

Generalizations of the Kervaire invariant

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1. Introduction

M. Kervaire in [4], F. Peterson and myself in [1], and W. Browder in [3] defined numerical invariants for various classes of $2n$ -manifolds roughly as follows: Suppose M is a closed compact $2n$ -manifold with some additional structure T . In [4] T is a framing of the normal bundle of M and n is odd, in [1] it is a spin structure and $n \equiv 1 \pmod{4}$ and in [3] it is a v_{n+1} structure. Using T one constructs a function

$$\varphi: H^n(M; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

satisfying

$$(1.1) \quad \varphi(u + v) = \varphi(u) + \varphi(v) + (u \cup v)(M).$$

The Kervaire invariant of (M, T) is defined to be the Arf invariant of φ . $\text{Arf } \varphi = 0$ or 1 and is 0 if and only if φ is zero on more than half the elements of $H^n(M; \mathbb{Z}_2)$.

In this paper we describe a general technique for constructing functions satisfying (1.1) and hence of obtaining generalizations of the Kervaire invariant. This technique gives, as special cases, the functions defined in [1], [3] and [4]. In the remainder of this section we outline in detail our techniques and state our results. Some of these results appear in [2]. Our algebraic results concerning the Arf invariant are stated in Theorem 1.20 and proved in § 3. The proofs of all other lemmas and theorems in this section are either given immediately or in § 2.

All homology and cohomology groups will be with $\mathbb{Z}_2$ coefficients unless otherwise stated. Usually spaces will have base points. $[X, Y]$ denotes the set of homotopy classes of maps from X to Y . S denotes suspension and

$$\{X, Y\} = \lim [S^k X, S^k Y].$$

K_n will denote $K(\mathbb{Z}_2, n)$.

We first describe how $\{ \}$ gives rise to quadratic functions. Let X be a CW complex with base point of dimension $2n$. We define a function

$$F: H^n(X) \times H^{2n}(X) \longrightarrow \{X, K_n\}$$

as follows:

¹ This work was carried out while the author was supported by NSF grant GP 21510.

LEMMA (1.2). $\{S^{2n}, K_n\} \approx \mathbf{Z}_2$.

Let $\mu \in \{S^{2n}, K_n\}$ be the generator. We view $u \in H^n(X)$ as a map $u: X \rightarrow K_n$. If $v \in H^{2n}(X)$, there is a map $g_v: X \rightarrow S^{2n}$ such that $g_v^*(s_{2n}) = v$, where $s_{2n} \in H^{2n}(S^{2n})$ is the generator. Define $F(u, v)$ by

$$F(u, v) = \{u\} + \{\mu g_v\}.$$

PROPOSITION (1.3). If X is a $2n$ -dimensional CW complex, then

$$F: H^n(X) \times H^{2n}(X) \longrightarrow \{X, K_n\}$$

is bijective and

$$F(u, v) + F(u', v') = F(u + u', u \cup u' + v + v').$$

Remark. If $u^2 \neq 0$, $F(u, 0)$ has order four.

Let $j: \mathbf{Z}_2 \rightarrow \mathbf{Z}_4$ be the homomorphism sending 1 to 2. Suppose $h: \{X, K_n\} \rightarrow \mathbf{Z}_4$ is a homomorphism. Since $hF(0, v)$ is linear in v , $hF(0, v) = jv(x)$ for some $x \in H_{2n}(X)$. Let $\varphi_h: H^n(X) \rightarrow \mathbf{Z}_4$ be given by

$$\varphi_h(u) = hF(u, 0).$$

Then (1.3) yields:

LEMMA (1.4). $\varphi_h(u + v) = \varphi_h(u) + \varphi_h(v) + j(u \cup v)(x)$. Furthermore, if φ and x satisfy the above formula, $\varphi = \varphi_h$ for some h .

Definition (1.5). An m -Poincaré triple (X, ζ, α) is

- (i) A CW complex X with finitely generated homology (X is without base point).
- (ii) A fibration ζ over X with fibres homotopy equivalent to S^{k-1} , k large, (e.g. $k > m + 1$).
- (iii) $\alpha \in \pi_{m+k}(T(\zeta))$ ($T(\zeta)$ = Thom space) such that an $m + k$ Spanier-Whitehead S -duality is given by

$$S^{m+k} \xrightarrow{\alpha} T(\zeta) \xrightarrow{\Delta} T(\zeta) \wedge X^+$$

where Δ is the diagonal map.

Recall $\Delta\alpha$ being an S -duality means that if $\bar{s}_{m+k} \in H_{m+k}(S^{m+k}; \mathbf{Z})$ is a generator and $y = (\Delta\alpha)_* \bar{s}_{m+k}$,

$$/y: H^{q+k}(T(\zeta); \mathbf{Z}) \approx H_{m-q}(X; \mathbf{Z})$$

for all q . Let $U_k \in H^k(T(\zeta))$ be the Thom class and let $x = U_k/y \in H_m(X)$. It follows from the above isomorphism and the Thom isomorphism that

$$x \cap : H^q(X) \approx H_{m-q}(X).$$

Furthermore, for any CW complex Y ,

$$(1.6) \quad A_\alpha: \{X^+, Y\} \approx \{S^{m+k}, T(\zeta) \wedge Y\},$$

where $A_\alpha\{f\} = \{(\text{id} \wedge f)\Delta\alpha\}$.

Suppose (X, ζ, α) is a $2n$ -Poincaré triple. Let X_i be the connected components of X and let $a_i: S^k \rightarrow T(\zeta|X_i) \subset T(\zeta)$ be the inