd. By working with categories in context $\{b\}$, we may assume that b is an absolute object. Write $b := v(b')$. We may now consider the following commutative diagram:

$$\begin{tikzcd} A'_{b'} \arrow[rr] \arrow[dr] \arrow[dd] & & A_b \arrow[dr] \arrow[dd] & \\ & A' \arrow[rr, "u"] \arrow[dd, "f'"'] & & A \arrow[dd, "f"] \\ * \arrow[rr, equal] \arrow[dr, "b'"'] & & * \arrow[dr, "b"] & \\ & B' \arrow[rr, "v"] & & B. \end{tikzcd}$$

Observe that all the non-horizontal faces are cartesian. We may now consider the following commutative diagram:

$$\begin{tikzcd} \operatorname{colim}_{A'_{b'}} i'^* u^* F \arrow[r] \arrow[d] & \operatorname{colim}_{A_b} i^* F \arrow[d] \\ f'_! u^*(F)(b') \arrow[r] & f_!(F)(b). \end{tikzcd}$$

The two vertical maps are equivalences by Lemma 8.4.5. But the top map is clearly an equivalence, since it is induced by the functor $A'_{b'} \to A_b$ which is a pullback of $\mathrm{id}_*\colon * \to *$ and hence an equivalence. $\square$

Corollary 8.4.7. *Let $p\colon A \to B$ be a proper (resp. smooth) functor and consider a commutative square of the form*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"'] & A \arrow[d, "p"] \\ Y \arrow[r, "\psi"] & B \end{tikzcd}$$

where f is initial (resp. terminal) and both φ and ψ are left (resp. right) fibrations. Then, for any functor $v\colon B' \to B$, the induced functor

$$g\colon B' \times_B X \to B' \times_B Y$$

is again initial (resp. terminal).

Proof. The left fibration φ corresponds to a functor $F\colon A \to \mathrm{Grpd}$:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ A \arrow[r, "F"] & \mathrm{Grpd}. \end{tikzcd}$$

The left fibration ψ is obtained by factorizing $p \circ \varphi$ into an initial functor and a left fibration, hence it corresponds to the functor

$$p_!(F) := F \circ p\colon A \to \mathrm{Grpd}.$$

If we now define the map $p'\colon A' \to B'$ via the pullback square

$$\begin{tikzcd} A' \arrow[r, "u"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ B' \arrow[r, "v"] & B, \end{tikzcd}$$

then the map g corresponds to the Beck-Chevalley map $p'_! u^*(F) \to v^* p_!(F)$, which is a natural isomorphism by Theorem 8.4.6. It follows that g is a contravariant equivalence over B'. Since the map $B' \times_B Y \to B'$ is a left fibration, it follows from Proposition 8.1.18 that g is in fact initial. $\square$

Lemma 8.4.8. *Consider a commutative square of the form*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \times C \arrow[r, "\mathrm{id}_A \times g"] & A \times B, \end{tikzcd}$$

where p and q are left fibrations. Assume that f is fiberwise initial over A, meaning that for every object a in A, f induces an initial functor

$$f_a\colon X_a \to Y_a.$$

Then f is initial.

Proof. Since q is a left fibration, it suffices to check that f is a covariant equivalence over $A \times B$, see Proposition 8.1.18. For any object b of B, we may consider the following commutative square

$$\begin{tikzcd} X_{/b} \arrow[r, "f_{/b}"] \arrow[d, "p_{/b}"'] & Y_{/b} \arrow[d, "q_{/b}"] \\ A \times C_{/b} \arrow[r, "\mathrm{id}_A \times g_{/b}"] & A \times B_{/b} \end{tikzcd}$$

obtained from the above commutative square by pulling back along $B_{/b} \to B$. For any object a of A, the commutative square

$$\begin{tikzcd} X_a \arrow[r, "f_a"] \arrow[d, "p_a"'] & Y_a \arrow[d, "q_a"] \\ C \arrow[r, "g"] & B \end{tikzcd}$$

induces a commutative square

$$\begin{tikzcd} (X_a)_{/b} \arrow[r, "f_a"] \arrow[d, "(p_a)_{/b}"'] & (Y_a)_{/b} \arrow[d, "(q_a)_{/b}"] \\ C_{/b} \arrow[r, "g_{/b}"] & B_{/b} \end{tikzcd}$$

by pulling back along $B_{/b} \to B$. The top map agrees with the map obtained from

$$\begin{tikzcd} X_{/b} \arrow[rr, "f_{/b}"] \arrow[dr] & & Y_{/b} \arrow[dl] \\ & A & \end{tikzcd}$$

by passing to fibers over a. In particular, $(f_{/b})_a = (f_a)_{/b}$ is a pullback of the initial functor f_a along a smooth functor $B_{/b} \to B$ and hence is initial itself. Since the functor $X_{/b} \to A$ induced by p is proper (being the composition of a left fibration with the projection of $C_{/b} \times A \to A$), pulling back the immersion of the terminal object in the slice $A_{/a}$ yields a terminal functor of the form

$$(X_{/b})_a = (X_a)_{/b} \to (X_{/b})_{/a} = X_{/(a,b)} \,,$$

hence a weak homotopy equivalence. Similarly, the functor $Y_{/b} \to A$ induced by q yields a weak homotopy equivalence

$$(Y_{/b})_a = (Y_a)_{/b} \to (Y_{/b})_{/a} = Y_{/(a,b)} \,.$$

This means that we have a commutative square of the form

$$\begin{tikzcd} (X_a)_{/b} \arrow[r, "(f_a)_{/b}"] \arrow[d, "(p_a)_{/b}"'] & (Y_a)_{/b} \arrow[d, "(q_a)_{/b}"] \\ X_{/(a,b)} \arrow[r, "f_{/(a,b)}"] & Y_{/(a,b)} \end{tikzcd}$$

in which the top horizontal map as well as the two vertical ones are weak homotopy equivalences. Therefore, the functor f is a covariant equivalence over $A \times B$ by Proposition 8.3.6. $\square$

9 The Yoneda lemma

The goal of this chapter is to prove a version of the Yoneda lemma in synthetic category theory. On the way, we will encounter various useful auxiliary notions:

- *corepresentable functors* $\mathrm{Hom}_C(x, -)\colon C \to \mathrm{Grpd}$;
- the *opposite category* C^{op} of a synthetic category C;
- the *twisted arrow category* $\mathrm{Tw}(C)$;
- *locally small* synthetic categories.

9.1 Opposite categories

In ordinary category, every category C gives rise to another category C^{op}, called the *opposite category* of C. While C^{op} has the same objects as C, the morphisms in C^{op} are different: a morphism from x to y in C^{op} is given by a morphism from y to x in C^{op}. Our goal in this section is to provide an analogous construction in the case where C is a *synthetic* category.

To motivate our definition of C^{op}, recall from ordinary category theory that for every ordinary category C, the functor

$$h_C\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}), \qquad x \mapsto \mathrm{Hom}_C(-, x)$$

is fully faithful; it is known as the *Yoneda embedding*. Applying this construction to C^{op}, this provides a fully faithful functor

$$h_{C^{\mathrm{op}}}\colon C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{Set}).$$

The essential image consists of precisely those functors $F\colon C \to \mathrm{Set}$ that are *corepresentable*, meaning that they are naturally isomorphic to a functor of the form $\mathrm{Hom}_C(x, -)\colon C \to \mathrm{Set}$ for some object x of C.

For our synthetic definition of the opposite category C^{op} of C, we will turn this logic on its head and *define* C^{op} as the full subcategory of $\mathrm{Fun}(C, \mathrm{Grpd})$ spanned by the corepresentable functors. Making this precise will take up the remainder of this section.

Construction 9.1.1 (Covariant Hom-functor)**.** Let C be a small category and let x be an object of C. The slice category $C_{x/}$ fits in a pullback square of the form

$$\begin{tikzcd} C_{x/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ C \arrow[r, "(x, \mathrm{id}_C)"'] & C \times C. \end{tikzcd}$$

The left vertical map is a small left fibration by Theorem 5.3.3, hence fits into a pullback square of the form

$$\begin{tikzcd} C_{x/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ C \arrow[r, dashed, "\mathrm{Hom}_C(x,-)"] & \mathrm{Grpd}. \end{tikzcd}$$

This defines the functor $\mathrm{Hom}_C(x, -) \colon C \to \mathrm{Grpd}$.

Definition 9.1.2 (Corepresentable functor)**.** We say that a functor $F \colon C \to \mathrm{Grpd}$ is *corepresentable* if we have an object x of C and an isomorphism $\mathrm{Hom}_C(x, -) \cong F$.

A priori the condition of a functor to be corepresentable seems to involve a choice of object x of C. It turns out that this object is uniquely determined if it exists, so that corepresentability is really a *property* of a functor F:

Proposition 9.1.3. *For a functor* $F \colon C \to \mathrm{Grpd}$*, let* $p_F \colon \int F \to C$ *denote the small cocartesian fibration classified by* F*. Then* F *is corepresentable if and only if* $\int F$ *admits an initial object.*

Proof. Let x be an object of C. Observe that there is an equivalence between the following three types of data:

(1) Objects y of $\int F$ satisfying $p_F(y) = x$;

(2) Functors $\varphi \colon C_{x/} \to \int F$ over C;

(3) Natural transformations $\mathrm{Hom}_C(x, -) \to F$ of functors $C \to \mathrm{Grpd}$.

The equivalence between (2) and (3) is an immediate consequence of directed univalence. For comparing (1) and (2), note that the slice $C_{x/}$ has an initial object (x, id_x), so that the functor $(x, \mathrm{id}_x) \colon * \to C_{x/}$ is initial by Proposition 8.1.13. It follows from Remark 8.1.5 that for every object y of $\int F$ satisfying $p_F(y) = x$ there exists a unique functor $\varphi \colon C_{x/} \to \int F$ fitting in the following commutative diagram:

$$\begin{tikzcd} & * \arrow[dl, "(x,\mathrm{id}_x)"'] \arrow[dr, "y"] & \\ C_{x/} \arrow[rr, dashed, "\varphi"] \arrow[dr] & & \int F \arrow[dl, "p_F"] \\ & C. & \end{tikzcd}$$

We claim that the object y is initial in $\int F$ if and only if the corresponding natural transformation $\mathrm{Hom}_C(x, -) \to F$ is a natural isomorphism of functors $C \to \mathrm{Grpd}$. Indeed, since directed univalence respects composition the latter is equivalent to the condition that $\varphi\colon C_{x/} \to C$ is an equivalence. Since φ is a morphism between left fibrations over C, it is an equivalence if and only if it is an initial functor, see Lemma 8.1.6. But by Proposition 8.1.8 this is in turn equivalent to $y\colon * \to \int F$ being an initial functor, which by Proposition 8.1.13 happens precisely whenever y is initial in $\int F$. This finishes the proof. $\square$

As a consequence of Proposition 9.1.3, we expect to be able to form the full subcategory C^{op} of $\mathrm{Fun}(C, \mathrm{Grpd})$ spanned by the coreprentable functors. In order to make this precise, we need to formulate the condition of being corepresentable in terms of a category. We start with a formulation of initiality.

Definition 9.1.4. For a small category C, we define

$$\mathtt{hasInitial}(C) := \sum_{x\colon * \to C} \mathtt{isEquiv}(C_{x/} \to C).$$

More precisely, we define this category via the following pullback square:

$$\begin{tikzcd} \mathtt{hasInitial}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(\mathrm{Cat}) \arrow[d, "\pi_{\mathrm{Iso}}"] \\ C \arrow[r, "x \mapsto (C_{x/} \to C)"] & \mathrm{Fun}([1], \mathrm{Cat}), \end{tikzcd}$$

where the bottom functor is obtained by applying directed univalence to the cocartesian functor

$$\begin{tikzcd} \mathrm{Fun}([1], C) \arrow[rr, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \arrow[dr, "\mathrm{ev}_0"'] & & C \times C \arrow[dl, "\mathrm{pr}_1"] \\ & C & \end{tikzcd}$$

over C.

Lemma 9.1.5. *The category* $\mathtt{hasInitial}(C)$ *is a proposition.*

Proof. Note that the type $\mathtt{isEquiv}(C_{x/} \to C)$ is a proposition for every x. So we get an embedding $\mathtt{hasInitial}(C) \to C^{\simeq}$, and thus we obtain a full subcategory $\langle\mathtt{hasInitial}(C)\rangle \hookrightarrow C$ consisting of all initial objects of C. Notice that every morphism in this subcategory is automatically an isomorphism, hence it is actually equivalent to the groupoid $\mathtt{hasInitial}(C)$. It will now remain to show that the functor $\mathtt{hasInitial}(C) \to *$ is fully faithful, since all fully faithful functors are embeddings. But this is clear: by definition of initial objects the mapping spaces on both sides are contractible. $\square$

Construction 9.1.6. Let C be a small category. We will construct a groupoid $(C^{\mathrm{op}})^{\simeq}$ equipped with an embedding

$$(C^{\mathrm{op}})^{\simeq} \hookrightarrow \mathrm{Map}(C, \mathrm{Grpd})$$

such that a functor $F\colon C \to \mathrm{Grpd}$ is contained in $(C^{\mathrm{op}})^{\simeq}$ if and only if it is corepresentable, or equivalently if the total space $\int F$ of the left fibration $p_F\colon \int F \to C$ associated to F has an initial object.

Consider the groupoid $\Gamma := \mathrm{Map}(C, \mathrm{Grpd})$. Then the inclusion $\Gamma \hookrightarrow \mathrm{Fun}(C, \mathrm{Grpd})$ gives rise via currying to a functor $\Gamma \times C \to \mathrm{Grpd}$, and hence to a left fibration $p\colon X \to \Gamma \times C$. Working in context Γ, we may then consider the statement $\mathtt{hasInitial}(X)$ in context Γ, which by Lemma 9.1.5 defines a proposition in context Γ, which translated back to the absolute context corresponds to the desired embedding of groupoids of the form

$$(C^{\mathrm{op}})^{\simeq} \hookrightarrow \Gamma = \mathrm{Map}(C, \mathrm{Grpd}).$$

For any point $\gamma \in \Gamma$, the fiber of this map is the proposition $\mathtt{hasInitial}(X_\gamma)$, which is precisely what we wanted.

Definition 9.1.7 (Opposite category)**.** For a small category C, we define the ∞-category C^{op} as the full subcategory

$$C^{\mathrm{op}} \subseteq \mathrm{Fun}(C, \mathrm{Grpd})$$

spanned by $(C^{\mathrm{op}})^{\simeq} \subseteq \mathrm{Map}(C, \mathrm{Grpd})$, and refer to it as the *opposite category* of C. We will write

$$h_{C^{\mathrm{op}}}\colon C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{Grpd})$$

for the inclusion functor called the *Yoneda embedding of the opposite*.

Remark 9.1.8. The definition of C^{op} does not depend on the universe for which C is small: if Grpd' is a larger universe of groupoids, we will have a full embedding of Grpd into Grpd', hence a full embedding

$$\mathrm{Fun}(C, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(C, \mathrm{Grpd}'),$$

and thus the definition of C^{op} as full subcategory of the domain or of the codomain will be canonically equivalent.

The opposite category of C is functorial in C:

Proposition 9.1.9. *Let $f\colon C \to D$ be a functor. Then the left Kan extension functor $f_!\colon$ $\mathrm{Fun}(C, \mathrm{Grpd}) \to \mathrm{Fun}(D, \mathrm{Grpd})$ restricts to a functor*

$$f^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}.$$

Proof. Consider the induced adjunction

$$f_!\colon \mathrm{Fun}(C, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}(D, \mathrm{Grpd}) : f^*.$$

For every object x of C, we obtain a commutative square

$$\begin{tikzcd} C_{x/} \arrow[r, "f_{x/}"] \arrow[d] & D_{f(x)/} \arrow[d] \\ C \arrow[r, "f"] & D, \end{tikzcd}$$

and it follows from the diagram

$$\begin{tikzcd} & * \arrow[dl, "{(x,\mathrm{id}_x)}"'] \arrow[dr, "{(f(x),\mathrm{id}_{f(x)})}"] & \\ C_{x/} \arrow[rr, "f_{x/}"] & & D_{f(x)/} \end{tikzcd}$$

that $f_{x/}$ is an initial functor, using the cancellation property (2) from Proposition 8.1.8. As a consequence, we obtain an equivalence

$$f_!(h_{C^{\mathrm{op}}}) \cong h_{D^{\mathrm{op}}}(f(x)),$$

and in particular the functor $f_!(h_{C^{\mathrm{op}}})\colon D \to \mathrm{Grpd}$ is corepresentable. □

Definition 9.1.10 (Hom-functor)**.** We define the *Hom-functor* of C

$$\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{Grpd}$$

as the functor obtained by uncurrying the inclusion functor $h_{C^{\mathrm{op}}}\colon C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{Grpd})$.

Lemma 9.1.11. *For a category C, there exists a preferred equivalence*

$$C^{\simeq} \xrightarrow{\sim} (C^{\mathrm{op}})^{\simeq}, \qquad x \mapsto \mathrm{Hom}_C(x, -).$$

Proof. Identifying $\mathrm{Fun}(C, S\,)\mathrm{LFib}(C)$ Let $(C^{\mathrm{op}})^{\simeq}$ can be described as the anima whose terms are tuples of the form (X, p, x, s) with $p\colon X \to C$ a left fibration with small fibers, $x\colon X$ an object of X and $s\colon \mathtt{hasInitial}(X)$ a proof that x is an initial object of X. The rule $(X, p, x, s) \mapsto p(x)$ defines a canonical projection

$$C^{\mathrm{op}} \to C\,.$$

There is a map in the other direction defined by the rule

$$x \mapsto (C_{x/}, p_x, (x, \mathrm{id}_x), s_x)$$

with $p_x\colon C_{x/} \to C$ the canonical projection, and s_x the canonical proof that the pair (x, id_x) is an initial object in $C_{x/}$ (for this to make sense, we use the fact that the formation of slices $C_{x/}$ is functorial in $x\colon C^{\simeq}$, which follows from Corollary 7.7.9: this allows to see the pullback of $\mathrm{ev}_0\colon \mathrm{Fun}([1], C) \to C$ over $C^{\simeq}$ as a functor from $C^{\simeq}$ to the universe of small groupoids). These two maps are inverse to each other: the second is obviously a section of the first whereas Proposition 9.1.3 ensures that the other composite is the identity. □

Lemma 9.1.12. *Let C be a category and $x, y\colon C^{\simeq}$ by two objects. There is a preferred equivalence*

$$C^{\mathrm{op}}(y, x) \xrightarrow{\sim} C(x, y)\,.$$

Proof. By definition, there is a canonical equivalence of the form

$$C^{\mathrm{op}}(y, x) \xrightarrow{\sim} \mathrm{Map}_C(C_{y/}, C_{x/})\,.$$

On the other hand, since the fiber of the canonical projection $C_{x/} \to C$ at y is $C(y, x)$ by definition, Corollary 8.1.14 provides a canonical equivalence

$$\mathrm{Map}_C(C_{y/}, C_{x/}) \simeq C(y, x)\,.$$

We may simply compose these two equivalences. □

Proposition 9.1.13. *For every small category C, the opposite category C^{op} is small.*

Proof. By the bicompleteness condition on Cat, it suffices to check that $\mathrm{Map}([1], C^{\mathrm{op}})$ is small. We may write this as

$$\mathrm{Map}([1], C^{\mathrm{op}}) \simeq \sum_{(x,y):(C^{\mathrm{op}})^{\simeq}\times(C^{\mathrm{op}})^{\simeq}} C^{\simeq}(x, y).$$

Since $C^{\simeq}$ is small, so is $(C^{\mathrm{op}})^{\simeq}$ by Lemma 9.1.11. Therefore, it suffice to prove that each $C^{\simeq}(x, y)$. But we have seen in Lemma 9.1.12 that that $C^{\simeq}(x, y) \simeq C(x, y)$, and since $C(x, y)$ is small for all x and y, this achieves the proof. □

Lemma 9.1.14. *If C is a category and $x\colon C^{\simeq}$ a terminal (initial) object of C, then interpreting x as an object of C^{op} by Lemma 9.1.11, it becomes an initial (terminal) object in the category C^{op}.*

Proof. If x is a terminal in C, then $C(y, x)$ is contractible for all $y\colon C^{\simeq}$. Therefore, by virtue of the preceding lemma, we see that $C^{\mathrm{op}}(x, y)$ is contractible for all $y\colon C^{\mathrm{op}}$, which proves that x is initial in C^{op}. The case of an initial object is done similarly. □

Lemma 9.1.15. *If $p\colon X \to C$ is a right fibration, then $p^{\mathrm{op}}\colon X^{\mathrm{op}} \to C^{\mathrm{op}}$ is a left fibration.*

Proof. We have to check that the square

$$\begin{tikzcd}
\mathrm{Map}([1], X^{\mathrm{op}}) \arrow[r, "(p^{\mathrm{op}})_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Map}([1], C^{\mathrm{op}}) \arrow[d, "\mathrm{ev}_0"] \\
(X^{\mathrm{op}})^{\simeq} \arrow[r, "(p^{\mathrm{op}})^{\simeq}"] & (C^{\mathrm{op}})^{\simeq}
\end{tikzcd}$$

is a pullback square. Identifying $C^{\simeq}$ and $(C^{\mathrm{op}})^{\simeq}$, this square is a diagram of the form bellow.

$$\begin{tikzcd}
\sum_{x,y\colon X^{\simeq}} X^{\mathrm{op}}(y, x) \arrow[r] \arrow[d] & \sum_{x,y\colon C^{\simeq}} C^{\mathrm{op}}(y, x) \arrow[d] \\
X^{\simeq} \arrow[r, "p^{\simeq}"] & C^{\simeq}
\end{tikzcd}$$

By virtue of Lemma 9.1.12, this square is equivalent to

$$\begin{tikzcd}
\sum_{x,y\colon X^{\simeq}} X(x, y) \arrow[r] \arrow[d] & \sum_{x,y\colon C^{\simeq}} C(x, y) \arrow[d] \\
X^{\simeq} \arrow[r, "p^{\simeq}"] & C^{\simeq}
\end{tikzcd}$$

which in turns is equivalent to the commutative square

$$\begin{tikzcd}
\mathrm{Map}([1], X) \arrow[r, "p_*"] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Map}([1], C) \arrow[d, "\mathrm{ev}_1"] \\
X^{\simeq} \arrow[r, "p^{\simeq}"] & C^{\simeq}
\end{tikzcd}$$

and the latter is a pullback square, since p is a right fibration. □

Proposition 9.1.16. *If C is a category and x an object of C, there is a canonical equivalence*

$$(C^{\mathrm{op}})_{x/} \xrightarrow{\sim} (C_{/x})^{\mathrm{op}}.$$

Proof. By virtue of Lemma 9.1.14, the pair

$$(x, \mathrm{id}_x)\colon (C_{/x})^{\simeq} \xrightarrow{\sim} ((C_{/x})^{\mathrm{op}})^{\simeq}$$

defines an initial object in $(C_{/x})^{\mathrm{op}}$. On the other hand, Lemma 9.1.15 ensures that the right fibration $C_{/x} \to C$ induces a left fibration of the form

$$(C_{/x})^{\mathrm{op}} \to C^{\mathrm{op}}\,.$$

the latter sends the initial object of $(C_{/x})^{\mathrm{op}}$ to x. Proposition 9.1.3 now readily implies that there is a canonical equivalence from $(C^{\mathrm{op}})_{x/}$ to $(C_{/x})^{\mathrm{op}}$ over C^{op}. □

9.2 The twisted arrow category

Definition 9.2.1 (Twisted arrow category)**.** For a category C, we define the left fibration $\mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ as the one classified by the Hom-functor:

$$\begin{tikzcd} \mathrm{Tw}(C) \arrow[r] \arrow[d, "{(s,t)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Grpd}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ C^{\mathrm{op}} \times C \arrow[r, "\mathrm{Hom}_C"] & \mathrm{Grpd}. \end{tikzcd}$$

We refer to its total category $\mathrm{Tw}(C)$ as the *twisted arrow category* of C. We denote the two projections to C^{op} and C by

$$s\colon\ \mathrm{Tw}(C) \to C^{\mathrm{op}} \qquad \text{and} \qquad t\colon\ \mathrm{Tw}(C) \to C.$$

Lemma 9.2.2. *If C is a small category, so is* $\mathrm{Tw}(C)$.

Proof. This is immediate from the fact that the left fibration $\mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ has small fibers and that $C^{\mathrm{op}} \times C$ is small by Proposition 9.1.13. □

The twisted arrow category is suitably functorial in C:

Construction 9.2.3. Consider a functor $f\colon C \to D$. We will construct a functor $\mathrm{Tw}(f)\colon\ \mathrm{Tw}(C) \to \mathrm{Tw}(D)$ fitting in a commutative square

$$\begin{tikzcd} \mathrm{Tw}(C) \arrow[r, dashed, "\mathrm{Tw}(f)"] \arrow[d] & \mathrm{Tw}(D) \arrow[d] \\ C^{\mathrm{op}} \times C \arrow[r, "f^{\mathrm{op}} \times f"] & D^{\mathrm{op}} \times D. \end{tikzcd}$$

Equivalently, we need to produce a morphism of left fibrations

$$\begin{tikzcd} \mathrm{Tw}(C) \arrow[rr, dashed] \arrow[dr] & & (f^{\mathrm{op}} \times f)^*(\mathrm{Tw}(D)) \arrow[dl] \\ & C^{\mathrm{op}} \times C & \end{tikzcd}$$

over $C^{\mathrm{op}} \times C$. Under unstraightening, this is in turn equivalent to producing a natural transformation $\mathrm{Hom}_C(-,-) \to \mathrm{Hom}_D(-,-) \circ (f^{\mathrm{op}} \times f)$ of functors $C^{\mathrm{op}} \to C \to \mathrm{Grpd}$. By currying, we may equivalently produce a natural transformation between the curried functors $C^{\mathrm{op}} \to \mathrm{Fun}(C, \mathrm{Grpd})$. For $\mathrm{Hom}_C(-,-)$, this curried functor is simply the inclusion. For $\mathrm{Hom}_D(-,-) \circ (f^{\mathrm{op}} \times f) = \mathrm{Hom}_D(-,-) \circ (f^{\mathrm{op}} \times \mathrm{id}_D) \circ (\mathrm{id}_{C^{\mathrm{op}}} \times f)$ it is given by the composite

$$C^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} D^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(D, \mathrm{Grpd}) \xrightarrow{f^*} \mathrm{Fun}(C, \mathrm{Grpd}),$$

which by definition of f^{op} is the same as the composite

$$C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{Grpd}) \xrightarrow{f_!} \mathrm{Fun}(D, \mathrm{Grpd}) \xrightarrow{f^*} \mathrm{Fun}(C, \mathrm{Grpd}).$$

The desired natural transformation is now simply given by whiskering the unit transformation $\mathrm{id}_{\mathrm{Fun}(C,\mathrm{Grpd})} \to f^* f_!$ with the inclusion $C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{Grpd})$.

Proposition 9.2.4. *The fibers of both projections $s\colon \mathrm{Tw}(C) \to C^{\mathrm{op}}$ and $t\colon \mathrm{Tw}(C) \to C$ have initial objects.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
C^{\simeq} \times C \arrow[d, "\simeq"'] \arrow[rrd, bend left, "\sum_{x:C^{\simeq}} \mathrm{Hom}_C(x,-)"] & & \\
(C^{\mathrm{op}})^{\simeq} \times C \arrow[r, hook] & C^{\mathrm{op}} \times C \arrow[r, "\mathrm{Hom}_C"'] & \mathrm{Grpd}.
\end{tikzcd}$$

So we obtain pullback squares

$$\begin{tikzcd}
\sum_{x:C^{\simeq}} C_{x/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Tw}(C) \arrow[d, "{(s,t)}"] \\
C^{\simeq} \times C \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner"] & C^{\mathrm{op}} \times C \arrow[d, "\mathrm{pr}_1"] \\
C^{\simeq} \simeq (C^{\mathrm{op}})^{\simeq} \arrow[r, hook] & C^{\mathrm{op}}.
\end{tikzcd}$$

In particular, the fiber of s over an object $x \in C^{\mathrm{op}}$ is the slice category $C_{x/}$, which has an initial object (x, id_x).

Now consider an object y of C and let $\mathrm{Tw}(C)_y$ denote the fiber of $t\colon \mathrm{Tw}(C) \to C$ at y:

$$\begin{tikzcd}
\mathrm{Tw}(C)_y \arrow[r] \arrow[d, "p_y"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Tw}(C) \arrow[d, "{(s,t)}"] \\
C^{\mathrm{op}} \arrow[r, "{(\mathrm{id},y)}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C^{\mathrm{op}} \times C \arrow[d, "\mathrm{pr}_2"] \\
* \arrow[r, "y"] & C.
\end{tikzcd}$$

Note that p_y is a left fibration whose fiber at y is given by $C^{\mathrm{op}}(y,y) \simeq C(y,y)$, hence we obtain an object (y, id_y) in this fiber, producing an object $\mathrm{id}_y \in \mathrm{Tw}(C)_y$. We then obtain a unique map

$$\begin{tikzcd} & * \arrow[dl, "{(y,\mathrm{id}_y)}"'] \arrow[dr, "\mathrm{id}_y"] & \\ C^{\mathrm{op}}_{y/} \arrow[rr, dashed] \arrow[dr] & & \mathrm{Tw}(C)_y \arrow[dl, "p_Y"] \\ & C^{\mathrm{op}}. & \end{tikzcd}$$

It will thus suffice to show that for every object x of C^{op} (i.e. every object x of C) the dashed map induces an equivalence

$$C^{\mathrm{op}}(y,x) \xrightarrow{\sim} C(x,y).$$

Under the equivalence from Lemma 9.1.12, we have to check in fact that the dashed map induces the identity of $C(x,y)$. A term f in $C(x,y)$ correspond to a functor

$$f \colon [1] \to C\,.$$

By functoriality of the formation of the twisted arrow category (Construction 9.2.3), we obtain a commutative square bellow.

$$\begin{tikzcd} \mathrm{Tw}([1]) \arrow[r, "\mathrm{Tw}(f)"] \arrow[d, "{(s,t)}"'] & \mathrm{Tw}(C) \arrow[d, "{(s,t)}"] \\ {[1]^{\mathrm{op}} \times [1]} \arrow[r, "f^{\mathrm{op}} \times f"] & C^{\mathrm{op}} \times C \end{tikzcd}$$

Since f sends the ordered pair $0,1$ to x,y, this induces a commutative diagram of the form

$$\begin{tikzcd} {[1](0,1)} \arrow[r, "\sim"] \arrow[d] & {[1]^{\mathrm{op}}(1,0)} \arrow[r, dashed] \arrow[d] & {[1](0,1)} \arrow[d] \\ C(x,y) \arrow[r, "\sim"] & C^{\mathrm{op}}(y,x) \arrow[r, dashed] & C(x,y) \end{tikzcd}$$

in which the vertical maps are induced by f^{op} and f. All the maps of the upper horizontal line are isomorphisms because they are maps between terminal objects (expressing that 1 is terminal in $[1]$, say). Interpreting this proof in context $C(x,y)$ for f the universal term of $C(x,y)$, this proves that the dashed map is indeed fiber-wise the identity. □

Remark 9.2.5. Given any object x of C seen as an object of C^{op}, the proof of the preceding proposition shows that the fiber of the target functor

$$t \colon \mathrm{Tw}(C) \to C$$

at x is equivalent to $(C^{\mathrm{op}})_{x/}$. Using Proposition 9.1.16, this means that we have a canonical equivalences over C^{op} of the form:

$$(C_{/x})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Tw}(C)_x\,.$$

Corollary 9.2.6. *The maps s and t induce equivalences*

$$\Pi_\infty(C^{\mathrm{op}}) \xleftarrow{\Pi_\infty(s)} \Pi_\infty(\mathrm{Tw}(C)) \xrightarrow{\Pi_\infty(t)} \Pi_\infty(C).$$

Proof. Let $f\colon X \to Y$ be a left fibration such that for every object y of Y the fiber X_y is weakly contractible (i.e. $\Pi_\infty(X_y)) \simeq *$.) Because the inclusion $X_y \hookrightarrow X_{/y}$ is terminal, it follows that also $\Pi_\infty(X_{/y}) \xrightarrow{\sim} *$. But then it follows that $\Pi_\infty(f)$ is an equivalence. Now apply this to both $f = s$ and $f = t$. □

Lemma 9.2.7. *Consider the functor $\tilde{u} = ((s,t), \mathrm{Tw}(u))$ defined by*

$$\begin{tikzcd}
\mathrm{Tw}(C) \arrow[r, dashed, "\tilde{u}"] \arrow[d, "(s,t)"'] & C^{\mathrm{op}} \times D \times_{D^{\mathrm{op}}\times D} \mathrm{Tw}(D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tw}(D) \arrow[d, "(s,t)"] \\
C^{\mathrm{op}} \times C \arrow[r, "\mathrm{id}_{C^{\mathrm{op}}} \times u"'] & C^{\mathrm{op}} \times D \arrow[r, "u^{\mathrm{op}}\times \mathrm{id}_D"'] & D^{\mathrm{op}} \times D.
\end{tikzcd}$$

Then the functor $\tilde{u}$ is initial.

Proof. By virtue of Lemma 8.4.8, it suffices to show that $\tilde{u}$ induces an initial functor fiberwise over each object of C^{op}. Let x be an object of C (and thus an object of C^{op}). The functor induced over x fits into a diagram of the form

$$\begin{tikzcd}
 & * \arrow[dl, "(x,\mathrm{id}_x)"'] \arrow[dr, "(u(x),\mathrm{id}_{u(x)})"] & \\
C_{x/} \arrow[rr, dashed] & & D_{u(x)/},
\end{tikzcd}$$

and thus by the cancellation property it follows that the bottom map is initial, as desired. □

Proposition 9.2.8. *Let $u\colon C \to D$ be a functor between small categories. The adjunction*

$$(\mathrm{id}_{C^{\mathrm{op}}} \times u)_! \colon \ \mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}(C^{\mathrm{op}} \times D, \mathrm{Grpd}) \ \colon (\mathrm{id}_{C^{\mathrm{op}}} \times u)^*.$$

induces a canonical isomorphism

$$(\mathrm{id}_{C^{\mathrm{op}}} \times u)_!(\mathrm{Hom}_C) \xrightarrow{\sim} (u^{\mathrm{op}} \times \mathrm{id}_D)^*(\mathrm{Hom}_D)$$

in $\mathrm{Fun}(C^{\mathrm{op}} \times D, \mathrm{Grpd})$.

Proof. This is a direct interpretation of the previous lemma in view of the explicit description of the functor $(\mathrm{id}_{C^{\mathrm{op}}} \times u)_!$ given by Theorem 8.1.19. □

Lemma 9.2.9. *Consider the functor $\bar{u} = ((s,t), \mathrm{Tw}(u))$ defined by*

$$\begin{tikzcd}
\mathrm{Tw}(C) \arrow[r, dashed, "\bar{u}"] \arrow[d, "(s,t)"'] & D^{\mathrm{op}} \times C \times_{D^{\mathrm{op}}\times D} \mathrm{Tw}(D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tw}(D) \arrow[d, "(s,t)"] \\
C^{\mathrm{op}} \times C \arrow[r, "u^{\mathrm{op}}\times \mathrm{id}_C"'] & D^{\mathrm{op}} \times C \arrow[r, "\mathrm{id}_{D^{\mathrm{op}}} \times u"'] & D^{\mathrm{op}} \times D.
\end{tikzcd}$$

Then the functor $\bar{u}$ is initial.

Proof. As in the proof of Lemma 9.2.9, it suffices to show that $\tilde{u}$ induces an initial functor fiberwise over each object of C. Let x be an object of C. By Remark 9.2.5, the functor induced on fibers over x fits into a diagram of the form

$$\begin{tikzcd} & * \arrow[dl, "{(x,\mathrm{id}_x)}"'] \arrow[dr, "{(u(x),\mathrm{id}_{u(x)})}"] & \\ (C^{\mathrm{op}})_{x/} \arrow[rr, dashed] & & (D^{\mathrm{op}})_{u(x)/}, \end{tikzcd}$$

and thus, by the cancellation property, is initial. □

Proposition 9.2.10. *Let $u\colon C \to D$ be a functor between small categories. The adjunction*

$$(u^{\mathrm{op}} \times \mathrm{id}_C)_! \colon \operatorname{Fun}(C^{\mathrm{op}} \times C, \mathrm{Grpd}) \rightleftarrows \operatorname{Fun}(D^{\mathrm{op}} \times C, \mathrm{Grpd}) : (u^{\mathrm{op}} \times \mathrm{id}_C)^*.$$

induces a canonical isomorphism

$$(u^{\mathrm{op}} \times \mathrm{id}_C)_!(\mathrm{Hom}_C) \xrightarrow{\sim} (\mathrm{id}_{D^{\mathrm{op}}} \times u)^*(\mathrm{Hom}_D)$$

in $\operatorname{Fun}(C^{\mathrm{op}} \times D, \mathrm{Grpd})$.

Proof. This is a reformulation of Lemma 9.2.9. □

Proposition 9.2.11. *For a functor $u\colon C \to D$ between small categories, the following conditions are equivalent:*

(1) The functor u is fully faithful;

(2) The functor $u_!\colon$ $\operatorname{Fun}(C, \mathrm{Grpd}) \to \operatorname{Fun}(D, \mathrm{Grpd})$ is fully faithful;

(3) The commutative square

$$\begin{tikzcd} \mathrm{Tw}(C) \arrow[r, dashed, "\mathrm{Tw}(u)"] \arrow[d, "{(s,t)}"'] & \mathrm{Tw}(D) \arrow[d, "{(s,t)}"] \\ C^{\mathrm{op}} \times C \arrow[r, "u^{\mathrm{op}} \times u"] & D^{\mathrm{op}} \times D \end{tikzcd}$$

is a pullback square;

(4) The functor $u^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ is fully faithful.

Proof. The equivalence between (1) and (3) is immediate, since both conditions ask the induced map $C(x, y) \to D(u(x), u(y))$ to be an equivalence for all objects x and y of C. The equivalence between (1) and (4) readily follows from Lemma 9.1.12. The fact that (4) follows from (2) holds by definition of u^{op}. It remains to show that (1) implies (2).

Let $p\colon X \to C$ be a small left fibration, classified by a functor $F\colon C \to \mathrm{Grpd}$. We may factor $u \circ p$ as

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ C \arrow[r, "u"] & D, \end{tikzcd}$$

where j is initial and q is a left fibration. Then q is a small left fibration and thus corresponds to a functor $u_!(F)\colon C \to \mathrm{Grpd}$. The natural transformation $F \to u^* u_! F$ corresponds to the functor $(p, j)\colon X \to C \times_D Y$, and we have to show this is an equivalence. Since it is a morphism of left fibrations over C, we may equivalently show that for every object x of C,the induced map

$$X_x \to Y_{u(x)}$$

is an equivalence. Now, this map fits into the following commutative square

$$\begin{tikzcd} X_x \arrow[r] \arrow[d] & Y_{u(x)} \arrow[d] \\ X_{/x} \arrow[r, "u_{/x}"] & Y_{/u(x)}, \end{tikzcd}$$

and since the vertical functors are both terminal (because left fibrations are proper, by Corollary 8.2.7), we may equivalently show that the functor $u_{/x}$ is a weak homotopy equivalence. Since u is fully faithful, we have a pullback square

$$\begin{tikzcd} C_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D_{/u(x)} \arrow[d] \\ C \arrow[r, "u"] & D, \end{tikzcd}$$

and hence an equivalence $C_{/x} \xrightarrow{\sim} C_{/u(x)}$. It follows that we also have an equivalence $X_{/x} \xrightarrow{\sim} X_{/u(x)}$, and so it will suffice to show that the map $j_{/u(x)}\colon X_{/u(x)} \to Y_{/u(x)}$ is a weak equivalence. But since this functor is a pullback of j along the smooth functor $D_{/u(x)} \to D$, $j_{/u(x)}$ is an initial functor, and hence in particular a weak equivalence. This finishes the proof. □

Corollary 9.2.12. *Consider the pushout square of categories bellow.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

If f is fully faithful, so is g.

Proof. We may assume that this is a commutative square of small categories (up to a choice of suitable universe). Let us assume that f is fully faithful. By virtue of Proposition 9.2.11 the functor

$$f_!\colon \ \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})$$

is fully faithful. In otherwords, if we consider the induced pullback square

$$\begin{tikzcd} \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "v^*"] \arrow[d, "g^*"'] & \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, "f^*"] \\ \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "u^*"] & \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \end{tikzcd}$$

the functor $f_!$ is a left adjoint section of f^*. Proposition 5.5.8 thus tells us that the functor g^* is has a left adjoint section, i.e. that the functor $g_!$ is fully faithful. A new application of Proposition 9.2.11 implies that the functor g itself is fully faithful. □

9.3 The Yoneda lemma

The goal of this section is to state and prove the Yoneda lemma. We start with the following general theorem:

Theorem 9.3.1. *Consider a category C equipped with a functor $\Phi\colon C \to \mathrm{Grpd}$, and let $q\colon \int \Phi \to C$ denote the associated left fibration:*

$$\begin{tikzcd} \int \Phi \arrow[r, "\Phi_\bullet"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ C \arrow[r, "\Phi"] & \mathrm{Grpd}. \end{tikzcd}$$

Then there exists a canonical pullback square of the form

$$\begin{tikzcd} \mathrm{Fun}(C, \mathrm{Grpd})_{\Phi/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\int \Phi, \mathrm{Grpd}_\bullet) \arrow[d, "(\pi_{\mathrm{univ}})_*"] \\ \mathrm{Fun}(C, \mathrm{Grpd}) \arrow[r, "q^*"'] & \mathrm{Fun}(\int \Phi, \mathrm{Grpd}). \end{tikzcd}$$

Proof. We form the following pullback square.

$$\begin{tikzcd} W \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\int \Phi, \mathrm{Grpd}_\bullet) \arrow[d, "(\pi_{\mathrm{univ}})_*"] \\ \mathrm{Fun}(C, \mathrm{Grpd}) \arrow[r, "q^*"'] & \mathrm{Fun}(\int \Phi, \mathrm{Grpd}) \end{tikzcd}$$

The pair $w = (q, \Phi_\bullet)$ determines an object $w : W^{\simeq}$ such that $p(w) = \Phi$. Since p is a left fibration, this determines a unique functor

$$f\colon\ \mathrm{Fun}(C, \mathrm{Grpd})_{\Phi/} \to W$$

over $\mathrm{Fun}(C, \mathrm{Grpd})$ such that $f(\Phi) = w$. To prove the theorem, it suffices to prove that f is an equivalence. Since this is a functor between two left fibrations over $\mathrm{Fun}(C, \mathrm{Grpd})$, it suffices to check this on fibers over any object of $\mathrm{Fun}(C, \mathrm{Grpd})$. Finally, up to replacing the underlying groupoidal context, it suffices to consider fibers defined in the absolute context. Consider an arbitrary object Ψ of $\mathrm{Fun}(C, \mathrm{Grpd})$ (in the absolute context). The fiber of $\mathrm{Fun}(C, \mathrm{Grpd})_{\Phi/}$ over $\mathrm{Fun}(C, \mathrm{Grpd})$ is by definition the mapping groupoid

$$\mathrm{Fun}(C, \mathrm{Grpd})(\Phi, \Psi).$$

By straightening/unstraightening (Theorem 7.3.16), this is equivalent to the relative mapping space $\mathrm{Map}_C(\int \Phi, \int \Psi)$ over C. Since $\int \Psi$ sits in a pullback square of the form

$$\begin{tikzcd} \int \Psi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ C \arrow[r, "\Psi"] & \mathrm{Grpd}, \end{tikzcd}$$

we have by adjunction the equivalence bellow, in which $\int \Phi$ is seen over Grpd with structural map $\Psi \circ q$.

$$\mathrm{Map}_C\left(\int \Phi, \int \Psi\right) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Grpd}}\left(\int \Phi, \mathrm{Grpd}_\bullet\right)$$

We also have the following canonical pullback squares (the first one by definition and the second one by conservativity of left fibrations).

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Grpd}}\left(\int \Phi, \mathrm{Grpd}_\bullet\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}\left(\int \Phi, \mathrm{Grpd}_\bullet\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}\left(\int \Phi, \mathrm{Grpd}_\bullet\right) \arrow[d, "(\pi_{\mathrm{univ}})_*"] \\
* \arrow[r, "\Psi\circ q"] & \mathrm{Map}\left(\int \Phi, \mathrm{Grpd}\right) \arrow[r] & \mathrm{Fun}\left(\int \Phi, \mathrm{Grpd}\right)
\end{tikzcd}$$

It remains to check that the map induced by f on fibers over Ψ simply is the composition

$$\mathrm{Fun}(C, \mathrm{Grpd})(\Phi, \Psi) \xrightarrow{\sim} \mathrm{Map}_C\left(\int \Phi, \int \Psi\right) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Grpd}}\left(\int \Phi, \mathrm{Grpd}_\bullet\right).$$

A map $u\colon \Phi \to \Psi$ can be seen as a functor $u\colon [1] \to \mathrm{Fun}(C, \mathrm{Grpd})$. Since $f(\Phi) = w$, there is a unique functor $v\colon [1] \to W$ such that $p(v) = u$. By definition, we then have $f(u) = v$. On the other hand the straightening/unstraightening equivalence consists in interpreting u as a functor $\tilde{u}\colon [1]\times C \to \mathrm{Grpd}$ corresponding to a left fibration $q\colon X \to [1]\times C$ and to apply Construction 7.2.1. In other words, we see $[1]\times\int \Phi$ over $[1]\times C$ by pulling back $\int \Phi$ along the projection $[1]\times C \to C$, which we interpret by transposition as a functor

$$\bar{u}\colon [1] \to \mathrm{Fun}(\int \Phi, [1]\times C)$$

and we consider the unique lifting of $\bar{u}$ of the form

$$\bar{v}\colon [1] \to \mathrm{Fun}(\int \Phi, X)$$

with source the composition of the embedding the fiber $X_0 \hookrightarrow X$ with the identification of $\int \Phi$ with the fiber X_0. In other words, we consider the commutative square

$$\begin{tikzcd}
\Phi \arrow[r, equal] \arrow[d, equal] & \Phi \arrow[d, "u"] \\
\Phi \arrow[r, "u"'] & \Psi
\end{tikzcd}$$

and pick the lower horizontal line in the image of this square by f. This is indeed the same as applying f directly. □

Corollary 9.3.2. *Let e be the terminal object of* Grpd. *Then there is an equivalence* $\mathrm{Grpd}_{e/} \xrightarrow{\sim} \mathrm{Grpd}_\bullet$ *over* Grpd.

Proof. We apply Theorem 9.3.1 with $C = *$ and with $\Phi = e\colon * \to \mathrm{Grpd}$. Since e is terminal, it represents the left fibration $* \to *$. Using the equivalence $\mathrm{Fun}(*, \mathrm{Grpd}) \xrightarrow{\sim} \mathrm{Grpd}$, we thus obtain a pullback square of the form

$$\begin{tikzcd}
\mathrm{Grpd}_{e/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Grpd} \arrow[r, equal] & \mathrm{Grpd},
\end{tikzcd}$$

and thus the top functor is an equivalence. $\square$

Corollary 9.3.3. *There is an equivalence of functors*

$$\mathrm{Hom}_{\mathrm{Grpd}}(e,-) \simeq \mathrm{id}_{\mathrm{Grpd}}$$

of functors Grpd $\to$ Grpd.

Proof. The equivalence $\mathrm{Grpd}_{e/} \simeq \mathrm{Grpd}_\bullet$ of left fibrations over Grpd corresponds under straightening/unstraightening to a natural isomorphism $\mathrm{Hom}_{\mathrm{Grpd}}(e,-) \simeq \mathrm{id}_{\mathrm{Grpd}}$. $\square$

Corollary 9.3.4. *Consider a small category C, and let e_C denote the terminal object of* $\mathrm{Fun}(C, \mathrm{Grpd})$ *(i.e. the constant value with value e). Then there is a pullback square*

$$\begin{tikzcd} \mathrm{Fun}(C,\mathrm{Grpd}_\bullet) \arrow[r] \arrow[d, "(\pi_{\mathrm{univ}})_*"'] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Fun}(C,\mathrm{Grpd}) \arrow[r, "\mathrm{Hom}(e_C,-)"] & \mathrm{Grpd} \end{tikzcd}$$

Proof. By definition of the functor $\mathrm{Hom}(e_C,-)$, there is a pullback square if we replace the left upper corner by the slice category $\mathrm{Fun}(C,\mathrm{Grpd})_{e_C/}$. By Lemma 5.3.2 this slice category is equivalent to $\mathrm{Fun}(C,\mathrm{Grpd}_{e/})$, and so the claim follows from the equivalence $\mathrm{Grpd}_{e/} \xrightarrow{\sim} \mathrm{Grpd}_\bullet$ from Corollary 9.3.2. $\square$

From Theorem 9.3.1 we may immediately deduce the following more general claim:

Corollary 9.3.5. *Consider a commutative diagram of the following form:*

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[dr, "f"'] & V \arrow[r, "\Psi"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ & C \arrow[r, "\Phi"] & \mathrm{Grpd}. \end{tikzcd}$$

Assume that i is an initial functor. Then the commutative square

$$\begin{tikzcd} \mathrm{Fun}(C,\mathrm{Grpd})_{\Phi/} \arrow[r] \arrow[d] & \mathrm{Fun}(U,\mathrm{Grpd}_\bullet) \arrow[d, "\pi_{\mathrm{univ}}"'] \\ \mathrm{Fun}(C,\mathrm{Grpd}) \arrow[r, "f^*"'] & \mathrm{Fun}(U,\mathrm{Grpd}) \end{tikzcd}$$

is a pullback square.

Proof. By Theorem 9.3.1 the diagram is a pullback square if we replace U by V and i by id_V. But applying part (3) of Proposition 8.1.4 to the universal left fibration $\pi_{\mathrm{univ}}\colon \mathrm{Grpd}_\bullet \to \mathrm{Grpd}$, we obtain a pullback square of the form

$$\begin{tikzcd} \mathrm{Fun}(V,\mathrm{Grpd}_\bullet) \arrow[r, "i^*"] \arrow[d, "(\pi_{\mathrm{univ}})_*"'] & \mathrm{Fun}(U,\mathrm{Grpd}_\bullet) \arrow[d, "(\pi_{\mathrm{univ}})_*"] \\ \mathrm{Fun}(V,\mathrm{Grpd}) \arrow[r, "i^*"'] & \mathrm{Fun}(U,\mathrm{Grpd}), \end{tikzcd}$$

and thus we may paste the two pullback squares to get the result. $\square$

Corollary 9.3.6. *Consider a category C with an object $x\colon C^{\simeq}$ defined in the absolute context. We assume that, for each object y of C, $C(x,y)$ is small (i.e. that the projection of $C_{x/}$ to C is a left fibration with small fibers). We consider the functor*

$$h_{C^{\mathrm{op}}}(x) = \mathrm{Hom}_C(x,-)\colon C \to \mathrm{Grpd}$$

determined by the following pullback square.

$$\begin{tikzcd} C_{x/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Grpd}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ C \arrow[r, "h_{C^{\mathrm{op}}}(x)"'] & \mathrm{Grpd}. \end{tikzcd}$$

Then we have a homotopy pullback square

$$\begin{tikzcd} \mathrm{Fun}(C,\mathrm{Grpd})_{h_{C^{\mathrm{op}}}(x)/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Grpd}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Fun}(C,\mathrm{Grpd}) \arrow[r, "\mathrm{ev}_x"] & \mathrm{Grpd}. \end{tikzcd}$$

In particular, we obtain an equivalence

$$\mathrm{ev}_x \simeq \mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})}(h_{C^{\mathrm{op}}}(x),-)$$

in the category $\mathrm{Fun}(\mathrm{Fun}(C,\mathrm{Grpd}),\mathrm{Grpd})$.

Proof. The map $U = * \xrightarrow{(x,\mathrm{id}_x)} C_{x/}$ is initial, and hence we may apply Corollary 9.3.5. □

We have thus provided an equivalence $F(x) \simeq \mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})}(h_{C^{\mathrm{op}}}(x),F)$ for all $F\colon C \to \mathrm{Grpd}$, functorially in F. This is a version of the Yoneda lemma. We would now like to make this equivalence natural in x as well.

For a small category C, choose a universe Cat' such that Cat (and hence Grpd) is small with respect to Cat'. We write Grpd' for the full subcategory of Cat' spanned by the groupoidal objects in Cat'. We then consider the functors

$$(s,t)\colon\ \mathrm{Tw}(\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})) \to \mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})$$

and

$$(s,t)\colon\ \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C.$$

We now define a category V and a functor $v\colon\ \mathrm{Tw}(C) \to V$ by means of the following commutative diagram:

$$\begin{tikzcd} \mathrm{Tw}(C) \arrow[r, dashed, "v"] \arrow[d, "(s,t)"'] \arrow[rr, bend left, "\mathrm{Tw}(h_C)"] & V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Tw}(\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Grpd}'_{\bullet} \arrow[d, "\pi'_{\mathrm{univ}}"] \\ C^{\mathrm{op}} \times C \arrow[r, "\mathrm{id}_{C^{\mathrm{op}}} \times h_C"'] & C^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd}) \arrow[r, "(h_C)^{\mathrm{op}} \times \mathrm{id}"] & \mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd}) \arrow[r, "\mathrm{Hom}"] & \mathrm{Grpd}'. \end{tikzcd}$$

We similarly define a category W and a functor $w\colon \mathrm{Tw}(C) \to W$ by means of the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Tw}(C) \arrow[r, dashed, "w"] \arrow[d, "{(s,t)}"'] \arrow[rr, bend left] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[r, hook] \arrow[d, "\pi_{\mathrm{univ}}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}'_\bullet \arrow[d, "\pi'_{\mathrm{univ}}"] \\
C^{\mathrm{op}} \times C \arrow[r, "\mathrm{id}_{C^{\mathrm{op}}} \times h_C"] \arrow[rr, bend right, "\mathrm{Hom}_C"'] & C^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, hook, "\mathrm{ev}"] & \mathrm{Grpd} \arrow[r, hook] & \mathrm{Grpd}'.
\end{tikzcd}$$

Lemma 9.3.7. *Both functors v and w are initial.*

Proof. For v, this is a particular case of Lemma 9.2.7. The proof for w is similar. Recall from Lemma 8.4.8 that for a commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\
C \times D \arrow[r, "\mathrm{id}_C \times g"'] & C \times C
\end{tikzcd}$$

such that p and q are left fibrations, if f is fiberwise initial over C then f is initial. Therefore, in order to show that w is initial, it will suffice to show that for every object x of C (seen as an object of C^{op}), the induced map $w_x\colon \mathrm{Tw}(C)_x \to W_x$ is initial:

$$\begin{tikzcd}
\mathrm{Tw}(C)_x \arrow[r, "w_x"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
C \arrow[r, "h_C"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "\mathrm{ev}_x"] & \mathrm{Grpd}.
\end{tikzcd}$$

But we already know from the previous corollary that $V_x \simeq W_x$ holds fiberwise, and on fibers the maps v_x and w_x fit into a commutative diagram over $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ as follows:

$$\begin{tikzcd}
C_{x/} \arrow[r, "v_x"] \arrow[d, "\simeq"'] & V_x \arrow[d, "\simeq"] \\
\mathrm{Tw}(C)_x \arrow[r, "w_x"] & W_x
\end{tikzcd}$$

Indeed, to see the commutativity of this square, since the projection of W_x to $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ is a left fibration, by virtue of Proposition 8.1.13, it suffices to check that the initial object of $C_{x/}$ is sent to the same object of W_x either way. This amounts to say that, in the equivalence

$$F(x) \simeq \mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})}(h_{C^{\mathrm{op}}}(x), F)$$

provided by Corollary 9.3.6, in the case where $F = h_{C^{\mathrm{op}}}(x)$, the identity of x is sent to the identity of $h_{C^{\mathrm{op}}}(x)$. This is true by construction.

Now, since v_x is initial (see the proof of Lemma 9.2.7), this implies that w_x is initial as well. This finishes the proof. □

Definition 9.3.8. A category C is *locally small* if the left fibration

$$(s,t)\colon \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$$

has small fibers. In this situation, we have a pullback square of the form

$$\begin{tikzcd} \mathrm{Tw}(C) \arrow[r] \arrow[d, "(s,t)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ C^{\mathrm{op}} \times C \arrow[r, "\mathrm{Hom}_C"] & \mathrm{Grpd} \end{tikzcd}$$

that induces the functor

$$h_C\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

called the *Yoneda embedding*, defined by the rule

$$h_C(y)(x) = \mathrm{Hom}_C(x,y)\,.$$

A *presheaf* on C is a functor of the form

$$F\colon C^{\mathrm{op}} \to \mathrm{Grpd}\,.$$

Such a presheaf is said to be *representable* if there exists an object x of C as well as an isomorphism of the form

$$\sigma\colon h_C(x) \xrightarrow{\sim} F\,.$$

Proposition 9.3.9. *The Yoneda embedding* $h_C\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ *is fully faithful.*

Proof. The proof of the previous lemma shows that the square

$$\begin{tikzcd} C_{x/} \simeq \mathrm{Tw}(C)_x \arrow[r, "v_x"] \arrow[d] & V_x \arrow[d] \\ C \arrow[r, "h_C"] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \end{tikzcd}$$

is a pullback for all x. It then follows from Proposition 9.2.11 (recalling the definition of V) that the functor $h_C\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ is fully faithful. □

Corollary 9.3.10. *For every category C, there exists a preferred equivalence*

$$C \xrightarrow{\sim} (C^{\mathrm{op}})^{\mathrm{op}}.$$

Proof. For any object x of C, the functor $h_C(x)$ is corepresentable in C^{op}, because, as explained in Remark 9.2.5, the corresponding left fibration is the functor $(C^{\mathrm{op}})_{x/} \cong (C_{/x})^{\mathrm{op}} \to C^{\mathrm{op}}$:

$$\begin{tikzcd} (C^{\mathrm{op}})_{x/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tw}(C) \arrow[d, "(s,t)"] \\ C^{\mathrm{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\mathrm{op}} \times C \arrow[d, "\mathrm{pr}_2"] \\ * \arrow[r, "x"] & C \end{tikzcd}$$

It follows that we obtain a factorization

$$\begin{tikzcd}
C^{\simeq} \arrow[r, dashed] \arrow[d, hook] & ((C^{\mathrm{op}})^{\mathrm{op}})^{\simeq} \arrow[d, hook] \\
C \arrow[r, "h_C"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}),
\end{tikzcd}$$

and hence we obtain a functor $C \to (C^{\mathrm{op}})^{\mathrm{op}}$. The composition of this functor with the full embedding of $(C^{\mathrm{op}})^{\mathrm{op}}$ into $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ is the Yoneda embedding that we already know to be fully faithful. It follows that the functor $C \to (C^{\mathrm{op}})^{\mathrm{op}}$ is fully faithful. But we also know by Lemma 9.1.11 that both C and $(C^{\mathrm{op}})^{\mathrm{op}}$ have the same objects, which implies (strong) essentially surjectivity. This finishes the proof. □

Remark 9.3.11. This corollary means that there exists an equivalence $\mathrm{Tw}(C^{\mathrm{op}}) \simeq \mathrm{Tw}(C)$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Tw}(C^{\mathrm{op}}) \arrow[rr, dashed, "\sim"] \arrow[d, "(s,t)"'] & & \mathrm{Tw}(C) \arrow[d, "(s,t)"] \\
(C^{\mathrm{op}})^{\mathrm{op}} \times C^{\mathrm{op}} & C \times C^{\mathrm{op}} \arrow[l, "\sim"'] \arrow[r, "\mathrm{swap}", "\sim"'] & C^{\mathrm{op}} \times C.
\end{tikzcd}$$

Remark 9.3.12. If C is locally small, $\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{Grpd}$ is a functor. In particular any morphism $f \colon x \to y$ in C, seen as functor from $[1]$ to C, induces for each object z of C a morphism

$$f_* \colon \mathrm{Hom}_C(z, x) \to \mathrm{Hom}_C(z.y)$$

defined as $f_* := \mathrm{Hom}_C(z, f)$. In other words, composition by f is now encoded in the functorial nature of Hom_C.

Corollary 9.3.13. *Let $f \colon x \to y$ be a morphism in a locally small category C. Assume that, for any object z of C, f induces an isomorphism*

$$f_* \colon \mathrm{Hom}_C(z, x) \xrightarrow{\sim} \mathrm{Hom}_C(z.y)\,.$$

Then f is an isomorphism in C.

Proof. Since isomorphisms in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ are detected object-wise, this corollary simply says that the Yoneda embedding is conservative. But any fully faithful functor is conservative, so that this corollary is a direct consequence of Proposition 9.3.9. □

Proposition 9.3.14. *Let us consider a pullback square of the form bellow.*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] & \mathrm{Grpd} \arrow[d, "\pi_{\mathrm{univ}}"] \\
C^{\mathrm{op}} \arrow[r, "F"] & \mathrm{Grpd}
\end{tikzcd}$$

Then F is representable if and only if the category X has an initial object.

Proof. In view of the preceding remark, this is a special instance of Proposition 9.1.3. □

Let us consider a locally small category C. Since $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ is small with respect to Cat', we have a Hom functor of the the form:

$$\begin{tikzcd}
\mathrm{Tw}(\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})) \arrow[r] \arrow[d, "(s,t)"'] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "\mathrm{Hom}"] & \mathrm{Grpd}' .
\end{tikzcd}$$

The Yoneda embedding of C being fully faithful, it follows that the functor $\mathrm{id}_{C^{\mathrm{op}}} \times h_C$ is fully faithful as well, and hence the left Kan extension functor

$$(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_! \colon \mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{Grpd}') \to \mathrm{Fun}(C^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}), \mathrm{Grpd}')$$

is fully faithful, by Proposition 9.2.11. On the other hand, by virtue of Proposition 9.2.8, if

$$i \colon \mathrm{Grpd} \to \mathrm{Grpd}'$$

denotes the obvious full embedding, we get the identification

$$(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!(i \circ \mathrm{Hom}_C) = h_C^{\mathrm{op}} \times \mathrm{id}^*_{\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})}(i \circ \mathrm{Hom})$$

and, by a similar argument, the initiality of the functor w from Lemma 9.3.7 means that

$$(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!(i \circ \mathrm{Hom}_C) = i \circ \mathrm{ev}$$

holds. Moreover, the full faithfulness of the Yoneda embedding can be expressed by saying that there is a pullback square of the form

$$\begin{tikzcd}
\mathrm{Tw}(C) \arrow[r, "h_C"] \arrow[d, "(s,t)"'] & \mathrm{Tw}(\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})) \arrow[d, "(s,t)"] \\
C^{\mathrm{op}} \times C \arrow[r, "(h_C)^{\mathrm{op}} \times h_C"] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})
\end{tikzcd}$$

inducing a pullback square of the form

$$\begin{tikzcd}
\mathrm{Tw}(C) \arrow[r, "v"] \arrow[d, "(s,t)"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d] \\
C^{\mathrm{op}} \times C \arrow[r, "\mathrm{id}_{C^{\mathrm{op}}} \times h_C"] & C^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) .
\end{tikzcd}$$

By construction, we have a pullback square of the form

$$\begin{tikzcd}
\mathrm{Tw}(C) \arrow[r, "w"] \arrow[d, "(s,t)"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d] \\
C^{\mathrm{op}} \times C \arrow[r, "\mathrm{id}_{C^{\mathrm{op}}} \times h_C"] & C^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) .
\end{tikzcd}$$

All in all, we obtain:

Theorem 9.3.15 (Yoneda lemma)**.** *Let C be a locally small category. There is an isomorphism*

$$\mathrm{Hom}(h_C^{\mathrm{op}}(x), F) \xrightarrow{\sim} F(x),$$

which is functorial in both variables $F\colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ and $x\colon C^{\mathrm{op}}$.
It is the unique functorial identification which extends the identification

$$\mathrm{Hom}_C(x, y) = \mathrm{Hom}(h_C^{\mathrm{op}}(x), h_C(y))$$

on representable functors expressing the full faithfulness of the Yoneda embedding h_C.

Proof. As already explained above, there are identifications:

$$(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!(i \circ \mathrm{Hom}_C) = h_C^{\mathrm{op}} \times \mathrm{id}^*_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})}(i \circ \mathrm{Hom}) \quad \text{and} \quad (\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!(i \circ \mathrm{Hom}_C) = i \circ \mathrm{ev}$$

This gives the first assertion. The restricton of this identification expresses the full faithfulness of the Yoneda embedding by construction (this is what the last two pullback squares before the statement of the theorem mean). Finally, the full faithfulness of h_C implies the full faithfulness of the functor $(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!$, which readily implies the unicity assertion. □

Corollary 9.3.16. *Let C and D be small categories. Then the Yoneda embedding of D induces a fully faithful functor*

$$\mathrm{Fun}(C, D) \hookrightarrow \mathrm{Fun}(C, \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})) \simeq \mathrm{Fun}(C \times D^{\mathrm{op}}, \mathrm{Grpd})$$

whose essential image is the collection of those functors $F\colon C \times D^{\mathrm{op}} \to \mathrm{Grpd}$ with the property that $F(x, -)\colon D^{\mathrm{op}} \to \mathrm{Grpd}$ is representable for any object x of C.

Proof. This is immediate from part (2) of Proposition 9.2.11. □

Theorem 9.3.17. *Let $f\colon C \to D$ be a functor between small categories. We consider the adjunction*

$$f_!\colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) : f^*,$$

where f^* is defined by the rule $G \mapsto G \circ f^{\mathrm{op}}$. Then the following assertions are equivalent:

(1) The functor f admits a right adjoint;

(2) There is a functor $g\colon D \to C$ together with a functorial identification

$$\mathrm{Hom}_D(f^{\mathrm{op}}(x), y) \xrightarrow{\sim} \mathrm{Hom}_C(x, g(y))$$

natural in $(x, y)\colon C^{\mathrm{op}} \times D$.

(3) For any object y in D, the functor $f^* h_D(y) = \mathrm{Hom}_D(f^{\mathrm{op}}(-), y)\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ is representable;

(4) There is a functor $g\colon D \to C$ together with a natural identification

$$g_! \xrightarrow{\sim} f^*$$

of functors from $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})$ to $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$.

Proof. It is clear that (2) implies (3), and the implication (3) $\implies$ (2) follows from Corollary 9.3.16. We have seen in Theorem 5.9.12 that (1) is equivalent to the statement that for any object y of D the relative slice $C_{/y} = C \times_D D_{/y}$ admits an initial object. Note that the left fibration $(C_{/y})^{\mathrm{op}} \to C^{\mathrm{op}}$ is represented by $f^* h_D(y)$:

$$\begin{tikzcd}
(C_{/y})^{\mathrm{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (D_{/y})^{\mathrm{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
C^{\mathrm{op}} \arrow[r, "f^{\mathrm{op}}"] \arrow[rr, bend right, "f^* h_D(y)"'] & D^{\mathrm{op}} \arrow[r, "\mathrm{Hom}_D(-,y)"'] & \mathrm{Grpd}.
\end{tikzcd}$$

It then follows from Proposition 9.3.14 that $(C_{/y})^{\mathrm{op}}$ has an initial object if and only if the presheaf $f^* h_D(y)$ is representable, proving the equivalence between (1) and (3).

To show that (1) implies (4), we observe first that, for any functor $g\colon D \to C$, will have an adjunction of the form:

$$g_! \colon \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) : g^*.$$

If g is right adjoint to f, then g^* is right adjoint to f^*, and then we have $g_! \xrightarrow{\sim} f^*$ since both are left adjoints to v^*.

Finally, assuming (4) we get $f^* h_D \simeq g_! h_D \simeq h_C g$, and thus this directly implies (3). This finishes the proof. □

Corollary 9.3.18. *Let $u\colon C \to D$ be a functor between small categories. We consider the adjunction*

$$u_! \colon \mathrm{Fun}(C, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}(D, \mathrm{Grpd}) \colon u^*$$

where u^* is the functor defined by the rule $F \mapsto F \circ u^{\mathrm{op}}$ and $u_!$ is the left Kan extension functor along u^{op}. Then there is a canonical commutative square of the form bellow.

$$\begin{tikzcd}
C \arrow[r, "h_C"] \arrow[d, "u"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, "u_!"] \\
D \arrow[r, "h_D"] & \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})
\end{tikzcd}$$

Proof. We will apply Corollary 9.3.16 for the Yoneda embedding of $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})$ (using a universe of groupoids Grpd' large enough so that it is well defined as done above multiple times already). In other words, it suffices to check that

$$\mathrm{Hom}((u_! \circ h_C)^{\mathrm{op}}(x), F) = \mathrm{Hom}((h_D \circ u)^{\mathrm{op}}(x), F))$$

functorially in x in C^{op} and F in $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})$. Applying Theorem 9.3.17 to the adjunction

$$u_! \colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) : u^*$$

itself, the left hand side can be identified as

$$\mathrm{Hom}((u_! h_C)^{\mathrm{op}}(x), F) = \mathrm{Hom}(h_C^{\mathrm{op}}(x), u^*(F))$$

and the Yoneda lemma implies that

$$\operatorname{Hom}(h_C^{\mathrm{op}}(x), u^*(F)) = u^*(F)(x)$$

holds, whereas another instance of the Yoneda lemma implies

$$\operatorname{Hom}(h_D^{\mathrm{op}}(u^{\mathrm{op}}(x), F)) = F(u^{\mathrm{op}}(x))\,.$$

Finally, $u^*(F)(x) = F(u^{\mathrm{op}}(x))$ holds functorially by definition. □

Corollary 9.3.19. *For any functor $f\colon C \to D$, there is a commutative square*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\simeq"'] & D \arrow[d, "\simeq"] \\ (C^{\mathrm{op}})^{\mathrm{op}} \arrow[r, "(f^{\mathrm{op}})^{\mathrm{op}}"'] & (D^{\mathrm{op}})^{\mathrm{op}}, \end{tikzcd}$$

(where the vertical maps are the equivalences given by Corollary 9.3.10).

Proof. This follows straight away from Corollary 9.3.18 and Proposition 9.1.9. □

Lemma 9.3.20. *Let C be a locally small category and $F\colon C \to \mathrm{Grpd}$ a presheaf on C. We consider an object y of C and suppose that the following holds.*

(i) For each object x of C, there is a morphism

$$s_x\colon \operatorname{Hom}_C(x, y) \to F(x)\,.$$

(ii) for each morphism $u\colon x' \to x$ in C, there is a commutative square in C of the form

$$\begin{tikzcd} \operatorname{Hom}_C(x, y) \arrow[r, "s_x"] \arrow[d, "u^*"'] & F(x) \arrow[d, "u^*"] \\ \operatorname{Hom}_C(x', y) \arrow[r, "s_{x'}"] & F(x') \end{tikzcd}$$

Then there is a unique pair (s, α) where s is a morphism of presheaves

$$s\colon \operatorname{Hom}_C(-, y) \to F$$

and $\alpha\colon s(\mathrm{id}_y) = s_y$. Moreover, for each object x of C, there is an isomorphism $s(x) = s_x$, where

$$s(x)\colon \operatorname{Hom}_C(x, y) \to F(x)$$

is the map induced by s. In particular, if moreover each morphism s_x is an isomorphism, then the presheaf F is representable.

Proof. By virtue of Theorem 9.3.15, there is a unique morphism of presheaves

$$s\colon \operatorname{Hom}_C(-, y) \to F$$

such that $s(\mathrm{id}_y) = s_y$. We will show that, for any object x of C, the induced map

$$s(x)\colon \mathrm{Hom}_C(x, y) \to F(x)$$

is isomorphic to s_x. Since isomorphisms can be detected object-wise in any functor category, this will prove the last assertion as well.

Let $p\colon E \to C^{\mathrm{op}}$ be the left fibration corresponding to F. Then s corresponds to a functor

$$\sigma\colon (C_{/y})^{\mathrm{op}} \to E$$

over C^{op} and we want to prove that σ induces an equivalence on fibers over any object x. A term in the fiber of the source is thus a map $u\colon x \to y$ in C, that we see as a map $u^{\mathrm{op}}\colon y \to x$ in C^{op}. We define

$$u^* = (u^{\mathrm{op}})_!\colon E_y \to E_x$$

to be the transport functor along u^{op}. We see through directed univalence that

$$\sigma_x(u) = u^*(\mathrm{id}_y)\,.$$

But condition (ii) means precisely that $u^*(\mathrm{id}_y)$ is the image of u by the functor corresponding to s_x by directed univalence. For u the universal term of the anima $C(x, y)$, this computation in context $C(x, y)$ proves that $s_x = s(x)$ holds, which achieves the proof. □

9.4 Functoriality of the opposite category

We fix a universe of small categories $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$.

We would know like to turn the assignment $C \mapsto C^{\mathrm{op}}$ into a functor $(-)^{\mathrm{op}}\colon \mathrm{Cat} \to \mathrm{Cat}$ at the level of the universe.

Construction 9.4.1. Let $\mathrm{ev}_1\colon \mathrm{Fun}([1], \mathrm{Cat}) \to \mathrm{Cat}$ be the cocartesian fibration exhibiting the assignment

$$S \mapsto \mathrm{Cat}_{/S}$$

as a functor (with values in a larger universe Cat', say). Given any functor between small categories $u\colon S \to T$, the induced functor

$$u_!\colon \mathrm{Cat}_{/S} \to \mathrm{Cat}_{/T}$$

is induced by composition with u, and since Cat has pullbacks, it has a right adjoint

$$u^*\colon \mathrm{Cat}_{/T} \to \mathrm{Cat}_{/S}$$

assigning to each functor $q\colon Y \to T$ the first projection $p\colon S \times_T Y \to S$. Therefore, the functor ev_1 above is a cartesian fibration as well.

Let LFib be the full subcategory of $\mathrm{Fun}([1], \mathrm{Cat})$ whose objects are the left fibrations between small categories. Since the full embedding

$$\mathrm{LFib}(S) \hookrightarrow \mathrm{Cat}_{/S}$$

is compatible with pullbacks, the restriction of the evaluation functor

$$\mathrm{ev}_1\colon \mathrm{LFib} \to \mathrm{Cat}$$

is a cartesian fibration and the embedding $\mathrm{LFib} \hookrightarrow \mathrm{Fun}([1], \mathrm{Cat})$ is a cartesian functor over Cat. On the other hand, since, for any functor between small categories $u \colon S \to T$, the pullback functor

$$u^* \colon \mathrm{LFib}(T) \to \mathrm{LFib}(S)$$

has a left adjoint

$$u_! \colon \mathrm{LFib}(S) \to \mathrm{LFib}(T)$$

the functor $\mathrm{ev}_1 \colon \mathrm{LFib} \to \mathrm{Cat}$ is a cocartesian fibration as well. Let CoRep be the full subcategory of LFib whose objects are those left fibrations between small categories whose domain has an initial object. Since, by virtue of Proposition 9.1.9, each functor

$$u_! \colon \mathrm{LFib}(S) \to \mathrm{LFib}(T)$$

preserves the property of being corepresentable, the restriction

$$\mathrm{ev}_1 \colon \mathrm{CoRep} \to \mathrm{Cat}$$

is a cocartesian fibration and the full embedding

$$\mathrm{CoRep} \hookrightarrow \mathrm{LFib}$$

is a cocartesian functor over Cat.

Proposition 9.4.2. *The assignment $C \mapsto C^{\mathrm{op}}$ can be turned canonically into a functor*

$$(-)^{\mathrm{op}} \colon \mathrm{Cat} \to \mathrm{Cat}$$

together with a natural transformation

$$h_{C^{\mathrm{op}}} \colon C^{\mathrm{op}} \to \mathrm{LFib}(C)$$

inducing the Yoneda embedding of the opposite after composition with the straightening equivalence $\mathrm{LFib}(C) = \mathrm{Fun}(C, \mathrm{Grpd})$.

Proof. The functor $(-)^{\mathrm{op}}$ is defined by the following pullback square.

$$\begin{tikzcd} \mathrm{CoRep} \arrow[r] \arrow[d, "\mathrm{ev}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Cat} \arrow[r, "(-)^{\mathrm{op}}"] & \mathrm{Cat} \end{tikzcd}$$

We may now apply straightening to the natural transformation associated to the full embedding

$$\mathrm{CoRep} \hookrightarrow \mathrm{LFib}$$

interpreted as a cocartesian functor between cocartesian fibrations with small fibers over Cat. $\square$

Definition 9.4.3. The functor

$$(-)^{\mathrm{op}} \colon \mathrm{Cat} \to \mathrm{Cat}, \qquad C \mapsto C^{\mathrm{op}}$$

obtained from Proposition 9.4.2 is called the *opposite category functor*.

Proposition 9.4.4. *The functor* $(-)^{\mathrm{op}}\colon \mathrm{Cat} \to \mathrm{Cat}$ *is an equivalence.*

Proof. It suffices to check that the functor is fully faithful and strongly essentially surjective- The second property follows immediately from Corollary 9.3.10. We will now focus our attention on the property of full faithfulness.

Fix categories C and D. We must show that the induced map

$$\mathrm{Cat}(C, D) \to \mathrm{Cat}(C^{\mathrm{op}}, D^{\mathrm{op}}), \quad f \mapsto f^{\mathrm{op}}$$

is an equivalence. We claim that an inverse is given by sending $g\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ to the composite

$$C \xrightarrow{\sim} (C^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{g^{\mathrm{op}}} (D^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} D.$$

To show that the composites are the respective identities, it will suffice to do so on objects (since $\mathrm{Cat}(C, D)$ and $\mathrm{Cat}(C^{\mathrm{op}}, D^{\mathrm{op}})$ are groupoids). The claim follows easily from Corollary 9.3.19. $\square$

Corollary 9.4.5. *The opposite category functor has the following properties.*

(i) It commutes with limits (in particular with pullbacks). In particular, we have $*^{\mathrm{op}} \xrightarrow{\sim} *$ *and for categories C and D there exists a canonical equivalence*

$$(C \times D)^{\mathrm{op}} \xrightarrow{\sim} C^{\mathrm{op}} \times D^{\mathrm{op}}.$$

(ii) For any categories C and D there exists a canonical equivalence

$$\mathrm{Fun}(C, D)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}}).$$

Proof. Equivalences of categories commute with limits and colimits, and this explains assertion (i). For any categories A, C and D, the evaluation functor

$$\mathrm{ev}\colon C \times \mathrm{Fun}(C, D) \to D$$

induces thanks to the compatibility of $(-)^{\mathrm{op}}$ with cartesian products a functor

$$\mathrm{ev}^{\mathrm{op}}\colon C^{\mathrm{op}} \times \mathrm{Fun}(C, D)^{\mathrm{op}} \to D^{\mathrm{op}}$$

hence a comparison functor

$$\gamma\colon \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}}) \to \mathrm{Fun}(C, D)^{\mathrm{op}}.$$

On the other hand, there are canonical equivalences:

$$\begin{aligned}
\mathrm{Map}(A\,\mathrm{Fun}(C, D)) &\xrightarrow{\sim} \mathrm{Map}(A \times C, D)\\
&\xrightarrow{\sim} \mathrm{Map}((A \times C)^{\mathrm{op}}, D^{\mathrm{op}})\\
&\xrightarrow{\sim} \mathrm{Map}(A^{\mathrm{op}} \times C^{\mathrm{op}}, D^{\mathrm{op}})\\
&\xrightarrow{\sim} \mathrm{Map}(A^{\mathrm{op}}, \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}}))\\
&\xrightarrow{\sim} \mathrm{Map}((A^{\mathrm{op}})^{\mathrm{op}}, \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}})^{\mathrm{op}})\\
&\xrightarrow{\sim} \mathrm{Map}(A, \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}})^{\mathrm{op}}).
\end{aligned}$$

where the final equivalence is induced by the equivalence $A \xrightarrow{\sim} (A^{\mathrm{op}})^{\mathrm{op}}$. Using Corollary 9.3.19, we see that the equivalence

$$\mathrm{Map}(A\,\mathrm{Fun}(C,D)) \xrightarrow{\sim} \mathrm{Map}(A,\mathrm{Fun}(C^{\mathrm{op}},D^{\mathrm{op}})^{\mathrm{op}})$$

obtained above as composition of equivalences is isomorphic to the map defined by the rule $u \mapsto \gamma\circ u$. Therefore, the comparison functor γ is an equivalence. □

Lemma 9.4.6. *There is a unique equivalence* $[1] \xrightarrow{\sim} [1]^{\mathrm{op}}$ *exchanging* 0 *and* 1.

Proof. Since $[1](0,1) = [1]^{\mathrm{op}}(1,0)$, there is a canonical morphism from 1 to 0 in $[1]^{\mathrm{op}}$, i.e. a functor

$$u\colon [1] \to [1]^{\mathrm{op}}$$

inducing the map

$$u^{\simeq}\colon \{0,1\} \xrightarrow{\sim} \{0,1\}$$

defined by the rule $0 \mapsto 1$ and $1 \mapsto 0$. The functor u must be fully faithful because it induces the endomorphism of the empty anima or of the terminal anima

$$[1](x,y) \to [1]^{\mathrm{op}}(u(x),u(y))$$

for all $x,y\colon [1]^{\simeq}$. Being strongly essentially surjective and fully faithful, the functor is thus an equivalence. This is clearly the only one possible (in any context), so that the anima of all equivalences from $[1]$ to $[1]^{\simeq}$ is contractible. □

Remark 9.4.7. The preceding lemma can also be proved by a functoriality argument. To construct an equivalence $[1]^{\mathrm{op}} \xrightarrow{\sim} [1]$, it will suffice to show that for every category C there is an equivalence $\mathrm{Map}([1],C) \simeq \mathrm{Map}([1]^{\mathrm{op}},C)$ which is natural in C. But we have

$$\mathrm{Map}([1],C) \simeq \sum_{(x,y):C^{\simeq}\times C^{\simeq}} C(x,y) \simeq \sum_{(x,y)\in(x,y):C^{\simeq}\times C^{\simeq}} C(y,x) \simeq \mathrm{Map}([1],C^{\mathrm{op}}) \simeq \mathrm{Map}([1]^{\mathrm{op}},C),$$

where the middle equivalence is given by the flip map $C^{\simeq}\times C^{\simeq} \to C^{\simeq}\times C^{\simeq}$ sending (x,y) to (y,x). Given a functor $f\colon C \to C'$, the above equivalences fit in a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r, "\simeq"] \arrow[d, "f\circ-"'] & \mathrm{Map}([1]^{\mathrm{op}},C) \arrow[d, "f\circ-"] \\
\mathrm{Map}([1],C') \arrow[r, "\simeq"] & \mathrm{Map}([1]^{\mathrm{op}},C').
\end{tikzcd}$$

We may now pick a map $f \in \mathrm{Map}([1],[1]^{\mathrm{op}})$ corresponding to the identity of $[1]^{\mathrm{op}}$ under the equivalence to $\mathrm{Map}([1]^{\mathrm{op}},[1]^{\mathrm{op}})$, and similarly we may pick its inverse $g \in \mathrm{Map}([1]^{\mathrm{op}},[1])$ as the map corresponding to the identity of $[1]$ under the equivalence to $\mathrm{Map}([1],[1])$. It follows from the above naturality that f is inverse to g. Tracing through the above equivalences shows that f exchanges the objects 0 and 1.

Proposition 9.4.8. *There is a preferred equivalence* $\mathrm{Fun}([1],C)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Fun}([1],C^{\mathrm{op}})$ *which fits functorially in a commutative diagram*

$$\begin{tikzcd}
\mathrm{Fun}([1],C)^{\mathrm{op}} \arrow[rr, "\simeq"] \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)^{\mathrm{op}}"'] & & \mathrm{Fun}([1],C)^{\mathrm{op}} \arrow[d, "(\mathrm{ev}_0^{\mathrm{op}},\mathrm{ev}_1^{\mathrm{op}})"] \\
(C\times C)^{\mathrm{op}} \arrow[r, "\simeq"] & C^{\mathrm{op}}\times C^{\mathrm{op}} \arrow[r, "\mathrm{flip}"] & C^{\mathrm{op}}\times C^{\mathrm{op}}.
\end{tikzcd}$$

Proof. Using Lemma 9.4.6 together with assertion (ii) of Corollary 9.4.5, we have

$$\mathrm{Fun}([1], C)^{\mathrm{op}} = \mathrm{Fun}([1], C)^{\mathrm{op}} .$$

This equivalence obviously fit in a commutative square as stated above. □

Corollary 9.4.9. *Let $f: C \to D$ be a functor.*

- *If f admits a right (left) adjoint section, then f^{op} admits a left (right) adjoint secion.*
- *If f is a cocartesian (resp. cartesian) fibration, then $f^{\mathrm{op}}: C^{\mathrm{op}} \to D^{\mathrm{op}}$ is a cartesian (resp. cocartesian) fibration.*
- *If f is a left (resp. right) fibration, then $f^{\mathrm{op}}: C^{\mathrm{op}} \to D^{\mathrm{op}}$ is a right (resp. left) fibration.*

Proof. This follows straight away from Corollary 9.4.5 and Proposition 9.4.8. □

Definition 9.4.10. Let S be a small category, corresponding to an object $s: * \to \mathrm{Cat}$. We let

$$\mathrm{Cart}(S) \subseteq \mathrm{Cat}_{/S} := \mathrm{Cat}_{/s}$$

denote the (non-full) subcategory of cartesian fibrations over C and cartesian functors over C (defined as an appropriate variation of Construction 7.7.3 and of Construction 7.7.4). We denote by

$$\mathrm{RFib}(S) \subseteq \mathrm{Cart}(S)$$

the full subcategory spanned by the right fibrations over S.

Lemma 9.4.11. *The equivalence* $(-)^{\mathrm{op}}: \mathrm{Cat} \xrightarrow{\sim} \mathrm{Cat}$ *induces an equivalence*

$$\mathrm{Cat}_{/S} \xrightarrow{\sim} \mathrm{Cat}_{/S^{\mathrm{op}}} .$$

This equivalence restricts to equivalences of the form

$$\mathrm{RFib}(S) \xrightarrow{\sim} \mathrm{LFib}(S^{\mathrm{op}}) \qquad \textit{and} \qquad \mathrm{Cart}(S) \xrightarrow{\sim} \mathrm{CoCart}(S^{\mathrm{op}}) .$$

Proof. The first assertion is obvious and is only stated for the record. The second follows right away from the first through Corollary 9.4.9. □

Theorem 9.4.12 (Straightening/unstraightening)**.** *For every small category S there are canonical equivalences*

$$\mathrm{Un}\colon \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Cat}) \xrightarrow{\sim} \mathrm{Cart}(S) \qquad \textit{and} \qquad \mathrm{Un}\colon \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Grpd}) \xrightarrow{\sim} \mathrm{RFib}(S).$$

Proof. In order to get these equivalences, simply combine Theorem 7.7.8 for S^{op} with Lemma 9.4.11. □

Corollary 9.4.13. *For any small category S, there is a canonical equivalence of the form*

$$\mathrm{Fun}(S, \mathrm{Cat}^{\mathrm{op}})^{\simeq} \xrightarrow{\sim} \mathrm{Cart}(S)^{\simeq} .$$

Proof. The functor $(-)^{\mathrm{op}}$ is an equivalence and thus has a right adjoint that we will denote by

$$A \mapsto A'$$

on objects. We will see later in the book that this adjoint is $(-)^{\mathrm{op}}$ itself (Proposition 9.4.18). However, we can see that $A' \simeq A^{\mathrm{op}}$ holds object-wise. This follows imediately from the explcit equivalence $A \xrightarrow{\sim} (A^{\mathrm{op}})^{\mathrm{op}}$ and from the commutative squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\simeq"'] & B \arrow[d, "\simeq"] \\ (A^{\mathrm{op}})^{\mathrm{op}} \arrow[r, "(f^{\mathrm{op}})^{\mathrm{op}}"'] & (B^{\mathrm{op}})^{\mathrm{op}}, \end{tikzcd}$$

associated to each functor $f\colon A \to B$ by Corollary 9.3.19.

We obtain the following chain of equivalences.

$$\begin{aligned} \mathrm{Fun}(S, \mathrm{Cat}^{\mathrm{op}})^{\simeq} &\xrightarrow{\sim} \mathrm{Fun}(S, \mathrm{Cat}') \\ &\xrightarrow{\sim} \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Cat})^{\simeq} \\ &\xrightarrow{\sim} \mathrm{Cart}(S)^{\simeq} \end{aligned}$$

using the fact that $\mathrm{Fun}(C, D)^{\simeq}$ may be identified with $\mathrm{Map}(C, D)$. □

Corollary 9.4.14. *Let $p\colon C \to S$ be a cartesian fibration. We assume given a subcategory $i\colon W \hookrightarrow C$ wuch that $p \circ i$ is a cartesian fibration and i is a cartesian functor over S. Then the functor $\eta_C\colon C \to L_S(C, W)$ exhibits $L_S(C, W)$ as the localization of C by* $\mathrm{Map}([1], W)$.

Proof. This follows from Theorem 7.6.20 for $p^{\mathrm{op}}\colon C^{\mathrm{op}} \to S^{\mathrm{op}}$ and applying the functor $(-)^{\mathrm{op}}$. □

Construction 9.4.15. Let RFib be the full subcategory of $\mathrm{Fun}([1], \mathrm{Cat})$ whose objects are the right fibrations between small categories. The functor

$$\mathrm{ev}_1\colon\ \mathrm{Fun}([1], \mathrm{Cat}) \to \mathrm{Cat}$$

is a cocartesian fibration. Since, for any functor between small synthetic categories $u\colon A \to B$, the functor

$$u_!\colon \mathrm{Cat}_{/A} \to \mathrm{Cat}_{/B}$$

induced by composition with u has a right adjoint (defined by pullbacks along u), the functor ev_1 is a cartesian fibration as well. The property of right fibrations to be stable under pullbacks in Cat precisely means that the restriction of ev_1 to RFib remains a cartesian fibration:

$$\mathrm{ev}_1\colon\ \mathrm{RFib} \to \mathrm{Cat}\,.$$

For any small category C

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathrm{Cat}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "c"'] & \mathrm{Cat} \end{tikzcd}$$

the fiber of the latter cartesian fibration at c is precisely $\mathrm{RFib}(C)$. But $\mathrm{ev}_1\colon \mathrm{RFib} \to \mathrm{Cat}$ is a cocartesian fibration as well, since the pullback functors

$$u^*\colon \mathrm{RFib}(B) \to \mathrm{RFib}(A)$$

have a left adjoint $u_!$ for each functor between small categories $u\colon A \to B$.

We define Rep as the full subcategory of RFib whose objects are the fibrations between small categories whose domain has a terminal object. Corollary 9.3.18 implies that taking the codomain of these right fibrations defines a functor

$$\mathrm{ev}_1\colon \mathrm{Rep} \to \mathrm{Cat}$$

that is a cocartesian fibration, so that the embedding of Rep into RFib defines a cocartesian functor over Cat.

Lemma 9.4.16. *The functor* $\mathrm{ev}_1\colon \mathrm{Rep} \to \mathrm{Cat}$ *is a directed univalent cocartesian fibration. More precisely, there is a canonical pullback square of the form*

$$\begin{tikzcd} \mathrm{Rep} \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Cat} \arrow[r, "((-)^{\mathrm{op}})^{\mathrm{op}}"] & \mathrm{Cat} \end{tikzcd}$$

Proof. By definition, there is a pullback square bellow.

$$\begin{tikzcd} \mathrm{CoRep} \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Cat} \arrow[r, "(-)^{\mathrm{op}}"] & \mathrm{Cat} \end{tikzcd}$$

There is also an equivalence $\mathrm{RFib}(A) \xrightarrow{\sim} \mathrm{LFib}(A^{\mathrm{op}})$ functorially in A given by Lemma 9.4.11. This yields a pullback square of the form

$$\begin{tikzcd} \mathrm{RFib} \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{LFib} \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Cat} \arrow[r, "(-)^{\mathrm{op}}"] & \mathrm{Cat}\,. \end{tikzcd}$$

The last assertion of theorem thus follows of the stability of pullback square by (horizontal) composition. The first assertion follows from the fact that any cocartesian fibration equivalent to a directed univalent one is directed univalent. □

Lemma 9.4.17. *Recall that there is an ambient universe of large categories* Cat'. *Let us consider an equivalence of the form* $\varphi'\colon \mathrm{Cat}' \to \mathrm{Cat}'$ *that we suppose to restrict to an equivalence of the form*

$$\varphi\colon \mathrm{Cat} \to \mathrm{Cat}\,.$$

We also assume that there is an equivalence $[1] = \varphi([1])$ *together with commutative triangles of the form*

$$\begin{tikzcd} \mathrm{Map}([1], C) \arrow[rr, "\simeq"] \arrow[dr, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & & \mathrm{Map}([1], \varphi(C)) \arrow[dl, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ & C \times C & \end{tikzcd}$$

where the horizontal map is the one induced by φ and by the identification $[1] = \varphi([1])$. *Then there is an isomorphism of functors* $\mathrm{id}_{\mathrm{Cat}} \xrightarrow{\sim} \varphi$.

Proof. We form the following pullback square.

$$\begin{tikzcd}
\varphi^*\mathrm{Cat}_\bullet \arrow[r, "\simeq"] \arrow[d, "\varphi^*\pi_{\mathrm{univ}}"'] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Cat} \arrow[r, "\varphi"] & \mathrm{Cat}
\end{tikzcd}$$

We can repeat all the constructions of straightening/unstraightening and of the opposite category functor with respect to $\varphi^*\pi_{\mathrm{univ}}$ instead of π_{univ}. Let $\psi \colon \mathrm{Cat} \to \mathrm{Cat}$ be the opposite category functor associated to $\varphi^*\pi_{\mathrm{univ}}$. The preceding theorem also holds replacing π_{univ} by $\varphi^*\pi_{\mathrm{univ}}$. However, our assumption on the compatibility of φ with the interval $[1]$ means that φ preserves the properties of being a right (left) fibration and of having a terminal object. In other words, the cocartesian fibration $\mathrm{ev}_1 \colon \mathrm{Rep} \to \mathrm{Cat}$ is independent of φ, and we get from the appropriate version of Lemma 9.4.16 a pullback square of the form:

$$\begin{tikzcd}
\mathrm{Rep} \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_1"'] & \varphi^*\mathrm{Cat}_\bullet \arrow[d, "\varphi^*\pi_{\mathrm{univ}}"] \\
\mathrm{Cat} \arrow[r, "\psi\circ\psi"] & \mathrm{Cat}
\end{tikzcd}$$

But, since π_{univ} is directed univalent, this means that the identity of Rep determines an isomorphism of functors of the form

$$\varphi \circ \psi \circ \psi \xrightarrow{\sim} ((-)^{\mathrm{op}})^{\mathrm{op}} .$$

But we have pullback squares of the form

$$\begin{tikzcd}
\mathrm{CoRep} \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_1"'] & \varphi^*\mathrm{Cat}_\bullet \arrow[r] \arrow[d, "\varphi^*\pi_{\mathrm{univ}}"] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Cat} \arrow[r, "\psi"] & \mathrm{Cat} \arrow[r, "\varphi"] & \mathrm{Cat}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{CoRep} \arrow[r] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Cat} \arrow[r, "(-)^{\mathrm{op}}"] & \mathrm{Cat}
\end{tikzcd}$$

and by univalence of π_{univ}, the identity of CoRep determines an isomorphism of functors

$$(-)^{\mathrm{op}} \xrightarrow{\sim} \varphi \circ \psi .$$

We obtain that

$$(-)^{\mathrm{op}} \circ \psi \xrightarrow{\sim} ((-)^{\mathrm{op}})^{\mathrm{op}} .$$

Therefore, we must have

$$\psi = (-)^{\mathrm{op}} .$$

This implies in turns that $\varphi \circ ((-)^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} ((-)^{\mathrm{op}})^{\mathrm{op}}$, hence $\varphi = \mathrm{id}_{\mathrm{Cat}}$. □

Proposition 9.4.18. *There is a functorial equivalence $C \xrightarrow{\sim} (C^{\mathrm{op}})^{\mathrm{op}}$ inducing a pullback square of the form*

$$\begin{tikzcd}
\mathrm{Rep} \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Cat} \arrow[r, "\mathrm{id}_{\mathrm{Cat}}"] & \mathrm{Cat} .
\end{tikzcd}$$

Proof. It follows from Proposition 9.4.8 that me may apply Lemma 9.4.17 to $\varphi = ((-)^{\mathrm{op}})^{\mathrm{op}}$. $\square$

Remark 9.4.19. The previous proposition allows a fairly explicit description of the objects and morphisms of $\mathrm{Cat}_\bullet$, using the equivalence with Rep. An object of Rep is a right fibration $p\colon X \to A$ together with a terminal object x of X. But then the induced functor $X_{/x} \to A_{/p(x)}$ preserves terminal objects and is a morphism of right fibrations over A. Therefore, it is an equivalence. Since the projection $X_{/x} \to X$ is an equivalence as well, this means that, up to equivalence, p is totally determined by the pair (A, a) consisting of a small synthetic category A and of an object a of A. A morphsim in Rep is by definition a commutative square in Cat of the form

$$\begin{tikzcd} A_{/a} \arrow[r, "u"] \arrow[d, "p"'] & B_{/b} \arrow[d, "q"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

where a is an object of A, b an object of B, and p and q are the canonical projections. The functor u is totally determined by the functor

$$\tilde{u}\colon A_{/a} \to A_{/b} = A \times_B B_{/b}$$

over A. Since q is a right fibration, this means that this functor is totally determined by the image of the terminal object (a, id_a) in the fiber product $A_{/b}$. This image must be a pair of the form (a, v) where $v\colon f(a) \to b$ is a morphism in B. In other words, a morphism in $\mathrm{Cat}_\bullet$ is a pair of the form

$$(f, v)\colon (A, a) \to (B, b)$$

where $f\colon A \to B$ is a functor between small categories, a is an object of A, b is an object of B, and $v\colon f(a) \to b$ is a morphism of B.

This allows to describe objects and morphisms in any Grothendieck construction (i.e. in the total category associated to a functor with values in Cat). Indeed, let $F\colon A \to \mathrm{Cat}$ be a functor from a category A to the category of small categories. We can then form the following pullback square.

$$\begin{tikzcd} \int_{a\colon A} F(a) \arrow[r] \arrow[d, "p_F"'] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ A \arrow[r] & \mathrm{Cat} \end{tikzcd}$$

Any object of $\int_{a\colon A} F(a)$ will have an image in A. If we also write $F(a)$ the category corresponding to $F(a)\colon \mathrm{Cat}^{\simeq}$, using the pullback square

$$\begin{tikzcd} F(a) \arrow[r] \arrow[d] & \int_{a\colon A} F(a) \arrow[d, "p_F"] \\ {[0]} \arrow[r, "a"] & A \end{tikzcd}$$

We see that an object of $\int_{a\colon A} F(a)$ is a pair (a, x) where a is an object of A and x an object of $F(x)$. A morphism in $\int_{a\colon A} F(a)$ is a pair of the form

$$(u, v)\colon (a, x) \to (b, y)$$

where $u\colon a \to b$ is a morphism in A and $v\colon F(u)(x) \to y$ is a morphism in $F(b)$ (since one can interpret the pair $(F(u), v)$ as a morphism in $\mathrm{Cat}_\bullet$).

9.5 Approximation of presheaves by representable ones

Let $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ a directed univalent cocartesian fibrations defining a universe of small categories with corresponding universe of small groupoids Grpd.

Remark 9.5.1. Recall that, for any small category C, the presheaf category $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ admits all small colimits: given a small category I, the colimit functor

$$\mathrm{colim}_I\colon\ \mathrm{Fun}(I, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

may be describes as follows: given the projection $\mathrm{pr}_2\colon I \times C^{\mathrm{op}} \to C^{\mathrm{op}}$ it is given by

$$\mathrm{Fun}(I, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})) \simeq \mathrm{Fun}(I \times C^{\mathrm{op}}, \mathrm{Grpd}) \xrightarrow{(\mathrm{pr}_2)_!} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}).$$

A fundamental computation of colimits in a presheaf category is the following one.

Theorem 9.5.2. *Let C be a small category. Then the colimit of the Yoneda embedding*

$$h_C\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

is the terminal object.

Proof. Let e be the terminal object of Grpd. We abuse notation by denoting by e also the constant presheaf on C with value e. We want to prove that the map $\mathrm{colim}_C\, h_C \to e$ is an isomorphism in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$.

Choose a universe Cat′ that contains Grpd both as an object and as a full subcategory, and write Grpd′ for the groupoids in Cat′. Write $i\colon \mathrm{Grpd} \hookrightarrow \mathrm{Grpd}'$ for the resulting inclusion. Due to the explicit construction of colimits in Grpd (in terms of factorizations of functors into initial functors and left fibrations) it follows that the inclusion i preserves colimits. In particular, the top square in the following diagram commutes up to preferred natural isomorphism:

$$\begin{tikzcd}
\mathrm{Fun}(C, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})) \arrow[r, "\mathrm{colim}_C"] \arrow[d, hook, "i_*"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, hook, "i_*"] \\
\mathrm{Fun}(C, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')) \arrow[r, "\mathrm{colim}_C"] \arrow[d, "(h_C)_!"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}') \\
\mathrm{Fun}(\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}), \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')). \arrow[ur, "\mathrm{colim}\,\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd}) = \mathrm{ev}_e"'] &
\end{tikzcd}$$

We thus see that

$$i\, \mathrm{colim}_C\, h_C = \mathrm{colim}_C\, i h_C = \mathrm{colim}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})}\, (h_C)_!(i h_C).$$

But, by virtue of Proposition 8.4.4, taking colimits indexed by $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ simply consists in evaluating at the terminal object. To conclude the proof, it suffices to prove that

$$(h_C)_!(i h_C) = i_*$$

holds as functors $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$. To see this, we observe that the previous diagram is equivalent to the diagram

$$\begin{tikzcd}
\mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{Grpd}) \arrow[r, "(\mathrm{pr}_1)_!"] \arrow[d, "i_*"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, "i_*"] \\
\mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{Grpd}') \arrow[r, "(\mathrm{pr}_1)_!"] \arrow[d, "(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}') \\
\mathrm{Fun}(C^{\mathrm{op}} \times \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}), \mathrm{Grpd}'). \arrow[ur, dashed, "(\mathrm{pr}_1)_!"'] &
\end{tikzcd}$$

and we recall the identification

$$(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!(i\,\mathrm{Hom}_C) = i\mathrm{ev}$$

used in the proof of the Yoneda Lemma (Theorem 9.3.15). □

Corollary 9.5.3. *Let C be a small category and let $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ be a functor corresponding to a right fibration p_F:*

$$\begin{tikzcd}
C_{/F} \arrow[r] \arrow[d, "p_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet^{\mathrm{op}} \arrow[d, "\pi_{\mathrm{univ}}^{\mathrm{op}}"] \\
C \arrow[r, "F^{\mathrm{op}}"'] & \mathrm{Grpd}^{\mathrm{op}}.
\end{tikzcd}$$

Then F is canonically a colimit of representable presheaves: there is a preferred isomorphism

$$\mathrm{colim}_{(x,s)\in C_{/F}} h_C(x) \xrightarrow{\sim} F$$

in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$.

Proof. The functor $p_F\colon C_{/F} \to C$ induces an adjunction

$$(p_F)_!\colon\ \mathrm{Fun}((C_{/F})^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})\ : p_F^*.$$

By Theorem 9.5.2 the colimit $\mathrm{colim}_{(x,s)\in C_{/F}} h_{C/F}(x,s)$ is a terminal object $e_{C_{/F}}$ in $\mathrm{Fun}((C_{/F})^{\mathrm{op}}, \mathrm{Grpd})$, hence it follows that

$$\mathrm{colim}_{(x,s)\in C_{/F}} h_C(x) = \mathrm{colim}(h_C \circ p_F) = \mathrm{colim}((p_F)_! h_{C_{/F}}) \xrightarrow{\sim} (p_F)_!(\mathrm{colim}\, h_{C_{/F}}) \xrightarrow{\sim} (p_F)_!(e_{C_{/F}}).$$

So it remains to show that there exists a canonical equivalence $(p_F)_!(e_{C_{/F}}) \simeq F$. To see this, consider the diagram

$$\begin{tikzcd}
C_{/F} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet^{\mathrm{op}} \arrow[d, "\pi_{\mathrm{univ}}^{\mathrm{op}}"] \\
C_{/F} \arrow[r] & * \arrow[r, "e"] & \mathrm{Grpd}^{\mathrm{op}}.
\end{tikzcd}$$

The bottom functor is the one corresponding to $e_{C_{/F}}\colon C_{/F}^{\mathrm{op}} \to \mathrm{Grpd}$, hence its associate right fibration is simply the identity functor on $C_{/F}$. The left Kan extension is obtained by factorizing its composite with p_F into a terminal functor followed by a right fibration, but this is clear since p_F is already a right fibration:

$$\begin{tikzcd} C_{/F} \arrow[r, equal] \arrow[d, equal] & C_{/F} \arrow[d, "p_F"] \\ C_{/F} \arrow[r] & C. \end{tikzcd}$$

Hence the functor $(p_F)_!(e_{C_{/F}})\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ is the functor classifying the right fibration $p_F\colon C_{/F} \to C$, which is F. □

Remark 9.5.4. The reason for the notation $C_{/F}$ in the previous statement is that the category $C_{/F}$ fits into a pullback square of the following form:

$$\begin{tikzcd} C_{/F} \arrow[r] \arrow[d, "p_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})_{/F} \arrow[d] \\ C \arrow[r, "h_C"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}). \end{tikzcd}$$

So objects in $C_{/F}$ are really pairs (x, s) where x is an object of C and $s\colon h_C(x) \to F$ is a morphism in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$.

Remark 9.5.5. Let I, J and C be categories, and let $F\colon I \to C$ be a functor. Assume that $\lim_I F$ exists in C. Then also $\lim_{i\in I} \mathrm{const}_{F(i)}$ exists in $\mathrm{Fun}(J, C)$ and is given by $\mathrm{const}_{\lim_I F}$. If we let $\nu\colon \mathrm{const}_{\lim_I F} \to F$ denote the counit exhibiting $\lim_I F$ as a limit in $\mathrm{Fun}(J, C)$, then we see that the pair $(\lim_I F, \eta)$ is a terminal object of $C_{/F}$:

$$\begin{tikzcd} C_{/F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(I, C)_{/F} \arrow[d] \\ C \arrow[r, "\mathrm{const}"'] & \mathrm{Fun}(I, C). \end{tikzcd}$$

Consider a small category A and consider a functor $F\colon A^{\mathrm{op}} \to \mathrm{Grpd}$, corresponding to a cocartesian fibration $E \to A^{\mathrm{op}}$. We write $A_{/F} := E^{\mathrm{op}}$, so that we may write E as $(A_{/F})^{\mathrm{op}}$:

$$\begin{tikzcd} (A_{/F})^{\mathrm{op}} \arrow[r] \arrow[d, "p^{\mathrm{op}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ A^{\mathrm{op}} \arrow[r, "F"] & \mathrm{Grpd}. \end{tikzcd}$$

The functor $(p^{\mathrm{op}})_!\colon \mathrm{Fun}((A_{/F})^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})$ sends the terminal functor e to F, hence we have

$$\mathrm{colim}_{A^{\mathrm{op}}} F = \mathrm{colim}_{A^{\mathrm{op}}} (p^{\mathrm{op}})_!(e) \simeq \mathrm{colim}_{(A_{/F})^{\mathrm{op}}} e.$$

To compute this, we must factor the map $(A_{/F})^{\mathrm{op}} \to e$ into an initial functor and a left fibration:

$$\begin{tikzcd} (A_{/F})^{\mathrm{op}} \arrow[r, "\mathrm{initial}"] \arrow[d, equal] & \Pi_\infty((A_{/F})^{\mathrm{op}}) \arrow[d, "\text{left fib}"] \\ (A_{/F})^{\mathrm{op}} \arrow[r] & e. \end{tikzcd}$$

So we see that $\mathrm{colim}_{A^{\mathrm{op}}} F \simeq \Pi_\infty((A_{/F})^{\mathrm{op}}) \simeq \Pi_\infty(A_{/F})^{\mathrm{op}} \simeq \Pi_\infty(A_{/F})$.

Corollary 9.5.6. *Let $F\colon A^{\mathrm{op}} \to \mathrm{Grpd}$ be a representable functor. Then $\operatorname{colim}_{A^{\mathrm{op}}} F$ is contractible.*

Proof. We have $\operatorname{colim}_{A^{\mathrm{op}}} F = \Pi_\infty(A_{/F})$. But since F is representable, the category $A_{/F}$ admits a terminal object, meaning that there is a terminal functor $* \to A_{/F}$. By Example 8.3.4 this implies that $\Pi_\infty(A_{/F})$ is contractible. □

Proposition 9.5.7. *Let A be a small category. The full embedding*

$$\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{RFib}(A) \hookrightarrow \mathrm{Cat}_{/A}$$

has a left adjoint L sending each functor $u\colon X \to A$ to the colimit of the functor

$$X \xrightarrow{u} A \xrightarrow{h_A} \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})\,.$$

Proof. Let $F\colon A^{\mathrm{op}} \to \mathrm{Grpd}$ be a presheaf with corresponding right fibration $p_F\colon A_{/F} \to A$. If e denotes the constant presheaf on X with value the terminal object of Grpd, we see that

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Cat}_{/A}}((X,u),(A_{/F},p_F)) &= \mathrm{Hom}_{\mathrm{Cat}_{/X}}((X,\mathrm{id}_X),(X\times_A A_{/F},\mathrm{pr}_1))\\
&= \mathrm{Hom}_{\mathrm{RFib}(A)}((X,\mathrm{id}_X),(X\times_A A_{/F},\mathrm{pr}_1))\\
&= \mathrm{Hom}(e, u^*(F))\\
&= \mathrm{Hom}(u_!(e), F)\,.
\end{aligned}$$

It suffices to check that $u_!(e)$ is the colimit of the composition of the Yoneda embedding h_A with the functor u. By virtue of Theorem 9.5.2, the colimit of the Yoneda embedding of X is e. Since $u_!$ commutes with colimits, and since $u_! \circ h_X = h_A \circ u$, this proves that

$$\mathrm{Hom}_{\mathrm{Cat}_{/A}}((X,u),(A_{/F},p_F)) = \mathrm{Hom}(\operatorname{colim}_{x\colon X} h_A(u(x))), F)$$

holds functorially in F. We conclude by applying Theorem 9.3.17 to the embedding of the category $\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})^{\mathrm{op}}$ into $(\mathrm{Cat}_{/A})^{\mathrm{op}}$. □

Remark 9.5.8. The left adjoint of the proposition above is natural with respect to functors $u_!$. In other words, for any functor between small categories

$$u\colon A \to B$$

there is a canonical commutative square

$$\begin{tikzcd}
\mathrm{Cat}_{/A} \arrow[r, "L"] \arrow[d, "u_!"'] & \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, "u_!"] \\
\mathrm{Cat}_{/B} \arrow[r, "L"] & \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})
\end{tikzcd}$$

corresponding to the commutative square of right adjoints bellow.

$$\begin{tikzcd}
\mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, hook] \arrow[d, "u^*"'] & \mathrm{Cat}_{/B} \arrow[d, "u^*"] \\
\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, hook] & \mathrm{Cat}_{/A}
\end{tikzcd}$$

Proposition 9.5.9. *Let $u\colon A \to B$ be a right fibration between small categories A corresponding to a presheaf $F\colon B^{\mathrm{op}} \to \mathrm{Grpd}$.*

$$\begin{tikzcd} A^{\mathrm{op}} \arrow[r] \arrow[d, "u^{\mathrm{op}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ B^{\mathrm{op}} \arrow[r, "F"] & \mathrm{Grpd} \end{tikzcd}$$

Then the canonical commutative square

$$\begin{tikzcd} A \arrow[r, "h_A"] \arrow[d, "u"'] & \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, "u_!"] \\ B \arrow[r, "h_B"] & \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}) \end{tikzcd}$$

is a pullback square in which all vertical maps are right fibrations with small fibers and the functor $u_!$ induces an equivalence

$$\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \xrightarrow{\sim} \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})_{/F}$$

over $\mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})$.

Proof. We observe that the functor

$$u_!\colon \mathrm{Cat}_{/A} \to \mathrm{Cat}_{/B}$$

sends the full subcategory $\mathrm{RFib}(A)$ to $\mathrm{RFib}(B)$. Since The composition

$$\mathrm{RFib}(B) \hookrightarrow \mathrm{Cat}_{/B} \xrightarrow{L} \mathrm{RFib}(B)$$

is the identity, this means that there is the canonical commutative diagram bellow.

$$\begin{tikzcd} \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "\sim"] \arrow[d, "u_!"'] & \mathrm{RFib}(A) \arrow[r, hook] \arrow[d, "u_!"'] & \mathrm{Cat}_{/A} \arrow[d, "u_!"] \\ \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "\sim"] & \mathrm{RFib}(B) \arrow[r, hook] & \mathrm{Cat}_{/B} \end{tikzcd}$$

The square on the right hand side is a pullback square: since the two horizontal maps are full embeddings, it suffices to check this on objects, which follows right away from Proposition 5.2.16. In particular, since

$$u_!\colon \mathrm{Cat}_{/A} \to \mathrm{Cat}_{/B}$$

is a right fibration, so is the functor

$$u_!\colon\ \mathrm{RFib}(A) \to \mathrm{RFib}(B)\,.$$

It is now clear that the equivalence

$$\mathrm{Cat}_{/A} \xrightarrow{\sim} (\mathrm{Cat}_{/B})_{/(A,u)}$$

canonically induced by $u_!$ restricts to an equivalence

$$\mathrm{RFib}(A) \xrightarrow{\sim} \mathrm{RFib}(B)_{/(A,u)}\,.$$

Therefore, since the functor

$$u_!\colon \operatorname{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \to \operatorname{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})$$

sends the terminal object to F, it induces an equivalence

$$\operatorname{Fun}(A, \mathrm{Grpd}) \xrightarrow{\sim} \operatorname{Fun}(B, \mathrm{Grpd})_{/F}\ .$$

over $\operatorname{Fun}(B, \mathrm{Grpd})$. □

9.6 Locally small categories

We will now discuss various consequences of the above results for locally small categories. Recall that C is locally small if, for all objects x and y the groupoid $\operatorname{Hom}_C(x, y)$ is small. Equivalently, C is locally small if and only if the functor $(s, t)\colon \operatorname{Tw}(C) \to C^{\mathrm{op}} \times C$ has small fibers.

Example 9.6.1. If C is small, then it is locally small.

Example 9.6.2. If C is small, then the category $\operatorname{Fun}(C, \mathrm{Grpd})$ is locally small: given functors $F_0, F_1\colon C \to \mathrm{Grpd}$ corresponding to left fibrations $p_0\colon X_0 \to C$ and $p_1\colon X_1 \to C$, we have an equivalence

$$\operatorname{Fun}(C, \mathrm{Grpd})(F_0, F_1) \simeq \operatorname{Map}_C(X_0, X_1),$$

and the right-hand side is small.

Theorem 9.6.3. *Let C be a locally small category and let x be an object of C. Then the functor*

$$\operatorname{Hom}_C(x, -)\colon C \to \mathrm{Grpd}$$

preserves limits.

Proof. We first prove the case where C is actually a small category. Let $F\colon I \to C$ be a functor such that $\lim_I F$ exists: we have a natural equivalence

$$\operatorname{Hom}_C(x, \lim_I F) \simeq \operatorname{Hom}_{\operatorname{Fun}(I,C)}(\mathrm{const}_x, F),$$

functorial in x. We have to show that the canonical map $h_C \lim_I F \to \lim_I h_C F$ is an isomorphism in $\operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. For this, we apply the Yoneda lemma: for every functor $\Phi\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ we will show that the induced map

$$\operatorname{Hom}(\Phi, h_C \lim_I F) \to \operatorname{Hom}(\Phi, \lim_I h_C F) = \operatorname{Hom}(\Phi_I, h_C F)$$

is an equivalence. Let us first prove this in the case where $\Phi = h_C(x)$ for some object $x \in C$. Then we have a chain of natural equivalences

$$\begin{aligned}
\operatorname{Hom}(\Phi, h_C \lim_I F) &= \operatorname{Hom}(h_C(x), h_C(\lim_I F)) \\
&\simeq \operatorname{Hom}(x, \lim_I F) \\
&\simeq \operatorname{Hom}(\mathrm{const}_x, F) \\
&\simeq \operatorname{Hom}(h_C\, \mathrm{const}_x, h_C F) \\
&\simeq \operatorname{Hom}(\mathrm{const}_{h_C(x)}, h_C F) \\
&\simeq \operatorname{Hom}(h_C(x), \lim_I h_C F).
\end{aligned}$$

For general Φ, we may choose a functor $x\colon J \to C$ such that $\Phi \simeq \operatorname{colim}_{j\in J} h_C(x_j)$. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\Phi, h_C \lim_I F) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}(\operatorname{colim}_{j\in J} h_C(x_j), h_C \lim_I F) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}(h_C(x), (h_C \lim_I F)_J) \arrow[d, "\simeq"] \\
\mathrm{Hom}(\Phi_I, h_C F) \arrow[r, "\simeq"] & \mathrm{Hom}(\operatorname{colim}_{j\in J} h_C(x_j)_I, h_C \lim_I F) \arrow[r, "\simeq"] & \mathrm{Hom}(h_C(x)_I, (h_C \lim_I F)_J).
\end{tikzcd}$$

This proves the claim for arbitrary Φ, finishing the proof when C is a small category.

For general C, we may choose a big enough universe Cat' containing Cat as a full subcategory, and such that C is Cat'-small. Let Grpd' be the subcategory of groupoids in Cat. Then the composite

$$\mathrm{Hom}_C(x, -)\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$$

preserves all limits. But since the second functor is fully faithful, the fact that the object $h_C \lim_I F$ in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ becomes a limit in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$ implies that it is already a limit in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. □

Theorem 9.6.4. *The category* Grpd *has small limits. More precisely, given a small category I and a functor $F\colon I \to$* Grpd*, the unique morphism to the terminal functor*

$$h_{I^{\mathrm{op}}} \to (e_I)_{I^{\mathrm{op}}} \qquad in \qquad \mathrm{Fun}(I^{\mathrm{op}}, \mathrm{Fun}(I, \mathrm{Grpd}))$$

induces through the Yoneda lemma a cone

$$c\colon\ \mathrm{Hom}(e_I, F)_I \to F = \mathrm{Hom}(h_{I^{\mathrm{op}}}(-), F)$$

which exhibits $\mathrm{Hom}(e_I, F)$ *as the limit of F in* Grpd*, i.e. the induced map*

$$\mathrm{Hom}_{\mathrm{Grpd}}(x, \mathrm{Hom}(e_I, F)) \to \mathrm{Hom}(x_I, F)$$

is an equivalence for all objects x of Grpd*.*

Proof. We saw in Theorem 9.5.2 that $\operatorname{colim}_{I^{\mathrm{op}}} h_{I^{\mathrm{op}}} \xrightarrow{\sim} e_I$, hence

$$\mathrm{Hom}(e_I, F) \simeq \mathrm{Hom}(\operatorname{colim}_{I^{\mathrm{op}}} h_{I^{\mathrm{op}}}, F) \simeq \lim_I \mathrm{Hom}(h_{I^{\mathrm{op}}}, F) \simeq \lim_I F,$$

where the middle equivalence holds by Theorem 9.6.3 and the last equivalence holds by Yoneda. □

Construction 9.6.5. Let C be a category. We choose a universe Cat' enlarging Cat such that both C and Cat are Cat'-small, and write Grpd' for the full subcategory of groupoids in Cat'. We let C' be the image of the map

$$\sum_{X\colon \mathrm{Cat}^{\simeq}} \mathrm{Hom}_{\mathrm{Cat}'}(A, C) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')\ , \quad (u\colon X \to C) \mapsto \underset{x\,\operatorname{colim} X}{h_C(u(x))}$$

where $h_C\colon c \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$ is the Yoneda embedding. A presheaf $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}'$ is said to be *small* if it is a term of C'. We define $\widehat{C}$ to be the full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$ spanned by small presheaves.

For instance, any representable presheaf is small, hence we have a Yoneda embedding

$$h_C\colon C \to \widehat{C}\,.$$

Lemma 9.6.6. *A category C has small colimits if and only if the small Yoneda embedding*

$$h_C\colon C \to \widehat{C}$$

has a left adjoint

$$L\colon \widehat{C} \to C\,.$$

Proof. Assume that C has small colimits. Given any small category X and any functor $u\colon X \to C$, we let a be the colimit of u in C and F be the small presheaf defined as the colimit of $h_C \circ u$. For any $b\colon C$ we thus have

$$\begin{aligned}
\mathrm{Hom}_C(a,b) &= \lim_{x\colon X^{\mathrm{op}}} \mathrm{Hom}_C(u(x)^{\mathrm{op}}, b) \\
&= \lim_{x\colon X^{\mathrm{op}}} \mathrm{Hom}_{\widehat{C}}(h_C(u(x))^{\mathrm{op}}, h_C(b)) \\
&= \mathrm{Hom}_{\widehat{C}}(\operatorname*{colim}_{x\colon X} h_C(u(x)), h_c(b)) \\
&= \mathrm{Hom}_{\widehat{C}}(F, h_C(b))\,.
\end{aligned}$$

functorially in $b\colon C$. Applying Theorem 11.5.12 to h_C^{op} now implies that h_C has a left adjoint.

Conversely, if L is a left adjoint of the small Yoneda embedding h_C, for any small category X and any functor $u\colon X \to C$, the colimit of $h_C \circ u$ in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$ yields by definition a small presheaf F, and $L(F)$ clearly is a colimit of u in C. □

Lemma 9.6.7. *If C is locally small, then so is $\widehat{C}$.*

Proof. Let G be a small presheaf on C. Then G is in fact a functor from C^{op} to Grpd: given any object $a\colon C^{\simeq}$, $G(a)$ is a small colimit of groupoids of the form $\mathrm{Hom}_C(a, u(x))$, hence is small. This means that the full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$ spanned by those F's such that $\mathrm{Hom}_C(F, G)$ is small contains all representable presheaves and is stable under small colimits (because the embedding of Grpd into Grpd′ preserves the formation of small limits). This implies the lemma right away. □

Lemma 9.6.8. *If C is small then $\widehat{C} = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ holds.*

Proof. We have seen in the proof of Lemma 9.6.7 that $\widehat{C}$ is a full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. the converse embedding follows from Corollary 9.5.3, since the domain of the right fibration associated to any functor from C^{op} to Grpd is small. □

Lemma 9.6.9. *For any functor $u\colon C \to D$, the associated left Kan extension functor*

$$u_!\colon\ \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}') \to \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}')$$

restricts to a functor of type

$$u_!\colon \widehat{C} \to \widehat{D}\,.$$

(a) If C and D have small colimits, there is a natural transformation of the form

$$comp_{u,F}\colon L(u_!(F)) \to u(L(F))$$

for all $F\colon \widehat{C}$. This natural transformation is invertible if and only if the functor u commutes with small colimits.

(b) *If furthermore C is small and D is locally small, then the functor $u_!$ is right adjoint to the functor*

$$u^*\colon \widehat{D} \to \widehat{C} = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})\,.$$

Proof. The property that $u_!$ preserves small presheaves follows right away from the fact the functor $u_!$ preserves (small) colimits and representable functors.

If C and D have small colimits, we produce a comparison map of the form

$$comp_{u,F}\colon L(u_!(F)) \to u(L(F))$$

as follows. Any $F\colon \widehat{C}$ correspond to a right fibration of the form $p_F\colon X \to C$ with the property that the collection of terminal functors of the form $f\colon I \to X$ with I small forms a cover of the context in which F is defined. This implies that any category with small colimits has colimits indexed by X even though the latter might not be small. There is a canonical identification of the form

$$\mathrm{colim}_{x\colon X}\ h_C(p(x)) = F$$

hence a canonical comparison map

$$L(u_!(F)) = \mathrm{colim}_{x\colon X}\ h_D(up(x)) \to u\big(\mathrm{colim}_{x\colon X}\ p(x)) = u(L(F)\big)\,.$$

The fact that this map is invertible for all $F\colon \widehat{C}^{\simeq}$ if and only if the functor u commutes with small colimits is a tautology.

If C is small and D locally small, Lemma 9.6.7 and Lemma 9.6.8 imply that the functor

$$u^*\colon\ \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}') \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$$

restricts to a functor of the form

$$u^*\colon \widehat{D} \to \widehat{C} = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})\,.$$

Therefore the canonical adjunction $(u_!, u^*)$ restricts to small presheaves. □

Proposition 9.6.10. *Let C be a locally small category. The locally small category $\widehat{C}$ of small presheaves on C has small colimits and the full embedding*

$$\widehat{C} \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

commutes with small colimits.

Proof. Let I be a small category and $F\colon I \to \widehat{C}$ a functor. For each $i\colon I^{\simeq}$, there is a small category X_i and a terminal functor of the form

$$u_i\colon X_i \to C_{/F_i}\,.$$

Using the projections $C_{/F_i} \to C$, these induce a map

$$u\colon \sum_{i\colon I^{\simeq}} X_i^{\simeq} \to C^{\simeq}\,.$$

We denote by B the full subcategory of C spanned by the image of u. Since C is locally small and I as well as each X_i are small, B is a small category. Let $v\colon B \hookrightarrow C$ be the associated full embedding. The induced functor

$$v_!\colon \operatorname{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}') \to \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$$

is fully faithful. By virtue of Lemma 9.6.9, it restricts to a fully faithful embedding

$$v_!\colon \operatorname{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \widehat{C}\,.$$

By construction, for each $i \in I^{\simeq}$, there is a functor

$$p_i\colon X_i \to B$$

such that $v \circ P_i$ agrees with the composition of u_i with the projection $C_{/F_i} \to C$. We define $\Phi_i\colon B^{\mathrm{op}} \to \mathrm{Grpd}$ as the presheaf corresponding to the factorization of p_i into a terminal functor followed by a right fibration (or, equivalently, as the colimit of the composition of p_i with the Yoneda embedding of B). By construction, $v_!(\Phi_i) = F_i$ for all $i\colon I^{\simeq}$. Since $v_!$ is fully faithful, there is a unique functor

$$\Phi\colon I \to \operatorname{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})$$

such that $v_! \circ \Phi = F$ holds true. Let Ψ be the colimit of Φ in $\operatorname{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})$. Then $v_!(\Psi)$ is small but it is also the colimit of F in $\operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$. Therefore, $v_!(\Psi)$ is the colimit of F in $\widehat{C}$. □

Theorem 9.6.11 (Kan)**.** *Let A be a locally small category and C any category with small colimits. There is a functor that commutes with small colimits*

$$Lu_!\colon \widehat{A} \to C$$

equipped with a canonical identification

$$Lu_! \circ h_A \cong u$$

of functors from A to C. Furthermore, if C is locally small, the functor $Lu_!$ is left adjoint to the functor

$$u^* h_C\colon C \to \operatorname{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})\,, \quad y \mapsto \operatorname{Hom}_C(u(-), y)\,.$$

Proof. The functor

$$u_!\colon \operatorname{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}') \to \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$$

commutes with (small) colimits (being a left adjoint). We have a commutative square of the following form.

$$\begin{tikzcd}
\widehat{A} \arrow[r, "u_!"] \arrow[d, hook] & \widehat{C} \arrow[d, hook] \\
\operatorname{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}') \arrow[r, "u_!"] & \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')
\end{tikzcd}$$

This implies that the induced functor

$$u_!\colon \widehat{A} \to \widehat{C}$$

preserves colimit cocones indexed by small categories. The functor

$$L\colon \widehat{C} \to C$$

is a left adjoint by Lemma 9.6.6, hence preserves all colimits. This implies that the composite $Lu_!$ preserves small colimits.

If furthermore A is small and C is locally small then Lemma 9.6.9 tells us that $u_!$ is a left adjoint to u^*. Since L is a left adjoint to the small Yoneda embedding, the composite $Lu_!$ is a left adjoint to $u^* h_C$. ◻

Corollary 9.6.12. *Let A be locally small and let C with small colimits. Then a functor*

$$\Phi\colon \widehat{A} \to C$$

commutes with small colimits if and only if there is an isomorphism of the of the form $\Phi \cong Lu_!$ for $u := \Phi \circ h_A$. In particular, if furthermore A is small and C is locally small, the property for a functor Φ as above to commutes with small colimits is equivalent to the property of having a right adjoint.

Proof. For $u = \Phi h_A$, the canonical adjunction

$$u_!\colon \operatorname{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}') \rightleftarrows \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}') : u^* .$$

induces an adjunction of the form:

$$u_!\colon \operatorname{Fun}(\widehat{A}, \operatorname{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}')) \rightleftarrows \operatorname{Fun}(\widehat{A}, \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')) : u^* .$$

For a presheaf $F\colon A^{\mathrm{op}} \to \mathrm{Grpd}$ we have

$$\operatorname{Hom}(h_A^{\mathrm{op}}(-), F) = F$$

by the Yoneda lemma. Then Φ induces a natural transformation

$$\operatorname{Hom}_{\widehat{A}}(-,-) \to \operatorname{Hom}_C(\Phi^{\mathrm{op}}(-), \Phi(-)),$$

and hence we obtain a canonical homomorphism

$$F = \operatorname{Hom}(h_A(-), F) \to \operatorname{Hom}_C(\Phi h_A(-), \Phi(F)) = \operatorname{Hom}_C(u(-), \Phi(F)) = u^* h_C \Phi(F),$$

which is functorial in F. In other words, we have constructed a natural transformation

$$\iota \to u^* h_C \Phi(-)$$

where $\iota\colon \operatorname{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \operatorname{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}')$ denotes the canonical full embedding. By passing to adjoints, this yields a canonical morphism

$$u_! \to h_C\,\Phi.$$

Applying the functor L and using that $L\,h_C \cong \mathrm{id}_C$, this yields a canonical map

$$Lu_! \to \Phi$$

inducing the identity of u when composing with h_A. Therefore, we know that the map $Lu_!(F) \to \Phi(F)$ is an isomorphism whenever F is a representable presheaf. But since both $Lu_!$ and Φ commute with colimits, the collection of objects on which the map $Lu_! \to \Phi$ is an isomorphism is closed under colimits in $\widehat{A}$, and so by Corollary 9.5.3 it is an equivalence for an arbitrary presheaf F.

The last assertion comes from the fact that any functor (isomorphic to a functor) of the form $Lu_!$ ◻

Remark 9.6.13. The property that C is locally small is necessary for the last assertion to hold. Indeed, assume that C is a category with small colimits. Taking $A = *$ and $y\colon * \to C$, we may construct a colimit-preserving functor $Ly_!\colon \mathrm{Grpd} \to C$, written also as $s \otimes y := Ly_!(s)$. If the previous corollary is satisfied for C, then then this functor has a right adjoint $R\colon C \to \mathrm{Grpd}$, hence we obtain a natural equivalence

$$\mathrm{Hom}_C(s \otimes y, x) \simeq \mathrm{Hom}_{\mathrm{Grpd}}(s, R(x)).$$

In particular, taking s to be the terminal groupoid we see that $\mathrm{Hom}_C(y, x) \simeq \mathrm{Hom}_{\mathrm{Grpd}}(*, R(x)) = R(x)$, and hence C is locally small.

Proposition 9.6.14. *Let $u\colon A \to B$ be a functor between small categories. Then the pullback functor*

$$u^*\colon \mathrm{Fun}(B, \mathrm{Grpd}) \to \mathrm{Fun}(A, \mathrm{Grpd})$$

admits a right adjoint

$$u_*\colon \mathrm{Fun}(A, \mathrm{Grpd}) \to \mathrm{Fun}(B, \mathrm{Grpd}).$$

Proof. By Corollary 9.6.12 it will suffice to show that u^* preserves small colimits. But this is an instance of Lemma 5.10.7. □

Notation 9.6.15. If C and D have small colimits,we write $\mathrm{LFun}(C, D)$ for the full subcategory of $\mathrm{Fun}(C, D)$ whose objects are the functors commuting with small colimits.

Notation 9.6.16. Given two categories C and D, we write $\mathrm{Fun}_!(C, D)$ for the full subcategory of $\mathrm{Fun}(C, D)$ spanned by the functors from C to D that preserve small colimits. We also write $\mathrm{Fun}^L(C, D)$ for the full subcategory of $\mathrm{Fun}(C, D)$ spanned by functors that admit a right adjoint. There are obvious full embeddings of the form:

$$\mathrm{Fun}^L(C, D) \hookrightarrow \mathrm{Fun}_!(C, D) \hookrightarrow \mathrm{Fun}(C, D)\,.$$

Corollary 9.6.17. *Let A be as locally small category and C a category with small colimits. Then the Yoneda embedding $h_A\colon A \to \widehat{A}$ induces an equivalence:*

$$\mathrm{Fun}_!(\widehat{A}, C) \cong \mathrm{Fun}(A, C)\,.$$

In particular, the rule assigning to a functor $u\colon A \to C$ the functor $Lu_!$ is itself a functor.

Furthermore, if A is small and C is locally small, then we have

$$\mathrm{Fun}^L(\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}), C) = \mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}), C)\,.$$

Proof. The last assertion is a reformulation of Corollary 9.6.12.

We observe that there is a canonical identification of the form:

$$\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}), \mathrm{Fun}([1], C)) = \mathrm{Fun}([1], \mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}), C))\,.$$

By virtue of Corollary 6.3.4 it suffices thus to check that the Yoneda embedding $h_A\colon A \to \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})$ induces an equivalence:

$$\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}), C)^{\simeq} \cong \mathrm{Fun}(A, C)^{\simeq}\,.$$

This follows immediately from the first parts of Theorem 9.6.11 and of Corollary 9.6.12. □

Theorem 9.6.18. *Let C be a locally small category and let I be a small category. Then the hom-groupoids of* $\mathrm{Fun}(I, C)$ *may be computed as*

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}(F, G) \simeq \lim_{u\colon i\to j} \mathrm{Hom}_C(F(i), G(j)),$$

where the right-hand side is defined as the limit of the composite functor

$$\mathrm{Tw}(I) \xrightarrow{(s,t)} I^{\mathrm{op}} \times I \xrightarrow{F^{\mathrm{op}}\times G} C^{\mathrm{op}} \times C \xrightarrow{\mathrm{Hom}_C} \mathrm{Grpd}.$$

In particular, the category $\mathrm{Fun}(I, C)$ *is again locally small.*

Proof. Choose a universe so that C is small, and consider the fully faithful functor $h_C\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. Then $h_C\colon \mathrm{Fun}(I, C) \to \mathrm{Fun}(I, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}))$ is fully faithful as well. Given $F, G\colon I \to C$, we have

$$\mathrm{Hom}(F, G) \simeq \mathrm{Hom}(h_C(F), h_C(G)).$$

We may now consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Tw}(I) \arrow[r, dashed, "w"] \arrow[rr, bend left, "\mathrm{Tw}(G)"] \arrow[d, "(s,t)"'] & W \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tw}(C) \arrow[d, "(s,t)"] \\
I^{\mathrm{op}} \times I \arrow[r, "\mathrm{id}\times G"] & I^{\mathrm{op}} \times C \arrow[r, "G^{\mathrm{op}}\times\mathrm{id}"] & C^{\mathrm{op}} \times C.
\end{tikzcd}$$

We showed in Lemma 9.2.7 that the unique functor w obtained from this diagram is an initial functor. Now, consider the terminal object e of $\mathrm{Fun}(\mathrm{Tw}(I), \mathrm{Grpd})$, so that we have a natural isomorphism

$$\mathrm{Hom}_I \cong (s, t)_!(e)$$

of functors $I^{\mathrm{op}}\times I \to \mathrm{Grpd}$. It then follows from the above diagram that there is a natural isomorphism

$$\mathrm{Hom}_C(G(-), -) = (G^{\mathrm{op}} \times \mathrm{id}_C)^* \,\mathrm{Hom}_C(-, -) \simeq (\mathrm{id}_{I^{\mathrm{op}}} \times G)_! \,\mathrm{Hom}_I(-, -) \simeq (\mathrm{id}_{I^{\mathrm{op}}} \times G)_!(s, t)_!(e)$$

of functors $I^{\mathrm{op}} \times C \to \mathrm{Grpd}$. It follows that

$$\begin{aligned}
\mathrm{Hom}(F, G) &\simeq \mathrm{Hom}(\mathrm{Hom}_C(G(-), -), \mathrm{Hom}_C(F(-), -)) \\
&\simeq \mathrm{Hom}((\mathrm{id}_{I^{\mathrm{op}}} \times G)_!(s, t)_!(e), \mathrm{Hom}_C(F(-), -)) \\
&\simeq \mathrm{Hom}(e, (s, t)^*(\mathrm{id}_{I^{\mathrm{op}}} \times G)^* \,\mathrm{Hom}_C(F(-), -)) \\
&\simeq \mathrm{Hom}(e, \mathrm{Hom}_C(F(-), G(-))) \\
&\simeq \lim_{\mathrm{Tw}(I)} (s, t)^*((i, j) \mapsto \mathrm{Hom}_C(F(i), G(j))).
\end{aligned}$$

□

Remark 9.6.19. Consider a category I and a functor $\Phi\colon I^{\mathrm{op}}\times I \to D$. If D admits limits indexed by $\mathrm{Tw}(I)$, then we may define the *end of* Φ as the limit in D of the composite $\mathrm{Tw}(I) \xrightarrow{(s,t)} I^{\mathrm{op}}\times I \xrightarrow{\Phi} D$. This is sometimes denoted $\int^{i\in I} \Phi(i, i)$, though we will mostly avoid this notation. In this language, the previous theorem may be rephrased as

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}(F, G) \simeq \int^{i\in I} \mathrm{Hom}_C(F(i), G(i)).$$

Corollary 9.6.20. *Let I be a small category and C a category with small colimits. For any functor $F\colon I \to C$, the colimit of the functor*

$$\mathrm{Tw}(I)^{\mathrm{op}} \to \mathrm{Fun}(I, C)\,, \quad (j, i) \mapsto j \otimes F(i)$$

is F itself.

Proof. For any functor $G\colon I \to C$, we have

$$\mathrm{Hom}(j \otimes X, G) = \mathrm{Hom}(X, G(j))$$

for any X in C. The corollary is thus a direct reformulation of the previous theorem. □

Construction 9.6.21. Let A be a small category and B a category with small colimits. Interpreting objects of A as functors of the form $a\colon \Gamma \to A$ for some groupoid Γ, we define the *relative Yoneda embedding* of A in B as the functor

$$h_{A/B}\colon A \times B \to \mathrm{Fun}(A^{\mathrm{op}}, B)\,, \quad (a, x) \mapsto a \otimes x = a_!(x)\,.$$

Given any functor of the form

$$\Phi\colon \mathrm{Fun}(A^{\mathrm{op}}, B) \to C$$

we get a functor

$$\Phi \circ h_{A/B}\colon A \times B \to C$$

which in turn yields by transposition a functor

$${}^t\Phi\colon B \to \mathrm{Fun}(A, C)\,.$$

Given $x\colon B$, the functor ${}^t\Phi(x)$ assigns to each $a\colon A$ the term $\Phi(a \otimes x)$. In particular, we see that, if ever Φ preserves small colimits, then so does ${}^t\Phi$.

Lemma 9.6.22. *Let A and B as above. For any functor $F\colon A^{\mathrm{op}} \to B$, the presheaf on $A \times B$ defined by the rule*

$$(a, x) \mapsto \mathrm{Hom}_B(x, F(a))$$

is small.

Proof. We observe first that a left fibration $p\colon X \to C$ correspond to a small presheaf on some category C if and only if there exists an initial functor from some small category to X. One can then consider the diagram

$$\begin{tikzcd}
\mathrm{Tw}(A)^{\mathrm{op}} \arrow[r] \arrow[d] & (\mathrm{Fun}(A,B)^{\mathrm{op}} \times A) \times_{(\mathrm{Fun}(A,B)^{\mathrm{op}} \times (\mathrm{Fun}(A,B))} \mathrm{Tw}(\mathrm{Fun}(A,B)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
A^{\mathrm{op}} \times A \arrow[r, "F \times \mathrm{id}"] & \mathrm{Fun}(A,B)^{\mathrm{op}} \times A \arrow[r, "\mathrm{Hom}(-, F(-))"] & \mathrm{Grpd}
\end{tikzcd}$$

in which the right hand square is cartesian and in which the upper horizontal map of the square on the left hand side is an initial functor by Lemma 9.2.7. This proves the lemma. □

Theorem 9.6.23. *Let A be a small category and B and C two categories with small colimits. The assignment $\Phi \mapsto {}^t\Phi$ defines an equivalence*

$$\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, B), C) \cong \mathrm{Fun}_!(B, \mathrm{Fun}(A, C))\,.$$

Proof. We observe that we have a canonical equivalence of the form

$$\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, B), \mathrm{Fun}([1], C)) = \mathrm{Fun}([1], \mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, B), C)))\,.$$

Similarly, we have

$$\mathrm{Fun}_!(B, \mathrm{Fun}(A, \mathrm{Fun}([1], C))) = \mathrm{Fun}([1], \mathrm{Fun}_!(B, \mathrm{Fun}(A, C)))\,.$$

Therefore, Corollary 6.3.4 informs us that it suffices to prove that the assignment $\Phi \mapsto {}^t\Phi$ induces an equivalence of the form

$$\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, B), C)^{\simeq} \cong \mathrm{Fun}_!(B, \mathrm{Fun}(A, C))^{\simeq}\,.$$

Given a functor $\Psi\colon B \to \mathrm{Fun}(A, C)$, we denote by $\bar{\Psi}\colon A \times B \to C$ the associated functor by transposition, which induces a functor

$$\bar{\Psi}_!\colon \widehat{A \times B} \to \widehat{C}$$

by Lemma 9.6.9. We define a functor Ψ' as the composite

$$\Psi'\colon\ \mathrm{Fun}(A^{\mathrm{op}}, B) \xrightarrow{\mathrm{Fun}(A^{\mathrm{op}}, h_B)} \widehat{A \times B} \xrightarrow{\bar{\Psi}_!} \widehat{C} \xrightarrow{L} C$$

where L is the left adjoint of the Yoneda embedding provided by Lemma 9.6.6. The functor Ψ' commutes with small colimits whenever the functor Ψ has this property. Indeed, the assignment $\Psi \mapsto \Psi'$ is functorial in C in the following sense: given any functor $u\colon C \to D$ commuting with small colimits, we have a canonical commutative square of the form

$$\begin{array}{ccc} \widehat{C} & \xrightarrow{L} & C \\ {\scriptstyle u_!}\downarrow & & \downarrow{\scriptstyle u} \\ \widehat{D} & \xrightarrow{L} & D \end{array}$$

by Lemma 9.6.9 (a), hence a canonical identification of the form

$$u(\Psi') = (u_*(\Psi))$$

for any functor $\Psi\colon B \to \mathrm{Fun}(A, C)$, where $u_*\colon\ \mathrm{Fun}(A, C) \to \mathrm{Fun}(A, D)$ denotes postcomposition by u. The particular case where

$$u := \operatorname*{colim}_{i\colon I}\colon\ \mathrm{Fun}(I, C) \to C$$

for some small category I yields the compatibility of the assignment $\Psi \mapsto \Psi'$ with small colimits.

We claim that the rule $\Psi \mapsto \Psi'$ is an inverse of the rule $\Phi \mapsto {}^t\Phi$. The main step will follow from the following assertion: if ever $\Psi = {}^t\Phi$ for some functor Φ as above, then there is a canonical comparison map of the form

$$c\colon \Psi' \to \Phi$$

and the latter is invertible whenever Φ commutes with small colimits.

Indeed, since Φ is a functor, there is a canonical map of the form

$$\mathrm{Hom}_B(x, f(a)) = \mathrm{Hom}_{\mathrm{Fun}(A^{\mathrm{op}},B)}(a \otimes x, f) \to \mathrm{Hom}_C(\Phi(a \otimes x), \Phi(f))$$

functorially in $a\colon A$, $x\colon B$ and $f\colon A^{\mathrm{op}} \to B$. By transposition, this defines a canonical map

$$\Psi'(f) = L\bar{\Psi}(h_{B,*}(f)) \to \Phi(f)$$

functorially in f. For $f = a \otimes x$, this map is an isomorphism: this comes from the canonical identification

$$\bar{\Psi}_!\, h_{A\times B} = h_C\, \bar{\Psi}$$

and from the computation (left as an exercise for the reader, using essentially that binary cartesian product of groupoids commutes with colimits in each variable):

$$h_{A\times B}(a, x) = a \otimes h_B(x)\,.$$

But, by virtue of Corollary 9.6.20 (and the proof of Lemma 9.6.22), any functor $f\colon A^{\mathrm{op}} \to B$, seen as a presheaf on $A \times B$ through the functor $h_{B,*}$, canonically is a small colimit in $\mathrm{Fun}(A^{\mathrm{op}}, B^{\mathrm{op}}, \mathrm{Grpd}')$, of presheaves of the form $a \otimes x$ (with Grpd' the category of groupoids in any large enough universe). Since the functor $\bar{\Psi}_!$ preserves such colimits, if ever Φ commutes with small colimits, the fact that $\Psi'(f) = \Phi(f)$ for all f follows from the special case where $f = a \otimes x$.

If we start from a functor $\Psi\colon\ \mathrm{Fun}(A, B) \to C$, then the functorial formula

$$\bar{\Psi}_!(a \otimes h_B(x)) = h_C(\Psi(x)(a))$$

to which we can apply the functor L shows that

$$\Psi'(a \otimes x) = \Psi(x)(a)$$

functorially in both variables $x\colon B$ and $a\colon A$. In other words ${}^t(\Psi') = \Psi$ holds. □

9.7 Revisiting smooth/proper base change

Let A and B be small categories and let $u\colon A \to B$ be a functor. Consider the restriction functor

$$u^*\colon\ \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}), \qquad F \mapsto F \circ u^{\mathrm{op}}.$$

By Proposition 9.6.14, u^* admits a right adjoint

$$u_*\colon\ \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}).$$

Theorem 9.7.1. *Consider a pullback square of small categories*

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\ B' \arrow[r, "g"'] & B. \end{tikzcd}$$

If u is proper or g is smooth, then for any presheaf $F\colon A^{\mathrm{op}} \to \mathrm{Grpd}$, the Beck-Chevalley map

$$\mathrm{BC}_*\colon g^*u_*(F) \to u'_*f^*(F)$$

is an isomorphism.

Proof. For every presheaf $\Phi\colon B'^{\mathrm{op}} \to \mathrm{Grpd}$, we need to show that the induced map

$$\mathrm{Hom}(\Phi, g^* u_*(F)) \to \mathrm{Hom}(\Phi, u'_* f^*(F))$$

is an equivalence. By adjunction, this corresponds to a map of the form

$$\mathrm{Hom}(u^* g_! \Phi, F) \to \mathrm{Hom}(f_! u'^* \Phi, F)\,.$$

One can show that this is precisely the map induced the Beck-Chevalley isomorphism $\mathrm{BC}_!\colon f_! u'^* \Phi \to u^* g_! \Phi$ from Theorem 8.4.1 or Theorem 8.4.6. This finishes the proof. □

Theorem 9.7.2. *Let $u\colon A \to B$ be an initial functor between small categories. For any functor $F\colon B \to \mathrm{Grpd}$ there is a canonical isomorphism*

$$\lim_B F \xrightarrow{\simeq} \lim_A u^* F.$$

Proof. Consider the functor $p\colon B \to *$, so that $\lim_B = p_*$ and $\lim_A = (pu)_* = p_* u_*$. We need to check that the unit map $\mathrm{id} \to u_* u^*$ induces an isomorphism $p_* \to p_* u_* u^*$. By passing to left adjoints, we may equivalently show that the map $u_! u^* p^* \to p^*$ is an isomorphism. To this end, consider a groupoid X. We compute both sides in terms of left fibrations over B. The object $p^*(X)$ corresponds to the projection $\mathrm{pr}_2\colon X \times B \to B$. Similarly, $u^* p^*(X)$ is the projection $\mathrm{pr}_2\colon X \times A \to A$. The object $u_! u^* p^*(X)$ is computed by factoring the composite $u \circ \mathrm{pr}_2\colon X \times A \to B$ into an initial functor followed by a left fibration. But for this we may simply take

$$\begin{tikzcd} X \times A \arrow[r, "\mathrm{id}_X \times u"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times B \arrow[d, "\mathrm{pr}_2"] \\ A \arrow[r, "u"] & B. \end{tikzcd}$$

So this diagram exhibits $\mathrm{pr}_2\colon X \times B \to B$ as an instance of $u_! u^* p^*(X)$. With these choices the map $u_! u^* p^*(X) \to p^*(X)$ simply corresponds to the identity map of $X \times B$ over B, hence is in particular an equivalence. This finishes the proof. □

Corollary 9.7.3. *Let $u\colon A \to B$ be a functor between categories. If A is small and u is initial, then* Grpd *admits B-indexed limits, and we have a natural isomorphism*

$$\lim_B = \lim_A u^*$$

as functors $\mathrm{Fun}(B, \mathrm{Grpd}) \to \mathrm{Grpd}$.

Theorem 9.7.4. *Let A and C be categories and assume that C admits A-indexed limits. Then, for any initial functor $u\colon A \to B$, C also admits B-indexed limits and we have a natural isomorphism*

$$\lim_B \cong \lim_A u^*$$

as functors $\mathrm{Fun}(B, C) \to C$.

Proof. Let $F\colon B \to C$ be a functor. We need to show that the functor $C^{\mathrm{op}} \to \mathrm{Grpd}$, $x \mapsto \mathrm{Hom}(x_B, F)$ is representable by some object $\lim_B F$. We compute:

$$\begin{aligned}
\mathrm{Hom}(x_B, F) &\simeq \mathrm{Hom}(h_C(x)_B, h_C(F)) \\
&\simeq \mathrm{Hom}(h_C(x), \lim_{b\in B} h_C(F(b))) \\
&\simeq \lim_{b\in B} \mathrm{Hom}(h_C(x), h_C(F(b))) \\
&\simeq \lim_{b\in B} \mathrm{Hom}_C(x, F(b)) \\
&\simeq \lim_{a\in A} \mathrm{Hom}_C(x, F(u(a))) \\
&\simeq \mathrm{Hom}_C(x, \lim_{a\in A} F(u(a))).
\end{aligned}$$

Hence the functor is represented by the limit $\lim_{a\in A} F(u(a))$. The isomorphism $\lim_B \cong \lim_A u^*$ then follows from the Yoneda lemma. □

Notation 9.7.5. Given a functor $u\colon A \to B$ between small categories A and B and an object b of B, we may consider the functors

$$\begin{tikzcd}
A_{/b} \arrow[r, "i_b"] \arrow[d, "u_{/b}"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "u"] \\
B_{/b} \arrow[r, "j_b"'] & B.
\end{tikzcd}$$

For a functor $F\colon A^{\mathrm{op}} \to C$, we write

$$F_{/b} := F \circ i_b \colon (A_{/b})^{\mathrm{op}} \to \mathrm{Grpd}.$$

Given a presheaf $F\colon A^{\mathrm{op}} \to \mathrm{Grpd}$, smooth base change provides us with an isomorphism

$$u_*(F)(b) = (j_b^* u_*(F))(b, \mathrm{id}_b) \xrightarrow{\simeq} (u_{/b})_*(i_b^* F)(b, \mathrm{id}_b) = \lim_{(B_{/b})^{\mathrm{op}}} (u_{/b})_*(F_{/b}) = \lim_{(A_{/b})^{\mathrm{op}}} F_{/b}.$$

Theorem 9.7.6 (Pointwise formula for right Kan extensions)**.** *Let $u\colon A \to B$ be a functor between categories. Let C be a category such that, for any object b of B, C admits $(A_{/b})^{\mathrm{op}}$-indexed limits. Then the functor $u^*\colon \mathrm{Fun}(B^{\mathrm{op}}, C) \to \mathrm{Fun}(A^{\mathrm{op}}, C)$ admits a right adjoint*

$$u_*\colon \mathrm{Fun}(A^{\mathrm{op}}, C) \to \mathrm{Fun}(B^{\mathrm{op}}, C)$$

satisfying the formula $u_*(F)(b) = \lim_{(A_{/b})^{\mathrm{op}}} F_{/b}$.

Proof. Choose a universe so that C is small. Then we may consider the Yoneda embedding $h_C\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$, which induces a fully faithful functor

$$h_C\colon \mathrm{Fun}(B^{\mathrm{op}}, C) \to \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})) \simeq \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})),$$

and similarly for B replaced by A. The induced restriction functor $u^*\colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}))$ admits a right adjoint

$$u_*\colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})),$$

and we need to show that it restricts to a functor $\mathrm{Fun}(A^{\mathrm{op}}, C) \to \mathrm{Fun}(B^{\mathrm{op}}, C)$. But for a presheaf $F\colon A^{\mathrm{op}} \to C$ and an object b of B we have

$$u_*(h_C(F))(b) = \lim_{(a,s)\in(A_{/b})^{\mathrm{op}}} h_C(F(a)) \simeq h_C(\lim_{(A_{/b})^{\mathrm{op}}} F(a)),$$

which does indeed lie in the essential image of the functor $C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. □

Corollary 9.7.7. *Let $u\colon A \to B$ be a functor and let $\Phi\colon C \to B$ be a functor such that C and D both admit $(A_{/b})^{\mathrm{op}}$-indexed limits for any object b of B, which are preserved by the functor Φ. Then for every presheaf $F\colon A^{\mathrm{op}} \to C$ the Beck-Chevalley map*

$$\Phi(u_*(F)) \to u_*(\Phi(F))$$

is a natural isomorphism.

Proof. It suffices to show for every object b of B the induced map $\Phi(u_(F))(b) \to u_*(\Phi(F))(b)$ is an isomorphism. This is immediate from the pointwise formula from Theorem 9.7.6. □

Lemma 9.7.8. *Let C be a small category. The following conditions are equivalent.*

(i) *C is weakly contractible;*

(ii) *C^{op} is weakly contractible;*

(iii) *taking the associated C-indexed constant functor defines a fully faithful functor*

$$\mathrm{Grpd} \hookrightarrow \mathrm{Fun}(C, \mathrm{Grpd})$$

and its essential image consists of those functors sending all maps of C to isomorphisms in Grpd.

(iv) *for any category D, taking the associated C-indexed constant functor defines a fully faithful functor*

$$D \hookrightarrow \mathrm{Fun}(C, D)\,.$$

Proof. Corollary 9.2.6 shows that conditions (i) and (ii) are equivalent. It is clear that condition (iv) implies condition (iii). Condition (iv) follows straight away from condition (i): if C is weakly contractible, then the unique functor $C \to [0]$ exhibits $[0]$ as the localization of C by all morphisms of C. In other words, for any category D, the associated constant functor $X \mapsto X_C$ defines a fully faithful functor of the form

$$D \hookrightarrow \mathrm{Fun}(C, D)$$

whose essential image consists of hose functors $F\colon C \to D$ with the property that $F(u)\colon F(x) \to F(y)$ is an isomorphism in D for any morphism $u\colon x \to y$ in C. It remains to prove that condition (iii) implies condition (i). Recall from Remark 8.4.3 that, if X is a small groupoid and if e is the terminal object of Grpd, then we have a canonical isomorphism

$$\mathrm{Hom}(\Pi_\infty(C), X) = \mathrm{Hom}(e_C, X_C)\,.$$

If the functor $X \mapsto X_C$ is fully faithful, the isomorphism

$$\mathrm{Hom}(e, X) = \mathrm{Hom}(e_C, X_C)$$

can thus be reinterpreted as asserting that the map $\Pi_\infty(C) \to e$ induces an isomorphism of the form

$$\mathrm{Hom}(e, X) = \mathrm{Hom}(\Pi_\infty(C), X)$$

for all X. Therefore, by virtue of Corollary 9.3.13, the map $\Pi_\infty(C) \to e$ must be an isomorphism in Grpd, which means that C is weakly contractible by definition. □

Definition 9.7.9. A functor $f\colon A \to B$ is *weakly proper* if, for any object b of B, the induced functor $A_b \to A_{/b}$ is terminal.
A functor $f\colon A \to B$ is *weakly smooth* if, for any object b of B, the induced functor $A_b \to A_{b/}$ is initial.

Example 9.7.10. Any proper functor is weakly proper. In particular, (locally) cocartesian functors are weakly proper. Dually, smooth functors are weakly smooth. In fact, a functor $u\colon A \to B$ is weakly proper if and only if $u^{\mathrm{op}}\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$ is weakly smooth. Examples of weakly proper functors that are not locally cocartesian will come from simplicial approximation; see Proposition 11.8.6.

Lemma 9.7.11. *Let A be a small category and X a small synthetic groupoid. Then $\mathrm{Hom}_{\mathrm{Grpd}}(\Pi_\infty(A), X)$ is the limit of the A-indexed constant functor with value X.*

Proof. We know that $\Pi_\infty(A)$ is the colimit of the constant A-indexed diagram with value the terminal object e of Grpd – see Remark 8.4.3. We thus have

$$\mathrm{Hom}_{\mathrm{Grpd}}(\Pi_\infty(A), X) = \mathrm{Hom}(e_A, X_A)$$

where X_A denotes the constant A-indexed diagram with value X. But we also know that $\mathrm{Hom}(e_A, -)$ is the limit functor for A-indexed diagrams. □

Lemma 9.7.12. *Let C be a category and $f\colon A \to B$ a weakly smooth functor. We assume that C admits colimits of type A_b for any object b of B. Then the functor*

$$u^*\colon \mathrm{Fun}(B^{\mathrm{op}}, C) \to \mathrm{Fun}(A^{\mathrm{op}}, C)$$

has a right adjoint

$$u_*\colon \mathrm{Fun}(A^{\mathrm{op}}, C) \to \mathrm{Fun}(B^{\mathrm{op}}, C)$$

and, for and object b of B and any functor $F\colon A \to C$, there is a canonical isomorphism

$$u_*(F)(b) = \lim_{A_b^{\mathrm{op}}} F_{|A_b^{\mathrm{op}}}$$

where $F_{|A_b}$ denotes the composition of F with the embedding $A_b^{\mathrm{op}} \hookrightarrow A^{\mathrm{op}}$.

Proof. Appyling Theorem 9.7.4, we see that C admits colmits of type $(A_{/b})^{\mathrm{op}}$ for each object b of B. This lemma follows now straightaway from Theorem 9.7.6. □

Proposition 9.7.13. *Let $f\colon A \to B$ be a functor between categories. We suppose that the following properties hold.*

(i) The functor f is either weakly proper or weakly smooth.

(ii) For each object b of B, the fiber A_b is weakly contractible.

(iii) For each object b of B there is an object a in A_b such that $f(a) = b$ holds.

Then f exhibits B as the localization of A by

$$W := \sum_{b\colon B^{\simeq}} \mathrm{Map}([1], A_b)$$

(the family of maps in A that are sent to identities in B).

Proof. We will prove the case where f is weakly proper (the weakly smooth case follows from the weakly proper one by considering $f^{\mathrm{op}}\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$).

We obviously have a commutative triangle of the form

$$\begin{tikzcd} A \arrow[rr, "\gamma"] \arrow[dr, "f"'] & & A[W^{-1}] \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

where $\gamma\colon A \to A[W^{-1}]$ is the localization of A by W. We will show that φ is an equivalence. Condition (iii) implies that proving full faithfulness will suffice.

We may choose a universe Cat so that A and B are small. The commutative triangle above thus gives rise to a commutative triangle of the form

$$\begin{tikzcd} \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) & & \mathrm{Fun}(A[W^{-1}]^{\mathrm{op}}, \mathrm{Grpd}) \arrow[ll, "\gamma^*"'] \\ & \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}) \arrow[ul, "f^*"] \arrow[ur, "\varphi^*"'] & \end{tikzcd}$$

in which γ^* is fully faithful, with essential image those functors that send any map in W to an isomorphism of Grpd. We claim that f^* is fully faithful. Equivalently, this means that, for any functor of the form $E\colon B \to \mathrm{Grpd}$, the unit map

$$\eta_E\colon E \to f_* f^*(E)$$

is an isomorphism. We observe that the restriction of $f^*(E)$ to the fiber A_b is the constant functor with value $E(b)$. By virtue of Lemma 9.7.11 and Lemma 9.7.12 this implies that, for each object b of B, there is a canonical isomorphism of the form:

$$\mathrm{Hom}(\Pi_\infty(A_b^{\mathrm{op}}), E(b)) = f_* f^*(E)(b)\,.$$

Since A_b is weakly contractible, so is A_b^{op} and, by Corollary 9.3.3, the previous identification can be rewritten as:

$$E(b) = f_* f^*(E)(b)\,.$$

This isomorphism agrees with the evaluation of η_E at b, and this proves that f^* is fully faithful. Since γ^* is fully faithful as well, the functor φ^* must also be fully faithful.

In order to prove that φ^* is an equivalence, it suffices to check that both fully faithful functors f^* and γ^* have the same essential image in $\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})$. Since f^* is the composition of φ^* and γ^*, it suffices to check that and object in the essential image of γ^* is in the essential image of f^*. Let $E\colon A \to \mathrm{Grpd}$ be a functor sending any map in W to an isomorphism of Grpd. We will prove that the co-unit map

$$\varepsilon_E\colon f^* f_*(E) \to E$$

is an isomorphism. Let a be an object of A. The restriction of E to each fiber $A_{f(a)}^{\mathrm{op}}$ sends each map to an isomorphism. Since the fiber is weakly contractible, this means that the restriction to the fiber is constant with value $E(a)$. In other words, the functor

$$a\colon [0] \to A_{f(a)}^{\mathrm{op}}$$

induces isomorphisms

$$\lim_{A^{\mathrm{op}}_{f(a)}} E_{|A^{\mathrm{op}}_{f(a)}} = \mathrm{Hom}(\Pi_\infty(A^{\mathrm{op}}_{f(a)}), E(a)) = E(a)\,.$$

By virtue of Lemma 9.7.12, this means that there is a canonical isomorphism of the form

$$f_*(E)(f(a)) = E(a)\,.$$

The composition of this isomorphism with the obvious isomorphism $f^* f_*(E)(a) = f_*(E)(f(a))$ coincides with the evaluation of ε_E at a. Therefore, $f^* f_*(E) = E$ and thus E lies in the essential image of f^*. Since φ^* is an equivalence, so is its left adjoint $\varphi_!$. This implies finally that the functor φ itself if fully faithful, by Proposition 9.2.11. □

Remark 9.7.14. Assumption (iii) in the previous proposition is not necessary: since each fibre A_b is weakly contractible, it must be non-empty. However, the proof of the proposition without assumption (iii) would require tools that will only be developed later in this book, through descent theory, using simplicial techniques.

9.8 Limits and colimits of categories

We will now show that the universe Cat admits all small limits and colimits.

Theorem 9.8.1. *The category* Cat *admits small colimits.*

Proof. Let I be a small category and let $F\colon A \to \mathrm{Cat}$ be a functor with associated cocartesian fibration $p\colon X \to I$:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ A \arrow[r, "F"] & \mathrm{Cat}. \end{tikzcd}$$

Now consider an object b of Cat with associated small category B:

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "b"] & \mathrm{Cat}. \end{tikzcd}$$

Then the composite $b_A\colon A \to * \xrightarrow{b} \mathrm{Cat}$ corresponds to the projection $\mathrm{pr}_1\colon A \times B \to A$. Natural transformations $F \to b_A$ corresponds to functors $(p, \varphi)\colon X \to A \times B$ over A that send p-cocartesian edges to pr_1-cocartesian edges.

A morphism in $A \times B$ is a pair (u, v), where $u\colon a \to a'$ is a morphism in A and $v\colon b \to b'$ is a morphism in B. The morphism (u, v) is (pr_1)-cocartesian fibration if and only if v is an isomorphism. So we see that (p, φ) is a cocartesian functor over A if and only if the functor $\varphi\colon X \to B$ sends p-cocartesian morphisms to isomorphisms in B.

Denote by $\widetilde{X}$ the localization of X at the p-cocartesian edges, and denote by

$$\gamma\colon X \to \widetilde{X}$$

the localization functor. We then define $\mathrm{colim}(F)\colon * \to \mathrm{Cat}$ as the functor corresponding to the cocartesian fibration $\widetilde{X} \to *$:

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "\mathrm{colim}(F)"'] & \mathrm{Cat}. \end{tikzcd}$$

The map γ gives rise to a functor

$$\begin{tikzcd} X \arrow[rr, "{(p,\gamma)}"] \arrow[dr, "p"'] & & A \times \widetilde{X} \arrow[dl, "\mathrm{pr}_1"] \\ & A & \end{tikzcd}$$

over A, which corresponds to a natural transformation $\eta_F\colon F \to (\mathrm{colim}(F))_A$ of functors $A \to \mathrm{Cat}$. We observe finally that η_F induces an isomorphism

$$\mathrm{Hom}(F, b_A) \xrightarrow{\simeq} \mathrm{Hom}(\mathrm{colim}(F), b)$$

by construction. □

Remark 9.8.2. The proof of the theorem in fact provides a stronger statement: not only does Cat admit colimits, we also have a concrete description of the colimits in Cat as suitable localizations.

Theorem 9.8.3. *The category* Cat *admits small limits. For a functor $F\colon I \to \mathrm{Cat}$, the limit $\lim_I F$ is computed as the category of cocartesian sections of the cocartesian fibration $p\colon X \to I$ classified by F.* □

Proof. Consider a cocartesian fibration $p\colon X \to I$, and let Y be a category. Cones $Y \to X_i$ correspond to maps $I \times Y \to X$ over I that preserve cocartesian edges. In other words, we consider the pullback

$$\begin{tikzcd} \mathrm{Fun}_I(I \times Y, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(I \times Y, X) \arrow[d, "p_*"] \\ * \arrow[r, "\mathrm{pr}_1"] & \mathrm{Fun}(I \times Y, I), \end{tikzcd}$$

and we consider the full subcategory $\mathrm{cocart}_I(I \times Y, X)$ of $\mathrm{Fun}_I(I \times Y, X)$ spanned by functors that preserve cocartesian edges over I.

The above pullback square is isomorphic to a pullback square of the form

$$\begin{tikzcd} \mathrm{Fun}(Y, \mathrm{Fun}_I(I, X)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(Y, \mathrm{Fun}(I, X)) \arrow[d, "(p_*)_*"] \\ * \arrow[r, "\mathrm{id}_I"] & \mathrm{Fun}(Y, \mathrm{Fun}(I, I)). \end{tikzcd}$$

Since the property of preserving cocartesian edges is an objectwise property, we discover that there is an equivalence of categories

$$\mathrm{cocart}_I(I \times Y, X) \simeq \mathrm{Fun}(Y, \mathrm{cocart}_I(I, X)).$$

Now assume that p and Y are small, i.e. come equipped with pullback squares of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ I \arrow[r, "F"] & \mathrm{Cat} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "y"] & \mathrm{Cat}. \end{tikzcd}$$

Using directed univalence, we then obtain identifications

$$\mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{Cat})}(y_I, F) \simeq \mathrm{cocart}_I(I \times Y, X)^{\simeq} \simeq \mathrm{Fun}(Y, \mathrm{cocart}_I(I, X))^{\simeq}.$$

If I is small, then also X and $\mathrm{cocart}_I(I, X)$ are small, so that we obtain an object $\lim_I F \colon \mathrm{Cat}^{\simeq}$ and a pullback square

$$\begin{tikzcd} \mathrm{cocart}_I(I, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, dashed, "\lim_I F"] & \mathrm{Cat}. \end{tikzcd}$$

The previous argument shows that $\lim_I F$ is a limit of F in Cat, as desired. ◻

Lemma 9.8.4. *Let C and D be categories. Then there exists an equivalence*

$$\lim_C D \simeq \mathrm{Fun}(\Pi_\infty(C), D).$$

Proof. For every other category E we have

$$\begin{aligned} \mathrm{Map}(E, \lim_C D) &\simeq \lim_C \mathrm{Map}(E, D) \simeq \lim_{\Pi_\infty(C)} \mathrm{Map}(E, D) \\ &\simeq \mathrm{Map}(\Pi_\infty(C), \mathrm{Map}(E, D)) \simeq \mathrm{Map}(E, \mathrm{Fun}(\Pi_\infty(C), D)), \end{aligned}$$

finishing the proof. ◻

Corollary 9.8.5. *Let $u \colon A \to B$ be a functor between small categories. Then the functor $u^* \colon \mathrm{Fun}(B, \mathrm{Cat}) \to \mathrm{Fun}(A, \mathrm{Cat})$ admits a left adjoint*

$$u_! \colon \mathrm{Fun}(A, \mathrm{Cat}) \to \mathrm{Fun}(B, \mathrm{Cat}).$$

Remark 9.8.6. The inclusion Grpd $\hookrightarrow$ Cat admits both a left adjoint Π_∞ and a right adjoint $(-)^{\simeq} \colon \mathrm{Cat} \to \mathrm{Grpd}$, hence this inclusion preserves both limits and colimits.

Proposition 9.8.7. *Let us consider a pullback square of categories of the form.*

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "u"] \\ B \arrow[r, "v"] & S \end{tikzcd}$$

We consider another category I and assume that all three of A, B and S has I-indexed limits and that both functors u and v commute with them. Then X has I-indexed limits and both functors p and q commute with them.

Proof. Up to enarging the universe Cat, we may assume that all categories involved here are small. We write $r\colon X \to S$ for the composite of u and p. Let $F\colon I \to X$ be a functor. We let a by the limit of $i \mapsto pF_i$, b for the limit of $i \mapsto qF_i$ and s for the limit of $i \mapsto rF_i$. We have $u(a) = s$ and $v(b) = s$, hence an object y in X with $p(y) = a$ and $p(y) = b$. Since the functor

$$\lim_I\colon \operatorname{Fun}(I, \mathrm{Grpd}) \to \mathrm{Grpd}$$

commutes with pullbacks (being a right adjoint), we have the following computations. For any object x of X, we have the following computations.

$$\begin{aligned}
\lim_i \operatorname{Hom}_X(x, F_i) &= \lim_i \left(\operatorname{Hom}_A(px, pF_i) \times_{\operatorname{Hom}_S(rx, rF_i)} \operatorname{Hom}_B(qx, qF_i)\right) \\
&= \lim_i \operatorname{Hom}_A(px, pF_i) \times_{\lim_i \operatorname{Hom}_S(rx, rF_i)} \lim_i \operatorname{Hom}_B(qx, qF_i) \\
&= \operatorname{Hom}_A(px, \lim_i pF_i) \times_{\operatorname{Hom}_S(rx, \lim_i rF_i)} \operatorname{Hom}_B(qx, \lim_i qF_i) \\
&= \operatorname{Hom}_A(px, a) \times_{\operatorname{Hom}_S(rx, s)} \lim_i \operatorname{Hom}_B(qx, b) \\
&= \operatorname{Hom}_X(x, y)\,.
\end{aligned}$$

This proves that y is the limit of $i \mapsto F_i$ in X. $\square$

Colimits indexed by colimits

We show that colimits indexed by a colimit category $C = \operatorname{colim}_{i \in I} C_i$ may be computed as iterated colimits.

Lemma 9.8.8. *Let C be a category. Let $F\colon I \to \mathrm{Cat}$ be a functor, and consider*

$$\begin{tikzcd}
F(i) \arrow[r, "l_i"] \arrow[d] & \int_{i \in I} F(i) \arrow[r] \arrow[d, "p_F"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\Gamma \arrow[r, "i"] & I \arrow[r, "F"] & \mathrm{Cat},
\end{tikzcd}$$

where i is any object of I (in any groupoidal context Γ). Let $\Phi\colon \int_{i \in I} F(i) \to C$ be a functor such that the composite

$$\Phi \circ l_i\colon F(i) \to C$$

admits a colimit in C in context Γ for every object i. Then there is a canonical functor

$$I \to C, \quad i \mapsto \operatorname{colim}_{F(i)} \Phi \circ l_i.$$

If, furthermore, C admits I-indexed colimits, then $\operatorname{colim} \Phi$ *exists in C and we have*

$$\operatorname{colim}_{i\colon I} \operatorname{colim}_{F(i)} l_i^* \Phi = \operatorname{colim}_{\int_{i \in I} F(i)} \Phi.$$

Proof. Since the functor p_F is a cocartesian fibration, it is in particular proper. By proper base change, we have[1]

$$\operatorname{colim}_{F(i)} l_i^* = i^*(p_F)_!.$$

[1] Whenever we write $\operatorname{colim}_{F(i)}$ we are talking about colimits in context Γ.

Using a Yoneda argument, the object $(p_F)_!(\Phi)$ exists in C, i.e. the functor

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}((p_F)_!(\Phi), G) \simeq \mathrm{Hom}_{\mathrm{Hom}(\int F, C)}(\Phi, p_F^* G)$$

is representable for some object $(p_F)_!(\Phi)$ and we have

$$(p_F)_!(\Phi)(i) = \mathrm{colim}_{F(i)}\, l_i^* \Phi$$

for any object i of I. It is then also clear that

$$\mathrm{colim}_{\int F} \Phi = \mathrm{colim}_I (p_F)_! \Phi,$$

by inspection of universal properties. □

Corollary 9.8.9. *Let I and C be categories with I small. We assume that I-indexed colimits exist in C. and we consider functors $F\colon I \to \mathrm{Cat}$ and $\psi\colon \mathrm{colim}_I F \to C$. Assume that for every object i of C the colimit of the composite $F(i) \xrightarrow{\lambda_i} \mathrm{colim}_i F \xrightarrow{\psi} C$ exists in C. Then also the colimit of ψ exists and*

$$\mathrm{colim}_{\mathrm{colim}_I F}\, \psi \simeq \mathrm{colim}_{i \in I}\, \mathrm{colim}_{F(i)}\, \gamma_i^* \psi.$$

Proof. Since I is small, the category $\mathrm{colim}_I F$ is a localization of $\int_I F$, and thus there is a localization functor $\gamma\colon \int_I F \to \mathrm{colim}_I F$. Since every localization is terminal by Proposition 8.1.9, it follows that there is an equivalence

$$\mathrm{colim}_{\mathrm{colim}_I F}\, \psi \simeq \mathrm{colim}_{\int_I F}\, \gamma^* \psi.$$

Byt this is equivalent to $\mathrm{colim}_{i\colon I}\, \mathrm{colim}_{F(i)}\, \gamma_i^* \psi$ by Lemma 9.8.8. □

Lemma 9.8.10. *Let I be small and $X\colon I \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. Then the canonical functor*

$$p\colon \int_{i\colon I} C_{/X_i} \to C$$

is a cartesian fibration.

Proof. Given a presheaf $X\colon C^{\mathrm{op}} \to \mathrm{Grpd}$, we will see $C_{/X}$ through the following pullback square

$$\begin{tikzcd} C_{/X} \arrow[r] \arrow[d] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})_{/X} \arrow[d] \\ C \arrow[r, "h_C"] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \end{tikzcd}$$

where h_C is the Yoneda embedding.

We will prove that p is a locally cartesian fibration and that p-cartesian morphisms are stable under composition. Let $u\colon i \to j$ be a morphism in I, with induced morphism $u_!\colon X_i \to X_j$. Given a presheaf $X\colon C^{\mathrm{op}} \to \mathrm{Grpd}$, we will see $C_{/X}$ through the following pullback square

$$\begin{tikzcd} C_{/X} \arrow[r] \arrow[d] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})_{/X} \arrow[d] \\ C \arrow[r, "h_C"] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \end{tikzcd}$$

where h_C is the Yoneda embedding. The explicit description of the Grothendieck construction given in Remark 9.4.19 means that a morphism in $\int_{i\colon I} C_{/X_i}$ over $f\colon a \to b$ in C amounts to a commutative square of the form

$$\begin{tikzcd} h_C(a) \arrow[r, "h_C(f)"] \arrow[d, "s"'] & h_C(b) \arrow[d, "t"] \\ X_i \arrow[r, "u_!"] & X_j \end{tikzcd}$$

in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$, where $u\colon i \to j$ is a morphism in I. We see that such a morphism is locally p-cartesian when u is invertible, and thus that locally cartesian lifts obviously exist. Proving that locally p-cartesian morphisms are stable under composition is straightforward. $\square$

Lemma 9.8.11. *Let I be small and $X\colon I \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. Consider the pullback*

$$\begin{tikzcd} C/X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet^{\mathrm{op}} \arrow[d, "\pi_{\mathrm{univ}}^{\mathrm{op}}"] \\ C \arrow[r, "X_i^{\mathrm{op}}"] & \mathrm{Cat}, \end{tikzcd}$$

which provides a functor $I \to \mathrm{Cat}, i \mapsto C/X_i$. Then there is an equivalence

$$\mathrm{colim}_{i\in I} C/X_i \simeq C/(\mathrm{colim}_{i\in I} X_i).$$

Proof. We start with some observations:

- The functor $\mathrm{Cat}_{/C} \to \mathrm{Cat}, (X, p) \mapsto X$ preserves colimits, since it admits a right adjoint $C \times -\colon \mathrm{Cat} \to \mathrm{Cat}_{/C}$. Furthermore this functor is conservative. Hence we may compute this colimit either in $\mathrm{Cat}_{/C}$ or Cat.
- The category $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ is equivalent to the full subcategory $\mathrm{RFib}(C) \subseteq \mathrm{Cat}_{/C}$ consisting of the right fibrations. This equivalence is given by sending X to $(C/X, p_X\colon C/X \to C)$.
- So it will suffice to show that $\mathrm{colim}_{i\in I} C/X_i$, computed in $\mathrm{Cat}_{/C}$, is still contained in $\mathrm{RFib}(C)$, since then it must also be a colimit in $\mathrm{RFib}(C)$ and hence correspond to the colimit of the X_i in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$.

By virtue of Lemma 9.8.10, the functor $\int_{i\in I} C/X_i \to C$ is a cartesian fibration. Since $\mathrm{colim}_i\, C/X_i$ is obtained by inverting all the morphisms that lie in a fiber over $C^{\simeq}$ and since fiberwise localizations of cartesian fibrations are still cartesian fibrations by Corollary 9.4.14, we see that the resulting functor $\mathrm{colim}_{i\in I} C/X_i \to C$ is a cartesian fibration as well. To finish the proof, it suffices to check that this functor is conservative. But since this localisation is formed fiberwise, it is compatible with pullbacks under any functor of codomain C. In other words, in order to prove conservativity, we may work objectwise in C. Therefore, we may now assume, without loss of generality, that C is a groupoid. Up to changing contexts, we may even assume that C is the point, in which case $\mathrm{colim}_i\, C/X_i$ coincides with $\mathrm{colim}_i\, X_i$, which is a groupoid by construction, so that conservativity is then tautological. $\square$

Proposition 9.8.12. *Let C be a small category. The embedding functor*

$$\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \simeq \mathrm{RFib}(C) \hookrightarrow \mathrm{Cat}_{/C}$$

has a right adjoint defined by assigning to each $f\colon X \to C$ the presheaf

$$c \mapsto \mathrm{Hom}_{\mathrm{Cat}_{/C}}(C_{/c}, X)\,.$$

Proof. This follows right away from Corollary 9.6.12 and Lemma 9.8.11. $\square$

9.9 Descent and Quillen's Theorem B

Consider a bigger universe Cat' so that Cat is Cat'-small and so that Cat is Cat'-small and there is a fully faithful functor $\mathrm{Cat} \hookrightarrow \mathrm{Cat}'$. Let Grpd' denote the corresponding universe of groupoids.

$$\begin{tikzcd}
\mathrm{Cat}_{/\mathrm{Cat}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}'_{/\mathrm{Cat}} \arrow[d] \\
\mathrm{Cat} \arrow[r, hook] & \mathrm{Cat}'.
\end{tikzcd}$$

The objects of the category $\mathrm{Cat}_{/\mathrm{Cat}}$ are pairs (X, F) where X is a small category and $F\colon X \to \mathrm{Cat}$ is a functor. The category $\mathrm{Fun}([1], \mathrm{Cat}')$ has a non-full subcategory $\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}')$ whose objects are cocartesian fibrations in Cat' and whose morphisms are the pullback squares in Cat':

$$\begin{tikzcd}
x \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & x' \arrow[d, "f'"] \\
y \arrow[r, "b"] & y'
\end{tikzcd}$$

with f and f' cocartesian fibrations. Let $\mathrm{univ} = (\mathrm{Cat}_\bullet, \pi_{\mathrm{univ}})$, where $\pi_{\mathrm{univ}} colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ is the universal cocartesian fibration with small fibers, seen as an object of $\mathrm{Fun}([1], \mathrm{Cat}')$.

There is an obvious full embedding of the form

$$\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}) \hookrightarrow \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}')$$

and we can form the relative slice category

$$\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})_{/\mathrm{univ}} = \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}) \times_{\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}')} \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}')_{/\mathrm{univ}}\,.$$

We observe that the forgetful functor

$$\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})_{/\mathrm{univ}} \to \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})$$

is an equivalence. Indeed, it suffices to check that it induces an equivalence of animae of the form

$$\mathrm{Map}([1],)\, \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})_{/\mathrm{univ}}) \to \mathrm{Map}([1],)\, \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}))$$

which amounts to see that any pullback square of small categories

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f'"] \\
Y \arrow[r, "b"] & Y'
\end{tikzcd}$$

extends uniquely into pullback squares of the form

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r] \arrow[d, "f'"] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ Y \arrow[r, "b"] & Y' \arrow[r] & \mathrm{Cat}. \end{tikzcd}$$

Evaluating at 1 induces a canonical functor

$$\mathrm{Fun}_{\mathrm{Cart}}^{\mathrm{coCart}}([1], \mathrm{Cat})_{/\mathrm{univ}} \to \mathrm{Cat})_{/\mathrm{Cat}}\,.$$

Theorem 9.9.1. *There is a canonical equivalence*

$$\mathrm{Cat}_{/\mathrm{Cat}} \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Cart}}^{\mathrm{coCart}}([1], \mathrm{Cat})$$

assigning to each pullback square of the form

$$\begin{tikzcd} \int F \arrow[r] \arrow[d, "p_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ X \arrow[r, "F"] & \mathrm{Cat}. \end{tikzcd}$$

the functor p_F, seen as a term of $\mathrm{Fun}_{\mathrm{Cart}}^{\mathrm{coCart}}([1], \mathrm{Cat})$.

Proof. We have a functor

$$\mathrm{Fun}_{\mathrm{Cart}}^{\mathrm{coCart}}([1], \mathrm{Cat}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Cart}}^{\mathrm{coCart}}([1], \mathrm{Cat})_{/\mathrm{univ}} \to \mathrm{Cat}_{/\mathrm{Cat}}\,.$$

We observe that the induced map

$$\mathrm{Map}([1], \mathrm{Fun}_{\mathrm{Cart}}^{\mathrm{coCart}}([1], \mathrm{Cat})) \to \mathrm{Map}([1], \mathrm{Cat}_{/\mathrm{Cat}})$$

is an equivalence: this means that pullback squares of the form

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ Y \arrow[r, "b"] & Y' \arrow[r] & \mathrm{Cat} \end{tikzcd}$$

with Y and Y' small categories are uniquely determined by diagrams of the form

$$\begin{tikzcd} & & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ Y \arrow[r, "b"] & Y' \arrow[r] & \mathrm{Cat}. \end{tikzcd}$$

It is clear the inverse of this equivalence is given by the rule $(X, F) \mapsto p_F$. □

The advantage of this theorem is that it provides a convenient way of computing colimits in $\mathrm{Fun}_{\mathrm{Cart}}^{\mathrm{coCart}}([1], \mathrm{Cat})$:

Theorem 9.9.2. *The category* $\mathrm{Fun}_{\mathrm{Cart}}^{\mathrm{coCart}}([1], \mathrm{Cat})$ *admits small colimits, and the inclusion* $\mathrm{Fun}_{\mathrm{Cart}}^{\mathrm{coCart}}([1], \mathrm{Cat}) \hookrightarrow \mathrm{Fun}([1], \mathrm{Cat})$ *preserves small colimits.*

Proof. Since the functor $\mathrm{Cat} \hookrightarrow \mathrm{Cat}'$ preserves the formation of small colimits and since the functor $\mathrm{Cat}'_{/\mathrm{Cat}} \to \mathrm{Cat}'$ commutes with large colimits (hence with small ones as well), it is clear that the pullback $\mathrm{Cat}_{/\mathrm{Cat}}$ has small colimits and that the forgetful functor

$$\mathrm{Cat}_{/\mathrm{Cat}} \to \mathrm{Cat}$$

commutes with colimits. The fact that $\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})$ admits small colimits is thus an immediate consequence of Theorem 9.9.1. To see that they are preserved by the inclusion, notice that the functors $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}([1], \mathrm{Cat}) \to \mathrm{Cat}$ preserve colimits and are jointly surjective. So it suffices to show that the functors $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}) \to \mathrm{Cat}$ preserve small colimits. This follows from the observation that the composite

$$\mathrm{Cat}_{/\mathrm{Cat}} \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}) \xrightarrow{\mathrm{ev}_1} \mathrm{Cat}$$

agrees with the forgetful functor $(X, F \colon X \to \mathrm{Cat}) \mapsto X$, which preserves colimits, and that the composite

$$\mathrm{Cat}_{/\mathrm{Cat}} \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}) \xrightarrow{\mathrm{ev}_0} \mathrm{Cat}$$

agrees with the functor $(X, F \colon X \to \mathrm{Cat}) \mapsto \int F = X \times_{\mathrm{Cat}} \mathrm{Cat}_{\bullet}$, which also commutes with colimits (since pulling back along an exponentiable functor commutes with colimits). □

Corollary 9.9.3. *Let A be a small category and $X, Y \colon A \to \mathrm{Cat}$ be two functors and $u \colon X \to Y$ a natural transformation. We assume that the following two conditions are satisfied.*

(i) For each object $a \colon A^{\simeq}$, the induced functor $u_a \colon X_a \to Y_a$ is a cocartesian fibration (a cartesian fibration, a right fibration, a left fibration).

(ii) For each morphism $\alpha \colon a_0 \to a_1$ in A, the induced commutative square

$$\begin{tikzcd} X_{a_0} \arrow[r, "\alpha_!"] \arrow[d, "u_{a_0}"'] & X_{a_1} \arrow[d, "u_{a_1}"] \\ Y_{a_0} \arrow[r, "\alpha_!"] & Y_{a_1} \end{tikzcd}$$

is a pullback square in Cat*.*

Then the induced functor $\mathrm{colim}_{a \colon A} X_a \to \mathrm{colim}_{a \colon A} Y_a$ *is a cocartesian fibration (a cartesian fibration, a right fibration, a left fibration, respectively) and the canonical commutative squares*

$$\begin{tikzcd} X_a \arrow[r] \arrow[d, "u_a"'] & \mathrm{colim}_{a \colon A} X_a \arrow[d] \\ Y_a \arrow[r] & \mathrm{colim}_{a \colon A} Y_a \end{tikzcd}$$

are pullback squares for any object $a \colon A^{\simeq}$.

Proof. Let us discuss the case of cocartesian fibrations. The natural transformation u can be interpreted an A-indexed diagram with values in $\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})$. Taking the colimit there and using the preceding theorem obviously yields the conclusions of the corollary.

Assuming that all functors u_a are left fibrations, since there is a full embedding

$$\mathrm{Fun}(A, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(A, \mathrm{Cat})$$

whose essential image consists of those functors taking values in Grpd object-wise, we can see the natural transformation from Y to the constant A-indexed functor with value Cat classifying the left fibrations u_a as natural transformation from Y to the constant functor with value Grpd. In other words, we get a commutative diagram of the form

$$\begin{tikzcd} \mathrm{colim}_{a\colon A} X_a \arrow[r, dashed] \arrow[d] & \mathrm{Grpd}_\bullet \arrow[r] \arrow[d] & \mathrm{Cat}_\bullet \arrow[d] \\ \mathrm{colim}_{a\colon A} Y_a \arrow[r] & \mathrm{Grpd} \arrow[r, hook] & \mathrm{Cat} \end{tikzcd}$$

in which all squares are pullback squares, proving that the functor induced on colimits is a left fibration.

The remaining cases of cartesian fibrations and of right fibrations are proved similarly by duality, or by applying the preceding cases to the diagram induced by composing with the equivalence $(-)^{\mathrm{op}}$, and then applying $(-)^{\mathrm{op}}$ again, using that the latter preserves both pullbacks and colimits. $\square$

Lemma 9.9.4. *Let A be a small category and $X, Y\colon A \to$ Grpd be two functors, with $u\colon X \to Y$ a natural transformation. We assume that, for each morphism $\alpha\colon a_0 \to a_1$ in A, the induced commutative square*

$$\begin{tikzcd} X_{a_0} \arrow[r, "\alpha_!"] \arrow[d, "u_{a_0}"'] & X_{a_1} \arrow[d, "u_{a_1}"] \\ Y_{a_0} \arrow[r, "\alpha_!"] & Y_{a_1} \end{tikzcd}$$

is a pullback square in Grpd*. Then the canonical commutative squares*

$$\begin{tikzcd} X_a \arrow[r] \arrow[d, "u_a"'] & \mathrm{colim}_{a\colon A} X_a \arrow[d] \\ Y_a \arrow[r] & \mathrm{colim}_{a\colon A} Y_a \end{tikzcd}$$

are pullback squares in Grpd *for any object $a\colon A^{\simeq}$.*

Proof. Since the embedding Grpd $\hookrightarrow$ Cat commutes with colimits, so does the induced embedding of categories of arrows

$$\mathrm{Fun}([1], \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}([1], \mathrm{Cat})\,.$$

We apply Corollary 9.9.3 noticing that any functor between groupoids is both a left and a right fibration. $\square$

Theorem 9.9.5 (Quillen's Theorem B)**.** *Let $u\colon A \to B$ be a functor between small categories. The following conditions are equivalent.*

(i) For any morphism $\beta\colon b \to b'$ in B, the induced functor

$$\beta_!\colon A_{/b} \to A_{/b'} \qquad (\beta^*\colon A_{b'/} \to A_{b/})$$

induces an equivalence $\Pi_\infty(A_{/b}) \xrightarrow{\sim} \Pi_\infty(A_{/b'})$ (an equivalence $\Pi_\infty(A_{b'/}) \xrightarrow{\sim} \Pi_\infty(A_{b/})$, respectively).

(ii) For any object b of B, the square

$$\begin{tikzcd} \Pi_\infty(A_{/b}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_\infty(A) \arrow[d, "\Pi_\infty(u)"] \\ e = \Pi_\infty(B_{/b}) \arrow[r] & \Pi_\infty(B) \end{tikzcd} \quad \textit{(the square} \quad \begin{tikzcd} \Pi_\infty(A_{b/}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_\infty(A) \arrow[d, "\Pi_\infty(u)"] \\ e = \Pi_\infty(B_{b/}) \arrow[r] & \Pi_\infty(B) \end{tikzcd} \quad \textit{resp.)}$$

is a pullback square in Grpd.

(iii) Any pullback square of the form

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

in which g is a right fibration (a left fibration, respectively) induces a pullback square

$$\begin{tikzcd} \Pi_\infty(A') \arrow[r, "\Pi_\infty(f)"] \arrow[d, "\Pi_\infty(u')"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_\infty(A) \arrow[d, "\Pi_\infty(u)"] \\ \Pi_\infty(B') \arrow[r, "\Pi_\infty(g)"] & \Pi_\infty(B) \end{tikzcd}$$

in Grpd.

(iv) Any pullback square of the form

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

in which g is a smooth functor (a proper functor, respectively) induces a pullback square

$$\begin{tikzcd} \Pi_\infty(A') \arrow[r, "\Pi_\infty(f)"] \arrow[d, "\Pi_\infty(u')"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_\infty(A) \arrow[d, "\Pi_\infty(u)"] \\ \Pi_\infty(B') \arrow[r, "\Pi_\infty(g)"] & \Pi_\infty(B) \end{tikzcd}$$

in Grpd.

Proof. We will prove the formulation in terms of slices, right fibrations and smooth functors. The respective case in terms of coslices, left fibrations and proper functors follows right away by the obvious duality.

Condition (ii) follows from condition (iii) since the canonical projection $B_{/b} \to B$ is a right fibration. Similarly, assuming condition (ii), any morphism $\beta\colon b \to b'$ induces a pullback square of the form

$$\begin{tikzcd} \Pi_\infty(A_{/b}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_\infty(A_{/b'}) \arrow[d] \\ \Pi_\infty(B_{/b}) \arrow[r, "\simeq"] & \Pi_\infty(B_{/b'}) \end{tikzcd}$$

in which the lower horizontal map is an endomorphism of the terminal object, so that the upper horizontal map must be invertible. We will now prove that condition (i) implies condition (ii).

Let e be the terminal object of Grpd. We will also write e for the constant presheaf with value e. We have $A = A_{/e}$ and $A_{/b} = A \times_B B_{/b} = A_{/u^*(\mathrm{Hom}_B(-,b))}$, where

$$\begin{tikzcd}
\mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "u^*"] \arrow[d, "\simeq"] & \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, "\simeq"] \\
\mathrm{RFib}(B) \arrow[r, "u^*"] & \mathrm{RFib}(A).
\end{tikzcd}$$

Since Π_∞ commutes with colimits (being a left adjoint), Lemma 9.8.11 implies that

$$X \mapsto \Pi_\infty(A/X)$$

preserves colimits. But the functor u^* also has a left adjoint and thus commutes with colimits. This proves that the assignment $Y \mapsto \Pi_\infty(A/u^*Y)$ commutes with colimits. Since we have

$$\mathrm{colim}_B\, h_B = e \qquad \text{and} \qquad u^*(e) = e\,,$$

by Theorem 9.5.2, we see that $\mathrm{colim}_{b\colon B}\, \pi_\infty(A_{/b}) \simeq \Pi_\infty(A_{/e}) \simeq \Pi_\infty(A)$. Since for every morphism $\beta\colon b \to b'$ we have a pullback square

$$\begin{tikzcd}
\Pi_\infty(A_{/b}) \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_\infty(A_{/b'}) \arrow[d] \\
\Pi_\infty(B_{/b}) \arrow[r, "\simeq"] & \Pi_\infty(B_{/b'})
\end{tikzcd}$$

it follows from Lemma 9.9.4 that the commutative square

$$\begin{tikzcd}
\Pi_\infty(A_{/b}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_\infty(A) \arrow[d, "\Pi_\infty(u)"] \\
e = \Pi_\infty(B_{/b}) \arrow[r] & \Pi_\infty(B)
\end{tikzcd}$$

induced by passing to colimits are pullback squares as desired.

We will finally prove that condition (ii) implies condition (iii). For any object b' of B', if we put $b = g(b')$, the induced functor

$$B'_{/b'} \to B_{/b}$$

is an equivalence over B (it is a terminal functor between right fibrations). The horizontal arrows in the commutative square

$$\begin{tikzcd}
A'_{/b'} \arrow[r] \arrow[d] & A_{/b} \arrow[d] \\
B'_{/b'} \arrow[r] & B_{/b}
\end{tikzcd}$$

are equivalences and we thus obtain from condition (ii) a pullback squares of the form

$$\begin{tikzcd}
\Pi_\infty(A'_{/b'}) \arrow[r] \arrow[d] & \Pi_\infty(A) \arrow[d, "\Pi_\infty(u)"] \\
\Pi_\infty(B'_{/b'}) \arrow[r] & \Pi_\infty(B)
\end{tikzcd}$$

for each object b' in B'. This means that the colimit of this B'-indexed diagram may be computed in $\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})$. Since, as explained above, the colimit of the $\Pi_\infty(A'_{/b'})$'s is $\Pi_\infty(A')$, this shows that condition (iii) follows from condition (ii).

Since any right fibration is smooth, it is clear that condition (iii) follows from condition (iv). For the converse, given a pullback square of the form

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

in which g is a smooth functor, we choose a factorization of g into $g = q \circ j$ with a terminal functor $j \colon B' \to B''$ and a right fibration $q \colon B'' \to B$. We define $A'' = B'' \times_B A$. The functor j induces a functor $i \colon A' \to A''$ that is terminal (we see this by applying Corollary 8.4.7 in the case where φ is the identity). If $p \colon A'' \to A$ denotes the base change of q, and $u'' \colon A'' \to B''$ the canonical projection, we obtain a commutative diagram of the form

$$\begin{tikzcd} \Pi_\infty(A') \arrow[r, "\Pi_\infty(i)"] \arrow[d, "\Pi_\infty(u')"'] & \Pi_\infty(A'') \arrow[r, "\Pi_\infty(p)"] \arrow[d, "\Pi_\infty(u'')"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_\infty(A) \arrow[d, "\Pi_\infty(u)"] \\ \Pi_\infty(B') \arrow[r, "\Pi_\infty(j)"] & \Pi_\infty(B'') \arrow[r, "\Pi_\infty(q)"] & \Pi_\infty(B) \end{tikzcd}$$

in which the two horizontal maps of the square at the left hand side are equivalences. Therefore, since the square at the right hand side is a pullback square, so is the composite. $\square$

10 Localizations

10.1 Cartesian products in localizations

We fix a universe of small categories Cat. Given a category C and a subcategory W, we will say *localization of C by W* instead of localization of C by $\mathrm{Map}([1], W)$. We will often denote by $L(C, W)$ the localization of C by W.

Proposition 10.1.1. *Let C be a category and $W \hookrightarrow C$ a subcategory. If C has a terminal object then so does the localization $L(C, W)$ and the localization functor $\gamma \colon C \to L(C, W)$ preserves terminal objects.*

Proof. Since terminal functors are stable under composition, this follows right away from the facts that localization functors always are terminal (Proposition 8.1.9) and that a terminal object is the same thing as a terminal functor out of the terminal category (Proposition 8.1.13). □

Definition 10.1.2. A subcategory W of a category C is *unital* if $C^{\simeq}$ is a subcategory of W (i.e. if $\mathrm{id}_x : W$ for any $x : C^{\simeq}$).

Proposition 10.1.3. *Let C and C' be two categories, with unital subcategories*

$$W \hookrightarrow C \quad \text{and} \quad W' \hookrightarrow C'$$

respectively, and associated localization functors

$$\gamma \colon C \to L(C, W) \quad \text{and} \quad \gamma' \colon C' \to L(C', W') .$$

Then the functor

$$\gamma \times \gamma' \colon C \times C' \to L(C, W) \times L(C', W')$$

induces an equivalence

$$L(C \times C', W \times W') \xrightarrow{\sim} L(C, W) \times L(C', W') .$$

Proof. Let $\varphi \colon C \times C' \to D$ be any functor. Given two maps $f : x \to y$ and $f' : x' \to y'$ in C and C' respectively, we have a canonical commutative square of the form below.

$$\begin{tikzcd}
(x, x') \arrow[r, "(\mathrm{id}_x, f')"] \arrow[d, "(f, \mathrm{id}_{x'})"'] & (x, y') \arrow[d, "(f, \mathrm{id}_y)"] \\
(y, x') \arrow[r, "(\mathrm{id}_y, f')"] & (y, y')
\end{tikzcd}$$

Since both W and W' are unital, all identities of C belong to W and all identities of C' belong to W'. This means that the property for the functor φ to send all maps in $W \times W'$ to isomorphisms is equivalent to the conjunction of the following two properties:

- for any map $f\colon x \to y$ in W and any object x' of C', the map $\varphi(f, \mathrm{id}_{x'})$ is an isomorphism in D;
- for any map $f'\colon x' \to y'$ in W' and any object x of C, the map $\varphi(\mathrm{id}_x, f')$ is an isomorphism in D.

In other words, the obvious equivalence

$$\mathrm{Fun}(C \times C', D) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{Fun}(C', D))$$

restricts to an equivalence

$$\mathrm{Fun}^{W\times W'}(C \times C', D) \xrightarrow{\sim} \mathrm{Fun}^{W}(C, \mathrm{Fun}^{W'}(C', D))$$

functorially in D. This implies that there is a canonical equivalence

$$\mathrm{Fun}^{W\times W'}(C \times C', D) \xrightarrow{\sim} \mathrm{Fun}(L(C,W), \mathrm{Fun}(L(C',W'), D))\,.$$

Using the equivalence

$$\mathrm{Fun}(L(C,W) \times L(C',W'), D) \xrightarrow{\sim} \mathrm{Fun}(L(C,W), \mathrm{Fun}(L(C',W'), D))$$

we see now that the functor $\gamma \times \gamma'$ exhibits $L(C,W) \times L(C',W')$ as a localization of $C \times C'$ by $W \times W'$. □

Lemma 10.1.4. *Let $F\colon C \rightleftarrows D\colon G$ be an adjunction. We consider two subcategories*

$$V \hookrightarrow C \quad \textit{and} \quad W \hookrightarrow D$$

respectively, and we assume that F sends any map in V to a map in W, and, conversely, that G sends any map in W to a map in V. Then there is an induced adjunction of the form

$$\bar{F}\colon L(C,V) \rightleftarrows L(D,W)\colon \bar{G}$$

with canonical commutative squares

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d] & D \arrow[d] \\ L(C,V) \arrow[r, "\bar{F}"] & L(D,W) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} D \arrow[r, "G"] \arrow[d] & C \arrow[d] \\ L(D,W) \arrow[r, "\bar{G}"] & L(C,V) \end{tikzcd}$$

in which the vertical maps are the localization functors. Furthermore, for any objects x and y in C and D, respectively, the unit and co-unit maps

$$x \to \bar{G}(\bar{F}(x)) \quad \textit{and} \quad \bar{F}(\bar{G}(y)) \to y$$

are the images of the unit and co-unit maps

$$x \to G(F(x)) \quad \textit{and} \quad F(G(y)) \to y$$

respectively.

Proof. For any category E, there is an induced adjunction of the form

$$G^*\colon \operatorname{Fun}(D,E) \rightleftarrows \operatorname{Fun}(C,E)\colon F^*\,.$$

Our hypotheses on F and G ensure that this restricts to an adjunction of the form

$$G^*\colon \operatorname{Fun}^W(D,E) \rightleftarrows \operatorname{Fun}^V(C,E)\colon F^*\,.$$

We thus get an adjunction of the form

$$\bar{G}^*\colon \operatorname{Fun}(L(D,W),E) \rightleftarrows \operatorname{Fun}(L(C,V),E)\colon \bar{F}^*\,.$$

The unit $\mathrm{id} \to \bar{F}^*\bar{G}^*$ may be seen as a functor

$$\bar{\eta}\colon L(C,V) \to \operatorname{Fun}([1],E)$$

which is determined by the functor

$$\eta\colon C \to \operatorname{Fun}([1],E)$$

corresponding to the unit $\mathrm{id} \to GF$. Similarly, the co-unit $\bar{G}^*\bar{F}^* \to \mathrm{id}$ may be interpreted as a functor

$$\bar{\varepsilon}\colon L(D,W) \to \operatorname{Fun}([1],E)$$

that is fully determined by the induced functor

$$\varepsilon\colon D \to \operatorname{Fun}([1],E)$$

corresponding to the co-unit $FG \to \mathrm{id}$. □

Proposition 10.1.5. *Let C be a category and $W \hookrightarrow C$ a unital subcategory. We assume that C has binary cartesian products and that W is stable under cartesian product in the sense that, for any maps f and g in W, the induced map $f \times g$ belongs to W. Then the localization $L(C,W)$ has binary cartesian products and the localization functor $\gamma\colon C \to L(C,W)$ commutes with them.*

Proof. There is an adjunction

$$\Delta\colon C \rightleftarrows C \times C\colon (-)\times(-)$$

where Δ is the functor defined by the rule $x \mapsto (x,x)$. The property that W is stable under binary cartesian product precisely means that Lemma 10.1.4 applies in this situation, yielding an adjunction of the form

$$\bar{\Delta}\colon L(C,W) \rightleftarrows L(C\times C, W\times W)\colon \overline{(-)\times(-)}\,.$$

But, since W is unital, Proposition 10.1.3 provides a canonical equivalence

$$L(C\times C, W\times W) \xrightarrow{\sim} L(C,W)\times L(C,W)$$

whose composition with $\bar{\Delta}$ simply is the diagonal functor

$$L(C,W) \to L(C,W)\times L(C,W)\,, \qquad x \mapsto (x,x)\,.$$

This proves the existence of cartesian products in $L(C,W)$. The explicit compatibilities between units and co-units in Lemma 10.1.4 show that the localization functor from C to $L(C,W)$ is compatible with the formation of cartesian products. □

10.2 Localizations and adjunctions

Proposition 10.2.1. *Let C be a category. If $f\colon C \to D$ is a localization of C by a subcategory $W \hookrightarrow C$ and admits a right (resp. left) adjoint $g\colon D \to C$, then g is fully faithful.*

Proof. We will consider the case of right adjoints (the case of left adjoints is done similarly, or follows from the case of right adjoints by applying the operator $(-)^{\mathrm{op}}$).
We choose a universe Cat so that C and D are small. It is clear that g is fully faithful if and only if g^{op} has the same property. In order to prove that g is fully faithful, it thus suffices to check that

$$g_!\colon \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) \to \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

is fully faithful by Proposition 9.2.11. But we have an adjunction of the form

$$f_!\colon \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})\colon f^*$$

and, by virtue of Theorem 9.3.17, the fact that g is right adjoint to f is equivalent to the property that there is an isomorphism of functors of the form

$$f^* \xrightarrow{\sim} g_!\,.$$

On the other hand, the functor f^* is fully faithful by definition of localizations, which proves that $g_!$ is fully faithful as well. □

Remark 10.2.2. If $f\colon C \to D$ has a fully faithful right adjoint $g\colon D \to C$, then we may consider the functor

$$f_!\colon \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})\,.$$

For any object x of C, the functor $\operatorname{Hom}_C(-, x)$ is sent through $f_!$ to $\operatorname{Hom}_D(-, f(x))$. On the other hand, for a functor $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$, the property that the unit map

$$\eta_f\colon F \to f^* f_!(F)$$

is invertible is equivalent to the property that F belongs to the essential image of f^*, which, by virtue of the preceding proposition, is equivalent to the property that F sends all maps inverted by f to an equivalence. For $F = \operatorname{Hom}_C(-, x)$ representable, the unit map takes the form of the canonical map

$$\operatorname{Hom}_C(-, x) \to \operatorname{Hom}_D(f(-), f(x)) = \operatorname{Hom}_D(-, gfx)$$

induced by the unit $\eta_x\colon x \to gfx$. In other words, the property that $\operatorname{Hom}_C(-, x)$ sends all maps of C that are inverted through f to equivalences is equivalent to the property that the unit of the adjunction induces an equivalence $x \xrightarrow{\sim} gfx$.

Proposition 10.2.3. *Let $f\colon C \to D$ be a functor with a fully faithful right adjoint $g\colon D \to C$. We consider a category I and a functor $F\colon I \to D$. If the colimit (the limit, respectively) of gF exists in C then the colimit (the limit, respectively) of F exists in D: more precisely, the colimit (the limit, resp.) of F is the image by f of the colimit (the limit, resp.) of gF.*

Proof. We may assume that there is a universe Cat so that C, D as well as I are all small. Let us assume that the colimit of gF exists in C. Since f is a left adjoint, it preserves colimit cocones:

$$\begin{aligned}\operatorname{Hom}_D(f(\operatorname{colim}_I gF), x) &\simeq \operatorname{Hom}_C(\operatorname{colim}_I gF, gx)\\ &\simeq \lim_{I^{\mathrm{op}}} \operatorname{Hom}_C((gF)^{\mathrm{op}}, gx)\\ &\simeq \lim_{I^{\mathrm{op}}} \operatorname{Hom}_D(F^{\mathrm{op}}, x)\end{aligned}$$

holds functorially in $x\colon D$.
Let us assume that gF has a limit in C. We claim that $f(\lim_I gF)$ is the limit of F in D. We observe first that, for each object i of I, the presheaf $\operatorname{Hom}_C(-, gF(i))$ sends any map inverted by f to an equivalence (see Remark 10.2.2 above). Therefore, the functor

$$\lim_I \operatorname{Hom}_C(-, gF) = \operatorname{Hom}_C(-, \lim_I gF)$$

has the same property. Since the functor

$$f_!\colon \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})$$

coincides with f on representable presheaves and the same is true for its right adjoint $f^* = g_!$ with respect to g, Remark 10.2.2 thus tells us that there is a canonical isomorphism of the form

$$\lim_I gF \xrightarrow{\sim} gf(\lim_I gF)\,.$$

This yields

$$\begin{aligned}\operatorname{Hom}_D(x, f(\lim_I gF)) &\simeq \operatorname{Hom}_C(gx, gf(\lim_I gF))\\ &\simeq \operatorname{Hom}_C(gx, \lim_I gF)\\ &\simeq \lim_I \operatorname{Hom}_C(gx, gF)\\ &\simeq \lim_I \operatorname{Hom}_D(x, F)\end{aligned}$$

functorially in x. □

Corollary 10.2.4. *Let $f\colon C \to D$ be a functor with a fully faithful right adjoint $g\colon D \to C$. We consider a category I and assume that colimits (limits, resp.) indexed by I exist in C. Then colimits (limits, resp.) indexed by I exist in D.*

10.3 Calculus of fractions

We fix a universe Cat and a small category C equipped with a subcategory

$$W \hookrightarrow C\,.$$

We denote by $\gamma\colon C \to L(C)$ the localization of C by $\operatorname{Map}([1], W)$. Our goal is to compute the functor

$$\operatorname{Hom}_{L(C)}(x, -)\colon L(C) \to \mathrm{Grpd}$$

for a fixed object x of $L(C)$. Since the map

$$\gamma^{\simeq}\colon C^{\simeq} \to L(C)^{\simeq}$$

is a cover by Corollary 11.5.14, we ought to understand functors of the form

$$\mathrm{Hom}_{L(C)}(\gamma(x), -)\colon L(C) \to \mathrm{Grpd}$$

for objects $x\colon C^{\simeq}$. Since functors out of $L(C)$ are the same thing as functors out of C sending morphisms of W to isomorphisms, we will in fact study functors of the form

$$\mathrm{Hom}_{L(C)}(\gamma(x), \gamma(-))\colon C \to \mathrm{Grpd}\,.$$

For any objects x and y in C, any map $s\colon z \to x$ that comes from W yields, for any map $f\colon z \to y$ in C, a map

$$\gamma(f) \circ \gamma(s)^{-1}\colon \gamma(x) \to \gamma(y)$$

in the localization $L(C)$. Many situations ensure that all maps in $L(C)$ are obtained in this way. This is what we are about to make precise.

Definition 10.3.1. A *putative right calculus of fractions* at an object $x\colon C^{\simeq}$ is a functor of the form

$$p\colon W(x) \to C_{/x}$$

with the following properties:

(i) For any object $w\colon W(x)^{\simeq}$, if we denote by $s_w\colon z_w \to x$ the map corresponding to $p(w)$, so that

$$p(w) = (z_w, s_w)\,,$$

the map s_w is sent to an isomorphism in $L(C)$.

(ii) There is an object i_x in $W(x)$ (in context $\{x\}$) and $p(i_x) = (x, \mathrm{id}_x)$ holds true.

(iii) The category $W(x)$ is weakly contractible.

Dually, a *putative left calculus of fractions* at $x\colon C^{\simeq}$ is a functor of the form $p\colon W(x) \to C_{x/}$ such that the induced functor

$$p^{\mathrm{op}}\colon W(x)^{\mathrm{op}} \to (C_{x/})^{\mathrm{op}} = (C^{\mathrm{op}})_{/x}$$

is a putative right calculus of fractions at $x\colon (C^{\mathrm{op}})^{\simeq}$.

Notation 10.3.2. If $x\colon C^{\mathrm{op}}$ any $y\colon C$, we define

$$\mathrm{Hom}_{L(C)}(x, y) := \mathrm{Hom}_{L(C)}(\gamma^{\mathrm{op}}(x), \gamma(y))\,.$$

Notation 10.3.3. There is a map

$$u\colon \mathrm{Hom}_C(x, y) \to \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y)$$

functorial in y, induced by $i_x\colon W(x)^{\simeq}$ and the identification $p(i_x) = x$. Similarly, there is a map of the form

$$v\colon \mathrm{Hom}_{L(C)}(x, y) \to \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_{L(C)}(z_w^{\mathrm{op}}, y)$$

functorially in $y\colon C$. We observe that v is always an equivalence: since $W(x)^{\mathrm{op}}$ is weakly contractible (because $W(x)$ has this property), assertion (iii) of Lemma 9.7.8 tells us that, for each y, the functor

$$w \mapsto \mathrm{Hom}_{L(C)}(z_w^{\mathrm{op}}, y)$$

is constant, with value its evaluation at the object i_x, and Lemma 9.7.11 implies that the colimit of this diagram coincides with its evaluation at i_x. Composing the obvious map

$$\operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \to \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_{L(C)}(z_w^{\mathrm{op}}, y)$$

with the inverse of v yields a map of the form

$$\operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \to \mathrm{Hom}_{L(C)}(x, y)$$

that is functorial in y. The latter map will be referred to as the *canonical map*.

Definition 10.3.4. A *right calculus of fractions* at x is a putative right calculus of fractions $p\colon W(x) \to C_{/x}$ such that the functor from C to Grpd defined by the rule

$$y \mapsto \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y)$$

sends any map coming from W to an equivalence in Grpd.

Dually a *left calculus of fractions* at x is a putative left calculus of fractions at x that becomes a right calculus of fractions at x after applying the operator $(-)^{\mathrm{op}}$.

Proposition 10.3.5. *Let $p\colon W(x) \to C_{/x}$ be a putative right calculus of fractions at an object x of C. A necessary and sufficient condition for this putative right calculus of fractions at x to be a right calculus of fractions at x is that the canonical map*

$$\operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \xrightarrow{\sim} \mathrm{Hom}_{L(C)}(x, y)$$

is invertible for all y. Furthermore, if this is the case, this isomorphism is uniquely characterized by its functoriality in y and by the property that it sends the identity of x, seen in C (and then as a term of the colimit through i_x), to the identity of x, seen as an object of $L(C)$.

Proof. We have an adjunction of the form

$$\gamma_!\colon \mathrm{Fun}(C, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}(L(C), \mathrm{Grpd})\colon \gamma^*$$

and the right adjoint γ^* is fully faithful, with essential image those functors from C to Grpd sending any map coming from W to an isomorphism. Furthermore, the functor γ^* is a left adjoint as well (by Proposition 9.6.14) and thus commutes with small colimits. Given any functor $F\colon C \to \mathrm{Grpd}$, the unit map

$$\eta_F\colon F \to \gamma^*\gamma_!(F)$$

is thus compatible with colimits in $\mathrm{Fun}(C, \mathrm{Grpd})$. Furthermore, η_F is an isomorphism if and only if F sends any map coming from W to isomorphisms in Grpd. For $F = \mathrm{Hom}_C(x, -)$, the unit map coincides with the map

$$\eta\colon \mathrm{Hom}_C(x, -) \to \mathrm{Hom}_{L(C)}(\gamma(x), \gamma(-))$$

induced by γ. For $F = \operatorname{colim}_{w\colon W(x)^{\mathrm{op}}} \operatorname{Hom}_C(z_w^{\mathrm{op}}, -)$, the unit map

$$\eta' \colon \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \operatorname{Hom}_C(z_w^{\mathrm{op}}, -) \to \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \operatorname{Hom}_{L(C)}(\gamma^{\mathrm{op}}(z_w^{\mathrm{op}}), \gamma(-))$$

is the colimit of unit maps for each z_w. As explained above, η' is an isomorphism if and only if its domain sends any map coming from W to an isomorphism. We consider the following commutative diagram, functorially in $y\colon C$,

$$\begin{tikzcd} \operatorname{Hom}_C(x, y) \arrow[r, "u"] \arrow[d, "\eta"'] & \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \operatorname{Hom}_C(z_w^{\mathrm{op}}, y) \arrow[d, "\eta'"] \\ \operatorname{Hom}_{L(C)}(x, y) \arrow[r, "v"] & \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \operatorname{Hom}_{L(C)}(z_w^{\mathrm{op}}, y) \end{tikzcd}$$

where both vertical maps are unit maps, whereas u and v are the maps introduced in Notation 10.3.3. The invertibility of η' is equivalent to the invertibility of the map $v^{-1} \circ \eta'$. The last assertion is a direct application of the Yoneda lemma applied to $L(C)^{\mathrm{op}}$. □

Corollary 10.3.6. *Let $p\colon W(x) \to C_{/x}$ be a right calculus of fractions at an object x. We consider the adjunction on presheaves categories*

$$\gamma_! \colon \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \operatorname{Fun}(L(C)^{\mathrm{op}}, \mathrm{Grpd}) \colon \gamma^*$$

induced by the localization functor $\gamma\colon C \to L(C)$. Then, for any presheaf $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ there is a canonical isomorphism of the form

$$\operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} F(z_w^{\mathrm{op}}) \xrightarrow{\sim} \gamma_!(F)(\gamma(x)) .$$

Proof. Since $W(x)^{\mathrm{op}}$ is weakly contractible, $i_x\colon W(x)^{\simeq}$ induces an isomorphism of the form

$$G(\gamma(x)) \xrightarrow{\sim} \operatorname{colim}_{w\colon W(x)^{\mathrm{op}}} G(\gamma^{\mathrm{op}}(z_w^{\mathrm{op}})) .$$

Taking the inverse of this isomorphism for $G = \gamma_!(F)$, together with the canonical map

$$\operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} F(z_w) \to \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} G(\gamma^{\mathrm{op}}(z_w^{\mathrm{op}}))$$

induced by the unit map $F \to \gamma^*(G)$, this yields a canonical map of the form:

$$\operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} F(z_w^{\mathrm{op}}) \xrightarrow{\sim} \gamma_!(F)(\gamma(x)) .$$

The formation of this map is compatible with colimits. Therefore, to prove that it is invertible, Corollary 9.5.3 tells us that it suffices to prove that it is invertible for $F = \operatorname{Hom}_C(-, y)$ for some object y of C. But then, $\gamma_!(F)$ is nothing else than $\operatorname{Hom}_{L(C)}(-, \gamma(y))$ by Corollary 9.3.18. In this case, the isomorphism is a characterization of right calculus of fractions at x, by Proposition 10.3.5. □

Remark 10.3.7. If we have a right calculus of fractions at x as in the previous corollary, the Yoneda lemma applied for C and for $L(C)$ respectively allows us to write

$$\operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \operatorname{Hom}(h_C^{\mathrm{op}}(z_w^{\mathrm{op}}), F) \xrightarrow{\sim} \operatorname{Hom}(h_{L(C)}(x), \gamma_!(F))$$

for any presheaf F on C. Using Corollary 9.3.18, one can recast this formula by saying that any right calculus of fractions at x of the form

$$p\colon W(x) \to C_{/x}$$

induces through the Yoneda embedding of C a right calculus of fractions at $h_C(x)$

$$W(x) \xrightarrow{p} C_{/x} \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})_{/h_C(x)}$$

with respect to the localization of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ by the subcategory of maps that become invertible through the functor $\gamma_!$.

Remark 10.3.8. Let $p\colon W(x) \to C_{/x}$ be a right calculus of fractions at $x\colon C^{\simeq}$. Given any other object $y\colon C^{\simeq}$, we may consider the associated right fibration

$$C_{/y} \to C$$

and form the pullback

$$W(x)_{/y} = W(x) \times_C C_{/y}\,.$$

We claim that there is a canonical equivalence of the form

$$\Pi_\infty(W(x)_{/y}) \xrightarrow{\sim} \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y)$$

and thus

$$\Pi_\infty(W(x)_{/y}) \xrightarrow{\sim} \mathrm{Hom}_{L(C)}(x, y)\,.$$

Indeed, the right fibration $W(x)_{/y} \to W(x)$ corresponds to the functor obtained by composing $\mathrm{Hom}_C(-, y)$ with the functor from $W(x)^{\mathrm{op}}$ to C^{op} induced by p^{op}. The explicit construction of colimits in Cat (as in the proof of Theorem 9.8.1) means that

$$\operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \xrightarrow{\sim} \Pi_\infty((W(x)_{/y})^{\mathrm{op}})\,.$$

Since there is a canonical equivalence

$$\Pi_\infty(W(x)_{/y}) \xrightarrow{\sim} \Pi_\infty((W(x)_{/y})^{\mathrm{op}})$$

by Corollary 9.2.6, this proves our claim above. By virtue of Corollary 11.5.14, we thus have a canonical cover of the form

$$(W(x)_{/y})^{\simeq} \to \mathrm{Hom}_{L(C)}(x, y)$$

sending pairs (w, f), with $w\colon W(x)^{\simeq}$ and $f\colon z_w \to y$ in C, to the map $\gamma(f) \circ \gamma(s_w)^{-1}$ in $L(C)$.

11 Peano arithmetic and simplices

11.1 Sets

We begin by discussing in more detail the behavior of sets in or synthetic framework. Recall from Definition 11.2.17 our synthetic definition of a set:

Definition 11.1.1. A *set* is a groupoid X such that the diagonal $\Delta\colon X \to X \times X$ is an embedding, in the sense of Definition 1.3.24. We denote by $\mathrm{Set} \subseteq \mathrm{Grpd}$ the full subcategory spanned by the sets.

Warning 11.1.2. The approach to sets presented here differs fundamentally from traditional axiomatic set theories like ZFC. Rather than working with a global membership relation $\in$, where for any two sets X and Y one can always ask whether $X \in Y$, we adopt a structural perspective in the tradition of Lawvere. This structural approach characterizes sets through their relationships and mappings rather than through membership. For a text book introduction to this structural viewpoint on sets, we refer the reader to Lawvere and Rosebrugh [LR03].

We want to think of a set as a groupoid in which it is a *property* whether two objects are equal or not: if an isomorphism exists, it is unique. We may formalize the notion of a 'property' using the following definition:

Definition 11.1.3. A groupoid P is called a *proposition* if the map $p_P\colon P \to *$ is an embedding, i.e. if the diagonal $\Delta\colon P \to P \times P$ is an equivalence.

Note that if a proposition P admits an absolute object $x\colon * \to P$, it determines a section of the embedding $p_P\colon P \to *$, hence this embedding is an equivalence by Lemma 1.3.28. So if an absolute object x exists, it is unique.

Lemma 11.1.4. *A groupoid X is a set if and only if for every two objects x and y of X the mapping groupoid $X(x, y)$ is a proposition.*

Proof. Since X is a groupoid, we have $X \xrightarrow{\sim} \mathrm{Fun}(\Delta^1, X)$, and consequently the mapping groupoid $X(x, y)$ fits in a pullback square as follows:

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\Delta"] \\ \{(x,y)\} \arrow[r, "{(x,y)}"] & X \times X. \end{tikzcd}$$

Since embeddings are closed under pullback, this implies one direction. For the other direction we consider the objects $x = \mathrm{pr}_1\colon X \times X \to X$ and $y = \mathrm{pr}_2\colon X \times X \to X$. $\square$

Notation 11.1.5. Let X be a set and let x and y be objects of X. We write $x = y$ if the mapping groupoid $X(x, y)$ is inhabited (hence contractible).

Let us now introduce some standard terminology regarding sets:

Terminology 11.1.6. Let X and Y be sets.

(i) An *element*[1] of X, written $x \in X$, is an object x of X;

(ii) A *function* $f: X \to Y$ is a functor from X to Y. Given an element x of X we write $f(x)$ for the resulting element in Y (given by composing x and f).

(iii) We say two functions $f, g: X \to Y$ are *equal*, written $f = g$, if there exists a (necessarily unique) natural isomorphism $f \cong g$.

(iv) We say that f is a *bijection* if it is an equivalence. Note that this is the case if and only if there exists a function $g: Y \to X$ satisfying $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

Observation 11.1.7. When X and Y are sets, the category $\mathrm{Fun}(X, Y)$ is again a set. Its elements are functions from X to Y.

Example 11.1.8. Note that the terminal category $*$ is a set: the diagonal $\Delta: * \to * \times *$ is even an equivalence. We refer to $*$ as the *one-point set*. Similarly the initial category $\emptyset$ is a set, which we will refer to as the *empty set*.

Example 11.1.9. The groupoid of natural numbers $\mathbb{N}$ is a set by Corollary 11.2.12. The injections $\{0\} \hookrightarrow \mathbb{N}$ and $\mathrm{succ}: \mathbb{N} \hookrightarrow \mathbb{N}$ induce a bijection $\{0\} \sqcup \mathbb{N} \xrightarrow{\cong} \mathbb{N}$ by Lemma 11.2.14.

Construction 11.1.10. We can define the category of small sets as follows. We fix a universe Cat and its full subcategory of groupoids. There is a functor

$$\delta: \mathrm{Grpd} \to \mathrm{Fun}([1], \mathrm{Grpd})$$

sending x to the diagonal map $(\mathrm{id}_x, \mathrm{id}_x): x \to x \times x$. The category Set of small sets is defined through the following pullback square.

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd} \arrow[d, "\delta"] \\
\mathrm{Grpd} \arrow[r, hook, "(x \mapsto \mathrm{id}_x)"] & \mathrm{Fun}([1], \mathrm{Grpd})
\end{tikzcd}$$

Lemma 11.1.11. *Let X be a groupoid. The following are logically equivalent:*

(1) X is a proposition.

(2) For any two $x, y : X$, the type $X(x, y)$ is contractible; in formulas $\prod_{x,y:X} \mathtt{is\_Contr}\, X(x, y)$.

(3) For any two $x, y : X$ we have an identification $p : X(x, y)$; in formulas $\prod_{x,y:X} X(x, y)$.

(4) If we have an inhabitant $x : X$, then X is contractible; in formulas $X \to \mathtt{is\_Contr}\, X$.

[1] In [LR03] this is called a *generalized element*.

Proof. 1 $\iff$ 2. This follows the characterization of embeddings between groupoids as fully faithful functors between groupoids.
2 $\implies$ 3. Trivial.
3 $\implies$ 4. If we have $f : ((x,y) : X \times X) \to X(x,y)$ and $c : X$, then X is contracible because $f(c,-)$ is a contraction of X onto c.
4 $\implies$ 2. Assume we have $x, y : X$. In particular, X is inhabited by $x : X$, so that it is contractible by 4. But then $X(x,y)$ is also contractible. □

We can characterize sets in a similar way.

Lemma 11.1.12. *Let X be a groupoid. The following are logically equivalent:*

(1) X is a set.

(2) For any $x, y : X$ and any two identities $p, q : x = y$, we have $p = q$.

(3) For every $x : X$ and every $p : X(x,x)$, we have $p = \mathrm{id}_x$.

Proof. The biimplication "1 $\iff$ 2" follows from Lemma 11.1.11 and the implication " 2 $\implies$ 3 " is trivial. Finally, 2 follows from 3 by path induction on q. In other words, if we have $p, q\colon x = y$ then the composition of the inverse of q with p must be the identity of x, and applying q again this yields $p = q$. □

11.1.1 Decidable equality

Recall that the *negation* of a groupoid Z is defined as $\neg Z :\equiv \mathrm{Map}(Z, \emptyset)$. For identity types, we abbreviate $x \neq y :\equiv \neg(x = y)$ (for $x, y : X$).

Definition 11.1.13. A groupoid X is said to have *decidable equality* if for each $x, y : X$ we either have $x = y$ or $x \neq y$. In formulas

$$\mathtt{hasDecidableEquality}X :\equiv \prod_{x,y:X} x = y \sqcup x \neq y$$

It turns out that having decidable equality is a very strong condition. Indeed, we will show that a groupoid X with decidable equality is automatically a set.

This argument goes back to Hedberg (add reference to Hedberg and maybe Escardó).

Definition 11.1.14. We say that a map $f\colon X \to Y$ of groupoids is *wildly constant* if we have a section of $X \times_Y X \to X \times X$, i.e., an inhabitant of

$$\mathtt{wconstant}(f) :\equiv \prod_{x,y:X} fx = fy.$$

.

Lemma 11.1.15 (Hedberg)**.** *Let X be a groupoid and assume that for each $x, y : X$ we have a wildly constant map $x = y \to x = y$. Then X is a set.*

Before we prove Lemma 11.1.15, we deduce the following

Corollary 11.1.16. *Let X be a groupoid with decidable equality. Then X is a set.*

Proof. By Hedberg's lemma we only have to show that for each $x = y$ we have a wildly constant map $f: x = y \to x = y$. Since X has decidable equality, we may do this by case distinction:

- If we have a $p : x = y$, then we may define f simply as the constant map with value p.
- It we have a $e : x \neq y$, then we may define f as the composite $x = y \xrightarrow{e} \emptyset \to x = y$.

In both cases it is immediate that f is wildly constant. $\square$

Proof of Lemma 11.1.15. Assume that for each x, y we have a wildly constant map $f_{x,y} : x = y \to x = y$. First we note that for each $p : x = y$ we have $f_{x,y}(p) = p \circ f_{x,x}(\mathrm{id}_x)$; indeed, by path induction on p it suffices to check the case $\mathrm{id} : x = x$ which is trivial. Then for two arbitrary $p, q : x = y$ we obtain an identification

$$p = f_{x,y}(p) \circ f_{x,x}(\mathrm{id}_x)^{-1} = f_{x,y}(q) \circ f_{x,x}(\mathrm{id}_x)^{-1} = q$$

where we use the equality $f_{x,y}(p) = f_{x,y}(q)$ coming from the fact that $f_{x,y}$ is wildly constant.

Since p, q were arbitrary, it follows from Lemma 11.1.12 that X is a set. $\square$

Exercise 11.1.17. Assuming the law of excluded middle (i.e. that there is a term $\mathrm{LEM}_E : E \sqcup \neg E$ for each set E), prove the converse of Corollary 11.1.16, i.e., that every set has decidable equality.

11.2 The natural numbers

The fact that the universe of categories Cat is cocomplete gives rise to a groupoid $\mathbb{N}$ of *natural numbers*, together with a canonical addition operation $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ satisfying the expected properties. In this section, we will introduce $\mathbb{N}$ and show that it satisfies the type theoretical version of Peano arithmetics.

Definition 11.2.1. We define the category $\mathrm{B}\,\mathbb{N}$ as the following pushout in Cat:

$$\begin{tikzcd} \partial\Delta^1 \arrow[r] \arrow[d] & \Delta^1 \arrow[d, "\mathrm{one}"] \\ * \arrow[r, "\mathrm{base}"'] \arrow[ur, phantom, "\ulcorner"] & \mathrm{B}\,\mathbb{N}\,. \end{tikzcd}$$

We define the groupoid $\mathbb{N}$ as

$$\mathbb{N} := \mathrm{Hom}_{\mathrm{B}\,\mathbb{N}}(\mathrm{base}, \mathrm{base}),$$

the groupoid of endomorphisms of the canonical basepoint of $\mathrm{B}\,\mathbb{N}$.

- The identity morphism of the base point will be denoted by $0 : \mathbb{N}$;
- The morphism $\mathrm{one} : \Delta^1 \to \mathrm{B}\,\mathbb{N}$ corresponds to an endomorphism of base in $\mathrm{B}\,\mathbb{N}$, resulting in an object $1 : \mathbb{N}$.
- A generic object of $\mathbb{N}$ will be denoted by n or m.

We write

$$+\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}, \qquad n+m := m\circ n$$

for the composition operation of endomorphisms in $\mathrm{B}\,\mathbb{N}$, called the *addition function*. Since composition is unital and associative, we obtain natural isomorphisms

$$(n+m)+k \cong n+(m+k), \qquad 0+n \cong n \cong n+0.$$

We define the *successor function* $\mathrm{succ}\colon \mathbb{N} \to \mathbb{N}$ by $\mathrm{succ}(x) = x+1$, i.e. as the composite

$$\mathrm{succ}\colon \mathbb{N} \cong \mathbb{N}\times\{1\} \to \mathbb{N}\times\mathbb{N} \xrightarrow{+} \mathbb{N}.$$

All in all, we get a triple $(\mathbb{N}, 0, \mathrm{succ})$. Our goal in this section is to show that this triple satisfies a form of Peano arithmetic, allowing us to prove things for natural numbers 'by induction'.

Lemma 11.2.2. *For any category C, there is an equivalence*

$$\mathrm{Map}(\mathrm{B}\,\mathbb{N}, C) \simeq \{f : \mathrm{Map}([1], C) \mid s(f) \cong t(g)\} \simeq \sum_{x:C^{\simeq}} C(x,x).$$

Proof. This is an immediate consequence of the universal property of the pushout. □

Construction 11.2.3. Let C be a category and let $\varphi\colon \mathrm{B}\,\mathbb{N} \to C$ be an endomorphism in C, corresponding to an object $x = \varphi(\mathrm{base})$ of C and a morphism $f = \varphi(\mathrm{one})\colon x \to x$. By passing to hom-groupoids, φ induces a morphism

$$\varphi\colon \mathbb{N} = \mathrm{B}\,\mathbb{N}(\mathrm{base}, \mathrm{base}) \to C(x,x)$$

which we will denote by $n \mapsto f^n$. Since φ preserves identities and composition, we have:

- The endomorphism $f^0\colon x \to x$ is the identity of x in C;
- For $n, m \in \mathbb{N}$ the endomorphism f^{n+m} of x agrees with $f^n \circ f^m$.

We may therefore think of f^n as obtained from f by 'iterating f a number of n times'.

Example 11.2.4. As a special case of Construction 11.2.3, take $C = \mathrm{Cat}$ to be the category of small categories, and let $x = P$ be a small category equipped with an endomorphism $f\colon P \to P$. Then the previous construction provides a map

$$\mathbb{N} \to \mathrm{Cat}(P,P) \simeq \mathrm{Map}(P,P), \qquad n \mapsto f^n.$$

Uncurrying this functor, we obtain a functor $\mathbb{N}\times P \to P$, $(n,x) \mapsto f^n(x)$.

Remark 11.2.5. Construction 11.2.3 is functorial in C: if $F\colon C \to D$ is a functor, then the endomorphism $F(f^n)\colon F(x) \to F(x)$ agrees with $F(f)^n\colon F(x) \to F(x)$ for every natural number $n \in \mathbb{N}$.

We will now show that for any sequence $(P_n)_{n\in\mathbb{N}}$ of categories P_n equipped with functors $f_n\colon P_n \to P_{n+1}$, and for any starting object x_0 of P_0, we obtain a sequence of objects $(x_n : P_n)_{n\in\mathbb{N}}$ satisfying $x_{n+1} := f_n(x_n)$.

Theorem 11.2.6 (Peano arithmetics)**.** *Let $p\colon P \to \mathbb{N}$ be a functor, and let $f\colon P \to P$ be a functor the following square commute:*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "p"'] & P \arrow[d, "p"] \\ \mathbb{N} \arrow[r, "\mathrm{succ}"] & \mathbb{N}. \end{tikzcd}$$

For every object x_0 of the fiber P_0, there exists a section $s\colon \mathbb{N} \to P$ of p satisfyng $s(0) = x_0$ and $s \circ \mathrm{succ} = fs$.

Proof. We may regard the functor $p\colon P \to \mathbb{N}$ as an object of $\mathrm{Fun}([1], \mathrm{Cat})$. The commutative square

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "p"'] & P \arrow[d, "p"] \\ \mathbb{N} \arrow[r, "\mathrm{succ}"] & \mathbb{N} \end{tikzcd}$$

then corresponds to an endomorphism of this object. Construction 11.2.3 thus provides a map

$$\mathbb{N} \to \mathrm{Fun}([1], \mathrm{Cat})(p, p),$$

which we may think of as sending a natural number n to the commutative square

$$\begin{tikzcd} P \arrow[r, "f^n"] \arrow[d, "p"'] & P \arrow[d, "p"] \\ \mathbb{N} \arrow[r, "\mathrm{succ}^n"] & \mathbb{N}, \end{tikzcd}$$

By precomposing with the map

$$\begin{tikzcd} * \arrow[r, "x_0"] \arrow[d, equal] & P \arrow[d, "p"] \\ * \arrow[r, "0"] & \mathbb{N}, \end{tikzcd}$$

we obtain a functor $s\colon \mathbb{N} \to \mathrm{Fun}([1], \mathrm{Cat})(*, p) \simeq \mathrm{Cat}(*, P) \simeq P^{\simeq} \hookrightarrow P$. We have

$$s(0) = f^0(x_0) = x_0$$

and

$$s(n+1) = f^{n+1}(x_0) = f(f^n(x_0)) = f(s(n)),$$

which are instances of the general relations from Construction 11.2.3. □

Observation 11.2.7. If for any object $n\colon \mathbb{N}$ there is a groupoid $Q(n)$ there there is a functor $q\colon Q \to \mathbb{N}$ whose fiber at each $n\colon \mathbb{N}$ is $Q(n)$. If there is, for each $n\colon \mathbb{N}$ a functor $i_n\colon Q(n) \to Q(n{+}1)$ we get a commutative diagram of the form

$$\begin{tikzcd} Q \arrow[r, "i"] \arrow[dr, bend right, "q"'] & \mathrm{succ}^* Q \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{succ}*q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "q"] \\ & \mathbb{N} \arrow[r, "\mathrm{succ}"] & \mathbb{N} \end{tikzcd}$$

where i is given termwise by i_n. Equivalently, this corresponds to a commutative square

$$\begin{array}{ccc} Q & \xrightarrow{j} & Q \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{N} & \xrightarrow{\text{succ}} & \mathbb{N}, \end{array}$$

$j = \text{can} = i$. Such a commutative square may be regarded as an endomorphism of the map $q\colon Q \to \mathbb{N}$, seen as an object of $\text{Fun}([1], \text{Cat})$.

Assuming given an object $x_0\colon Q(0)$ by the universal property of $B\,\mathbb{N}$, we obtain for every $n \in \mathbb{N}$ a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{j^n} & Q \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{N} & \xrightarrow{\text{succ}^n} & \mathbb{N}, \end{array}$$

hence a map $i^{(n)}\colon Q \to (\text{succ}^n)^*Q$ which induces morphisms $i^{(n)}_m\colon Q(m) \to Q(m+n)$. We may then define

$$x_n := (i^{(n)})_0(x_0)\colon Q(n).$$

Since we formalize statements as groupoids and their proofs as their objects, the more pedestrian description above of the proof of the previous theorem explains how to see the latter as establishing the usual induction principle.

11.2.1 Decidability of the natural numbers

We show that the groupoid $\mathbb{N}$ is a set by showing that it has decidable equality. The only thing we use is the Peano induction principle (Theorem 11.2.6).

Lemma 11.2.8. *For all $n : \mathbb{N}$ we have $0 \neq \text{succ}\, n$.*

Proof. By the recursion we can construct a function $f : \mathbb{N} \to \text{Grpd}$ with $f(0) := *$ and $f(\text{succ}\, n) := \emptyset$. Thus for each $n : \mathbb{N}$ we get the desired map $e\colon (0 = \text{succ}\, n) \to \emptyset$ defined as $e(p) := f(p)(*)$, where $f(p)\colon * \to \emptyset$ is the map induced by the equality $p : 0 = \text{succ}\, n$. □

Construction 11.2.9. (Predecessor function) The function $\text{pred}\colon \mathbb{N} \to \mathbb{N}$ is defined by recursion with $\text{pred}(0) := 0$ and $\text{pred}(\text{succ}\, n) := n$.

Lemma 11.2.10. *For all $n, m : N$ we have a biimplication $(n = m) \leftrightarrow (\text{succ}\, n = \text{succ}\, m)$.*

Proof. The function $(n = m) \to (\text{succ}\, n = \text{succ}\, m)$ is induced on identity types by $\text{succ} : \mathbb{N} \to \mathbb{N}$. The converse implication $(\text{succ}\, n = \text{succ}\, m) \to (n = m)$ is induced by $\text{pred} : \mathbb{N} \to \mathbb{N}$. □

Proposition 11.2.11. *The groupoid $\mathbb{N}$ of natural numbers has decidable equality.*

Proof. We need to show that for all $n, m : N$ we have $n = m$ or $n \neq m$. By double induction on n and m it suffices to observe the following:

- We have $\text{id}_0 : 0 = 0$.

- We have $0 \neq \operatorname{succ} m$ and $\operatorname{succ} n \neq 0$ by Lemma 11.2.8.
- If we assume that $n = m$ then we also have $\operatorname{succ} n = \operatorname{succ} m$, and similarly $n \neq m$ implies $\operatorname{succ} n \neq \operatorname{succ} m$; both by Lemma 11.2.10. □

Corollary 11.2.12. *The groupoid $\mathbb{N}$ is a set.*

Proof. Immediate from Proposition 11.2.11 and Corollary 11.1.16. □

Example 11.2.13. Consider $\{-1\} = \Delta^0$. Define the map $\sigma\colon \{-1\} \sqcup \mathbb{N} \to \{-1\} \sqcup \mathbb{N}$ by

$$\sigma(-1) = 0, \qquad \text{and} \qquad \sigma(n) = \operatorname{succ}(n),\ \text{for } n \in \mathbb{N}\,.$$

We may also consider the functor $\pi\colon \{\underline{0}\} \sqcup \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(-1) = 0 \qquad \text{and} \qquad \pi(n) = \operatorname{succ}(n) \text{ for } n \in \mathbb{N}\,.$$

We then have a commutative diagram

$$\begin{tikzcd}
\{-1\} \sqcup \mathbb{N} \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \{-1\} \sqcup \mathbb{N} \arrow[d, "\pi"] \\
\mathbb{N} \arrow[r, "\operatorname{succ}"] & \mathbb{N}\,.
\end{tikzcd}$$

Since the fiber over $0 \in \mathbb{N}$ contains the object $x_0 = -1 \in \{-1\} \sqcup \mathbb{N}$, we obtain by Theorem 11.2.6 a section $i\colon\ \mathbb{N} \to \{-1\} \sqcup \mathbb{N}$ of π, so that $\pi i = \mathrm{id}_{\mathbb{N}}$. Now consider the pullbacks

$$\begin{tikzcd}
\mathbb{N}^{\geq 1} \arrow[r, hook] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{N} \arrow[d, "i"] \\
\mathbb{N} \arrow[r, hook, "\mathrm{incl}"] & \{-1\} \sqcup \mathbb{N}
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathbb{N}^{\leq 0} \arrow[r, hook, "j"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{N} \arrow[d, "i"] \\
\{-1\} \arrow[r, hook, "\mathrm{incl}"] & \{-1\} \sqcup \mathbb{N}
\end{tikzcd}$$

By universality of binary coproducts it follows that we obtain an equivalence $\mathbb{N}_{\leq 0} \sqcup \mathbb{N}_{\geq 1} \xrightarrow{\sim} \mathbb{N}$. We then define the map $p\colon\ \{0\} \sqcup \mathbb{N}^{\geq 1} \to \mathbb{N}$ by $p(0) = 0$ and $p(n) = j(n)$ for $n \in \mathbb{N}^{\geq 1}$. lemma 11.2.8 implies right away that the successor map $\operatorname{succ}\colon\ \mathbb{N} \to \mathbb{N}$ restricts to a map $\operatorname{succ}\colon\ \mathbb{N}^{\geq 1} \to \mathbb{N}^{\geq 1}$, i.e. we have a square

$$\begin{tikzcd}
\mathbb{N}^{\geq 1} \arrow[r, dashed, "\operatorname{succ}"] \arrow[d, hook, "j"'] & \mathbb{N}^{\geq 1} \arrow[d, hook, "j"] \\
\mathbb{N} \arrow[r, "\operatorname{succ}"] & \mathbb{N}\,.
\end{tikzcd}$$

We may also consider $f\colon\ \{0\} \sqcup \mathbb{N}^{\geq 1} \to \{0\} \sqcup \mathbb{N}^{\geq 1}$ by

$$f(0) = 1 \qquad \text{and} \qquad f(n) = \operatorname{succ}(n) \text{ for } n \in \mathbb{N}^{\geq 1}\,.$$

This fits into a commutative square

$$\begin{tikzcd}
\{0\} \sqcup \mathbb{N}^{\geq 1} \arrow[r, "f"] \arrow[d, "p"'] & \{0\} \sqcup \mathbb{N}^{\geq 1} \arrow[d, "p"] \\
\mathbb{N} \arrow[r, "\operatorname{succ}"] & \mathbb{N}\,.
\end{tikzcd}$$

The fiber over 0 has an object $x_0 = 0$, so by the induction principle we obtain a section $s\colon \mathbb{N} \to \{0\} \sqcup \mathbb{N}^{\geq 1}$ of p.

Using the maps $\mathbb{N} \xrightarrow{i} \{0\} \sqcup \mathbb{N} \to * \sqcup *$, we observe that the section s of p is defined over $* \sqcup *$, hence it induces sections $s_{\leq 0}\colon \mathbb{N}_{\leq 0} \to *$ and $s^{\geq 1}\colon \mathbb{N}^{\geq 1} \to \mathbb{N}^{\geq 1}$ of the restrictions $p_{\geq 0}\colon \mathbb{N}_{\geq 0} \to *$ and $p_{\geq 1}\colon \mathbb{N}^{\geq 1} \to \mathbb{N}^{\geq 1}$. Note that the restriction $p_{\geq 1}$ is just the identity map, hence an equivalence, and the fact that $p_{\geq 0}\colon * \to \mathbb{N}_{\geq 0}$ admits a section means that it is in fact an equivalence as well. We conclude that $p\colon \{0\} \sqcup \mathbb{N}^{\geq 1} \to \mathbb{N}$ is an equivalence.

Lemma 11.2.14. *The functor* $\mathrm{succ}\colon \mathbb{N} \to \mathbb{N}$ *induces an equivalence* $\mathbb{N} \to \mathbb{N}^{\geq 1}, n \mapsto n+1$.

Proof. Let $p\colon \mathbb{N}^{\geq 1} \to \mathbb{N}$ be the restriction of $\mathrm{pred}\colon \mathbb{N} \to \mathbb{N}$, and $s\colon \mathbb{N} \to \mathbb{N}^{\geq 1}$ the map defined by $j(s(n)) = n+1$ for all $n\colon \mathbb{N}$. It is clear p and s are termwise inverse to each other. ◻

11.2.2 Posets

Lemma 11.2.15. *For any category C, the canonical functor $C \to C \star [0]$ is a full embedding.*

Proof. We have a pullback square of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C \star [0] \arrow[d] \\ \{0\} \arrow[r, hook] & [1] \end{tikzcd}$$

in which the lower horizontal map is a full embedding (since 0 is an initial object in $[1]$). Since full embeddings are stable under pullbacks, this proves the lemma. ◻

Lemma 11.2.16. *For any category C,* $0\colon [0] \hookrightarrow C \star [0]$ *induces a terminal object of C.*

Proof. If we see 0 as an object of $C \star [0]$, for any object x of C, seen as an object of $C \star [0]$, a morphism from x to 0 amounts to a section of the canonical projection

$$C \star [0] \to [0] \star [0] = [1]\,.$$

The universal property of the join as dependent product tells us that such a section is totally determined by its restriction to $\{0\} \sqcup \{1\} = [1]^{\simeq}$. In other words, once x is fixed there is exactly one such section sending the initial object of $[1]$ to x. In other words, there is exactly one morphism from x to 0 in $C \star [0]$. The embedding $[0] \hookrightarrow C \star [0]$ is fully faithful (by an argument similar to the preceding lemma). Therefore, for any object x of $C \star [0]$, there is exactly one morphism from x to 0. ◻

Definition 11.2.17. A category C is calleda *partially ordered set*, or a *poset* for short, if, for any objects $x, y\colon C^{\simeq}$, $C(x, y)$ is a proposition. For a fixed universe Cat, we denote by Poset the full subcategory spanned by those c : Cat that correspond to a category C that is a poset.

Notation 11.2.18. When dealing with sets or posets, we sometimes use special notation.

- We may write $x \in S$ instead of $x : S$ for an object of a set S.

- If P is a poset, we may write $p \leq q$ to say that the proposition $P(p, q)$ is inhabited, hence contractible, as shown in Lemma 11.1.11.

Example 11.2.19. For each $n\colon \mathbb{N}$, the category $[n]$ is a poset. This is proven by induction using the fact that, for any poset P, the join $P \star [0]$ is a poset. Indeed we have $P^{\simeq} \sqcup [0] = (P \star [0])^{\simeq}$ by Proposition 3.6.8 and the canonical functor $P \hookrightarrow P \star [0]$ is a full embedding. On the other hand, for any $x\colon P$, the collection of morphisms from 0 to x in $P \star [0]$ is empty whereas 0 is a terminal object of $P \star [0]$, which implies that $P \star [0]$ a poset.

By Proposition 11.6.9, to check that a groupoid X is a proposition, we have to show that for each $x, y : X$ the identity type $X(x, y)$ is contractible. It turns out that it suffices to show that it is inhabited. In other words, a groupoid is a proposition if all its elements are pairwise equal.

Next, we want to characterize posets. For this we need a preliminary lemma.

Lemma 11.2.20. *Let Q and A be categories and $\Gamma \subset Q^{\simeq} \times A^{\simeq}$ be a subgroupoid satisfying*

- *for each $q : Q^{\simeq}$ and every arrow $a \to a'$ in $\mathrm{Map}(\Delta^1, A)$, we have that $\Gamma(q, a)$ implies $\Gamma(q, a')$.*

Then the projection

$$R \coloneqq \langle \Gamma \subseteq Q^{\simeq} \times A^{\simeq} \rangle \to Q \times A \to A \tag{11.1}$$

is a cocartesian fibration.

Proof. Since $Q \times A \to A$ is tautologically a cocartesian fibration and $R \subseteq Q \times A$ is a full subcategory, this just amounts to checking that each cocartesian lift of an arrow $a \to a' : \mathrm{Map}(\Delta^1, A)$ lies in R. This follows immediately from the assumption. □

Lemma 11.2.21. *Let P be a category. The following assertions are logically equivalent.*

(1) *P is a poset, i.e., for each $p, q : P^{\simeq}$, the mapping space $P(p, q)$ is a proposition.*

(2) *The map*

$$\mathrm{Map}(\Delta^1, P) \to P^{\simeq} \times P^{\simeq} \tag{11.2}$$

is an embedding of groupoids.

(3) *The functor*

$$\mathrm{Fun}(\Delta^1, P) \to P \times P \tag{11.3}$$

is fully faithful.

(4) *For every category A, the induced map*

$$\mathrm{Fun}(A, P) \to \mathrm{Fun}(A^{\simeq}, P) \tag{11.4}$$

is fully faithful.

(5) *For every category A, the induced map* (11.4) *is an embedding.*

(6) *For any object $x\colon P^{\simeq}$, the canonical projection*

$$P_{/x} \to P \tag{11.5}$$

is fully faithful.

(7) *For any object $x\colon P^{\simeq}$, the canonical projection* (11.5) *is an embedding.*

(8) *P^{op} is a poset.*

Moreover, if P is a poset, then the essential image of the map (11.4) *consists of those collections $(p_a)_{a:A^{\simeq}}$ for which $p_a \le p_b$ whenever we have an arrow $a \to b$ in A.*

Proof. (1) $\iff$ (8). This is obvious and only stated for the record.
(1) $\iff$ (2). Clear.
(1) $\implies$ (3). Assume that P is a poset and fix objects $f = (f_0 \to f_1)$ and $g = (g_0 \to g_1)$ of $\mathrm{Fun}(\Delta^1, P)$. We have to show that the map

$$\mathrm{Fun}(\Delta^1, P)(f, g) \to P(f_0, g_0) \times P(f_1, g_1) \tag{11.6}$$

of groupoids is an equivalence. But the fiber over some pair (α_0, α_1) in the target is identified with the groupoid $P(f_0, g_1)(g \circ \alpha_0, \alpha_1 \circ f)$ which is contractible by the assumption that $P(f_0, g_1)$ is a proposition.
(3) $\implies$ (4). If the functor (11.3) is fully faithful, then we may identify $\mathrm{Fun}(\Delta^1, P)$ as a full subcategory of $P \times P$, which means that the following square is cartesian

$$\begin{array}{ccc} \mathrm{Map}(A, \mathrm{Fun}(\Delta^1, P)) & \longrightarrow & \mathrm{Map}(A^{\simeq}, \mathrm{Fun}(\Delta^1, P)^{\simeq}) \\ \downarrow & & \downarrow \\ \mathrm{Map}(A, P \times P) & \longrightarrow & \mathrm{Map}(A^{\simeq}, (P \times P)^{\simeq}) \end{array} \tag{11.7}$$

Note that we may replace $\mathrm{Fun}(\Delta^1, P)^{\simeq}$ by $\mathrm{Fun}(\Delta^1, P)$ in the upper right because $A^{\simeq}$ is a groupoid, so that this square is identified with the square

$$\begin{array}{ccc} \mathrm{Map}(\Delta^1, \mathrm{Fun}(A, P)) & \longrightarrow & \mathrm{Map}(\Delta^1, \mathrm{Fun}(A^{\simeq}, P)) \\ \downarrow & & \downarrow \\ \mathrm{Map}(A, P) \times \mathrm{Map}(A, P) & \longrightarrow & \mathrm{Map}(A^{\simeq}, P) \times \mathrm{Map}(A^{\simeq}, P) \end{array} \tag{11.8}$$

after curry-uncurry. This cartesian square expresses precisely that the functor (11.4) is fully faithful.

It remains to compute the essential image of the functor $\mathrm{Fun}(A, P) \to \mathrm{Fun}(A^{\simeq}, P)$. Clearly each $f^{\simeq}\colon A^{\simeq} \to P$ obtained from a functor $f\colon A \to P$ needs to satisfy $f(a) \le f(a')$ for each arrow $a \to a'$. To prove the converse, assume that we have an assignment on objects $a : A^{\simeq} \mapsto f_a : P$ satisfying $f_a \le f_{a'}$ whenever there is an arrow $a \to a'$ in A. Our task is to extend this assignment to a genuine functor $f\colon A \to P$.

Note that since P is a poset, we have the subgroupoid

$$\Gamma \coloneqq \{(p, a) \mid p \le f_a\} \subseteq P^{\simeq} \times A^{\simeq}, \tag{11.9}$$

which by assumption satisfies the condition of Lemma 11.2.20, so that the projection

$$\langle (p,a) \mid p \leq f_a \rangle \subseteq P \times A \to A \tag{11.10}$$

is a cocartesian fibration. Trivially each object (f_a, a) is a terminal object in its fiber over $a : A^{\simeq}$, so that by functoriality of universals they assemble to a functor $f\colon A \to P$ extending the original family $(f_a)_{a:A^{\simeq}}$. (4) $\implies$ (5). Fully faithful functors induce embeddings on groupoid cores.
(5) $\implies$ (2). Just specialize $A \coloneqq \Delta^1$ and use the identification $(\Delta^1)^{\simeq} = \partial\Delta^1$.
(3) $\implies$ (6). This follows from the stability of full embeddings by pullbacks.
(6) $\implies$ (7). Any fully faithful functor is an embedding.
(7) $\implies$ (1). If $P_{/x} \to P$ is an embedding, then, for each object y of P, we get by pullback an embedding of the form

$$P(y,x) \hookrightarrow * .$$

Doing this for all $x\colon P^{\simeq}$ proves that P is a poset. □

Corollary 11.2.22. *Let P be a poset and A any category. Then* $\mathrm{Fun}(A,P)$ *is a poset. For two functors $f, g\colon A \to P$ we have $f \leq g$ if and only if for all $a : A^{\simeq}$ we have $f(a) \leq g(a)$.*

Proof. That the category $\mathrm{Fun}(A,P)$ is again a poset follows immediately from the characterization 5 of Lemma 11.2.21 because $\mathrm{Fun}(A,-)$ preserves embeddings and commutes with the operation $P \mapsto (\mathrm{Fun}(\Delta^1,P) \to P \times P)$. The second statement is just a reformulation of the characterization 4. □

11.3 The simplex category

We have given an external definition of the simplices $[n]$ for every natural number n in **??** using the join operation. We may now internalize this definition.

Construction 11.3.1. Consider the functor

$$- \star [0]\colon \mathrm{Cat} \to \mathrm{Cat}, \qquad C \mapsto C \star [0]$$

together with the empty object $\emptyset\colon \mathrm{Cat}$. By iterating, we obtain a functor $[-]\colon \mathbb{N} \to \mathrm{Cat}$, sending $n\colon \mathbb{N}$ to a category $[n]$ defined by the formula

$$[0] = \emptyset \star [0] \quad \text{and} \quad [n+1] = [n] \star [0] \quad \text{for } n\colon \mathbb{N}.$$

Proposition 11.3.2. *For any posets C and D, the rule $\varphi \mapsto \varphi \star \mathrm{id}_{[0]}$ defines an equivalence*

$$\mathrm{Equiv}(C,D) \xrightarrow{\sim} \mathrm{Equiv}(C \star [0], D \star [0]) .$$

Proof. Let $\psi : C \star [0] \xrightarrow{\sim} D \star [0]$ be an equivalence. Since 0 is a terminal object of $C \star [0]$ by the previous lemma, so is $\psi(0)$, hence $\psi(0) = 0$.

Recall from Proposition 3.6.8 that we have

$$C^{\simeq} \sqcup [0] = (C \star [0])^{\simeq} .$$

Since coproducts of categories are universal, this shows that $\psi(x)\colon D^{\simeq}$ must hold for any $x\colon C^{\simeq}$ (otherwise, if $\psi(x) = 0$, then x is a terminal object in $C \star [0]$ and thus $x = 0$, which is a contradiction if we assume $x\colon C^{\simeq}$). Since C is a full subcategory in $C \star [0]$, this shows that ψ restricts to a fully faithful functor from C to D. Applying what precedes to the inverse of ψ shows that this restriction is in fact an equivalence:

$$\psi_{|C}\colon C \xrightarrow{\sim} D$$

(since full faithfulness is not an issue, it suffices to see what happens on objects, which we have explained above). Since functors between posets are determined by their values on objects by Lemma 11.2.21, we see immediately that the rule $\psi \mapsto \psi_{|C}$ is an inverse of the rule $\varphi \mapsto \varphi \star \mathrm{id}_{[0]}$. $\square$

Theorem 11.3.3. *The functor* $[-]\colon \mathbb{N} \to \mathrm{Cat}^{\simeq}$ *is an embedding.*

Proof. We see that $\mathrm{Cat}^{\simeq}([n], [n]) \simeq *$ for all $n \in N$: This is clear for $n = 0$ and the preceding proposition provides the induction step. It remains to check that $[m] = [n]$ implies $m = n$ for any $m, n\colon \mathbb{N}$. We prove first that $[n] = [0]$ implies $n = 0$ for all $n\colon \mathbb{N}$. We observe next that the type $[n+1] = [0]$ is empty for all $n\colon \mathbb{N}$. For $n = 0$, this comes from the fact that that there is a pullback square of the form

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & {[0]} \arrow[d, "0"] \\
{[0]} \arrow[r, "1"] & {[1]}
\end{tikzcd}$$

providing a map from the type $[1] = [0]$ to $\emptyset = [0]$, hence to $\emptyset$ (and any map to the latter is invertible). Since $[n]$ is a retract of $[n + 1]$, we have a map from $[n + 1] = [0]$ to $[n] = [0]$, which implies by induction that the type $[n+1] = [0]$ is always empty. On the other hand the functoriality of the operator $C \mapsto C \star [0]$ induces a map from the type $[m] = [n]$ to the type $[m + 1] = [n + 1]$ and it follows from Proposition 11.3.2 that this map is invertible. $\square$

Exercise 11.3.4. Show that if C is 0-truncated, then also $C \star [0]$ is 0-truncated. Also, show that if C has no other automorphisms other than identities, then also $C \star [0]$ has no other automorphisms other than identities.

Definition 11.3.5 (Simplex category)**.** We define the *simplex category*

$$\mathbf{\Delta} \subseteq \mathrm{Cat}$$

as the full subcategory of Cat whose objects are $[n]$ for every $n \in \mathbb{N}$. Given a category C, a functor $F\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ is called a *simplicial object* of C. We write $\mathrm{s}C := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ for the category of simplicial objects in C.

Definition 11.3.6 (Geometric realization)**.** For a simplicial object $X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to C$, we define its *geometric realization* $|X_\bullet|$ as

$$|X_\bullet| := \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n,$$

whenever this colimit exists in C.

Lemma 11.3.7. *The diagonal functor* $\delta\colon \mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$, $[n] \mapsto [n] \times [n]$ *is an initial functor.*

Proof. Let $m, n \in \mathbb{N}$ be two natural numbers. By Corollary 8.3.7, we have to show that the slice category

$$\mathbf{\Delta}_{/([m],[n])} = \mathbf{\Delta} \times_{\mathbf{\Delta}\times\mathbf{\Delta}} (\mathbf{\Delta}\times\mathbf{\Delta})_{/([m],[n])}$$

is weakly contractible. The forgetful functor $\mathbf{\Delta}_{/([m],[n])} \to \mathbf{\Delta}$ is a right fibration, hence there exists a pullback square

$$\begin{tikzcd} (\mathbf{\Delta}_{/([m],[n])})^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Grpd}_\bullet \arrow[d] \\ \mathbf{\Delta}^{\mathrm{op}} \arrow[r, dashed] & \mathrm{Grpd}. \end{tikzcd}$$

Exercise: the horizontal functor is the product $\mathbf{\Delta}^m \times \mathbf{\Delta}^n$ of the functors $\mathbf{\Delta}^m$ and $\mathbf{\Delta}^n$ represented by $[m]$ and $[n]$.

Since $- \star [0] \colon \mathrm{Cat} \to \mathrm{Cat}$ preserves simplices by definition, it restricts to an endofunctor on $\mathbf{\Delta}$, hence we may consider the maps

$$\mathrm{id}_{[n]} \to (-) \star [0] \leftarrow \mathrm{const}_{[0]}$$

of endofunctors of $\mathbf{\Delta}$.

Now consider the product $[m]\times[n]$ in Cat. Observe that we have $N([n]) = \mathbf{\Delta}^n$, hence $N([m]\times[n]) = \mathbf{\Delta}^m \times \mathbf{\Delta}^n$. The inclusion

$$[m]\times[n] \hookrightarrow ([m]\times[n]) \star [0]$$

admits a right adjoint

$$r\colon ([m]\times[n]) \star [0] \to [m]\times[n]$$

satisfying $r(a,b) = (a,b)$ for $(a,b) \in [m]\times[n]$ and $r(0) = (m,n)$. So given a functor $(a,b)\colon [p]\colon [m] \to [n]$, we may consider

$$\begin{tikzcd} {[0]} \arrow[r, hook] \arrow[d, "(a,b)"'] & {[p] \star [0]} \arrow[d, "(a,b)\star[0]"] & \\ {[m]\times[n]} \arrow[r, hook] \arrow[dr, equal] & {([m]\times[n]) \star [0]} \arrow[d, "r"] & {[0]} \arrow[l, hook'] \arrow[dl, "(m,n)"] \\ & {[m]\times[n].} & \end{tikzcd}$$

This provides homotopies between $\mathrm{id}_{\mathbf{\Delta}_{/([m],[n])}}$ and φ, and between φ and $\mathrm{const}_{(m,n)}$, where

$$\varphi([p],(a,b)) = ([p+1],(a,b)\star[0]).$$

It follows that $\mathbf{\Delta}_{/([m],[n])}$ is weakly contractible. □

Corollary 11.3.8. *The functor $\delta^{\mathrm{op}}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}$ is a terminal functor.* □

Corollary 11.3.9. *Assume that C is a category with small colimits, and consider a functor $\Phi\colon \mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}} \to C$. Then there is an equivalence*

$$\operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Phi_{n,n} \xrightarrow{\sim} \operatorname{colim}_{([m],[n])\in\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} \Phi_{m,n}.$$

Proposition 11.3.10. *Let C be a category C admitting geometric realizations and finite products, and assume that the functor $- \times -\colon C \times C \to C$ preserves geometric realizations in both variables. Then the colimit functor $|-|\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \to C$ preserves finite products.*

Proof. Consider simplicial objects $X, Y\colon \mathbf{\Delta}^{\mathrm{op}} \to C$. We have to show that the map $|X \times Y| \to |X| \times |Y|$ induced by the two projection maps is an equivalence. To this end, consider the *box product* $X \boxtimes Y\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to C$, defined as the composite

$$\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X \times Y} C \times C \xrightarrow{- \times -} C.$$

By Theorem 11.3.9, we have

$$|X \times Y| \simeq \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} (X \boxtimes Y)_{n,n} \simeq \mathrm{colim}_{([m],[n]) \in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} X_m \times Y_n.$$

Since for a fixed object T of C, the functor $T \times -\colon C \to C$ preserves geometric realizations by assumption, we get

$$\mathrm{colim}_{([m],[n]) \in \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} X_m \times Y_n \simeq \mathrm{colim}_{[m] \in \mathbf{\Delta}^{\mathrm{op}}} X_m \times \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} Y_n = |X| \times |Y|,$$

as desired. □

11.4 Images and Čech complexes

Recall from Definition 1.3.24 the notion of an *embedding*, which we may think of as a monomorphism of categories. This notion is the first in a hierarchy of notions of *n-truncation* for every integer $n \geq -1$. In particular, we obtain notions of *n-truncated categories* for every n. For example, a (-1)-truncated groupoid are categories P such that the map $P \to *$ is an embedding; we will refer to such categories as *propositions* and denote their full subcategory by Prop.

In this chapter, we will explore the notion of truncation in more depth. We will start in Section 11.4 by introducing the *Čech nerve construction*, which shows that every morphism $f\colon X \to Y$ of groupoids may canonically be factored as a surjection followed by an embedding; we may think of this procedure as a form of '(-1)-truncation' of the morphism. The general framework of truncated categories will be introduced in Section 11.6; this in particular leads to notions of 'propositions', 'sets' and 'posets'. The notion of truncation of categories will be introduced in Section 11.6, where we will also explain how the Čech nerve construction may be interpreted as (-1)-truncation. This in particular leads to a functor $\tau_{\leq -1}\colon \mathrm{Grpd} \to \mathrm{Prop}$ which is left adjoint to the inclusion $\mathrm{Prop} \hookrightarrow \mathrm{Grpd}$. Finally, we will give explicit descriptions of the set-truncation of groupoids and the homotopy category of a category in Section 11.7.

Before defining the general hierarchy of truncation levels, we will discuss one special case: the (-1)-truncation operation. We will see that there is a very explicit description of this operation in terms of the *Čech nerve* of a morphism.

Definition 11.4.1. Since Grpd admits small limits, the functor

$$\mathrm{ev}_{[0]}\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Grpd}\,, \quad X \mapsto X_0 = X([0])$$

admits a right adjoint

$$\check{C}\colon \mathrm{Grpd} \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$$

given by right Kan extension along the functor $i = [0]\colon * \to \mathbf{\Delta}^{\mathrm{op}}$. We refer to the simplicial groupoid $\check{C}(X)$ as the *Čech nerve* of X.

Remark 11.4.2. Using the pointwise formula for right Kan extensions, we may concretely describe the evaluation of $\check{C}(X)$ at $[n]\colon \mathbf{\Delta}^{\mathrm{op}}$ as follows.

$$\check{C}(X)_n \simeq X^{n+1}\,.$$

Indeed, pulling back the projection $\mathbf{\Delta}_{/[n]} \to \mathbf{\Delta}$ along the embedding $[0]\colon * \to \mathbf{\Delta}^{\mathrm{op}}$, we obtain an anima canonically equivalent to

$$(h_{[n]})_0 = \mathrm{Hom}_{\mathbf{\Delta}}([0],[n]) = [n]^{\simeq}\,;$$

using that $(C \star [0])^{\simeq} \simeq C^{\simeq} \sqcup e$, by Proposition 3.6.8, we observe furthermore by induction on n that

$$[n]^{\simeq} = \{0,\dots,n\}\,,$$

where $\{0,\dots,n\}$ is defined inductively by

$$\{0,\dots,0\} = [0] \quad \text{and} \quad \{0,\dots,n+1\} = \{0,\dots,n\} \sqcup [0]\,.$$

It follows that, for $X \in \mathrm{Grpd}^{\simeq}$, we have

$$\check{C}(X)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Grpd}, \qquad \check{C}(X)_n = \mathrm{Hom}_{\mathrm{Grpd}}(\{0,\dots,n\}, X)\,.$$

By a new induction, one can show that this simplifies to

$$\check{C}(X)_n \simeq X^{n+1}\,,$$

where the right-hand side is defined inductively via $X^0 = e$ and $X^{n+1} = X^n \times X$.

Remark 11.4.3. Since $i\colon * \to \mathbf{\Delta}^{\mathrm{op}}$ is fully faithful, also $i_!$ is fully faithful, so the unit $\mathrm{id} \xrightarrow{\sim} i^* i_!$ is an isomorphism. Passing to right adjoints, also the counit $i^* i_* \to \mathrm{id}$ is an isomorphism, so that i_* is fully faithful.

Lemma 11.4.4. *For any functor $Y\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Grpd}$, the morphism $|\mathrm{pr}_2|\colon |\Delta^1 \times Y| \to |Y|$ is an isomorphism.*

Proof. Under the equivalence $|\Delta^1 \times Y| \simeq |\Delta^1| \times |Y|$ from Theorem 11.3.10, the map $|\mathrm{pr}_2|\colon |\Delta^1 \times Y| \to |Y|$ corresponds to the projection map $\mathrm{pr}_2\colon |\Delta^1| \times |X| \to |X|$. So it remains to show that $|\Delta^1| \simeq *$. Since Δ^1 is the presheaf on $\mathbf{\Delta}$ represented by $[1] \in \mathbf{\Delta}$, this is an instance of Corollary 9.5.6. □

Corollary 11.4.5. *Given simplicial groupoids $X, Y\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Grpd}$ and given a simplicial homotopy $h\colon \Delta^1 \times X \to Y$ (i.e. a morphism in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$.) Then we have $|h_0| = |h_1|$ as maps $|X| \to |Y|$.*

Proof. Consider the diagram

$$\begin{tikzcd}
 & {|X|} \arrow[dl, equal] \arrow[d, "i_0", "\simeq"'] \arrow[dr, "|h_0|"] & \\
{|X|} & {|\Delta^1 \times X|} \arrow[l, "\simeq", "\mathrm{pr}_2"'] \arrow[r, "|h|"] & {|Y|} \\
 & {|X|.} \arrow[ul, equal] \arrow[u, "i_1"', "\simeq"] \arrow[ur, "|h_1|"'] &
\end{tikzcd}$$

□

Consider morphisms $f\colon X \to Y$ and $g\colon Y \to X$ in Grpd. Then there is a one-to-one correspondence between commutative diagrams

$$\begin{tikzcd} \check{C}(X) \arrow[d, "0"] \arrow[dr, "\check{C}(f)"] & \\ \Delta^1 \times \check{C}(X) \arrow[r, "h"] & \check{C}(Y) \\ \check{C}(X) \arrow[u, "1"'] \arrow[ur, "\check{C}(g)"'] & \end{tikzcd}$$

and extensions

$$\begin{tikzcd} X \sqcup X \arrow[r, "\langle f,g \rangle"] \arrow[d] & Y \\ (\Delta^1 \times \check{C}(X))_0. \arrow[ur, dashed, "\tilde{h}"] & \end{tikzcd}$$

Corollary 11.4.6. *If there is a map $x\colon e \to X$, then $|\check{C}(X)|$ is contractible.*

Proof. Apply the above observation to the maps $f\colon X \to e \xrightarrow{x} X$ and $g = \mathrm{id}_X\colon X \to X$. $\square$

Construction 11.4.7. All of the above applies in any groupoidal context. Let Y be a small groupoid and let $f\colon X \to Y$ be amorphism of small groupoids, i.e. a groupoid (X, f) in context Y. We have an adjunction

$$\begin{tikzcd} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}_{/Y}) \arrow[r, shift left, "\mathrm{ev}_0"] \arrow[d, "\mathrm{fgt}"'] & \mathrm{Grpd}_{/Y} \arrow[l, shift left, "\check{C}_Y"] \arrow[dl, bend left, "(X,f)\mapsto \check{C}(f)"] \\ \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}). & \end{tikzcd}$$

Here $\check{C}(f)$ is the groupoid $\pi_f\colon \check{C}(X/Y) \to Y$ in context Y, obtained by applying the Čech nerve construction in context Y to the groupoid $f\colon X \to Y$ in context Y. We will refer to $\check{C}(f)$ as the *Čech nerve of f*.

Remark 11.4.8. Let C be a category that admits small colimits. Then also the slice $C_{/y}$ admits small colimits for any object y of C, and the forgetful functor $C_{/y} \to C$ preserves small colimits.

So for any morphism $f\colon X \to Y$ of small groupoids we obtain a map $|\pi_f|\colon |\check{C}(X/Y)| \to Y$.

Theorem 11.4.9. *Consider a morphism $f\colon X \to Y$ of small groupoids. If f admits a section, then the map $|\pi_f|\colon |\check{C}(X/Y)| \to Y$ is an equivalence.*

Proof. This is an instance of Corollary 11.4.6 applied in context Y. $\square$

Let X and T be objects of Grpd. We may consider the projection map $f = \mathrm{pr}_2\colon X \times T \to T$. Note that this is the image of the map $X \to *$ under the pullback functor $\mathrm{Grpd} \to \mathrm{Grpd}_{/T}$, and it follows that

$$\check{C}(X \times T/T) \simeq \check{C}(X) \times T$$

as simplicial objects in $\mathrm{Grpd}_{/T}$. We then get

$$|\check{C}(X \times T/T)| \simeq |\check{C}(X) \times T| \simeq |\check{C}(X)| \times T.$$

Taking $T = X$, the projection map $\mathrm{pr}_2 \colon X \times X \to X$ admits a section, given by the diagonal. By Theorem 11.4.9, it follows that

$$|\check{C}(X \times X/X)| \xrightarrow{\sim} X.$$

This proves:

Corollary 11.4.10. *For every small groupoid X the projection map $\mathrm{pr}_2 \colon |\check{C}(X)| \times X \to X$ is an equivalence.* □

Theorem 11.4.11. *Let X be a small groupoid. Then the groupoid $|\check{C}(X)|$ is a proposition, in the sense that $|\check{C}(X)| \to *$ is an embedding.*

Proof. Consider the groupoid $T = |\check{C}(X)|$. By the previous corollary, the projection map $f = \mathrm{pr}_1 \colon X \times T \to X$ is an equivalence. We thus get

$$\check{C}(X \times T/T) \xrightarrow{\sim} \check{C}(X) \times T \xrightarrow{\sim} \check{C}(X),$$

hence

$$T \times T \xrightarrow{\sim} |\check{C}(X \times T/T)| \xrightarrow{\sim} T.$$

It follows that T is a proposition. □

Corollary 11.4.12. *Let $f \colon X \to Y$ be any morphism in* Grpd*. Then the morphism $|\pi_f| \colon |\check{C}(X/Y)| \to Y$ is an embedding.*

Proof. Apply Theorem 11.4.11 in context Y. □

11.5 Covers

The notion of a Čech cover leads to the definition of a cover.

Definition 11.5.1. A morphism $f \colon X \to Y$ in Grpd is said to be a *cover* if the induced map $|\pi_f| \colon |\check{C}(X/Y)| \to Y$ is an isomorphism.

In homotopy type theory, covers are usually called *surjections*.

Remark 11.5.2. For any pullback square

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$

there is a canonical isomorphism

$$\check{C}(X/Y) \times_Y Y' \simeq \check{C}(X'/Y')$$

and since $(-) \times_Y Y'$ commutes with small colimits, this induces a canonical isomorphism

$$|\check{C}(X/Y)| \times_Y Y' \simeq |\check{C}(X'/Y')| \,.$$

This implies in particular that covers are stable under pullbacks. This also tells us that if u is a cover, then f is a cover if and only if f' is a cover.

Definition 11.5.3 (Essential surjectivity)**.** A functor $F\colon C \to D$ of categories is called *essentially surjective* if the induced map $F^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ on groupoid cores is a cover.

Proposition 11.5.4. *For any morphism $f\colon X \to Y$, the map $X \to |\check{C}(X/Y)|$ induced by the identification $\check{C}(X/Y) = X$ is a cover. In particular, f factors as a composite*

$$X \xrightarrow{\text{cover}} |\check{C}(X/Y)| \xrightarrow{\text{embedding}} Y.$$

Proof. This is a reformulation of Corollary 11.4.12. $\square$

Definition 11.5.5. We refer to the groupoid $|\check{C}(X/Y)|$ as the *image* of f, and denote it by $\mathrm{Im}(f)$.

Proposition 11.5.6. *Let $f\colon X \to Y$ be a morphism in* Grpd*. The following conditions are equivalent.*

(i) The map f is a cover.

(ii) The pullback functor

$$f^*\colon \mathrm{Cat}_{/Y} \to \mathrm{Cat}_{/X}$$

is conservative.

(iii) The pullback functor

$$f^*\colon \mathrm{Grpd}_{/Y} \to \mathrm{Grpd}_{/X}$$

is conservative.

Proof. Assume that f is a cover and consider a morphism $\varphi\colon Z \to W$ over Y:

$$\begin{tikzcd} Z \arrow[rr, "\varphi"] \arrow[dr, "g"'] & & W \arrow[dl, "h"] \\ & Y, & \end{tikzcd}$$

and assume that the induced map $X \times_Y \varphi\colon X \times_Y Z \to X \times_Y W$ is an isomorphism. Then φ induces a map

$$\check{C}(X/Y) \times_Y Z \to \check{C}(X/Y) \times_Y W$$

of simplicial objects in $\mathrm{Grpd}_{/Y}$ which is a levelwise isomorphism, hence an isomorphism in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}_{/Y})$. Applying the functor $|-|\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}_{/Y}) \to \mathrm{Grpd}_{/Y}$ thus produces an isomorphism

$$|\check{C}(X/Y) \times_Y Z| \xrightarrow{\sim} |\check{C}(X/Y) \times_Y W| \,.$$

On the other hand, for any map $T \to Y$ in Grpd, since the functor $(-) \times_Y T$ commutes with colimits, we have a canonical equivalence of the form

$$|\check{C}(X/Y) \times_Y T| \xrightarrow{\sim} |\check{C}(X/Y)| \times_Y T = T \,.$$

We obtain a commutative diagram of the form

$$\begin{tikzcd} {|\check{C}(X/Y)\times_Y Z|} \arrow[r] \arrow[d, "\simeq"'] & {|\check{C}(X/Y)\times_Y W|} \arrow[d, "\simeq"'] \\ Z \arrow[r, "\varphi"] & W \end{tikzcd}$$

proving that φ is an equivalence. This proves that (ii) follows from (i).

For each small groupoid X, the full embedding $\mathrm{Grpd}_{/X} \hookrightarrow \mathrm{Cat}_{/X}$ is conservative and is compatible with pullbacks. This implies right away that (iii) follows from (ii).

Finally, observe that the pullback of $|\check{C}(X/Y)|$ along f is $|\check{C}(X \times_Y X/X)|$, through the first projection of $X \times_Y X$ to X, say. Since the first projection of $X \times_Y X$ to X has a section, this means that f^* sends the embedding of the image of f into Y to an equivalence in $\mathrm{Grpd}_{/X}$. Therefore, assertion (i) follows from assertion (iii). $\square$

Corollary 11.5.7. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two functors between small groupoids.*

(a) If f and g are covers, so is $g \circ f$.

(b) If $g \circ f$ is a cover, so is g.

Proof. We will use characterization (iii) of Proposition 11.5.6. To prove assertion (a) we observe that if g^* and f^* are conservative functors, so is the composite $f^* \circ g^* \simeq (g \circ f)^*$. Similarly, to prove assertion (b), we check that if the composite $f^* \circ g^* \simeq (g \circ f)^*$ is concervative, then so is g^*. $\square$

Corollary 11.5.8. *Any retract of a cover is a cover.*

Proof. We will use again characterization (iii) of Proposition 11.5.6. Any retract of a conservative functors is conservative (this follows from Proposition 6.2.2, since any retract of a pullback square is a pullback square). Therefore, if f is a retract of g then f^* is a retract of g^*. $\square$

The following is an important consequence of Proposition 11.5.6.

Proposition 11.5.9. *Let $f\colon X \to Y$ be amap between groupoids that is both a cover and an embedding. Then f is an equivalence.*

Proof. In general, in order to check that f is an equivalence, we may regard it as a morphism in $\mathrm{Grpd}_{/Y}$. By Proposition 11.5.6, it will suffice to show that its pullback $\mathrm{pr}_2\colon X \times_Y X \to X$ along itself is an equivalence. In ther words, without any loss of generality, we may assume further more that f has a section and apply Lemma 1.3.28. $\square$

We now finally get the strengthening

Proposition 11.5.10. *Let $f\colon A \to B$ be a functor. If f is fully faithful and if the map $f^{\simeq}\colon A^{\simeq} \to B^{\simeq}$ is a cover, then the functor f is an equivalence.*

Proof. Again we only show one direction. Since f is fully faithful, the functor $f^{\simeq}$ is an embedding, hence by Proposition 11.5.9 it is an equivalence. In particular, $f^{\simeq}$ admits a section, hence by Theorem 6.3.3 the functor f is an equivalence. $\square$

Proposition 11.5.11. *Let $f\colon X \to Y$ be a morphism in* Grpd*. Then the following are equivalent:*

(1) The morphism f is a cover;

(2) For every embedding $Z \to Y$, if we have $X \times_Y Z \xrightarrow{\sim} X \times_Y Y = X$ then $Z \xrightarrow{\sim} Y$.

Proof. The fact that (1) implies (2) is an instance of Proposition 11.5.6. The fact that (2) implies (1) follows by taking $Z = |\check{C}(X/Y)| \hookrightarrow Y$, so that $X \times_Y Z = |\check{C}(X \times_Y X/X)| \xrightarrow{\sim} X$. □

Theorem 11.5.12. *Let $u\colon A \to B$ be a functor. Assume there exists a cover $i\colon X \to B^{\simeq}$ such that for every object x of X, the relative slice category $A_{/i(x)} = A \times_B B_{/i(x)}$ admits a terminal object. Then u admits a right adjoint.*

Proof. For any groupoid Γ, we can consider the situation in context Γ: we may ask whether the resulting functor $u_\Gamma\colon A_\Gamma \to B_\Gamma$ admits a right adjoint in context Γ. If this holds for $\Gamma = B^{\simeq}$, it also holds in the absolute context.

If Y is a groupoid equipped with a map $b\colon Y \to B^{\simeq}$, then in context $B^{\simeq}$ this looks like

$$\begin{tikzcd}
Y \arrow[r, "{(b,b)}"] & B^{\simeq} \times B^{\simeq} \arrow[rr] \arrow[dr] & & B \times B^{\simeq} \arrow[dl, "\mathrm{pr}_2"] \\
& & B^{\simeq}. &
\end{tikzcd}$$

So we may express that $A_{/b}$ has a terminal object for every object b of B as a proposition P in context $B^{\simeq}$. The existence of a terminal object is then expressed as the assertion that this proposition P is equivalent to the terminal anima in context $B^{\simeq}$. To prove such an equivalence, by virtue of Proposition 11.5.6, it suffices to check that the structural map $P \to B^{\simeq}$ becomes an equivalence after pulling back along the cover $i\colon X \to B^{\simeq}$. □

Proposition 11.5.13. *Let $u\colon A \to B$ be a functor. The following conditions are equivalent.*

(a) The induced map $u^{\simeq}\colon A^{\simeq} \to B^{\simeq}$ is a cover.

(b) For any category C, the induced functor

$$u^*\colon \operatorname{Fun}(B, C) \to \operatorname{Fun}(A, C)$$

is conservative.

(c) There is a universe Cat *for which A and B are small, such that the induced functor*

$$u^*\colon \operatorname{Fun}(B, \mathrm{Cat}) \to \operatorname{Fun}(A, \mathrm{Cat})$$

is conservative.

(d) There is a universe Cat *for which A and B are small, with corresponding universe of small groupoids* Grpd*, such that the induced functor*

$$u^*\colon \operatorname{Fun}(B, \mathrm{Grpd}) \to \operatorname{Fun}(A, \mathrm{Grpd})$$

is conservative.

Proof. We consider the commutative diagram bellow.

$$\begin{tikzcd}
\mathrm{Fun}(B, \mathrm{Cat}) \arrow[r, "u^*"] \arrow[d] & \mathrm{Fun}(A, \mathrm{Cat}) \arrow[d] \\
\mathrm{Fun}(B^{\simeq}, \mathrm{Cat}) \arrow[r, "(u^{\simeq})^*"] \arrow[d, equal] & \mathrm{Fun}(A^{\simeq}, \mathrm{Cat}) \arrow[d, equal] \\
\mathrm{Cat}_{/B^{\simeq}} \arrow[r, "(u^{\simeq})^*"] & \mathrm{Cat}_{/A^{\simeq}}
\end{tikzcd}$$

all the vertical functors are known to be conservative, and, by virtue of Proposition 11.5.6, the conservativity of $(u^{\simeq})^*$ is equivalent to the property that $u^{\simeq}$ is a cover. This implies right away that conditions (a) and (c) are equivalent. A similar reasoning replacing Cat by Grpd above shows that assertions (a) and (d) are equivalent. It is clear that (c) and (d) also follow from (b). It suffices now to check that condition (a) implies condition (b). We may assume that we work in a universe so that A, B and C are small. Condition (d) implies that, for any category X, the functor

$$u^* \colon \mathrm{Fun}(B, \mathrm{Fun}(X, \mathrm{Grpd})) \to \mathrm{Fun}(A, \mathrm{Fun}(X, \mathrm{Grpd}))\,.$$

is conservative (because one can check equivalences objectwise). We consider finally the commutative square

$$\begin{tikzcd}
\mathrm{Fun}(B, C) \arrow[r, hook] \arrow[d, "u^*"'] & \mathrm{Fun}(B, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})) \arrow[d, "u^*"] \\
\mathrm{Fun}(A, C) \arrow[r, hook] & \mathrm{Fun}(A, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}))
\end{tikzcd}$$

in which the horizontal arrows are the full embeddings induced by the Yoneda embedding of C. This commutative diagram thus shows that (b) follows from (a). □

Corollary 11.5.14. *For any category C and any family of morphisms W stable under composition in C, the localization functor $\gamma \colon C \to C[W^{-1}]$ induces a cover*

$$\gamma^{\simeq} \colon C^{\simeq} \to (C[W^{-1}])^{\simeq}\,.$$

Proof. The functor

$$\gamma^* \colon \mathrm{Fun}(C[W^{-1}], D) \to \mathrm{Fun}(C, D)$$

is fully faithful and thus conservative for any D. Proposition 11.5.13 thus ends the proof. □

Corollary 11.5.15. *Let A be a small category and $F \colon A \to$ Cat a functor. The canonical functors $i_a \colon F(a) \to \mathrm{colim}_{a \colon A} F(a)$ induce a cover of the form*

$$\sum_{a \colon A^{\simeq}} F(a)^{\simeq} \to (\mathrm{colim}_{a \colon A} F(a))^{\simeq}\,.$$

Proof. We know that $\mathrm{colim}_{a \colon A} F(a)$ is a localization of $\int_{a \colon A} F(a)$, where the latter is defined by the pullback bellow.

$$\begin{tikzcd}
\int_{a \colon A} F(a) \arrow[r] \arrow[d, "p_F"'] & \mathrm{Cat}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\
A \arrow[r] & \mathrm{Cat}
\end{tikzcd}$$

By virtue of Corollary 11.5.14, we thus have a cover

$$(\int_{a\colon A} F(a))^{\simeq} \to (\operatorname{colim}_{a\colon A} F(a))^{\simeq}\,.$$

But, as explain in Remark 9.4.19, objects of $(\int_{a\colon A} F(a))^{\simeq}$ are known to be pairs of the form (a, x) where $a\colon A^{\simeq}$ and $x\colon F(a)^{\simeq}$. In other words, there is a cover (in fact, an equivalence) of the form

$$\sum_{a\colon A^{\simeq}} F(a)^{\simeq} \to (\int_{a\colon A} F(a))^{\simeq}\,.$$

This corollary thus follows from the fact that covers are stable under composition. □

Corollary 11.5.16. *Let A be a small category and $F\colon A \to \mathrm{Grpd}$ a functor. The canonical maps $i_a\colon F(a) \to \operatorname{colim}_{a\colon A} F(a)$ induce a cover of the form*

$$\sum_{a\colon A^{\simeq}} F(a) \to \operatorname{colim}_{a\colon A} F(a)\,.$$

Proof. This is a particular case of the previous corollary since the embedding of Grpd into Cat commutes with colimits (being a left adjoint). □

Proposition 11.5.17. *Let C be a locally small category and Γ a small groupoid. We consider a functor $f\colon \Gamma \to C^{\simeq}$. Then $|\check{C}(\Gamma/C^{\simeq})|$ is small. In particular, the full subcategory of C with groupoid of objects $|\check{C}(\Gamma/C^{\simeq})|$ is small.*

Proof. For any $x, y\colon C^{\simeq}$, we have an embedding

$$C^{\simeq}(x, y) \hookrightarrow C(x, y)$$

with small codomain. Therefore, the groupoid $C^{\simeq}$ is locally small. In other words, there is a pullback square of the form

$$\begin{tikzcd} C^{\simeq} \arrow[r] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\ C^{\simeq} \times C^{\simeq} \arrow[r] & \mathrm{Cat}. \end{tikzcd}$$

Since cocartesian fibrations with small fibers are stable under composition, the diagonal $C^{\simeq} \to C^{\simeq} \times \cdots \times C^{\simeq}$ is a cocartesian fibration with small fibers for any finitely many cartesian product of copies of $C^{\simeq}$. This implies that, for any $n\colon \mathbb{N}$, the embedding

$$\underbrace{\Gamma \times_{C^{\simeq}} \cdots \times_{C^{\simeq}} \Gamma}_{n \text{ times}} \hookrightarrow \underbrace{\Gamma \times \cdots \times \Gamma}_{n \text{ times}}$$

is a cocartesian fibration with small fibers and with small codomain. Therefore, $\check{C}(\Gamma/C^{\simeq})$ is a functor that is objectwise with values in small groupoids. Therefore, since the indexing category Δ^{op} is small as well, its colimit $|\check{C}(\Gamma/C^{\simeq})|$ is a small groupoid. □

11.6 Truncated categories and functors

We define $\mathbb{Z}_{\geq -2} = [0] \sqcup [0] \sqcup \mathbb{N}$. The two objects obtained by the contribution of $[0] \sqcup [0]$ are called -2 and -1. We define $-2+1=-1$ and $-1+1=0$. Since $n+1$ is already defined for $n\colon \mathbb{N}$, this defines a successor operation on $\mathbb{Z}_{\geq -2}$ starting at -2. This induces an equivalence $\mathbb{N} \xrightarrow{\sim} \mathbb{Z}_{\geq -2}$. In particular, we can do induction starting from -2. We will often write $n \geq -2$ instead of $n\colon \mathbb{Z}_{\geq -2}$.

We will now discuss the hierarchy of n-truncated functors for $n \geq -2$.

Definition 11.6.1 (n-truncated functors)**.** We fix a category U with pullbacks as well as with a terminal object $*$. For an integer $n \geq -2$, we inductively define what it means for a morphism $f\colon A \to B$ in U to be *n-truncated*:

- When $n=-2$, we say f is (-2)-truncated if it is an equivalence;
- For $n\colon \mathbb{Z}_{\geq -2}$, we say f is $(n+1)$-truncated if the map $\Delta_f := (1_A, 1_A)\colon A \to A \times_B A$ to the pullback of f along itself is n-truncated.

An objet A of C is called *n-truncated* if the map $A \to *$ is n-truncated.

Definition 11.6.2 (Locally n-truncated functors)**.** A functor $f\colon A \to B$ is *locally n-truncated* if for each pair of objects $x, y : A^{\simeq}$, the map of groupoids $A(x,y) \to B(fx, fy)$ is n-truncated. A category A is called locally n-truncated, if every hom-groupoid $A(x,y)$ is n-truncated.

Remark 11.6.3.

- A category C is (-2)-truncated if and only if it is contractible (Definition 1.2.3)
- A functor $F\colon C \to D$ is (-1)-truncated if and only if it is an embedding (Definition 1.3.24)
- A functor $F\colon C \to D$ is locally (-2)-truncated if and only if it is fully faithful.

The following proposition is obviously true.

Proposition 11.6.4. *Any functor that commutes with pullbacks preserves n-truncated morphisms. Any functor that preserves pullbacks and terminal objects preserves n-truncated objects.*

Proposition 11.6.5. *Assume that the ambiant category U is locally small. A morphism $f\colon A \to B$ is n-truncated in a category U if and only if, for any object $X\colon U^{\simeq}$, the induced map*

$$f_*\colon \operatorname{Hom}_U(X,A) \to \operatorname{Hom}_U(X,B)$$

is n-truncated in Grpd.

Proof. This follows right away from the fact that the Yoneda embedding of U is conservative and commutes with limits, whereas equivalences in a presheaf category are detected object-wise. □

Lemma 11.6.6 (Truncations levels are cumulative)**.** *Let $-2 \leq n \leq m$. Then every n-truncated morphism is also m-truncated.*

Proof. This follows by induction on n; the base case is the observation that if $C \to D$ is an equivalence then also $C \to C \times_D C$ is an equivalence. □

Proposition 11.6.7 (Closure properties of n-truncated maps)**.** *Let $n \geq -2$.*

(1) Consider morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ and assume that g is n-truncated. Then f is n-truncated if and only if gf is n-truncated.

(2) n-truncated functors are closed under fiber products, retracts and base change.

(3) For every map of groupoids $u\colon \Gamma \to \Gamma'$, n-truncated functors are preserved by dependent sum and dependent product along u.

Proof. todo □

Lemma 11.6.8 (Truncation is fiberwise)**.** *Consider functors $C \xrightarrow{f} D \to \Gamma$ with groupoid Γ and categories C and D. Then f is n-truncated if and only if f_γ is n-truncated for each object $\gamma : \Gamma$.*

Proof. The statement "if" follows from the closure of n-truncated functors under base change. The converse is tautological because γ can be chosen to be the universal object $\gamma = \mathrm{id}_\Gamma$. □

For groupoids, the truncation level can be fully determined by looking at the identity types.

Proposition 11.6.9. *Let $f\colon X \to Y$ be a map of groupoids and $n \geq -1$. Then f is n-truncated if and only if for all objects $x, y : X$, the induced map $X(x, y) \to Y(fy, fx)$ is $(n-1)$-truncated.*

Proof. Apply Lemma 11.6.8 to $X \xrightarrow{f} X \times_Y X \to X \times X$ and observe that base-changing the diagonal Δ_f to $(x, y) : X \times X$ precisely yields the map $X(x, y) \to Y(fy, fx)$. □

Remark 11.6.10. Let X and Y be groupoids. In type theoretic language we can therefore define

$$\mathtt{isTruncated}(-2, X) := \mathtt{is\_Contr}(X)$$

and

$$\mathtt{isTruncated}(n+1, X) := \prod_{x,y:X} \mathtt{isTruncated}(n, X(x, y))$$

for all $n \geq -2$. Moreover, it follows from Lemma 11.6.8 (with $X = C$ and $Y = D = \Gamma$) that for any map $f\colon X \to Y$ we have the bi-implication

$$\mathtt{isTruncated}(n, f) \leftrightarrow \prod_{y:Y} \mathtt{isTruncated}(n, X_y).$$

Lemma 11.6.11 (Truncation of functor categories)**.** *For every n-truncated functor $C \to D$, the induced map*

$$\mathrm{Fun}(T, C) \to \mathrm{Fun}(T, D) \tag{11.11}$$

is again n-truncated for every category T. If C is n-truncated, then so is $\mathrm{Fun}(T, C)$.

Proof. This is a particular case of Proposition 11.6.4. □

For categories, the situation is slightly more complicated, since being truncated involves a condition both on identity types and on hom groupoids.

Proposition 11.6.12 (Characterization of n-truncated functors)**.** *Let $f\colon C \to D$ be a functor and $n \geq -2$. The following are equivalent:*

(1) The functor f is n-truncated.

(2) The map of groupoids $\mathrm{Map}([1], f)\colon\ \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$ is n-truncated.

(3) The functor f is locally n-truncated and induces an n-truncated map of groupoids $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$.

Proof.

$1 \iff 2$ Follows by induction. The base case ($n = -2$) is Corollary 6.3.4. The induction step follows from the fact that $\mathrm{Map}([1], -)$ preserves pullbacks.

$2 \iff 3$ Condition 2 implies that $f^{\simeq}$ is n-truncated because $f^{\simeq}$ is a retract of $\mathrm{Map}([1], f)$. Hence we may assume that $f^{\simeq}$ is n-truncated to prove the bi-implication.

Applying Lemma 11.6.8 to

$$\mathrm{Map}([1], C) \xrightarrow{\alpha} \mathrm{Map}([1], D) \times_{D^{\simeq}\times D^{\simeq}} C^{\simeq} \times C^{\simeq} \to C^{\simeq} \times C^{\simeq}$$

shows that f being locally n-truncated is equivalent to saying that α is n-truncated. Moreover, $\mathrm{Map}([1], f)$ is the composite

$$\mathrm{Map}([1], C) \xrightarrow{\alpha} \mathrm{Map}([1], D) \times_{D^{\simeq}\times D^{\simeq}} C^{\simeq} \times C^{\simeq} \to \mathrm{Map}([1], D)$$

where the second map is n-truncated as a base-change of the n-truncated map $f^{\simeq} \times f^{\simeq}$. We conclude that $\mathrm{Map}([1], f)$ is n-truncated if and only if α is n-truncated if and only if n is locally n-truncated, as desired.

□

Proposition 11.6.13. *Let $n \geq -2$. Every locally n-truncated category is $n+1$-truncated.*

Proof. The case $n = -2$ is exactly the statement that every fully faithful functor is an embedding; this follows from Proposition 6.4.2 and Corollary 3.2.3. Now let $n \geq -1$ and $f\colon C \to D$ be n-truncated. By Lemma 11.6.12 we only have to show that $f^{\simeq}$ is $n+1$-truncated. Let $x, y : C^{\simeq}$. Then we have a commutative diagram of groupoids

$$\begin{tikzcd} C^{\simeq}(x, y) \arrow[r] \arrow[d, "f^{\simeq}"] & C(x, y) \arrow[d, "f"] \\ D^{\simeq}(fx, fy) \arrow[r] & D(fx, fy) \end{tikzcd}$$

where the two horizontal maps are embeddings (i.e., (-1)-truncated) by Corollary 2.2.12 and the right vertical map is n-truncated by assumption. Since $n \geq -1$, it follows that the left vertical map is also n-truncated. Since $x, y : C^{\simeq}$ were arbitrary, we are done by Proposition 11.6.9. □

Corollary 11.6.14. *Every (-1)-truncated functor is conservative and every (-1)-truncated category is a groupoid.*

Proof. Let $f: C \to D$ be a (-1)-truncated functor. Let $x, y : C^{\simeq}$ and $u: C(x, y)$ such that $f(u)$ is invertible. Since the map $C^{\simeq}(x, y) \to D^{\simeq}(fx, fy)$ is an equivalence we find an invertible arrow $v : C^{\simeq}(x, y)$ with $f(v) = f(u)$. But that implies $v = u$ in $C(x, y)$ because f is an embedding. It follows that u is invertible. Since u was arbitary, f is conservative.

The second statement is the special case where $D = *$. □

11.7 Truncation

Definition 11.7.1 (Induction principle of n-truncation)**.** Let X be a groupoid. We say that a map $t: X \to X'$ is an *n-approximation* if it satisfyies the following *induction principle of n-truncation*:

- For every n-truncated map $p: Y \to X$ of groupoids, every section of $X \times_{X'} Y \to X$ can be lifted to a section of $Y \to X'$. Explicitly this means that for every commutative triangle

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ X \arrow[ur, "f"] \arrow[r, "t"'] & X' \end{tikzcd}$$

  where p is n-truncated we have a section $f': X' \to Y$ of p and an identification $f = f't$ over X'.

If additionally X' is n-truncated, we say that t exhibits X' as the *n-truncation* of X. If an n-truncation of X exist, we denote it by $t: X \to \|X\|_n$ and its action on objects by $t: x \mapsto [x]$.

Remark 11.7.2. A more type-theoretic way to think about the induction principle of n-truncation is as follows: Whenever we have a family of n-truncated groupoids $Y_{x'}$ indexed by $x' : X'$, to produce a section $x' : X' \mapsto f'(x') : Y_{x'}$ it suffices to do so on terms of the form $x' = tx$, by specifying the value $f'(x') \coloneqq f(x)$.

Lemma 11.7.3. *In the setting of Definition 11.7.1, assume that we have two lifts $f_1', f_2': X' \to Y$ of the section f. Then we have an identification $h': f_1' = f_2'$ with $h't = \mathrm{id}_f$*

Proof. Consider the family $Z: (x' : X) \mapsto Y_{x'}(f_1'x', f_2'x')$. Since $Y \to X'$ is n-truncated, each $Z_{x'}$ is $(n-1)$-truncated; in particular, the projection $Z \to X'$ is n-truncated. Moreover, for each $x : X$ we have $h(x) \coloneqq \mathrm{id}_{f(x)} : Y_x(fx, fx) = Z(tx)$ (using that $f_i'tx = fx$). Therefore, we may apply the induction principle to $Z \to X'$ to obtain a lift $h': X' \to Z$ of h, which is precisely the datum of an identification $f_1' = f_2'$ extending the identity of f. □

Exercise 11.7.4. Show that n-truncation of a groupoid is unique if it exists. (Soon we will show that it always does.)

Exercise 11.7.5. Let $X \to X'$ be an n-approximation of groupoids. Show that for every n-truncated map $Y \to X'$ of groupoids, the canonical map

$$\mathrm{Map}_{X'}(X', Y) \to \mathrm{Map}_{X'}(X, Y) \tag{11.12}$$

is an equivalence.

Remark 11.7.6. Matching the terminology of Definition 11.2.17, the -1-truncation and 0-truncations are also called propositional truncation and set truncation, respectively.

We conclude by defining the local analog of truncation.

Definition 11.7.7. A functor $t\colon C \to C'$ is called a *local n-approximation* if for every locally n-truncated functor $p\colon D \hookrightarrow C'$, the induced map

$$\mathrm{Map}_{C'}(C', D) \to \mathrm{Map}_{C'}(C, D) \tag{11.13}$$

is an equivalence. If C' is additionally locally n-truncated, then we say that t exhibits C' as the *local n-truncation* of C, and write $C' = \|C\|_n^{\mathrm{loc}}$.

11.7.1 Propositional truncation

Let us refer to (-1)-truncation as *propositional truncation*. We will now show that the Čech nerve construction from Section 11.4 computes the propositional truncation.

Lemma 11.7.8. *A map of groupoids $t\colon X \to X'$ is an (-1)-approximation if and only if t is a cover.*

Proof. In the case $n = -1$ the induction principle of the n-truncation can be simplified by recalling that an (-1)-truncated map has a section if and only if it is an equivalence (and any two sections are isomorphic in that case), see Lemma 1.3.28. Thus in the case $n = -1$, the condition of Definition 11.7.1 can be reformulated as saying that an embedding $Y \to X'$ is an equivalence if and only if the pulled back map $X \times_{X'} Y \to X'$ is an equivalence. This is equivalent to $X \to X'$ being a cover by Proposition 11.5.11. □

Corollary 11.7.9 (Existence of propositional truncation)**.** *For every groupoid X, the map $X \to |\check{C}(X)|$ is a propositional truncation. (Here the Čech nerve is computed in any universe in which X is small.)*

Proof. The groupoid $|\check{C}(X)|$ is a proposition by Theorem 11.4.11 and the map $X \to |\check{C}(X)|$ is a cover by Proposition 11.5.4. □

As a consequence of the corollary, the (-1)-truncation $\|X\|_{-1}$ exists and is given by $|\check{C}(X)|$.

Definition 11.7.10. Let X be a groupoid. We say that there *merely exists* an object of X if $\|X\|_{-1}$ is inhabited.

Lemma 11.7.11. *A map $f\colon Y \to X$ of groupoids is a cover if and only if for each $x : X$ there merely exists an $y : Y$ with $fy = x$, i.e., if we have an inhabitant of*

$$\prod_{x:X} \left\| \sum_{y:Y} fy = x \right\|_{-1}.$$

Proof. If $x\colon X$ is a term defined in context Γ (with Γ a groupoid), we see that $\left\|\sum_{y:Y} fy = x\right\|_{-1}$ literally is $|\check{C}(x^*(f))|$ for $x^*(f)$ the pullback of f along $x\colon \Gamma \to X$. Therefore, if we have a term in

$$\prod_{x:X} \left\| \sum_{y:Y} fy = x \right\|_{-1},$$

we have a term in $|\check{C}(x^*(f))|$ for all x, in particular for x the universal term of X, also known as the identity. This means that $|\check{C}(X/Y)| \hookrightarrow Y$ has a section and is thus an equivalence. The converse follows from the fact that covers are stable under base change. □

Remark 11.7.12. What the proof of Lemma 11.7.11 actually does is the construction of an equivalence

$$|\check{C}(X/Y)| \xrightarrow{\sim} \prod_{x:X} \left\| \sum_{y:Y} f y = x \right\|_{-1}$$

in context Y (since these are propositions in context Y, such an equivalence is unique in context Y).

Proposition 11.7.13. *Propositional truncation $X \mapsto \|X\|_{-1}$ yields a left adjoint to the inclusion* Prop $\hookrightarrow$ Grpd.

Proof. By the pointwise construction of adjoints and directed univalence we need to show the following: for each groupoid X there exists a proposition X' and a map $t\colon X \to X'$ such that for all propositions Z the induced map

$$\mathrm{Map}(X', Z) \to \mathrm{Map}(X, Z) \tag{11.14}$$

is an equivalence. Choosing $X \to X' :\equiv \|X\|_{-1}$ to be a propositional truncation, this is a special case of Theorem 11.7.5 for the constant family $Y :\equiv X' \times Z \to X'$. □

11.7.2 Set and Poset truncation

We saw that one can construct propositional truncations using the Čech nerve. We will now see that one can bootstrap to set and poset truncations.

Lemma 11.7.14. *Let $p\colon A \to B$ be a functor that is a cover on objects. For every poset P, the induced functor*

$$\mathrm{Fun}(B, P) \to \mathrm{Fun}(A, P) \tag{11.15}$$

is fully faithful. Moreover a functor $f\colon A \to P$ lies in the essential image (i.e., factors through p if and only if it satisfies $f(a) \leq f(a')$ whenever we have an arrow $pa \to pa'$ in B.

Proof.

- To establish fully faithfulness, Corollary 11.2.22 it suffices to show for all $f, g\colon B \to P$ the logical equivalence between
  - for all $b : B^{\simeq}$ we have $f(b) \leq g(b)$ and
  - for all $a : A^{\simeq}$ we have $f(pa) \leq g(pa)$,

  with the forward direction already given by functoriality of g. The converse follows from the induction principle for propositional truncation which holds because $p^{\simeq}\colon A^{\simeq} \to B^{\simeq}$ is a (-1)-approximation by assumption. Indeed it tells us that to show the proposition $f(b) \leq g(b)$ for all objects $b : B^{\simeq}$ it suffices to do so for those of the form $b :\equiv pa$.
- Let $f\colon A \to P$ satisfy $fa \leq fa'$ whenever we have an arrow $pa \to pa'$. We have to show that f factors through p. (The converse is obvious.)

We start with the special case where $P = S$ is a set, and where $A = A^{\simeq}$ and $B = B^{\simeq}$ are groupoids. Let $R \to B \times S$ be the fiberwise -1-truncation of the map $(p, f)\colon A \to B \times S$; in other words, R consists of those pairs (b, s) for which there merely exists an $a : A$ with $pa = b$ and $fa = s$. We claim that the composite

$$R \subseteq B \times S \to B \tag{11.16}$$

is an embedding. To show this we have to construct, for any $b : B$ and any two $(b, s), (b, s') : R_b$, an equality $s = s'$ in S; but since S is a set, hence $S(s, s')$ is a proposition, mere existence is enough. By definition of R there merely exist $a, a' : A$ with $fa = s$ and $fa' = s$ but also $pa = b = pa'$. By the assumption on the map f, we thus obtain $s = fa = fa' = s'$ as claimed. Now we may use that every cover is a (-1)-approximation (Theorem 11.7.8) to obtained the dashed filler

$$\begin{tikzcd} & R \arrow[r, hook] & B \times S \arrow[d] \\ A \arrow[ur, "{(p,f)}"] \arrow[rr, "p"] & & B \arrow[ul, dashed] \end{tikzcd} \tag{11.17}$$

whose second component yields the desired factorization of f through p.

- To prove the general case consider the commutative square

$$\begin{tikzcd} \mathrm{Map}(B, P) \arrow[r] \arrow[d] & \mathrm{Map}(A, P) \arrow[d] \\ \mathrm{Map}(B^{\simeq}, P) \arrow[r] & \mathrm{Map}(A^{\simeq}, P) \end{tikzcd} \tag{11.18}$$

where all maps are embeddings; the horizontal ones by what we proved earlier, and the vertical ones by Lemma 11.2.21. Thus it suffices to show that $f^{\simeq}\colon A^{\simeq} \to P^{\simeq}$ factors through $p^{\simeq}\colon A^{\simeq} \to B^{\simeq}$ and that the resulting map $g^{\simeq}\colon B^{\simeq} \to P^{\simeq}$ extends to $g\colon B \to P$. We have the first fact by applying the step we already proved; indeed, $P^{\simeq}$ is a set and $A^{\simeq}, B^{\simeq}$ are groupoids. For the second step we have to show $g(b) \leq g(b')$ whenever we have an arrow $b \to b'$ in B (by Lemma 11.2.21). But since $p\colon A \to B$, hence also $A \times A \to B \times B$ is a (-1)-approximation, it suffices to show this in the case where $b = pa$ and $b' = pa'$ for $a, a' : A^{\simeq}$, in which case it is precisely the assumption on f. □

Corollary 11.7.15. *Let $p\colon A \to B$ be a functor which is a cover both on objects and on each mapping groupoid. Then the induced restriction functor*

$$\mathrm{Fun}(B, P) \to \mathrm{Fun}(A, P) \tag{11.19}$$

is an equivalence for each poset P.

In other words, every functor that is a cover both on objects and on mapping groupoids is a poset-approximation. Next we produce an explicit poset-truncation for each category C.

Construction 11.7.16. Let C be a category and choose a universe Cat where C is small. Consider the composite

$$h_{-1}\colon C \xrightarrow{h} \mathrm{PSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \xrightarrow{L} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Prop}) =: \mathrm{PSh}_{-1}(C) \tag{11.20}$$

of the Yoneda embedding with the functor L given by postcomposition with the propositional truncation functor $\|-\|_{-1}\colon \mathrm{Grpd} \to \mathrm{Prop}$.

Denote by $\|C\|_{\mathrm{pos}} \subseteq \mathrm{PSh}_{-1}(C)$ the essential image of h_{-1}, which is tautologically equipped with a map $C \to \|C\|_{\mathrm{pos}}$.

Proposition 11.7.17. *The category $\|C\|_{\mathrm{pos}}$ is a poset and the functor $C \to \|C\|_{\mathrm{pos}}$ is a cover both on objects and on mapping groupoids.*

Proof. The category Prop is a poset ref, hence also $\mathrm{PSh}_{-1}\, C = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Prop})$ by Corollary 11.2.22 and its full subcategory $\|C\|_{\mathrm{pos}}$. Moreover, the functor $C \to \|C\|_{\mathrm{pos}}$ is an objectwise cover by construction. It remains to show that for objects $c, d : C^{\simeq}$, the map

$$C(c,d) \to \|C\|_{\mathrm{pos}}([c],[d]) = \mathrm{PSh}_{-1}(C)(h_{-1}c, h_{-1}c) \tag{11.21}$$

is a cover. By Yoneda and unraveling the construction, this map is identified with the propositional truncation map

$$C(c,d) \to \|C(c,d)\|_{-1}, \tag{11.22}$$

which is a cover by Theorem 11.7.8. □

Corollary 11.7.18. *The assignment $C \mapsto \|C\|_{\mathrm{pos}}$ assembles to a left adjoints to the inclusions*

$$\mathrm{Poset} \hookrightarrow \mathrm{Cat} \quad \textit{and} \quad \mathrm{Set} \hookrightarrow \mathrm{Grpd}. \tag{11.23}$$

Proof. The first statement follows directly from Corollary 11.7.15 and Proposition 11.7.17 by the usual pointwise construction of adjoints. For the second statement it suffices to note that the poset truncation of a groupoid is again a groupoid, hence a set; this follows immediately from the fact that $C \to \|C\|_{\mathrm{pos}}$ is a cover on mapping groupoids. □

11.8 Rezk nerves and complete Segal animae

The reader may have observed that a lot of the structure of a category C is already contained in the information of the groupoids $\mathrm{Map}([n], C)$ for $n = 0, 1, 2$ together with their comparison maps:

- The objects of C are the objects of $C^{\simeq} = \mathrm{Map}([0], C)$;
- The morphisms of C are the objects of $\mathrm{Map}([1], C)$;
- Composition in C is controlled by $\mathrm{Map}([2], C)$, which by the Segal axiom is equivalent to $\mathrm{Map}([1], C) \times_{C^{\simeq}} \mathrm{Map}([1], C)$.

It turns out that C can in fact be fully recovered from the simplicial groupoid $[n] \mapsto \mathrm{Map}([n], C)$. This is what we will establish in this section.

Definition 11.8.1 (Rezk Nerve)**.** Let $i\colon \mathbf{\Delta} \hookrightarrow \mathrm{Cat}$ denote the inclusion. We define the *Rezk nerve functor* as the composite functor

$$N\colon \mathrm{Cat} \xrightarrow{h_{\mathrm{Cat}}} \mathrm{Fun}(\mathrm{Cat}^{\mathrm{op}}, \mathrm{Grpd}) \xrightarrow{i^*} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}).$$

By Theorem 9.6.11, this functor admits a left adjoint

$$\mathrm{ac} = Li_! \colon \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Cat}$$

satisfying $\mathrm{ac}\,\Delta^n = \mathrm{ac}\,h_{\mathbf{\Delta}}([n]) = [n]$, functorially in $[n] \in \mathbf{\Delta}$. We refer to ac as the *associated category* functor.

Construction 11.8.2. We fix once and for all a small category A and a functor

$$i \colon A \to \mathrm{Cat}\,.$$

We assume that, for each object a of A, there is a given terminal object e_a in $i(a)$.

Let C be small category. We form the following pullback square

$$\begin{tikzcd} A_{/C} \arrow[r, "i_{/C}"] \arrow[d] & \mathrm{Cat}_{/C} \arrow[d] \\ A \arrow[r, "i"] & \mathrm{Cat} \end{tikzcd}$$

The projection $A_{/C} \to A$ is the right fibration associated to the presheaf

$$a \mapsto \operatorname{Hom}_{\mathrm{Cat}}(i(a), C)\,.$$

Recall from Proposition 9.5.7 that the embedding of $\operatorname{RFib}(C)$ into $\mathrm{Cat}_{/C}$ has a left adjoint

$$L \colon \mathrm{Cat}_{/C} \to \operatorname{RFib}(C)$$

Composing L with $i_{/C}$ yields a functor

$$\tilde{\tau}_C \colon A_{/C} \to \operatorname{RFib}(C)\,.$$

This functor sends each functor $\gamma \colon i(a) \to C$ to the right fibration $C_{/\gamma(e_a)} \to C$. Indeed, in terms of right fibrations, we can factor γ into a terminal functor $u \colon i(a) \to X$ followed by a right fibrations $q \colon X \to C$. Since e_a is a terminal object of $i(a)$, , $u(e_a)$ is a terminal object of X whose image in C is $\gamma(e_a)$ by definition. This means that $u(e_a)$ determines a unique equivalence $C_{/\gamma(e_a)} \xrightarrow{\sim} X$ over C. But the functor L will send the morphism u to an isomorphism, which explains the identification

$$L((i(a), \gamma)) = C_{/\gamma(e_a)}\,.$$

In other words, the functor $\tilde{\tau}$ determines through the equivalence $\operatorname{RFib}(C) = \operatorname{LFib}(C^{\mathrm{op}})$ a functor

$$\tilde{\tau} \colon A_{/C} \to (C^{\mathrm{op}})^{\mathrm{op}}$$

hence a functor

$$\tau_C \colon A_{/C} \to C$$

naturally in C. The functor τ_C is very explicit on objects: it sends $\gamma \colon i(a) \to C$ to $\gamma(e_a)$. On morphisms, it sends a commutative triangle of the form

$$\begin{tikzcd} i(a) \arrow[rr, "i(u)"] \arrow[dr, "\beta"'] & & i(b) \arrow[dl, "\gamma"] \\ & C & \end{tikzcd} \qquad \text{(with } u \colon a \to b \text{ a morphism in } A\text{)}$$

to the image of the map $u(e_a) \to e_b$ in $i(b)$ through γ.

Since the functor L is compatible with left adjoints of pullbacks (see Remark 9.5.8), for any functor $f\colon C \to D$, there is a canonical commutative square of the form

$$\begin{tikzcd} A_{/C} \arrow[r, "\tau_C"] \arrow[d, "f_!"'] & C \arrow[d, "f"] \\ A_{/D} \arrow[r, "\tau_D"] & D \end{tikzcd}$$

where $f_!$ is the functor defined by composition with f.

Lemma 11.8.3. *Let $f\colon C \to D$ be a right fibration. Then*

$$\begin{tikzcd} A_{/C} \arrow[r, "\tau_C"] \arrow[d, "f_!"'] & C \arrow[d, "f"] \\ A_{/D} \arrow[r, "\tau_D"] & D \end{tikzcd}$$

is a pullback square.

Proof. There is a canonical commutative triangle of the form

$$\begin{tikzcd} A_{/C} \arrow[rr, "f_!"] \arrow[dr] & & A_{/C} \arrow[dl] \\ & A & \end{tikzcd}$$

and each projection $A_{/C} \to A$ is a right fibration. Therefore, $f_!$ is a right fibration. This means that the comparison map

$$A_{/C} \to A_{/D} \times_D C$$

is a morphism of right fibrations over $A_{/D}$. In order to prove that it is an equivalence, it suffices to check that it induces equivalences on fibers. In other words, we may fix a functor $\delta\colon i(a) \to D$, and we have to compare the collection of commutative diagrams in Cat of the form

$$\begin{tikzcd} & C \arrow[d, "f"] \\ i(a) \arrow[r, "\delta"] \arrow[ur, dashed, "\gamma"] & D \end{tikzcd}$$

with the collection of commutative diagrams of the form

$$\begin{tikzcd} {[0]} \arrow[r, dashed, "\gamma'"] \arrow[d, "e_a"'] & C \arrow[d, "f"] \\ i(a) \arrow[r, "\delta"] & D \end{tikzcd}$$

through the rule $\gamma \mapsto \gamma' = \gamma(e_a)$. But e_a being a terminal object of $i(a)$ and f a right fibration, the fact that this rule defines an equivalence obviously holds. □

Definition 11.8.4. Let A be a small category and $i\colon A \to \mathrm{Cat}$ be a functor. A *décalage structure* on the pair (A, i) consists of the following data.

(i) For each object a of A, there is a terminal object e_a in $i(a)$.

(ii) There is an object ω of A and an endofunctor $D\colon A \to A$.

(iii) There are natural transformations of the form

$$d\colon \mathrm{id}_A \to D \qquad \text{and} \qquad b\colon \omega_A \to D$$

(where ω_A is the constant A-indexed functor with value ω)

(iv) For each object a of A, there is a commutative diagram

$$\begin{tikzcd} i(a) \arrow[r, "i(d_a)"] \arrow[d, equal] & i(D(a)) \arrow[d, equal] & i(\omega) \arrow[l, "i(b_a)"'] \arrow[d, equal] \\ i(a) \arrow[r, hook] & i(a) \star [0] & {[0]} \arrow[l, hook'] \end{tikzcd}$$

in which the vertical maps are equivalences and the horizontal maps of the bottom line are the structural embeddings.

In particular, for each small category C, we can thus perform Construction 11.8.2 and produce a canonical functor

$$\tau\colon A_{/C} \to C$$

sending each functor $\gamma\colon i(a) \to C$ to the object $\gamma(e_a)$ in C.

Lemma 11.8.5. *Let $i\colon A \to \mathrm{Cat}$ be a functor indexed by a small category and equipped with a décalage structure. We consider a small category C with a terminal object t, and let $A^t_{/C}$ be the fiber of the functor $\tau\colon A_{/C} \to C$ at t. Then $A^t_{/C}$ is weakly contractible and the full embedding $A^t_{/C} \to A_{/C}$ is a terminal functor.*

Proof. We will prove that the décalage structure on (A, i) lifts to a décalage structure on the pair consisting of $A_{/C}$ and of the composite

$$A_{/C} \to A \xrightarrow{i} \mathrm{Cat}\,.$$

We form the following pullback squares.

$$\begin{tikzcd} \tilde{A} \arrow[rr] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & & {[0]} \arrow[d, "t"] \\ D^*A_{/C} \arrow[r] \arrow[d, "D^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_{/C} \arrow[r, "\tau_C"] \arrow[d, "p"] & C \\ A \arrow[r, "D"] & A & \end{tikzcd}$$

where p simply is the structural map. There is canonical commutative triangle of the form

$$\begin{tikzcd} \tilde{A} \arrow[rr, "\varphi"] \arrow[dr, "\tilde{p}"'] & & A_{/C} \arrow[dl, "p"] \\ & A & \end{tikzcd}$$

where $\tilde{p}$ is the composition of D^*p with the embedding of $\tilde{A}$ into D^*A, and φ is induced by composing with the functorial maps $d_a\colon a \to D(a)$. We claim that $\tilde{p}$ is a right fibration and that φ is an equivalence.

To prove that $\tilde{p}$ is a right fibration, we observe first that D^*p is a right fibration and that $\tilde{A}$ is a full subcategory of D^*A (because t is a terminal object, so that, in particular, $t\colon [0] \to C$ is fully faithful). Given any map $u\colon x \to y$ in A and any functor $\gamma\colon i(D(y)) \to C$, the property that the pair (a, γ) belongs to $\tilde{A}$ simply means that γ preserves terminal objects. There is a unique lift of u in D^*A, whose source is essentially determined by the composite

$$\gamma \circ i(D(u))\colon i(D(x)) \to C\,.$$

We want to prove that this lift determines an object of $\tilde{A}$. In other words, we have to prove that this composite sends the terminal object ot $i(D(x))$ to t. But the functor $i(D(u))$ preserves the terminal object by condition (iv) of Definition 11.8.4. This proves that $\tilde{p}$ is a right fibration. Proving that φ is an equivalence is similar: being a morphism of right fibrations over A, it suffices to check that φ induces an equivalence fiber-wise. This amounts to prove that any functor of the form

$$\gamma\colon i(a) \to C$$

extends uniquely into a functor of the form

$$\gamma'\colon i(D(a)) \to C$$

that preserves terminal objects. Condition (iv) of Definition 11.8.4 implies that it suffices to observe the following fact: given any category B an extension of a functor of the form

$$u\colon B \to C$$

into a functor of the form

$$v\colon B \star [0] \to C$$

sending 0 to t is the same thing as a commutative diagram of the following form.

$$\begin{tikzcd} & C_{/t} \arrow[d] \\ B \arrow[r, "u"] \arrow[ur, dashed, "\bar{v}"] & C \end{tikzcd}$$

But, since t is a terminal object, the projection $C_{/t} \to C$ is an equivalence, hence such an extension exists and is unique.

We define a functor $D_C\colon A_{/C} \to A_{/C}$ as the composite

$$A_{/C} \xrightarrow{\varphi^{-1}} \tilde{A} \hookrightarrow D^*A \to A_{/C}\,.$$

This yields a commutative square of the form bellow.

$$\begin{tikzcd} A_{/C} \arrow[r, "D_C"] \arrow[d, "p"'] & A_{/C} \arrow[d, "p"] \\ A \arrow[r, "D"] & A \end{tikzcd}$$

There are natural transformations

$$\mathrm{id}_{A_{/C}} \to D_C \leftarrow \omega_C$$

lifting their analogues in A: there is a functor

$$\delta\colon \tilde{A} \to \mathrm{Fun}([1], A_{/C})$$

defined by the rule

$$(a, \gamma\colon i(D(a)) \to C) \mapsto (d_a, \mathrm{id}_{\gamma\circ d_a})\colon (a, \gamma \circ d_a) \to (D(a), \gamma)$$

and $d_C\colon \mathrm{id}_{A_{/C}} \to D_C$ is defined as $\delta \circ \varphi^{-1}$. We define ω_C as the object of $A_{/C}$ associated to the unique functor $t\colon i(\omega) \to C$ reaching the object t. We observe that, for any object a of A and any functor $\gamma\colon i(D(a)) \to C$ that preserve the terminal object, the anima of isomorphism $\gamma \circ i(b_a) \cong t$ is contractible. There is thus a unique way to promote the natural transformation b into a natural transformation $b_C\colon \omega_C \to D_C$ (using $\tilde{A}$ and φ^{-1} as above).

We observe that ω_C is in fact an object of $A^t_{/C}$ and that D_C takes its values in $A^t_{/C}$. Since $A^t_{/C}$ is a full subcategory of $A_{/C}$, if we denote by D^t_C the restriction of D_C to $A^t_{/C}$, we obtain natural transformations of the form:

$$\mathrm{id}_{A^t_{/C}} \to D^t_C \leftarrow \omega_C\ .$$

This proves that $A^t_{/C}$ is weakly contractible. It remains to prove that the embedding of $A^t_{/C}$ into $A_{/C}$ is a terminal functor.

Let $\xi = (a, \gamma\colon i(a) \to C)$ be an object of $A_{/C}$. We want to prove that $(A^t_{/C})_{\xi/}$ is weakly contractible. A term of $(A^t_{/C})_{\xi/}$ essentially consists of

$$u\colon a \to b \qquad \text{together with} \qquad \begin{tikzcd} i(a) \arrow[rr, "i(u)"] \arrow[dr, "\gamma"'] & & i(b) \arrow[dl, "\delta"] \\ & C & \end{tikzcd}$$

The diagram

$$\begin{tikzcd} a \arrow[r, "d_a"] \arrow[d, "u"'] & D(a) \arrow[d, "D(u)"] \\ b \arrow[r, "d_b"] & D(b) \end{tikzcd}$$

together with the diagram

$$\begin{tikzcd} i(a) \arrow[r, "i(d_a)"] \arrow[d, "i(u)"'] & i(D(a)) \arrow[r, "\gamma'"] \arrow[d, "i(D(u))"] & C \arrow[d, equal] \\ i(b) \arrow[r, "i(d_b)"] & i(D(b)) \arrow[r, "\delta'"] & C \end{tikzcd}$$

defines natural transformations of the form

$$\mathrm{id}_{(A^t_{/C})_{\xi/}} \to D_\xi \leftarrow \omega_\xi$$

where ω_ξ is the constant functor with value the object determined by the map $d_a\colon a \to D(a)$ together with the functor $\gamma'\colon i(D(a)) \to C$. This shows that $(A^t_{/C})_{\xi/}$ is weakly contractible and achieves the proof of the lemma. □

Proposition 11.8.6. *Let $i\colon A \to \mathrm{Cat}$ be a functor indexed by a small category and equipped with a décalage structure. For any small category C, the evaluation at the terminal object of $i(a)$ defines a weakly proper functor with weakly contractible fibers*

$$\tau_C\colon A_{/C} \to C\,.$$

In particular, the category C is the localization of $A_{/C}$ by the class of morphisms that are sent to identites by τ_C in C.

Proof. Let x be an object of C. Then $C_{/x} \to C$ is a right fibration and Lemma 11.8.3 tells us that there is a canonical equivalence of the form

$$A_{/(C_{/x})} \xrightarrow{\sim} (A_{/C})_{/x} := (A_{/C}) \times_C C_{/x}\,.$$

By virtue of Lemma 11.8.5, the functor τ_C is weakly proper and has weakly contractible fibers. The last assertion about C being a localization of $A_{/C}$ is a particular case of Proposition 9.7.13. □

Construction 11.8.7. Let us consider a fully faithful functor $i\colon A \to \mathrm{Cat}$ such that (A, i) is equipped with a décalage structure. We form the following pullback square.

$$\begin{tikzcd} A_\bullet \arrow[r, "i_A"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ A \arrow[r, "i"] & \mathrm{Cat} \end{tikzcd}$$

The functor π_A is a cocartesian fibrations whose fiber over each object a it the category with terminal object $i(a)$. In particular, it has a fully faithful right adjoint

$$e\colon A \to A_\bullet$$

defined by $s(a) = e_a$ on objects.

We have a pair of adjoint functors of the form

$$Li_!\colon \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \mathrm{Cat}\colon i^*$$

Given a presheaf $X\colon A^{\mathrm{op}} \to \mathrm{Grpd}$, we form the pullback squares bellow.

$$\begin{tikzcd} A_{\bullet/X} \arrow[r, "\pi_{A/X}"] \arrow[d, "p_{\bullet/X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_{/X} \arrow[r] \arrow[d, "p_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet^{\mathrm{op}} \arrow[d] \\ A_\bullet \arrow[r, "\pi_A"] & A \arrow[r, "X^{\mathrm{op}}"] & \mathrm{Grpd}^{\mathrm{op}} \end{tikzcd}$$

In other words, we have:

$$A_{\bullet/X} = \int_{(a,\gamma)\colon A_{/C}} i(a)\,.$$

Observe that the functor $\pi_{A_{/X}}$ has a right adjoint section, being the pullback of a functor with this property. The assumption of full faithfulness on i implies that there are canonical isomorphisms of the form

$$A_{/a} \xrightarrow{\sim} A_{/i(a)}$$

defined by sending $(a', u\colon a' \to a)$ to $(a', i(u)\colon i(a') \to i(a))$. The canonical isomorphism deduced from Lemma 9.8.11

$$\operatorname{colim}_{(a,\gamma)\colon A_{/X}} A_{/a} \xrightarrow{\sim} A_{/X}$$

can thus be rewritten as

$$\operatorname{colim}_{(a,\gamma)\colon A_{/X}} A_{/i(a)} \xrightarrow{\sim} A_{/X}\,.$$

We observe that we have a functorially given family of functors $\tau_{i(a)}\colon A_{/i(a)} \to i(a)$. Since

$$\operatorname{colim}_{(a,\gamma)\colon A_{/X}} i(a) = Li_!(X)$$

by construction, we end up with the following commutative diagram

$$\begin{tikzcd}
\int_{(a,\gamma)\colon A_{/X}} A_{/i(a)} \arrow[r] \arrow[d] & A_{/X} \arrow[d] \\
\int_{(a,\gamma)\colon A_{/X}} i(a) \arrow[r] & Li_!(X)
\end{tikzcd}$$

in which both horizontal maps are localizations at cocartesian morphisms with respect to the projection to $A_{/X}$. On the other hand, the vertical arrow on the left is fiberwise a localization as well as a cocartesian functor over $A_{/X}$. Therefore, it is also a localization by Theorem 7.6.20. Furthermore, for each object (a,γ) of $A_{/X}$, $\gamma\colon h_A(a) \to X$, there are functorial equivalences of the form

$$(A_{/X})_{/(a,\gamma)} \simeq A_{/a} \simeq A_{/i(a)}$$

and we have an equivalence

$$\operatorname{Fun}([1], A_{/X}) \simeq \int_{(a,\gamma)\colon A_{/X}} (A_{/X})_{/(a,\gamma)}$$

over $A_{/X}$, with $\mathrm{ev}_1\colon \operatorname{Fun}([1], A_{/X}) \to A_{/X}$ as structural map. In other words, we end up with the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Fun}([1], A_{/X}) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "q_X"'] & A_{/X} \arrow[d, "\tau'_X"] \\
A_{\bullet/X} \arrow[r, "j_X"'] & Li_!(X)
\end{tikzcd}$$

Since i is fully faithful, the terminal object of $i(a)$ is determined by a unique map $e_a\colon \omega \to a$ in A. We will say that a map $u\colon a \to b$ in A is a *last vertex morphism* if $u \circ e_a \simeq e_b$ holds (such an isomorphism, if it exists, is necessarily unique).

Lemma 11.8.8. *Let W be the of morphisms in $A_{/X}$ whose image in A is a last vertex morphism. We also denote by $\tilde{W}$ the family of morphisms in* $\mathrm{Fun}([1], A_{/X})$ *whose image by the functor* ev_0 *is in W. Then the functor* $\mathrm{ev}_{[0]}$ *induces an equivalence of localizations*

$$\mathrm{Fun}([1], A_{/X})[\tilde{W}^{-1}] \xrightarrow{\sim} (A_{/X})[W^{-1}] .$$

Furthermore, the functor q_X sends morphisms in $\tilde{W}$ to cocartesian morphisms relatively to the projection of $A_{\bullet/X}$ to $A_{/X}$, and induces an equivalence

$$\mathrm{Fun}([1], A_{/X})[\tilde{W}^{-1}] \xrightarrow{\sim} Li_!(X) .$$

Proof. The functor ev_0 has a fully faithful left adjoint sending an object of $A_{/X}$ to its identity (the latter being seen as an object of $\mathrm{Fun}([1], A_{/X})$). The co-unit of this adjunction consists of squares of the form bellow.

$$\begin{tikzcd} a \arrow[r, equal] \arrow[d, equal] & a \arrow[d, "f"] \\ a \arrow[r, "f"] & b \end{tikzcd}$$

In particular, the co-units of this adjunctions all belong to $\tilde{W}$. The units belongs to W because they are isomorphisms. The functor ev_0 sends W into $\tilde{W}$ and its adjoint sends $\tilde{W}$ into W. It thus follows right away from Lemma 10.1.4 that ev_0 induces a functor of the form

$$\mathrm{Fun}([1], A_{/X})[\tilde{W}^{-1}] \to (A_{/X})[W^{-1}]$$

that has a left adjoint and such that both the unit and co-unit morphisms are objectwiwe invertible. In other words, this is an equivalence.

Using Remark 9.4.19, we can describe objects of $A_{\bullet/X}$ as triples (a, γ, x) with a an object of A, $\gamma\colon X_a$, and $x\colon i(a)^{\simeq}$. A morphism from (a, γ, x) to (b, δ, y) is a pair (f, u) with $f\colon a \to b$ a morphism in A and $u\colon i(f)(x) \to y$ a morphism in $i(b)$. One checks that such a morphism is cocartesian with respect to projection of $A_{\bullet/X}$ to $A_{/X}$ if and only if u is an isomorphism. We also know that the localization of $A_{\bullet/X}$ by cocartesian morphisms precisely is $Li_!(X)$. A morphism in $A_{/X}$ essentially is a pair (u, γ) with $u\colon a \to b$ a morphism in A and $\gamma\colon X_a$. The functor q_X sends such a pair (u, γ) on the triple (b, γ, x), where x is the image of the terminal object $e_a\colon i(a)^{\simeq}$ in $i(b)$ through u. Under this explicit description, it is clear that $\tilde{W}$ consists precisely of those morphisms sent by q_X to cocartesian morphisms.

Since the functor i is fully faithful, we may think of $x\colon i(a)$ as a map from ω to a in A. Since $[1] = i(D(\omega))$ we can represent the data of a morphism in $A_{\bullet/X}$ as a commutative diagram of the form

$$\begin{tikzcd} \omega \arrow[r, "d_\omega"] \arrow[d, "x"'] & D(\omega) \arrow[d, "v"] & \omega \arrow[l, "b_\omega"'] \arrow[dl, "y"] \\ a \arrow[r, "u"] & b & \end{tikzcd}$$

together with a term $\gamma\colon X_b$. This presentation makes clear that any morphism in $A_{\bullet/X}$ comes from a morphism in $\mathrm{Fun}([1], A_{/X})$. This implies right away that the localization of $\mathrm{Fun}([1], A_{/X})$ by $\tilde{W}$ coincides with the localization of $A_{\bullet/X}$ by cocartesian morphisms, by a simple inspection of universal properties. □

Theorem 11.8.9. *Let $i\colon A \to \mathrm{Cat}$ be a functor indexed by a small category. We assume the following to hold:*

(i) The functor i is fully faithful.

(ii) The pair (A, i) is equipped with a décalage structure.

Then the functor

$$i^*\colon \mathrm{Cat} \to \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})$$

defined by the rule $C \mapsto (a \mapsto \mathrm{Hom}_{\mathrm{Cat}}(i(a), C))$ is fully faithful.

Proof. Let C be a small category. It suffice to prove that the co-unit functor

$$Li_!i^*(C) \to C$$

is an equivalence. We have $A_{/C} = A_{/i^*(C)}$ functorially in C by construction. We thus observe from Corollary 9.5.3 that $i^*(C)$ is a colimit of the form

$$\mathrm{colim}_{(a,\gamma)\colon A_{/C}} h_A(a) \xrightarrow{\sim} i^*(C)\,.$$

The functorially given family of functors $\tau_{i(a)}\colon A_{/i(a)} \to i(a)$. yields the following commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_{(a,\gamma)\colon A_{/C}} A_{/i(a)} \arrow[r, equal] \arrow[d] & A_{/C} \arrow[d, "\tau_C"] \\
\mathrm{colim}_{(a,\gamma)\colon A_{/C}} i(a) \arrow[r] & C\,.
\end{tikzcd}$$

In other words, taking into account the commutative diagram at the end of Construction 11.8.7, we obtain a commutative diagram of the following form (where $A_{\bullet/C} = A_{\bullet/i^*(C)}$).

$$\begin{tikzcd}
A_{/C} \arrow[d, "\tau'_C"'] \arrow[dr, "\tau_C"] & \\
Li_!(i^*(C)) \arrow[r, "\varepsilon_C"'] & C
\end{tikzcd}$$

Using furthermore Proposition 11.8.6, we observe that C is the localization of $A_{/C}$ by the family of morphisms whose images by τ_C are identities. On the other hand, Lemma 11.8.8 tells us that τ'_C exhibits $Li_!(i^*(C))$ by the family of morphisms in $A_{/C}$ whose image in A is a last vertex morphism. In order to prove the theorem, is suffices to prove that any morphism in $A_{/C}$ inducing an identity in C is sent to an isomorphism in $Li_!(i^*(C))$.

We can represent the data of a morphism in $A_{/C}$ as a commutative diagram of the form

$$\begin{tikzcd}
\omega \arrow[r, "d_\omega"] \arrow[d, "e_a"'] & D(\omega) \arrow[d, "v"] & \omega \arrow[l, "b_\omega"'] \arrow[dl, "e_b"] \\
a \arrow[r, "u"] & b &
\end{tikzcd}$$

together with a functor $\gamma\colon i(b) \to C$. The property that such a morphism is sent to an identity in C means that $\gamma \circ i(v)$ factorizes through $\gamma(e_a)\colon [0] \to C$. By functoriality in C, this means that we only need to understand the case $C = [0]$. In this case, $A_{/C} = A$ has a terminal object that remains terminal in any of its localizations by Proposition 10.1.1. This means that the image of ω in $Li_!(i^*(C))$ is a terminal object. But the image of $D(\omega)$ is the same as the image of ω through the map b_ω. Therefore, v corresponds to a map between terminal objects in $Li_!(i^*(C))$ and is thus invertible. □

Corollary 11.8.10. *The Rezk nerve functor* $N\colon \mathrm{Cat} \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$ *is fully faithful.*

Proof. We apply Theorem 11.8.9 in the case $A = \mathbf{\Delta}$ with the canonical embedding $i\colon \mathbf{\Delta} \hookrightarrow \mathrm{Cat}$. The fact that $(\mathbf{\Delta}, i)$ has a décalage structure holds by definition of $\mathbf{\Delta}$. □

Construction 11.8.11 (unions in categories of presheaves)**.** Let C be a small category and $X\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ a presheaf. We assume given a family of embeddings $I^{\simeq} \hookrightarrow \mathrm{Emb}(\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}))^{\simeq}$ (see Construction 7.6.13) of the form

$$U \hookrightarrow X$$

and we assume that the following conditions hold: for any $U \hookrightarrow X$ and $V \hookrightarrow X$ in I, the pullback $U \times_X V \hookrightarrow X$ is either in $I^{\simeq}$ or is empty (i.e. $\emptyset \simeq U \times_X V$).
We form the full subcategory I of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})_{/X}$ spanned by $I^{\simeq}$. We thus have a canonical functor

$$\varphi\colon I \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

defined by $\varphi(U \hookrightarrow X) = U$ and a cocone $c\colon \varphi \to \mathrm{const}_I(X)$ induced by the embeddings $U \hookrightarrow X$, hence a canonical map

$$j\colon \operatorname*{colim}_{U \hookrightarrow X\colon I} U \to X\,.$$

We observe that I is a small category: it is a full subcategory of the category of presheaves on $C_{/X}$ with values in the small category of propositions. The colimit of the U's above is called the *union of the family I in X*. This terminology is justified by the following lemma.

Lemma 11.8.12. *The map $j\colon \operatorname{colim}_I \varphi \to X$ constructed above is an embedding. More precisely, it induces an isomorphism of the form*

$$\operatorname*{colim}_{U \hookrightarrow X\colon I} U \simeq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \check{C}(T/X)$$

with $T = \sum_{U \hookrightarrow X\colon I^{\simeq}} U \to X$ the canonical map and $\check{C}(T/X)$ defined as $\check{C}(T/X)_c = \check{C}(T_c/X_c)$ for all $c\colon C$.

Proof. Let $Z = \operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \check{C}(T/X)$. For each object c of C, since the evaluation at c commutes with small limits and with small colimits, the evaluation of Z at c is the colimit of $\check{C}(T_c/X_c)$, hence the canonical map $Z_c \to X_c$ is an embedding, by Corollary 11.4.12. Since evaluation at $C^{\simeq}$ is conservative, this shows that we have a canonical embedding $Z \hookrightarrow X$. Replacing X by Z, we may assume, without loss of generality, that $Z = X$ (or, equivalently, that the map $T \to X$ is a cover fiberwise).

Let c be an object of C. We want to prove that the colimit of the $U(c)$'s is $X(c)$. We consider then the following pullback squares.

$$\begin{tikzcd} J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ I \arrow[r, "\varphi(c)"] & \mathrm{Cat} \end{tikzcd}$$

We now have

$$\operatorname*{colim}_{U\hookrightarrow X\colon I} U(c) \simeq \|J\| \,.$$

The embeddings $U(c) \hookrightarrow X(c)$ induce a functor $J \to I \times X(c)$ hence a projection $\mathrm{pr}\colon J \to X(c)$. It suffices to check that J is weakly contractible in context $X(c)$. Since $T(c) \to X(c)$ is a cover, it is in fact sufficient to prove it in context $T(c)$. It is thus sufficient to prove this in each context $U(c)$ for each embedding $U(c) \hookrightarrow X(c)$. The fiber F of the projection pr over $U(c)$ is the full subcategory of J spanned by those pairs consisting of an embedding $V \hookrightarrow X$ to gether with a term of $U(c) \times_{X(c)} V(c)$ in context $U(c)$, i.e. a proof that the first projection $U(c) \times_{X(c)} V(c) \xrightarrow{\sim} U(c)$ is an equivalence. We thus have a functor

$$\varphi\colon F \to F\,, \quad U \to U \times_X V$$

and the second projection $U \times_X V \to V$ defines a map from id_F to φ. On the other hand, the first projection $U \times_X V \to U$ defines a map from φ to the constant functor with value U (together with the proof that $U(c) \times_{X(c)} U(c) \xrightarrow{\sim} U(c)$), proving that F is weakly contractible. □

Example 11.8.13. Let C be a small category and $X\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ a presheaf. We consider two embeddings $U \hookrightarrow X$ and $V \hookrightarrow X$. We denote by

$$U \cap V := U \times_X V \hookrightarrow X$$

the embedding of the fiber product of U and V over X. The *union of U and V in X*, denoted by $U \cup V$, is the colimit of the diagram of immersions

$$U \hookrightarrow X\,,\ V \hookrightarrow X \text{ and } U \cap V \hookrightarrow X\,.$$

The previous lemma tells us that we have a canonical embedding

$$U \cup V \hookrightarrow X\,.$$

In fact, one can check that the induced commutative square

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\ U \arrow[r, hook] & U \cup V \end{tikzcd}$$

is both cartesian and cocartesian. Indeed, since $U \cup V$ embedds in X, to check that it is a pullback square, it suffices to do it after replacing $U \cup V$ by X, and this holds then by definition. To prove that it is cocartesian, we observe that the full subcategory I of all immersions of codomain X spanned by immersions with domain U, V or $U \cap V$ is equivalent to the category $\ulcorner$, so that the presentation

$$U \cup V = \operatorname*{colim}_{W\hookrightarrow X\colon I} W$$

literally says that the square above is a pushout square.

Construction 11.8.14. We construct an embedding

$$i_n\colon\ \mathrm{Seg}^n \hookrightarrow \Delta^n$$

in Fun($\mathbf{\Delta}^{\mathrm{op}}$, Grpd) for each $n\colon \mathbb{N}$ as the union of all the embeddings of the form

$$v_i^n\colon \Delta^0 \hookrightarrow \Delta^n \quad \text{and} \quad w_j^n\colon \Delta^1 \to \Delta^n$$

with $i\colon \{0,\dots,n\}$ and $j\colon \{1,\dots,n\}$, defined inductively as follows. For $n\colon \mathbb{N}$, we see $[n+1]$ as $[n] \star [0]$ and the embeddings

$$[n] \hookrightarrow [n] \star [0] \hookleftarrow [0]$$

thus define two embeddings

$$\Delta^n \xrightarrow{\delta_{n+1}^{n+1}} \Delta^{n+1} \xleftarrow{v_{n+1}^{n+1}} \Delta^0 .$$

We define w_{n+1}^{n+1} as the map corresponding to the unique section of the structural map $[n] \star [0] \to [0] \star [0] = [1]$. We define finally:

$$\begin{aligned} v_{n+1}^i &= \delta_{n+1}^{n+1} \circ v_n^i && \text{if } i\colon \{0,\dots,n\}, \\ w_{n+1}^j &= \delta_{n+1}^{n+1} \circ w_n^j && \text{if } j\colon \{1,\dots,n\}. \end{aligned}$$

Since functors between posets are fully determined by their effect on objects, the maps above are fully determined by their induced maps on vertices. We see by induction that v_i^n sends 0 to i and w_j^n sends 1 to j and 0 to $j-1$.

Definition 11.8.15. A *Segal anima* is a functor $C\colon \mathbf{\Delta} \to$ Grpd such that, for each $n\colon \mathbb{N}$, the embedding $\mathrm{Seg}^n \hookrightarrow \Delta^n$ induces an isomorphism

$$\mathrm{Hom}(\Delta^n, C) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Seg}^n, C) .$$

Lemma 11.8.16. *The union of* $\mathrm{Seg}^n \hookrightarrow \Delta^n \overset{\delta_{n+1}^{n+1}}{\hookrightarrow} \Delta^{n+1}$ *and* $\Delta^1 \overset{w_{n+1}^{n+1}}{\hookrightarrow} \Delta^{n+1}$ *is* $\mathrm{Seg}^{n+1} \hookrightarrow \Delta^{n+1}$ *for any* $n\colon \mathbb{N}$.

Proof. The image of

$$\mathrm{Seg}^n \sqcup \Delta^1 \to \Delta^{n+1} .$$

coincides with the image of the map

$$\sum_{j\colon \{1,\dots n+1\}} \Delta^1 \to \Delta^{n+1}$$

induced by the maps $w_j^{n+1}\colon \Delta^1 \to \Delta^{n+1}$, which, by virtue of Lemma 11.8.12, precisely is Seg^{n+1}. □

Remark 11.8.17. Given $X\colon \mathbf{\Delta}^{\mathrm{op}} \to$ Grpd and $n\colon \mathbb{N}$, we define

$$\underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$$

as follows:

$$\underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{0 \text{ times}} := X_0$$

and for any $n\colon \mathbb{N}$

$$\underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n+1 \text{ times}} := \Big(\underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}\Big) \times_{X_0} X_1$$

where the structural maps to X_0 are induced by the last projection to X_1 composed with the map $d^1 \colon X_1 \to X_0$ on the fist factor, and by the map $d^0 \colon X_1 \to X_0$ on the second factor. The previous lemma provides the induction step to prove that

$$\mathrm{Hom}(\mathrm{Seg}^n, X) \simeq \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$$

for any $n \colon \mathbb{N}$. The case $n = 2$ is already interesting: there is a canonical pullback square of the form

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{Seg}^2, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "d^0"] \\ X_1 \arrow[r, "d^1"] & X_0 . \end{tikzcd}$$

Construction 11.8.18. Let C be a Segal anima. An object of C is a term of C_0. A *morphism* of C simply is a term of C_1. We define the composition law of morphisms in C as the composition of the inverse of the equivalence

$$C_2 = \mathrm{Hom}(\Delta^2, C) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Seg}^2, C) \simeq C_1 \times_{C_0} C_1$$

with the map

$$d^1 \colon C_2 \to C_1 \, .$$

This defines a morphism

$$C_1 \times_{C_0} C_1 \to C_1 \, , \quad (f, g) \mapsto g \circ f \, .$$

It is characterized by the property that, there is a unique term $t \colon C_2$ together with identifications of the form $e_2 \colon d^0(t) = f$, $e_0 \colon d^2(t) = g$, and $e_1 \colon d^1(t) = g \circ f$.

A morphism $f \colon x \to y$ in C is a term $f \colon C_1$ with identifications of the form $s \colon d^1(f) = x$ and $t \colon d^0(f) = y$. For instance, given any object x of C, there is the morphism $\mathrm{id}_x \colon x \to x$ defined as the image of $x \colon C_0$ through the map $C_0 \to C_1$ induced by $\Delta^1 \to \Delta^0$. One checks that the associativity equivalence

$$(C_1 \times_{C_0} C_1) \times_{C_0} C_1 \simeq C_1 \times_{C_0} (C_1 \times_{C_0} C_1)$$

induces a canonical identification

$$a_{f,g,h}(h \circ g) \circ f = h \circ (g \circ f)$$

for any maps $f \colon x \to y$, $g \colon y \to z$ and $h \colon z \to t$ in C.

A morphism $f \colon x \to y$ in C is an *isomorphism* if there are morphisms $g \colon y \to x$ and $h \colon y \to x$ together with identifications of the form

$$l \colon g \circ f = \mathrm{id}_x \quad \text{and} \quad r \colon f \circ h = \mathrm{id}_y \; .$$

Isomorphisms form an anima $\mathrm{Iso}(C)$ equipped with a forgetful map

$$\mathrm{Iso}(C) \to C_1 \, , \quad (f, g, h, l, r) \mapsto f$$

and there is a canonical proof that id_x is an isomorphism in x, yielding a canonical map

$$\mathrm{id} \colon C_0 \to \mathrm{Iso}(C) \, .$$

More precisely, $\mathrm{Iso}(C)$ is defined through the following diagram.

$$\begin{tikzcd}
\mathrm{Iso}(C) \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner"] & C_2 \times_{d^0, C_1, d^2} C_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_1 \arrow[d, "\Delta"] \\
& C_2 \times C_2 \arrow[r, "d^0 \times d^2"] \arrow[d, "d^1 \times d^1"] & C_1 \times C_1 \\
C_0 \times C_0 \arrow[r, "{(x,y) \mapsto (\mathrm{id}_y, \mathrm{id}_x)}"] & C_1 \times C_1. &
\end{tikzcd}$$

Given $x, y \colon C_0$, we denote by $x \xrightarrow{\simeq} y$ the fiber of the map $\mathrm{Iso}(C) \to C_0 \times C_0$ at (x, y).

Definition 11.8.19. A Segal anima C is *complete* if the map $\mathrm{id}\colon C_0 \to \mathrm{Iso}(C)$ is an equivalence.

Example 11.8.20. For any small category $C\colon \mathrm{Cat}^{\simeq}$, the Rezk nerve $N(C)$ is a complete Segal anima.

Definition 11.8.21. A morphism of simplicial animae $f\colon C \to D$ is *fully faithful* if the induced square

$$\begin{tikzcd}
C_1 \arrow[r, "f_1"] \arrow[d, "{(d^0, d^1)}"'] & D_1 \arrow[d, "{(d^0, d^1)}"] \\
C_0 \times C_0 \arrow[r, "f_0 \times f_0"] & D_0 \times D_0
\end{tikzcd}$$

is cartesian. A morphism of simplicial animae $f\colon C \to D$ is *essentially surjective* if the the collection of pairs of the form (x, u) with $x\colon C_0$ and $u\colon f(x) \xrightarrow{\simeq} y$ covers D_0 through the map $(x, u) \mapsto y$.

A morphism of Segal animae is a *Dwyer–Kan equivalence* if it is both fully faithful and essentially surjective.

Remark 11.8.22. If C is a Segal anima and $x, y\colon C$ are two objects, the collection $C(x, y)$ of morphisms from x to y in C fits in the pullback square bellow.

$$\begin{tikzcd}
C(x,y) \arrow[r] \arrow[d] & C_1 \arrow[d, "{(d^0, d^1)}"] \\
* \arrow[r, "{(x,y)}"] & C_0 \times C_0
\end{tikzcd}$$

A morphism of Segal animae $f\colon C \to D$ is fully faithful if any only if, for any objects x, y of C (in some groupoidal context Γ), the induced map

$$C(x, y) \to D(f(x), f(y))$$

is an equivalence (in context Γ).

The proof of the following lemma is straightforward.

Lemma 11.8.23. *If $f\colon C \to D$ is a fully faithful morphism of Segal animae, then there is a canonical pullback square of the form:*

$$\begin{tikzcd}
\mathrm{Iso}(C) \arrow[r, "\mathrm{Iso}(f)"] \arrow[d] & \mathrm{Iso}(D) \arrow[d] \\
C_1 \arrow[r, "f_1"] & D_1
\end{tikzcd}$$

Proposition 11.8.24. *A morphism of Segal animae $f\colon X \to Y$ is an equivalence if and only if it is fully faithful and induces an equivalence $f_0\colon X_0 \xrightarrow{\sim} Y_0$.*

Proof. This is clearly a necessary condition. Conversely, if $f\colon X \to Y$ is fully faithful and induces an equivalence $f_0\colon X_0 \xrightarrow{\sim} Y_0$, then it induces an equivalence

$$f_1\colon X_1 \xrightarrow{\sim} Y_1$$

as well: f_1 is then a pullback of the equivalence $f_0 \times f_0$. This implies right away that f induces an equivalence of the form

$$\underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}} \xrightarrow{\sim} \underbrace{Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1}_{n \text{ times}}.$$

The commutative square

$$\begin{tikzcd}
\mathrm{Hom}(\Delta^n, X) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}(\Delta^n, Y) \arrow[d, "\simeq"] \\
\mathrm{Hom}(\mathrm{Seg}^n, X) \arrow[r, "\simeq"] & \mathrm{Hom}(\mathrm{Seg}^n, Y)
\end{tikzcd}$$

thus implies that $f_n\colon X_n \xrightarrow{\sim} Y_n$ is an equivalence for all $n\colon \mathbb{N}$. □

Corollary 11.8.25. *A morphism of complete Segal animae is an isomorphism if and only if it is a Dwyer–Kan equivalence.*

Proof. This is clearly a necessary condition. Let $f\colon C \to D$ be a a Dwyer–Kan equivalence between complete Segal animae. By virtue of Proposition 11.8.24, it suffices to prove that $f_0\colon C_0 \to D_0$ is an equivalence. It suffices to check that f_0 is both a cover and an embedding (see Proposition 11.5.9). We consider the following commutative diagram.

$$\begin{tikzcd}
C_0 \arrow[r, "f_0"] \arrow[d, "\simeq"'] & D_0 \arrow[d, "\simeq"] \\
\mathrm{Iso}(C) \arrow[r, "\mathrm{Iso}(f)"] \arrow[d] & \mathrm{Iso}(D) \arrow[d] \\
C_1 \arrow[r, "f_1"] \arrow[d] & D_1 \arrow[d] \\
C_0 \times C_0 \arrow[r, "f_0 \times f_0"] & D_0 \times D_0
\end{tikzcd}$$

The upper square is cartesian because its two vertical maps are equivalences (since C and D are complete). The lower square is cartesian because f is fully faithful. The middle square is a pullback square as well by Lemma 11.8.23. Therefore, the outer square is cartesian, which proves that f_0 is an embedding. Finally, if $C_0 \times_{D_0} \mathrm{Iso}(D)$ denotes the pullback of C_0 along the projection of $\mathrm{Iso}(D)$ to D_0 that pick the codomain of isomorphisms in D, we have an obvious commutative triangle of the form

$$\begin{tikzcd}
 & C_0 \times_{D_0} \mathrm{Iso}(D) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
C_0 \arrow[rr, "f_0"] & & D_0
\end{tikzcd}$$

with $p(x, u\colon x \xrightarrow{\simeq} y) = x$ and $q(x, u\colon x \xrightarrow{\simeq} y) = x$. Since q is a cover by assumption, so is f_0. $\square$

Construction 11.8.26. Let $E\colon \mathrm{Grpd}^{\simeq}$ be a small groupoid. We form the following pullback square.

$$\begin{tikzcd} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})_E \arrow[r] \arrow[d] & * \arrow[d, "E"] \\ \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "\mathrm{ev}_0"] & \mathrm{Grpd} \end{tikzcd}$$

Alternatively, one can see E as a constant presheaf on $\mathbf{\Delta}$ and form the co-slice category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})_{E/}$. the category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})_E$ is the the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})_{E/}$ whose objects are pairs (X, e), with X a presheaf on $\mathbf{\Delta}$ and $e\colon E \xrightarrow{\sim} X_0$ an equivalence.

Proposition 11.8.27. *The full embedding $i_E\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})_E \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})_{E/}$ has a right adjoint $(X, e) \mapsto (e^*(X), e_X)$.*

Proof. Let $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Grpd}$ be a simplicial groupoid equipped with a map $e\colon E \to X_0$. Since $[0]$ is an initial object of $\mathbf{\Delta}^{\mathrm{op}}$, X induces a functor $\tilde{X}$ with values in $\mathrm{Grpd}_{X_0/}$.

$$\begin{tikzcd} (\mathbf{\Delta}^{\mathrm{op}})_{[0]/} \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Grpd}_{X_0/} \arrow[d] \\ \mathbf{\Delta}^{\mathrm{op}} \arrow[r, "X"'] \arrow[ur, dashed, "\tilde{X}"] & \mathrm{Grpd} \end{tikzcd}$$

Pulling back along e induces a functor

$$e^*\colon \mathrm{Grpd}_{X_0/} \to \mathrm{Grpd}_{E/}\,.$$

We define $e^*(X)$ to be the composite

$$e^*(X)\colon \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\tilde{X}} \mathrm{Grpd}_{X_0/} \xrightarrow{e^*} \mathrm{Grpd}_{E/} \xrightarrow{\text{forget}} \mathrm{Grpd}\,.$$

For each $n\colon \mathbb{N}$, we have a canonical pullback square

$$\begin{tikzcd} e^*(X)_n \arrow[r, "\mathrm{pr}_{E,X}"] \arrow[d] & X_n \arrow[d] \\ E \arrow[r, "e"] & X_0 \end{tikzcd}$$

hence, in particular, an equivalence $e_X\colon E \xrightarrow{\sim} e^*(X)_0$ and a morphism

$$\mathrm{pr}_{E,X}\colon (e^*(X), e_X) \to (X, e)\,.$$

One checks immediately that, for any $Y\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})^{\simeq}$ and any equivalence $f\colon E \xrightarrow{\sim} Y_0$, post-composing with $\mathrm{pr}_{E,X}$ induces an equivalence:

$$\mathrm{Hom}((Y, f), (e^*(X), e_X)) \xrightarrow{\sim} \mathrm{Hom}((Y, f), (X, e))\,.$$

Therefore, the assignment $(X, e) \mapsto (e^*(X), e_X)$ is the restriction on objects of a right adjoint to i_E. $\square$

Remark 11.8.28. In the construction of the right adjoint in the proof of Proposition 11.8.27, we see that if X is a Segal anima, then so is $e^*(X)$: this follows right away from the fact that the functor $(-) \times_{X_0} E$ preserves pullback squares.

Remark 11.8.29. Let $f\colon X \to Y$ be a morphism of simplicial groupoids. We have a unique factorization of f of the form

$$X \xrightarrow{\tilde{f}} f_0^*(Y) \xrightarrow{\mathrm{pr}_{X_0,Y}} Y$$

where $\tilde{f}$ is a morphism in $\mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Grpd})_{X_0}$. furthermore, the projection $\mathrm{pr}_{X_0,Y}$ is in particular fully faithful.

Assuming that both X and Y are Segal animae, we observe the following properties:

(i) The morphism f is fully faithful if and only if $\tilde{f}$ is an isomorphism.

(ii) The morphism f is essentially surjective if and only if the projection $\mathrm{pr}_{X_0,Y}$ is a Dwyer–Kan equivalence.

Indeed, property (i) is a direct consequence of Proposition 11.8.24 and property (ii) is obvious.

Lemma 11.8.30. *Let us consider two morphisms of Segal animae $f\colon C \to D$ and $g\colon D \to E$. If two out of the three morphisms f, g or $g \circ f$ are Dwyer–Kan equivalences, so is the third.*

Proof. Since one can compose pulback squares as well as covers, it is clear that Dwyer–Kan equivalences are stable under composition.

Let us assume that f and $g \circ f$ are Dwyer–Kan equivalences. Since covers are stable under pullbacks and compositions, it follows from Corollary 11.5.7 that g is essentially surjective. We have to prove that the map

$$D_1 \to (D_0 \times D_0) \times_{(E_0 \times E_0)} E_1$$

induced by g_1 is an equivalence over $D_0 \times D_0$. Let $X = C_0 \times_{D_0} \mathrm{Iso}(D)$ and $q\ X \to D_0$ be the map picking the codomain of isomorphisms. Since $q \times q$ is a cover, it suffice to prove that g_1 is an equivalence after pulling back along $q \times q$ by Proposition 11.5.4. But after such a pullback, $q \times q$ has a section. Furthermore, given $u\colon a \xrightarrow{\simeq} b$ in a Segal anima C, we get equivalences of the form

$$u^*\colon C(b,t) \to C(a,t) \quad \text{and} \quad u_*\colon C(t,a) \to C(t,b)$$

for any $t\colon C_0$ (exercise left to the reader). Therefore, it suffices to prove that g induces an equivalence

$$D(f(x), f(y)) \xrightarrow{\simeq} E(g(f(x)), g(f(y)))$$

for any terms $x, y\colon C_0$. This follows from the commutative triangles

$$\begin{tikzcd}
& C(x,y) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\
D(f(x), f(y)) \arrow[rr] & & E(g(f(x)), g(f(y))).
\end{tikzcd}$$

Let us assume now that g and $g \circ f$ are Dwyer–Kan equivalences. The pasting lemma for pullback squares (1.3.16) tells us that f is fully faithful. We define $E'_0 = C_0$ and $w = g_0 \circ f_0$, and form the pullback squares bellow.

$$\begin{tikzcd} C'_0 \arrow[r, "f'_0"] \arrow[d, "u"] & D'_0 \arrow[r, "g'_0"] \arrow[d, "v"] & E'_0 \arrow[d, "w"] \\ C_0 \arrow[r, "f_0"] & D_0 \arrow[r, "g_0"] & E_0 \end{tikzcd}$$

Note that all the vertical maps in the diagram above are covers. We define $C' = u^*(C)$, $D' = v^*(D)$ and $E' = w^*(E)$. We obtain a commutative diagram of the form

$$\begin{tikzcd} C' \arrow[r, "f'"] \arrow[d] & D' \arrow[r, "g'"] \arrow[d] & E' \arrow[d] \\ C \arrow[r, "f"] & D \arrow[r, "g"] & E \end{tikzcd}$$

In which all vertical maps are Dwyer–Kan equivalences. The morphism g' and $g' \circ f'$ are Dwyer–Kan equivalences as well: they are essentially surjective by construction and the full faithfulness of g and $g \circ f$ obviously implies the same property for any of their their pullbacks. To prove that f is a Dwyer–Kan equivalence, it suffices now to prove that f' is a Dwyer–Kan equivalences. In other words, without loss of generality, we may assume that the composite $g \circ f$ has a section. Let $y \colon D_0$ such that there is $x \colon C$ with an isomorphism $g(y) \xrightarrow{\sim} g(f(x))$. Then there is an isomorphism of the form $f(x) \xrightarrow{\sim} y$ by Lemma 11.8.23. Using the fully faithfulness of f, and the given section of $g \circ f$, this implies that the induced map

$$\mathrm{Iso}(f) \colon \mathrm{Iso}(C) \to \mathrm{Iso}(D)$$

is a cover. But f_0 obviously is a retract of $\mathrm{Iso}(f)$ and is thus a cover as well by Corollary 11.5.8. □

Theorem 11.8.31. *For any Segal anima X, the co-unit map*

$$\varepsilon_X \colon X \to N(\mathrm{ac}(X))$$

a Dwyer–Kan equivalence.

The proof of Theorem 11.8.31 will require a few preparation. We fix a simplicial anima X.

Construction 11.8.32. Let W the subcategory of lax vertex maps in $\mathbf{\Delta}$ (see the end of Construction 11.8.7). In other words, W is the synthetic subcategory of $\mathbf{\Delta}$ defined by $W^{\simeq} = \mathbf{\Delta}^{\simeq}$ with morphisms those morphisms

$$f \colon [m] \to [n]$$

such that $f(m) = n$. We form the pullback square bellow.

$$\begin{tikzcd} W_X \arrow[r] \arrow[d, hook] & W \arrow[d, hook] \\ \mathbf{\Delta}_{/X} \arrow[r] & \mathbf{\Delta} \end{tikzcd}$$

Inverting W_X in $\mathbf{\Delta}_{/X}$ yields $\mathrm{ac}(X)$: by virtue of Lemma 11.8.8, there is a canonical equivalence

$$\mathbf{\Delta}_{/X}[W_X^{-1}] \xrightarrow{\sim} \mathrm{ac}(X)\,.$$

We observe that $W_X^{\simeq} = (\mathbf{\Delta}_{/X})^{\simeq}$. Given an object d of $\mathbf{\Delta}_{/X}$, we write $W(d)$ for the co-slice of d, seen as an object of W_X. There is thus a canonical functor

$$p_d \colon W(d) \to (\mathbf{\Delta}_{/X})_{d/}\,.$$

This is obviously a putative left calculus of fractions at d. For $c \colon \mathbf{\Delta}_{/X}$ we write

$$W(d)_{c/} := (\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}_{/X}} W(d)\,.$$

The dual version of Remark 10.3.8 tells us that p_d is a left calculus of fractions if and only if the functor

$$(\mathbf{\Delta}_{/X})^{\mathrm{op}} \to \mathrm{Grpd}\,, \quad c \mapsto \Pi_\infty(W(d)_{c/})$$

sends all morphisms of W_X to isomorphisms.

From now on, we fix a term $y \colon X_0$. We interpret the latter as the object $d = ([0], y)$ in $\mathbf{\Delta}_{/X}$.

An arbitrary object c of $\mathbf{\Delta}_{/X}$ is a pair of the form $c = ([n], x)$ for some $n \colon \mathbb{N}$ and $x \colon X_n$. We define $P_X(x, y)$ through the pullback square

$$\begin{tikzcd}
P_X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+1} \arrow[d, "{(d^{n+1}, \mathrm{ev}_{n+1})}"] \\
* \arrow[r] & X_n \times X_0
\end{tikzcd}$$

where d^{n+1} is the operator induced by the canoncal embedding $[n] \hookrightarrow [n] \star [0] = [n+1]$ and ev_{n+1} the operator induced by the canoncal embedding $[0] \hookrightarrow [n] \star [0] = [n+1]$.

The projection $\mathbf{\Delta}_{/X} \to \mathbf{\Delta}$ induces a functor

$$(\mathbf{\Delta}_{/X})_{c/} \to \mathbf{\Delta}_{[n]/}$$

inducing with the projection $W_X \to W$ a canonical functor

$$q \colon W(d)_{c/} \to W_{[n]/} = \mathbf{\Delta}_{[n]/} \times_{\mathbf{\Delta}} W\,.$$

Let i_n be the object of $W_{[n]/}$ corresponding to the diagram

$$[n] \hookrightarrow [n] \star [0]\,.$$

Lemma 11.8.33 (Hebestreit–Steinebrunner)**.** *Under the assumptions of Construction 11.8.32, the functor $q \colon W(d)_{c/} \to W_{[n]/}$ is a right fibration with fiber at i_n canonically isomorphic to $P_X(x, y)$. Furthermore, the embedding of $P_X(x, y)$ into $W(d)_{c/}$ as fiber of q is an initial functor, inducing a canonical equivalence*

$$P_X(x, y) \xrightarrow{\sim} \Pi_\infty(W(d)_{c/})\,.$$

If furthermore X is a Segal anima, then the presheaf

$$c \mapsto \Pi_\infty(W(d)_{c/})$$

sends all morphisms of W_X to isomorphisms, so that

$$p_d \colon W(d) \to (\mathbf{\Delta}_{/X})_{d/}$$

is a left calculus of fractions at $d = ([0], y)$ in $\mathbf{\Delta}_{/X}$ with respect to W_X.

Proof. Since $\mathbf{\Delta}_{/X} \to \mathbf{\Delta}$ is a right fibration, so is the induced functor

$$(\mathbf{\Delta}_{/X})_{c/} \to \mathbf{\Delta}_{[n]/}$$

by Proposition 6.4.5. Right fibrations being stable under pullbacks, this yields a right fibration of the form

$$(\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W \to \mathbf{\Delta}_{[n]/} \times_{\mathbf{\Delta}} W\,.$$

We observe that $[0]$ is an initial object of W and that $W_X \to W$ is a right fibration (as pullback of the right fibration $\mathbf{\Delta}_{/X} \to \mathbf{\Delta}$ along the embedding $W \hookrightarrow \mathbf{\Delta}$), hence we get another right fibration through another application of Proposition 6.4.5:

$$W(d) = (W_X)_{d/} \to W_{[0]/} \simeq W\,.$$

Therefore, we get by pullback of the latter along the second projection

$$(\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W \to W$$

a right fibration of the form

$$(\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} (W_X)_{d/} = ((\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W) \times_W (W_X)_{d/} \to (\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W\,.$$

Composing with the right fibration $(\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W \to \mathbf{\Delta}_{[n]/} \times_{\mathbf{\Delta}} W$, this proves that the functor

$$q\colon W(d)_{c/} \to W_{[n]/}$$

is a right fibration as composition of two right fibrations. We observe that $i_n\colon (W_{[n]/})^{\simeq}$ is an initial object. Indeed, an object in $W_{[n]/}$ is a morphism $u\colon [n] \to [m]$. A morphism from $u\colon [n] \to [m]$ to $u'\colon [n] \to [m']$ is a morphism $f\colon [m] \to [m']$ such that $f \circ u = u'$ and $f(m) = m'$. For $u\colon [n] \to [m]$, there is a unique map $\tilde{u}\colon [n+1] \to [m]$ such that $\tilde{u}(i) = u(i)$ for $i\colon [n]^{\simeq}$ and $\tilde{u}(n+1) = m$: since objects of $\mathbf{\Delta}$ are posets (Example 11.2.19), this follows from the last assertion of Lemma 11.2.21. The morphism $\tilde{u}$ is the unique morphism from i_n to the object determined by u. Right fibrations being smooth, since $i_n\colon * \to W_{[n]/}$ is an initial functor, the embedding of the fiber of q at i_n is an initial functor.

Since q is a right fibration, it is conservative and its fiber at i_n is a groupoid. The terms of this groupoid tautologically consist of those of $P_X(x, y)$.

It remains to prove the last assertion of the lemma, assuming furthermore that X is a Segal anima. In this case, we observe, using Lemma 11.8.16, that there is a pullback square of the following form.

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[d] & X_1 \arrow[d, "d^0"] \\ X_n \arrow[r, "\mathrm{ev}_n"] & X_0 \end{tikzcd}$$

The latter is the outer square in the commutative diagram

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[d, "{(d^{n+1},\mathrm{ev}_{n+1})}"'] & X_1 \arrow[d, "{(d^0,d^1)}"] \\ X_n \times X_0 \arrow[r, "\mathrm{ev}_n \times \mathrm{id}_{X_0}"'] \arrow[d, "\mathrm{pr}_1"'] & X_0 \times X_0 \arrow[d, "\mathrm{pr}_1"] \\ X_n \arrow[r, "\mathrm{ev}_n"] & X_0 \end{tikzcd}$$

in which the lower square is cartesian. In other words, we have a pullback square of the form

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[d, "(d^{n+1}, \mathrm{ev}_{n+1})"'] & X_1 \arrow[d, "(d^0, d^1)"] \\ X_n \times X_0 \arrow[r, "\mathrm{ev}_n \times \mathrm{id}_{X_0}"'] & X_0 \times X_0. \end{tikzcd}$$

This means that there is a canonical isomorphism of the form

$$P_X(x, y) \simeq P_X(x_n, y)$$

where $x_n\colon X_0$ is the evaluation of $x\colon X_n$ at $n\colon [n]^{\simeq}$. Let $c = ([n], x)$ and $c' = ([n'], x')$ be two objects of $\mathbf{\Delta}_{/X}$. A morphism $u\colon c \to c'$ in $\mathbf{\Delta}_{/X}$, is a morphism $u\colon [n] \to [n']$ in $\mathbf{\Delta}$ together with an identification $s\colon u^*(x') = x$. saying that u belongs to W_X means that $u(n) = n'$, which induces through s an identification $\sigma\colon x_n = x'_{n'}$. Let $c_0 = ([0], x_n)$ and $c'_0 = ([0], x'_{n'})$. We have a commutative square in $\mathbf{\Delta}_{/X}$ of the form

$$\begin{tikzcd} c_0 \arrow[r, equal] \arrow[d] & c'_0 \arrow[d] \\ c \arrow[r, "u"] & c' \end{tikzcd}$$

inducing a commutative diagram of the following form

$$\begin{tikzcd} P_X(x', y) \arrow[r, "\simeq"] \arrow[d, equal] & \Pi_\infty(W(d)_{c'/}) \arrow[r] \arrow[d] & \Pi_\infty(W(d)_{c/}) \arrow[d] & P_X(x, y) \arrow[l, "\simeq"'] \arrow[d, equal] \\ P_X(x'_{n'}, y) \arrow[r, "\simeq"] & \Pi_\infty(W(d)_{c'_0/}) \arrow[r, equal] & \Pi_\infty(W(d)_{c_0/}) & P_X(x_n, y). \arrow[l, "\simeq"'] \end{tikzcd}$$

This achieves the proof of the lemma. □

Exercise 11.8.34. Under the conventions and definitions of Construction 11.8.32, check that the proof of Lemma 11.8.33 shows in fact that the property for X to be a Segal anima is equivalent to the property that, for any $y\colon X_0$,

$$p_d\colon W(d) \to (\mathbf{\Delta}_{/X})_{d/}$$

is a left calculus of fractions at $d = ([0], y)$ in $\mathbf{\Delta}_{/X}$.

Proof Theorem 11.8.31. Let X be a Segal anima. As explained in Construction 11.8.7, we have canonical commutative square

$$\begin{tikzcd} \mathrm{Fun}([1], \mathbf{\Delta}_{/X}) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "q_X"'] & \mathbf{\Delta}_{/X} \arrow[d, "\tau'_X"] \\ \mathbf{\Delta}_{\bullet/X} \arrow[r, "j_X"'] & \mathrm{ac}(X) \end{tikzcd}$$

and Lemma 11.8.8 tells us that τ'_X exhibits $\mathrm{ac}(X)$ as the localization of $\mathbf{\Delta}_{/X}$ by last vertex maps. Given $n\colon \mathbb{N}$, the map

$$(\varepsilon_X)_n\colon X_n \to N(\mathrm{ac}(X))_n$$

sends $\gamma\colon X_n$ to the composition of the functor j_X with the embedding

$$[n] \hookrightarrow \mathbf{\Delta}_{\bullet/X}$$

corresponding to the fiber of the cartesian fibration

$$\mathbf{\Delta}_{\bullet/X} \to \mathbf{\Delta}_{/X}$$

at the object $([n], \gamma)$.

An object of $\mathrm{Fun}([1], \mathbf{\Delta}_{/X})$ essentially is a pair (u, γ) with $u\colon [m] \to [n]$ a morphism of $\mathbf{\Delta}$ and $\gamma\colon X_n$. For any $i\colon [n]^{\simeq}$, the projection

$$[n]_{/i} \to [n]$$

is a full embedding and induces an equivalence $[n]_{/i} \xrightarrow{\sim} [i]$. From there, we observe that, since m is a terminal object of $[m]$, the property for u to be a right fibration is equivalent to the property that u induces an equivalence $[m] \xrightarrow{\sim} [n]_{/u(m)}$. Using Remark 9.4.19, this implies that One can decribe $\mathbf{\Delta}_{\bullet/X}$ as the full subcategory of $\mathrm{Fun}([1], \mathbf{\Delta}_{/X})$ whose objects are pairs (u, γ) as above, such that u is a right fibration. Let

$$r\colon \mathbf{\Delta}_{\bullet/X} \to \mathrm{Fun}([1], \mathbf{\Delta}_{/X})$$

defined by this description of $\mathbf{\Delta}_{\bullet/X}$. The functor q_X assigns to each pair (u, γ) in $\mathrm{Fun}([1], \mathbf{\Delta}_{/X})$ the right fibration $[u(m)] = [n]_{/u(m)} \hookrightarrow [n]$ together with the same $\gamma\colon X_n$. The canonical commutative square

$$\begin{tikzcd}
{[m]_{/m}} \arrow[r] \arrow[d, equal] & {[n]_{/u(m)}} \arrow[d, equal] \\
{[m]} \arrow[r] \arrow[d, "u"'] & {[u(m)]} \arrow[d, hook] \\
{[n]} \arrow[r, equal] & {[n]}
\end{tikzcd}$$

define a map $\eta_{(u,\gamma)}\colon (u, \gamma) \to r(q_X(u, \gamma))$ in $\mathrm{Fun}([1], \mathbf{\Delta}_{/X})$. One checks right away that this turns r into a right adjoint section of q_X with $\eta_{(u,\gamma)}$ as unit maps.

The functor $j_X = j_X \circ q_X \circ r$ is isomorphic to the functor $\tau'_X \circ \mathrm{ev}_0 \circ r$.

This implies that the map

$$(\varepsilon_X)_0\colon X_0 \to N(\mathrm{ac}(X))_0 = \mathrm{ac}(X)^{\simeq}$$

is a cover. Indeed, we know that

$$(\mathbf{\Delta}_{\bullet/X})^{\simeq} = \sum_{n\colon \mathbb{N}} X_n \to \mathrm{ac}(X)^{\simeq}$$

is a cover by Corollary 11.5.14. But, for any $\gamma\colon X_n$, since

$$n\colon [0] \hookrightarrow [n]$$

is a last vertex map, it induces an isomorphism from the image of $([0], \gamma(n))$ to the image of $([n], \gamma)$ in $\mathrm{ac}(X)$, which implies that the restricted map $X_0 \to \mathrm{ac}(X)^{\simeq}$ is already a cover.

It remains to prove that the morphism $\varepsilon_X\colon X \to N(\mathrm{ac}(X))$ is fully faithful. Let us consider given $x, y\colon X_0$. We have to prove that the induced map

$$(\varepsilon_X)_{x,y}\colon X(x,y) \to N(\mathrm{ac}(X))(\varepsilon_X(x), \varepsilon_X(y)) = \mathrm{ac}(X)(\varepsilon_X(x), \varepsilon_X(y))$$

is an equivalence. Thanks to the last assertion of Lemma 11.8.33 together with the dual version of Remark 10.3.8, there is a canonical isomorphism of the form

$$\Pi_\infty(W(d)_{c/}) \xrightarrow{\sim} \mathrm{ac}(X)(\varepsilon_X(x), \varepsilon_X(y))$$

with $c = ([0], x)$ and $d = ([0], y)$. Under the obvious identification $P_X(x, y) = X(x, y)$, using that $j_X \simeq \tau'_X \circ \mathrm{ev}_0 \circ r$, we see that the map $(\varepsilon_X)_{x,y}$ coincides with the equivalence of Lemma 11.8.33

$$X(x,y) \xrightarrow{\sim} \Pi_\infty(W(d)_{c/}) = \mathrm{ac}(X)(\varepsilon_X(x), \varepsilon_X(y))\,.$$

In other words, it sends $u\colon X_1$ with $u(0) = x$ and $u(1) = y$ to the image through τ'_X of the morphism $([0], x) \to ([1], u)$ induced by the initial embedding $[0] \hookrightarrow [1]$ (and we can apply this pedestrian description to the universal term of $X(x, y)$). This achieves the proof. □

Corollary 11.8.35. *As simplicial groupoid X is a complete Segal anima if and only if the co-unit map $\varepsilon_X\colon X \to N(\mathrm{ac}(X)$ is invertible.*

Proof. This is a necessary condition since the Rezk nerve of any small category is a complete Segal anima. Conversely, if X is a complete Segal anima, then by Theorem 11.8.31, the map $\varepsilon_X\colon X \to N(\mathrm{ac}(X)$ is a Dwyer–Kan equivalence between complete Segal animae. Corollary 11.8.25 thus ends the proof. □

Corollary 11.8.36. *A morphism of Segal animae $f\colon C \to D$ is a Dwyer–Kan equivalence if and only if $\mathrm{ac}(f)\colon\ \mathrm{ac}(C) \to \mathrm{ac}(D)$ is an equivalence.*

Proof. The Rezk nerve functor being fully faithful, it is conservative. Therefore, we might as well prove that Dwyer–Kan equivalence are exactly those morphisms sent by the functor $N \circ \mathrm{ac}$ to equivalences. For any morphism of Segal animae $f\colon C \to D$ we have a commutative square

$$\begin{tikzcd} C \arrow[r, "\varepsilon_C"] \arrow[d, "f"'] & N(\mathrm{ac}(C)) \arrow[d, "N(\mathrm{ac}(f))"] \\ D \arrow[r, "\varepsilon_D"] & N(\mathrm{ac}(D)) \end{tikzcd}$$

in which the two horizontal maps are Dwyer–Kan equivalences by Theorem 11.8.31. Since the right vertical map is a morphism of complete Segal animae, we see that this corollary follows immediately from the combination of Corollary 11.8.25 and Lemma 11.8.30. □

Corollary 11.8.37. *The functor* ac *exhibits the category of small categories as the localization of the category of small Segal animae by Dwyer–Kan equivalences.*

Proof. Let SegCat be the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$ whose objects are Segal animae. We obtain an adjunction of the form

$$\mathrm{ac}\colon \mathrm{SegCat} \rightleftarrows \mathrm{Cat}\colon N$$

with N fully faithful by Corollary 11.8.10. This corollary is now a direct consequence of Corollary 11.8.36 and of Proposition 5.5.10. □

Corollary 11.8.38. *Let C be a small category. Then $\Pi_\infty(C)$ is the colimit of $N(C)\colon \Delta^{\mathrm{op}} \to \mathrm{Grpd}$.*

Proof. Identifying the universe Cat with the category of complete Segal animae, this follows immediately from the fact that composing the full embedding

$$\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$$

with the nerve functor yields the functor sending each groupoid X to the constant functor indexed by Δ^{op} with value X. □

12 Filtered colimits of animae

12.1 The poset of natural numbers

In the previous section, we defined the *set* of natural numbers. The goal of this section is to equip $\mathbb{N}$ with the structure of a poset: we will define a category ω with objects given by the natural numbers, and with morphisms given by maps $n \to m$ whenever $n \leq m$.

Rather than directly defining ω, we will first define what it means to take "limits and colimits over ω" by writing down an explicit formula. This will require us to first discuss some preliminaries on sequences in a category C.

Definition 12.1.1. Let C be a category. We define the category $\mathrm{Seq}_\omega(C)$ of *ω-indexed sequences in C* as the following pullback:

$$\begin{tikzcd}
\mathrm{Seq}_\omega(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{n\in\mathbb{N}} \mathrm{Fun}([1], C) \arrow[d, "p"] \\
\prod_{n\in\mathbb{N}} C \arrow[r, "\Delta"] & \prod_{n\in\mathbb{N}}(C \times C),
\end{tikzcd}$$

where the functor p is defined by $p((f_n)_{n\in\mathbb{N}})_i := (t(f_i), s(f_{i+1}))$. Dually, we define the category $\mathrm{Seq}_{\omega^{\mathrm{op}}}(C)$ of *ω^{op}-indexed sequences in C* as

$$\mathrm{Seq}_{\omega^{\mathrm{op}}}(C) := \mathrm{Seq}_\omega(C^{\mathrm{op}})^{\mathrm{op}}.$$

Explicitly, an object of $\mathrm{Seq}_\omega(C)$ consists of a collection of objects $(X_n)_{n\in\mathbb{N}}$ of C together with a sequence of morphisms $f_n \colon X_n \to X_{n+1}$ between them:

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

We will denote such an object by $(X_n, f_n)_{n\in\mathbb{N}}$. Dually, an object of $\mathrm{Seq}_{\omega^{\mathrm{op}}}(C)$ consists of a collection of objects $(X_n)_{n\in\mathbb{N}}$ with morphisms $f_n \colon X_{n+1} \to X_n$ going in the other direction.

Definition 12.1.2 (Milnor (co)limit)**.** Let $(X_n, f_n)_{n\in\mathbb{N}^{\mathrm{op}}}$ be an ω^{op}-indexed sequence in a category C with pullbacks as well as countable products (i.e. $\mathbb{N}$-indexed limits). We define the *Milnor limit* "$\lim_n$" X_n of the sequence above as the pullback

$$\begin{tikzcd}
\text{“}\lim_n\text{''}\, X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_n X_n \arrow[d, "q"] \\
\prod_n X_n \arrow[r, "\Delta"] & \prod_n (X_n \times X_n),
\end{tikzcd}$$

where for $i \in \mathbb{N}$ the i-th component of the map q is given by the map $(\mathrm{pr}_i, f_i \circ \mathrm{pr}_{i+1})\colon \prod_n X_n \to X_i \times X_i$.

Dually, if $(X_n, f_n)_{n\in\mathbb{N}}$ is an ω-indexed sequence in a category C with colimits, we define its *Milnor colimit* "colim_n" X_n as the Milnor limit of the associated sequence in C^{op}.

The Milnor limit "$\lim_n$" X_n comes equipped with projection maps $\mathrm{pr}_i\colon$ "$\lim_n$" $X_n \to X_i$ for every $i \in \mathbb{N}$ that satisfy the condition that the following triangles commute:

$$\begin{tikzcd} & X_{i+1} \arrow[dr, "f_i"] & \\ \text{“}\lim_n\text{”}\ X_n \arrow[ur, "\mathrm{pr}_{i+1}"] \arrow[rr, "\mathrm{pr}_i"'] & & X_i . \end{tikzcd}$$

Moreover, it is *universal* with this property: for any other object T equipped with maps $t_i\colon T \to X_i$ satisfying $f_i \circ t_{i+1} \cong t_i$ we obtain a unique map $t\colon T \to$ "$\lim_n$" X_n such that $t_i \cong \mathrm{pr}_i \circ t$.

The proof of the following lemma is obvious.

Lemma 12.1.3. *If $F\colon C \to D$ is a functor between categories with pullbacks and countable products, and if F commutes with pullbacks and countable products, then for any object X of $\mathrm{Seq}_{\omega^{\mathrm{op}}}(C)$, there is a canonical equivalence of the form*

$$F(\text{“}\lim_n\text{”}\ X_n) \xrightarrow{\sim} \text{“}\lim_n\text{”}\ F(X_n)\,.$$

Construction 12.1.4. Let C be a category. We define for every $n \in \mathbb{N}$ a restriction functor

$$\mathrm{res}_n\colon\ \mathrm{Seq}_\omega(C) \to \mathrm{Fun}([n], C)$$

which informally sends an ω-indexed sequence $(X_i, f_i)_{i\in\mathbb{N}}$ to the diagram $X_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} X_n$. We define this functor by induction. For $n = 0$, we define res_0 to be the composite

$$\mathrm{Seq}_\omega(C) \to \prod_{n\in\mathbb{N}} \mathrm{Fun}([1], C) \xrightarrow{\mathrm{pr}_0} \mathrm{Fun}([1], C) \xrightarrow{s} C.$$

For $n \in \mathbb{N}$ we define res_{n+1} as the unique map determined by the following commutative diagram:

$$\begin{tikzcd} \mathrm{Seq}_\omega(C) \arrow[d] \arrow[dr, dashed, "\mathrm{res}_{n+1}"] \arrow[drr, bend left, "\mathrm{res}_n"] & & \\ \prod_i \mathrm{Fun}([1], C) \arrow[dr, "\mathrm{pr}_n"'] & \mathrm{Fun}([n+1], C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([n], C) \arrow[d, "\mathrm{ev}_n"] \\ & \mathrm{Fun}([1], C) \arrow[r, "s"] & C. \end{tikzcd}$$

In light of the top commutative triangles, these functors determine a comparison functor

$$\mathrm{res}\colon\ \mathrm{Seq}_\omega(C) \to \text{“}\lim_n\text{”}\ \ \mathrm{Fun}([n], C),$$

where the right-hand side denotes the Milnor limit in Cat of the ω^{op}-indexed sequence in Cat given by the categories $\mathrm{Fun}([n], C)$ and the restriction functors $\mathrm{Fun}([n+1], C) \to \mathrm{Fun}([n], C)$ along the inclusions $[n] \hookrightarrow [n+1]$.

Proposition 12.1.5. *For a category C, the comparison functor*

$$\mathrm{Seq}_\omega(C) \to \text{“}\lim_n\text{”}\ \mathrm{Fun}([n],C)$$

is an equivalence.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Seq}_\omega(C) \arrow[r] \arrow[d] & \prod_n \mathrm{Fun}([n],C) \arrow[r] \arrow[d] & \prod_n \mathrm{Fun}([1],C) \arrow[d] \\
\prod_n \mathrm{Fun}([n],C) \arrow[r, "\Delta"] \arrow[d, "\prod_n s"] & \prod_n(\mathrm{Fun}([n],C)\times \mathrm{Fun}([n],C)) \arrow[r] & \prod_n(\mathrm{Fun}([n],C)\times C) \arrow[d, "\prod_n \mathrm{id}\times s"] \\
\prod_n C \arrow[rr, "\Delta"'] & & \prod_n(C\times C).
\end{tikzcd}$$

Since the outer square, bottom rectangle and upper right square are pullback squares, it follows from the pasting law of pullback squares that also the upper left square is a pullback square. Since this is precisely the diagram inducing the map $\mathrm{Seq}_\omega(C) \to \text{“}\lim_n\text{”}\ \mathrm{Fun}([n],C)$, the claim follows. □

Definition 12.1.6 (Poset of natural numbers)**.** We define the poset ω as the Milnor colimit

$$\omega := \text{“}\mathrm{colim}_n\text{”}\ [n] : \mathrm{Cat}^{\simeq}$$

of the simplices $[n]$ along their inclusion maps $[n] \hookrightarrow [n] \star [0] = [n+1]$.

Proposition 12.1.7. *For a category C, there is a preferred equivalence*

$$\mathrm{Seq}_\omega(C) \xrightarrow{\sim} \mathrm{Fun}(\omega, C)\,.$$

Proof. Since $\mathrm{Fun}(-,C)\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{Cat}$ preserves limits, this proposition follows immediately from Lemma 12.1.3 and Proposition 12.1.5. □

Proposition 12.1.8. *For a functor $X_\bullet\colon \omega \to C$, regarded via Proposition 12.1.7 as an ω-indexed sequence in C, there is a preferred equivalence*

$$\text{“}\mathrm{colim}_n\text{”}\ X_n \xrightarrow{\sim} \mathrm{colim}_{n\colon \omega} X_n.$$

Similarly we have $\text{“}\mathrm{colim}_n\text{”}\ X_n \xrightarrow{\sim} \lim_{n\colon \omega^{\mathrm{op}}} X_n$ *for every ω^{op}-indexed sequence in C.*

Proof. By definition of ω, this is a straightforward combination of three suitable applications of Corollary 9.8.9 put together. □

Lemma 12.1.9. *For an object $X \in C$, the canonical maps $X \to \mathrm{colim}_{n\colon \omega} X$ and $X \to \lim_{n\colon \omega^{\mathrm{op}}} X$ are equivalences.*

Proof. We prove the case for limits; the case for colimits is dual. By the previous proposition it suffices to show that the map $X \to \text{“}\lim_n\text{”}\ X$ is an equivalence. We claim that an inverse is given by the map $\text{“}\lim_n\text{”}\ X \to X$ given by the zeroth projection map. It is clear that the composite $X \to \text{“}\lim_n\text{”}\ X \xrightarrow{\mathrm{pr}_0} X$ is the identity on X, so it suffices to show that the composite

$$\text{“}\lim_n\text{”}\ X \xrightarrow{\mathrm{pr}_0} X \to \text{“}\lim_n\text{”}\ X$$

is the identity as well. In light of the universal property of the pullback square defining “$\lim''_n X$”, it suffices to show that we get the correct map after composing with the projection map “$\lim''_n X \to \prod_n X$ and that the homotopy filling the square matches up:

- We need to show that for every $i \in \mathbb{N}$ the maps “$\lim''_n X \to X$ given by $(x_n)_{n\colon \mathbb{N}} \mapsto x_0$ and $(x_n)_{n\colon \mathbb{N}} \mapsto x_i$ are naturally isomorphic. We do this by induction, the case $i = 0$ being clear. For $i \geq 0$, we may use the natural isomorphism $x_i \cong x_{i+1}$ provided as part of the data of an object in “$\lim''_n X$. Composing this with the natural isomorphism $x_0 \cong x_i$ we have already constructed inductively gives the desired isomorphism $x_0 \cong x_{i+1}$.
- We need to show that for every $i \in \mathbb{N}$ the isomorphisms constructed above fit into a commutative square

$$\begin{tikzcd} x_0 \arrow[r, "\cong"] \arrow[d, equal] & x_i \arrow[d, "\cong"] \\ x_0 \arrow[r, "\cong"] & x_{i+1}. \end{tikzcd}$$

  But this holds true by construction.

□

Corollary 12.1.10. *The category ω is weakly contractible.*

Proof. We need to show that for every groupoid X the canonical map $X \to \mathrm{Map}(\omega, X)$ is an equivalence. By the previous lemma we have

$$\mathrm{Map}(\omega, X) \simeq \lim_{n\colon \omega^{\mathrm{op}}} \mathrm{Map}([n], X) \simeq \lim_{n\colon \omega^{\mathrm{op}}} X,$$

using that X is a groupoid. The claim then follows from the previous lemma. □

Lemma 12.1.11. *The structural functors $i_n\colon [n] \to \omega$ induce a cover*

$$i\colon \mathbb{N} \to \omega^{\simeq}$$

defined by the rule $i(n) = i_n(n)$.

Proof. Let $j\colon \mathbb{N} \to \omega$ be the composition of i with the embedding $\omega^{\simeq} \hookrightarrow \omega$. There is an obvious conservative functor

$$\mathrm{Seq}_\omega(C) \to \prod_{n\colon \mathbb{N}} C$$

defined by sending a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

to the $\mathbb{N}$-indexed family $(X_n)_{n\colon \mathbb{N}}$. Under the equivalence of Proposition 12.1.7, this functor corresponds to the functor

$$j^*\colon \mathrm{Fun}(\omega, C) \to \mathrm{Fun}(\mathbb{N}, C) = \prod_{n\colon \mathbb{N}} C\,,$$

proving that the latter is conservative. Applying Proposition 11.5.13 with $A = \mathbb{N}$ and $B = \omega$, this shows that i is a cover. □

Construction 12.1.12. There is a functor

$$\mathrm{succ}^*\colon \mathrm{Seq}_\omega(C) \to \mathrm{Seq}_\omega(C)$$

sending a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

to the sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \to X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to \cdots$$

The functor succ^* above has a right adjoint

$$\mathrm{succ}_*\colon \mathrm{Seq}_\omega(C) \to \mathrm{Seq}_\omega(C)$$

defined by sending a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

to the sequence

$$X_0 \xrightarrow{\mathrm{id}_{X_0}} X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

The co-unit of this adjunction obviously is the identity $\mathrm{succ}^* \mathrm{succ}_* = \mathrm{id}$, so that, in particular, the functor succ_* is fully faithful.

Taking $C = \omega$, we define a functor

$$\mathrm{succ}\colon \omega \to \omega$$

as the image of the identity of ω through the functor

$$\mathrm{Fun}(\omega,\omega) \xrightarrow{\sim} \mathrm{Seq}_\omega(\omega) \xrightarrow{\mathrm{succ}^*} \mathrm{Seq}_\omega(\omega) \xrightarrow{\sim} \mathrm{Fun}(\omega,\omega)\,.$$

Lemma 12.1.13. *The functor* $\mathrm{succ}\colon \omega \to \omega$ *induces, for each small category C, a functor*

$$(-)\circ \mathrm{succ}\colon \mathrm{Fun}(\omega, C) \to \mathrm{Fun}(\omega, C)$$

which agrees with the functor

$$\mathrm{succ}^*\colon \mathrm{Seq}_\omega(C) \to \mathrm{Seq}_\omega(C)$$

through the equivalence of Proposition 12.1.7. In particular, for each $n\colon \mathbb{N}$, there is an identification of the form

$$\mathrm{succ}(i(n)) = i(n+1)\,.$$

Furthermore the functor $\mathrm{succ}\colon \omega \to \omega$ *is fully faithful and terminal.*

Proof. Applying the Yoneda lemma under the form of Lemma 9.3.20, we see that there are equivalences

$$\mathrm{Hom}_{\mathrm{Cat}}(\omega, C) \simeq \mathrm{Fun}(\omega, C)^{\simeq} \simeq \mathrm{Seq}_\omega(C)^{\simeq}$$

functorially in C. This implies that composing with $\mathrm{succ}\colon \omega \to \omega$ agrees with the map

$$(\mathrm{succ}^*)^{\simeq}\colon \mathrm{Seq}_\omega(C)^{\simeq} \to \mathrm{Seq}_\omega(C)^{\simeq}$$

functorially in C. On the other hand, for any small category A, there is a canonical equivalence of the form

$$\mathrm{Fun}(A, \mathrm{Fun}(\omega, C)) = \mathrm{Fun}(\omega, \mathrm{Fun}(A, C)) \quad \text{and} \quad \mathrm{Fun}(A, \mathrm{Seq}_\omega(C)) = \mathrm{Seq}_\omega(\mathrm{Fun}(A, C)) .$$

This shows that the agreement between succ^* and $(-)\circ\mathrm{succ}$ holds for all terms of the type $\mathrm{Fun}(\omega, C)$, proving the first assertion of the lemma. The formula

$$\mathrm{succ}(i(n)) = i(n+1)$$

is a straightforward application of the Yoneda lemma as well, and is left as an exercise for the reader. The functor

$$\mathrm{succ}^* \colon \mathrm{Fun}(\omega, \mathrm{Grpd}) \to \mathrm{Fun}(\omega, \mathrm{Grpd})$$

has both a left adjoint $\mathrm{succ}_!$ and a right adjoint succ_*. But the latter, seen through the equivalence of Proposition 12.1.7, is known to be fully faithful. Therefore, the functor succ_* is fully faithful on $\mathrm{Fun}(\omega, \mathrm{Grpd})$. By transposition of the co-unit, this implies that the unit of the adjunction given by $\mathrm{succ}_!$ and succ^* is invertible, hence that $\mathrm{succ}_!$ is fully faithful. This implies that the functor $\mathrm{succ} \colon \omega \to \omega$ is fully faithful.

In order to prove that the functor succ is terminal, we will use the characterization given by Proposition 8.4.4, albeit in the language of Milnor colimits, which makes sense thanks to Proposition 12.1.8. In other words, we consider a sequence of maps of small animae

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

and we want to prove that the map

$$a \colon \text{“colim”}_n\, X_{n+1} \xrightarrow{\sim} \text{“colim”}_n\, X_n$$

induced by the commutative diagram

$$\begin{tikzcd}
 & X_1 \arrow[r, "f_1"] \arrow[d, equal] & \cdots \arrow[r] & X_n \arrow[r, "f_n"] \arrow[d, equal] & X_{n+1} \arrow[r] \arrow[d, equal] & \cdots \\
X_0 \arrow[r, "f_0"] & X_1 \arrow[r, "f_1"] & \cdots \arrow[r] & X_n \arrow[r, "f_n"] & X_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

is an equivalence. The commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "f_0"] & X_1 \arrow[r, "f_1"] \arrow[d, "f_1"] & \cdots \arrow[r] & X_n \arrow[r, "f_n"] \arrow[d, "f_n"] & X_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & \cdots \\
X_1 \arrow[r, "f_1"] & X_2 \arrow[r, "f_2"] & \cdots \arrow[r] & X_{n+1} \arrow[r, "f_{n+1}"] & X_{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

defines a map

$$b \colon \text{“colim”}_n\, X_n \to \text{“colim”}_n\, X_{n+1} .$$

Let

$$k_n \colon X_n \to \text{“colim”}_n\, X_n \quad \text{and} \quad l_n \colon X_{n+1} \to \text{“colim”}_n\, X_{n+1}$$

be the structural maps, for $n \colon \mathbb{N}$. By construction, there are terms

$$c_n \colon k_n = k_{n+1} \circ f_n \quad \text{and} \quad d_n \colon l_n = l_{n+1} \circ f_{n+1}$$

for each $n\colon \mathbb{N}$ and the collections $(k_n, c_n)_{n\colon \mathbb{N}}$ and $(l_n, d_n)_{n\colon \mathbb{N}}$ determine the natural transformations exhibiting "colim_n" X_n and "colim_n" X_{n+1} as the colimits of the sequences $n \mapsto X_n$ and $n \mapsto X_{n+1}$, respectively. By construction of a and b there are terms of the form

$$c'_n\colon k_{n+1} \circ f_n = a \circ b \circ k_n \quad \text{and} \quad d'_n\colon l_{n+1} \circ f_{n+1} = b \circ a \circ l_n$$

for all terms n in $\mathbb{N}$. Composing c_n with c'_n as well as d_n with d'_n, this provides terms of the form

$$c''_n\colon k_n = a \circ b \circ k_n \quad \text{and} \quad d''_n\colon l_n = b \circ a \circ l_n$$

for all terms n of $\mathbb{N}$. We also have commutative squares of the form

$$\begin{tikzcd} k_{n+1} \circ f_n \arrow[r, equal, "c''_{n+1} f_n"] \arrow[d, equal, "c_n"] & a \circ b \circ k_{n+1} \circ f_n \arrow[d, equal, "(a \circ b) c_n"] \\ k_n \arrow[r, equal, "c''_n"] & a \circ b \circ k_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} l_{n+1} \circ f_{n+1} \arrow[r, equal, "d''_{n+1} f_{n+1}"] \arrow[d, equal, "d_n"] & b \circ a \circ l_{n+1} \circ f_{n+1} \arrow[d, equal, "(b \circ a) d_n"] \\ l_n \arrow[r, equal, "d''_n"] & b \circ a \circ l_n \end{tikzcd}$$

exhibiting finally $a \circ b$ and $b \circ a$ as endomorphisms of colimit co-cones, hence as equivalences. □

Lemma 12.1.14. *The colimit of the sequence*

$$[0] \star [0] \hookrightarrow [0] \star [1] \hookrightarrow \cdots \hookrightarrow [0] \star [n] \hookrightarrow [0] \star [n+1] \hookrightarrow \cdots$$

is canonically equivalent to ω.

Proof. Applying Theorem 9.7.4 for $C = \mathrm{Cat}^{\mathrm{op}}$, we see that, by virtue of the last assertion of Lemma 12.1.13, for any functor $F\colon \omega \to \mathrm{Cat}$, the canonical functor

$$\mathrm{colim}_\omega\, \mathrm{succ}^* F \to \mathrm{colim}_\omega F$$

is an equivalence. In the case where F corresponds to the sequence

$$[0] \hookrightarrow [1] \hookrightarrow \cdots \hookrightarrow [n] \hookrightarrow [n+1] \hookrightarrow \cdots$$

this equivalence may be written as

$$\text{“}\mathrm{colim}_n\text{”}\ [n+1] \xrightarrow{\sim} \text{“}\mathrm{colim}_n\text{”}\ [n] = \omega\,.$$

The lemma simply comes from the identification $[n+1] = [0] \star [n]$. □

Proposition 12.1.15. *For any $n\colon \mathbb{N}$, the structural functor $i_n\colon [n] \to \omega$ is a fully faithful right fibration. Furthermore, there is an equivalence*

$$i\colon \mathbb{N} \xrightarrow{\sim} \omega^{\simeq}$$

defined by the rule $i(n) = i_n(n)$. In particular, ω is a poset.

Proof. Each canonical embedding $[n] \hookrightarrow [n] \star [0] = [n+1]$ is a fully faithful right fibration: since $[n+1]$ is a poset, it is easy to see from Lemma 11.2.21 that this corresponds to the functor

$$P_n\colon [n+1]^{\mathrm{op}} \to \mathrm{Grpd}$$

defined by the rule

$$P_n(i) = \begin{cases} [0] & \text{if } i : [n]^{\simeq}, \\ \emptyset & \text{else.} \end{cases}$$

By induction, using that fully faithful right fibrations are stable under composition, we deduce that, for any $k : \mathbb{N}$, the canonical embedding

$$[n] \hookrightarrow [n] \star [k]$$

is a fully faithful right fibration.
On the other hand, for any functor $F : \omega \to \mathrm{Cat}$ and any $k : \mathbb{N}$, the canonical functor

$$\mathrm{colim}_{\omega}(\mathrm{succ}^k)^* F \to \mathrm{colim}_{\omega} F$$

is an equivalence. The case where $k = 0$ is clear and the case where $k = 1$ has been explained in the proof of Lemma 12.1.13. This provides the induction step once we know that $(\mathrm{succ}^{k+1})^* = \mathrm{succ}^* \circ (\mathrm{succ}^k)^*$, using that $\mathrm{succ}^{k+1} = \mathrm{succ}^k \circ \mathrm{succ}$. This means that one can see ω as the sequential colimit

$$\omega = \text{“}\mathrm{colim}_k\text{''}\ [n] \star [k] \,.$$

This is proven by induction: the case $n = 0$ precisely is the previous lemma and the induction step is provided by the cofinality formula above. We can see $[n]$ as the sequential colimit of the constant sequence with value $[n]$, by Corollary 12.1.10. The diagram

$$\begin{tikzcd} {[n]} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[n]} \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & {[n]} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[n]} \arrow[r, equal] \arrow[d] & \cdots \\ {[n] \star [0]} \arrow[r] & {[n] \star [1]} \arrow[r] & \cdots \arrow[r] & {[n] \star [k]} \arrow[r] & {[n] \star [k+1]} \arrow[r] & \cdots \end{tikzcd}$$

is a sequence of pullback squares (because all maps are (full) embeddings) and all vertical maps are right fibrations. By virtue of Corollary 9.9.3, we obtain pullback squares

$$\begin{tikzcd} {[n]} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[n]} \arrow[d] \\ {[n] \star [k]} \arrow[r] & \omega \end{tikzcd}$$

where the lower horizontal maps are part of the co-cone exhibiting ω as colimit of the $[n] \star [k]$'s and the vertical arrow at the right hand side is the functor induced by the sequence of commutative squares above. Corollary 9.9.3 also tells us that the induced functor $[n] \to \omega$ is a right fibration. It must be fully faithful, since its fibers are propositions. This implies that the map $i : \mathbb{N} \to \omega^{\simeq}$ is an embedding: for each $a, b : \mathbb{N}$ we can choose $n = a + b$ so that both a and b are terms of $[n]$. This implies that

$$[n](a, b) \xrightarrow{\sim} \omega(i(a), i(b))$$

is a proposition. In particular, ω is a poset. Since $i : \mathbb{N} \to \omega^{\simeq}$ is also a cover by Lemma 12.1.11, this proves that $i : \mathbb{N} \xrightarrow{\sim} \omega^{\simeq}$. □

Lemma 12.1.16. *The functor* $\mathrm{succ} : \omega \to \omega$ *is the unique functor so that* $\mathrm{succ}^{\simeq} : \omega^{\simeq} \to \omega^{\simeq}$ *agrees with the successor operation on* $\mathbb{N}$ *via the equivalence* $i : \mathbb{N} \xrightarrow{\sim} \omega^{\simeq}$.

Proof. The fact that the successor operations on $\mathbb{N}$ and ω agree holds by construction. The unicity statement comes from Lemma 11.2.21 because ω is a poset. □

Notation 12.1.17. Given $n\colon \mathbb{N}$ we write $\omega_{/n} = \omega_{/i(n)}$ and similarly $\omega_{n/} = \omega_{i(n)/}$.

Lemma 12.1.18. *For any $n\colon \mathbb{N}$, the functor*

$$\mathrm{succ}^n\colon \omega \to \omega$$

is a left fibration and induces an equivalence

$$\omega \xrightarrow{\sim} \omega_{n/}\,.$$

Proof. The case $n = 0$ simply means that 0 defines an initial object in ω: in other words, we have to check that $\omega(i(0), i(n))$ is contractible for all $n\colon \mathbb{N}$. This follows immediately form the fact that 0 is initial in $[n]$ for all n.
Recall from Example 11.2.13 that one can decompose $\mathbb{N} = \{0\} \sqcup \mathbb{N}^{\geq 1}$ so that $\mathrm{succ}\colon \mathbb{N} \to \mathbb{N}$ induces an equivalence $\mathbb{N} \xrightarrow{\sim} \mathbb{N}^{\geq 1}$. One defines $\omega^{\geq 1}$ as the full subcategory of ω with objects those of $\mathbb{N}^{\geq 1}$. We prove by induction that $1 \leq n + 1$ for any $n\colon \mathbb{N}$. In other words, $1 \leq n$ for any $n\colon \mathbb{N}^{\geq 1}$. On the other hand, the type $1 \leq 0$ is empty. Since ω is a poset, the co-slice $\omega_{1/}$ is a full subcategory of ω by Lemma 11.2.21. This readily imply that $\omega^{\geq 1} = \omega_{1/}$ as full subcategories of ω. Since succ induces an equivalence $\mathbb{N} \xrightarrow{\sim} \mathbb{N}^{\geq 1}$, this implies that it induces an equivalence $\omega \xrightarrow{\sim} \omega_{1/}$. The equivalence $\omega \xrightarrow{\sim} \omega_{n/}$ is deduced from the case $n = 1$ by an obvious induction. □

Lemma 12.1.19. *The poset ω has binary coproducts (i.e. the maximum of two natural numbers always exists).*

Proof. It suffices to prove that each $[n]$ has binary coproducts and that each inclusion $[n] \hookrightarrow [n]\star[0]$ commutes with them. This is easily proven by induction on n: for $a, b\colon [n]$ either $a \leq b$ or $b \leq a$ is inhabited and the maximum of a and b is b or a, respectively. □

Proposition 12.1.20. *For any $m, n\colon \mathbb{N}$ with maximum $k = \max(m, n)$, the functor succ^k induces an equivalence*

$$\omega \xrightarrow{\sim} \omega_{m/} \times_\omega \omega_{n/}\,.$$

Proof. There is a canonical equivalence of full subcategories of ω:

$$\omega_{k/} \simeq \omega_{m/} \times_\omega \omega_{n/}\,.$$

We conclude with Lemma 12.1.18. □

12.2 Filtered categories

Definition 12.2.1. A category A is *confluent* if, for any morphism $u\colon x \to y$ in A, the functor $u^*\colon A_{y/} \to A_{x/}$ defined by composition with u is a terminal functor.

Remark 12.2.2. The property for A to be confluent is equivalent to the following one: given any morphisms in A of the form $u\colon x \to y$ and $v\colon x \to z$, the category $A_{y/} \times_{A_{x/}} A_{z/}$ is weakly contractible.

$$\begin{tikzcd} A_{y/} \times_{A_{x/}} A_{z/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & A_{z/} \arrow[d, "v^*"] \\ A_{y/} \arrow[r, "u^*"] & A_{x/} \end{tikzcd}$$

Indeed, an object of $A_{x/}$ is a pair (z, v) with $v\colon x \to z$ a morphism in A and since $v^*\colon A_{z/} \to A_{x/}$ is a left fibration whose domain has an initial object sent to (z, v), there is a canonical equivalence of the form

$$(A_{x/})_{(z,v)/} \xrightarrow{\sim} A_{z/}$$

over $A_{x/}$, so that we have a canonical equivalence

$$(A_{y/})_{(z,v)/} \xrightarrow{\sim} A_{y/} \times_{A_{x/}} A_{z/}\ .$$

Our claim now follows by applying Corollary 8.3.8 with $C = (A_{x/})^{\mathrm{op}}$.

Example 12.2.3. If A is a small category with pushouts, then it is confluent. Indeed, since the functor $\mathrm{Hom}_A(-, a)\colon A^{\mathrm{op}} \to \mathrm{Grpd}$ preserves pullbacks, for any morphisms in A of the form $u\colon x \to y$ and $v\colon x \to z$, if $p = y \sqcup_x z$ denotes the corresponding pushout, then

$$A_{p/} \xrightarrow{\sim} A_{y/} \times_{A_{x/}} A_{z/}$$

has an initial object and is thus weakly contractible.

Proposition 12.2.4. *Let $p\colon X \to A$ be a left fibration. If A is confluent, then so is X.*

Proof. If p is a left fibration, then, for any object x of X, the induced functor

$$X_{x/} \to A_{p(x)/}$$

is an equivalence (this is a morphism of left fibrations over A and it preserves the initial objects). Therefore, for any morphism $u\colon x \to y$ in X, we obtain a commutative square of the form

$$\begin{tikzcd} X_{y/} \arrow[r, "u^*"] \arrow[d, "\simeq"'] & X_{x/} \arrow[d, "\simeq"] \\ A_{p(y)/} \arrow[r, "p(u)^*"] & A_{p(x)/} \end{tikzcd}$$

in which the vertical maps are equivalences. Since $p(u)^*$ is a terminal functor, the same is true about u^*. □

Theorem 12.2.5 (Sattler–Wärn)**.** *A small category A is confluent if and only if the colimit functor*

$$\mathrm{colim}_A\colon\ \mathrm{Fun}(A, \mathrm{Grpd}) \to \mathrm{Grpd}$$

commutes with pullbacks.

Proof. Assume that A is confluent. Under the unstraightening equivalence

$$\mathrm{Un}\colon \operatorname{Fun}(A, \mathrm{Grpd}) \xrightarrow{\sim} \mathrm{LFib}(A)\,,$$

the colimit functor colim_A corresponds to the functor

$$\mathrm{LFib}(A) \to \mathrm{Grpd}$$

defined by assigning to each left fibration $p\colon X \to A$ the anima $\Pi_\infty(X)$. In other words, it suffices to prove that, for any pullback square of small categories of the form

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$

in which all functors are left fibrations, and such that there is a left fibration $p\colon Y \to A$, the induced commutative square

$$\begin{tikzcd} \Pi_\infty(X') \arrow[r, "\Pi_\infty(u)"] \arrow[d, "\Pi_\infty(f')"'] & \Pi_\infty(X) \arrow[d, "\Pi_\infty(f)"] \\ \Pi_\infty(Y') \arrow[r, "\Pi_\infty(v)"] & \Pi_\infty(Y) \end{tikzcd}$$

is a pullback square. In order to prove this, we start by observing that, by virtue of Proposition 12.2.4, Y itself is confluent. For each morphism $w\colon y_0 \to y_1$ in Y, the induced functor

$$w^*\colon Y_{y_1/} \to Y_{y_0/}$$

is thus terminal. Since f is a left fibration, it is in particular proper. Therefore, the pullback squares

$$\begin{tikzcd} X_{y_1/} \arrow[r] \arrow[d] & X_{y_0/} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y_{y_1/} \arrow[r, "w^*"] & Y_{y_0/} \arrow[r] & Y \end{tikzcd}$$

show that w^* induces a terminal functor $X_{y_1/} \to X_{y_0/}$. Since terminal functors are weak homotopy equivalences, the equivalence of properties (i) and (iii) in Quillen's Theorem B (Theorem 9.9.5) implies that we have a canonical equivalence

$$\Pi_\infty(X') \xrightarrow{\sim} \Pi_\infty(Y') \times_{\Pi_\infty(Y)} \Pi_\infty(X)\,,$$

which proves that the functor colim_A preserves pullbacks of animae.

Conversely, let us assume that the functor colim_A preserves pullbacks in Grpd. Observe that, for any pullback square of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ A \arrow[r, "F"] & \mathrm{Grpd} \end{tikzcd}$$

if we see $\Pi_\infty(X)$ as an object of Grpd, we have a canonical equivalence

$$\operatorname{colim}_A F = \Pi_\infty(X)$$

in Grpd. Given two morphisms $u\colon x \to y$ and $v\colon x \to z$ in A, we thus have

$$\operatorname{colim}_{a\colon A}(\operatorname{Hom}_A(y,a) \times_{\operatorname{Hom}_A(x,a)} \operatorname{Hom}_A(z,a)) = \Pi_\infty(A_{y/} \times_{A_{x/}} A_{z/})\,.$$

Similarly we have:

$$\operatorname{colim}_{a\colon A} \operatorname{Hom}_A(y,a) \times_{\operatorname{colim}_{a\colon A} \operatorname{Hom}_A(x,a)} \operatorname{colim}_{a\colon A} \operatorname{Hom}_A(z,a) = \Pi_\infty(A_{y/}) \times_{\Pi_\infty(A_{x/})} \Pi_\infty(A_{z/})$$

and since any category with an initial object is weakly contractible, this yields

$$\operatorname{colim}_{a\colon A} \operatorname{Hom}_A(y,a) \times_{\operatorname{colim}_{a\colon A} \operatorname{Hom}_A(x,a)} \operatorname{colim}_{a\colon A} \operatorname{Hom}_A(z,a) = e$$

where e is the terminal anima. The property that colim_A commutes with pullbacks of animae thus implies that

$$\Pi_\infty(A_{y/} \times_{A_{x/}} A_{z/}) = e\,.$$

As explained in Remark 12.2.2, this proves that A is confluent. □

Definition 12.2.6. A functor is *left exact* if it preserves terminal objects as well as pullback squares. A functor is *right exact* if it preserves initial objects as well as pushout squares.
A functor is *exact* if it is both left exact and right exact.

Definition 12.2.7. A category is *filtered* if it is weakly contractible and confluent. A category A is *cofiltered* if A^{op} is filtered.

Example 12.2.8. If A is confluent, then, for any object x of A, the co-slice $A_{x/}$ is filtered: the projection $A_{x/} \to A$ being a left fibration, we know that $A_{x/}$ is confluent as well, by Proposition 12.2.4, and any category with an initial object is weakly contractible.

Theorem 12.2.9. *A small category A is filtered if and only if the colimit functor*

$$\operatorname{colim}_A\colon \operatorname{Fun}(A, \mathrm{Grpd}) \to \mathrm{Grpd}$$

is left exact.

Proof. The functor $\operatorname{colim}_A\colon \operatorname{Fun}(A, \mathrm{Grpd}) \to \mathrm{Grpd}$ sends the terminal object of $\operatorname{Fun}(A, \mathrm{Grpd})$ to $\Pi_\infty(A)$. Therefore the property that the A-indexed colimit functor preserves terminal objects of animae is equivalent to the property that A is weakly contractible. Theorem 12.2.5 thus ends the proof. □

Definition 12.2.10. A category A is *sifted* if it is weakly contractible and, for any objects $x, y\colon A^{\simeq}$, the category $A_{x/} \times_A A_{y/}$ is weakly contractible.

Proposition 12.2.11. *Let A be a weakly contractible small category. The following conditions are equivalent.*

(i) The category A is sifted.

(ii) For any object $x\colon A^{\simeq}$, the projection

$$A_{x/} \to A$$

is a terminal functor.

(iii) The diagonal functor $\Delta\colon A \to A \times A$ is a terminal functor.

(iv) For any functors $X, Y\colon A \to \mathrm{Grpd}$, the canonical map

$$\operatorname*{colim}_{a\colon A} X_a \times Y_a \xrightarrow{\sim} \operatorname*{colim}_{a\colon A} X_a \times \operatorname*{colim}_{a\colon A} Y_a$$

is an equivalence.

(v) For any objects $x, y, z\colon A^{\simeq}$, the category $A_{x/} \times_A A_{y/} \times_A A_{z/}$ is weakly contractible.

(vi) For any objects $x, y\colon A^{\simeq}$, the projection

$$A_{x/} \times_A A_{y/} \to A$$

is a terminal functor.

Proof. Quillen's Theorem A (more precisely, the dual version of Corollary 8.3.8) implies that conditions (i) and (ii) are equivalent. Similarly, it shows that conditions (v) and (vi) are equivalent.

Given any object (x, y) in $A \times A$ we have

$$A \times_{A \times A} (A \times A)_{(x,y)/} = A_{x/} \times_A A_{y/}\,.$$

Quillen's Theorem A thus proves that conditions (i) and (iii) are equivalent.

Since A is weakly contractible, it is clear that condition (i) follows from condition (vi).

By Theorem 9.7.4 for $C = \mathrm{Grpd}^{\mathrm{op}}$, we see that condition (iii) implies that, for any functors $X, Y\colon A \to \mathrm{Grpd}$, we have:

$$\operatorname*{colim}_{a\colon A} X_a \times Y_a \xrightarrow{\sim} \operatorname*{colim}_{(a,b)\colon A \times A} X_a \times Y_b\,.$$

There is furthermore a canonical equivalence

$$\operatorname*{colim}_{(a,b)\colon A \times A} F(a, b) \xrightarrow{\sim} \operatorname*{colim}_{a\colon A} \operatorname*{colim}_{b\colon A} F(a, b)$$

for any functor $F\colon A \times A \to \mathrm{Grpd}$. Since the functor $T \times (-)$ commutes with small colimits in Grpd for any small groupoid T, we also have

$$\operatorname*{colim}_{a\colon A} \operatorname*{colim}_{b\colon A} X_a \times Y_b \xrightarrow{\sim} \operatorname*{colim}_{a\colon A} X_a \times \operatorname*{colim}_{b\colon A} Y_b\,.$$

Putting all this together shows that condition (iv) follows from condition (iii).

It suffices now to prove that condition (iv) implies condition (v).

Condition (iv) means that the functor

$$\operatorname{colim}_A\colon \operatorname{Fun}(A, \mathrm{Grpd}) \to \mathrm{Grpd}$$

commutes with binary cartesian products. This implies that it commutes with ternary cartesian products. For $x, y, z\colon A^{\simeq}$, we have

$$\operatorname*{colim}_{a\colon A} \operatorname{Hom}_A(x, a) \times \operatorname{Hom}_A(y, a) \times \operatorname{Hom}_A(z, a) = \Pi_\infty(A_{x/} \times_A A_{y/} \times_A A_{z/})\,.$$

We also have

$$\operatorname*{colim}_{a\colon A} \operatorname{Hom}_A(t, a) = \Pi_\infty(A_{t/}) = *$$

for any $t\colon A^{\simeq}$. Therefore,

$$\operatorname*{colim}_{a\colon A} \operatorname{Hom}_A(x, a) \times \operatorname*{colim}_{a\colon A} \operatorname{Hom}_A(y, a) \times \operatorname*{colim}_{a\colon A} \operatorname{Hom}_A(z, a) = *$$

and thus condition (iv) implies that $A_{x/} \times_A A_{y/} \times_A A_{z/}$ is weakly contractible. □

Example 12.2.12. The category $\mathbf{\Delta}^{\mathrm{op}}$ is sifted, by Theorem 11.3.8.

Corollary 12.2.13. *A category is filtered if and only if it is sifted and confluent.*

Proof. Any left exact functor preserves terminal objects as well as binary cartesian products. Theorem 12.2.5 and Proposition 12.2.11 thus imply that any filtered category is sifted. The converse implication holds for free, since any sifted category is weakly contractible, by definition. □

Proposition 12.2.14. *Let A be a category. Then A is sifted if and only if the following two conditions hold true.*

(i) The map $A^{\simeq} \to [0]$ is a cover.

(ii) For any object $x\colon A^{\simeq}$, the projection $A_{x/} \to A$ is terminal.

Proof. We observe that the canonical functor $A^{\simeq} \to \Pi_\infty(A)$ is always a cover by Corollary 11.5.14. Since any equivalence is a cover, Proposition 12.2.11 shows that these are necessary conditions. Conversely, let us assume that conditions (i) and (ii) above hold. In order to prove that A is sifted, it suffices to prove that A is weakly contractible. For any groupoid Γ, the localization of A by all its maps in context Γ simply is $\Pi_\infty(A) \times \Gamma$ (with structural map to Γ the second projection). But since $A^{\simeq} \to [0]$ is a cover, the functor $(-) \times A^{\simeq}$ is conservative. Therefore, it suffices to prove that A is weakly contractible in context $A^{\simeq}$. In other words, without any loss of generality, it suffices to prove the proposition under the extra assumption that there is a given object $x\colon A^{\simeq}$. The co-slice $A_{x/}$ being weakly contractible, since it has an initial object, and terminal functors being weak equivalences, we observe that A must be weakly contractible. □

Proposition 12.2.15. *A poset P is filtered if and only if it is sifted.*

Proof. Corollary 12.2.13 shows that this is a necessary condition. For the converse, assuming P to be sifted, in order to prove that it is filtered, it suffices to check that it is confluent. Let us consider $x, y, z\colon P^{\simeq}$ such that $x \leq y$ and $x \leq z$. Since the projection $P_{x/} \to P$ is an embedding by Lemma 11.2.21, the canonical map

$$P_{y/} \times_{P_{x/}} P_{z/} \to P_{y/} \times_P P_{z/}$$

is an equivalence. This proves that P is confluent whenever it is sifted, by Remark 12.2.2. □

Proposition 12.2.16. *The poset ω is filtered.*

Proof. This follows from Corollary 12.1.10, Proposition 12.1.20 and Proposition 12.2.15. □

Corollary 12.2.17. *The formation of sequential colimits is compatible with terminal objects and pullbacks of animae. In other words, the functor*

$$\operatorname{colim}_\omega\colon\ \operatorname{Fun}(\omega, \mathrm{Grpd}) \to \mathrm{Grpd}$$

is left exact.

Proof. This follows from Theorem 12.2.9 and from the previous proposition. □

Proposition 12.2.18. *Let $u\colon A \to B$ be a terminal functor with A filtered (sifted, respectively). Then B is filtered (sifted, respectively).*

Proof. The functor

$$u^*\colon\ \operatorname{Fun}(B, \mathrm{Grpd}) \to \operatorname{Fun}(A, \mathrm{Grpd})$$

commutes with small limits and, for any functor $F\colon B \to \mathrm{Grpd}$ there is a functorial isomorphism

$$\operatorname{colim}_A u^*(F) \xrightarrow{\sim} \operatorname{colim}_B F\,.$$

Therefore, any exactness property of colim_A induces a similar property for colim_B. □

Proposition 12.2.19. *Let $p\colon A \to B$ be a functor between small categories. We assume that the following conditions hold:*

(i) p is proper (e.g. a cocartesian fibration);

(ii) B is filtered (sifted, respectively);

(iii) for any $b\colon B^{\simeq}$, the corresponding fiber A_b is filtered (sifted, respectively).

Then A is filtered (sifted, respectively).

Proof. By virtue of Lemma 8.4.5, for any functor $F\colon A \to \mathrm{Grpd}$, condition (i) implies that we have canonical isomorphisms

$$\operatorname{colim}_{a\colon A} F(a) \simeq \operatorname{colim}_{b\colon B}\ \operatorname{colim}_{x\colon A_b} F(x)\,.$$

Since the operation of composing with each embedding $A_b \hookrightarrow A$ commutes with small limits, conditions (ii) and (iii) thus prove through Theorem 12.2.9 (through Proposition 12.2.11, resp.) that the formation of A-indexed colimits defines a left exact functor (a functor that preserves finite products, resp.). Therefore, a new application of Theorem 12.2.9 (of Proposition 12.2.11, resp.) implies that A is filtered (sifted, resp.). □

Corollary 12.2.20. *Let B be a small filtered (sifted) category and $A\colon B \to \mathrm{Cat}$ a functor such that A_b is filtered (sifted, resp.) for any $b\colon B^{\simeq}$. Then $\operatorname{colim}_B A$ is filtered (sifted, resp.).*

Proof. We know that $\operatorname{colim}_B A$ is a localization of $\int_B A$. Since any localization is a terminal functor, it suffices to check that $\int_B A$ is filtered (sifted), by Proposition 12.2.18. This follows by applying Proposition 12.2.19 with p the projection of $\int_B A$ to B. □

12.3 Filtered colimits of categories

We fix a universe Cat.

Definition 12.3.1. A functor is said to *commute with small filtered colimits* if it commutes with colimits of type I for any filtered category $I \colon \mathrm{Cat}^{\simeq}$.

A full subcategory C of a category D is said to be stable under small filtered colimits if the embedding $C \hookrightarrow D$ commutes with filtered colimits.

Example 12.3.2. The functor

$$\lim_{\lrcorner} \colon \mathrm{Fun}(\lrcorner, \mathrm{Grpd}) \to \mathrm{Grpd}\,, \quad (X \to Z \leftarrow Y) \mapsto X \times_Z Y$$

commutes with small filtered colimits. Indeed, if I is a small filtered category, for any functor

$$F \colon I \times \lrcorner \to \mathrm{Grpd}\,,$$

there is a Beck-Chevalley comparison map

$$\mathrm{colim}_I \lim_{\lrcorner} F \to \lim_{\lrcorner} \mathrm{colim}_I F$$

that turns out to be invertible because colim_I is a left exact functor. This literally means that the functor $\lim_{\lrcorner}$ commutes with I-indexed colimits.

Lemma 12.3.3. *The full subcategory of synthetic Segal animae is stable under small filtered colimits in* $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$.

Proof. For each $n \colon \mathbb{N}$, the functor

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Grpd}\,, \quad X \mapsto X_n = \mathrm{Hom}(\Delta^n, X)$$

commutes with colimits (being a left adjoint) hence with small filtered ones. This implies that the functor

$$X \mapsto \mathrm{Hom}(\mathrm{Seg}^n, X) = \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$$

commutes with small filtered colimits: by an obvious induction on n, this amounts to the property that the formation of pullbacks commutes with small filtered colimits in Grpd (Example 12.3.2). Given a small filtered category I and a functor

$$I \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})\,, \quad i \mapsto X_i\,,$$

and assuming that each X_i is a synthetic Segal anima, we obtain for each $n \colon \mathbb{N}$ a commutative square

$$\begin{tikzcd}
\mathrm{colim}_{i \colon I} \mathrm{Hom}(\Delta^n, X_i) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathrm{colim}_{i \colon I} \mathrm{Hom}(\mathrm{Seg}^n, X_i) \arrow[d, "\simeq"] \\
\mathrm{Hom}(\Delta^n, \mathrm{colim}_{i \colon I} X_i) \arrow[r] & \mathrm{Hom}(\mathrm{Seg}^n, \mathrm{colim}_{i \colon I} X_i)
\end{tikzcd}$$

in which the vertical maps are isomorphisms because of the arguments above, whereas the upper horizontal one is an isomorphism as a colimit of isomorphisms. This proves that $\mathrm{colim}_{i \colon I} X_i$ is a synthetic Segal anima. □

Lemma 12.3.4. *The full subcategory of complete synthetic Segal animae is stable under small filtered colimits in* $\mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$.

Proof. The diagram of pullback squares

$$\begin{tikzcd} \mathrm{Iso}(X) \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner" very near start] & X_2 \times_{X_1} X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X_1 \arrow[d] \\ & X_2 \times X_2 \arrow[r] \arrow[d] & X_1 \times X_1 \\ X_0 \times X_0 \arrow[r] & X_1 \times X_1 & \end{tikzcd}$$

shows that the assignment $X \mapsto \mathrm{Iso}(X)$ may be seen as a functor from $\mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$ to Grpd that commutes with small filtered colimits. The map of type $X_0 \to \mathrm{Iso}(X)$ sending each $x \colon X_0$ to (x, x, id_x) is functorial and the full subcategory of those X's such that $X \xrightarrow{\sim} \mathrm{Iso}(X)$ through this map is clearly stable under small filtered colimits. Therefore, we conclude the proof with Lemma 12.3.3. $\square$

Proposition 12.3.5. *The Rezk nerve functor* $N \colon \mathrm{Cat} \to \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$ *commutes with small filtered colimits.*

Proof. Let cSegCat be the full subcategory of $\mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$ whose objects are the complete synthetic Segal animae. Corollary 11.8.10 and Corollary 11.8.35 imply that the nerve functor induces an equivalence

$$N \colon \mathrm{Cat} \xrightarrow{\sim} \mathrm{cSegCat}\,.$$

On the other hand, Lemma 12.3.4 implies that the embedding

$$\mathrm{cSegCat} \hookrightarrow \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$$

commutes with small filtered colimits. $\square$

Definition 12.3.6. Let C be a locally small category with small filtered colimits. An object X of C is *compact* if the functor

$$C \to \mathrm{Grpd}\,, \quad Y \mapsto \mathrm{Hom}_C(X, Y)$$

commutes with small filtered colimits.

Example 12.3.7. Let A be a small category. For any $a \colon A^{\simeq}$, the associated representable presheaf $h_A(a) = \mathrm{Hom}_A(-, a)$ is compact in $\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})$: the functor

$$Y \mapsto \mathrm{Hom}(h_A(a), Y) \simeq Y(a)$$

commutes with small colimits (being a right adjoint), hence with filtered ones in particular.

Proposition 12.3.8. *If C is a locally small category with small filtered colimits and pushouts, then the full subcategory of compact objects is stable under pushouts.*

Proof. For any object y of C, the functor $\mathrm{Hom}_C(-, y)$ will turn pushouts in C into pullbacks in Grpd. This proposition is thus a direct consequence of the fact that filtered colimits are compatible with pullbacks in Grpd. $\square$

Theorem 12.3.9. *For any $n\colon \mathbb{N}$, the category $[n]$ is a compact object of* Cat*.*

Proof. For any $n\colon \mathbb{N}$, we have $\Delta^n = N([n])$ that is representable, hence compact in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$ (Example 12.3.7). Let I be a small filtered category and

$$X\colon I \to \mathrm{Cat}$$

a functor. Since the Rezk nerve functor is fully faithful (Corollary 11.8.10) and commutes with small filtered colimits (Proposition 12.3.5), we obtain the following computation

$$\begin{aligned}
\operatorname*{colim}_{i\colon I} \mathrm{Hom}_{\mathrm{Cat}}([n], X_i) &\simeq \operatorname*{colim}_{i\colon I} \mathrm{Hom}(\Delta^n, N(X_i)) \\
&\simeq \mathrm{Hom}(\Delta^n, \operatorname*{colim}_{i\colon I} N(X_i)) \\
&\simeq \mathrm{Hom}(\Delta^n, N(\operatorname*{colim}_{i\colon I} X_i)) \\
&\simeq \mathrm{Hom}_{\mathrm{Cat}}([n], \operatorname*{colim}_{i\colon I} X_i)
\end{aligned}$$

thus proving that $[n]$ is compact in Cat. □

Corollary 12.3.10. *The functor*

$$\mathrm{Cat} \to \mathrm{Grpd}\,, \quad C \mapsto C^{\simeq}$$

commutes with small filtered colimits.

Proof. The preceding theorem tells us that $[0]$ is compact and $C^{\simeq} = \mathrm{Hom}([0], C)$. □

Corollary 12.3.11. *Let A be a small filtered category, and $C, D\colon A \to$* Cat *two functors. We assume given a natural transformation $u_a\colon C_a \to D_a$, $a\colon A$, such that u_a is an embedding (a full embedding) for all $a\colon A^{\simeq}$. Then the induced functor*

$$\mathrm{colim}_A\ C \to \mathrm{colim}_A\ D$$

is an embedding (a full embedding, respectively).

Proof. Any left exact functor preserves pullback squares of the form

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

hence preserves embeddings. For the case of full embeddings, we use the characterization of fully faithful functors provided by Remark 6.1.2, together with Theorem 12.3.9 for $n = 1$ and $n = 0$. □

Theorem 12.3.12. *For any small filtered category A, the functor*

$$\mathrm{colim}_A\colon \mathrm{Fun}(A, \mathrm{Cat}) \to \mathrm{Cat}$$

is left exact.

Proof. The nerve functor is fully faithful (Corollary 11.8.10), commutes with small limits (being a right adjoint) and commutes with small filtered colimits (Proposition 12.3.5). Therefore, it suffices to prove that taking A-indexed colimits in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd})$ is a left exact functor. We know that the functor

$$\mathrm{colim}_A \colon \mathrm{Fun}(A, \mathrm{Grpd}) \to \mathrm{Grpd}$$

is left exact, by Theorem 12.2.9. This implies that, for any small category C, the functor

$$\mathrm{colim}_A \colon \mathrm{Fun}(A, \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

is left exact (since both limits and colimits are computed objectwise in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$). This applies in particular for $C = \mathbf{\Delta}$ and proves the theorem. $\square$

Remark 12.3.13 (Morphisms in a filtered colimit of categories)**.** Let A be a small filtered category and $C \colon A \to \mathrm{Cat}$ a functor. One can understand the colimit of C as follows. Objects are easy to understand by Corollary 12.3.10:

$$\operatorname*{colim}_{a\colon A} C_a^{\simeq} = \Big(\operatorname*{colim}_{a\colon A} C_a\Big)^{\simeq} .$$

Assume given $a_0, a_1 \colon A^{\simeq}$ and two objects $x_{a_0} \colon C_{a_0}^{\simeq}$ and $y_{a_1} \colon C_{a_1}^{\simeq}$. We will write x and y for the object of $\mathrm{colim}_A\, C$ obtained as the images of x_{a_0} and y_{a_1} through the canonical functors

$$C_{a_0} \to \mathrm{colim}_A\, C \quad \text{and} \quad C_{a_1} \to \mathrm{colim}_A\, C$$

respectively.

For any map $u \colon a_0 \to a$ in A, we will write $x_a \colon C_a^{\simeq}$ for the image of x_{a_0} through the functor induced by u. Similarly, for any map $v \colon a_1 \to a$ in A, we will write $y_a \colon C_a^{\simeq}$ for the image of y_{a_1} through the functor induced by v.

We observe that the projection $A_{a_0/} \times_A A_{a_1/} \to A$ is terminal (since A is sifted, using Proposition 12.2.11). The terms of $A_{a_0/} \times_A A_{a_1/}$ literally are of the form (a, u, v) with $a \colon A$ together with $u \colon a_0 \to a$ and $v \colon a_1 \to a$ in A. In other words, we have an equivalence

$$\operatorname*{colim}_{(a,u,v)\colon A_{a_0/}\times_A A_{a_1/}} C_a \xrightarrow{\sim} \operatorname*{colim}_{a\colon A} C_a$$

The category $A_{a_0/} \times_A A_{a_1/}$ is filtered as well: it is weakly contractible, and it is confluent, since the projection to A is a left fibration, so that we can apply Proposition 12.2.4. Therefore, the colimit of the $A_{a_0/} \times_A A_{a_1/}$-indexed diagram of cartesian squares

$$(a,u,v) \mapsto \begin{tikzcd} \mathrm{Hom}_{C_a}(x_a, y_a) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}([1], C_a) \arrow[d] \\ {[0]} \arrow[r] & C_a^{\simeq} \times C_a^{\simeq} \end{tikzcd}$$

yields a cartesian square after applying the colimit functor. Using Theorem 12.3.9 for $n = 1$ and $n = 0$, this produces a canonical equivalence of the form

$$\operatorname*{colim}_{(a,u,v)\colon A_{a_0/}\times_A A_{a_1/}} \mathrm{Hom}_{C_a}(x_a, y_a) = \mathrm{Hom}_{\mathrm{colim}_A C}(x, y)\,,$$

expressing morphisms from x to y as a filtered colimit.

12.4 Finite categories

We will construct the universe of finite categories. We fix a universe Cat.

Construction 12.4.1. Given a full embedding $i\colon C \hookrightarrow \mathrm{Cat}$, we consider the functor

$$\operatorname{colim}_{\ulcorner}\colon \operatorname{Fun}(\ulcorner, \mathrm{Cat}) \to \mathrm{Cat}$$

and define $E(C)^{\simeq} \hookrightarrow \mathrm{Cat}^{\simeq}$ as the image of the induced map

$$\operatorname{Fun}(\ulcorner, C)^{\simeq} \xrightarrow{i^{\simeq}} \operatorname{Fun}(\ulcorner, \mathrm{Cat})^{\simeq} \xrightarrow{\operatorname{colim}_{\ulcorner}^{\simeq}} \mathrm{Cat}^{\simeq}\,.$$

We define $E(C)$ as the full subcategory of Cat with collection of objects $E(C)^{\simeq}$, with corresponding embedding

$$i_{E(C)}\colon E(C) \hookrightarrow \mathrm{Cat}\,.$$

Since the pushout of any diagram of the form $x \xleftarrow{\mathrm{id}_x} x \xrightarrow{\mathrm{id}_x} x$ is x itself, the functor $\mathrm{const}_{\ulcorner}\colon C \to \operatorname{Fun}(\ulcorner, C)$ induces a canonical functor

$$e_C\colon C \to E(C)$$

such that $i_{E(C)} \circ e_C \simeq i$. We observe that, by virtue of Proposition 11.5.17, if C is small, then so is $E(C)$ (as full subcategory of a locally small category with a small collection of objects).

We define C_0 as the full subcategory of Cat with objects $\emptyset$, $[0]$ and $[1]$, with

$$i_0\colon C_0 \hookrightarrow \mathrm{Cat}$$

the embedding. We define for any $n\colon \mathbb{N}$

$$e_n\colon C_n \hookrightarrow C_{n+1}$$

through $C_{n+1} = E(C_n)$ and $e_n = e_{C_n}$. There is also a full embedding

$$i_{n+1}\colon C_{n+1} \to \mathrm{Cat}$$

defined by $i_{n+1} = i_{E(C_n)}$. The sequence of maps e_n, $n\colon \mathbb{N}$, define a functor $C_\bullet\colon \omega \to \mathrm{Cat}$.

Definition 12.4.2. We define the category FinCat of *finite categories* as the colimit

$$\mathrm{FinCat} = \operatorname*{colim}_{n\colon \omega} C_n\,.$$

Remark 12.4.3. If follows from Corollary 12.3.11 that FinCat is the full subcategory of Cat induced by the embedding

$$\operatorname*{colim}_{n\colon \omega} C_n^{\simeq} \hookrightarrow \mathrm{Cat}^{\simeq}\,.$$

Since ω as well as each C_n are small, so is FinCat.

Lemma 12.4.4. *The category* $\ulcorner$ *is a finite category.*

Proof. We know that $[n]$ belongs to C_0 for $n = 0$ and $n = 1$ and we have a pushout square of the form below.

$$\begin{tikzcd} {[0]} \arrow[r] \arrow[d] & {[1]} \arrow[d] \\ {[1]} \arrow[r] & \ulcorner \end{tikzcd}$$

This shows that $\ulcorner$ belongs to C_1. □

Proposition 12.4.5. *Any finite category is compact in* Cat*.*

Proof. Let Cat_ω be the full subcategory of compact objects of Cat. By virtue of Proposition 12.3.8, the full embedding

$$\mathrm{Cat}_\omega \hookrightarrow \mathrm{Cat}$$

commutes with pushouts. In other words, for any full subcategory C of Cat_ω, $E(C)$ is a full subcategory of Cat_ω as well. We know that C_0 is contained in Cat_ω. Therefore, C_n is a full subcategory of Cat_ω for all $n\colon \mathbb{N}$. The ω-indexed colimit of full embeddings

$$C_n \hookrightarrow \mathrm{Cat}_\omega\,, \quad n\colon \omega\,,$$

thus provides a full embedding

$$\mathrm{FinCat} \hookrightarrow \mathrm{Cat}_\omega$$

by Corollary 12.3.11. □

Lemma 12.4.6. *The category* FinCat *is the smallest full subcategory of* Cat *stable under pushouts containing the objects* $\emptyset$ *and* $[1]$*. In other words, given a full subcategory* C *of* Cat*, if the embedding* $C \hookrightarrow \mathrm{Cat}$ *is a right exact functor, then the following conditions are equivalent.*

(i) The embedding $\mathrm{FinCat} \hookrightarrow \mathrm{Cat}$ *factors through the embedding* $C \hookrightarrow \mathrm{Cat}$*.*

(ii) The embedding $\mathrm{FinCat}^{\simeq} \hookrightarrow \mathrm{Cat}^{\simeq}$ *factors through the embedding* $C^{\simeq} \hookrightarrow \mathrm{Cat}^{\simeq}$*.*

(iii) The object $[1]$ *belongs to* C*.*

Proof. It follows from Lemma 12.4.4 that any functor from $\ulcorner$ to FinCat factors through C_n for some n. Therefore, the colimit of such a diagram in Cat belongs to C_{n+1}, hence to FinCat. This proves that FinCat is stable under pushouts in Cat.

Conditions (i) and (ii) are obviously equivalent by the universal property of full embeddings. Therefore, it suffices to prove that (i) follows from (iii). With the notation of Construction 12.4.1, we know that the embedding $e_C\colon C \to E(C)$ is an equivalence and that condition (iii) implies that C_0 is a full subcategory of C. But if ever $C_n \hookrightarrow \mathrm{Cat}$ factors through $C \hookrightarrow \mathrm{Cat}$, we get an induced embedding

$$E(C_n) \hookrightarrow E(C) = C$$

for all $n\colon \mathbb{N}$. This proves by induction that C_n is a full subcategory of C for all $n\colon \mathbb{N}$ and provides a commutative diagram of the form

$$\begin{tikzcd} C_n \arrow[rr, "e_n"] \arrow[dr] & & C_{n+1} \arrow[dl] \\ & C & \end{tikzcd}$$

hence a morphism from the sequence

$$C_0 \hookrightarrow C_1 \hookrightarrow \cdots \hookrightarrow C_n \hookrightarrow C_{n+1} \hookrightarrow \cdots$$

to the sequence

$$C \hookrightarrow C \hookrightarrow \cdots \hookrightarrow C \hookrightarrow C \hookrightarrow \cdots$$

and taking the associated Milnor colimits, this yields an embedding $\mathrm{FinCat} \hookrightarrow C$. $\square$

Proposition 12.4.7. *The opposite of a finite category is finite: for any $C\colon \mathrm{FinCat}$, we have $C^{\mathrm{op}}\colon \mathrm{FinCat}$.*

Proof. We form the pullback below.

$$\begin{tikzcd} \mathrm{FinCat}' \arrow[r, hook] \arrow[d, hook] & \mathrm{FinCat} \arrow[d, hook] \\ \mathrm{Cat} \arrow[r, "(-)^{\mathrm{op}}"] & \mathrm{Cat} \end{tikzcd}$$

Since the functor $C \mapsto C^{\mathrm{op}}$ is an equivalence, we see that the embedding of FinCat' into Cat is a fully faithful right exact functor. Therefore, by virtue of Lemma 12.4.6, it suffices to check that $[1]$ is an object of FinCat', which follows right away from the identification $[1] = [1]^{\mathrm{op}}$. $\square$

Proposition 12.4.8. *If A and B are finite categories, then so is $A \times B$.*

Proof. Let B be any small category. We form the pullback square below.

$$\begin{tikzcd} \mathrm{Cat}'_B \arrow[r] \arrow[d, hook] & \mathrm{FinCat} \arrow[d, hook] \\ \mathrm{Cat} \arrow[r, "(-)\times B"] & \mathrm{Cat} \end{tikzcd}$$

We thus have a full embedding of the category Cat'_B whose objects are the categories A such that $A \times B$ is finite. Since the functor $(-) \times B$ commutes with small colimits (being left adjoint to $\mathrm{Fun}(B, -)$), we observe that the embedding of Cat'_B into Cat commutes with pushouts. In particular, if ever C_0 is a full subcategory of Cat'_B, then FinCat is contained in Cat'_B, by Lemma 12.4.6.

For $B = \emptyset$, we have $A \times \emptyset \xrightarrow{\sim} \emptyset$ for all A, hence $\mathrm{Cat}'_\emptyset = \mathrm{Cat}$. For $B = [0]$, we have $A \times [0] \xrightarrow{\sim} A$ for all A, hence $\mathrm{Cat}'_{[0]} = \mathrm{FinCat}$. For $B = [1]$, we observe that $A \times [1]$ is finite for $A = \emptyset$, $A = [0]$, as well as for $A = [1]$. In the last case, this comes from the fact that $[1] \times [1]$ is a pushout of a diagram of the form

$$[2] \leftarrow [1] \rightarrow [2]$$

and that $[2]$ itself is finite, being a pushout of a diagram of the form

$$[1] \leftarrow [0] \rightarrow [1]\,.$$

In other words, C_0 is a full subcategory of $\mathrm{Cat}'_{[1]}$, hence FinCat as well, by Lemma 12.4.6.

Using the equivalences $A \times B \simeq B \times A$, this implies right away that, for any B finite, C_0 is a full subcategory of Cat'_B, so that $A \times B$ is finite for any finite category A. $\square$

Corollary 12.4.9. *For any finite category A, the functor*

$$\mathrm{Fun}(A, -) \colon \mathrm{Cat} \to \mathrm{Cat}$$

commutes with small filtered colimits.

Proof. Let I be a small filtered category and $C \colon I \to \mathrm{Cat}$ a functor. In order to prove that the canonical functor

$$\operatorname*{colim}_{i \colon I} \mathrm{Fun}(A, C_i) \to \mathrm{Fun}(A, \operatorname*{colim}_{i \colon I} C_i)$$

is an equivalence, it suffices to check that the induced map

$$\mathrm{Hom}_{\mathrm{Cat}}([1], \operatorname*{colim}_{i \colon I} \mathrm{Fun}(A, C_i)) \to \mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Fun}(A, \operatorname*{colim}_{i \colon I} C_i))$$

is an equivalence (by Corollary 6.3.4, taking into account the correspondence between Map and $\mathrm{Hom}_{\mathrm{Cat}}$ through directed univalence). Since $[1]$ and $[1] \times A$ are compact, we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Cat}}([1], \operatorname*{colim}_{i \colon I} \mathrm{Fun}(A, C_i)) &= \operatorname*{colim}_{i \colon I} \mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Fun}(A, C_i)) \\
&= \operatorname*{colim}_{i \colon I} \mathrm{Hom}_{\mathrm{Cat}}([1] \times A, C_i) \\
&= \mathrm{Hom}_{\mathrm{Cat}}([1] \times A, \operatorname*{colim}_{i \colon I} C_i) \\
&= \mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Fun}(A, \operatorname*{colim}_{i \colon I} C_i))\,.
\end{aligned}$$

□

Proposition 12.4.10. *The embedding* $\mathrm{FinCat} \hookrightarrow \mathrm{Cat}$ *commutes with pushouts.*

Proof. Since $\ulcorner$ is compact in Cat, we have by Corollary 12.4.9 a canonical equivalence

$$\operatorname*{colim}_{n \colon \omega} \mathrm{Fun}(\ulcorner, C_n) \xrightarrow{\sim} \mathrm{Fun}(\ulcorner, \mathrm{FinCat})\,.$$

By construction, we have functors $p_n \colon \mathrm{Fun}(\ulcorner, C_n) \to C_{n+1}$ for each $n \colon \mathbb{N}$ fitting in a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{Fun}(\ulcorner, C_n) \arrow[r, hook] \arrow[d, dashed, "p_n"] & \mathrm{Fun}(\ulcorner, C_{n+1}) \arrow[r, hook] \arrow[d, dashed, "p_{n+1}"] & \mathrm{Fun}(\ulcorner, \mathrm{Cat}) \arrow[d, "\text{pushout}"] \\
C_{n+1} \arrow[r, hook] & C_{n+2} \arrow[r, hook] & \mathrm{Cat}
\end{tikzcd}$$

for each $n \colon \mathbb{N}$. Since the map $n \mapsto n + 1$ is a terminal functor from ω to itself, this provides a commutative diagram of the form below, in which all horizontal maps are full embeddings.

$$\begin{tikzcd}
\operatorname*{colim}_{n \colon \omega} \mathrm{Fun}(\ulcorner, C_n) \arrow[r, equal] \arrow[d, dashed] & \mathrm{Fun}(\ulcorner, \mathrm{FinCat}) \arrow[r, hook] \arrow[d, dashed] & \mathrm{Fun}(\ulcorner, \mathrm{Cat}) \arrow[d, "\text{pushout}"] \\
\mathrm{FinCat} \arrow[r, equal] & \mathrm{FinCat} \arrow[r, hook] & \mathrm{Cat}
\end{tikzcd}$$

This is literally a proof that the pushout functor in Cat preserves the property of being finite. □

Corollary 12.4.11. *If A and B are finite categories, then so is $A \star B$.*

Proof. There is a pushout of the form below.

$$\begin{tikzcd}
A \times [1]^{\simeq} \times B \arrow[r, hook] \arrow[d] & A \times [1] \times B \arrow[d] \\
A \sqcup B \arrow[r, hook] & A \star B
\end{tikzcd}$$

The previous proposition together with Proposition 12.4.8 thus imply that $A \star B$ is finite. □

Proposition 12.4.12. *Let C be a category. If C has an initial (terminal) object as well as pushouts (pullbacks), then C has colimits (limits, resp.) of type A for any finite category A.*

Proof. Since C has an initial object, it has colimits of type $\emptyset$. It also has colimits of type $[0]$ and $[1]$ (more generally, of type A for any A with a terminal object). The full subcategory of Cat spanned by categories A such that C has A-indexed colimits is stable under pushouts, by Corollary 9.8.9. This full subcategory also contains C_0. Therefore, it contains FinCat as well, by Lemma 12.4.6. □

Proposition 12.4.13. *Let $f : C \to D$ be a functor. We assume that C and D both have initial (terminal) objects and pushouts (pullbacks). If f is right (left) exact then it commutes with colimits (limits) of type A for any finite category A.*

Proof. We will consider the case of finite colimits (the case of finite limits will follow by replacing C by C^{op}).

The full subcategory T of Cat spanned by categories A such that f commutes with A-indexed colimits is stable under pushouts. Indeed, if $A = A_1 \sqcup_{A_0} A_2$ and f commutes with colimits indexed by A_0, A_1 or A_2, for any functor $X : A \to C$, if we denote by $X_{|E}$ the composition of X with the structural functor $E \to A$ for $E = A_0, A_1, A_2$, then we have canonical isomorphisms

$$\operatorname{colim}_E \; f X_{|E} \xrightarrow{\sim} f(\operatorname{colim}_E \; X_{|E}) \,.$$

If f is right exact, a double application of Corollary 9.8.9 yields isomorphisms

$$\begin{aligned}
\operatorname{colim}_A fX &\simeq (\operatorname{colim}_{A_1} \; fX_{|A_1}) \sqcup_{(\operatorname{colim}_{A_0} \; fX_{|A_0})} (\operatorname{colim}_{A_2} \; fX_{|A_2}) \\
&\simeq f(\operatorname{colim}_{A_1} \; X_{|A_1}) \sqcup_{f(\operatorname{colim}_{A_0} \; X_{|A_0})} f(\operatorname{colim}_{A_2} \; X_{|A_2}) \\
&\simeq f\big((\operatorname{colim}_{A_1} \; X_{|A_1}) \sqcup_{(\operatorname{colim}_{A_0} \; X_{|A_0})} (\operatorname{colim}_{A_2} \; X_{|A_2})\big) \\
&\simeq f(\operatorname{colim}_A \; X) \,.
\end{aligned}$$

If f preserves initial objects, this full subcategory T contains $\emptyset$. It clearly contains $[0]$ and $[1]$. Therefore, by Lemma 12.4.6, T contains all finite categories. □

Recall that a poset P is called *finite* if its underlying set $P^{\simeq}$ is finite (meaning that there is a cover of the form $f : \{1, \ldots, n\} \to P^{\simeq}$ for some $n : \mathbb{N}$).

Proposition 12.4.14. *Any finite poset is a finite category.*

Proof. If P is a finite poset, there is a unique term $n : \mathbb{N}$ such that there exists an isomorphism from $\{1, \ldots, n\}$ to $P^{\simeq}$. We call such n the cardinality of P. The proof proceeds by induction on n. For

$n = 0$, this is clear. We assume the proposition to be true for all posets of cardinality $k \leq n$ and we will prove it for posets of cardinal $n + 1$.

Let us define a maximal object $m \colon P^{\simeq}$ by the property that for any other $x \colon P^{\simeq}$, either $x \leq m$ or $P(m, x) = \emptyset$. Let Γ be the collection of maximal objects of P. By induction on n, one proves that P has maximal objects (i.e. that Γ covers the terminal anima). Therefore, it suffices to prove that P is a finite category in context Γ. In other words, without any loss of generality, we may assume that P actually has a maximal object m. We let $P_{\leq m}$ be the full subcategory of P whose objects are those x with $x \leq m$, and $P_{<m}$ the full subcategory of $P_{\leq m}$ whose objects are those x with x not isomorphic to m. Finally, we define Q as the full subcategory of P whose objects are those $x \colon P^{\simeq}$ that are not isomorphic to m in P (so that $P_{<m}$ is the fiber product of $P_{\leq m}$ and Q over P). We claim that the induced pullback square

$$\begin{tikzcd}
P_{<m} \arrow[r, hook] \arrow[d, hook] & P_{\leq m} \arrow[d, hook] \\
Q \arrow[r, hook] & P
\end{tikzcd}$$

is also a pushout square in Cat. Indeed, we check immediately that all four maps in this square are right fibrations, so that this is a particular case of Lemma 9.8.11 in disguise (with $C = P$ and $I = \ulcorner$). Therefore, it suffices to prove that $P_{<m}$, $P_{\leq m}$ and Q are finite categories. The cases of $P_{<m}$ and of Q are settled by induction hypothesis. As for $P_{\leq m}$ we check that we have

$$P_{\leq m} = P_{<m} \star [0] \, .$$

Therefore, Corollary 12.4.11 ends the proof. □

13 Ind-objects

13.1 Left exact localizations

We consider a category C.

Definition 13.1.1. We say that a family of morphisms $F \hookrightarrow \mathrm{Map}([1], C)$ is *closed under pullbacks* if the following property is verified: for any map $f\colon x \to y$ in F, and for any map $v\colon y' \to y$ in C, there exists a cartesian square of the form

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d, "f'"'] & x \arrow[d, "f"] \\ y' \arrow[r, "v"] & y \end{tikzcd}$$

such that f' belongs to F.

If $W \hookrightarrow C$ is a subcategory of C, we say that W is closed under pullbacks in C if the associated family of morphisms $\mathrm{Map}([1], W)$ is closed under pullbacks in C.

Lemma 13.1.2. *Assume that F is a family of morphisms in C that is* closed under pullbacks. *For any map $v\colon y' \to y$ in F, and any object x of C, the induced functor*

$$v_! \times_C \mathrm{id}_{C_{/x}}\colon C_{/y'} \times_C C_{/x} \to C_{/y} \times_C C_{/x}$$

has a right adjoint, which we can describe as follows. It sends an object (z, s, f) of $C_{/y} \times_C C_{/x}$, seen as a diagram

$$\begin{tikzcd} & z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ x & & y \end{tikzcd}$$

to the object (z', s', f') obtained by forming the cartesian square

$$\begin{tikzcd} z' \arrow[r, "w"] \arrow[d, "f'"'] & z \arrow[d, "f"] \\ y' \arrow[r, "v"] & y \end{tikzcd}$$

and by choosing a composition s' of s and w.

Proof. Since the Yoneda embedding is fully faithful, continuous, and compatible with slices, we see that proving the lemma for $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ implies that it holds for C itself. In other words, we may assume, without loss of generality, that C has finite products and pullbacks. But then we have

$$C_{/y} \times_C C_{/x} = C_{/y \times x}$$

and we observe that pulling back along $v \times \mathrm{id}_x$ in C produces the expected right adjoint. □

Proposition 13.1.3. *If the subcategory $W \hookrightarrow C$ is closed under pullbacks in C, for any object $x \colon C^{\simeq}$, forming the pullback square*

$$\begin{tikzcd} W(x)^{\simeq} \arrow[r, hook, "p^{\simeq}"] \arrow[d] & (C_{/x})^{\simeq} \arrow[d] \\ \mathrm{Fun}([1], W)^{\simeq} \arrow[r, hook] & \mathrm{Fun}([1], C)^{\simeq} \end{tikzcd}$$

defines a full embedding $p \colon W(x) \to C_{/x}$ that is a right calculus of fractions at x.

Proof. Let $v \colon y' \to y$ be a map in W. With the notation of the previous lemma, we have a functor

$$v_! \colon C_{/y'} \to C_{/y}$$

which encodes the operation of composing with v, and we know that the induced functor

$$v_! \times_C \mathrm{id}_{C_{/x}} \colon C_{/y'} \times_C C_{/x} \to C_{/y} \times_C C_{/x}$$

has a right adjoint by Lemma 13.1.2. The explicit description of the latter shows that, when we restrict ourselves to the subcategories in the diagram

$$v_! \times_C \mathrm{id}_{W(x)} \colon C_{/y'} \times_C W(x) \to C_{/y} \times_C W(x)\,,$$

we still get a functor with a right adjoint. But functors with right adjoints are initial, hence are weak homotopy equivalences, by Quillen's Theorem A. Applying Π_∞ yields the map induced by applying the functor

$$y \mapsto \operatorname*{colim}_{w \colon W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, -)$$

to v. □

Definition 13.1.4. We say that $W \hookrightarrow C$ has the 2-*out-of*-3 *property* if, for any commutative triangle in C of the form

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "h"'] & & y \arrow[dl, "g"] \\ & z & \end{tikzcd}$$

if two out of f, g and h are in W, then so is the third.

Theorem 13.1.5. *Let C be a category and $W \hookrightarrow C$ a subcategory. We denote by $\gamma \colon C \to L(C)$ the localization of C by W. We assume that the category C has pullbacks and that the subcategory W is closed under pullbacks and has the 2-out-of-3 property. Then the following properties hold.*

(i) For any object x of C, the full subcategory $W(x)$ of $C_{/x}$ spanned by pairs (y, f) such that $f: y \to x$ is a morphism in W is a right calculus of fractions at x.

(ii) For any object x of C, the category $W(x)^{\mathrm{op}}$ is filtered.

(iii) The category $L(C)$ has pullbacks and the functor γ commutes with pullbacks.

(iv) Given any category D with pullbacks and any functor $f: C \to D$ sending any map in W to an isomorphism in D, the induced functor $\bar{f}: L(C) \to D$ commutes with pullbacks if and only if the functor f itself has the same property.

(v) For any object x of C, the functor γ induces a functor

$$\gamma_{/x}: C_{/x} \to L(C)_{/\gamma(x)}$$

that exhibits $L(C)_{/\gamma(x)}$ as the localization of $C_{/x}$ at the subcategory $W(x) \hookrightarrow C_{/x}$.

Proof. Property (i) is a particular case of Proposition 13.1.3. Let x be an object of C. We check immediately that the collection of objects in $W(x)$ is stable under pullbacks in $C_{/x}$. Therefore, $W(x)$ has pullbacks, which implies right away that $W(x)^{\mathrm{op}}$ is confluent (see Example 12.2.3). But $W(x)^{\mathrm{op}}$ has an initial object, hence is weakly contractible, which shows property (ii).

In order to establish (iii), we want to prove that, for each pair of morphisms

$$x \xrightarrow{f} y \xleftarrow{v} y'$$

in $L(C)$, there exists a pullback square in $L(C)$ of the form

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d, "f'"] & x \arrow[d, "f"] \\ y' \arrow[r, "v"] & y \end{tikzcd}$$

in $L(C)$. It suffices to prove such a statement in context

$$\Gamma = \sum_{x,y,y' : L(C)^{\simeq}} L(C)(x, y) \times L(C)(y', y)$$

for the universal term $\gamma : \Gamma$ (in context Γ). Since

$$\gamma^{\simeq}: C^{\simeq} \to L(C)^{\simeq}$$

is a cover by Corollary 11.5.14, so is

$$\gamma^{\simeq} \times \gamma^{\simeq} \times \gamma^{\simeq}: C^{\simeq} \times C^{\simeq} \times C^{\simeq} \to L(C)^{\simeq} \times L(C)^{\simeq} \times L(C)^{\simeq} .$$

We thus have a cover

$$\sum_{x,y,y' : C^{\simeq}} L(C)(\gamma(x), \gamma(y)) \times L(C)(\gamma(y'), \gamma(y)) \to \Gamma .$$

In other words, it suffices to prove the existence of pullbacks in the case where all objects x, y, y' come from C. Furthermore, for all objects x and y in C, by virtue of Corollary 11.5.14, we have a cover of the form

$$(W(x)_{/y})^{\simeq} \to \Pi_\infty(W(x)_{/y}) = L(C)(\gamma(x), \gamma(y))\,.$$

This yields a cover

$$\sum_{x,y,y' \colon C^{\simeq}} (W(x)_{/y})^{\simeq} \times (W(y')_{/y})^{\simeq} \to \Gamma\,.$$

This means that it suffices to prove that the formation of pullbacks from diagrams as above is possible in $L(C)$ in the case where f and v are of the form

$$f = \gamma(s)^{-1} \circ \gamma(p) \quad \text{and} \quad v = \gamma(t)^{-1} \circ \gamma(q) \quad \text{respectively,}$$

with $p \colon x_0 \to y$ and $q \colon y'_0 \to y$ any maps in C, and $s \colon x_0 \to x$ and $t \colon y'_0 \to y'$ any maps in W. Therefore, it suffices to check that any pullback square in C of the form

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d, "f'"] & x \arrow[d, "f"] \\ y' \arrow[r, "v"] & y \end{tikzcd}$$

in C yields a pullback square of the form

$$\begin{tikzcd} \gamma(x') \arrow[r, "\gamma(u)"] \arrow[d, "\gamma(f')"] & \gamma(x) \arrow[d, "\gamma(f)"] \\ \gamma(y') \arrow[r, "\gamma(v)"] & \gamma(y) \end{tikzcd}$$

in $L(C)$. For this, it suffices to prove that, given any object t in $L(C)$, we have a pullback square of the form

$$\begin{tikzcd} \mathrm{Hom}_{L(C)}(t, x') \arrow[r, "u_*"] \arrow[d, "f'_*"] & \mathrm{Hom}_{L(C)}(t, x) \arrow[d, "f_*"] \\ \mathrm{Hom}_{L(C)}(t, y') \arrow[r, "v_*"] & \mathrm{Hom}_{L(C)}(t, y) \end{tikzcd}$$

in Grpd. It suffices to prove this for the universal term t in context $L(C)^{\simeq}$. Using the cover $C^{\simeq} \to L(C)^{\simeq}$, we may assume, without loss of generality, that t comes from C. In other words, in order to prove property (iii), it suffices to prove that any functor of type $C \to \mathrm{Grpd}$ given by the rule

$$x \mapsto \mathrm{Hom}_{L(C)}(t, x)$$

commutes with pullbacks for any $t \colon C^{\simeq}$. But property (i) ensures that such a functor can be computed through the rule of the form

$$x \mapsto \operatorname*{colim}_{w \colon W(t)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, x)\,.$$

Since each functor $\mathrm{Hom}_C(z_w, -)$ commutes with pullbacks, and since $W(t)^{\mathrm{op}}$ is filtered, we see that, for any finite category A and any A-indexed functor with values in C

$$a \mapsto x_a$$

the canonical map

$$\operatorname*{colim}_{w\colon W(t)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, \lim_{a\colon A} x_a) = \operatorname*{colim}_{w\colon W(t)^{\mathrm{op}}} \lim_{a\colon A} \mathrm{Hom}_C(z_w^{\mathrm{op}}, x_a) \to \lim_{a\colon A} \operatorname*{colim}_{w\colon W(t)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, x_a)$$

is an equivalence, by Theorem 12.2.9 and Proposition 12.4.13.

The proof of property (iv) is similar. First, if $\bar{f}$ preserves pullback squares, since γ preserves pullback squares as well, so does the composite $f = \bar{f} \circ \gamma$. Conversely, assuming that f preserves pullbacks, if Γ denotes the collection of all pullback squares in $L(C)$, it suffices to prove that the pullback square in $L(C)$ corresponding to the universal term of Γ yields a pullback square in D in context Γ. Since, by right calculus of fractions, as seen in the proof of property (iii), the collection of pullback squares in C covers Γ, it suffices to prove that the functor $\bar{f}$ sends the image of any pullback square of C to a pullback square of D. Since $\bar{f} \circ \gamma = f$, this amounts to the property that f preserves pullback squares.

Finally, let us prove property (v). We consider an object $x\colon C^{\simeq}$ and we want to prove that the induced functor

$$\bar{\gamma}_{/x}\colon L(C_{/x}) \to L(C)_{/\gamma(x)}$$

is an equivalence. It suffices to check that the functor $\bar{\gamma}_{/x}$ is fully faithful and induces a cover

$$(\gamma_{/x})^{\simeq}\colon (C_{/x})^{\simeq} \to (L(C)_{/\gamma(x)})^{\simeq}\,.$$

The latter property comes from the fact that there is a commutative diagram of the form

$$\begin{tikzcd} & \sum_{t\colon C^{\simeq}} (W(t)_{/x})^{\simeq} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ (C_{/x})^{\simeq} \arrow[rr] & & (L(C)_{/\gamma(x)}) \end{tikzcd}$$

with p and q defined explicitly as follows. A term in the top vertex of the triangle above is a triple of the form $t \xleftarrow{s} z \xrightarrow{g} x$ in C, with s in W. The map p sends such a triple to the pair (z, g) whereas the map q sends it to $(t, \gamma(s)^{-1} \circ \gamma(g))$. The commutativity of this triangle is provided by the identification in $L(C)_{/\gamma(x)}$ provided by the invertible morphism induced termwise by s. Therefore, in order to prove essential surjectivity, it suffices to check that the map q is a cover, which follows right away from the fact that it is the composition of the cover

$$\sum_{t\colon C^{\simeq}} (W(t)_{/x})^{\simeq} \to \sum_{t\colon C^{\simeq}} L(C)(\gamma(t), \gamma(x))$$

with the cover

$$\sum_{t\colon C^{\simeq}} L(C)(\gamma(t), \gamma(x)) \to \sum_{t\colon L(C)^{\simeq}} L(C)(t, \gamma(x))\,.$$

It remains to prove that, for any maps $u\colon y \to x$ and $v\colon z \to x$ in C, the induced map

$$L(C_{/x})((y,u),(z,v)) \to L(C)_{/\gamma(x)}((\gamma(y),\gamma(u)),(\gamma(z),\gamma(v)))$$

is an equivalence. We observe that property (i) applies to $C_{/x}$ with respect to the full subcategory $W(x) \hookrightarrow C_{/x}$. We thus have a right calculus of fractions at (y, u): taking into account the identification of $C_{/y}$ with the slice of $C_{/x}$ over (y, u), we see that the collection of morphisms in $L(C_{/x})$ from (y, u) to (z, v) is thus the colimit

$$\operatorname*{colim}_{w\colon W(y)^{\mathrm{op}}} \mathrm{Hom}_{C_{/x}}((t_w, u_w)^{\mathrm{op}}, (z, v)) ,$$

where we see each term $w\colon W(y)$ as a pair consisting of $s_w : t_w \to y$ in W and define $u_w : t_w \to x$ as the composition of s_w and u. In other words, we can consider the diagram assigning to each term w in $W(y)^{\mathrm{op}}$ the diagram

$$[0] \xrightarrow{u_w} \mathrm{Hom}_C(t_w, x) \xleftarrow{v_*} \mathrm{Hom}_C(t_w, z) .$$

Forming first the pullback and then the $W(y)^{\mathrm{op}}$-indexed colimit yields $L(C_{/x})((y, u), (z, v))$, whereas, by right calculus of fractions at y in C, we see that taking the $W(y)^{\mathrm{op}}$-indexed colimit first and then forming the pullback produces the collection of morphisms from $\gamma(y)$ to $\gamma(z)$ over $\gamma(x)$ in $L(C)$. Therefore, the property of fully faithfulness of the functor $\bar{\gamma}_{/x}$ follows from property (ii) and another application of Theorem 12.2.9. $\square$

Corollary 13.1.6. *Let C be a small category with finite limits, and $W \hookrightarrow C$ a subcategory closed under pullbacks, with associated localization functor $\gamma\colon C \to L(C)$. Then $L(C)$ has finite limits and γ is left exact.*

Proof. This follows right away from Theorem 13.1.5 (iii) and from Theorem 13.1.5 (v) in the case where x is the terminal object of C. $\square$

Definition 13.1.7. Let C be a category with pushouts. A subcategory $W \hookrightarrow C$ is closed under pushouts in C if $W^{\mathrm{op}} \hookrightarrow C^{\mathrm{op}}$ is closed under pullbacks.

Corollary 13.1.8. *Let C be a category and $W \hookrightarrow C$ a subcategory. We denote by $\gamma\colon C \to L(C)$ the localization of C by W. We assume that the category C has pushouts and that the subcategory W is closed under pushouts and has the 2-out-of-3 property. Then the following properties hold.*

(i) For any object x of C, the full subcategory $W(x)$ of $C_{x/}$ spanned by pairs (y, f) such that $f : x \to y$ is a morphism in W is a left calculus of fractions at x.

(ii) For any object x of C, the category $W(x)$ is filtered.

(iii) The category $L(C)$ has pushouts and the functor γ commutes with them.

(iv) Given any category D with pushouts and any functor $f : C \to D$ sending any map in W to an isomorphism in D, the induced functor $\bar{f} : L(C) \to D$ commutes with pushouts if and only if the functor f itself has the same property.

(v) For any object x of C, the functor γ induces a functor

$$\gamma_{x/} : C_{x/} \to L(C)_{\gamma(x)/}$$

that exhibits $L(C)_{\gamma(x)/}$ as the localization of $C_{x/}$ at the subcategory $W(x) \hookrightarrow C_{x/}$.

Proof. Apply Theorem 13.1.5 to C^{op}. $\square$

Corollary 13.1.9. *Let C be a small category with finite colimits, and $W \hookrightarrow C$ a subcategory closed under pushouts, with associated localization functor $\gamma\colon C \to L(C)$. Then $L(C)$ has finite colimits and γ is right exact.*

Proof. Apply Corollary 13.1.6 to C^{op}. □

Definition 13.1.10. A subcategory $W \hookrightarrow C$ is *saturated* if the obvious commutative square

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & C \arrow[d, "\gamma"] \\ L(C)^{\simeq} \arrow[r, hook] & L(C) \end{tikzcd}$$

is cartesian.

The next proposition shows that left exact localizations can always be described by localizing a saturated subcategory.

Proposition 13.1.11. *Let C be a category and $W \hookrightarrow C$ a subcategory. We denote by $\gamma\colon C \to L(C)$ the localization of C by W. We assume that the category C has pullbacks and that the subcategory W is closed under pullbacks and has the 2-out-of-3 property. We define $\bar{W}$ through the following pullback square.*

$$\begin{tikzcd} \bar{W} \arrow[r, hook] \arrow[d] & C \arrow[d, "\gamma"] \\ L(C)^{\simeq} \arrow[r, hook] & L(C) \end{tikzcd}$$

Then $\bar{W}$ is closed under pullbacks, has the 2-out-of-3 property, and the functor γ exhibits $L(C)$ as the localization of C by $\bar{W}$.

Proof. Assertion (iii) of Theorem 13.1.5 implies that $\bar{W}$ is closed under pullbacks. The rest of the proposition is obvious. □

Lemma 13.1.12. *Let $W \hookrightarrow C$ be a subcategory closed under pullbacks and with the 2-out-of-3 property. Given two objects x and y in C and two diagrams of the form*

$$x \xleftarrow{s} z \xrightarrow{p} y \quad \text{and} \quad x \xleftarrow{s'} z' \xrightarrow{p'} y$$

with s and s' in W, the anima $\gamma(s)^{-1} \circ \gamma(p) = \gamma(s')^{-1} \circ \gamma(p')$ is covered by the collection of commutative diagrams in C of the form

$$\begin{tikzcd} & z \arrow[dl, "s"'] \arrow[dr, "p"] & \\ x & z'' \arrow[l, "s''"'] \arrow[r, "p''"] \arrow[u, "\varphi"] \arrow[d, "\psi"] & y \\ & z' \arrow[ul, "s'"] \arrow[ur, "p'"'] & \end{tikzcd}$$

with s'' in W.

In particular, the collection of morphisms of the form $f\colon x \to y$ in C inducing an isomorphism in $L(C)$ is covered by the collection of diagrams of the form

$$x'' \xrightarrow{p'} x' \xrightarrow{p} x \xrightarrow{f} y$$

with the property that each composite $f \circ p$ and $p \circ p'$ is in W.

Proof. We observe that $W(x)_{/y}$ always has pullbacks (for any objects x and y) and thus that the localization $\Pi_\infty(W(x)_{/y})$ is obtained by a right calculus of fractions at any object of $W(x)_{/y}$. This implies the first assertion right away. The last assertion follows from the first by interpreting the fact that a map in $L(C)$ is invertible if and only if it has a right inverse which itself has a right inverse. $\square$

Proposition 13.1.13. *Let $W \hookrightarrow C$ be a subcategory closed under pullbacks and with the 2-out-of-3 property. Then W is saturated if and only if the collection of diagrams of the form*

$$x''' \xrightarrow{p''} x'' \xrightarrow{p'} x' \xrightarrow{p} x$$

with the property that each composite $p' \circ p''$ and $p \circ p'$ is in W coincides with the collection of diagrams of the same shape with the property that all three morphisms p, p' and p'' are in W.

Proof. It suffices to prove that the collection of diagrams of the form

$$x''' \xrightarrow{p''} x'' \xrightarrow{p'} x' \xrightarrow{p} x$$

with the property that each composite $p' \circ p''$ and $p \circ p'$ is in W is covered by the collection of diagrams of the same shape with the property that all three morphisms p, p' and p'' are in W. This follows right away from Lemma 13.1.12. $\square$

Proposition 13.1.14. *Let $W \hookrightarrow C$ be a saturated subcategory closed under pullbacks. Then the localization functor $C \to L(C)$ restricted to W exhibits $L(C)^\simeq$ as the localization of W by all its maps:*

$$\Pi_\infty(W) \cong L(C)^\simeq .$$

Proof. Let $x\colon C^\simeq$. Since $W^\simeq = C^\simeq$, we may see x as an object of W. We observe that W is closed under pullbacks with the 2-out-of-3 property as a subcategory of W. If we define $W(x)$ as the full subcategory of $C_{/x}$ with objects those pairs (y, s) with $s\colon y \to x$ a map in W, we thus have a right calculus of fractions at x

$$W(x) \hookrightarrow C_{/x} \quad \textit{in } C$$

and a right calculus of fractions at x

$$W(x) = W_{/x} \quad \textit{in } W.$$

Given any other object y, the map

$$\Pi_\infty(W)(x, y) \to L(C)(x, y)$$

may thus be described as the canonical map

$$\operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_W(z_w^{\mathrm{op}}, y) \to \operatorname*{colim}_{w\colon W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) .$$

It suffices now to check that the map above is an embedding with image $L(C)^{\simeq}(x, y)$. We observe right away that $W(x)$ has pullbacks, hence that $W(x)^{\mathrm{op}}$ is confluent. Since the functor

$$\operatorname*{colim}_{W(x)^{\mathrm{op}}}\colon \ \mathrm{Fun}(W(x)^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Grpd}$$

preserves pullbacks, it preserves embeddings. Therefore, since each map

$$\mathrm{Hom}_W(z_w^{\mathrm{op}}, y) \hookrightarrow \mathrm{Hom}_C(z_w^{\mathrm{op}}, y)$$

is an embedding, this proves that $\Pi_\infty(W)(x, y)$ embeds into $L(C)(x, y)$. In order to prove that the image consists exactly of isomorphisms from x to y in the localization $L(C)$, it suffices to work in the context of such equivalences that come from an object of $W(x)_{/y}$. But if $s\colon z \to x$ is in W and $p\colon z \to y$ is a map in C such that $\gamma(s)^{-1} \circ \gamma(p)$ is invertible in $L(C)$, then so is $\gamma(p)$ itself. Since W is saturated, this implies right away that p is in W as well. □

13.2 Compactly generated left Bousfield localizations

Recall the notion of compact object (Definition 12.3.6).

Definition 13.2.1. Let C be a locally small category with small filtered colimits. An object $x\colon C^{\simeq}$ is *compact* if the functor

$$\mathrm{Hom}_C(x, -)\colon C \to \mathrm{Grpd}$$

commutes with small filtered colimits.

We denote by C_ω the full subcategory of C whose collection of objects consists of the compact objects of C.

Example 13.2.2. If A is a small category and $a\colon A^{\simeq}$, then $\mathrm{Hom}_A(-, a)$ is a compact object of $C = \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})$. Indeed, by virtue of the Yoneda Lemma, this amounts to claiming that the functor $F \mapsto F(a)$ commutes with filtered colimits, which is due to the fact that it commutes with all colimits.

Lemma 13.2.3. *Let C be a locally small category with small colimits. The full subcategory C_ω has finite colimits and the full embedding $C_\omega \hookrightarrow C$ is a right exact functor.*

Proof. By virtue of Proposition 12.4.12 and Proposition 12.4.13, this amounts to saying that the initial object is compact and that, for any pushout square of C of the form

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "f"'] & b \arrow[d, "g"] \\ a' \arrow[r, "u'"] & b' \end{tikzcd}$$

with a, b and a' compact, the object b is compact. Indeed, if $c\colon C$ is initial, then the functor $\mathrm{Hom}_C(c, -)$ is the constant functor with value the terminal object of Grpd. Claiming that c is compact amounts to claiming that, for any small filtered category I, the I-indexed colimit functor preserves terminal objects in Grpd, which follows from the fact that, by definition, filtered categories are weakly contractible. As for the second assertion, we observe that, for any small filtered category

I, and for any functor $i \mapsto x_i$ with values in C, we have the following immediate computations, using the very definition of compactness together with Theorem 12.2.9.

$$\begin{aligned}
\operatorname*{colim}_{i\colon I} \operatorname{Hom}_C(b', x_i) &= \operatorname*{colim}_{i\colon I}\Big(\operatorname{Hom}_C(a', x_i) \times_{\operatorname{Hom}_C(a,x_i)} \operatorname{Hom}_C(b, x_i)\Big) \\
&= \operatorname*{colim}_{i\colon I} \operatorname{Hom}_C(a', x_i) \times_{\operatorname*{colim}_{i\colon I} \operatorname{Hom}_C(a,x_i)} \operatorname*{colim}_{i\colon I} \operatorname{Hom}_C(b, x_i) \\
&= \operatorname{Hom}_C(a', \operatorname*{colim}_{i\colon I} x_i) \times_{\operatorname{Hom}_C(a,\operatorname*{colim}_{i\colon I} x_i)} \operatorname{Hom}_C(b, \operatorname*{colim}_{i\colon I} x_i) \\
&= \operatorname{Hom}_C(b', \operatorname*{colim}_{i\colon I} x_i)
\end{aligned}$$

□

Definition 13.2.4. Let C be a locally small category and $S \hookrightarrow \operatorname{Map}([1], C)$ a family of morphisms in C. An object $x\colon C^{\simeq}$ is said to be *S-local* if, for any map $f\colon a \to b$ belonging to S, the induced map

$$f^*\colon C(b, x) \to C(a, x)$$

is an equivalence.

We denote by $L_S(C)$ the full subcategory of C spanned by S-local objects.

A morphism $f\colon a \to b$ in C is called an *S-equivalence* if, for any S-local object x, the induced map

$$f^*\colon C(b, x) \to C(a, x)$$

is an equivalence.

Proposition 13.2.5. *Let C be a locally small category with small filtered colimits. If S is a family of morphisms between compact objects of C, then $L_S(C)$ has small filtered colimits and the full embedding $L_S(C) \hookrightarrow C$ commutes with small filtered colimits. In other words: any small filtered colimit of S-local objects in C is an S-local object.*

Proof. This is straightforward. □

The main goal of this section is to prove the theorem below.

Theorem 13.2.6. *Let C be a locally small category with small colimits and $S \hookrightarrow \operatorname{Map}([1], C_\omega)$ a small family of maps between compact objects of C. Then the full embedding $L_S(C) \hookrightarrow C$ has a left adjoint*

$$loc_S\colon C \to L_S(C)$$

exhibiting $L_S(C)$ as the localization of C by the family of S-equivalences. Furthermore, the functor loc_S preserves compact objects.

The proof of this theorem will need a little bit of preparation. Fix C locally small with filtered colimits.

Definition 13.2.7. A family of morphisms $S \hookrightarrow \operatorname{Map}([1], C)$ is *codiagonal-stable* if, for any map $u\colon a \to b$ in S, if we form the pushout square

$$\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "u"'] & b \arrow[d] \\
b \arrow[r] & b \amalg_a b
\end{tikzcd}$$

then the codiagonal map $v\colon b \amalg_a b \to b$ induced by the obvious commutative square

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "u"'] & b \arrow[d, equal] \\ b \arrow[r, equal] & b \end{tikzcd}$$

belongs to S.

Lemma 13.2.8. *If S is codiagonal-stable, then an object $x\colon C^{\simeq}$ is S-local if and only if, for any map $u\colon a \to b$ in S, the induced map*

$$u^*\colon \operatorname{Hom}_C(b,x) \to \operatorname{Hom}_C(a,x)$$

has a section.

Proof. This is clearly a necessary condition. Conversely, let us consider an object x with the property that u^* has a section for any $u\colon S$. We observe that, for any map $u\colon a \to b$ in S, the codiagonal map $v\colon b \amalg_a b \to b$ has a section, hence

$$v^*\colon \operatorname{Hom}_C(b,x) \to \operatorname{Hom}_C(b \amalg_a b, x)$$

is a section. In other terms, v^* has a left inverse as well as a right inverse and is thus an equivalence. In other words, for any $u\colon a \to b$ in S, the induced commutative square

$$\begin{tikzcd} \operatorname{Hom}_C(b,x) \arrow[r, equal] \arrow[d, equal] & \operatorname{Hom}_C(b,x) \arrow[d, "u^*"] \\ \operatorname{Hom}_C(b,x) \arrow[r, "u^*"] & \operatorname{Hom}_C(a,x) \end{tikzcd}$$

is a pullback square. This means that the map $u^*\colon \operatorname{Hom}_C(b,x) \to \operatorname{Hom}_C(a,x)$ is an embedding. Since u^* also is a cover, it must be an isomorphism, by Proposition 11.5.9. ◻

Lemma 13.2.9. *The family of S-equivalences has the following stability properties.*

(i) It has the 2-out-of-3 property.

(ii) It is stable under small colimits: for any small category I and any functors $a, b\colon I \to C$ and any natural transformation

$$u\colon a \to b$$

if $u_i\colon a_i \to b_i$ is an S-equivalence for any object $i\colon I^{\simeq}$, then the induced morphism

$$\operatorname*{colim}_{i\colon I} u_i\colon \operatorname*{colim}_{i\colon I} a_i \to \operatorname*{colim}_{i\colon I} b_i$$

is an S-equivalence.

(iii) For any pushout square in C of the form

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "f"'] & b \arrow[d, "g"] \\ a' \arrow[r, "u'"] & b' \end{tikzcd}$$

with u an S-equivalence, the morphism u' is an S-equivalence.

Proof. Properties (i) and (ii) are straightforward (they follow right away from the analogous properties for the family of isomorphisms in Grpd, using the fact that the functor $\mathrm{Hom}_C(x,-)$ commutes with small limits for any object x). As for property (iii), we simply observe that we have an induced pullback square of the form

$$\begin{tikzcd} \mathrm{Hom}_C(b',x) \arrow[r, "g^*"] \arrow[d, "(u')^*"'] & \mathrm{Hom}_C(b,x) \arrow[d, "u^*"] \\ \mathrm{Hom}_C(a',x) \arrow[r, "f^*"] & \mathrm{Hom}_C(a,x) \end{tikzcd}$$

proving that u' is an S-equivalence whenever u has this property. □

Construction 13.2.10. Let $S \hookrightarrow \mathrm{Map}([1],C)$ be a family of maps in C. We define $\nabla(S)$ as the union in $\mathrm{Map}([1],C)$ of S with the image of the map

$$\kappa\colon S \to \mathrm{Map}([1],C)$$

that assigns to each $u\colon a \to b$ in S the associated codiagonal $\kappa(u) = v\colon b \amalg_a b \to b$. We define a sequence of embeddings of anima

$$S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow S_{n+1} \hookrightarrow \cdots$$

by defining $S_0 = S$ and $S_{n+1} = \nabla(S_n)$ for all $n\colon \mathbb{N}$. We put

$$\bar{S} = \operatorname*{colim}_{n\colon \omega} S_n\,.$$

Lemma 13.2.11. *Assume that $S \hookrightarrow \mathrm{Map}([1],C)$ is a small family of maps between compact objects. Then the following properties hold.*

(i) The family $\bar{S}$ is a small family of maps between compact objects.

(ii) The family $\bar{S}$ is codiagonal-stable.

(iii) An object $x\colon C^{\simeq}$ is S-local if and only if it is $\bar{S}$-local.

Proof. It is clear that $\nabla(S)$ is small whenever S is small. Therefore, each S_n is small. Since small anima are stable under ω-indexed colimits, this proves that $\bar{S}$ is small. The second part of property (i) follows right away from Lemma 13.2.3. The property for $\bar{S}$ to be codiagonal-stable amounts to the property that $\bar{S}$ contains $\nabla(\bar{S})$, which holds by construction. Finally, it is clear that any $\bar{S}$-local object is S-local because S is a subfamily of $\bar{S}$ by construction. Conversely, it suffices to prove that any S-local object x is also $\nabla(S)$-local, which follows right away from properties (i) and (iii) of Lemma 13.2.9. □

Proof of Theorem 13.2.6 through the small object argument. By virtue of Lemma 13.2.11, we may assume that S is codiagonal-stable. We define a functor

$$L\colon C \to C$$

as well as a natural transformation $l\colon \mathrm{id}_C \to L$ as follows. There is a functor

$$\mathrm{Grpd} \times C \to C\,, \quad (a,x) \mapsto a \otimes x$$

defined by the functorial formula

$$\mathrm{Hom}_C((a \otimes x)^{\mathrm{op}}, y) = \mathrm{Hom}_{\mathrm{Grpd}}(a^{\mathrm{op}}, \mathrm{Hom}_C(x^{\mathrm{op}}, y))$$

for $a\colon \mathrm{Grpd}$ and $x, y\colon C$. More precisely, if a is an anima and x an object of C, then $a \otimes x$ is the a-indexed colimit of the constant diagram with value x.

Given $x\colon C$ and an object a of C, there is thus a canonical map

$$c_{a,x}\colon\ \mathrm{Hom}_C(a, x) \otimes a \to x$$

corresponding to the identity of $\mathrm{Hom}_C(a, x)$, functorially in x. We define $A(x)$ as the colimit indexed by S of the diagram that assigns $\mathrm{Hom}_C(a, x) \otimes a$ for each term $u\colon a \to b$ of S. Similarly, we define $B(x)$ as the colimit indexed by S of the diagram that assigns $\mathrm{Hom}_C(a, x) \otimes b$ for each term $u\colon a \to b$ of S. We define

$$\alpha_x\colon A(x) \to x$$

as the map induced by the family $c_{a,x}$ indexed by each term $u\colon a \to b$ of S. We form the pushout square

$$\begin{tikzcd}
A(x) \arrow[r, "\alpha_x"] \arrow[d, "i_x"'] & x \arrow[d, "l_x"] \\
B(x) \arrow[r, "\beta_x"] & L(x)
\end{tikzcd}$$

where i_x is the S-indexed colimit of the maps

$$\mathrm{id} \otimes u\colon\ \mathrm{Hom}_C(a, x) \otimes a \to \mathrm{Hom}_C(a, x) \otimes b\,.$$

We now define a sequence of functors

$$L^0 \xrightarrow{l^1} L^1 \xrightarrow{l^2} \cdots \to L^n \xrightarrow{l^{n+1}} L^{n+1} \to \cdots$$

by induction through the formulas

$$L^0 = \mathrm{id}_C \quad \text{and} \quad L^{n+1} = L \circ L^n\,, \quad l^{n+1} = l_{L^n}\,.$$

We define

$$\lambda_x\colon x \to loc_S(x)$$

functorially in x by the formula

$$loc_S(x) = \operatorname*{colim}_{n\colon \omega} L^n(x)$$

with λ_x the map induced by the component of the colimit cocone at $0\colon \mathbb{N}$. It follows from Lemma 13.2.9 that l_x is an S-equivalence for all x. We can also see the map λ_x as the ω-indexed colimit of the natural transformation

$$\begin{tikzcd}
x \arrow[r, equal] \arrow[d, equal] & x \arrow[r, equal] \arrow[d, "l_x"] & \cdots \arrow[r, equal] & x \arrow[r, equal] \arrow[d] & x \arrow[r, equal] \arrow[d] & \cdots \\
x \arrow[r, "l_x"] & L(x) \arrow[r] & \cdots \arrow[r] & L^n(x) \arrow[r, "l_{L^n(x)}"] & L^{n+1}(x) \arrow[r] & \cdots
\end{tikzcd}$$

in which the vertical maps are S-equivalences (this is proven by induction, as finite compositions of maps of the form l_y for various y's). Therefore, Lemma 13.2.9 (ii) implies that λ_x is an S-equivalence for any x.

We claim that $loc_S(x)$ is S-local. Indeed, for any $u\colon a \to b$ in S, and for any $n\colon \mathbb{N}$, there is a canonical commutative diagram of the form

$$\begin{tikzcd}
\mathrm{Hom}_C(b, L^n(x)) \arrow[r, "(l_x^{n+1})_*"] \arrow[d, "u^*"'] & \mathrm{Hom}_C(b, L^{n+1}(x)) \arrow[r] \arrow[d, "u^*"] & \mathrm{Hom}_C(b, loc_S(x)) \arrow[d, "u^*"] \\
\mathrm{Hom}_C(a, L^n(x)) \arrow[r, "(l_x^{n+1})_*"] \arrow[ur, dashed, "\lambda_n"] & \mathrm{Hom}_C(a, L^{n+1}(x)) \arrow[r] & \mathrm{Hom}_C(a, loc_S(x))
\end{tikzcd}$$

in which the dashed arrow λ_n is induced by the map $\beta_{L^n(x)}$. We can think of the collection of the maps λ_n as a morphism of sequences of morphisms of anima:

$$\begin{tikzcd}
\mathrm{Hom}_C(a, x) \arrow[r] \arrow[d, "\lambda_0"'] & \cdots \arrow[r] & \mathrm{Hom}_C(a, L^n(x)) \arrow[r, "(l_x^{n+1})_*"] \arrow[d, "\lambda_n"] & \mathrm{Hom}_C(a, L^{n+1}(x)) \arrow[r] \arrow[d, "\lambda_{n+1}"] & \cdots \\
\mathrm{Hom}_C(b, L(x)) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_C(b, L^{n+1}(x)) \arrow[r] & \mathrm{Hom}_C(b, L^{n+2}(x)) \arrow[r] & \cdots
\end{tikzcd}$$

Taking the associated Milnor colimits yields a morphism

$$\lambda'\colon \operatorname*{colim}_{n\colon \omega} \mathrm{Hom}_C(a, L^n(x)) \to \operatorname*{colim}_{n\colon \omega} \mathrm{Hom}_C(b, L^{n+1}(x))\,.$$

Since ω is filtered, the embedding $\omega_{1/} \to \omega$ is terminal, hence we have

$$\operatorname*{colim}_{n\colon \omega} \mathrm{Hom}_C(b, L^{n+1}(x)) = \operatorname*{colim}_{n\colon \omega} \mathrm{Hom}_C(b, L^n(x))\,.$$

The map λ' thus yields a map

$$\lambda\colon \operatorname*{colim}_{n\colon \omega} \mathrm{Hom}_C(a, L^n(x)) \to \operatorname*{colim}_{n\colon \omega} \mathrm{Hom}_C(b, L^n(x))\,.$$

Since $\mathrm{Hom}_C(t, loc_S(x))$ canonically is the colimit of the $\mathrm{Hom}_C(t, L^n(x))$'s for any compact object t, the map λ determines a map

$$\bar{\lambda}\colon\ \mathrm{Hom}_C(a, loc_S(x)) \to \mathrm{Hom}_C(b, loc_S(x))$$

that is a section of the map

$$u^*\colon\ \mathrm{Hom}_C(b, loc_S(x)) \to \mathrm{Hom}_C(a, loc_S(x))$$

by construction. Lemma 13.2.8 thus proves that $loc_S(x)$ is S-local.

Consider now an S-local object y in C. Then, for any object $x\colon C^{\simeq}$, the S-equivalence λ_x induces an isomorphism

$$\lambda_x^*\colon\ \mathrm{Hom}_C(loc_S(x), y) \xrightarrow{\simeq} \mathrm{Hom}_C(x, y)\,.$$

Since this equivalence is obviously functorial in y, this shows that the full embedding of $L_S(C)$ into C is a right adjoint by Theorem 9.3.17.

The characterization of $L_S(C)$ as localization of C by S-equivalences follows from Proposition 5.5.10 and from Corollary 9.3.13 applied to $L_S(C)$.

It remains to prove that the left adjoint loc_S preserves compact objects. Let $x\colon C^{\simeq}$ be a compact object. We consider a small filtered category I as well as an I-indexed functor $i \mapsto y_i$ with values in $L_S(C)$. We know from Proposition 13.2.5 that the colimit of the diagram of the y_i's can be computed independently in $L_S(C)$ or in C itself. In other words, we have

$$\begin{aligned}\operatorname*{colim}_{i\colon I} \operatorname{Hom}_{L_S(C)}(loc_S(x), y_i) &= \operatorname*{colim}_{i\colon I} \operatorname{Hom}_C(x, y_i) \\ &= \operatorname{Hom}_C(x, \operatorname*{colim}_{i\colon I} y_i) \\ &= \operatorname{Hom}_{L_S(C)}(loc_S(x), \operatorname*{colim}_{i\colon I} y_i)\,,\end{aligned}$$

thus proving that $loc_S(x)$ is a compact object of $L_S(C)$. □

Recall that $\operatorname{Fun}_!(C, D)$ is the full subcategory $\operatorname{Fun}(C, D)$ spanned by functors commuting with small colimits.

Corollary 13.2.12. *Let C be a locally small category with small colimits and S a small family of morphisms between compact objects of C, with associated localization functor*

$$loc_S\colon C \to L_S(C)\,.$$

Given any category D with small colimits, we denote by $\operatorname{Fun}_!^S(C, D)$ the full subcategory of $\operatorname{Fun}(C, D)$ spanned by colimit preserving functors sending any morphism in S to an isomorphism of D. Then pre-composing with the functor loc_S induces an equivalence

$$\operatorname{Fun}_!(L_S(C), D) \cong \operatorname{Fun}_!^S(C, D)$$

for any category D with small colimits.

Proof. By virtue of Theorem 13.2.6 pre-composing with loc_S induces a full embedding of the form

$$\operatorname{Fun}_!(L_S(C), D) \hookrightarrow \operatorname{Fun}_!^S(C, D)$$

It suffices to prove that the induced map

$$\operatorname{Fun}_!(L_S(C), D)^{\simeq} \hookrightarrow \operatorname{Fun}_!^S(C, D)^{\simeq}$$

is a cover. Let $F\colon C \to D$ be a small colimits preserving functor sending all maps of S to isomorphisms of D. We define $F_S\colon L_S(C) \to D$ as the following composite:

$$F_S\colon L_S(C) \hookrightarrow C \xrightarrow{F} D\,.$$

It suffices to prove two properties:

(i) $F_S \circ loc_S = F$ holds (i.e. is inhabited);

(ii) F_S commutes with small colimits.

The second property follows from the first: we know that small colimits in $L_S(C)$ can always be computed by forming first the colimits in C and then applying loc_S. Let W be the family of morphisms of C that are sent to isomorphisms of D. Keeping track of the explicit construction of the functor loc_S through the small object argument as in the proof of Theorem 13.2.6, we observe that, for any object x of C, the unit map $x \to loc_S(x)$ belongs to W. This implies right away that $F_S \circ loc_S = F$ holds true. □

13.3 Left exact presheaves

Notation 13.3.1. Let C and D be two categories. We denote by $\mathrm{Lex}(C, D)$ the full subcategory of $\mathrm{Fun}(C, D)$ whose objects are the left exact functors from C to D.

Proposition 13.3.2. *Let C be a small category with finite colimits. The full embedding*

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

commutes with filtered colimits and has a left adjoint.

Proof. Let $h\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ be the Yoneda embedding. If $\emptyset_C$ denotes the initial object of C, there is a canonical comparison map

$$i\colon \emptyset \to h(\emptyset_C)\,.$$

Similarly, given any pushout square in C of the form

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "f"'] & b \arrow[d, "g"] \\ a' \arrow[r, "u'"] & b' \end{tikzcd}$$

there is a canonical comparison map

$$comp\colon h(a') \amalg_{h(a)} h(b) \to h(b')\,.$$

This defines a map from the collection of pushout squares of C into $\mathrm{Map}([1], \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}))$, the image of which we will denote by S'. We write S for the union of S' with the image of the map from $[0]$ to $\mathrm{Map}([1], \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}))$ corresponding to i above. We observe the following properties:

(a) An object $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ is S-local if and only if it is a left exact functor;

(b) The anima S is small and consists of morphisms between compact objects of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$.

Indeed, property (a) is a direct translation of the definitions through the Yoneda lemma (under the form of Corollary 9.3.13). The smallness of S follows from Proposition 11.5.17. The last assertion is a direct consequence of Example 13.2.2 and Lemma 13.2.3.

The proposition is now a particular case of Proposition 13.2.5 and Theorem 13.2.6. □

Corollary 13.3.3. *If C is a small category with finite colimits, then the category* $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ *of left exact functors from C^{op} to* Grpd *has small colimits as well as small limits.*

Proof. Since small colimits and small limits exist in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$, this follows from the previous theorem, by Corollary 10.2.4. □

Proposition 13.3.4. *Let C be a small category with finite colimits. For any object $x\colon C^{\simeq}$, the representable presheaf* $\mathrm{Hom}_C(-, x)$ *is a left exact functor from C^{op} to* Grpd*. Furthermore, the Yoneda embedding*

$$h\colon C \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$$

is a right exact functor.

Proof. The property for $\mathrm{Hom}_C(-, x)$ to be left exact follows from Theorem 9.6.3 applied to C^{op}. Proposition 10.2.3 tells us that colimits and limits in $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ are always computed by first forming them in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ and then applying the left adjoint to the embedding $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. This left adjoint being also the localization functor by S-equivalences for S defined in the proof of Proposition 13.3.2, this immediately proves that the Yoneda embedding is right exact as functor from C to $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$, by definition of S. □

Corollary 13.3.5. *If C is a small category with finite colimits, then, for any object $x\colon C^{\simeq}$, the associated representable presheaf $h_C(x) = \mathrm{Hom}_C(-, x)$ is a compact object of* $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$.

Proof. For any left exact functor $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Lex}(C^{\mathrm{op}},\mathrm{Grpd})}(h_C(x), F) &= \mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd})}(h_C(x), F) \\ &= F(x)\,. \end{aligned}$$

Since the functor

$$\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Grpd}\,, \quad F \mapsto F(x)$$

commutes with colimits and since the embedding of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ into $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ commutes with filtered colimits (see Proposition 13.3.2), this implies that $h_C(x)$ is a compact object of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$. □

Lemma 13.3.6. *Let C be a small category with finite colimits with Yoneda embedding $h\colon C \to$* $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. *We consider a right fibration with small fibers $q\colon E \to C$ classified by a presheaf* $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$. *The following conditions are equivalent:*

(a) F is a left exact functor;

(b) E is a filtered category;

(c) There is a small filtered category X, a functor $p\colon X \to C$, and an isomorphism of the form

$$\operatorname*{colim}_{x\colon X} h_C(p(x)) \cong F\,.$$

Proof. Corollary 9.5.3 shows that (c) is a consequence of (b). Let us assume that condition (c) holds. If I is a finite category and $i \mapsto y_i$ is an I-indexed functor with values in C, using the fact that colimits are computed object-wise in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$, since I^{op} is finite as well and since small filtered colimits commute with finite ones in Grpd, we have canonical isomorphisms:

$$\begin{aligned} F(\operatorname*{colim}_{i\colon I} y_i) &= \operatorname*{colim}_{x\colon X} \mathrm{Hom}_C(\operatorname*{colim}_{i\colon I} y_i, p(x)) \\ &= \operatorname*{colim}_{x\colon X} \lim_{i\colon I^{\mathrm{op}}} \mathrm{Hom}_C(y_i^{\mathrm{op}}, p(x)) \\ &= \lim_{i\colon I^{\mathrm{op}}} \operatorname*{colim}_{x\colon X} \mathrm{Hom}_C(y_i^{\mathrm{op}}, p(x)) \\ &= \lim_{i\colon I^{\mathrm{op}}} F(y_i^{\mathrm{op}})\,. \end{aligned}$$

In other words, F is left exact. Finally, let us assume that F is left exact. In order to prove that E is filtered, it suffices to prove that E has finite colimits: the existence of pushouts ensures the property of being confluent (Example 12.2.3) and the existence of initial objects implies the property of weak contractibility. Since the embedding

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

is fully faithful and contains representable presheaves by Proposition 13.3.4, we see that there are canonical pullback squares of the following form.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})_{/F} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})_{/F} \arrow[d] \\
C \arrow[r, "h"] & \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, hook] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})
\end{tikzcd}$$

Since $h\colon C \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ is right exact and since the projection of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})_{/F}$ to $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ commutes with small colimits, we see that E has finite colimits (applying Proposition 9.8.7 to E^{op}). □

Definition 13.3.7. Let C be a category. We let Γ be the collection of all functors of the form $p\colon X \to C$ where X is a small filtered category. The category of (small) *ind-objects* in C is the full subcategory of the category of presheaves on C (with values in the category of large groupoids for a large enough universe) spanned by the image of the map

$$p\colon X \to C \quad \mapsto \quad \operatorname*{colim}_{x\colon X} h_C(p(x)) \,.$$

The Yoneda embedding of C induces a full embedding of the form

$$C \hookrightarrow \mathrm{Ind}(C) \,.$$

Remark 13.3.8. Lemma 13.3.6 tells us that, for any small category with finite colimits, we have:

$$\mathrm{Ind}(C) = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \,.$$

Proposition 13.3.9. *Let C be a small category. A presheaf $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ is an ind-object in C if and only if the domain of the associated right fibration $C_{/F}$ is a filtered category. Furthermore, the category $\mathrm{Ind}(C)$ has small filtered colimits and the full embedding $\mathrm{Ind}(C) \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ commutes with small filtered colimits.*

Proof. Since any presheaf F is a colimit of the composition of the projection of $C_{/F}$ to C with the Yoneda embedding, it is clear that F is an ind-object whenever $C_{/F}$ is filtered. For the converse, if X is a small filtered category and $p\colon X \to C$ any functor, with F the presheaf obtained by producing the colimit of the composition of p with the Yoneda embedding, one can construct F by choosing a factorization of p into a terminal functor $i\colon X \to Y$ followed by a right fibration $\pi\colon Y \to C$. The presheaf F is then the one classifying π, hence $C_{/F} = Y$. Proposition 12.2.18 thus ensures that $C_{/F}$ is filtered.

Let us consider a small filtered category X and a functor

$$\varphi\colon X \to \mathrm{Ind}(C) \,.$$

We let F be the colimit of φ in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. Since, by virtue of Lemma 9.8.11, $C_{/F}$ is the colimit of the diagram

$$x \mapsto C_{/\varphi(x)}$$

that takes values in filtered categories, Corollary 12.2.20 implies that $C_{/F}$ itself is filtered. □

Theorem 13.3.10. *Let $u\colon C \to D$ be a right exact functor between small categories with finite colimits. Then the adjunction*

$$u_!\colon \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})\colon u^*$$

restricts to an adjunction of the form

$$u_!\colon \operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \operatorname{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})\colon u^*\,.$$

Proof. We know that the composition of two left exact functors is left exact. Since u^* precisely is the operation of composing with the left exact functor u^{op}, this implies right away that the functor u^* sends $\operatorname{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})$ into $\operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$. As for the functor $u_!$, recall first that, by virtue of Corollary 9.3.18, there is a canonical commutative square of the form

$$\begin{tikzcd} C \arrow[r, "h_C"] \arrow[d, "u"'] & \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, "u_!"] \\ D \arrow[r, "h_D"] & \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) \end{tikzcd}$$

where the two horizontal arrows are the Yoneda embeddings. Let $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ be a left exact functor and let $q\colon E \to C$ be the corresponding right fibration. By virtue of Corollary 9.5.3, there is a canonical isomorphism of the form:

$$\operatorname*{colim}_{x\colon E} h_C(q(x)) \cong F\,.$$

Hence we have:

$$\begin{aligned} u_!(F) &= u_!\Big(\operatorname*{colim}_{x\colon E} h_C(q(x))\Big) \\ &= \operatorname*{colim}_{x\colon E} u_!(h_C(q(x))) \\ &= \operatorname*{colim}_{x\colon E} h_D(u(q(x)))\,. \end{aligned}$$

Moreover, since F is left exact, Lemma 13.3.6 tells us that E is a small filtered category, and therefore $u_!(F)$ is a left exact functor. □

Notation 13.3.11. Given two categories with finite colimits C and D, we denote by $\operatorname{Rex}(C, D)$ the full subcategory of $\operatorname{Fun}(C, D)$ spanned by finite colimit preserving functors.

Theorem 13.3.12. *Let C be a small category with finite colimits. For any category D with small colimits, precomposing with the Yoneda embedding*

$$h_C\colon C \to \operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$$

induces an equivalence of the form

$$\operatorname{Fun}_!(\operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}), D) \cong \operatorname{Rex}(C, D)\,.$$

Proof. By virtue of Corollary 9.6.17, precomposing with the Yoneda embedding of C induces an equivalence of the form

$$\mathrm{Fun}_!(\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}), D) \cong \mathrm{Fun}(C, D)\,.$$

On the other hand, Proposition 13.3.2 tells us that the full embedding

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

has a left adjoint

$$l\colon\ \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$$

and, by Corollary 13.2.12, pre-composing with l yields an equivalence of the form

$$\mathrm{Fun}_!(\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}), D) \cong \mathrm{Fun}_!^S(\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}), D)$$

for S the family of morphisms so that S-local objects correspond precisely to left exact presheaves on C (see the proof of Proposition 13.3.2). In other words, there is a pullback square of the form below.

$$\begin{tikzcd}
\mathrm{Fun}_!^S(\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}), D) \arrow[r] \arrow[d] & \mathrm{Fun}_!(\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}), D) \arrow[d, "h_C^*"] \\
\mathrm{Rex}(C, D) \arrow[r, hook] & \mathrm{Fun}(C, D)
\end{tikzcd}$$

Since the vertical map on the right hand side of the square is an equivalence, this proves the theorem. □

Remark 13.3.13. Let Cat′ be a universe of large categories (i.e. such that Cat is both an object of Cat′ and a full subcategory of Cat′). If Grpd′ denotes the category of large groupoids, given a small category with finite colimits C, we then have a full embedding

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}')\,.$$

The latter functor commutes with small colimits as well as with small limits. For the case of limits, this follows from the analogous property for $\mathrm{Grpd} \hookrightarrow \mathrm{Grpd}'$. For small colimits, the argument is analogous once we notice that the respective left adjoints L and L' of the embeddings

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \quad \text{and} \quad \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}') \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}')$$

induce the following commutative square

$$\begin{tikzcd}
\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, hook] \arrow[d, "L"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}') \arrow[d, "L'"] \\
\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, hook] & \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}')
\end{tikzcd}$$

(this is easily seen using the description of L with the small object argument).

13.4 Compactly generated categories

Construction 13.4.1. Let C be a category with finite colimits, and let $i\colon E \hookrightarrow C$ be a full subcategory. We consider the functor

$$\mathrm{colim}_{\ulcorner}\colon \mathrm{Fun}(\ulcorner, E) \to C$$

and define $U(E)^{\simeq} \hookrightarrow C^{\simeq}$ as the image of the induced map

$$\mathrm{Fun}(\ulcorner, E)^{\simeq} \xrightarrow{i^{\simeq}} \mathrm{Fun}(\ulcorner, C)^{\simeq} \xrightarrow{\mathrm{colim}_{\ulcorner}^{\simeq}} C^{\simeq}\,.$$

We define $U(E)$ as the full subcategory of C with collection of objects $U(E)^{\simeq}$, with corresponding embedding

$$i_{U(E)}\colon U(E) \hookrightarrow C\,.$$

Since the pushout of any diagram of the form $x \xleftarrow{\mathrm{id}_x} x \xrightarrow{\mathrm{id}_x} x$ is x itself, the functor $\mathrm{const}_{\ulcorner}\colon E \to \mathrm{Fun}(\ulcorner, E)$ induces a canonical functor

$$u_E\colon E \to U(E)$$

such that $i_{U(E)} \circ u_E \simeq i$. We observe that, by virtue of Proposition 11.5.17, if C is locally small and if E is small, then $U(E)$ is small as well (as full subcategory of a locally small category with a small collection of objects).

We define $U_0(E)$ as the full subcategory of C with objects the initial object as well as the terms of $E^{\simeq}$. We denote by

$$i_0\colon U_0(E) \hookrightarrow C$$

the embedding. We define for any $n\colon \mathbb{N}$

$$u_n(E)\colon U_n(E) \hookrightarrow U_{n+1}(E)$$

through $U_{n+1}(E) = U(U_n(E))$ and $u_n(E) = u_{U_n(E)}$. There is also a full embedding

$$i_{n+1}\colon U_{n+1}(E) \to C$$

defined by $i_{n+1} = i_{U(U_n(E))}$. The sequence of maps $u_n(E)$, $n\colon \mathbb{N}$, defines a functor $U_\bullet(E)\colon \omega \to \mathrm{Cat}$. We denote by $\langle E\rangle_{\mathrm{fp}}$ the colimit of $U_\bullet(E)$. The family of full embeddings $i_n\colon U_n(E) \hookrightarrow C$ induces a canonical full embedding $\langle E\rangle_{\mathrm{fp}} \hookrightarrow C$ by Corollary 12.3.11. It follows from Lemma 12.4.4 that any functor from $\ulcorner$ to $\langle E\rangle_{\mathrm{fp}}$ factors through $U_n(E)$ for some n, hence that the colimit of such a diagram belongs to $U_{n+1}(E)$, hence to $\langle E\rangle_{\mathrm{fp}}$. This implies that $\langle E\rangle_{\mathrm{fp}}$ has finite colimits and that its embedding into C is right exact.

Definition 13.4.2. Given a full subcategory E of a category C with finite colimits, we call $\langle E\rangle_{\mathrm{fp}}$ the *smallest finitely cocomplete full subcategory of C generated by E*.

Remark 13.4.3. This terminology is justified by the fact that, given any full subcategory D of C, if D is closed under finite colimits in C (i.e. D has finite colimits and the embedding of D into C is right exact), the property that all objects of E are objects of D is equivalent to the property that all objects of $\langle E\rangle_{\mathrm{fp}}$ are objects of D (exercise left to the reader). We also observe that, if C is locally small and if $E^{\simeq}$ is small, then $\langle E\rangle_{\mathrm{fp}}$ is small.

Definition 13.4.4. A *compactly generated category* is a category D with the following properties:

(i) D is locally small;

(ii) D has small colimits;

(iii) there is given a small family of compact objects $E \hookrightarrow D^{\simeq}$ that generates D in the sense that, for any morphism $f: x \to y$ in D, if the induced map

$$f_*: \operatorname{Hom}_D(e, x) \to \operatorname{Hom}_D(e, y)$$

is an isomorphism for any $e: E$, then f is an isomorphism in D.

Example 13.4.5. The category Grpd is compactly generated with generating family of compact objects the terminal objects of Grpd.

Example 13.4.6. The category Cat of small categories is compactly generated: a family of generators is given by the single object $[1]$ by Corollary 6.3.4 and Theorem 12.3.9.

Example 13.4.7. If C is a small category with finite colimits, then $\operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ is compactly generated, the representable presheaves forming a generating family of compact objects by Corollary 13.3.5. This class of examples covers all cases, as expressed by the following theorem.

Theorem 13.4.8. *Let D be a compactly generated category. We consider a small family of compact objects $E \hookrightarrow D^{\simeq}$ that generates D and form the smallest finitely cocomplete full subcategory $C = \langle E \rangle_{\mathrm{fp}}$ of D generated by E. Then there is a canonical equivalence*

$$\operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \cong D$$

whose composition with the Yoneda embedding yields the embedding $C \hookrightarrow D$.

Proof. Let $i: C \hookrightarrow D$ be the inclusion. By virtue of Theorem 9.6.11, we then have an adjunction of the form

$$i_!: \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows D : i^*$$

where i^* is the functor defined by the formula

$$i^*(y)(x) = \operatorname{Hom}_D(i(x), y)$$

for any $x: C^{\mathrm{op}}$ and $y: D$. Let S be defined as in the proof of Proposition 13.3.2. We deduce right away from the fact that i is right exact that, for any object y of D, the presheaf $i^*(y)$ is S-local. Applying Corollary 9.3.13 to D, we see that the functor $i_!$ thus sends S-equivalences to isomorphisms. Let

$$loc_S: \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$$

be the left adjoint to the embedding $\operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$. We know from Proposition 13.3.2 that loc_S exhibits $\operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ as the localization of $\operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ by S-equivalences. Therefore, there is a unique functor

$$L: \operatorname{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \to D$$

together with an isomorphism $L \circ loc_S \cong i_!$. The functor L has a right adjoint

$$R\colon D \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$$

characterized by the property that composing R with the embedding of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ into $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ is isomorphic to i^*. We will now prove that the functor L is fully faithful. We will prove first that the restriction of L to representable presheaves is fully faithful. Indeed, if x and x' are objects of C, then we have:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Lex}(C^{\mathrm{op}},\mathrm{Grpd})}(h_C(x), h_C(x')) &= \mathrm{Hom}_C(x, x')\\ &= \mathrm{Hom}_D(i(x), i(x'))\\ &= \mathrm{Hom}_{\mathrm{Lex}(C^{\mathrm{op}},\mathrm{Grpd})}(L(h_C(x)), L(h_C(x')))\,.\end{aligned}$$

This implies that, for $x\colon C^{\simeq}$ and $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ left exact, the canonical map

$$\mathrm{Hom}_{\mathrm{Lex}(C^{\mathrm{op}},\mathrm{Grpd})}(h_C(x), F) \to \mathrm{Hom}_D(L(h_C(x)), L(F)) = \mathrm{Hom}_D(i(x), F)$$

is an isomorphism. Indeed since $h_C(x)$ is compact in $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ and since $i(x)$ is compact in D, both functors $\mathrm{Hom}_{\mathrm{Lex}(C^{\mathrm{op}},\mathrm{Grpd})}(h_C(x), -)$ and $\mathrm{Hom}_D(i(x), -)$ commute with filtered colimits. But Lemma 13.3.6 tells us that any left exact functor of the form $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$ is a small filtered colimit of representable presheaves, which implies our claim. We can finally prove that the functor L is fully faithful: given two left exact functors

$$F, G\colon C^{\mathrm{op}} \to \mathrm{Grpd}\,,$$

we write F as a small (filtered) colimit of representable presheaves

$$\operatorname*{colim}_{i\colon I} h_C(x_i) \cong F$$

so that we get canonical identifications of the form

$$\begin{aligned}\mathrm{Hom}(F, G) &= \mathrm{Hom}(\operatorname*{colim}_{i\colon I} h_C(x_i), G)\\ &= \lim_{i\colon I^{\mathrm{op}}} \mathrm{Hom}(h_C(x_i)^{\mathrm{op}}, G)\\ &= \lim_{i\colon I^{\mathrm{op}}} \mathrm{Hom}(L(h_C(x_i))^{\mathrm{op}}, L(G))\\ &= \mathrm{Hom}(\operatorname*{colim}_{i\colon I} L(h_C(x_i))^{\mathrm{op}}, L(G))\\ &= \mathrm{Hom}(L\big((\operatorname*{colim}_{i\colon I} h_C(x_i)\big)^{\mathrm{op}}, L(G))\\ &= \mathrm{Hom}(L(F), L(G))\,.\end{aligned}$$

The fully faithfulness of the functor L can be reformulated by saying that the unit map is an isomorphism $F \cong R(L(F))$ for all F. To conclude, we observe that the functor R is conservative: this amounts to saying that isomorphisms in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ can be detected object-wise and to the assumption that E generates D in the sense of Definition 13.4.4. Therefore, in order to prove that the co-unit map $L(R(y)) \to y$ is an isomorphism for any $y\colon D^{\simeq}$, it suffices to prove that its image by R is an isomorphism, which follows right away from the invertibility of the unit map. □

Corollary 13.4.9. *If D is compactly generated, then D has small limits.*

Proof. This follows from Corollary 13.3.3 and Theorem 13.4.8. □

Corollary 13.4.10. *Any small category is a small filtered colimit of finite categories. More precisely, there is a canonical equivalence* $\mathrm{Lex}(\mathrm{FinCat}^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{Cat}$.

Proof. We know that Cat is compactly generated by virtue of Corollary 6.3.4 and Theorem 12.3.9. This yields the equivalence $\mathrm{Lex}(\mathrm{FinCat}^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{Cat}$ by Theorem 13.4.8. The first assertion is then a particular case of Lemma 13.3.6 with $C = \mathrm{FinCat}$. □

Corollary 13.4.11. *A category has small colimits if and only if it has finite colimits as well as small filtered colimits. Similarly, if C and D are categories with small colimits, a functor $f: C \to D$ commutes with small colimits if and only if it commutes with finite colimits as well as with small filtered colimits.*

Proof. This is a direct application of the first assertion of the previous corollary and of Lemma 9.8.8. □

Proposition 13.4.12. *Let D be a compactly generated category and E a locally small category with small colimits. We consider a functor*

$$f^*: D \to E$$

admitting a conservative right adjoint

$$f_*: E \to D\,.$$

Then the following conditions are equivalent.

(i) The left adjoint functor f^* preserves compact objects.

(ii) There is a generating family of compact objects of D that are sent to compact objects of E through f^*.

(iii) The right adjoint functor f_* commutes with small filtered colimits.

In particular, if one of the equivalent conditions above is fulfilled, then E is compactly generated.

Proof. It is clear that (ii) follows from (i). Let $X \hookrightarrow D^{\simeq}$ be a small family of compact objects generating D. Let us assume furthermore that $f^*(x)$ is compact in E for all $x: X$. We consider a small filtered category I and an I-indexed diagram $i \mapsto y_i$ taking values in E. We want to prove that the canonical map

$$\operatorname*{colim}_{i:I} f_*(y_i) \to f_*(\operatorname*{colim}_{i:I} y_i)$$

is an isomorphism. This follows from the computations below, for each $x: X$.

$$\begin{aligned}
\mathrm{Hom}_D(x, \operatorname*{colim}_{i:I} f_*(y_i)) &= \operatorname*{colim}_{i:I} \mathrm{Hom}_D(x, f_*(y_i)) \\
&= \operatorname*{colim}_{i:I} \mathrm{Hom}_E(f^*(x), y_i) \\
&= \mathrm{Hom}_E(f^*(x), \operatorname*{colim}_{i:I} y_i) \\
&= \mathrm{Hom}_D(x, f_*(\operatorname*{colim}_{i:I} y_i))\,.
\end{aligned}$$

This implies right away that assertion (iii) is a consequence of assertion (ii). Finally, let us assume that assertion (iii) holds true, and let $x\colon D^{\simeq}$ be a compact object of D. We want to prove that $f^*(x)$ is compact. We consider again a small filtered category I and an I-indexed diagram $i \mapsto y_i$ taking values in E and compute:

$$\begin{aligned}\operatorname*{colim}_{i\colon I} \operatorname{Hom}_E(f^*(x), y_i) &= \operatorname*{colim}_{i\colon I} \operatorname{Hom}_D(x, f_*(y_i))\\ &= \operatorname{Hom}_D(x, \operatorname*{colim}_{i\colon I} f_*(y_i))\\ &= \operatorname{Hom}_D(x, f_*(\operatorname*{colim}_{i\colon I} y_i))\\ &= \operatorname{Hom}_E(f^*(x), \operatorname*{colim}_{i\colon I} y_i)\,.\end{aligned}$$

This completes the proof. □

13.5 Left Bousfield localization of compactly generated categories

Proposition 13.5.1. *Let D be a compactly generated category and $S \hookrightarrow \mathrm{Map}([1], D)$ a small family of morphisms between compact objects. Then the category of S-local objects $L_S(D)$ is compactly generated.*

Proof. By virtue of Theorem 13.2.6, the full embedding

$$L_S(D) \hookrightarrow D$$

has a left adjoint

$$loc_S\colon D \to L_S(D)$$

that preserves compact objects. Let $E \hookrightarrow D^{\simeq}$ be a small family of compact objects in D. We denote by $E_S \hookrightarrow L_S(D)^{\simeq}$ the image of the composed map

$$E \hookrightarrow D^{\simeq} \xrightarrow{loc_S^{\simeq}} L_S(D)^{\simeq}\,.$$

Proposition 11.5.17 ensures that E_S is small and all of its terms are compact objects of $L_S(D)$. By construction, the map $p\colon E \to E_S$ is a cover. It suffices now to check that the family of functors of the form

$$\operatorname{Hom}_{L_S(D)}(y, -) \quad \text{for all } y\colon E_S$$

is conservative. Let $f\colon a \to b$ be a morphism in $L_S(D)$ such that

$$f_*\colon \operatorname{Hom}_{L_S(D)}(y, a) \xrightarrow{\simeq} \operatorname{Hom}_{L_S(D)}(y, b)$$

is an isomorphism for all $y\colon E_S^{\simeq}$. This implies that

$$f_*\colon \operatorname{Hom}_{L_S(D)}(p(x), a) \xrightarrow{\simeq} \operatorname{Hom}_{L_S(D)}(p(x), b)$$

is an isomorphism for all $x\colon E^{\simeq}$. But $p(x) = loc_S(x)$ and this means that

$$f_*\colon \operatorname{Hom}_D(x, a) \xrightarrow{\simeq} \operatorname{Hom}_D(x, b)$$

is an isomorphism for all $x\colon E^{\simeq}$. In other words, the image of f through the full embedding of $L_S(D)$ into D is an isomorphism. Since full embeddings are conservative, this proves that f is an isomorphism in $L_S(D)$. □

Proposition 13.5.2. *Let D be a compactly generated category and $C \hookrightarrow D$ a small full subcategory such that $C^{\simeq}$ is a generating family of compact objects in D. We consider a family of morphisms*

$$S \hookrightarrow \mathrm{Map}([1], C)$$

with associated localization functor by S-equivalences

$$loc_S \colon D \to L_S(D)$$

We let $W \hookrightarrow C$ be the subcategory of morphisms in C that become invertible in $L_S(D)$ through the functor loc_S, with associated localization functor $\gamma \colon C \to L(C)$, and we define C_S as the full subcategory of $L_S(D)$ spanned by the image of the map

$$C^{\simeq} \hookrightarrow D^{\simeq} \xrightarrow{loc_S^{\simeq}} L_S(D)^{\simeq} .$$

Then there is a canonical equivalence

$$L(C) \cong C_S .$$

Proof. The functor loc_S obviously restricts to a right exact functor

$$l \colon C \to C_S$$

that sends maps in W to isomorphisms. Corollary 13.1.8 (iv) implies that there is a unique right exact functor

$$\lambda \colon L(C) \to C_S$$

equipped with an identification $\lambda \circ \gamma = l$. We will prove that λ is an equivalence. Theorem 13.4.8 provides a canonical equivalence

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \cong D$$

extending the embedding $C \hookrightarrow D$. The proof of Proposition 13.5.1 allows us to apply Theorem 13.4.8 once again to provide an equivalence of the form

$$\mathrm{Lex}(C_S^{\mathrm{op}}, \mathrm{Grpd}) \cong L_S(D)$$

extending $C_S \hookrightarrow L_S(D)$. Under these equivalences, the functor

$$l^* \colon \mathrm{Lex}(C_S^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$$

corresponds to the full embedding $L_S(D) \hookrightarrow D$.

On the other hand the functor

$$\gamma^* \colon \mathrm{Fun}(L(C)^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

is fully faithful with essential image consisting of those presheaves on C which send maps in W to isomorphisms of Grpd. Corollary 13.1.8 (iii) means that there is a canonical pullback square of the form below.

$$\begin{tikzcd} \mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "\gamma^*"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, hook] \\ \mathrm{Fun}(L(C)^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "\gamma^*"] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \end{tikzcd}$$

In particular the upper horizontal arrow above is a full embedding whose essential image consists of those left exact presheaves on C sending maps in W to isomorphisms in Grpd. Under the equivalence $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \cong D$, those correspond to T-local objects where

$$T = \mathrm{Map}([1], W) \hookrightarrow \mathrm{Map}([1], C) \hookrightarrow \mathrm{Map}([1], D)\,.$$

Since $S \hookrightarrow T$, it is clear that any T-local object is S-local. Conversely, any S-local object y is T-local: indeed, for any map $u\colon a \to b$ in W, since $loc_S(u)$ is an isomorphism by definition of W, the map

$$u^*\colon\ \mathrm{Hom}_D(b, y) \to \mathrm{Hom}_D(a, y)$$

is isomorphic to the isomorphism

$$u^*\colon\ \mathrm{Hom}_D(loc_S(b), y) \xrightarrow{\simeq} \mathrm{Hom}_D(loc_S(a), y)\,.$$

We obtain a commutative square of the form

$$\begin{tikzcd}
\mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "\lambda_!"] \arrow[d, "\simeq"'] & \mathrm{Lex}(C_S^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, "\simeq"] \\
L_T(D) \arrow[r, equal] & L_S(D)
\end{tikzcd}$$

thus proving that the upper horizontal map is an equivalence. In other words, the adjunction provided by Theorem 13.3.10

$$\lambda_!\colon \mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \mathrm{Lex}(C_S^{\mathrm{op}}, \mathrm{Grpd})\colon \lambda^*$$

consists of equivalences. Since the restriction of $\lambda_!$ to representable presheaves is isomorphic to the composition of λ with the Yoneda embedding of C_S, this proves that the functor λ is fully faithful. Since

$$\lambda^{\simeq}\colon L(C)^{\simeq} \to C_S^{\simeq}$$

is a cover by construction, this proves that λ is an equivalence. □

Notation 13.5.3. Recall that, for any category D with small colimits and any functor $f\colon A \to B$ between small categories, the functor

$$f^*\colon\ \mathrm{Fun}(B, D) \to \mathrm{Fun}(A, D)$$

has a left adjoint

$$f_!\colon\ \mathrm{Fun}(A, D) \to \mathrm{Fun}(B, D)\,.$$

In particular, given any small category A and object $a\colon A^{\simeq}$ the evaluation functor $F \mapsto F_a$ has a left adjoint

$$D \to \mathrm{Fun}(A, D)\,, \quad x \mapsto a \otimes x$$

defined by $a \otimes x = a_!(x)$, where we see a as an A-valued functor. If D is locally small, we thus have the formula

$$\mathrm{Hom}(x, F_a) = \mathrm{Hom}(a \otimes x, F)$$

functorially in x and F, which determines $a \otimes x$. Given a family of objects $E \hookrightarrow D^{\simeq}$, we define E_A as the image of the map

$$A^{\simeq} \times E \to \mathrm{Map}(A, D)\,, \quad (a, x) \mapsto a \otimes x\,.$$

Proposition 13.5.4. *Let D be a compactly generated category. For any small category A, the functor category* $\mathrm{Fun}(A, D)$ *is compactly generated. More precisely, given a small family of compact objects $E \hookrightarrow D^{\simeq}$ that generates D, the family E_A is a family of compact objects that generates* $\mathrm{Fun}(A, D)$.

Proof. Given $a : A^{\simeq}$, the assignment $F \mapsto F_a$ commutes with colimits and therefore, for any compact object x of D, the object $a \otimes x$ is compact: the functor $\mathrm{Hom}(a \otimes x, -)$ is the composition of the evaluation at a with $\mathrm{Hom}(x, -)$. One can detect isomorphisms in $\mathrm{Fun}(A, D)$ object-wise, and can detect isomorphisms in D by evaluating on terms of E. This shows that the family of functors of the form $\mathrm{Hom}(a \otimes x, -)$ is jointly conservative. □

Proposition 13.5.5. *Let D be a compactly generated category. For any finite category A, any functor $F : A \to D$ taking values (objectwise) in compact objects of D is compact as an object of* $\mathrm{Fun}(A, D)$.

Proof. The assertion is clearly true if $A = \emptyset$ or $A = [0]$. Let us prove it for $A = [1]$. Given an object x of D, and a morphism $u : a \to b$ in D, there is a correspondence of the form

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] & a \arrow[d, "u"] \\ x \arrow[r, "u \circ f"] & b \end{tikzcd} \quad \Leftrightarrow \quad f : x \to a\,.$$

In other words, the object $0 \otimes x$ can be described as the morphism

$$\mathrm{id}_x : x \to x$$

seen as a functor from $[1]$ to D. Similarly, if $\emptyset$ denotes the initial object of D, we see that the object $1 \otimes x$ corresponds to the unique morphism of the form

$$\emptyset \to x\,.$$

Given any morphism $u : a \to b$ in D there is a functor $\varphi : \ulcorner \to \mathrm{Fun}([1], D)$ yielding a diagram of the form

$$0 \otimes a \leftarrow 1 \otimes a \to 1 \otimes b$$

corresponding to the commutative diagram below.

$$\begin{tikzcd} a \arrow[d, "\mathrm{id}_a"'] & \emptyset \arrow[l] \arrow[r, "\mathrm{id}_\emptyset"] \arrow[d] & \emptyset \arrow[d] \\ a & a \arrow[l, "\mathrm{id}_a"'] \arrow[r, "u"] & b \end{tikzcd}$$

A simple inspection of universal properties shows that the colimit of this functor φ is nothing else than $u : a \to b$ (seen as a functor from $[1]$ to D). By virtue of Lemma 13.2.3, this proves that, if ever both a and b are compact in D, so is u in $\mathrm{Fun}([1], D)$.

Let $C \hookrightarrow \mathrm{Cat}$ be the full subcategory of Cat whose objects are those small categories such that any A-indexed functor with values in D_ω is a compact object of $\mathrm{Fun}(A, D)$. We know that C contains $\emptyset$, $[0]$ as well as $[1]$. By virtue of Lemma 12.4.6, it suffices now to prove that C is stable under pushouts in Cat.

Let us consider a pushout square in Cat of the form

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "u'"] & B' \end{tikzcd}$$

with A, A' as well as B in C. Given a functor $F\colon B' \to D$, we write $F_V\colon V \to D$ for the composition of F with the functor $v\colon V \to B'$, for v being either u, a or u'. We observe that the assignment $F \mapsto F_V$ commutes with colimits (in particular with small filtered ones). For any functors $F, G\colon B' \to D$, since we have a pullback square of the form

$$\begin{tikzcd} \mathrm{Fun}(B', D) \arrow[r, "b^*"] \arrow[d, "(u')^*"'] & \mathrm{Fun}(B, D) \arrow[d, "u^*"] \\ \mathrm{Fun}(A', D) \arrow[r, "a^*"] & \mathrm{Fun}(A, D) \end{tikzcd}$$

we have a canonical identification of the form

$$\mathrm{Hom}(F, G) = \mathrm{Hom}(F_{A'}, G_{A'}) \times_{\mathrm{Hom}(F_A, G_A)} \mathrm{Hom}(F_B, G_B)\,.$$

Since filtered colimits commute with pullbacks in Grpd, this shows that F is compact whenever F_V is compact in $\mathrm{Fun}(V, D)$ for $V = A$, $V = A'$, and $V = B$. This implies right away that C is stable under pushouts in Cat and completes the proof of the proposition. □

Lemma 13.5.6. *Let P be a finite poset and C a small category with finite colimits. The restriction of the functor*

$$P \times \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}))\,, \quad (a, x) \mapsto a \otimes x$$

to $P \times C$ via the Yoneda embedding of C takes values in the full subcategory $\mathrm{Fun}(P^{\mathrm{op}}, C)$. This induces an equivalence $\langle P \times C\rangle_{\mathrm{fp}} = \mathrm{Fun}(P^{\mathrm{op}}, C)$ (in other words, $\mathrm{Fun}(P^{\mathrm{op}}, C)$ is the smallest finitely cocomplete full subcategory of $\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}))$ generated by objects of the form $a \otimes x$ for $a\colon P^{\simeq}$ and $x\colon C^{\simeq}$). In particular, we have

$$\mathrm{Ind}(\mathrm{Fun}(P^{\mathrm{op}}, C)) = \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C))\,.$$

Proof. The evaluation of $a \otimes x$ at an object e of P is

$$\sum_{u\colon \mathrm{Hom}_P(e,a)} x\,.$$

Since $\mathrm{Hom}_P(e, a)$ is a finite set, this implies the first claim.

Therefore, in order to prove the second assertion, it suffices to know that any object of $\mathrm{Fun}(P^{\mathrm{op}}, C)$ is a finite colimit of objects of the form $a \otimes x$. Since P^{op} is a finite poset, so is its twisted arrow category $\mathrm{Tw}(P^{\mathrm{op}})$. In particular, a new application of Proposition 12.4.14 tells us that $\mathrm{Tw}(P^{\mathrm{op}})^{\mathrm{op}}$ is a finite category. Therefore, Corollary 9.6.20 concludes the proof of the second assertion. The third one follows right away from the second one by Theorem 13.4.8. □

Theorem 13.5.7. *Let C be a small category with finite colimits and $W \hookrightarrow C$ a subcategory closed under pushouts and with the 2-out-of-3 property, with associated localization functor $\gamma\colon C \to L(C)$. Given a small category A, we denote by $L(\mathrm{Fun}(A, C))$ the localization of $\mathrm{Fun}(A, C)$ by the family of natural transformations $u\colon F \to G$ such that $u_a\colon F_a \to G_a$ is a map of W for any object $a\colon A^{\simeq}$. Then, for any finite poset P, the functor*

$$\gamma_*\colon \mathrm{Fun}(P, C) \to \mathrm{Fun}(P, L(C))$$

induces an equivalence

$$L(\mathrm{Fun}(P, C)) \cong \mathrm{Fun}(P, L(C)) .$$

Proof. Recall that any finite poset is finite as a category (Proposition 12.4.14). Let us define $D = \mathrm{Fun}(P, \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}))$ and let $S \hookrightarrow \mathrm{Map}([1], D)$ be the family of morphisms in $\mathrm{Fun}(P, C)$ that are objectwise in W. We see right away that S-local objects simply are left exact functors from $L(C)^{\mathrm{op}}$ to $\mathrm{Fun}(P, \mathrm{Grpd})$, hence that

$$D_S = \mathrm{Fun}(P, \mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{Grpd})) .$$

Since, by virtue of Proposition 13.5.5 the Yoneda embedding

$$h_C\colon C \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$$

induces a full embedding

$$\mathrm{Fun}(P, C) \hookrightarrow D$$

taking values in compact objects, Proposition 13.5.2 implies that the functor

$$loc_S = (\gamma_!)_*\colon D \to D_S$$

induces an equivalence

$$L(\mathrm{Fun}(P, C)) = \mathrm{Fun}(P, C)_S$$

where $\mathrm{Fun}(P, C)_S$ is the full subcategory of $\mathrm{Fun}(P, \mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{Grpd}))$ spanned by the essential image of the functor

$$\mathrm{Fun}(P, C) \xrightarrow{\gamma_*} \mathrm{Fun}(P, L(C)) \hookrightarrow D_S .$$

This implies that the induced functor

$$L(\mathrm{Fun}(P, C)) \to \mathrm{Fun}(P, L(C))$$

is right exact and fully faithful (right exactness was already known by Corollary 13.1.8 anyway). For any $a\colon P^{\simeq}$ and any object x of C, we see that the formation of $a \otimes x$ commutes with γ by right exactness. Lemma 13.5.6 thus implies that the comparison functor is essentially surjective: its essential image is stable under finite colimits and contains all objects of the form $a \otimes x$, hence everything. □

Theorem 13.5.8. *Let P be a finite poset. For any categories C and D with finite colimits, the canonical equivalence from Theorem 9.6.23*

$$\mathrm{Fun}_!(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)), \mathrm{Ind}(D)) \cong \mathrm{Fun}_!(\mathrm{Ind}(C), \mathrm{Fun}(P, \mathrm{Ind}(D)))$$

restricts to a functorial equivalence

$$\mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) \cong \mathrm{Rex}(C, \mathrm{Fun}(P, D)) .$$

Proof. The equivalence is defined by sending any functor $\Phi\colon \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)) \to \mathrm{Ind}(D)$ to the transposition of the functor obtained by composing Φ with the functor

$$P \times \mathrm{Ind}(C) \to \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C))\,, \quad (a, x) \mapsto a \otimes x\,.$$

On the other hand, since $\mathrm{Fun}(P^{\mathrm{op}}, C)$ is the smallest full subcategory of $\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C))$ stable under finite colimits and containing objects of the form $a \otimes x$ for $a\colon P^{\simeq}$ and $x\colon C^{\simeq}$, yielding the equivalence

$$\mathrm{Ind}(\mathrm{Fun}(P^{\mathrm{op}}, C)) = \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C))\,,$$

by Lemma 13.5.6, we may see $\mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D)$ as the full subcategory of the category

$$\mathrm{Fun}_!(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)), \mathrm{Ind}(D))$$

spanned by those functors that send objects of the form $a \otimes x$ to representable presheaves on D by Theorem 13.4.8. We also have similarly

$$\mathrm{Fun}_!(\mathrm{Ind}(C), \mathrm{Fun}(P, \mathrm{Ind}(D))) = \mathrm{Rex}(C, \mathrm{Fun}(P, \mathrm{Ind}(D)))$$

by another application of Theorem 13.4.8.

In other words, we have a pullback square of the form:

$$\begin{tikzcd}
\mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) \arrow[r] \arrow[d, hook] & \mathrm{Rex}(C, \mathrm{Fun}(P, D)) \arrow[d, hook] \\
\mathrm{Fun}_!(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)), \mathrm{Ind}(D)) \arrow[r, "\simeq"] & \mathrm{Fun}_!(\mathrm{Ind}(C), \mathrm{Fun}(P, \mathrm{Ind}(D)))
\end{tikzcd}$$

in which the upper horizontal map must be an equivalence. □

Remark 13.5.9. We can also give an alternative proof of Theorem 13.5.7 using Theorem 13.5.8 instead of Proposition 13.5.2 (exercise left to the reader).

Remark 13.5.10. One can understand the functoriality of the equivalence prodided by the previous theorem. Given a functor between finite posets $u\colon P \to Q$, if C has finite colimits, if $u^*(F) = F \circ u$, there is an adjunction

$$u_!\colon \mathrm{Fun}(P, \mathrm{Ind}(C)) \rightleftarrows \mathrm{Fun}(Q, \mathrm{Ind}(C))\colon u^*$$

and the formula

$$u_!(F)(y) = \operatorname*{colim}_{x\colon P\,,\, u(x) \leq y} F(x)$$

shows that this adjunction resticts to an adjunction of the form

$$u_!\colon \mathrm{Fun}(P, C) \rightleftarrows \mathrm{Fun}(Q, C)\colon u^*\,.$$

There is thus a functor

$$(u^{\mathrm{op}})_!\colon \mathrm{Fun}(P^{\mathrm{op}}, C) \to \mathrm{Fun}(Q^{\mathrm{op}}, C)$$

inducing a functor

$$((u^{\mathrm{op}})_!)^*\colon \mathrm{Rex}(\mathrm{Fun}(Q^{\mathrm{op}}, C), D) \to \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D)\,.$$

For $a\colon P$ and $x\colon C$, we have
$$(u^{\mathrm{op}})_!(a \otimes x) = u(a) \otimes x\,.$$
This means that the functor $((u^{\mathrm{op}})_!)^*$ above corresponds under the correspondence of the previous theorem to the functor of type
$$\mathrm{Rex}(C, \mathrm{Fun}(Q, D)) \to \mathrm{Rex}(C, \mathrm{Fun}(P, D))$$
defined by $\Psi \mapsto u^*\Psi$.

On the other hand, since both $(u^{\mathrm{op}})_!$ and its right adjoint $(u^{\mathrm{op}})^*$ are right exact, the induced adjunction
$$((u^{\mathrm{op}})^*)^*\colon\ \mathrm{Fun}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) \rightleftarrows \mathrm{Fun}(\mathrm{Fun}(Q^{\mathrm{op}}, C), D)\colon\ ((u^{\mathrm{op}})_!)^*$$
resticts to an adjunction
$$((u^{\mathrm{op}})^*)^*\colon \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) \rightleftarrows \mathrm{Rex}(\mathrm{Fun}(Q^{\mathrm{op}}, C), D)\colon\ ((u^{\mathrm{op}})_!)^*\,.$$
Therefore, $((u^{\mathrm{op}})^*)^*$ corresponds to the functor of type
$$\mathrm{Rex}(C, \mathrm{Fun}(P, D)) \to \mathrm{Rex}(C, \mathrm{Fun}(Q, D))$$
defined by $\Phi \mapsto u_!\Phi$.

If furthermore u has a right adjoint $v\colon Q \to P$, then
$$(v^{\mathrm{op}})_! = (u^{\mathrm{op}})^*\,.$$
This means that $(u^{\mathrm{op}})^*$ has a furhter right adjoint, namely $(v^{\mathrm{op}})^*$, that one can interpret through the correspondence of Theorem 13.5.8 as well.

13.6 Free completion by finite colimits

Definition 13.6.1. Let C be a small category. We define $\mathrm{rex}(C)$ as the smallest finitely cocomplete full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ generated by the family of representable presheaves. The Yoneda embedding thus yields a canonical fully faithful functor $h_C\colon C \to \mathrm{rex}(C)$.

Theorem 13.6.2. *For any category D with finite colimits, pre-composing with the Yoneda embedding yields an equivalence of the form*
$$\mathrm{Rex}(\mathrm{rex}(C), D) \cong \mathrm{Fun}(C, D)\,.$$

Proof. We observe that, by virtue of Proposition 13.5.4 and Theorem 13.4.8,
$$\mathrm{Lex}(\mathrm{rex}(C)^{\mathrm{op}}, \mathrm{Grpd}) \cong \mathrm{Fun}(C, \mathrm{Grpd})\,.$$
Given a small category D with finite colimits, the Yoneda embedding
$$h_D\colon D \to \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})$$
induces a full embedding
$$\mathrm{Rex}(\mathrm{rex}(C), D) \hookrightarrow \mathrm{Rex}(\mathrm{rex}(C), \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}))\,.$$

But the codomain of this embedding can be identified with

$$\begin{aligned}\mathrm{Rex}(\mathrm{rex}(C), \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})) &= \mathrm{Fun}_!(\mathrm{Lex}(\mathrm{rex}(C)^{\mathrm{op}}, \mathrm{Grpd}), \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})) \\ &= \mathrm{Fun}_!(\mathrm{Fun}(C, \mathrm{Grpd}), \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})) \\ &= \mathrm{Fun}(C, \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}))\ .\end{aligned}$$

This implies that pre-composing with the Yoneda embedding of C yields a full embedding of the form

$$h_C^*\colon \mathrm{Rex}(\mathrm{rex}(C), D) \hookrightarrow \mathrm{Fun}(C, D)$$

because the latter functor fits in a commutative square of the form

$$\begin{tikzcd} \mathrm{Rex}(\mathrm{rex}(C), D) \arrow[r, "h_C^*"] \arrow[d, hook] & \mathrm{Fun}(C, D) \arrow[d, hook] \\ \mathrm{Rex}(\mathrm{rex}(C), \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})) \arrow[r, equal] & \mathrm{Fun}(C, \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})) \end{tikzcd}$$

in which the two vertical maps are induced by the Yoneda embedding of D.

Let $F\colon C \to D$ be any functor. The functor

$$h_D \circ F\colon C \to \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})$$

extends uniquely to a right exact functor

$$\varphi\colon \mathrm{rex}(C) \to \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})\ .$$

We then form the pullback square below.

$$\begin{tikzcd} C' \arrow[r, "\varphi'"] \arrow[d, hook] & D \arrow[d, hook, "h_D"] \\ \mathrm{rex}(C) \arrow[r, "\varphi"] & \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}) \end{tikzcd}$$

We know that all functors in this square are right exact (applying the dual version of Proposition 9.8.7), and that h_D is fully faithful, hence that C' is a full subcategory of $\mathrm{rex}(C)$ stable under finite colimits and containing representable presheaves on C. Therefore, the full embedding $C' \hookrightarrow \mathrm{rex}(C)$ is an equivalence (see Remark 13.4.3). This shows that the functor F is the restriction of a right exact functor from $\mathrm{rex}(C)$ to D. □

Notation 13.6.3. We define RexCat as the subcategory of Cat whose objects are the small categories with finite colimits and whose morphisms are the right exact functors. Dually, we define LexCat as the subcategory of Cat whose objects are the small categories with finite limits and whose morphisms are the left exact functors. There is a canonical equivalence

$$\mathrm{RexCat} \cong \mathrm{LexCat}\ , \quad C \mapsto C^{\mathrm{op}}\ .$$

Corollary 13.6.4. *The canonical embedding* RexCat $\hookrightarrow$ Cat *has a left adjoint given by the rule* $C \mapsto \mathrm{rex}(C)$.

Lemma 13.6.5. *Let C be a small category with finite colimits and let $\varphi\colon \mathrm{rex}(C) \to C$ be the unique right exact functor extending the identity of C through the correspondence of Theorem 13.6.2. Then φ is a left adjoint of the Yoneda embedding $h_C\colon C \hookrightarrow \mathrm{rex}(C)$. In particular, C canonically is a right exact localization of* $\mathrm{rex}(C)$.

Proof. We have

$$\mathrm{Lex}(\mathrm{rex}(C)^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{Fun}(C, \mathrm{Grpd})$$

by Theorem 13.4.8 for $D = \mathrm{Fun}(C, \mathrm{Grpd})$. Therefore, the functor φ is the restriction of a functor preserving small colimits of the form

$$\varphi_!\colon\ \mathrm{Fun}(C, \mathrm{Grpd}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})\,.$$

The last assertion of Corollary 9.6.12 even tells us that $\varphi_!$ must have a right adjoint φ^*. But, by definition, $\varphi_!$ is the extension by small colimits of the Yoneda embedding

$$C \hookrightarrow \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})\,.$$

In other words, φ^* is nothing else than the canonical inclusion

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(C, \mathrm{Grpd})$$

and $\varphi_!$ is the left adjoint of the latter provided by Proposition 13.3.2.

In particular both functors $\varphi_!$ and φ^* preserve representable left exact presheaves on $\mathrm{rex}(C)$ and C, thus inducing an adjunction at the level of representable (left exact) presheaves. □

Remark 13.6.6. Let C be a small category with finite colimits and W a family of maps in C stable under pushouts and with the 2-out-of-3 property. For a category with finite colimits D, we denote by $\mathrm{Rex}^W(C, D)$ the full subcategory of $\mathrm{Rex}(C, D)$ spanned by right exact functors from C to D sending any map in W to an isomorphism of D. It follows then from assertions (iii) and (iv) of Corollary 13.1.8 that the localization functor $\gamma\colon C \to L(C)$ with respect to W induces an equivalence

$$\mathrm{Rex}^W(C, D) \cong \mathrm{Rex}(L(C), D)\,.$$

Lemma 13.6.7. *The subcategory* RexCat *is stable under small filtered colimits in* Cat*.*

Proof. Let I be a small filtered category and $i \mapsto C_i$ an I-indexed functor with values in RexCat. Let us denote by C the colimit of this diagram in Cat.

We know that $\ulcorner = [0] \star [1]^{\simeq}$ is compact in Cat. Therefore the collection of functors $u\colon\ \ulcorner \to C_i$ forms a cover of the collection of functors of the form $v\colon\ \ulcorner \to C$. Therefore, in order to prove that C has pushouts, it suffices to prove that, for each object i of I, the canonical functor $C_i \to C$ preserves pushout squares. This is an immediate consequence of the description of morphisms in C given in Remark 12.3.13 and of the fact that small filtered colimits are compatible with the formation of pullback squares in Grpd. Similarly, Remark 12.3.13 implies right away that each canonical functor $C_i \to C$ preserves initial objects (in particular, C has an initial object).

Finally, given a small category D and a functor $f\colon C \to D$, the property that f is right exact is equivalent to the property that each restriction $f_i\colon C_i \to D$ is right exact for all $i\colon I^{\simeq}$. This comes from the fact that initial objects of C as well as pushout squares of C come from initial objects of C_i and from pushout squares of C_i for some $i\colon I^{\simeq}$, since $\emptyset$ and $\square = [1] \times [1]$ both are compact in Cat (being finite categories). This implies that C is the colimit of the C_i's in RexCat. □

Theorem 13.6.8. *The category* RexCat *of small categories with finite colimits is compactly generated (in particular, it has both small colimits and small limits).*

Proof. We will prove first that RexCat has small colimits. By virtue of Corollary 13.4.11 and of Lemma 13.6.7, it suffices to prove that it has an initial object as well as pushouts. We check immediately that $[0] = \mathrm{rex}(\emptyset)$ is an initial object (by Theorem 13.6.2, say). Let us consider two right exact functors

$$B \xleftarrow{u} A \xrightarrow{v} C$$

with A, B and C small categories with finite colimits and let us form the following pushout in Cat.

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By virtue of Theorem 13.6.2, for any small category T with finite colimits, the pullback square

$$\begin{tikzcd} \mathrm{Fun}(D,T) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(C,T) \arrow[d] \\ \mathrm{Fun}(B,T) \arrow[r] & \mathrm{Fun}(A,T) \end{tikzcd}$$

yields a pullback square of the form below.

$$\begin{tikzcd} \mathrm{Rex}(\mathrm{rex}(D),T) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Rex}(\mathrm{rex}(C),T) \arrow[d] \\ \mathrm{Rex}(\mathrm{rex}(B),T) \arrow[r] & \mathrm{Rex}(\mathrm{rex}(A),T) \end{tikzcd}$$

Given any small category with finite colimits E, we let $\varphi_E \colon \mathrm{rex}(E) \to E$ be the left adjoint of the Yoneda embedding $E \hookrightarrow \mathrm{rex}(E)$ provided by Lemma 13.6.5. We let W_E be the family of maps in $\mathrm{rex}(E)$ that are sent to isomorphisms in E through φ_E. The family W_E is stable under pushouts in $\mathrm{rex}(E)$ and has the 2-out-of-3 property, exhibiting E as a right exact localization of $\mathrm{rex}(E)$ by W_E. By virtue of Remark 13.6.6, there is thus a canonical equivalence

$$\mathrm{Rex}(E,T) = \mathrm{Rex}^{W_E}(\mathrm{rex}(E),T)\,.$$

We let W' be the image of the map

$$W_B \sqcup W_C \to \mathrm{Map}([1],\mathrm{rex}(D))$$

induced by $\mathrm{rex}(f)$ and $\mathrm{rex}(g)$. We define W as the smallest family of maps in $\mathrm{rex}(D)$ stable under pushouts and satisfying the 2-out-of-3 property containing W'. Remark 13.6.6 yields a canonical equivalence of the form

$$\mathrm{Rex}(\bar{D},T) = \mathrm{Rex}^{W}(\mathrm{rex}(D),T)\,,$$

where $\bar{D}$ denotes the localization of $\mathrm{rex}(D)$ by W. Furthermore, there is a pullback square of the form

$$\begin{tikzcd} \mathrm{Rex}^{W}(\mathrm{rex}(D),T) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Rex}^{W_C}(\mathrm{rex}(C),T) \arrow[d] \\ \mathrm{Rex}^{W_B}(\mathrm{rex}(B),T) \arrow[r] & \mathrm{Rex}^{W_A}(\mathrm{rex}(A),T) \end{tikzcd}$$

hence a canonical pullback square of the following form

$$\begin{tikzcd} \mathrm{Rex}(\bar{D},T) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Rex}(C,T) \arrow[d] \\ \mathrm{Rex}(B,T) \arrow[r] & \mathrm{Rex}(A,T) \end{tikzcd}$$

thus proving that $\bar{D}$ is the pushout of B and C under A in RexCat.

It is clear that the forgetful functor RexCat $\hookrightarrow$ Cat is conservative. Therefore, Corollary 6.3.4, Theorem 13.6.2 and Corollary 13.6.4 tell us that the functor

$$\mathrm{Hom}_{\mathrm{RexCat}}(\mathrm{rex}([1]), -)\colon \mathrm{RexCat} \to \mathrm{Grpd}$$

is conservative. To finish the proof, it suffices to check that $\mathrm{rex}([1])$ is compact in RexCat. Since $[1]$ is compact in Cat, this amounts to checking that the functor RexCat $\hookrightarrow$ Cat commutes with filtered colimits by Proposition 13.4.12, which is precisely what Lemma 13.6.7 does. □

Remark 13.6.9. Let Cat′ be a universe of large categories as in Remark 13.3.13. Let RexCat′ be the category of large categories with finite colimits. Then the embedding

$$\mathrm{RexCat} \hookrightarrow \mathrm{RexCat}'$$

commutes with small colimits as well as with small limits. Indeed, the explicit description of pushouts given in the proof of Theorem 13.6.8 shows that it preserves pushouts. If we consider the smallest full subcategory RexCat_f of RexCat stable under pushouts and containing objects of the form $\mathrm{rex}(A)$ for any finite category A, we then have

$$\mathrm{Lex}(\mathrm{RexCat}_f^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{RexCat} \quad \text{and} \quad \mathrm{Lex}(\mathrm{RexCat}_f^{\mathrm{op}}, \mathrm{Grpd}') = \mathrm{RexCat}'\,,$$

so that we can apply Remark 13.3.13.

Remark 13.6.10. The functor

$$\mathrm{Rex}\colon \mathrm{RexCat}^{\mathrm{op}} \times \mathrm{RexCat} \to \mathrm{Cat}$$

preserves small limits in each variable. To see this, we use the fact that, for any small category A and any small categories with finite colimits B and C, we have

$$\mathrm{Map}(A, \mathrm{RexCat}(B,C)) = \mathrm{Rex}(B, \mathrm{Fun}(A,C))^{\simeq}\,.$$

It thus suffices to recall that

$$\mathrm{Hom}_{\mathrm{RexCat}}(-,-) = \mathrm{RexCat}(-,-)^{\simeq}\colon \mathrm{RexCat}^{\mathrm{op}} \times \mathrm{RexCat} \to \mathrm{Grpd}$$

preserves small limits in each variable and to convince ourselves that

$$\mathrm{Fun}(A,-)\colon \mathrm{RexCat} \to \mathrm{RexCat}$$

commutes with limits. For the latter, this follows from the fact that the embedding

$$\mathrm{RexCat} \hookrightarrow \mathrm{Cat}$$

is conservative and commutes with small limits as well as with the operator $\mathrm{Fun}(A,-)$. Therefore, it suffices to check that

$$\mathrm{Fun}(A,-)\colon \mathrm{Cat} \to \mathrm{Cat}$$

commutes with small limits, which holds true since it is right adjoint to the functor

$$B \mapsto A \times B\,.$$

13.7 Tensor product of finitely cocomplete categories

Construction 13.7.1. Let C and D be two small categories with finite colimits. We have a Yoneda embedding:

$$h\colon C \times D \to \mathrm{rex}(C \times D)\,.$$

We let W be the smallest family of morphisms in $\mathrm{rex}(C \times D)$ stable under pushouts and satisfying the 2-out-of-3 property containing the following kind of maps:

(i) $h(x, y) \to *$ if x is an initial object of C;

(ii) $h(x, y) \to *$ if y is an initial object of D;

(iii) the canonical comparison map $h(x_1, y) \amalg_{h(x_0,y)} h(x_2, y) \to h(x, y)$ for any pushout square

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & x \end{tikzcd}$$

in C and any object y in D;

(iv) the canonical comparison map $h(x, y_1) \amalg_{h(x,y_0)} h(x, y_2) \to h(x, y)$ for any pushout square

$$\begin{tikzcd} y_0 \arrow[r] \arrow[d] & y_2 \arrow[d] \\ y_1 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & y \end{tikzcd}$$

in D and any object x in C.

We define finally

$$C \otimes D := L(\mathrm{rex}(C \times D), W)$$

to be the localization of $\mathrm{rex}(C \times D)$ by W, with localization functor

$$\gamma\colon \mathrm{rex}(C \times D) \to C \otimes D\,.$$

By construction, there is a canonical functor

$$\otimes\colon C \times D \to C \otimes D\,, \quad (x, y) \mapsto x \otimes y := \gamma(h_{C\times D}(x, y))\,.$$

Definition 13.7.2. Let C, D and E be categories with finite colimits. A functor

$$\varphi\colon C \times D \to E$$

is *right exact in each variable* if, for each object $x\colon C^{\simeq}$, the functor

$$\varphi(x, -)\colon D \to E$$

is right exact, and, for each object $y\colon D^{\simeq}$, the functor

$$\varphi(-, y)\colon C \to E$$

is right exact.

We denote by $\mathrm{Fun}^{rex,rex}(C, D; E)$ the full subcategory of $\mathrm{Fun}(C \times D, E)$ spanned by functors that are right exact in each variable.

Example 13.7.3. By construction, for any two categories with finite colimits C and D, the canonical functor

$$\otimes\colon C \times D \to C \otimes D\,, \quad (x, y) \mapsto x \otimes y$$

is right exact in each variable.

Remark 13.7.4. For any B, C and D with finite colimits the identification

$$\mathrm{Fun}(B, \mathrm{Fun}(C, D)) = \mathrm{Fun}(B \times C, D)$$

restricts to:

$$\mathrm{Rex}(B, \mathrm{Rex}(C, D)) = \mathrm{Fun}^{rex,rex}(B, C; D)\,.$$

Theorem 13.7.5. *Given C and D two categories with finite colimits, the assignment*

$$\varphi \mapsto ((x, y) \mapsto \varphi(x \otimes y))$$

defines, for any category E with finite colimits, an equivalence

$$\mathrm{RexCat}(C \otimes D, E) \cong \mathrm{Fun}^{rex,rex}(C, D; E)$$

Proof. As explained in Remark 13.6.6, there is a canonical equivalence:

$$\mathrm{Rex}(C \otimes D, E) \cong \mathrm{RexCat}^{W}(\mathrm{rex}(C \times D), E)\,.$$

On the other hand, we have

$$\mathrm{RexCat}(\mathrm{rex}(C \times D), E) = \mathrm{Fun}(C \times D, E)$$

by Theorem 13.6.2. This produces a canonical full embedding

$$\mathrm{Rex}(C \otimes D, E) \hookrightarrow \mathrm{Fun}(C \times D, E)$$

whose image at the level of objects obviously is $\mathrm{Fun}^{rex,rex}(C, D; E)$, by definition of W. □

Corollary 13.7.6. *Let $u\colon A \to B$ be a right exact functor between small categories with finite colimits. Then, for any compactly generated category C, the functor*

$$u^*\colon \mathrm{Lex}(B^{\mathrm{op}}, C) \to \mathrm{Lex}(A^{\mathrm{op}}, C)$$

defined by composition with u^{op} has a left adjoint

$$u_!\colon \mathrm{Lex}(A^{\mathrm{op}}, C) \to \mathrm{Lex}(B^{\mathrm{op}}, C)\,.$$

Proof. We may assume that $C = \mathrm{Lex}(E^{\mathrm{op}}, \mathrm{Grpd})$ for some small category with finite colimits E by Theorem 13.4.8. We observe that, for any category D with finite limits, we have

$$\mathrm{Lex}(A^{\mathrm{op}}, D) = \mathrm{Rex}(A, D^{\mathrm{op}})^{\mathrm{op}}$$

which allows us to reinterpret Theorem 13.7.5 through

$$\mathrm{Lex}(A^{\mathrm{op}}, C) = \mathrm{Lex}((A \otimes E)^{\mathrm{op}}, \mathrm{Grpd})\,.$$

Doing the same for B, replacing u by the induced functor

$$u \otimes \mathrm{id}_E\colon A \otimes E \to B \otimes E$$

we see that it suffices to consider the case where $C = \mathrm{Grpd}$. This is then a particular case of Theorem 13.3.10. □

Example 13.7.7. We define $\mathrm{FinAn} = \mathrm{rex}([0])$. We thus have

$$\mathrm{RexCat}(\mathrm{FinAn}, C) = C$$

for any category C with finite colimits. For any finite poset P we have

$$\begin{aligned}\mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}), C) &= \mathrm{Rex}(\mathrm{FinAn}, \mathrm{Fun}(P, C)) \\ &= \mathrm{Fun}(P, C) \\ &= \mathrm{Fun}(\mathrm{rex}(P), C)\,.\end{aligned}$$

In other words, we have:

$$\mathrm{rex}(P) = \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn})$$

In general, for any categories with finite colimits C and D, we have:

$$\begin{aligned}\mathrm{Rex}(\mathrm{rex}(P) \otimes C, D) &= \mathrm{Rex}(\mathrm{rex}(P), \mathrm{Rex}(C, D)) \\ &= \mathrm{Fun}(P, \mathrm{Rex}(C, D)) \\ &= \mathrm{Rex}(C, \mathrm{Fun}(P, D)) \\ &= \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D)\,.\end{aligned}$$

In other words, we have

$$\mathrm{rex}(P) \otimes C = \mathrm{Fun}(P^{\mathrm{op}}, C)$$

for all C with finite colimits.

Remark 13.7.8. If A and B are pointed right exact categories, the functor

$$A \times B \to B \times A\,, \quad (a, b) \mapsto (b, a)$$

induces a unique right exact functor

$$\theta_{A,B} \colon A \otimes B \to B \otimes A$$

such that $\theta_{A,B}(a \otimes b) = b \otimes a$ for all terms $a \colon A$ and $b \colon B$. This comes from the fact that the canonical equivalence

$$\mathrm{Fun}(A \times B, C) = \mathrm{Fun}(B \times A, C)$$

restricts to an equivalence of the form

$$\mathrm{Fun}^{rex,rex}(A, B; C) = \mathrm{Fun}^{rex,rex}(B, A; C)\,.$$

The unicity of the functor $\theta_{A,B}$ above implies that it must be an equivalence with inverse $\theta_{B,A}$.

Similarly, for A, B and C pointed right exact categories, the identification

$$(A \times B) \times C = A \times (B \times C)\,, \quad ((x, y), z) \mapsto (x, (y, z))$$

yields a unique identification of the form

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)\,, \quad (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)\,.$$

13.8 Idempotent completion

Definition 13.8.1. Let x and y be two objects of a category C. We say that x is a *retract* of y if there is a morphism of the form $q\colon y \to x$ that admits a section.

A *projector* is a morphism of the form $p\colon y \to y$ for some object y in a category C equipped with an identification $\pi\colon p \cong p^2$. We say that such a projector *has an image* if there exists an object x together with morphisms $q\colon y \to x$ and $s\colon x \to y$ such that s is a section of q, $p \cong s \circ q$ and $\pi\colon p \cong p^2$ is isomorphic to the identification coming from

$$\begin{aligned} p &= s \circ q \\ &= s \circ \mathrm{id}_x \circ q \\ &= s \circ q \circ s \circ q \\ &= p \circ p\,. \end{aligned}$$

Lemma 13.8.2. *If C is a category with ω-indexed colimits, then any projector has an image in C.*

Proof. Let $p\colon y \to y$ be a projector. We denote by Y the ω-indexed diagram corresponding to the sequence

$$y \xrightarrow{\mathrm{id}_y} y \xrightarrow{\mathrm{id}_y} y \to \cdots \to y \xrightarrow{\mathrm{id}_y} y \xrightarrow{\mathrm{id}_y} y \to \cdots$$

We then define an ω-indexed diagram X by forming the sequence of morphisms below.

$$y \xrightarrow{p} y \xrightarrow{p} y \to \cdots \to y \xrightarrow{p} y \xrightarrow{p} y \to \cdots$$

We clearly have

$$y = \operatorname*{colim}_{\omega} Y$$

and we define

$$x := \operatorname*{colim}_{\omega} X\,.$$

The sequence of commutative squares

$$\begin{tikzcd} y \arrow[r, "\mathrm{id}_y"] \arrow[d, "p"] & y \arrow[r, "\mathrm{id}_y"] \arrow[d, "p"] & y \arrow[r, "\mathrm{id}_y"] \arrow[d, "p"] & y \arrow[r] \arrow[d, "p"] & \cdots \\ y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r] & \cdots \end{tikzcd}$$

defines a map $Q\colon Y \to X$ and similarly,

$$\begin{tikzcd} y \arrow[r, "p"] \arrow[d, "p"] & y \arrow[r, "p"] \arrow[d, "p"] & y \arrow[r, "p"] \arrow[d, "p"] & y \arrow[r] \arrow[d, "p"] & \cdots \\ y \arrow[r, "\mathrm{id}_y"] & y \arrow[r, "\mathrm{id}_y"] & y \arrow[r, "\mathrm{id}_y"] & y \arrow[r] & \cdots \end{tikzcd}$$

defines a map $S\colon X \to Y$. This yields two maps

$$q\colon y \to x \quad \text{and} \quad s\colon x \to y$$

by passing to colimits. It remains to prove that s is a section of q. The composition $q \circ s$ is obtained from $Q \circ S$ by taking the colimit. But $Q \circ S$ simply is the sequence of maps

$$\begin{tikzcd}
y \arrow[r, "p"] \arrow[d, "p^2"] & y \arrow[r, "p"] \arrow[d, "p^2"] & y \arrow[r, "p"] \arrow[d, "p^2"] & y \arrow[r] \arrow[d, "p^2"] & \cdots \\
y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r] & \cdots
\end{tikzcd}$$

and $\pi\colon\ p \cong p^2$ yields an isomorphism from $Q \circ S$ to the sequence of maps

$$\begin{tikzcd}
y \arrow[r, "p"] \arrow[d, "p"] & y \arrow[r, "p"] \arrow[d, "p"] & y \arrow[r, "p"] \arrow[d, "p"] & y \arrow[r] \arrow[d, "p"] & \cdots \\
y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r] & \cdots
\end{tikzcd}$$

But the colimit of the latter sequence clearly is the identity of x, since $\omega_{1/} \to \omega$ is a terminal functor. □

Lemma 13.8.3. *Let C be a category and $q\colon\ y \to x$ a map in C equipped with a section $s\colon\ x \to y$. If we define $p := s \circ q$, then x canonically is the colimit of the diagram*

$$y \xrightarrow{p} y \xrightarrow{p} y \to \cdots \to y \xrightarrow{p} y \xrightarrow{p} y \to \cdots$$

Proof. We may assume that C is small. It suffices to prove the lemma for the image of the diagram through the Yoneda embedding of C: since the Yoneda embedding is fully faithful, if an object of C happens to be a colimit in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ of a diagram taking values in C, it must be a colimit in C as well. Therefore, we may assume that C has ω-indexed colimits. Let x' be the colimit of the diagram as stated in the lemma. The sequence of commutative squares defined by $q \circ p = q \circ s \circ q = q$

$$\begin{tikzcd}
y \arrow[r, "p"] \arrow[d, "q"] & y \arrow[r, "p"] \arrow[d, "q"] & y \arrow[r, "p"] \arrow[d, "q"] & y \arrow[r] \arrow[d, "q"] & \cdots \\
x \arrow[r, "\mathrm{id}_x"] & x \arrow[r, "\mathrm{id}_x"] & x \arrow[r, "\mathrm{id}_x"] & x \arrow[r] & \cdots
\end{tikzcd}$$

induces a map $u\colon\ x' \to x$. Similarly, the sequence of commutative squares associated to the relation $p \circ s = s \circ q \circ s = s$

$$\begin{tikzcd}
x \arrow[r, "\mathrm{id}_x"] \arrow[d, "s"] & x \arrow[r, "\mathrm{id}_x"] \arrow[d, "s"] & x \arrow[r, "\mathrm{id}_x"] \arrow[d, "s"] & x \arrow[r] \arrow[d, "s"] & \cdots \\
y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r] & \cdots
\end{tikzcd}$$

induces a map $v\colon\ x \to x'$. It is clear that $u \circ v = \mathrm{id}_x$ by functoriality, since $q \circ s = \mathrm{id}_x$. Finally, the composition $v \circ u$ is the colimit of the diagram

$$\begin{tikzcd}
y \arrow[r, "p"] \arrow[d, "p"] & y \arrow[r, "p"] \arrow[d, "p"] & y \arrow[r, "p"] \arrow[d, "p"] & y \arrow[r] \arrow[d, "p"] & \cdots \\
y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r, "p"] & y \arrow[r] & \cdots
\end{tikzcd}$$

that is the identity of x', by a cofinality argument as at the end of the proof of Lemma 13.8.2. □

Proposition 13.8.4. *Let C be a small category with finite colimits. Any retract of a representable presheaf on C is a compact object in* $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$. *Conversely, the collection of retracts of representable presheaves covers the collection of compact objects in* $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$.

Proof. Let y be an object of C and $q\colon \mathrm{Hom}_C(-, y) \to F$ a map in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ equipped with a section $s\colon F \to \mathrm{Hom}_C(-, y)$. Using Lemma 13.8.3 and the stability of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ by small filtered colimits, we see that F is a left exact functor (one can also see this rather directly from the definitions, using that any retract of an equivalence is an equivalence). Let us prove that F is a compact object. Let I be a small filtered category and $i \mapsto G_i$ an I-indexed functor with values in $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$. Then the canonical comparison map

$$\operatorname*{colim}_{i\colon I} \mathrm{Hom}(F, G_i) \to \mathrm{Hom}(F, \operatorname*{colim}_{i\colon I} G_i)$$

is a retract of

$$\operatorname*{colim}_{i\colon I} \mathrm{Hom}(h_C(y), G_i) \to \mathrm{Hom}(h_C(y), \operatorname*{colim}_{i\colon I} G_i)$$

(by a straightforward functoriality argument). The latter is an equivalence by the Yoneda lemma and the fact that small filtered colimits in $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ are computed objectwise (since they can be computed in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$).

Conversely, let F be a compact object of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$. By virtue of Lemma 13.3.6, there is a small filtered category I, a functor $\varphi\colon I \to C$ and an isomorphism of the form

$$\operatorname*{colim}_{i\colon I} h_C(\varphi(i)) \cong F\,.$$

Since F is compact, we have an equivalence:

$$\operatorname*{colim}_{i\colon I} \mathrm{Hom}(F, h_C(\varphi(i))) \cong \mathrm{Hom}(F, F)\,.$$

Since we have a canonical cover of the form

$$\sum_{i\colon I^{\simeq}} \mathrm{Hom}(F, h_C(\varphi(i))) \to \operatorname*{colim}_{i\colon I} \mathrm{Hom}(F, h_C(\varphi(i)))$$

this means that there exists $i\colon I^{\simeq}$ and a map $q\colon F \to h_C(\varphi(i))$ such that its composition with the structural map

$$h_C(\varphi(i)) \to \operatorname*{colim}_{i\colon I} h_C(\varphi(i)) \cong F$$

is the identity of F. □

Corollary 13.8.5. *Let D be a compactly generated category. Then the full subcategory D_ω of compact objects of D is small.*

Proof. By virtue of Theorem 13.4.8, we may assume that $D = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ for some small category C with finite colimits. Since the collection of retracts of representable presheaves forms a small family, we conclude with Proposition 13.8.4 and Proposition 11.5.17. □

Definition 13.8.6. A category C is called *idempotent complete* if any projector in C has an image.

Proposition 13.8.7. *A small category with finite colimits C is idempotent complete if and only if any compact object of* $\mathrm{Ind}(C)$ *is representable.*

Proof. If C is idempotent complete, then it follows from Proposition 13.8.4 that any compact object of $\mathrm{Ind}(C) = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ is representable. Conversely, any projector in C has an image in $\mathrm{Ind}(C)$ by Lemma 13.8.2, and such an image will be a compact object of $\mathrm{Ind}(C)$ by Proposition 13.8.4. Therefore, if any compact object of $\mathrm{Ind}(C)$ is representable, any projector of C has an image in C. □

Definition 13.8.8. Let us choose a universe of large categories Cat' (so that Cat is both an object and a full subcategory of Cat'). We define $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ as the subcategory of Cat' whose objects are the compactly generated categories and whose morphisms are the functors that commute with small colimits and that preserve compact objects.

Given A and B two compactly generated categories, we denote by $\mathrm{Fun}^{\mathrm{L}}_{\omega}(A, B)$ the full subcategory of $\mathrm{Fun}(A, B)$ spanned by the functors that preserve small colimits as well as compact objects. In particular, we have:

$$\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_{\omega}}(A, B) = \mathrm{Fun}^{\mathrm{L}}_{\omega}(A, B)^{\simeq} .$$

Theorem 13.8.9. *Given a small category with finite colimits C and a compactly generated category D, there is a canonical equivalence of categories of the form*

$$\mathrm{Rex}(C, D_{\omega}) \cong \mathrm{Fun}^{\mathrm{L}}_{\omega}(\mathrm{Ind}(C), D) .$$

In particular, assigning the full subcategory of compact objects defines a fully faithful functor

$$\mathrm{Pr}^{\mathrm{L}}_{\omega} \to \mathrm{RexCat} , \quad D \mapsto D_{\omega}$$

that is right adjoint to the functor

$$\mathrm{Ind} \colon \mathrm{RexCat} \to \mathrm{Pr}^{\mathrm{L}}_{\omega} , \quad C \mapsto \mathrm{Ind}(C) .$$

Proof. Corollary 13.8.5 and Theorem 13.4.8 ensure that

$$\mathrm{Lex}(D_{\omega}^{\mathrm{op}}, \mathrm{Grpd}) = D .$$

By definition, there is pullback square of the form

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{L}}_{\omega}(\mathrm{Ind}(C), D) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(C), D) \arrow[d, "\simeq"] \\
\mathrm{Rex}(C, D_{\omega}) \arrow[r, hook] & \mathrm{Rex}(C, D)
\end{tikzcd}$$

in which, by virtue of Theorem 13.3.12 and Remark 13.3.8, the vertical arrow on the right side is an equivalence. This proves the first assertion of the theorem. The second one follows from the first by applying the operator $(-)^{\simeq}$. □

Corollary 13.8.10. *The category $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ is locally small and has small limits as well as small colimits.*

Lemma 13.8.11. *The full embedding $\mathrm{Pr}^{\mathrm{L}}_{\omega} \hookrightarrow \mathrm{RexCat}$ commutes with small filtered colimits.*

Proof. Let I be a small filtered category and $D \colon I \to \mathrm{Pr}^{\mathrm{L}}_{\omega}$ a functor. We define

$$C := \operatorname*{colim}_{i \colon I} D(i)_{\omega} .$$

For any compactly generated category E we have:

$$\begin{aligned}
\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(C), E) &= \mathrm{Rex}(C, E) \\
&= \mathrm{Rex}(\operatorname*{colim}_{i\colon I} D(i)_\omega, E) \\
&= \lim_{i\colon I^{\mathrm{op}}} \mathrm{Rex}(D(i)_\omega, E) \\
&= \lim_{i\colon I^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{L}}(D(i), E)
\end{aligned}$$

hence an equivalence

$$\mathrm{Fun}^{\mathrm{L}}_\omega(\mathrm{Ind}(C), E) \cong \lim_{i\colon I^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{L}}_\omega(D(i), E)$$

In particular, the colimit of $i \mapsto D(i)$ in $\mathrm{Pr}^{\mathrm{L}}_\omega$ simply is $\mathrm{Ind}(C)$. The lemma thus simply means that C is idempotent complete by Proposition 13.8.7. But, using that $[2]$ is compact in Cat, since I is filtered, we see that the collection of projectors in C is the colimit of the diagram assigning to each $i\colon I$ the collection of projectors in $D(i)$. The fact that each projector has an image in each $D(i)$ thus implies that each projector in C has an image in C as well. □

Proposition 13.8.12. *The category* $\mathrm{Pr}^{\mathrm{L}}_\omega$ *is compactly generated:*

$$\mathrm{Pr}^{\mathrm{L}}_\omega \colon (\mathrm{Pr}^{\mathrm{L}}_\omega)^{\simeq}\,.$$

Proof. The preceding lemma allows us to apply Proposition 13.4.12. □

Remark 13.8.13. The functor $C \mapsto \mathrm{Ind}(C)$ restricts to an equivalence from the full subcategory of RexCat spanned by small idempotent complete categories with small colimits to $\mathrm{Pr}^{\mathrm{L}}_\omega$: this is what the combination of Theorem 13.8.9 and of Proposition 13.8.7 means. In other words, for any small category with finite colimits C, the Yoneda embedding induces a fully faithful right exact functor $C \hookrightarrow \mathrm{Ind}(C)_\omega$ that is the universal right exact functor of C to a small idempotent complete category with finite colimits. We call $\mathrm{Ind}(C)_\omega$ the *idempotent completion* of C.

Lemma 13.8.14. *For any finite poset P, the category* $\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn})$ *is a compact object of* RexCat*.*

Proof. For any small category with finite colimits C, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{RexCat}}(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}), C) &= \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}), C)^{\simeq} \\
&= \mathrm{Rex}(\mathrm{FinAn}, \mathrm{Fun}(P, C))^{\simeq} \\
&= \mathrm{Fun}(P, C)^{\simeq} \\
&= \mathrm{Hom}_{\mathrm{Cat}}(P, C)\,.
\end{aligned}$$

Since the forgetful functor RexCat $\hookrightarrow$ Cat commutes with small filtered colimits, the lemma follows from the fact that finite posets are compact objects of Cat (since they are finite categories). □

Proposition 13.8.15. *If A and B are compact in* RexCat*, then so is $A \otimes B$.*

Proof. If there are finite posets P and Q such that $A = \mathrm{rex}(P)$ and $B = \mathrm{rex}(Q)$ respectively, then we have

$$A \otimes B = \mathrm{rex}(P \times Q)$$

and since $P \times Q$ is a finite poset, this particular case follows from the previous lemma.

On the other hand, we know that $\mathrm{rex}([1])$ is a compact generator of RexCat. Therefore, applying Proposition 13.8.4 in the case where $C = \mathrm{RexCat}$, for a full subcategory of RexCat to contain all compact objects of RexCat, it suffices that it is stable under finite colimits, under the formation of images of projectors, and contains $\mathrm{rex}([1])$.

Given A in RexCat, since $A \otimes (-)$ commutes with small colimits, the full subcategory X_A of RexCat spanned by those small categories with finite colimits B such that $A \otimes B$ is compact is stable under finite colimits as well as under the formation of images of projectors (these being a particular kind of colimits, as explained in the proof of Lemma 13.8.2). If $A = \mathrm{rex}(P)$ for some finite poset P, we deduce that any compact object of RexCat belongs to X_A. Since $A \otimes B = B \otimes A$, this means that, for A an arbitrary compact object of RexCat, X_A contains all categories of the form $\mathrm{rex}(P)$ for any finite poset. Therefore, for any compact object A of RexCat, X_A contains all compact objects of RexCat, and this proves the proposition. $\square$

Corollary 13.8.16. *A small category with finite colimits A is compact in* RexCat *if and only if the functor*

$$\mathrm{Rex}(A, -) \colon \mathrm{RexCat} \to \mathrm{Cat}$$

commutes with small filtered colimits.

Proof. Since RexCat is compactly generated, this follows straight away from the formula

$$\mathrm{Hom}_{\mathrm{RexCat}}(A \otimes B, C) = \mathrm{Hom}_{\mathrm{RexCat}}(A, \mathrm{Rex}(B, C))$$

and from Proposition 13.8.15. $\square$

13.9 Pointed categories with pushouts

Definition 13.9.1. A small *pointed right exact category* is a small category C with pushouts as well as with an initial object that is also terminal. We denote by $\mathrm{RexCat}_\bullet$ the full subcategory of RexCat spanned by small pointed right exact categories.

Construction 13.9.2. Let $*$ be the terminal object of Grpd. We know that $\mathrm{Grpd}_\bullet = \mathrm{Grpd}_{*/}$ and that forgetting the base point is the universal left fibration with small fibers:

$$\pi_{\mathrm{univ}} \colon \mathrm{Grpd}_\bullet \to \mathrm{Grpd}\,.$$

This functor has a left adjoint

$$\mathrm{Grpd} \to \mathrm{Grpd}_\bullet\,, \quad x \mapsto x_+$$

where $x_+ = x \amalg *$ is equipped with the canonical embedding $* \to x \amalg *$. We define the *pointed* 0*-sphere* as $S^0 := *_+$. We observe that the functor π_{univ} commutes with small filtered colimits and is conservative (being a left fibration). Since Grpd is compactly generated by the point $*$, this shows that $\mathrm{Grpd}_\bullet$ is compactly generated by S^0, by virtue of Proposition 13.4.12.

Let C be a small category with finite colimits. We let $*$ be the terminal object of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ and observe that the canonical equivalence

$$\mathrm{Grpd}_{*/} = \mathrm{Grpd}_\bullet$$

induces an equivalence:

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})_{*/} = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)\,.$$

In particular, $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$ is compactly generated by Proposition 13.5.4. The forgetful functor

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$$

has a left adjoint

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)\,, \quad x \mapsto x_+\,.$$

We let $C_\bullet^\simeq$ be the image of the map

$$C^\simeq \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)^\simeq\,, \quad x \mapsto h_C(x)_+$$

and we define $C_\bullet$ as the smallest full subcategory of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$ stable under finite colimits containing the terms of $C_\bullet^\simeq$. There is a canonical right exact functor

$$C \to C_\bullet\,, \quad x \mapsto h_C(x)_+$$

with $C_\bullet$ pointed.

Proposition 13.9.3. *Given a small category with finite colimits C and a pointed right exact category D, pre-composing with the canonical right exact functor $C \to C_\bullet$ induces an equivalence of the form*

$$\mathrm{Rex}(C_\bullet, D) \cong \mathrm{Rex}(C, D)\,.$$

In particular, the full embedding $\mathrm{RexCat}_\bullet \hookrightarrow \mathrm{RexCat}$ *has a left adjoint*

$$C \mapsto C_\bullet\,.$$

Proof. It suffices to prove that, for any category A, we have an equivalence of the form

$$\mathrm{Map}(A, \mathrm{Rex}(C_\bullet, D)) \cong \mathrm{Map}(A, \mathrm{Rex}(C, D))\,.$$

Using the identification

$$\mathrm{Map}(A, \mathrm{Rex}(C, D)) = \mathrm{Rex}(C, \mathrm{Fun}(A, D))^\simeq\,,$$

we see that it suffices to prove that we get an equivalence of the form:

$$\mathrm{Rex}(C_\bullet, D)^\simeq \cong \mathrm{Rex}(C, D)^\simeq\,.$$

In other words, it suffices to prove the last assertion of the proposition.

We may assume that D is small (up to enlarging the ambient universe). Since D is also pointed, we observe that the canonical right exact functor from D to $D_\bullet$ is an equivalence. Indeed, there is an equivalence

$$\mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \cong \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})$$

due to the fact that the Yoneda embedding is exact, hence sends the initial object of D to an object $*$ that is both initial and terminal, which allows us to use the canonical identification

$$\mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd})_{*/} = \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}_\bullet)\,.$$

This implies that the canonical functor $D \to D_\bullet$ is fully faithful (being the restriction of an equivalence of categories). On the other hand, we know that its essential image is stable under finite colimits and contains the image of any object of D, which proves the essential surjectivity, by definition of $D_\bullet$. Any right exact functor $f\colon C \to D$ extends to small colimit preserving functors

$$f_!\colon \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}) \text{ and } f_!\colon \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \to \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$$

by Corollary 13.7.6. Since the functor $x \mapsto x_+$ commutes with colimits, we have a commutative square of the form

$$\begin{tikzcd} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r] \arrow[d, "f_!"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \arrow[d, "f_!"] \\ \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r] & \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \end{tikzcd}$$

by Corollary 9.7.7 and this restricts to a commutative square of the form

$$\begin{tikzcd} \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r] \arrow[d, "f_!"'] & \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \arrow[d, "f_!"] \\ \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, equal] & \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \end{tikzcd}$$

through an easy application of Theorem 13.3.10. By definition of $C_\bullet$, this induces a commutative square of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"'] & C_\bullet \arrow[d, "f_\bullet"] \\ D \arrow[r, equal] & D_\bullet \end{tikzcd}$$

This shows that the map

$$\mathrm{Rex}(C_\bullet, D)^{\simeq} \to \mathrm{Rex}(C, D)^{\simeq}$$

has a section. We will now prove that this is an embedding. Observe first that, by virtue of Theorem 13.3.12, the factorization of f through $C_\bullet$ is totally determined by the factorization of

$$f_!\colon \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$$

through $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$. The right adjoint of the latter is the composition of the functor

$$f^*\colon \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$$

with the forgetful functor

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})\,.$$

The uniqueness of left adjoints and the compatibility of the formation of adjoints with respect to composition shows that the factorization through $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$ is unique. This implies that the factorization of f through $C_\bullet$ is unique. □

Corollary 13.9.4. *The category* RexCat$_\bullet$ *has small colimits as well as small limits.*

Proof. This follows from the preceding proposition and from Corollary 10.2.4. □

Proposition 13.9.5. *The full embedding* RexCat$_\bullet \hookrightarrow$ RexCat *commutes with small filtered colimits.*

Proof. Let I be a small filtered category and $i \mapsto C_i$ an I-indexed diagram with values in RexCat. We assume that, for any object i of I, the initial object $*$ of C_i is also a terminal object of C_i. We thus have pullback squares of the form

$$\begin{tikzcd} C_i \arrow[r, "x\mapsto \mathrm{id}_x"] \arrow[d] & \mathrm{Fun}([1], C_i) \arrow[d, "\mathrm{ev}_1"] \\ {[0]} \arrow[r, "*"] & C_i \end{tikzcd}$$

functorially in i. Since the functor $\mathrm{Fun}([1], -)$ commutes with small filtered colimits in Cat and since small filtered colimits are compatible with the formation of pullbacks in Cat, Lemma 13.6.7 implies that the colimit C of the diagram $i \mapsto C_i$ in RexCat yields a pullback square in Cat of the form

$$\begin{tikzcd} C \arrow[r, "x\mapsto \mathrm{id}_x"] \arrow[d] & \mathrm{Fun}([1], C) \arrow[d, "\mathrm{ev}_1"] \\ {[0]} \arrow[r, "*"] & C \end{tikzcd}$$

thus proving that the initial object $*$ of C induces an equivalence $C_{/*} \cong C$. This proves that C belongs to $\mathrm{RexCat}_\bullet$ and implies that C is a colimit of diagram $i \mapsto C_i$ in $\mathrm{RexCat}_\bullet$ as well. □

Corollary 13.9.6. *The category* $\mathrm{RexCat}_\bullet$ *is compactly generated.*

Proof. This follows from Proposition 13.4.12 and from the preceding proposition. □

Proposition 13.9.7. *Let C be a small category. For any pointed right exact category D, there is a canonical equivalence*

$$\mathrm{Rex}(\mathrm{rex}(C)_\bullet, D) \cong \mathrm{Fun}(C, D)\,.$$

Proof. This is the conjunction of Theorem 13.6.2 and of Proposition 13.9.3. □

Remark 13.9.8. Let C be a category with pushouts. We denote by

$$1^* = \mathrm{ev}_1 \colon \mathrm{Fun}([1], C) \to C$$

the functor assigning to a map of C its codomain. The existence of an initial object $*$ in C amounts to the existence of a left adjoint

$$1_! \colon C \to \mathrm{Fun}([1], C)$$

of 1^*. Indeed, if $*$ is initial in C, given a morphism $u \colon x \to y$ in C and a term a in C, there is an obvious correspondence of the form

$$\begin{tikzcd} * \arrow[r] \arrow[d] & x \arrow[d, "u"] \\ a \arrow[r, "f"] & y \end{tikzcd} \quad \Leftrightarrow \quad f \colon a \to y\,.$$

Conversely, a correspondence as above for some object $* \colon C^{\simeq}$ implies in the particular case $a := y$ and $u := \mathrm{id}_y$ that there is a unique map from $*$ to y for any $y \colon C^{\simeq}$. This computation also shows that the left adjoint $1_!$ sends an object x to the unique map of type $* \to x$.

The property that the initial object $*$ of C is also terminal is equivalent to the property that the functor $1_!$ itself has a left adjoint

$$1^? \colon \operatorname{Fun}([1], C) \to C\,.$$

Indeed, if $*$ is also terminal, the dual version of the preceding observation tells us that the functor

$$0^* = \mathrm{ev}_0 \colon \operatorname{Fun}([1], C) \to C$$

has a right adjoint

$$0_* \colon C \to \operatorname{Fun}([1], C)$$

assigning to each term x of C the unique map of type $x \to *$. The identity of $*$ being both an initial object and a terminal object of $\operatorname{Fun}([1], C)$, the functor

$$0_* \colon \operatorname{Fun}([1], C) \to \operatorname{Fun}([1], \operatorname{Fun}([1], C))$$

yields the assignment

$$\begin{tikzcd} x \arrow[d, "u"'] & & x \arrow[r] \arrow[d, "u"'] & * \arrow[d, equal] \\ y & \mapsto & y \arrow[r] & * \end{tikzcd}$$

Since C has pushouts, the full embedding $i \colon \ulcorner \hookrightarrow \square$ induces an adjunction

$$i_! \colon \operatorname{Fun}(\ulcorner, C) \rightleftarrows \operatorname{Fun}(\square, C) : i^*$$

where $i_!$ sends a diagram of the form $b \leftarrow a \to c$ in C to the corresponding pushout square

$$\begin{tikzcd} a \arrow[r] \arrow[d] & c \arrow[d] \\ b \arrow[r] & b \amalg_a c\,. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

With these conventions, the functor $1^?$ is constructed as $i_! \circ i^* \circ 0_*$. In other words, it sends a map $u \colon x \to y$ in C to the pushout square

$$\begin{tikzcd} x \arrow[r] \arrow[d, "u"'] & * \arrow[d] \\ y \arrow[r] & \operatorname{cofib}(u)\,. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The object $1^?(u) = \operatorname{cofib}(u)$, also known as the *cofiber* of the morphism u, is thus the evaluation at the terminal object of $\square = [1] \times [1]$ of $i_!(0_*(u))$.

Conversely, if C has an initial object and if $1_!$ has a further left adjoint $1^?$, since $1_!$ is fully faithful with image those maps of the form $* \to x$, applying 0^* to the unit map

$$\eta_u \colon u \to 1_!\, 1^?(u)$$

yields for $u = \mathrm{id}_x$ a unique map $x \to *$ for all $x \colon C^{\simeq}$, hence $*$ is a terminal object of C.

Definition 13.9.9. Let C be a pointed right exact category with initial object $*$. The *suspension functor*

$$\Sigma\colon C \to C$$

is defined by forming for each term $x\colon C$ the following pushout square

$$\begin{tikzcd} x \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma(x)\,. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We warn the reader that, despite what the notation suggests, although the two maps from $*$ to $\Sigma(x)$ are the same (since $*$ is initial), the commutativity constraint making this square commutative is not the identity in general.

Proposition 13.9.10. *The suspension functor* $\Sigma\colon C \to C$ *is right exact.*

Proof. We adopt the notation of Remark 13.9.8. The functor 0_* is right exact because the functor $x \mapsto *$ is a right exact functor and so is the identity functor $x \mapsto x$. The cofiber functor $1^? = \mathrm{cofib}$ is right exact because it is a left adjoint. Therefore, $\Sigma = 1^? \circ 0_*$ is right exact. □

Proposition 13.9.11. *For any right exact functor between pointed right exact categories* $f\colon C \to D$*, there is a canonical isomorphism* $\Sigma f(x) \cong f(\Sigma(x))$ *functorially in* $x\colon C$*.*

Proof. The functor f preserves initial objects hence preserves terminal objects as well and thus commutes with the operator 0_*. Since it is right exact, it is compatible with the formation of pushouts, hence of cofibers. □

Remark 13.9.12. We define $\mathrm{FinAn} := \mathrm{rex}([0])$, so that

$$\mathrm{Ind}(\mathrm{FinAn}) = \mathrm{Grpd}\,.$$

For any category C with finite colimits, we thus have

$$\mathrm{Rex}(\mathrm{FinAn}, C) = C$$

by Theorem 13.6.2. If C is pointed, we even get by Proposition 13.9.7:

$$\mathrm{Rex}(\mathrm{FinAn}_\bullet, C) = C\,.$$

The suspension functor

$$\Sigma\colon \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet$$

thus induces a functor

$$\Sigma^*\colon \mathrm{Rex}(\mathrm{FinAn}_\bullet, C) \to \mathrm{Rex}(\mathrm{FinAn}_\bullet, C)\,.$$

Lemma 13.9.13. *Given a fixed pointed right exact category* C*, under the identification* $\mathrm{Rex}(\mathrm{FinAn}_\bullet, C) = C$*, the two functors*

$$\Sigma^*\colon \mathrm{Rex}(\mathrm{FinAn}_\bullet, C) \to \mathrm{Rex}(\mathrm{FinAn}_\bullet, C) \quad \textit{and} \quad \Sigma\colon C \to C$$

agree. More generally, for any pointed right exact category A*, the suspension functor*

$$\Sigma\colon \mathrm{Rex}(A, C) \to \mathrm{Rex}(A, C)$$

agrees with the operation of pre-composition with the suspension functor $\Sigma\colon A \to A$*.*

Proof. The last assertion means that, for any right exact functor $\varphi\colon A \to C$, we have

$$\Sigma(\varphi(a)) \cong \varphi(\Sigma(a))$$

functorially in $a\colon A$. This is precisely what the preceding proposition claims to be true. □

Proposition 13.9.14. *If A and B are small right exact categories, and if either A or B is pointed, then $A \otimes B$ is pointed.*

Proof. Being pointed for a small right exact category C is equivalent to the property that the canonical functor $C \to C_\bullet$ is fully faithful. Indeed, if this is the case, then its essential image is the smallest full subcategory of $C_\bullet$ stable under pushouts containing objects coming from C, which is $C_\bullet$ by definition. Equivalently, we deduce right away from Theorem 13.4.8 that this means that the functor $x \mapsto x_+$

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$$

is an equivalence. If B is pointed, we have

$$\begin{aligned}\mathrm{Lex}((A \otimes B)^{\mathrm{op}}, \mathrm{Grpd}) &= \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Grpd}))\\ &= \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Grpd}_\bullet))\\ &= \mathrm{Lex}((A \otimes B)^{\mathrm{op}}, \mathrm{Grpd}_\bullet)\ .\end{aligned}$$

Therefore, $A \otimes B$ is pointed whenever B is pointed. The case where A is pointed is either proved similarly or deduced from what precedes using Remark 13.7.8. □

Proposition 13.9.15. *Let A and B be two small categories with finite colimits. The canonical right exact functor $(-)_+\colon B \to B_\bullet$ induces a functor*

$$A \otimes B \to A \otimes B_\bullet\ , \quad x \otimes y \mapsto x \otimes y_+$$

that exhibits $A \otimes B_\bullet$ as the free pointed right exact category associated to $A \otimes B$ and thus induces a canonical equivalence

$$(A \otimes B)_\bullet \cong A \otimes B_\bullet\ .$$

Proof. We have

$$\begin{aligned}\mathrm{Rex}(A \otimes B, (A \otimes B)_\bullet) &= \mathrm{Rex}(A, \mathrm{Rex}(B, (A \otimes B)_\bullet))\\ &= \mathrm{Rex}(A, \mathrm{Rex}(B_\bullet, (A \otimes B)_\bullet))\\ &= \mathrm{Rex}(A \otimes B_\bullet, (A \otimes B)_\bullet)\ .\end{aligned}$$

There is thus a unique right exact functor $u\colon A \otimes B_\bullet \to (A \otimes B)_\bullet$ sending $x \otimes y_+$ to $(x \otimes y)_+$ for all $x\colon A$ and $y\colon B$. Similarly, the universal property of $(A \otimes B)_\bullet$ and Proposition 13.9.14 provide a functor $v\colon (A \otimes B)_\bullet \to A \otimes B_\bullet$ uniquely determined by the property that it sends $(x \otimes y)_+$ to $x \otimes y_+$ for all $x\colon A$ and $y\colon B$. Therefore u and v must be inverse to each other. □

Remark 13.9.16. Given a pointed right exact category C, the suspension functor

$$\Sigma\colon C \to C$$

agrees with the functor

$$\Sigma \otimes \mathrm{id}_C : \mathrm{FinAn}_\bullet \otimes C \to \mathrm{FinAn}_\bullet \otimes C$$

induced by the suspension $\Sigma: \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet$ up to the canonical equivalences

$$C \cong \mathrm{FinAn} \otimes C \cong \mathrm{FinAn}_\bullet \otimes C\,.$$

Indeed, for $S^0 := *_+$ and $S^1 := \Sigma(S^0)$, we see with Proposition 13.9.11 that

$$S^1 \otimes x = S^0 \otimes \Sigma(x)$$

for all $x : C$. Since $\mathrm{Rex}(\mathrm{FinAn}_\bullet, C) = C$ through evaluation at S^0, this proves the claim. This identification is useful to prove the following proposition.

Proposition 13.9.17. *The suspension functor can be promoted to an endomorphism of the identity of* $\mathrm{RexCat}_\bullet$ *in the category* $\mathrm{Fun}(\mathrm{RexCat}_\bullet, \mathrm{RexCat}_\bullet)$. *In other words, we get a commutativity constraint*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Sigma"'] & D \arrow[d, "\Sigma"] \\ C \arrow[r, "f"] & D \end{tikzcd} \qquad \kappa_f : \Sigma \circ f \xrightarrow{\simeq} f \circ \Sigma$$

for any right exact functor f, functorially in f.

Proof. There is a functorial identification

$$a_C : \mathrm{FinAn}_\bullet \otimes C \cong C$$

and the suspension $\Sigma: \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet$. We will see that the assignment

$$C \mapsto a_C \circ \Sigma \otimes \mathrm{id}_C \circ a_C^{-1}$$

provides a natural transformation with the expected properties. □

Exercise 13.9.18. Prove that, for any finite poset P, one has

$$\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}_\bullet) = \mathrm{rex}(P)_\bullet\,.$$

Deduce that, for any C pointed and right exact, the suspension functor

$$\Sigma: \ \mathrm{Fun}(P, C) \to \mathrm{Fun}(P, C)$$

is induced through the identification above by pre-composition with the suspension functor

$$\Sigma: \ \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}_\bullet) \to \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}_\bullet)\,.$$

14 Elements of stable category theory

14.1 Stable categories

Definition 14.1.1. A *stable category* is a pointed right exact category such that the suspension functor

$$\Sigma\colon C \to C$$

is an equivalence.

We denote by StabCat the full subcategory of RexCat spanned by small stable categories.

Lemma 14.1.2. *Let S be the image of the map* $[0] \to \mathrm{Map}([1], \mathrm{RexCat})$ *corresponding to the right exact functor*

$$\Sigma\colon \mathrm{rex}([1])_\bullet \to \mathrm{rex}([1])_\bullet\,.$$

Then S is a collection of morphisms between compact objects and the S-local objects in $\mathrm{RexCat}_\bullet$ *precisely are the small stable categories. Moreover, for any small pointed right exact category C, the suspension functor $\Sigma\colon C \to C$ is an S-equivalence.*

Proof. The suspension functor $\Sigma\colon C \to C$ is an equivalence if and only if the induced map

$$\Sigma_*\colon\ \mathrm{Map}([1], C) \to \mathrm{Map}([1], C)$$

is an equivalence. We have:

$$\begin{aligned}\mathrm{Map}([1], C) &= \mathrm{Fun}([1], C)^{\simeq}\\ &= \mathrm{Rex}(\mathrm{rex}([1])_\bullet, C)^{\simeq}\end{aligned}$$

The map Σ_* coincides with the map defined by pre-composition with $\Sigma\colon \mathrm{rex}([1])_\bullet \to \mathrm{rex}([1])_\bullet$ by the first part of Lemma 13.9.13. This proves the first assertion. The last one follows right away from the last part of Lemma 13.9.13. □

Proposition 14.1.3. *The category* StabCat *is compactly generated and the full embedding*

$$\mathrm{StabCat} \hookrightarrow \mathrm{RexCat}_\bullet$$

commutes with filtered colimits and has a left adjoint.

Proof. Since $\mathrm{RexCat}_\bullet$ is compactly generated by Corollary 13.9.6, the preceding lemma says that we can apply Proposition 13.5.1. □

Remark 14.1.4. One can prove directly that (small) stable categories are stable under (small) colimits in pointed right exact categories. Indeed, by virtue of Proposition 13.9.17, if $i \mapsto C_i$ is a diagram in StabCat indexed by a small category I, with colimit C in $\mathrm{RexCat}_\bullet$, the suspension $\Sigma\colon C \to C$ is the colimit of the suspension functors $\Sigma\colon C_i \to C_i$, hence is an equivalence as well. One can construct the left adjoint of the embedding of StabCat into $\mathrm{RexCat}_\bullet$ explicitly as follows.

Remark 14.1.5. Let C be a small pointed right exact category. One constructs a sequence of right exact functors

$$C_0 \xrightarrow{f_1} C_1 \to \cdots \to C_n \xrightarrow{f_{n+1}} C_{n+1} \to \cdots$$

by $C_n = C$ for all $n\colon \mathbb{N}$ and $f_{n+1} = \Sigma$ for all $n\colon \mathbb{N}$. We define the *Spanier-Whitehead stabilization of* C as the Milnor colimit in $\mathrm{RexCat}_\bullet$ of this sequence of functors (one can compute this Milnor colimit as an ω-indexed colimit, hence as an ω-indexed colimit in Cat itself):

$$\mathrm{SW}(C) := \text{“}\mathrm{colim}\text{”}_{n\colon \mathbb{N}}\ C_n\,.$$

By construction there is a cover

$$\sum_{n\colon \mathbb{N}} C_n^{\simeq} \to \mathrm{SW}(C)^{\simeq}\,.$$

For x an object of C and $n\colon \mathbb{N}$, we write $\Sigma^{\infty-n}(x)$ for the image of x in $\mathrm{SW}(C)$, where x is seen as a term of C_n. Given another object y of C and a natural number $p\colon \mathbb{N}$, the general explicit description of mapping spaces in filtered colimits of categories given in Remark 12.3.13 yields for $m := \max\{n, p\}$:

$$\operatorname*{colim}_{k\colon \omega_{m/}} \mathrm{Hom}_C(\Sigma^{k-n}(x), \Sigma^{k-p}(y)) = \mathrm{Hom}_{\mathrm{SW}(C)}(\Sigma^{\infty-n}(x), \Sigma^{\infty-p}(y))\,.$$

The component $C_0 = C$ yields a canonical right exact functor

$$\Sigma^\infty\colon C \to \mathrm{SW}(C)\,, \quad x \mapsto \Sigma^{\infty-0}(x)\,.$$

For each $k \geq \max n, p$, there is a structural map

$$can_k\colon\ \mathrm{Hom}_C(\Sigma^{k-n}(x), \Sigma^{k-p}(y)) \to \mathrm{Hom}_{\mathrm{SW}(C)}(\Sigma^{\infty-n}(x), \Sigma^{\infty-p}(y))\,.$$

We have a morphism of sequences of the form

$$\begin{tikzcd}
C \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & C \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & \cdots \arrow[r] & C \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & C \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & \cdots \\
C \arrow[r, "\Sigma"] & C \arrow[r, "\Sigma"] & \cdots \arrow[r] & C \arrow[r, "\Sigma"] & C \arrow[r, "\Sigma"] & \cdots
\end{tikzcd}$$

in which each commutative square is given by the commutativity constraint of Proposition 13.9.17 for $f = \Sigma$. The induced functor on the Milnor colimits simply is the suspension functor

$$\Sigma\colon \mathrm{SW}(C) \to \mathrm{SW}(C)\,.$$

We define for each $k \geq \max n, p$ the map

$$can'_k\colon\ \mathrm{Hom}_C(\Sigma^{k-n}(\Sigma(x)), \Sigma^{k-p}(\Sigma(y))) \to \mathrm{Hom}_{\mathrm{SW}(C)}(\Sigma(\Sigma^{\infty-n}(x)), \Sigma(\Sigma^{\infty-p}(y)))$$

defined as the composition of

$$can_k\colon \operatorname{Hom}_C(\Sigma^{k-n}(\Sigma(x)), \Sigma^{k-p}(\Sigma(y))) \to \operatorname{Hom}_{\mathrm{SW}(C)}(\Sigma^{\infty-n}(\Sigma(x)), \Sigma^{\infty-p}(\Sigma(y)))$$

with the isomorphism

$$\operatorname{Hom}_{\mathrm{SW}(C)}(\Sigma^{\infty-n}(\Sigma(x)), \Sigma^{\infty-p}(\Sigma(y))) = \operatorname{Hom}_{\mathrm{SW}(C)}(\Sigma(\Sigma^{\infty-n}(x)), \Sigma(\Sigma^{\infty-p}(y)))$$

assigning to each map

$$u\colon \Sigma^{\infty-n}(\Sigma(x)) \to \Sigma^{\infty-p}(\Sigma(y))$$

the map

$$\Sigma(\Sigma^{\infty-n}(x)) \cong \Sigma^{\infty-n}(\Sigma(x)) \xrightarrow{u} \Sigma^{\infty-p}(\Sigma(y)) \cong \Sigma(\Sigma^{\infty-p}(y))$$

where the signs $\cong$ stand for the canonical equivalences expressing that $\Sigma^{\infty-n}$ and $\Sigma^{\infty-p}$ commute with the suspension by right exactness. There are canonical commutative triangles of the form

$$\begin{tikzcd}
\operatorname{Hom}_C(\Sigma^{k-n}(\Sigma(x)), \Sigma^{k-p}(\Sigma(y))) \arrow[r, "can'_k"] \arrow[d, "\Sigma"'] & \operatorname{Hom}_{\mathrm{SW}(C)}(\Sigma(\Sigma^{\infty-n}(x)), \Sigma(\Sigma^{\infty-p}(y))) \\
\operatorname{Hom}_C(\Sigma^{k+1-n}(\Sigma(x)), \Sigma^{k+1-p}(\Sigma(y))) \arrow[ur, "can'_{k+1}"'] &
\end{tikzcd}$$

inducing an isomorphism of the form

$$\operatorname*{colim}_k \operatorname{Hom}_C(\Sigma^{k-n}(\Sigma(x)), \Sigma^{k-p}(\Sigma(y))) \cong \operatorname{Hom}_{\mathrm{SW}(C)}(\Sigma^{\infty-n}(\Sigma(x)), \Sigma^{\infty-p}(\Sigma(y)))\,.$$

Given $f\colon \operatorname{Hom}_C(\Sigma^{k-n}(\Sigma(x)), \Sigma^{k-p}(\Sigma(y)))$, since

$$\operatorname{Hom}_C(\Sigma^{k-n}(\Sigma(x)), \Sigma^{k-p}(\Sigma(y))) = \operatorname{Hom}_C(\Sigma^{k+1-n}(x), \Sigma^{k+1-p}(y))$$

it makes sense to form $can_{k+1}(f)\colon \operatorname{Hom}_{\mathrm{SW}(C)}(\Sigma^{\infty-n}(x), \Sigma^{\infty-p}(y))$. We observe that there is a canonical equivalence

$$\Sigma(can_{k+1}(f)) = can'_k(f)\,.$$

Theorem 14.1.6. *For any pointed right exact category C, the Spanier-Whitehead stabilization* $\mathrm{SW}(C)$ *is a stable category.*

Proof. We will prove that the functor

$$\Sigma\colon \mathrm{SW}(C) \to \mathrm{SW}(C)$$

induces an equivalence

$$\Sigma^{\simeq}\colon \mathrm{SW}(C)^{\simeq} \to \mathrm{SW}(C)^{\simeq}\,.$$

Indeed, we see that this is a cover: we know that the collection of objects of the form $\Sigma^{\infty-i}(x)$ covers $\mathrm{SW}(C)^{\simeq}$. But we have

$$\begin{aligned}\Sigma^{\infty-i}(x) &= \Sigma^{\infty-i-1}(\Sigma(x))\\ &= \Sigma(\Sigma^{\infty-i-1}(x))\,.\end{aligned}$$

Formula $\Sigma(can_{k+1}(f)) = can'_k(f)$ at the end of Remark 14.1.5 implies that $\Sigma^{\simeq}$ is an embedding. Since Fun([1]), −) preserves small filtered colimits, we observe finally that

$$\mathrm{Fun}([1], \mathrm{SW}(C)) = \mathrm{SW}(\mathrm{Fun}([1], C)) .$$

The preceding discussion appied to $\mathrm{Fun}([1], C)$ thus implies that $\Sigma\colon \mathrm{SW}(C) \to \mathrm{SW}(C)$ induces an automorphism of the collection of morphisms $\mathrm{Map}([1], \mathrm{SW}(C))$ hence is an equivalence of categories. □

Remark 14.1.7. Using the last assertion of Lemma 14.1.2, we see that, for any pointed right exact category C and any stable category D, the canonical functor

$$\Sigma^{\infty}\colon C \to \mathrm{SW}(C)$$

induces an equivalence

$$\mathrm{RexCat}(\mathrm{SW}(C), D)^{\simeq} \cong \mathrm{Rex}(C, D)^{\simeq} .$$

In particular, Theorem 14.1.6 yields another proof that the full embedding $\mathrm{StabCat} \hookrightarrow \mathrm{RexCat}_{\bullet}$ has a left adjoint given on objects by the Spanier-Whitehead construction. Applying Proposition 13.4.12, we see that StabCat is compactly generated. Overall, this yields an alternative proof of Proposition 14.1.3, arguably more explicit.

14.2 Spectra

Lemma 14.2.1. *Let A and B be pointed right exact categories. If either A or B is stable, then both $A \otimes B$ and $\mathrm{Rex}(A, B)$ are stable.*

Proof. Assume that A is stable. The suspension functor $\Sigma\colon A \to A$ induces a functor

$$\Sigma \otimes \mathrm{id}_B\colon A \otimes B \to A \otimes B$$

that is isomorphic to the suspension of $A \otimes B$. To see this, one can for instance use the canonical equivalence

$$(\mathrm{FinAn}_{\bullet} \otimes A) \otimes B \cong \mathrm{FinAn}_{\bullet} \otimes (A \otimes B)$$

and conclude with Remark 13.9.16. This implies that $A \otimes B$ is stable. The case where B is stable is deduced from the equivalence $A \otimes B \cong B \otimes A$.

Similarly, if A is stable, Lemma 13.9.13 implies that the suspension on $\mathrm{Rex}(A, B)$ is induced from the one on A and thus that it is an equivalence. The fact that $\mathrm{Rex}(A, B)$ is stable whenever B has this property is even more tautological. □

Theorem 14.2.2. *The embedding* $\mathrm{StabCat} \hookrightarrow \mathrm{RexCat}_{\bullet}$ *has a right adjoint given by the assignment*

$$B \mapsto \mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_{\bullet}), B) .$$

In particular, it commutes with small colimits.

Proof. Given A a small stable category and B a small pointed right exact category, since $(-) \otimes A$ commutes with colimits in $\mathrm{RexCat}_\bullet$ (being a left adjoint, by definition in RexCat, but then also in $\mathrm{RexCat}_\bullet$, by Proposition 13.9.15), and since the category ω is weakly contractible, we see that

$$\begin{aligned} A &= \mathrm{FinAn} \otimes A \\ &= \mathrm{FinAn}_\bullet \otimes A \\ &= \mathrm{SW}(\mathrm{FinAn}_\bullet) \otimes A\,. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \mathrm{Rex}(A, B) &= \mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_\bullet) \otimes A, B) \\ &= \mathrm{Rex}(A, \mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_\bullet), B))\,. \end{aligned}$$

Since, by virtue of Lemma 14.2.1 and Theorem 14.1.6, $\mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_\bullet), B)$ is stable, this implies the theorem right away. □

Construction 14.2.3. Let C be a pointed left exact category. We denote by $\mathrm{Sp}(C)$ the Milnor limit of the diagram

$$\cdots \to C \xrightarrow{\Omega} C \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} C \xrightarrow{\Omega} C\,.$$

Corollary 14.2.4. *For any pointed left exact category C, $\mathrm{Sp}(C)$ is a stable category.*

Proof. We have $\mathrm{Sp}(C) = \mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_\bullet), C^{\mathrm{op}})^{\mathrm{op}}$. We conclude with Lemma 14.2.1 and Theorem 14.1.6. □

Definition 14.2.5. Let $\mathrm{Grpd}_\bullet = \mathrm{Grpd}_{*/}$ be the category of small pointed anima. The stable category Sp of (small) *spectra* is defined as

$$\mathrm{Sp} := \mathrm{Sp}(\mathrm{Grpd}_\bullet)\,.$$

Lemma 14.2.6. *Let C be a pointed right exact category with pullbacks. Then C is a stable category if and only if C^{op} is a stable category.*

Proof. Since C has pullbacks with a terminal object that is also initial, one can apply Remark 13.9.8 to C^{op} and translate it to properties of C: this means that the functor

$$0_*\colon C \to \mathrm{Fun}([1], C)$$

has a right adjoint

$$0^!\colon \mathrm{Fun}([1], C) \to C$$

that takes a map $u\colon a \to b$ in C to its fiber $0^!(u) = \mathrm{fib}(u)$ appearing in the pullback below.

$$\begin{tikzcd} \mathrm{fib}(u) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & a \arrow[d, "u"] \\ * \arrow[r] & b \end{tikzcd}$$

We define the loop space functor

$$\Omega\colon C \to C$$

with the formula $\Omega := 0^! \circ 1_!$. Recall that $\Sigma = 1^? \circ 0_*$ so that Ω obviously is a right adjoint functor to Σ. On the other hand, we see through the equivalence $[1]^{\mathrm{op}} = [1]$ that

$$\Omega^{\mathrm{op}} \colon C^{\mathrm{op}} \to C^{\mathrm{op}}$$

coincides with the suspension functor on C^{op}. Since a left adjoint functor is an equivalence if and only if its right adjoint functor is an equivalence, this shows that C is stable if and only if C^{op} has the same property. □

Proposition 14.2.7. *For any (small) pointed right exact category C, we have*

$$\mathrm{Lex}(\mathrm{SW}(C)^{\mathrm{op}}, \mathrm{Grpd}) \cong \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})\,.$$

Proof. Since C is pointed we have

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$$

functorially in C. It is clear from Lemma 13.9.13 that the functor $\mathrm{Lex}((-)^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$ takes the diagram

$$C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} \cdots \to C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} \cdots$$

to the diagram

$$\cdots \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \xrightarrow{\Omega} \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet) \xrightarrow{\Omega} \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$$

itself isomorphic to the image of

$$\cdots \to \mathrm{Grpd}_\bullet \xrightarrow{\Omega} \mathrm{Grpd}_\bullet \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \mathrm{Grpd}_\bullet \xrightarrow{\Omega} \mathrm{Grpd}_\bullet$$

through the functor $\mathrm{Lex}(C^{\mathrm{op}}, -)$. We conclude by observing that the functor $\mathrm{Lex}(C^{\mathrm{op}}, -)$ preserves limits. □

Corollary 14.2.8. *For any small stable category C, there is a canonical equivalence*

$$\mathrm{Ind}(C) \cong \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})\,.$$

Proof. Since C is stable, the canonical functor $C \to \mathrm{SW}(C)$ is an equivalence. We conclude with Proposition 14.2.7 above and Remark 13.3.8. □

Corollary 14.2.9. *The idempotent completion of a small stable category is stable.*

Proof. The idempotent completion of a small category C with finite colimits is obtained by taking the full subcategory of compact objects in $\mathrm{Ind}(C)$. If C is stable, then so is $\mathrm{Ind}(C)$ (by Lemma 14.2.1, Lemma 14.2.6 and Corollary 14.2.8), and the suspension functor on the idempotent completion of C is obtained from the one on $\mathrm{Ind}(C)$ by functoriality, hence is an equivalence. □

Corollary 14.2.10. *The category of spectra is stable and compactly generated. More precisely, there is a canonical equivalence*

$$\mathrm{Lex}(\mathrm{SW}(\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{Grpd}) \cong \mathrm{Sp}\,.$$

Proof. We already know that Sp is stable by Corollary 14.2.4. On the other hand, Proposition 14.2.7 for $C = [0]$ yields

$$\mathrm{Lex}(\mathrm{SW}(\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{Lex}((\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{Sp})\,.$$

But since Sp is pointed, we also have

$$\mathrm{Lex}((\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{Sp}) = \mathrm{Lex}((\mathrm{FinAn})^{\mathrm{op}}, \mathrm{Sp}) = \mathrm{Sp}\,. \qquad \square$$

Notation 14.2.11. The preceding corollary together with Theorem 13.3.10 implies that the functor

$$\mathrm{FinAn}_\bullet \to \mathrm{SW}(\mathrm{FinAn}_\bullet)\,, \quad x \mapsto \Sigma^\infty(x)$$

extends to a small colimit preserving functor

$$\Sigma^\infty \colon \mathrm{Grpd}_\bullet \to \mathrm{Sp}$$

that has a right adjoint

$$\Omega^\infty \colon \mathrm{Sp} \to \mathrm{Grpd}_\bullet\,.$$

We observe that if $S^0 = *_+$, then $\Omega^\infty = \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty(S^0), -)$ (see Remark 9.6.13).

Proposition 14.2.12. *A morphism of spectra $u\colon E \to F$ is an isomorphism if and only if, for any integer $n\colon \mathbb{N}$, the induced map*

$$\Omega^\infty(\Sigma^n(u))\colon \Omega^\infty(\Sigma^n(E)) \to \Omega^\infty(\Sigma^n(F))$$

is an equivalence.

Proof. Let C be the full subcategory of Sp whose objects are those spectra X such that

$$u_*\colon\ \mathrm{Hom}_{\mathrm{Sp}}(X, E) \to \mathrm{Hom}_{\mathrm{Sp}}(X, F)$$

is an isomorphism. Assuming that $\Omega^\infty(\Sigma^n(u))$ is an equivalence for all n means that C contains $\Omega^n(\Sigma^\infty(S^0))$ for all $n\colon \mathbb{N}$. On the other hand, C clearly is stable under small colimits in Sp. There is a unique right exact functor

$$\mathrm{FinAn}_\bullet \to C$$

sending the point to $\Sigma^\infty(S^0)$. Since C is stable, this extends to a unique right exact functor

$$i\colon \mathrm{SW}(\mathrm{FinAn}_\bullet) \to C$$

sending $\Sigma^\infty(S^0)$ to itself. Theorem 13.3.12 implies that there is a unique small colimits preserving functor

$$i_!\colon \mathrm{Lex}(\mathrm{SW}(\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{Grpd}) \to C$$

extending i. Corollary 14.2.10 together with a new application of Theorem 13.3.12 tell us that composing $i_!$ with the full embedding $C \hookrightarrow \mathrm{Sp}$ is the identity. Therefore, $C = \mathrm{Sp}$. This proves that u is an isomorphism. $\square$

14.3 Exact functors between stable categories

Definition 14.3.1. Let C be a pointed right exact category with initial object 0. A *cofiber sequence* in C is a pushout square of the form

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d] & x \arrow[d, "v"] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & x'' \end{tikzcd} .$$

Let C be a pointed left exact category (i.e. such that C^{op} is a pointed right exact category). A *fiber sequence* in C is a pullback square of the form

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "v"] \\ 0 \arrow[r] & x'' \end{tikzcd} .$$

Lemma 14.3.2. *Let C be a category with a terminal object 0 that is also an initial object. We assume that C has both pullbacks and pushouts. The following assertions are equivalent.*

(i) The category C is stable.

(ii) The category C^{op} is stable.

(iii) A commutative square of C is a pullback square if and only if it is a pushout square.

(iv) A commutative square of the form

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "v"] \\ 0 \arrow[r] & x'' \end{tikzcd}$$

in C is a fiber sequence if and only if it is a cofiber sequence.

Proof. We already know that assertions (i) and (ii) are equivalent by Lemma 14.2.6. Let 0 be the initial object of C. It is clear that condition (iv) follows from condition (iii). Let us show that condition (i) follows from condition (iv). For any object $x\colon C$, the cofiber sequence

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma(x) \end{tikzcd}$$

is a fiber sequence as well, which means that there is a functorial equivalence

$$x \cong \Omega(\Sigma(x)) .$$

The same argument applies to C^{op} and thus shows that there is a functorial equivalence of the form

$$\Sigma(\Omega(x)) \cong x .$$

This shows that both Σ and Ω are equivalences of categories.

Let us assume now that assertion (iv) holds and consider a pullback square in C of the form

$$\begin{tikzcd}
x' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\
y' \arrow[r, "v"] & y\,.
\end{tikzcd}$$

We complete this diagram by forming further pullback squares below

$$\begin{tikzcd}
\Omega(x') \arrow[r, "\Omega(f')"] \arrow[d, "\Omega(u)"'] & \Omega(y') \arrow[r] \arrow[d, "\Omega(v)"] & 0 \arrow[d] & \\
\Omega(x) \arrow[r, "\Omega(f)"] \arrow[d] & \Omega(y) \arrow[r] \arrow[d] & \mathrm{fib}(v) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{fib}(f) \arrow[r] \arrow[d] & x' \arrow[r, "u"] \arrow[d, "f'"'] & x \arrow[d, "f"] \\
 & 0 \arrow[r] & y' \arrow[r, "v"] & y
\end{tikzcd}$$

and observe that the upper left square is the image of the lower left one through the functor Ω up to a flip along the diagonal. Since fiber sequences are cofiber sequences as well, the two squares

$$\begin{tikzcd}
\Omega(x') \arrow[r] \arrow[d, "\Omega(u)"'] & 0 \arrow[d] \\
\Omega(x) \arrow[r] & \mathrm{fib}(v)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Omega(y') \arrow[r] \arrow[d, "\Omega(v)"] & 0 \arrow[d] \\
\Omega(y) \arrow[r] & \mathrm{fib}(v)
\end{tikzcd}$$

are pushout squares, hence that the image of

$$\begin{tikzcd}
x' \arrow[r, "u"] \arrow[d, "f'"'] & x \arrow[d, "f"] \\
y' \arrow[r, "v"] & y\,.
\end{tikzcd}$$

by the functor Ω is a pushout square. Since Ω is an equivalence, this says that the latter square is a pushout square as well. Applying the same argument to C^{op} shows that condition (iii) follows from condition (iv).

It remains to check that condition (iv) follows from condition (i). Let

$$\begin{tikzcd}
x' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "v"] \\
0 \arrow[r] & x''
\end{tikzcd}$$

be a fiber sequence. We also form the following cofiber sequence.

$$\begin{tikzcd}
x \arrow[r] \arrow[d, "v"'] \arrow[dr, phantom, "\ulcorner", very near end] & 0 \arrow[d] \\
x'' \arrow[r, "w"] & y
\end{tikzcd}$$

Composing the squares

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d] & x \arrow[r] \arrow[d, "v"'] & 0 \arrow[d] \\ 0 \arrow[r] & x'' \arrow[r, "w"] & y \end{tikzcd}$$

defines canonical a map $a\colon x' \to \Omega(y)$. On the other hand, there is a canonical pushout square of the form

$$\begin{tikzcd} y \arrow[r] \arrow[d] & \Sigma(x) \arrow[d, "\Sigma(v)"] \\ 0 \arrow[r] & \Sigma(x'') \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

hence also a pushout of the form

$$\begin{tikzcd} \Omega(y) \arrow[r] \arrow[d] & x \arrow[d, "v"] \\ 0 \arrow[r] & x'' \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

inducing a canonical map $b\colon \Omega(y) \to x'$. We see that $b \circ a$ is the identity of x' because its image through Σ is part of an endomorphism of the pushout square

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma(x') \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

that is the identity on the shape $\ulcorner$. The same observation in C^{op} shows that $a \circ b$ is the identity. $\square$

Theorem 14.3.3. *If C is a stable category then so is C^{op}. In other words, any stable category has pullbacks.*

Proof. Let C be a stable category. The Yoneda embedding is a fully faithful right exact functor

$$h_C\colon C \hookrightarrow \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) \cong \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})$$

into a stable compactly generated category. Since those have small limits, Lemma 14.3.2 applies to them. In other words, the family of pullback squares in $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$ agrees with the family of pushout squares. This will be used to prove the existence of pullbacks in C as follows. Let us consider a diagram of the form

$$\begin{tikzcd} & x \arrow[d, "f"] \\ y' \arrow[r, "v"] & y \end{tikzcd}$$

in C. We complete this diagram through further pushout squares as follows.

$$\begin{tikzcd}
 & x \arrow[r] \arrow[d, "f"] & 0 \arrow[d] & \\
y' \arrow[r, "v"] \arrow[d] & y \arrow[r] \arrow[d] & \mathrm{cofib}(f) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{cofib}(v) \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner", very near start, shift left=0] & X' \arrow[r, "F'"] \arrow[d, "U"'] & \Sigma(y') \arrow[d, "\Sigma(v)"] \\
 & 0 \arrow[r] & \Sigma(x) \arrow[r, "\Sigma(f)"'] \arrow[ur, phantom, "\ulcorner", very near start] & \Sigma(y)
\end{tikzcd}$$

We define x' through $\Sigma(x') := X'$ and similarly f' through $\Sigma(f') := F'$ and u through $\Sigma(u) := U$. We obtain a pushout square of the form

$$\begin{tikzcd}
x' \arrow[r, "u"] \arrow[d, "f'"'] & x \arrow[d, "f"] \\
y' \arrow[r, "v"] & y
\end{tikzcd}$$

whose image through Σ is the square in the lower right of the big diagram above. The image of the latter square through h_C is thus a pushout in $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd})$, hence a pullback square as well. The functor h_C being fully faithful, this implies right away that this is a pullback square in C. $\square$

Theorem 14.3.4. *Let A and B be two stable categories, and $\varphi\colon A \to B$ be a functor. The following assertions are equivalent.*

(i) The functor φ is right exact.

(ii) The functor φ is left exact.

(iii) The functor φ preserves initial objects and cofiber sequences.

(iv) The functor φ preserves terminal objects and fiber sequences.

Proof. It is clear that any left exact functor preserves fiber sequences by definition. Let us assume condition (iv) to hold and prove that φ is left exact. We just have to prove that it preserves pullbacks. Let

$$\begin{tikzcd}
x' \arrow[r, "u"] \arrow[d, "f'"'] & x \arrow[d, "f"] \\
y' \arrow[r, "v"] & y
\end{tikzcd}$$

be a pullback square in A. We extend it by forming the fiber of f'.

$$\begin{tikzcd}
\mathrm{fib}(f') \arrow[r, "i"] \arrow[d] & x' \arrow[d, "f'"] \\
0 \arrow[r] & y'
\end{tikzcd}$$

We then get a commutative diagram in B of the form

$$\begin{tikzcd}
\varphi(\mathrm{fib}(f')) \arrow[r, "\varphi(i)"] \arrow[d] & \varphi(x') \arrow[r, "\varphi(u)"] \arrow[d, "\varphi(f')"'] & \varphi(x) \arrow[d, "\varphi(f)"] \\
0 \arrow[r] & \varphi(y') \arrow[r, "\varphi(v)"] & \varphi(y)
\end{tikzcd}$$

in which the outer square as well as the square on the left both are pullback squares. But B being stable these are pushout squares as well. This implies that the square on the right is a pushout square, hence a pullback square as well. In other words conditions (i) and (iii) are equivalent and conditions (ii) and (iv) are equivalent.

It suffices now to prove that conditions (iii) and (iv) are equivalent. This follows from the fact that in any stable category, the collection of initial objects coincides with the collection of terminal objects, and that the collection of cofiber sequences agrees with the collection of fiber sequences. □

Proposition 14.3.5. *Let $u\colon A \to B$ be an exact functor between small stable categories. Then the induced functor*

$$u^*\colon \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Sp})$$

has a right adjoint

$$u_*\colon \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Sp}) .$$

Proof. Since we have

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}) = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})$$

for any small stable category C by Proposition 14.2.7, Theorem 13.3.12 tells us that it suffices to prove that the functor u^* commutes with small colimits. Since it is left exact (being a right adjoint by Theorem 13.3.10), it is an exact functor, by Theorem 14.3.4. Therefore, by virtue of Corollary 13.4.11, it suffices to prove that u^* commutes with small filtered colimits. But Theorem 13.3.10 tells us that there is a commutative square of the form

$$\begin{tikzcd}
\mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "u^*"] \arrow[d, hook] & \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, hook] \\
\mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Grpd}) \arrow[r, "u^*"] & \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Grpd})
\end{tikzcd}$$

in which the two vertical maps are fully faithful embeddings that commute with small filtered colimits by Proposition 13.3.2. Since the lower horizontal map in the square above commutes with small colimits, this implies that the upper horizontal one commutes with small filtered colimits. □

14.4 Representable presheaves on stable categories

Construction 14.4.1. Let C be a small stable category. Recall from Corollary 14.2.8 that

$$\mathrm{Ind}(C) = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp}) .$$

The Yoneda embedding $C \hookrightarrow \mathrm{Ind}(C)$ thus yields a fully faithful exact functor

$$h_C^{\mathrm{Sp}}\colon C \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})$$

hence an exact functor of the form

$$(h_C^{\mathrm{Sp}})^{\mathrm{op}} \colon C^{\mathrm{op}} \to \mathrm{Rex}(C, \mathrm{Sp}^{\mathrm{op}})$$

corresponding to a functor

$$\mathrm{RHom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{Sp}$$

that is (right) exact in each variable. For x in C^{op} and y in C there is a functorial identification

$$\Omega^\infty(\mathrm{RHom}_C(x, y)) = \mathrm{Hom}_C(x, y) .$$

Therefore, if C is locally small, $\mathrm{RHom}_C(x, y)$ belongs to the category Sp of small spectra.

Lemma 14.4.2. *If C is a stable locally small category, then* RHom_C *takes its values in* Sp. *Moreover, the induced functor*

$$\mathrm{RHom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{Sp}$$

preserves small limits in each variable.

Proof. We can choose a larger universe for which C is small, with corresponding category of large spectra Sp' and thus a functor

$$\mathrm{RHom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{Sp}' .$$

An object of Sp' is by definition an infinite sequence of possibly large pointed groupoids x_n equipped with isomorphisms $x_n \cong \Omega(x_{n+1})$ for all $n \colon \mathbb{N}$. In the case of $\mathrm{RHom}_C(a, b)$ with a and b objects of C, by construction, this sequence may be obtained through

$$x_n = \mathrm{Hom}_C(a, \Sigma^n(b))$$

together with the isomorphisms

$$\begin{aligned} \mathrm{Hom}_C(a, \Sigma^n(b)) &= \mathrm{Hom}_C(a, \Omega(\Sigma^{n+1}(b))) \\ &= \Omega(\mathrm{Hom}_C(a, \Sigma^{n+1}(b))) . \end{aligned}$$

The last assertion of the lemma follows from the analogous property for the ordinary Hom functor Hom_C. □

Theorem 14.4.3. *Let $f \colon C \to D$ be an exact functor between locally small stable categories. Then f has a right adjoint $g \colon D \to C$ if and only if there is a functorial isomorphism of spectra*

$$\mathrm{RHom}_D(f(x), y) = \mathrm{RHom}_C(x, g(y))$$

for any $x \colon C$ and any $y \colon D$.

Proof. This is clearly a sufficient condition since one can recover Hom from RHom by applying Ω^∞.

For the converse, we may choose to work in a universe for which both C and D are small. The property that g is right adjoint to f is that $g_! = f^*$, where

$$f^* \colon \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

is the functor defined by precomposition by f^{op} whereas

$$g_! \colon \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) \to \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

extends g by small colimits along the Yoneda embedding. By virtue of Theorem 13.3.10, these identifications remain true when we restrict ourselves to ind-objects. We thus get identifications of the form $g_! = f^*$ where this time we consider functors

$$f^*, g_! \colon \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Sp}) \to \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Sp})\,.$$

Since we have a canonical commutative square of the form

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "g"'] & \operatorname{Ind}(D) \arrow[r, hook] \arrow[d, "g_!"] & \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Sp}) \arrow[d, "g_!"] \\ C \arrow[r, hook] & \operatorname{Ind}(C) \arrow[r, hook] & \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Sp}) \end{tikzcd}$$

we must have

$$g_!(\mathrm{RHom}_D(-, y)) = \mathrm{RHom}_C(-, g(y))$$

for all $y \colon D$. Since

$$f^*(\mathrm{RHom}_D(-, y)) = \mathrm{RHom}_D(f(-), y)$$

holds by definition, this proves the theorem. □

Proposition 14.4.4. *Let C be a locally small stable category with small colimits. Given an object x of C, the following conditions are equivalent.*

(i) The object x is compact in C.

(ii) The functor $\mathrm{RHom}_C(x, -) \colon C \to \mathrm{Sp}$ *commutes with filtered colimits.*

(iii) The functor $\mathrm{RHom}_C(x, -) \colon C \to \mathrm{Sp}$ *commutes with small colimits.*

Proof. Corollary 13.4.11 implies that conditions (ii) and (iii) are equivalent. We have an adjunction

$$\Sigma^\infty((-)_+) \colon \mathrm{Grpd} \rightleftarrows \mathrm{Sp} \colon \Omega^\infty$$

with the left adjoint preserving compact objects, hence the right adjoint Ω^∞ commutes with small filtered colimits. We know from Proposition 14.2.12 that the collection of functors $\Omega^\infty \circ \Sigma^n$ indexed by $n \colon \mathbb{N}$ is jointly conservative. We observe that

$$\Omega^\infty(\Sigma^n(\mathrm{RHom}_C(x, y))) = \operatorname{Hom}_C(x, \Sigma^n(y))$$

for all y. Therefore, condition (ii) simply means that each functor

$$y \mapsto \operatorname{Hom}_C(x, \Sigma^n(y))$$

commutes with filtered colimits. Since the functors Ω^n commute with all colimits, this is clearly equivalent to the property of compactness for x. □

14.5 ω-Presentable stable categories

Definition 14.5.1. We define $\mathrm{StabPr}^{\mathrm{L}}_{\omega}$ as the full subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ whose objects are the stable compactly generated categories.

Remark 14.5.2. If D and E both are compactly generated, and if E is stable, then $\mathrm{Fun}^{\mathrm{L}}(D, E)$ is stable as well.

Proposition 14.5.3. *A compactly generated category D is stable if and only if there is a small stable category C and an equivalence of the form* $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp}) \cong D$.

Proof. It is clear that $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})$ is stable and compactly generated. Conversely, we can consider a small category C such that

$$\mathrm{Ind}(C) = D\,.$$

We observe that, for any compact object x of D, $\Omega(x)$ is still compact: since Ω is left adjoint to the equivalence Σ, and since Σ commutes with small colimits, it is clear that the functor

$$\mathrm{Hom}_D(\Omega(x), -) = \mathrm{Hom}_D(x, \Sigma(-))$$

preserves small filtered colimits. Let $\bar{C}$ be the smallest full subcategory of D stable under pushouts and containing any object of the form $\Omega^n(h_C(x))$ for $x\colon C^{\simeq}$ and $n\colon \mathbb{N}$. It is clear that $\bar{C}$ is small and it follows from Theorem 13.4.8 that

$$\mathrm{Ind}(\bar{C}) = D\,.$$

But then it is clear that $\Sigma\colon \bar{C} \to \bar{C}$ is an equivalence with inverse the restriction to $\bar{C}$ of the loop functor of D. Therefore, $\bar{C}$ is stable. We conclude the proof with Corollary 14.2.8. □

Theorem 14.5.4. *The functor*

$$\mathrm{StabCat} \to \mathrm{StabPr}^{\mathrm{L}}_{\omega}\,, C \mapsto \mathrm{Ind}(C)$$

is left adjoint to the fully faithful functor defined by the rule $D \mapsto D_{\omega}$.

Proof. This follows right away from Theorem 13.8.9 since both functors $\mathrm{Ind}(-)$ and $(-)_{\omega}$ preserve the property of being stable. □

Corollary 14.5.5. *The category* $\mathrm{StabPr}^{\mathrm{L}}_{\omega}$ *is compactly generated.*

Proof. We know that $D \mapsto D_{\omega}$ defines a fully faithful functor from $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ to RexCat that commutes with filtered colimits. Since the full embedding of StabCat into RexCat also commutes with small filtered colimits, this implies that both full embeddings

$$\mathrm{StabPr}^{\mathrm{L}}_{\omega} \hookrightarrow \mathrm{Pr}^{\mathrm{L}}_{\omega} \quad \text{and} \quad \mathrm{StabPr}^{\mathrm{L}}_{\omega} \hookrightarrow \mathrm{StabCat}$$

commute with small filtered colimits. Proposition 13.4.12 completes the proof. □

14.6 Verdier quotients

Proposition 14.6.1. *Let C be a stable category and W a family of maps in C with the 2-out-of-3 property and stable under pullbacks. We also assume that one of the following conditions is verified:*

(i) W is also stable under pushouts;

(ii) W is stable under the suspension functor $\Sigma\colon C \to C$.

The localization $L(C)$ of C by W is then stable and the localization functor $\gamma\colon C \to L(C)$ is exact.

Proof. We know that $L(C)$ is pointed left exact and that the localization functor

$$\gamma\colon C \to L(C)$$

is left exact by Theorem 13.1.5.

Under condition (i), we can apply dually Corollary 13.1.8 and see that $L(C)$ has pushouts and that γ is right exact (Proposition 10.1.1 and its dual version ensure that $L(C)$ has a terminal object that is also initial and that γ preserves those). The proof of Theorem 13.1.5 (iv) tells us that the collection of pullback squares of C covers the collection of pullback squares of $L(C)$. Since all pullback squares of C also are pushout squares, and since γ is exact, this implies that all pullback squares of $L(C)$ are pushouts. Applying the previous argument to C^{op} shows that all pushout squares of $L(C)$ are pullback squares as well.

Under condition (ii), Lemma 10.1.4 tells us that the adjunction equivalence

$$\Sigma\colon C \rightleftarrows C : \Omega$$

induces an adjunction equivalence after localization, which implies right away that $\Omega\colon L(C) \to L(C)$ is an equivalence of categories. Therefore, $L(C)^{\mathrm{op}}$ is stable, and thus so is $L(C)$ itself. □

Definition 14.6.2. Let $i\colon A \to B$ be an exact full embedding between stable categories. The *Verdier quotient* of B by A is the stable category B/A defined by forming the pushout square

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & B/A \end{tikzcd}$$

in the category StabCat of stable categories (that are small for a big enough universe), where 0 denotes the terminal stable category.

Remark 14.6.3. In the commutative square defining B/A above, the commutativity constraint expressing that $p \circ i$ is the constant functor taking the value 0 in B/A is unique: the 0 functor is a terminal object in the category of exact functors from A to B/A.

Lemma 14.6.4. *In a stable category C, for any morphism $u\colon x \to y$ in C, the cofiber of u is canonically isomorphic to the suspension of its fiber.*

Proof. This follows right away from the fact that in the commutative diagram

$$\begin{tikzcd}
\mathrm{fib}(u) \arrow[r] \arrow[d] & x \arrow[r] \arrow[d, "u"] & 0 \arrow[d] \\
0 \arrow[r] & y \arrow[r] & \mathrm{cofib}(u)
\end{tikzcd}$$

both squares are pullbacks as well as pushouts. □

Lemma 14.6.5. *Let C be a stable category and*

$$\begin{tikzcd}
x' \arrow[r, "f"] \arrow[d, "u'"'] & x \arrow[d, "u"] \\
y' \arrow[r, "g"] & y
\end{tikzcd}$$

a pullback (hence also a pushout) square in C. Then u is an isomorphism if and only if u' is an isomorphism.

Proof. In a pullback square as in the lemma, if u is invertible, so is u'. The dual statement for pushouts implies the converse. □

Lemma 14.6.6. *Let $u\colon x \to y$ be a morphism in a stable category C. The following assertions are equivalent.*

(i) The morphism u is an isomorphism.

(ii) The fiber of u is isomorphic to 0.

(iii) The cofiber of u is isomorphic to 0.

Proof. It follows immediately from Lemma 14.6.4 that conditions (ii) and (iii) are equivalent. We deduce right away from Lemma 14.6.5 that conditions (i) and (ii) are equivalent. □

Proposition 14.6.7. *Let $i\colon A \to B$ be a fully faithful exact functor between stable categories. The family W of morphisms of B with fibers in A agrees with the family of morphisms of B with cofibers in A. Moreover W is stable under pullbacks as well as under pushouts, and the resulting localization of B by W is nothing else than the Verdier quotient B/A.*

Proof. The first assertion follows right away from Lemma 14.6.4. The stability of W under pullbacks is obvious, since fibers remain constant through pullbacks. Stability under pushouts follows by a dual argument.

Given an exact functor $\varphi\colon B \to C$, using Lemma 14.6.6, we see that the property for φ to send any map in W to an isomorphism in C is equivalent to the property of sending any object of A to 0 in C. In other words, by Remark 13.6.6, there is a pullback square of the form

$$\begin{tikzcd}
\mathrm{RexCat}^W(B, C) \arrow[r] \arrow[d] & \mathrm{RexCat}(B, C) \arrow[d, "i^*"] \\
0 \arrow[r] & \mathrm{RexCat}(A, C)
\end{tikzcd}$$

which shows that B/A literally has the universal property of the localization of B by W. □

Corollary 14.6.8. *Let $i\colon A \to B$ be an exact full embedding between small stable categories with corresponding Verdier quotient $p\colon B \to B/A$. The induced functor*

$$p^*\colon \mathrm{Lex}((B/A)^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Sp})$$

is fully faithful. Given an exact functor of the form

$$F\colon B^{\mathrm{op}} \to \mathrm{Sp}\,,$$

the following assertions are equivalent.

(i) The functor F belongs to the essential image of p^*.

(ii) The functor $i^*(F) = F \circ i^{\mathrm{op}}$ is constant with value 0.

(iii) We have $F(i(a)) \cong 0$ in Sp for any object a of A.

(iv) We have $\Omega^\infty(F(i(a))) \cong *$ in $\mathrm{Grpd}_\bullet$ for any object a of A.

Proof. The fact that the functor p^* is fully faithful with essential image described by condition (ii) is a direct consequence of the description of the universal property exhibiting B/A as a localization provided by Proposition 14.6.7, using Remark 13.6.9 and Remark 13.6.6. Conditions (ii) and (iii) obviously are equivalent. Since

$$\Omega^\infty\colon \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Grpd}_\bullet)$$

is an equivalence for any small stable category C, all of the above may be repeated replacing Sp by $\mathrm{Grpd}_\bullet$ and this implies that conditions (iii) and (iv) are equivalent. □

Remark 14.6.9. Let $i\colon A \to B$ be an exact fully faithful functor between small stable categories, with Verdier quotient. Let W be the family of maps with fibers in A.

Using the Yoneda embedding

$$B \hookrightarrow \mathrm{Ind}(B) = \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Sp})$$

we may see W as a small family of maps in $\mathrm{Ind}(B)$. The W-local objects are thus those presheaves satisfying condition (iii) of Corollary 14.6.8. More precisely, if $L_W \mathrm{Ind}(B)$ denotes the full subcategory of W-local objects in $\mathrm{Ind}(B)$, the functor p^* induces a fully faithful and essentially surjective functor from $\mathrm{Ind}(B/A)$ to $L_W \mathrm{Ind}(B)$. Since p^* corresponds to the full embedding of $L_W \mathrm{Ind}(B)$ to $\mathrm{Ind}(B)$, its left adjoint

$$p_!\colon \mathrm{Ind}(B/A) \to \mathrm{Ind}(B)$$

correspond to the W-localization functor

$$loc_W\colon L_W \mathrm{Ind}(B) \to \mathrm{Ind}(B)\,.$$

One can then use Proposition 13.5.2 to identify the localization of B by W with the smallest full subcategory of $\mathrm{Ind}(B/A)$ stable under pushouts and containing objects of the form $p(b)$ for $b\colon B^{\simeq}$. Let $\gamma\colon B \to L(B)$ be this localization.

We also get a Verdier quotient of the form

$$\begin{tikzcd} \mathrm{Ind}(B/A) \arrow[r, "p^*"] \arrow[d] & \mathrm{Ind}(B) \arrow[d, "i^*"] \\ 0 \arrow[r] & \mathrm{Ind}(A) \end{tikzcd}$$

since i^* is a localization (having a fully faithful left adjoint $i_!$) and that the objects of $\mathrm{Ind}(B)$ that are sent to 0 through i^* precisely are the objects in the essential image of p^* by Corollary 14.6.8, so that we can apply Proposition 14.6.7 (for large stable categories, i.e. seeing $\mathrm{Ind}(B)$ as a stable category in larger universe).

Definition 14.6.10. A *six gluing functors datum* is a diagram of stable categories of the form

$$\begin{tikzcd} U \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & X \arrow[l, "j^*"'] \arrow[r, bend left, "i^*"] & Z \arrow[l, "i_*"'] \end{tikzcd}$$

in which all functors are exact, with the following properties.

(i) We have $i^* \circ j_! \cong 0$.

(ii) Both functors $j_!$ and i_* are fully faithful.

(iii) The functor j^* is right adjoint to $j_!$ and left adjoint to j_*.

(iv) The functor i^* is left adjoint to i_*.

(v) The pair (j^*, i^*) is jointly conservative: for any object $x \colon X^{\simeq}$, we have $x = 0$ if and only if $j^*(x) = 0$ and $i^*(x) = 0$.

Remark 14.6.11. If we have a six gluing functors datum as above, then, for any object x of X the following square induced by the co-unit of the adjunction $(j_!, j^*)$ and by the unit of the adjunction (i^*, i_*)

$$\begin{tikzcd} j_!(j^*(x)) \arrow[r] \arrow[d] & x \arrow[d] \\ 0 \arrow[r] & i_*(i^*(x)) \end{tikzcd}$$

is a cofiber sequence in X (there is a unique way to make this square commutative by condition (i)). Indeed, it suffices to prove that the induced map from the cofiber of the co-unit map $j_!(j^*(x)) \to x$ to $i_*(i^*(x))$ is an isomorphism. By condition (v) of the definition, this means that it suffices to check that the two commutative squares

$$\begin{tikzcd} j^*(j_!(j^*(x))) \arrow[r] \arrow[d] & j^*(x) \arrow[d] \\ 0 \arrow[r] & j^*(i_*(i^*(x))) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} i^*(j_!(j^*(x))) \arrow[r] \arrow[d] & i^*(x) \arrow[d] \\ 0 \arrow[r] & i^*(i_*(i^*(x))) \end{tikzcd}$$

are cofiber sequences. But they are isomorphic to

$$\begin{tikzcd} j^*(x) \arrow[r, equal] \arrow[d] & j^*(x) \arrow[d] \\ 0 \arrow[r, equal] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & i^*(x) \arrow[d, equal] \\ 0 \arrow[r] & i^*(x) \end{tikzcd}$$

respectively, which are obviously cocartesian squares.

We define then a functor

$$i^! \colon X \to Z$$

by assigning to each $x \colon X$ a fiber sequence of the form

$$\begin{tikzcd} i^!(x) \arrow[r] \arrow[d] & i^*(x) \arrow[d] \\ 0 \arrow[r] & i^*(j_*(j^*(x))) \end{tikzcd}$$

where the vertical map on the right side is the image by i^* of the unit map of the adjunction (j^*, j_*). We can define $I(x)$ through the pullback square

$$\begin{tikzcd} I(x) \arrow[r] \arrow[d] & x \arrow[d] \\ 0 \arrow[r] & j_*(j^*(x)) \end{tikzcd}$$

and observe that since j^* sends the vertical map on the right side to an isomorphism, we have $j^*(I(x)) = 0$. In other words the cofiber sequence

$$\begin{tikzcd} j_!(j^*(I(x))) \arrow[r] \arrow[d] & I(x) \arrow[d] \\ 0 \arrow[r] & i_*(i^*(I(x))) \end{tikzcd}$$

shows that the unit map $I(x) \to i_*(i^*(I(x)))$ is an isomorphism. Since $i^*(I(x)) = i^!(x)$ essentially by definition, we thus have

$$I(x) = i_*(i^!(x))$$

and this yields a canonical fiber sequence of the form below.

$$\begin{tikzcd} i_*(i^!(x)) \arrow[r] \arrow[d] & x \arrow[d] \\ 0 \arrow[r] & j_*(j^*(x)) \end{tikzcd}$$

Remark that since i_* is fully faithful, $i^* \circ i_*$ is the identity and thus the fiber sequence above may be taken as a definition of $i^!(x)$ as well. The following lemma explains why we speak of six gluing functors.

Lemma 14.6.12. *With a six gluing functors datum as above, the functor $i^! \colon X \to Z$ introduced in the previous remark is right adjoint to i_*.*

Proof. Given x in X and z in Z, since i_* is fully faithful, it suffices to prove that

$$\mathrm{Hom}(i_*(z), i_*i^!(x)) = \mathrm{Hom}(i_*(z), x)$$

By construction of $i^!$, for any a and x in X, we have a pullback square of the form below.

$$\begin{tikzcd}
\mathrm{Hom}(a, i_*i^!(x)) \arrow[r] \arrow[d] & \mathrm{Hom}(a, x) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(a, j_*j^*(x))
\end{tikzcd}$$

By virtue of Lemma 14.6.6 it suffices to prove that

$$\mathrm{Hom}(i_*(z), j_*(u)) = 0$$

for all z in Z and all u in U. But j^* is left adjoint to j_* so that this amounts to proving the relation $j^* \circ i_* = 0$. This follows from the relation $i^* \circ j_! = 0$ by transposition:

$$\mathrm{Hom}(a, j^*\, i_*(b)) = \mathrm{Hom}(i^*\, j_!(a), b)\ . \qquad \square$$

Proposition 14.6.13. *We consider a six gluing functors datum of the form below, with X, U and Z small.*

$$\begin{tikzcd}
U \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & X \arrow[l, "j^*"'] \arrow[r, bend left, "i^*"] \arrow[r, bend right, "i^!"'] & Z \arrow[l, "i_*"']
\end{tikzcd}$$

Then all commutative squares below are both pullbacks and pushouts in StabCat.

$$\begin{tikzcd}
U \arrow[r, "j_!"] \arrow[d] & X \arrow[d, "i^*"] \\
0 \arrow[r] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "i_*"] \arrow[d] & X \arrow[d, "j^*"] \\
0 \arrow[r] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "j_*"] \arrow[d] & X \arrow[d, "i^!"] \\
0 \arrow[r] & Z
\end{tikzcd}$$

Proof. We first prove that each square is a pullback square of stable categories. Since all functors $j_!$, j_* and i_* are fully faithful, it suffices to check this on objects. For the first two squares, this follows right away from the pushout squares of the form

$$\begin{tikzcd}
j_!(j^*(x)) \arrow[r] \arrow[d] & x \arrow[d] \\
0 \arrow[r] & i_*(i^*(x))
\end{tikzcd}$$

for each $x \colon X^{\simeq}$. For the third one, this follows from the pullback squares of the form

$$\begin{tikzcd}
i_*(i^!(x)) \arrow[r] \arrow[d] & x \arrow[d] \\
0 \arrow[r] & j_*(j^*(x))
\end{tikzcd}$$

for each $x \colon X^{\simeq}$.

Finally, each functor with a fully faithful left or right adjoint is a localization. In the case of exact functors between stable categories, an exact localization is always obtained by inverting the morphisms whose fiber becomes zero in the localization. Therefore, the description of Verdier quotients as localizations provided by Proposition 14.6.7 ends the proof. $\square$

Construction 14.6.14. Given a small stable category C, we will write $C' = \mathrm{Ind}(C)_\omega$ for its idempotent completion. Each exact functor $u\colon C \to D$ between small stable categories induces an exact functor

$$u_!\colon \mathrm{Ind}(C) \to \mathrm{Ind}(D)$$

that preserves compact objects and thus restricts to an exact functor after idempotent completions:

$$u'\colon C' \to D' .$$

The naturality of the Yoneda embedding yields a commutative square of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d] & D \arrow[d] \\ C' \arrow[r, "u'"] & D' \end{tikzcd}$$

in which the two vertical maps are fully faithful exact functors.

Theorem 14.6.15 (Thomason)**.** *Let $i\colon A \to B$ be a fully faithful exact functor between small stable categories, and let B/A be the associated Verdier quotient.*

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & B/A \end{tikzcd}$$

By passing to idempotent completions, we obtain a cartesian square in StabCat *of the form*

$$\begin{tikzcd} A' \arrow[r, hook, "i'"] \arrow[d] & B' \arrow[d, "p'"] \\ 0 \arrow[r] & (B/A)' \end{tikzcd}$$

with i' fully faithful, and the induced functor

$$B'/A' \to (B/A)'$$

is an exact fully faithful functor that induces an equivalence after idempotent completion

$$(B'/A')' \cong (B/A)' .$$

Moreover, the induced commutative square of (large) stable categories

$$\begin{tikzcd} \mathrm{Ind}(A) \arrow[r, hook, "i_!"] \arrow[d] & \mathrm{Ind}(B) \arrow[d, "p_!"] \\ 0 \arrow[r] & \mathrm{Ind}(B/A) \end{tikzcd}$$

has the property that $i_!$ is fully faithful and that $p_!$ induces an equivalence

$$\mathrm{Ind}(B)/\mathrm{Ind}(A) \cong \mathrm{Ind}(B/A) .$$

Proof. Since i is fully faithful, the induced functor

$$i_! \colon \operatorname{Fun}(A^{\mathrm{op}}, \mathrm{Grpd}) \to \operatorname{Fun}(B^{\mathrm{op}}, \mathrm{Grpd})$$

is fully faithful, hence its restriction

$$i_! \colon \operatorname{Ind}(A) \to \operatorname{Ind}(B)$$

is fully faithful as well. Restricting further to compact objects thus yields an exact fully faithful functor

$$i' \colon A' \to B' .$$

Since $\operatorname{Ind}(C) = \operatorname{Ind}(C')$ for any small stable category C, Remark 14.6.9 implies right away that the comparison functor

$$B/A \to B'/A'$$

is fully faithful and induces an equivalence

$$\operatorname{Ind}(B/A) = \operatorname{Ind}(B'/A') .$$

Passing to compact objects, this proves that B/A and B'/A' have the same idempotent completion. Let us consider finally the commutative square in $\mathrm{StabPr}^{\mathrm{L}}_{\omega}$ given by

$$\begin{tikzcd}
\operatorname{Ind}(A) \arrow[r, hook, "i_!"] \arrow[d] & \operatorname{Ind}(B) \arrow[d, "p_!"] \\
0 \arrow[r] & \operatorname{Ind}(B/A)
\end{tikzcd}$$

In order to prove that this is both a pullback square and a pushout square of (large) stable categories, we will apply Proposition 14.6.13. In other words, it suffices to check that we have a six gluing functors datum of the form

$$\begin{tikzcd}
\operatorname{Ind}(A) \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \operatorname{Ind}(B) \arrow[l, "i^*"'] \arrow[r, bend left, "p_!"] \arrow[r, bend right, "p_*"'] & \operatorname{Ind}(B/A) \arrow[l, "p^*"']
\end{tikzcd}$$

The existence of adjoint functors follows from Theorem 13.3.10 and Proposition 14.3.5. The only non-trivial property to be checked is the fact that i^* and $p_!$ form a jointly conservative family of functors. In other words, we have to check that, for an object X of $\operatorname{Ind}(B)$ with $i^*(X) = 0$ and $p_!(X) = 0$ we must have $X = 0$. But by Corollary 14.6.8, the condition that $i^*(X) = 0$ is equivalent to the condition that X is in the essential image of p^*. If $X = p^*(Y)$ for some Y, since p^* is fully faithful, we then have $p_!(X) = Y$. Therefore, if $p_!(X) = 0$ we must have $Y = 0$ hence $X = 0$ as well. Since $(-)_{\omega}$ preserves pullback squares (being a right adjoint), this proves the theorem. □

Definition 14.6.16. Let B be a stable category. A *thick subcategory* of B is a fully faithful exact functor $i \colon A \to B$ with the property that, for any projector of the form $p \colon x \to x$ in A, if $i(p)$ has an image in B, then p has an image in A. We will then say by a slight abuse that A is a thick subcategory of B.

Proposition 14.6.17. *Let $i \colon A \to B$ be a fully faithful exact functor between stable categories. We denote by $\gamma \colon B \to B/A$ the associated Verdier quotient. The following conditions are equivalent.*

(i) *The commutative square*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "\gamma"] \\ 0 \arrow[r] & B/A \end{tikzcd}$$

is a pullback square.

(ii) *A is a thick subcategory of B.*

Proof. If (i) holds and $p\colon x \to x$ is a projector in A that has an image y in B, then $\gamma(y) = 0$ because $\gamma(y)$ is the image of a projector on $\gamma(i(p)) = 0$. Therefore, y is in the essential image of i. Conversely, if we denote by C' the idempotent completion of any stable category C, we know from the previous theorem that we have a pullback square of the form

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d] \\ 0 \arrow[r] & B'/A' \end{tikzcd}$$

Since $B/A \to B'/A'$ is fully faithful, this readily implies that the full subcategory of B whose objects are sent to 0 in B/A precisely is the intersection of A' with B. The explicit description of idempotent completions obtained from Proposition 13.8.4 shows that condition (ii) follows from condition (i). $\square$

15 K-Theory, after Quillen, Waldhausen and Thomason

15.1 Completion by small Sifted colimits

Notation 15.1.1. Given a small category with finite coproducts C we denote by $P_\Sigma(C)$ the full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ spanned by finite products preserving functors.

Proposition 15.1.2. *Let C be a small category with finite coproducts. The category $P_\Sigma(C)$ is compactly generated. Furthermore, the embedding*

$$P_\Sigma(C) \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

commutes small sifted colimits.

Proof. The fact that $P_\Sigma(C)$ has small sifted colimits and that the embedding of $P_\Sigma(C)$ into $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ commutes with them is immediate, since sifted colimits are compatible with finite products of groupoids, by Proposition 12.2.11. Let S be the family of maps consisting of

$$\emptyset \to h_C(*)$$

with $*$ the initial object of C and of the canonical comparison maps

$$h_C(x) \amalg h_C(y) \to h_C(x \amalg y)$$

for any objects x and y of C. It is clear that the S-local objects precisely are the objects of $P_\Sigma(C)$. Therefore, Proposition 13.5.1 ensures that $P_\Sigma(C)$ is compactly generated. □

Remark 15.1.3. If C is small with finite coproducts, the Yoneda embedding of C takes values in $P_\Sigma(C)$ and the induced functor

$$h_C : C \hookrightarrow P_\Sigma(C)$$

commutes with finite coproducts. Indeed, for any object x of C, the functor $\mathrm{Hom}_C(-, x)$ takes coproducts in C to products in Grpd.

Lemma 15.1.4. *Any small category with finite coproducts is sifted.*

Proof. Let A be a small category with an initial object $*$ as well as binary coproducts. Since $A = A_{*/}$, A is weakly contractible. We also have

$$A_{(x \amalg y)/} = A_{x/} \times_A A_{y/}$$

for all $x, y : A^{\simeq}$, proving that $A_{x/} \to A$ is a terminal functor for all x. □

Lemma 15.1.5. *Let C be a small category with finite coproducts. Given a presheaf $F\colon C^{\mathrm{op}} \to \mathrm{Grpd}$, the following assertions are equivalent.*

(i) The functor F preserves finite products.

(ii) The category $C_{/F}$ has finite coproducts.

(iii) The category $C_{/F}$ is sifted.

(iv) There is a small sifted category A, a functor $\varphi\colon A \to C$ and an isomorphism of type

$$\operatorname*{colim}_{a\colon A} h_C(\varphi(a)) \cong F$$

in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$.

Proof. It is clear that (iii) follows from (ii) by Lemma 15.1.4. Corollary 9.5.3 implies that (iv) follows from (iii). Assuming that F preserves finite products, we see that $C_{/F}$ has finite coproducts and that the projection $C_{/F} \to C$ preserves them as follows: The category $P_\Sigma(C)_{/F}$ has finite coproducts and the forgetful functor to $P_\Sigma(C)$ commutes with them. We have a pullback square

$$\begin{tikzcd} C_{/F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & P_\Sigma(C)_{/F} \arrow[d] \\ C \arrow[r, "h_C"] & P_\Sigma(C) \end{tikzcd}$$

in which the Yoneda embedding h_C preserves finite coproducts by Remark 15.1.3. Therefore, the dual version of Proposition 9.8.7 implies that $C_{/F}$ has finite coproducts. It follows from the last assertion of Proposition 15.1.2 and from Remark 15.1.3 that assertion (i) is a consequence of assertion (iv). □

Proposition 15.1.6. *Let $u\colon C \to D$ be a finite coproducts preserving functor between small categories with finite coproducts. Then the adjunction*

$$u_!\colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Grpd})\colon u^*$$

restricts to an adjunction of the form

$$u_!\colon P_\Sigma(C) \rightleftarrows P_\Sigma(D)\colon u^*\,.$$

Proof. Since the property for a functor to preserve finite products is compatible with composition of functors, it is clear that u^* sends $P_\Sigma(D)$ to $P_\Sigma(C)$. The fact that $u_!$ sends $P_\Sigma(C)$ to $P_\Sigma(D)$ follows from the fact that $u_! \circ h_C = h_D \circ u$ and from the characterization of presheaves that preserve finite products given by assertion (iv) of Lemma 15.1.5. □

Remark 15.1.7. The previous proposition is also a way to interpret localizations that are compatible with finite coproducts. Given C a small category with finite coproducts and $W \hookrightarrow \mathrm{Map}([1], C)$ a family of morphisms containing all identites and stable under binary coproducts, let us denote by

$$\gamma\colon C \to L(C)$$

the localization of C by W. By virtue of Proposition 10.1.1 and Proposition 10.1.5, the category $L(C)$ has an initial object as well as binary coproducts and the functor γ commutes with them. The preceding proposition thus provides an adjunction of the form

$$\gamma_! \colon P_\Sigma(()C) \rightleftarrows P_\Sigma(()L(C)) \colon \gamma^*$$

with γ^* full faithful. The essential image of γ^* consists precisely of the functors that preserve finite products and send maps in W to isomorphims. One ca identify $L(C)$ as the full subcategory of $P_\Sigma(()C)$ with objects the terms of the image of the map

$$C^{\simeq} \to P_\Sigma(()C) \quad x \mapsto \gamma_!(\mathrm{Hom}_C(-,x))\,.$$

Proposition 15.1.8. *Let C be a small category with finite coproducts and D a category with small colimits. The equivalence*

$$\mathrm{Fun}_!(\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Grpd}),D) = \mathrm{Fun}(C,D)$$

restricts to an equivalence from the category of small colimits preserving functors from $P_\Sigma(()C)$ to D and the category of finite coproducts preserving functors from C to D.

The proof is totally similar to the proof of Theorem 13.3.12 and is left to the reader.

Proposition 15.1.9. *Let D be a locally small category with small colimits. We assume that there is a small family of objects E in D with the following properties.*

(i) For any $x \colon E$, the functor
$$\mathrm{Hom}_D(x,-) \colon D \to \mathrm{Grpd}$$
commutes with small sifted colimits.

(ii) The family E generates D.

We let C be the smallest full subcategory of D containing the initial object as well as all the terms of E closed under binary coproducts. Then C is small and the full embedding

$$i \colon C \hookrightarrow D$$

extends to an equivalence
$$P_\Sigma(C) = D\,.$$

In particular, any objects of D canonically is a small sifted colimits of a functor with values in C.

The proof is entirely similar to the proof of Theorem 13.4.8 and is left as an exercise to the reader.

The preceding definition applies to $D = \mathrm{Grpd}$ with E the collection of terminal objects. We let FinSet be the smallest full subcategory of Grpd containing the empty anima as well as the terminal one and stable under binary coproducts, so that

$$P_\Sigma(\mathrm{FinSet}) = \mathrm{Grpd}\,.$$

The objects of FinSet are called *finite sets* (they indeed are sets).

15.2 Additive categories

Lemma 15.2.1 (the Eckmann-Hilton argument, le lemme de Godement)**.** *Let A be an anima together with a term $e\colon X$ as well as two maps*

$$\circ\colon A\times A\to A\,,\quad (a,b)\mapsto a\circ b\quad \textit{and}\quad \bullet\colon A\times A\to A\,,\quad (a,b)\mapsto a\bullet b\,.$$

We asume that the following rules hold.

(i) For every term $a\colon A$, there are terms of the following types:

$$\begin{aligned} u^l_a &\colon a = e\circ a\\ u^r_a &\colon a\circ e = a\\ v^l_a &\colon a = e\bullet a\\ v^r_a &\colon a\bullet e = a\,. \end{aligned}$$

(ii) For every terms $a,b,c,d\colon X$, there is a term of type

$$\gamma_{a,b,c,d}\colon (a\bullet b)\circ(c\bullet d) = (a\circ c)\bullet(b\circ d)\,.$$

Then, for any terms $a,b\colon A$, there are terms

$$\kappa_{a,b}\colon a\bullet b = a\circ b\quad \textit{and}\quad \kappa'_{a,b}\colon a\circ b = b\bullet a\,.$$

Proof. The term $\kappa_{a,b}$ is obtained as follows.

$$\begin{aligned} a\bullet b &= (a\circ e)\bullet(e\circ b)\\ &= (a\bullet e)\circ(e\bullet b)\\ &= a\circ b\,, \end{aligned}$$

Similarly, the sequence

$$\begin{aligned} a\circ b &= (e\bullet a)\circ(b\bullet e)\\ &= (e\circ b)\bullet(a\circ e)\\ &= b\bullet a \end{aligned}$$

yields $\kappa'_{a,b}$. □

Example 15.2.2. Given an anima X and a term $x\colon X$, we can form

$$\Omega(X,x) := \mathrm{Hom}_X(x,x)\,.$$

there is thus a composition law

$$c_{X,x}\colon \Omega(X,x)\times\Omega(X,x)\to\Omega(X,x)\,.$$

This operation defines a functor

$$\Omega\colon \mathrm{Grpd}_\bullet\to\mathrm{Grpd}_\bullet$$

that commutes with products. We define

$$\bullet\colon \Omega^2(X,x)\times\Omega^2(X,x) = \Omega(\Omega(X,\mathrm{id}_x)\times\Omega(X,\mathrm{id}_x)) \xrightarrow{\Omega(c_{X,x})} \Omega^2(X,x)$$

where $\Omega^2(X,x) := \Omega(\Omega(X,x),\mathrm{id}_x)$. There is also the composition law

$$\circ := c_{\Omega(X,x),\mathrm{id}_x}\colon \Omega^2(X,x)\times\Omega^2(X,x)\to\Omega^2(X,x)\,.$$

One checks immediately that Lemma 15.2.1 applies to the anima $A=\Omega^2(X,x)$ with these definitions of $\circ$ and $\bullet$.

Definition 15.2.3. A category C is *semiadditive* if its has an initial object 0 as well as binary coproducts with the following properties:

(i) The initial object 0 turns out to be a terminal object as well. In particular, for any object x of C, there is a map $p_x\colon x\to 0$.

(ii) For any objects x and y, the maps

$$p_x \amalg \mathrm{id}_y\colon x\amalg y\to x\amalg 0 = x \quad\text{ and }\quad \mathrm{id}_x \amalg p_y\colon x\amalg y\to 0\amalg y = y$$

exhibit $x\amalg y$ as a product of x and y in C.

Construction 15.2.4. Let C be a semiadditive category. For any objects $x,y\colon C^{\simeq}$, there is an operation

$$\mathrm{Hom}_C(x,y)\times\mathrm{Hom}_C(x,y)\to\mathrm{Hom}_C(x,y)\,,\quad \mapsto (f,g)\mapsto f+g$$

defined by the following composition

$$\mathrm{Hom}_C(x,y)\times\mathrm{Hom}_C(x,y) = \mathrm{Hom}_C(x\amalg x,y)\xrightarrow{\Delta^*}\mathrm{Hom}_C(x,y)$$

with $\Delta\colon x\to x\amalg x$ the map defined by the pair $(\mathrm{id}_X,\mathrm{id}_x)$ using the description of $x\amalg x$ as a product. This construction is functorial in the sense that for any functor between semiadditive categories

$$f\colon C\to D$$

that preserves finite coproducts (hence finite products as well), we have a canonical commutative square of the form

$$\begin{tikzcd}
\mathrm{Hom}_C(x,y)\times\mathrm{Hom}_C(x,y) \arrow[r, "+"] \arrow[d] & \mathrm{Hom}_C(x,y) \arrow[d] \\
\mathrm{Hom}_D(f(x),f(y))\times\mathrm{Hom}_D(f(x),f(y)) \arrow[r, "+"] & \mathrm{Hom}_D(f(x),f(y))
\end{tikzcd}$$

We observe furthermore that $\mathrm{Fun}(I,C)$ is semiadditive for any category I whenever C is pre-additive. This expresses how functorial is this construction in the variables $x,y\colon C$.

Example 15.2.5. The category RexCat is semiadditive. The initial object is the category [0] and the coproduct is given by the cartesian product of the underlying categories.

Construction 15.2.6 (2×2 matrices)**.** In a semiadditive category C, given for objects x, y, z, w a map

$$\varphi\colon x \amalg y \to z \amalg w$$

amounts to a pair of maps of type $u\colon x \amalg y \to z$ and $v\colon x \amalg y \to w$ (seeing $z \amalg w$ as a product) and seeing $x \amalg y$ as a coproduct, u correspond to a pair of maps of the form

$$a\colon x \to z \quad \text{and} \quad b\colon y \to z$$

whereas v corresponds to a pair of maps of the form

$$c\colon x \to w \quad \text{and} \quad d\colon y \to w\,.$$

In other words, defining a map φ as above is equivalent to defining a 2×2-matrix:

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The map φ is defined in formula by $\varphi(x, y) = (a(x) + b(y), c(x) + d(y))$.

Definition 15.2.7. An *additive* category is a semiadditive category C with the property that, for any object x of C, the matrix

$$\begin{pmatrix} \mathrm{id}_x & \mathrm{id}_x \\ \mathrm{id}_x & 0 \end{pmatrix}$$

(with $0\colon x \to x$ the unique map factorizing through 0) defines an automorphism of $x \amalg x$.

Proposition 15.2.8. *Any stable category is additive.*

Proof. This means that, for any objects x and y in a stable category C, for any maps $f, g\colon x \to y$ in C, one can reconstruct caonically the pair (f, g) from the pair $(f + g, f)$. For this, it suffices to produce a map $-g\colon x \to y$ such that $(f + g) + (-g) = f$ holds true. Since we have

$$\mathrm{Hom}_C(x, y) = \Omega^2(\mathrm{Hom}_C(x, \Sigma^2(y)), 0)\,,$$

Lemma 15.2.1 together with Example 15.2.2 tell us that one can see the operation $(f, g) \mapsto f + g$ as the composition law in the groupoid $\Omega(\mathrm{Hom}_C(x, \Sigma(y)), 0)$. Therefore, inverses for this composition law exist and this proves the proposition. ◻

Corollary 15.2.9. *Any full subcategory closed under finite (co)products of a stable category is additive.*

15.3 Waldhausen categories

In this chapter, we fix once and for all a universe of small categories Cat.

Definition 15.3.1. A *pre-Waldhausen* category is a small category A equipped with two subcategories

$$wA \hookrightarrow A \quad \text{and} \quad cA \hookrightarrow A$$

with the following properties, where morphisms of A that belong to wA are called *weak equivalences*, whereas those belonging to cA are called *cofibrations*.

W0. The category A has an initial object $*$. Furthermore, for any object x of A, the map $* \to a$ is a cofibration.

W1. For any object x of A, the identity of a is both a weak equivalence and a cofibration.

W2. The family of cofibrations is stable under pushouts in A.

W3. The family of trivial cofibrations (i.e. of maps that are both weak equivalences and cofibrations) is stable under pushouts.

W4 For any commutative cube in A of the form

$$\begin{tikzcd}
& x_0 \arrow[rr] \arrow[dl, "u"'] \arrow[dd, "i_0"' near start] & & x'_0 \arrow[dl, "u'"'] \arrow[dd] \\
x_1 \arrow[rr] \arrow[dd, "i_1"'] & & x'_1 \arrow[dd] & \\
& y_0 \arrow[rr] \arrow[dl, "v"] & & y'_0 \arrow[dl, "v'"] \\
y_1 \arrow[rr] & & y'_1 &
\end{tikzcd}$$

with both maps i_0 and i_1 cofibrations and both the front face and the back face cocartesian, if the three maps u, u' and v are weak equivalences, then so is v'.

A Waldhausen category is a pre-Waldhausen such that its initial object happens to be a terminal object as well.

Pre-Waldhausen categories naturally form a subgroupoid of $\mathrm{Fun}(\ulcorner, \mathrm{Cat})^{\simeq}$. A morphism of pre-Waldhausen categories is a morphism in $\mathrm{Fun}(\ulcorner, \mathrm{Cat})$, or equivalently, a functor

$$f\colon A \to B$$

preserving weak equivalences as well as cofibrations, with the additional property that, for any pushout square in A the form

$$\begin{array}{ccc}
x & \xrightarrow{u} & x' \\
{\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i'} \\
y & \xrightarrow{v} & y'
\end{array}$$

with i (hence also i') a cofibration, the associated commutative square

$$\begin{array}{ccc}
f(x) & \xrightarrow{f(u)} & f(x') \\
{\scriptstyle f(i)}\big\downarrow & & \big\downarrow{\scriptstyle f(i')} \\
f(y) & \xrightarrow{f(v)} & f(y')
\end{array}$$

is a pushout square in B.

Pre-Waldhausen categories naturally form a category $\mathrm{WaldCat}^{\mathrm{pre}}$, or rather a subcategory of $\mathrm{Fun}(\ulcorner, \mathrm{Cat})$.

Lemma 15.3.2. *Let A be a category equipped with a subcategory $cA \hookrightarrow A$ containing all identities and corresponding to a family of morphisms stable under pushouts in A. Define a cA-*cofibrant square *as a commutative square in A of the form*

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "i"'] & x' \arrow[d, "i"] \\ y \arrow[r, "v"] & y' \end{tikzcd}$$

in which i (hence i' as well) is a map in cA and such that the induced map $y \amalg_x x' \to y'$ is a map in cA as well. Then the family of cA-cofibrant squares form a family of maps in $\mathrm{Fun}([1], A)$ *that is stable under composition as well as under pushouts.*

Proof. Let us consider a composable pair of squares of the form

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "i"'] & x' \arrow[r, "u'"] \arrow[d, "i"'] & x'' \arrow[d, "i''"] \\ y \arrow[r, "v"] & y' \arrow[r, "v'"] & y'' \end{tikzcd}$$

each of then being cA-cofibrant. In the induced commutative diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & x'' \arrow[d] \\ y \amalg_x x' \arrow[r] \arrow[d] & y \amalg_x x'' \arrow[d] \\ y' \arrow[r] & y' \amalg_{x'} x'' \end{tikzcd}$$

the upper square is cocartesian and so is the outer square, so that the lower square is cocartesian as well. In particular, the vertical map at the lowest right corner is in cA. This implies that the map $y \amalg_x x'' \to y''$ is in cA, since it is the composition of two maps in cA.

Let us consider a commutative cube of the form

$$\begin{tikzcd} & x_0 \arrow[rr] \arrow[dl] \arrow[dd] & & x'_0 \arrow[dl] \arrow[dd] \\ x_1 \arrow[rr, crossing over] \arrow[dd] & & x'_1 & \\ & y_0 \arrow[rr] \arrow[dl] & & y'_0 \arrow[dl] \\ y_1 \arrow[rr] & & y'_1 \arrow[from=uu, crossing over] & \end{tikzcd}$$

in which the back faces is cA-cofibrant and the two vertical faces on the left and right hand side are cocartesian. We observe that the induced commutative square

$$\begin{tikzcd} y_0 \amalg_{x_0} x'_0 \arrow[r] \arrow[d] & y_1 \amalg_{x_1} x'_1 \arrow[d] \\ y'_0 \arrow[r] & y'_1 \end{tikzcd}$$

is cocartesian. This implies right away that the front face is cA-cofibrant. $\square$

Construction 15.3.3. Given a pre-Waldhausen categories A and B, we write $\mathrm{Wald}(A, B)$ for the full subcategory of $\mathrm{Fun}(A, B)$ whose objects are morphisms of pre-Waldhausen categories. In fact $\mathrm{Wald}(A, B)$ is a Waldhausen category itself: one defines weak equivalences as the natural tranformations between morphims of Waldhausen categories

$$\alpha\colon f \to g$$

such that $\alpha_x\colon f(x) \to g(x)$ is a weak equivalence for any object $x\colon A^{\simeq}$. Cofibrations are those natural tranformations between morphims of Waldhausen categories

$$\alpha\colon f \to g$$

with the property that, for any cofibration $i\colon x \to y$ in A, the induced commutative square

$$\begin{tikzcd} f(x) \arrow[r, "\alpha_x"] \arrow[d, "f(i)"'] & g(x) \arrow[d, "g(i)"] \\ f(y) \arrow[r, "\alpha_y"] & g(y) \end{tikzcd}$$

induces a cofibration of type $f(y) \amalg_{f(x)} g(x) \to g(y)$ (the stability of these cofibrations under pushouts follows straight away from the preceding lemma).

A particular case of interest is $A = [0] \star [1]$ for some $n\colon \mathbb{N}$. A morphism $[0] \star [1] \to B$ literally is a commutative triangle of the form

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ x \arrow[rr, "u"] & & y \end{tikzcd}$$

with u a cofibration. There is thus an equivalence from $\mathrm{Wald}([0] \star [1], B)$ to the full subcategory of $\mathrm{Fun}([1], B)$ whose objects are the cofibrations of B. More generally, given $n\colon \mathbb{N}$, we define $B^{[n]}$ as the full subcategory of $\mathrm{Fun}([n], B)$ whose objects are sequences of cofibrations of the form

$$x_0 \to \cdots \to x_n$$

in B, the assignment

$$(x_0 \to \cdots \to x_n) \mapsto (0 \to x_0 \to \cdots \to x_n)$$

defines an equivalence

$$B^{[n]} \cong \mathrm{Wald}([0] \star [n], B)\,.$$

Example 15.3.4. Any pointed small category with finite colimits is a Waldhausen category in which the weak equivalences are the isomorphisms and all morphisms are cofibrations. This observation defines a full embedding of the form

$$\mathrm{RexCat}_\bullet \hookrightarrow \mathrm{WaldCat}\,.$$

Particular cases of interest are stable categories.

Example 15.3.5. If A is a Waldhausen category the so is $\mathrm{Fun}(I, A)$ for any small category I: we define weak equivalences (cofibration, resp.) as those natural transformations $u\colon F \to G$ such that $u_i\colon F_i \to G_i$ is a weak equivalence (a cofibration, resp.) for all objects $i\colon I^{\simeq}$. We define $w(I, A)$ as the full subcategory of $\mathrm{Fun}(I, A)$ whose objects are the functors sending all maps in I to weak equivalences in A. We define weak equivalences and cofibrations in $w(I, A)$ as those maps that weka equivalences and cofibrations, respectively, in $\mathrm{Fun}(I, A)$. This turns $w(I, A)$ into a Waldhausen category.

Notation 15.3.6. For $A\colon$ WaldCat and $[n]\colon \mathbb{N}$ we write

$$w_n(A) := w([n], A)\,.$$

15.4 Additive functors

Definition 15.4.1. A *short exact sequence* in a Waldhausen category A is a pushout square of the form

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d] & x \arrow[d, "v"] \\ * \arrow[r] & x'' \end{tikzcd}$$

in which u is a cofibration. We denote by $\mathrm{ExSeq}(A)$ the full subcategory of $\mathrm{Fun}(\square, A)$ whose objects are short exact sequences. The assignment

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d] & x \arrow[d, "v"] \\ * \arrow[r] & x'' \end{tikzcd} \quad \mapsto (u\colon x' \to x)$$

defines an equivalence

$$\mathrm{ExSeq}(A) \cong \mathrm{Wald}([0] \star [1], A)\,.$$

Since $\mathrm{Wald}([0] \star [1], A)$ is a Waldhausen category (as a particular case of Construction 15.3.3), this turns $\mathrm{ExSeq}(A)$ into a Waldhausen category.

The proofs of the following lemma are obvious and they are only stated for the record.

Lemma 15.4.2. *We denote by* 0 *the Waldhausen category with underlying category* $[0]$*, Then* 0 *is both an initial and a terminal object of* WaldCat

Lemma 15.4.3. *If A and B are Waldhausen categories, then the cartesian product of the underlying small categories $A \times B$ is canonically endowed with the structure of a Waldhausen category through*

$$w(A \times B) = wA \times wB \quad \textit{and} \quad c(A \times B) = cA \times cB$$

The projections $\mathrm{pr}_1\colon A \times B \to A$ *and* $\mathrm{pr}_2\colon A \times B \to B$ *are morphisms of Waldhausen categories and exhibit $A \times B$ as the cartesian product of A and B in* WaldCat*. Dually, the two functors*

$$i_1\colon A \to A \times B\,, \quad x \mapsto (x, *) \quad \textit{and} \quad i_2\colon B \to A \times B\,, \quad y \mapsto (*, y)$$

are morphisms of Waldhausen categories and exhibit $A \times B$ as the coproduct of A and B in WaldCat.

In other words:

Proposition 15.4.4. *The category* WaldCat *is semiadditive.*

In particular, any full subcategory of WaldCat stable under finite coproducts is semiadditive as well.

Definition 15.4.5. An *admissible class of Waldhausen categories* is a family of Waldhausen categories stable under finite products as well as under the operators $\mathrm{Wald}([n], -)$ and $w([n], -)$ for all $n\colon \mathbb{N}$. Given such a class C, we define $\mathrm{WaldCat}_C$ as the full subcategory of WaldCat with objects the terms of C.

Example 15.4.6. Stable categories form an admissible class. In this case, we simply have $\mathrm{StabCat} = \mathrm{WaldCat}_C$. Therefore, any family of stable categories closed under finite products and under the operator $\mathrm{Fun}([n], -)$ is admissible. For instance, compact objects in $\mathrm{StabPr}^{\mathrm{L}}_{\omega}$ for a small admissible class.

Definition 15.4.7. Let C be an admissible class of Waldhausen categories and D a pointed category with finite limits. A functor $E\colon \mathrm{WaldCat}_C \to D$ is *additive* it satisfies the following properties.

(1) It preserves finite products.

(2) For any Waldhausen category A, the functor

$$A \to w([1], A)\,, \quad x \mapsto \mathrm{id}_x$$

induces an equivalence of type

$$E(A) \cong E(w([1], A))\,.$$

(3) For any Waldhausen category A the functor

$$(\ker, \mathrm{coker})\colon \mathrm{ExSeq}(A) \to A \times A$$

defined by the assignment

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d] & x \arrow[d, "v"] \\ * \arrow[r] & x'' \end{tikzcd} \quad \mapsto (x', x'')$$

induces an equivalence of type

$$E(\mathrm{ExSeq}(A)) \cong E(A) \times E(A)\,.$$

if it preserves finite products and if, for any Waldhausen category A the functor

$$(\ker, \mathrm{coker})\colon \mathrm{ExSeq}(A) \to A \times A$$

defined by the assignment

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d] & x \arrow[d, "v"] \\ * \arrow[r] & x'' \end{tikzcd} \quad \mapsto (x', x'')$$

induces an equivalence of type

$$E(\mathrm{ExSeq}(A)) \cong E(A) \times E(A)\,.$$

Proposition 15.4.8. *The property of being additive is preserved by small sifted colimits in the category of functors from* $\mathrm{WaldCat}_C$ *to* $\mathrm{Grpd}_\bullet$.

Proof. Since colimits in $\mathrm{Fun}(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$ are computes pointwise and since the property of being additive is expressed by the invertibility of functorially given maps between finite products of evaluations, this follows right away from the fact that small sifted colimits commute with finite products in $\mathrm{Grpd}_\bullet$. □

Lemma 15.4.9. *Let C be an admissible class of Waldhausen categories. For any finite products preserving functor*

$$F\colon \mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet\,,$$

if F is additive then the functor

$$A \mapsto \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(S_n(A))$$

is additive as well.

Proof. This follows immediately from Proposition 15.4.8, once we have noticed that Δ^{op} if sifted, and that the operators S_n and w_m commute with finite products as well as with the operator ExSeq. □

Proposition 15.4.10. *Let $F\colon \mathrm{WaldCat}_C \to D$ be an additive functor. We consider two morphisms*

$$p\colon A \to A' \quad \text{and} \quad q\colon A \to A''$$

in $\mathrm{WaldCat}_C$ *together with the following extra data.*

(i) There are given maps $i\colon A' \to A$ and $j\colon A'' \to A$ such that i is a section of p and j a section of q.

(ii) Each composition $p \circ j$ and $q \circ i$ vanishes (i.e. is isomorphic to the constant functor with value the terminal object).

(iii) There is given a map

$$\varphi\colon A \to \mathrm{ExSeq}(A)$$

so that $\varphi(x)$ is an exact sequence of the form

$$\begin{tikzcd} i(p(x)) \arrow[r] \arrow[d] & x \arrow[d] \\ * \arrow[r] & j(q(x)) \end{tikzcd}$$

for all $x\colon A$. Then p and q together induce an equivalence of type

$$F(A) = F(A') \times F(A'')\,.$$

Proof. Since p and q commute with finite coproducts, condition (ii) ensures that the functor

$$u\colon A' \times A'' \to A\,, \quad (x, y) \mapsto i(x) \amalg j(y)$$

defines a section of the functor

$$(p, q)\colon A \to A' \times A''\,.$$

It suffices to check that the composition of $u \circ (p, q)$ becomes the identity of $F(A)$ after applying F. We define a functor

$$\psi\colon A \to \mathrm{ExSeq}(A)$$

by the rule bellow.

$$\psi(x) = \begin{tikzcd} i(p(x)) \arrow[r] \arrow[d] & i(p(x)) \amalg j(q(x)) \arrow[d] \\ * \arrow[r] & j(q(x)) \end{tikzcd}$$

Since postcomposing with (ker, coker) turns φ and ψ equal, We have an equality of type $F(\varphi) = F(\psi)$. Let $p\colon \mathrm{ExSeq}(A) \to A$ the functor defined by the rule

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & x \arrow[d] \\ * \arrow[r] & x'' \end{tikzcd} \mapsto x\,.$$

Since $p \circ \varphi = \mathrm{id}_A$ and $p \circ \psi = u \circ (p, q)$, this shows that $F(p, q)$ is invertible. □

Corollary 15.4.11. *We consider an additive functor of the form*

$$F\colon \mathrm{WaldCat}_C\colon \ \to D\,.$$

Let $n\colon \mathbb{N}$ and $i_n\colon [n] \hookrightarrow [n+1]$ the map defined by $i_n(k) = k + 1$. Then, for any Waldhausen category A, the induced functor

$$i_n^*\colon F(S_{n+1}(A)) \to F(S_n(A))$$

is equivalent over $F(S_n(A))$ to the projection

$$\mathrm{pr}_1\colon F(S_n(A)) \times F(S_1(A)) \to F(S_n(A))\,.$$

Proof. The case $n = 0$ is clear so that we may assume $n > 0$. If we identify $S_n(A)$ with $A^{[n-1]}$, the functor i_n^* simply restricts from $[n]$ to $[n-1]$ in the following way: to each sequence of cofibrations

$$x_1 \to x_2 \to \cdots \to x_{n+1}$$

we form the sequence of cofibrations

$$x_2/x_1 \to \cdots \to x_{n+1}/x_1$$

where we have formed the pushout squares

$$\begin{tikzcd} x_1 \arrow[r] \arrow[d] & x_k \arrow[d] \\ * \arrow[r] & x_k/x_1 \end{tikzcd}$$

in A for all $k \geq 1$. There is also a functor from $A^{[n]}$ to A sending a sequence of shape

$$x_1 \to x_2 \to \cdots \to x_{n+1}$$

to x_1. The corollary will follow from the fact that the induced functor

$$A^{[n]} \to A^{[n-1]} \times A$$

is sent to an equivalence through any additive functor F.
There is an obvious functor

$$A^{[n-1]} \to A^{[n]}$$

sending a sequence

$$y_1 \to \cdots \to y_n$$

to the sequence

$$0 \to y_1 \to \cdots \to y_n$$

and a functor

$$A \to A^{[n]}$$

sending a term $z\colon A$ to the sequence

$$z \xrightarrow{\mathrm{id}_z} \cdots \xrightarrow{\mathrm{id}_z} z\,.$$

One checks immediately that the assumptions of Proposition 15.4.10 are fulfilled and this implies this corollary. □

Lemma 15.4.12. *Let X be a simplicial anima. Then X_0 is canonically isomorphic to the colimit of the composition*

$$\Delta^{\mathrm{op}} \xrightarrow{dec^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{X} \mathrm{Grpd}$$

where $dec\colon \Delta \to \Delta$ is the functor $[n] \mapsto [0] \star [n]$.

Proof. We form the following pullback square.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \Delta_{/X} \arrow[d] \\ \Delta \arrow[r, "dec"] & \Delta \end{tikzcd}$$

The colimit we want to compute is $\Pi_\infty(C)$. Sending a pair $([n], x)$, with $x\colon [n+1] \to X$, to x_0 yields a functor

$$\pi\colon C \to X_0$$

(the evaluation at the first vertex of $[n+1]$). We will prove that this turns C in to a contractible category in context X_0. There is indeed a section

$$s\colon X_0 \to C$$

of the functor π: sendings any term $x\colon X_0$ to the pair $([0], \mathrm{id}_x)$ where $\mathrm{id}_x\colon [1] \to X$ is the composition of $x\colon [0] \to X$ with the unique map $[1] \to [0]$. There is a functorial map

$$r_n\colon [n+2] \to [n+1]$$

induced by $[1] \to [0]$ and the presentations $[n+2] = [1] \star [n]$ and $[n+1] = [0] \star [n]$. There is similarly a map $i_n\colon [n+1] \to [n+2]$ induced by the map $1\colon [0] \to [1]$. We associate to each pair $([n], x)$ in C the pair $([n+1], x \circ r_n)$, which defines a functor

$$d\colon C \to C$$

over X_0. The embeddings i_n define a natural transformation $\mathrm{id}_C \to d$ and the embeddings $[1] \hookrightarrow [1] \star [n]$ a natural transformation from s to d, all of this in context X_0. This proves that C is weakly contractible in context X_0. □

Proposition 15.4.13. *Let C be an admissible class of Waldhausen categories. We consider a finite products preserving functor*

$$F\colon \mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet\,.$$

There is a canonical commutative square

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(S_n(A)) \end{tikzcd}$$

functorially in A and F, that is cartesian whenever F is additive.

Proof. We observe that $F(S_0(A))$ is contractible for all A and thus, by virtue of Lemma 15.4.12 there is a canonical equivalence of the form

$$\operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(S_{n+1}(A)) = *$$

The functorial embedding $i_n\colon [n] \hookrightarrow [n+1]$ defined through the identification of $[n+1]$ with $[0] \star [n]$ defines a functor

$$i_n^*\colon S_{n+1}(A) \to S_n(A)$$

functorially in $[n]$ and in A. We obtain a canonical commutative square of the form

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(S_{n+1}(A)) \arrow[d] \\ F(S_0(A)) \arrow[r] & \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(S_n(A)) \end{tikzcd}$$

where the vertical maps are induced by the functors i_n^* above and the horizontal maps are instances of the canonical map

$$t_0 \to \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} t_n$$

for any functor $t\colon \Delta^{\mathrm{op}} \to \mathrm{Grpd}_\bullet$.

It suffices now to prove that the commutative square above is cartesian whenever F is additive. By virtue of Lemma 9.9.4, it suffices to prove that, for any additive functor $F\colon \mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet$, and any map $u\colon [m] \to [n]$ in Δ, the functor F sends the commutative square of Waldhausen categories

$$\begin{tikzcd} S_{n+1}(A) \arrow[r] \arrow[d, "i_n^*"'] & S_{m+1}(A) \arrow[d, "i_m^*"] \\ S_n(A) \arrow[r] & S_m(A) \end{tikzcd}$$

to a pullback square. We conclude this proof by observing thanks to Corollary 15.4.11 that this commutative square is equivalent to the commutative square

$$\begin{tikzcd} S_n(A) \times S_1(A) \arrow[r, "\mathrm{id} \times u^*"] \arrow[d, "\mathrm{pr}_1"'] & S_n(A) \times S_1(A) \arrow[d, "\mathrm{pr}_1"] \\ S_n(A) \arrow[r, "u^*"] & S_m(A) \end{tikzcd}$$

which is obviously cartesian. □

Corollary 15.4.14. *For any admissible class of Waldhausen categories, the category of additive functors from* $\mathrm{WaldCat}_C$ *to* $\mathrm{Grpd}_\bullet$ *is additive.*

Proof. Since any object of $\mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$ if of the form $\Omega^2(F)$ by the preceding proposition, we can repeat the proof of Corollary 15.2.9. □

15.5 Construction of K-theory

Definition 15.5.1. A *region* is a finite poset R with pushouts.

If R is a region, we can define the *length* of a map $x \leq y$ in R: this is the maximal $n\colon \mathbb{N}$ such that there exists an embedding $u\colon [n] \hookrightarrow R$ with $u(0) = x$ and $u(n) = y$. An *elementary square* in R is a pushout square in R in which all maps are of length 1. We observe that any pushout square in R in which all maps are of positive length can be uniquely presented as the composition of elementary squares. An *edge map* in R is a map of length at least 2 that does not appear as the diagonal of an elementary square. An *edge object* of R is an objet that is the source or the target of an edge map.

A morphism of regions is a functors (between regions) of the form $f\colon R \to S$ that preserves pushout squares as well as edge objects.

A *Waldhausen region* is a region equipped with a family of maps closed under composition and under pushouts, called cofibrations (in a pedantic way, we could say that $[0] \star R$ is a pre-Waldhauesen category). A morphism of Waldhausen regions is a morphism of regions that preserves cofibrations.

Example 15.5.2. For any $m, n\colon \mathbb{N}$ the product $[m] \times [n]$ is a region with no edge object. On the other hand, $[0] \star ([m] \times [n])$ is a region whose unique edge object is the initial object. The elementary squares in $[m] \times [n]$ are those of the form

$$\begin{tikzcd} (i,j) \arrow[r] \arrow[d] & (i,j+1) \arrow[d] \\ (i+1,j) \arrow[r] & (i+1,j+1) \end{tikzcd}$$

for all $0 \leq i, j < n$. Unless stated otherwise, we will consider $[m] \times [n]$ as a Waldhausen region with cofibrations those maps of the form $(i,j) \leq (k,j)$.

Example 15.5.3. For $n\colon \mathbb{N}$, the category $\mathrm{Fun}([1],[n])$ may be identified with the full subposet of $[n]\times[n]$ whose objects are terms of the form (i,j) with $i \leq j$. This description turns $\mathrm{Fun}([1],[n])$ into a region whose edge objects are the identities, i.e. the pairs of the form (i,i) for any $i\colon [n]^{\simeq}$. The elementary squares in $\mathrm{Fun}([1],[n])$ are those of the form

$$\begin{tikzcd}
(i,j) \arrow[r] \arrow[d] & (i,j+1) \arrow[d] \\
(i+1,j) \arrow[r] & (i+1,j+1)
\end{tikzcd}$$

for all $0 \leq i < j < n$. Unless stated explicitly otherwise, we will consider $\mathrm{Fun}([1],[n])$ as Waldhausen region with cofibrations those maps of the form $(i,j) \leq (k,j)$. This defines a functor

$$[n] \mapsto \mathrm{Fun}([1],[n])$$

from Δ to the category of Waldhausen regions.

Definition 15.5.4. Let A be a Waldhausen category and R a Waldhausen region. We denote by $S(R,A)$ the full subcategory of $\mathrm{Fun}(R,A)$ whose objects are those functors of the form $x\colon R \to A$ with the following properties.

(i) The functor x preserves cofibrations.

(ii) The functor x preserves pushout squares.

(iii) The functor x sends edge objects to initial objects.

Remark 15.5.5. Any morphism of regions $f\colon R \to Q$ induces a right exact functor

$$f^*\colon S(Q,A) \to S(R,A)$$

obtained as the restriction of $f^*\colon\ \mathrm{Fun}(Q,A) \to \mathrm{Fun}(R,A)$.

Construction 15.5.6. For any Waldhausen category A and any region R, the category $S(R,A)$ has a natural structure of Waldhausen category. More precisely, we have The canonical embedding $R \hookrightarrow [0] \star R$ induces an equivalence

$$S([0]\star R, A) \cong S(R,A)$$

and $S([0]\star R, A)$ may be seen as a full subcategory of $\mathrm{Wald}([0]\star R, A)$ whose objects ar the functors sending edge objects to initial objects. We define weak equivalences and cofibrations in $S(R,A)$ as those maps that have these properties when seen in $\mathrm{Wald}([0]\star R, A)$.

Definition 15.5.7. We define a functor

$$S\colon \mathrm{WaldCat} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{WaldCat})$$

assigning to each $A\colon \mathrm{WaldCat}$ and $[n]\colon \Delta$ the Waldhausen category

$$S_n(A) := S(\mathrm{Fun}([1],[n]), A)\,.$$

We call $S(A)\colon\ \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{WaldCat})$ *Waldhausen's S-construction* associated to A.

Lemma 15.5.8. *Let A be a Waldhausen category and $n\colon \mathbb{N}$. The fully faithful right exact functor $u\colon [n] \hookrightarrow \mathrm{Fun}([1],[n])$ defined by $u(i) = (0,i)$ induces an equivalence*

$$u^*\colon S_n(A) \cong \mathrm{Wald}([n], A)\,.$$

Proof. It suffices to check that

$$u^*\colon \ \mathrm{Fun}([1], S_n(A))^{\simeq} \to \mathrm{Fun}([1], \mathrm{Wald}([n], A))^{\simeq}$$

is an equivalence. Given any Waldhausen category B, we can equipped $\mathrm{Fun}([1], B)$ with a structure of Waldhausen categories in which the weak equivalences and the cofibrations are defined termwise. We then observe that

$$\mathrm{Fun}([1], \mathrm{Wald}([n], A)) = \mathrm{Wald}([n], \mathrm{Fun}([1], A)) \quad \text{and} \quad \mathrm{Fun}([1], S_n(A))) = S_n(\mathrm{Fun}([1], A))\,.$$

Therefore, we see that it suffices to prove that

$$u^*\colon S_n(A)^{\simeq} \to \mathrm{Wald}([n], A)^{\simeq}$$

is an equivalence. This is done by induction on n. For $n = 0$, this is clear: both sides are contractible. For $n = 1$, both sides are equivalent to $A^{\simeq}$ and u^* is equivalent to the identity of $A^{\simeq}$. Given $n > 0$ for which the statements is known to be true, we deduce the case of $n+1$ as follows. Let R_n the full subcategory of $\mathrm{Fun}([1], [n+1])$ whose objects are the pairs of the form (i, j) with $j \leq n$ or $i = 0$. The embedding

$$v\colon R_n \hookrightarrow \mathrm{Fun}([1], [n+1])$$

induces an equivalence

$$v^*\colon S(\mathrm{Fun}([1], [n+1]), A)^{\simeq} \cong S(R_n, A)^{\simeq}$$

whose inverse is given by completing each diagram of the form

$$\begin{tikzcd}
* \arrow[r] & x_{0,1} \arrow[r] \arrow[d] & \cdots \arrow[r] & x_{0,n} \arrow[r] \arrow[d] & x_{0,n+1} \\
 & * \arrow[r] & \cdots \arrow[r] & x_{1,n} \arrow[d] & \\
 & & & \vdots \arrow[d] & \\
 & & & * &
\end{tikzcd}$$

(in which $x_{0,n} \to x_{0,n+1}$ is a cofibration) through pushouts (and a map to $*$).

$$
\begin{tikzcd}
* \arrow[r] & x_{0,1} \arrow[r] \arrow[d] & \cdots \arrow[r] & x_{0,n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & x_{0,n+1} \arrow[d] \\
 & * \arrow[r] & \cdots \arrow[r] & x_{1,n} \arrow[r] \arrow[d] & x_{1,n+1} \arrow[d] \\
 & & & \vdots \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & \vdots \arrow[d] \\
 & & & * \arrow[r] & x_{n,n+1} \arrow[d] \\
 & & & & *
\end{tikzcd}
$$

We see that there is a pullback square of the form

$$
\begin{tikzcd}
S(R_n, A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbb{S}_n(A) \arrow[d] \\
\mathrm{Wald}([2], A) \arrow[r] & A
\end{tikzcd}
$$

induced by the obvious embedding $\mathrm{Fun}([1],[n]) \hookrightarrow R_n$ and the map $w\colon [2] \hookrightarrow R_n$ defined by $w(0) = (0,0)$, $w(1) = (0,n)$ and $w(2) = (0,n+1)$. Since there is a pullback of the form

$$
\begin{tikzcd}
\mathrm{Wald}([n+1], A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Wald}([n], A) \arrow[d] \\
\mathrm{Wald}([2], A) \arrow[r] & A
\end{tikzcd}
$$

the induction hypothesis implies that there is a canonical equivalence

$$S(R_n, A) \cong \mathrm{Wald}([n+1], A)\,. \qquad \square$$

Remark 15.5.9. We may see the assignment $A \mapsto A^{\simeq}$ as a functor of type

$$\mathrm{WaldCat} \to \mathrm{Grpd}_\bullet\,.$$

Using the canonical equivalence of type $[m] \times \mathrm{Fun}([1],[n]) = \mathrm{Fun}([1],[n]) \times [m]$, we get a canonical equivalence of type

$$S_m(w_m(A)) = w_m(S_n(A))$$

for all $A\colon \mathrm{WaldCat}$. We will simply write

$$w_m S_n(A) := w_m(S_n(A))$$

and denote

$$wS(A)\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Grpd}_\bullet\,.$$

We thus get a functor

$$wS\colon \mathrm{WaldCat} \to \mathrm{Fun}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathrm{Grpd}_\bullet)\,.$$

Recall that the universal left fibration $\mathrm{Grpd}_\bullet \to \mathrm{Grpd}$ commutes with colimits indexed by small weakly contractible categories, in particular with small sifted colimits.

Definition 15.5.10. Given a Waldhausen category A, we define the *K-theory* of A as the pointed anima

$$K(A) := \Omega\Big(\operatorname*{colim}_{[m]\colon \Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} w_m S_n(A)^{\simeq}\Big)\,.$$

15.6 The additivity theorem

Recall that $\operatorname*{colim}_{\Delta^{\mathrm{op}}}\colon \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grpd}) \to \mathrm{Grpd}$ is called the geometric realization functor.

Lemma 15.6.1. *Let*

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative triangle of simplicial animae. We assume that, in any groupoidal context, for any map $\sigma\colon \Delta^n \to S$, the induced map of type

$$\Delta^n \times_S X \to \Delta^n \times_S Y$$

induces an isomorphism in Grpd *after geometric realization. Then the geometric realization of u is an isomorphism.*

Proof. Our assumptions mean that, for any small anima Γ and any term $n\colon \mathbb{N}$, for any map of the form $\sigma\colon \Gamma\times\Delta^n \to S$, the induced map of type

$$\sigma^*(u)\colon (\Gamma\times\Delta^n)\times_S X \to (\Gamma\times\Delta^n)\times_S Y$$

induces an isomorphism in Grpd after geometric realization. But any simplicial anima canonically is a colimit of simplicial animae of the form $\Gamma\times\Delta^n$. Furthermore, for any morphism $f\colon A\to S$ in $\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grpd})$, the pullback

$$f^*\colon \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grpd})_{/S} \to \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grpd})_{/A}$$

has a right adjoint: we can identify

$$\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grpd})_{/S} = \operatorname{Fun}((\Delta_{/S})^{\mathrm{op}}, \mathrm{Grpd})$$

so that f^* is the pullback functor along the functor

$$f_!\colon \Delta_{/A} \to \Delta_{/S}\,.$$

In particular, the functor f^* commutes with colimits. Therefore, the map u canonically is a colimit of morphisms of the form $\sigma^*(u)$ induced by pullbacks from maps σ as above. Since the realization functor commutes with colimits (this is a colimit functor itself by definition), the geometric realization of u is the colimit of the geometric realization of each of the $\sigma^*(u)$'s. We conclude by the obvious observation that colimits preserve isomorphism. $\square$

Remark 15.6.2. Let $n\colon \mathbb{N}$ and A a small category. Let $h\colon \Delta^1 \times \Delta^n \to \Delta^n$ be the map induced by the functor expressing the fact that n is a terminal object of $[n]$.

The map h induces a functor

$$\eta\colon \Delta_{/[1]} \times_\Delta \Delta_{/[n]} \to \Delta_{/\mathrm{Fun}([1],[n])}$$

sending $u\colon [n] \to [m]$ and $v\colon [m] \to [1]$ to the unique natural transformation

$$\kappa_{u,v}\colon u \to \bar{u}$$

where $\bar{u}\colon [m] \to [n]$ is defined as the composition of (v, u) with h. Since $u(i) \leq \bar{u}(i)$, this defines $\kappa_{u,v}$.

The evaluation functor

$$\mathrm{ev}\colon [1] \times \mathrm{Fun}([1],[m]) \to [m]$$

is functorial in $[m]$ hence so is its product with the identity of $[1]$

$$\mathrm{id}_{[1]} \times \mathrm{ev}\colon [1] \times [1] \times \mathrm{Fun}([1],[m]) \to [1] \times [m]\,.$$

This defines a functor

$$c_{[m]}\colon [1] \times \mathrm{Fun}([1],[m]) \to \mathrm{Fun}([1],[1] \times [m])$$

functorially in $[m]$. With the identification

$$\mathrm{Fun}([1] \times \mathrm{Fun}([1],[m]), A) = \mathrm{Fun}(\mathrm{Fun}([1],[m]), \mathrm{Fun}([1], A))$$

this defines a functor

$$c^*_{[m]}\colon \mathrm{Fun}(\mathrm{Fun}([1],[1] \times [m]), A) \to \mathrm{Fun}(\mathrm{Fun}([1],[m]), \mathrm{Fun}([1], A))$$

functorially in $[m]$ and A. Assigning each $u\colon [m] \to \mathrm{Fun}([1],[n])$ to the corresponding map $[1] \times [m] \to [n]$ defines a functor

$$u^*\colon \mathrm{Fun}(\mathrm{Fun}([1],[n]), A) \to \mathrm{Fun}(\mathrm{Fun}([1],[1] \times [m]), A)$$

functorially in u, seen as a term of $\Delta_{/\mathrm{Fun}([1],[n])}$. Composing with the functor η above, this means that, for any functor

$$x\colon \mathrm{Fun}([1],[n]) \to A$$

if, for any map $w\colon [m] \to [n]$ one writes

$$w^*(x)\colon \mathrm{Fun}([1],[n]) \to A$$

for the composition of x with the functor induced by w by applying $\mathrm{Fun}([1], -)$, then for any maps $u\colon [m] \to [n]$ and $v\colon [m] \to [1]$, one has a canonical map

$$\kappa_{u,v,x}\colon u^*(x) \to \bar{u}^*(x)$$

functorially in (u, v).

Lemma 15.6.3. *Let A be a Waldhausen category in which weak equivalences all are isomorphisms. The functor*

$$(\ker, \operatorname{coker}) \colon \operatorname{ExSeq}(A) \to A \times A$$

induces a morphism of simplicial animae

$$S(\operatorname{ExSeq}(A))^{\simeq} \to S(A)^{\simeq} \times S(A)^{\simeq}$$

that yields an isomorphism after geometric realization.

Proof. We have a commutative triangle of the form

$$\begin{tikzcd} S(\operatorname{ExSeq}(A))^{\simeq} \arrow[rr] \arrow[dr, "S(\ker)^{\simeq}"'] & & S(A)^{\simeq} \times S(A)^{\simeq} \arrow[dl, "\mathrm{pr}_1"] \\ & S(A)^{\simeq} & \end{tikzcd}$$

and we will apply Lemma 15.6.1 to it. We consider a map

$$x \colon \Delta^n \to S(A)$$

i.e. a functor $x \colon \operatorname{Fun}([1], [n]) \to A$ satisying the conditions of Definition 15.5.4. We have to prove that the induced map

$$\Delta^n \times_{S(A)^{\simeq}} S(\operatorname{ExSeq}(A))^{\simeq} \to \Delta^n \times_{S(A)^{\simeq}} (S(A)^{\simeq} \times S(A)^{\simeq})$$

induces an isomorphism after geometric realization. We will consider first the case where $n = 0$. In that case x simply is the initial object $*$ in A. It suffices to check that the comparison map between pullbacks above is an equivalence of simplicial animae, which can be checked objectswise. If we evaluate at $[p]$, this is the same as evaluating at 0, up to replacing A by $\operatorname{Wald}([p+1], A)$. Therefore, it suffices to check that we have an equivalence when we evaluate at $[0]$. The left hand side is then the collection of pushout squares of the form

$$\begin{tikzcd} * \arrow[r] \arrow[d] & x \arrow[d, "v"] \\ * \arrow[r] & x'' \end{tikzcd}$$

whereas the right hand side is the collection of pairs of objects of the form $(*, x'')$. It is clear that assinging x'' to each pushout square as above is an equivalence. Let us consider the general case. Let us write $x_0 \colon \Delta^0 \to S(A)$ for the restriction of x along $\{0\} \hookrightarrow \Delta^n$. This induces a pullback square of the form

$$\begin{tikzcd} \Delta^0 \times_{S(A)^{\simeq}} S(\operatorname{ExSeq}(A))^{\simeq} \arrow[r, "\simeq"] \arrow[d, "i"'] & \Delta^0 \times_{S(A)^{\simeq}} (S(A)^{\simeq} \times S(A)^{\simeq}) \arrow[d, "j"] \\ \Delta^n \times_{S(A)^{\simeq}} S(\operatorname{ExSeq}(A))^{\simeq} \arrow[r] & \Delta^n \times_{S(A)^{\simeq}} (S(A)^{\simeq} \times S(A)^{\simeq}) \end{tikzcd}$$

in which the upper horizontal map is an equivalence (by the preceding discussion). Therefore, it suffices to check that the two vertical maps induce an isomorphism after geometric realization. But the vertical map j on the right is equivalent to

$$S(A)^{\simeq} \cong \{0\} \times S(A)^{\simeq} \hookrightarrow \Delta^n \times S(A)^{\simeq} .$$

Since the geometric realization functor commutes with finite products (because Δ^{op} is sifted), it suffices to check that any map of the form $\Delta^0 \to \Delta^n$ induces an isomorphism after geometric realization. This follows from the fact that the geometric realization of Δ^n is contractible (as is the colimit of any representable presheaf on a small category).

What precedes tells us that

$$\Delta^0 \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq} \cong S(A)^{\simeq} .$$

There is a canonical map

$$p\colon \Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq} \to S(A)^{\simeq})$$

obtained through the second projection

$$\Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq} \to S(\mathrm{ExSeq}(A))^{\simeq}$$

followed from the map

$$S(\mathrm{ExSeq}(A))^{\simeq} \to S(A)^{\simeq}$$

induced by the functor

$$\mathrm{coker}\colon \mathrm{ExSeq}(A) \to A .$$

It is immediate to see that the vertical map i on the left hand side of the square above is a section of p. It suffices now to prove that the other composite $i \circ p$ simplicially homotopic to the identity.

Let $h\colon \Delta^1 \times \Delta^n \to \Delta^n$ be the map induced by the functor expressing the fact that n is a terminal object of $[n]$. Keeping track of the notations of Remark 15.6.2, for any maps $u\colon [m] \to [n]$ and $v\colon [m] \to [1]$ to the unique natural transformation

$$\kappa_{u,v}\colon u \to \bar{u}$$

produces a canonical map

$$\kappa_{u,v,x}\colon u^*(x) \to \bar{u}^*(x)$$

functorially in (u, v). A term of the category $\Delta_{/X}$ for

$$X = \Delta^1 \times \left(\Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq}\right)$$

literally is a tuple of the form (u, v, ξ) with (u, v) as above any ξ a tern of the category of pushout square in A of the form bellow.

$$\xi \quad = \quad \begin{tikzcd} u^*(x) \arrow[r] \arrow[d] & b \arrow[d] \\ * \arrow[r] & c \end{tikzcd}$$

Such a term ξ is thus canonically the back face of a term of the category of commutative cubes in A of the form

$$\begin{tikzcd} & u^*(x) \arrow[rr] \arrow[dl, "\kappa_x"'] \arrow[dd] & & b \arrow[dl] \arrow[dd] \\ \bar{u}^*(x) \arrow[rr] \arrow[dd] & & \bar{b} \arrow[dd] & \\ & * \arrow[rr] \arrow[dl] & & c \arrow[dl, "\mathrm{id}_c"] \\ * \arrow[rr] & & c & \end{tikzcd}$$

in which the back face, the front face, as well as the top face are pushout squares. We define

$$\bar{\xi} \quad := \quad \begin{array}{ccc} \bar{u}^*(x) & \longrightarrow & b \\ \downarrow & & \downarrow \\ * & \longrightarrow & c \end{array}$$

and the rule

$$H(u, v, \xi) = (\bar{u}, \bar{\xi})$$

produces a well edfined functor over Δ hence a morphism of simplicial animae

$$H\colon \Delta^1 \times \left(\Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq}\right) \to \Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq}$$

which is a homotopy between the identity and $i \circ p$. □

Lemma 15.6.4. *The functor* $\operatorname*{colim}_{[m]\colon \Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} w_m S_n(-)^{\simeq}\colon \mathrm{WaldCat} \to \mathrm{Grpd}_{\bullet}$ *is additive.*

Proof. We observe that

$$w_m(\mathrm{ExSeq}(A)) = \mathrm{ExSeq}(w_m(A))$$

functorially in $[m]$ and A. Therefore, it follows from Lemma 15.6.3, it suffices to prove that the functor

$$\Phi\colon A \mapsto \operatorname*{colim}_{[m]\colon \Delta^{\mathrm{op}}} w_m(A)^{\simeq}$$

sends the functor

$$i\colon A \to w_1(A)\,, \quad x \mapsto \mathrm{id}_x$$

to an equivalence. Let us consider the functor

$$\mathrm{ev}_0\colon w_1(A) \to A\,.$$

It suffices to check that $i \circ \mathrm{ev}_0$ becomes simplicially homotopic to the identity after applying the functor

$$w(-)^{\simeq}\colon \mathrm{WaldCat} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grpd})\,.$$

The map $\min\colon [1] \times [1] \to [1]$ induces a functor

$$h\colon [1] \times \mathrm{Fun}([1], A) \to \mathrm{Fun}([1], A)$$

corresponding by transposition to the functor

$$\mathrm{ev}_0^*\colon \mathrm{Fun}([1], A) \to \mathrm{Fun}([1] \times [1], A)\,.$$

Taking the Rezk nerve yields a map

$$N(h)\colon \Delta^1 \times N(\mathrm{Fun}([1], A)) \to N(\mathrm{Fun}([1], A))\,.$$

We observe that there is a canonical embedding

$$w(A)^{\simeq} \hookrightarrow N(\mathrm{Fun}([1], A))$$

so that the map $N(h)$ restricts to a map

$$H\colon \Delta^1 \times w(A)^{\simeq} \to w(A)^{\simeq}$$

that is the homotopy we were looking for. □

Theorem 15.6.5 (Waldhausen)**.** *The K-theory functor $K\colon \mathrm{WaldCat} \to \mathrm{Grpd}_\bullet$ is additive.*

Proof. This follows from Lemma 15.6.4 since the loop functor commutes with finite prducts. □

Notation 15.6.6. Given an admissible class of Waldhausen categories C, and D a pointed category with finite limits, we denote by

$$\mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, D)$$

the full subcategory of $\mathrm{Fun}(\mathrm{WaldCat}_C, D)$ spanned by finite product preserving functors and we write

$$\mathrm{Add}(\mathrm{WaldCat}_C, D)$$

for the full subcategory of $\mathrm{Fun}(\mathrm{WaldCat}_C, D)$ spanned by additive functors.

Up to size issues one can enforce the additivity property for formal reasons, as explained in the next proposition.

Proposition 15.6.7. *If D is a pointed compactly generated category, then so is $\mathrm{Add}(\mathrm{WaldCat}_C, D)$ for any small admissible class of Waldhausen categories C. More precisely, $\mathrm{Add}(\mathrm{WaldCat}_C, D)$ is a left Bousfield localization of the compactly generated category $\mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, D)$ by a small family of maps between compact objects. In particular, the full embedding*

$$\mathrm{Add}(\mathrm{WaldCat}_C, D) \hookrightarrow \mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, D)$$

commutes with filtered colimits and has a left adjoint. Furthermore, in the case where $D = \mathrm{Grpd}_\bullet$, the full embedding

$$\mathrm{WaldCat}_C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$$

commutes small with sifted colimits.

Proof. The is a Yoneda embedding

$$h\colon \mathrm{WaldCat}_C^{\mathrm{op}} \to \mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$$

We define the family of maps S in $\mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$ explicitly as follows:

(i) For any $A\colon \mathrm{WaldCat}^{\simeq}$, the image of

$$A \mapsto w([1], A)\,, \quad x \mapsto \mathrm{id}_x$$

through h is in S;

(ii) for any $A\colon \mathrm{WaldCat}^{\simeq}$, the image of

$$(\ker, \mathrm{coker})\colon \mathrm{ExSeq}(A) \to A \times A$$

through h is in S.

It follows right away from the Yoneda lemma applied to $\mathrm{WaldCat}_C^{\mathrm{op}}$ that the S-local objects in the compactly generated category $\mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$ precisely are the additive functors. The first part of the proposition thus follows from Proposition 13.5.1 and Proposition 13.2.5. Proposition 15.4.8 ensures the last part to be true. □

Remark 15.6.8. Given any category C with an initial object $*$, for any integer $n\colon \mathbb{N}$, let us denote by $i\colon [n] \to [n+1]$ the upper inclusion $k \mapsto k+1$. The functor

$$i^*\colon \operatorname{Fun}([n+1], C) \to \operatorname{Fun}([n], C)$$

has a right adjoint i_* ad a left adjoint $i_!$ sending a sequence of the form

$$a_1 \to \cdots \to a_n$$

to

$$a_1 \xrightarrow{\mathrm{id}_{a_1}} a_1 \to \cdots \to a_n \quad \text{and} \quad * \to a_1 \to \cdots \to a_n \quad \text{respectively.}$$

Both functors $i_!$ and i_* are fully faithful. There is an obvious natural transformation $i_! \to i_*$ corresponding to the inverse of $i^*\, i_* \cong \mathrm{id}$. Let

$$\kappa\colon \operatorname{Fun}([n], C) \to \operatorname{Fun}([n], C)$$

be the functor sending

$$a_0 \to \cdots \to a_n$$

to

$$a_0 \xrightarrow{\mathrm{id}} a_0 \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} a_0$$

with the obvious natural transformation $\kappa \to \mathrm{id}$. Given any $a\colon \operatorname{Fun}([n], C)$, there is a canonical (and functorial) pushout square of the form bellow.

$$\begin{tikzcd}
i_!\, i^*\, \kappa(a) \arrow[r] \arrow[d] & i_!\, i^*(a) \arrow[d] \\
i_*\, i^*\, \kappa(a) \arrow[r] & a
\end{tikzcd}$$

Lemma 15.6.9. *For each $n\colon \mathbb{N}$, there is an equivalence*

$$\operatorname{Wald}(S_n(C), D) = \operatorname{Wald}(C, S_n(D))$$

functorially in C and D.

Proof. We proceed by induction on n. The case $n = 0$ being obvious.

If we assume C and D to have finite colimits, we know that there is an explicit equivalence of the form

$$\operatorname{Rex}(\operatorname{Fun}(P^{\mathrm{op}}, C), D) \cong \operatorname{Rex}(C, \operatorname{Fun}(P, D))$$

for any finite poset P by Theorem 13.5.8, which is functorial in a way that is explained in Remark 13.5.10. Given a general Waldhausen category C, with Yoneda embedding

$$h\colon C \to \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$$

we can consider the left Bousfield localization of $\operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ be the family of maps S of the form:

(i) $h(*) \to *$;

(ii) $h(y) \amalg_{h(x)} h(z) \to h(y \amalg_x z)$ for any cofibration $x \to y$ and any any map $x \to z$;

There is a further left Bousfield localization that invert furthermore all maps of the form $h(u)$ for u a weak equivalence of C.

We define weak equivalences of $C' := L_S \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd})$ to be the map that become isomorphisms after applying the left adjoint to the further left Bousfield localization alluded to above. Cofibrations in C' simply are defined to be all maps. The Yoneda embedding of C defined a fully faithful functor that is also a morphism of Waldhausen categories $C \to C'$.

The functor $u^*\colon D' \to C'$ defined by $F \mapsto F \circ^{\mathrm{op}} u$ has a left adjoint $u_!$ defined by applying the left adjoint of u^* at the level of ordinary presheaves and then apply the S-localization functor. Since representable presheaves are S-local and since $u_!$ preserves representable presheaves, the commutative square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Grpd}) \arrow[d, "u_!"] \\ D \arrow[r] & \operatorname{Fun}(D^{\mathrm{op}}, \mathrm{Grpd}) \end{tikzcd}$$

restricts to a commutative square of the form bellow.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[d, "u_!"] \\ D \arrow[r] & D' \end{tikzcd}$$

Using the isomorphism $[n] \cong [n]^{\mathrm{op}}$, we have a canonical equivalence of the form:

$$\operatorname{Rex}(\operatorname{Fun}([n], C'), D') \cong \operatorname{Rex}(C', \operatorname{Fun}([n], D'))\,.$$

We want to show that this restricts to an equivalence

$$\operatorname{Wald}(C^{[n]}, D) \cong \operatorname{Wald}(C, D^{(n)})\,.$$

This is the precise form of the induction we want to establish. This is easily seen to be implied by the preceding remark together with Remark 13.5.10. □

Essentially the same argument proves the following.

Lemma 15.6.10. *For each $n\colon \mathbb{N}$, there is an equivalence*

$$\operatorname{Wald}(w_n(C), D) = \operatorname{Wald}(C, w_n(D))$$

functorially in C and D.

Definition 15.6.11. Let C be a small admissible class of Waldhausen categories. We denote by

$$i\colon \operatorname{Add}(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet) \hookrightarrow \operatorname{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$$

the canonical embedding and by

$$\alpha\colon \operatorname{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet) \to \operatorname{Add}(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$$

its left adjoint. A morphism of finite products preserving presheaves $u\colon F \to G$ is said to be an *additive equivalence* if $\alpha(u)\colon \alpha(F) \to \alpha(G)$ is an isomorphism.

Lemma 15.6.12. *Let C be a small admissible class of Waldhausen categories. The subcategory of additive equivalences has the following stability properties.*

(i) Additive equivalences are stable under small sifted colimits: for any small sifted category and any natural transformation $u\colon F \to G$ between I-indexed functors with values in $\mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$, if $u_i\colon F_i \to G_i$ is an additive equivalence for all $i\colon I^\simeq$, then the induced map

$$\operatorname*{colim}_{i\colon I} u_i\colon \operatorname*{colim}_{i\colon I} F_i \to \operatorname*{colim}_{i\colon I} G_i$$

is an additive equivalence.

(ii) Additive equivalences are stable under finite products.

(iii) The assignment $F \mapsto F \circ S_n$ preserves additive equivalences for all $n\colon \mathbb{N}$.

(iv) The assignent $F \mapsto F \circ w_n$ preserves additive equivalences for all $n\colon \mathbb{N}$.

Proof. any morphism $u\colon F \to G$ in $\mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$ fist functorially in a commutative square of the form

$$\begin{tikzcd} F \arrow[r] \arrow[d, "u"'] & i\alpha(F) \arrow[d, "i\alpha(u)"] \\ G \arrow[r] & i\alpha(G) \end{tikzcd}$$

in which the two horizontal maps are additive equivalences, being the unit the adjunction (α, i) with i fully faithful. Since u is an additive equivalence if and only if $i\alpha(u)$ is an isomorphism, property (i) follows from the fact that the functor $i\alpha$ commutes with small sifted colimits (since i has this property whereas α commutes with all colimits).

Similarly, in order to prove assertion (ii), it suffices to check that, for any functor

$$F\colon \mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet$$

that commutes with finite products and any $n\colon \mathbb{N}$, the canonical map

$$F^n \to i\alpha(F)^n$$

is an additive equivalence. Equivalently, we have to see that for any additive functor

$$E\colon : \mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet$$

the map above induces an identification of type

$$\mathrm{Hom}(i\alpha(F)^n, E) = \mathrm{Hom}(F^n, E)\,.$$

Since both assignments

$$F \mapsto F^n \quad \text{and} \quad F \mapsto i\alpha(F)^n$$

commute with small sifted colimits and since any such F is a small sifted colimit of corepresentable functors, we may assume that there is a Waldhausen category A in the class C such that $F = \mathrm{Wald}(A, -)^\simeq$. Since the category $\mathrm{WaldCat}_C$ is semi-additive, we then have

$$F^n = \mathrm{Wald}(A^n, -)\,.$$

Since E commutes with finite products then have by the Yoneda lemma:

$$\mathrm{Hom}(F^n, E) = E(A^n) = E(A)^n \,.$$

But since the category $\mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$ is semi-additive as well, we also have

$$\mathrm{Hom}(i\alpha(F)^n, E) = \mathrm{Hom}(i\alpha(F), E)^n \,.$$

Since $\mathrm{Hom}(i\alpha(F), E) = \mathrm{Hom}(F, E)$, this proves property (ii).

The proof of assertion (iii) is similar. This is obvious for $n \leq 1$ so that we may assume $n > 1$. Evaluating at each vertex of shape $(0, i)$ for $1 < i \leq n$, yields a natural transformation of the form

$$S_n(A) \to A^{n-1}$$

hence a functorial morphism of type

$$F(S_n(A)) \to F(A^{n-1}) = F(A)^{n-1}$$

for any functor F that commutes with finite products. Since $F \mapsto F^{n-1}$ preserves additive equivalences, it suffices to prove that the map

$$F \circ S_n \to F^{n-1}$$

is an additive equivalence for all F. Since this is a map between constructions that are compatible with small sifted colimits, it suffices to prove this in the case where $F = \mathrm{Wald}(A, -)^{\simeq}$ for some Waldhausen category A in the class C. By virtue of Lemma 15.6.9, there is then a canonical equivalence of type

$$F \circ S_n = \mathrm{Wald}(S_n(A), -)^{\simeq} \,.$$

Therefore, for any additive functor E we have

$$\mathrm{Hom}(F, E) = E(S_n(A)) = E(A)^{n-1}$$

by Corollary 15.4.11 (and an easy induction). We also have $F^{n-1} = \mathrm{Wald}(A^{n-1}, -)$ by semi-additivity of $\mathrm{WaldCat}_C$, hence $\mathrm{Hom}(F^{n-1}, E) = E(A)^{n-1}$. This implies that $\mathrm{Hom}(-, E)$ turns the map from $F \circ S_n$ to F^{n-1} to an equivalence and thus proves assertion (iii).

Finally, the natural map $A \to w_n(A)$ induces a map

$$F \to F \circ w_n$$

and it suffices to prove that the latter is an additive equivalence. Since this is a map between functors that preserve small sifted colimits and since and F is a small sifted colimits of corepresentable functors, it suffices to prove this for F of the form $\mathrm{Wald}(A, -)^{\simeq}$ for some A in the class C. By virtue of Lemma 15.6.10, we then have

$$F = \mathrm{Wald}(w_n(A), A) \,.$$

Therefore, for any additive functor E, the map $A \to w_n(A)$ induces an equivalence

$$E(A) \cong E(w_n(A))$$

which is equivalent to $\mathrm{Hom}(F \circ w_n, E) \to \mathrm{Hom}(F, E)$. □

Theorem 15.6.13. *Let C be any admissible class of Waldhausen categories. Then the full embedding*

$$\mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet) \hookrightarrow \mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet)$$

has a left adjoint defined explicitely by assigning to each finite products preserving functor

$$F\colon \mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet$$

the functor defined by the rule

$$A \mapsto \Omega\Big(\operatorname*{colim}_{[m]\colon \Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(w_m S_n(A))\Big)$$

Proof. Given a finite products functor $F\colon \mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet$, we define, for $A\colon \mathrm{WaldCat}_C$,

$$PF(A) := \operatorname*{colim}_{[m]\colon \Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(w_m S_n(A))\,.$$

We will establish first that PF is additive. For this, we observe that the formation of PF commutes with the operaton of restriction to $\mathrm{WaldCat}_{C'}$ for any embedding $C' \hookrightarrow C$, with C' an admissble class of Waldhausen categories. Furthermore, a finite product preserving functor from $\mathrm{WaldCat}_C$ to $\mathrm{Grpd}_\bullet$ is additive if and only if its restriction to $\mathrm{WaldCat}_{C'}$ for any small C' is additive. Therefore, in order to prove that PF is additive, without any loss of generality, we may assume that C is small. Since F commutes with finite products, F is then a small sifted colimit of corepresentable functors by Lemma 15.1.5. By virtue of the last assertion of Proposition 15.1.2, we see that the functor $F \mapsto PF$ commutes with sifted colimits (as a small sifted colimit of functors that preserves sifted colimits). Therefore, we deduce from Proposition 15.4.8 that it suffices to prove that PF is additive whenever F is corepresentable by a Waldhausen category B in the class C. But then, we observe that

$$w_m S_n \mathrm{WaldCat}(B, A)^{\simeq} = \mathrm{WaldCat}(B, w_m S_n(A))^{\simeq}\,.$$

for all m, n and A functorially. In other words we then have

$$PF(A) = \operatorname*{colim}_{[m]\colon \Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} w_m S_n \mathrm{WaldCat}(B, A)^{\simeq}\,.$$

Using the fact that

$$\mathrm{WaldCat}(B, \mathrm{ExSeq}(A)) = \mathrm{ExSeq}(\mathrm{WaldCat}(B, A))$$
$$\text{and}\quad w([1], \mathrm{Wald}(B, A)) = \mathrm{Wald}(B, w([1], A))\,,$$

we conclude from Lemma 15.6.4 that PF is additive.

To prove the second part, we also reduce to the case where C is small: we choose a larger universe Cat' containing Cat so that C is small. If Grpd' is the full subcategory of groupoids in Grpd, we know that the embedding

$$\mathrm{Grpd} \hookrightarrow \mathrm{Grpd}'$$

commutes with small colimits and with small limits. This means that the induced full embedding

$$\mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet) \hookrightarrow \mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}'_\bullet)$$

also commutes with small limits and with small colimits. In particular, it commutes with the loop functor Ω as well as with the operator P defined above. This means that it suffices to prove the universal property of the construction ΩP in the larger universe Cat' for which C is small.

From now on, we focus on the case where C is small and we keep track of the notations of Definition 15.6.11.

Now, Proposition 15.4.13 tells us that the operator P comes equipped with a natural map of the form

$$c_F \colon F \to \Omega(PF)$$

that is invertible whenever F is additive. Furthermore, since the category Δ^{op} is small and sifted, Lemma 15.6.12 tells us that the operator P preserves additive equivalences. Let $G = i\alpha(F)$ be the universal additive functor associated to F. The unit map $\eta \colon F \to G$ is then an additive equivalence with additive codomain. We obtain a commutative square

$$\begin{tikzcd}
F \arrow[r, "\eta"] \arrow[d, "c_F"'] & G \arrow[d, "c_G"] \\
\Omega(PF) \arrow[r, "\Omega P\eta"] & \Omega(PG)
\end{tikzcd}$$

with $P\eta$ an additive equivalence between additive functors, hence an equivalence. Since c_G is an equivalence, this proves that c_F is an additive equivalence with additive codomain, which implies the theorem right away. □

Remark 15.6.14. In the case where the weak equivalences of the Waldhausen category A all are isomorphisms, the simplicial object

$$[m] \mapsto w_m(S_n(A))$$

is constant for all n and thus have a simplification:

$$\operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(S_n(A)) \operatorname*{colim}_{[m]\colon \Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(w_m S_n(A))$$

Corollary 15.6.15. *Let C be any admissible class of Waldhauen categories. Then the universal additive functor associated to the functor*

$$\mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet\,, \quad A \mapsto A^{\simeq}$$

is the additive functor

$$\mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet\,, \quad A \mapsto K(A)\,.$$

Corollary 15.6.16. *Let C be any admissible class of Waldhauen categories and A a Waldhausen category that belongs to C. Then the universal additive functor associated to the functor*

$$\mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet\,, \quad B \mapsto \mathrm{Wald}(A,B)^{\simeq}$$

is the additive functor

$$\mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet\,, \quad B \mapsto K(\mathrm{Wald}(A,B))\,.$$

Examples of admissible classes of Waldhausen categories are the one of small stable categories as well as the one of small pointed categories with finite colimits. We might in either cases above restrict ourselves to the idempotent-complete categories.

Remark 15.6.17. Let C be an admissible class of Waldhausen categories. We let W be the smallest family of morphisms in $\mathrm{WaldCat}_C$ stable under finite products containing any map of the form:

1. $A \to w_1(A)$, defined by $x \mapsto \mathrm{id}_x$, for any A in C;
2. $(\ker, \mathrm{coker})\colon \mathrm{ExSeq}(A) \to A \times A$ for any A in C.

We observe that W contains all identities since the identity of A is a retract of the map from A to $w_1(A)$ above. We denote by

$$\gamma\colon \mathrm{WaldCat}_C \to L_{\mathrm{Add}}(\mathrm{WaldCat}_C)$$

the localization of $\mathrm{WaldCat}_C$ by W. We know that this localization has finite products and that the functor γ preserves them by Proposition 10.1.1 and Proposition 10.1.5. By virtue of Proposition 15.1.6, there is thus an adjunction

$$\gamma_!\colon \mathrm{Fun}_\Sigma(\mathrm{WaldCat}_C, \mathrm{Grpd}) \rightleftarrows \mathrm{Fun}_\Sigma(L_{\mathrm{Add}}(\mathrm{WaldCat}_C), \mathrm{Grpd})\colon \gamma^*$$

with γ^* fully faithful. The previous corollary means that, for any A and B in C, the map

$$\mathrm{Wald}(A, -)^{\simeq} \to \gamma^* \gamma_!(\mathrm{Wald}(A, -)^{\simeq})$$

evaluated at B literally is the canonical map

$$\mathrm{Wald}(A, B)^{\simeq} \to K(\mathrm{Wald}(A, B))\,.$$

In other words we have a canonical identification

$$\mathrm{Hom}_{L_{\mathrm{Add}}(\mathrm{WaldCat}_C)}(\gamma(A), \gamma(B)) = K(\mathrm{Wald}(A, B))$$

functorially in (A, B) seen in $\mathrm{WaldCat}_C^{\mathrm{op}} \times \mathrm{WaldCat}_C$.
We also have a canonical identification

$$\mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{Grpd}_\bullet) = \mathrm{Fun}_\Sigma(L_{\mathrm{Add}}(\mathrm{WaldCat}_C), \mathrm{Grpd})\,.$$

In particular, if C is small, any additive functor $E\colon \mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet$ canonically is a small sifted colimit of functors of the form $K(\mathrm{Wald}(A, -))$ with A in C.

15.7 The localization theorem

Construction 15.7.1. Let $f\colon A \to B$ be a morphism of Waldhausen categories. Recall that $i_n\colon [n] \to [n+1]$ is the upper embedding defined through $i \mapsto i+1$. We form the pullback square bellow.

$$\begin{tikzcd} S_n(f\colon A \to B) \arrow[r, "q_n"] \arrow[d, "p_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & S_{n+1}(B) \arrow[d, "i_n^*"] \\ S_n(A) \arrow[r, "S_n(f)"] & S_n(B) \end{tikzcd}$$

This construction clearly is functorial in $[n]\colon \Delta^{\mathrm{op}}$.

Proposition 15.7.2 (Waldhausen)**.** *Let C be an admissible class of Waldhausen categories. We consider a morphism $f\colon A \to B$ in* $\mathrm{WaldCat}_C$*. For any additive functor $E\colon \mathrm{WaldCat}_C \to \mathrm{Grpd}_\bullet$, there is a canonical pullback square of the form bellow.*

$$\begin{tikzcd} \operatorname*{colim}_{[n]\colon\Delta^{\mathrm{op}}} E(S_n(f\colon A\to B)) \arrow[r] \arrow[d] & 0 \arrow[d] \\ \operatorname*{colim}_{[n]\colon\Delta^{\mathrm{op}}} E(S_n(A)) \arrow[r] & \operatorname*{colim}_{[n]\colon\Delta^{\mathrm{op}}} E(S_n(B)) \end{tikzcd}$$

In particular, there is a canonical fiber sequence of the form bellow.

$$\begin{tikzcd} E(A) \arrow[r, "E(f)"] \arrow[d] & E(B) \arrow[d] \\ 0 \arrow[r] & \operatorname*{colim}_{[n]\colon\Delta^{\mathrm{op}}} E(S_n(f\colon A\to B)) \end{tikzcd}$$

Proof. The last assertion follows from the first by Proposition 15.4.13. The proof of this proposition is in fact a variation on the proof of the latter. Since

$$\operatorname*{colim}_{[n]\colon\Delta^{\mathrm{op}}} E(S_{n+1}(B)) = 0\,,$$

by Lemma 9.9.4, it suffices to prove that, for each $n\colon\ \mathbb{N}$, the induced commutative square

$$\begin{tikzcd} E(S_n(f\colon A\to B)) \arrow[r] \arrow[d] & E(S_{n+1}(B)) \arrow[d] \\ E(S_n(A)) \arrow[r] & E(S_n(B)) \end{tikzcd}$$

is a pullback square. Through the identifications $S_n(A) = A^{[n-1]}$ for $n > 0$, one can identify $S_n(f\colon A \to B)$ as a category with terms the pairs (x, y) where x is a term in $A^{[n-1]}$ and y a commutative diagram in B of the form

$$\begin{tikzcd} y_1 \arrow[r] \arrow[d] & y_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & y_n \arrow[d] \\ 0 \arrow[r] & f(x_1) \arrow[r] & \cdots \arrow[r] & f(x_{n-1}) \end{tikzcd}$$

in which all squares are pushout squares.

Let $v\colon S_n(B) \hookrightarrow S_{n+1}(B)$ be the full embedding defined by sending a sequence

$$x_1 \to \cdots \to x_n$$

to the sequence

$$0 \to x_1 \to \cdots \to x_n\,.$$

Since u is a section of i_n^*, there is a canonical pullback square of the form bellow

$$\begin{tikzcd} S_n(A) \arrow[r] \arrow[d, "u"'] & S_n(B) \arrow[d, "v"] \\ S_n(f\colon A\to B) \arrow[r] & S_{n+1}(B) \end{tikzcd}$$

(in particular u is fully faithful and a secion of p_n). There is also a full embedding

$$j\colon B \hookrightarrow S_n(f\colon A \to B)$$

defined by sending x to the pair consisting of the sequence of length n

$$x \xrightarrow{\mathrm{id}_x} \cdots \xrightarrow{\mathrm{id}_x} x$$

and of the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & f(x) \arrow[r] \arrow[d] & \cdots \arrow[r] & f(x) \arrow[d] \\ 0 \arrow[r] & f(x) \arrow[r] & \cdots \arrow[r] & f(x) \end{tikzcd}$$

in which all the vertical maps are identities. The functor j is a section of the functor π sending a pair (x, y) to the object y_1. There is an obvious functorial exact sequence of the form

$$\begin{tikzcd} j(\pi(x,y)) \arrow[r] \arrow[d] & (x,y) \arrow[d] \\ 0 \arrow[r] & v(p_n(x,y)) \end{tikzcd}$$

and therefore a direct application of Proposition 15.4.10 shows that the pair of functors

$$(p_n, \pi)\colon S_n(f\colon A \to B) \to S_n(A) \times B$$

induces an equivalence

$$E(S_n(f\colon A \to B)) = E(S_n(A)) \times E(B)\,.$$

It follows that the squares

$$\begin{tikzcd} E(S_n(f\colon A \to B)) \arrow[r] \arrow[d] & E(S_{n+1}(B)) \arrow[d] \\ E(S_n(A)) \arrow[r] & E(S_n(B)) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E(S_n(A)) \times E(B) \arrow[r, "S_n(f)\times\mathrm{id}"] \arrow[d, "\mathrm{pr}_1"'] & E(S_n(B)) \times E(B) \arrow[d, "\mathrm{pr}_1"] \\ E(S_n(A)) \arrow[r, "S_n(f)"] & E(S_n(B)) \end{tikzcd}$$

are equivalent, hence are both cartesian, which achieves this proof. □

Theorem 15.7.3 (Thomason-Trobaugh)**.** *Let B a small stable category and A a thick subcategory of B. Then the pullback square*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & B/A \end{tikzcd}$$

induces a fiber sequence

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] & K(B) \arrow[d] \\ 0 \arrow[r] & K(B/A) \end{tikzcd}$$

in $\mathrm{Grpd}_\bullet$.

Proof. We let C be the Waldhausen category defined by $C = B$ as categories, with cofibrations being all maps but with weak equivalences being the maps whose cofiber lies in A. We observe then that the equivalence

$$S_{n+1}(B) = B^{[n]}$$

induces an equivalence of the form

$$S_n(f\colon A \to B) = w_n(C)\,.$$

Applying Proposition 15.7.2 to the addmissible class of Waldhausen categories given by small stable categories, with $E = K$, and taking into account Remark 15.6.14, we see that there is a canonical fiber sequence of the form bellow.

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] & K(B) \arrow[d] \\ 0 \arrow[r] & K(C) \end{tikzcd}$$

It suffices now to see that the morphism of Waldhausen categories $C \to B/A$ induced by the functor $B \to B/A$ yields an equivalence of the form

$$K(C) = K(B/A)\,.$$

We want to understand the map

$$\operatorname*{colim}_{[m]\colon\Delta^{\mathrm{op}}}\ \operatorname*{colim}_{[n]\colon\Delta^{\mathrm{op}}} w_m(S_n(C))^{\simeq} \to \operatorname*{colim}_{[m]\colon\Delta^{\mathrm{op}}}\ \operatorname*{colim}_{[n]\colon\Delta^{\mathrm{op}}} \mathrm{Iso}_m(S_n(B/A))$$

where $\mathrm{Iso}_n = \mathrm{Fun}([n], (-)^{\simeq})$. The assignment

$$[m] \mapsto w_m(S_n(C))^{\simeq}$$

is the nerve of the category of weak equivalences $w(S_n(C))$, whereas

$$[m] \mapsto \mathrm{Fun}([m], S_n(B/A)^{\simeq})$$

is the nerve of the groupoid $S_n(B/A)^{\simeq}$. Taking the colimit in the variable $[m]$ amounts to consider the map

$$\Pi_\infty(w(S_n(C))) \to S_n(B/A)^{\simeq}\,.$$

It suffices to check that the latter is an equivalence for all n. The case $n = 0$ is obvious (we are comparing two terminal categories). For $n > 0$, identifying S_n with $(-)^{[n-1]}$, taking into account Theorem 13.5.7 with $P = [n-1]$, this is a particular case of Proposition 13.1.14. □

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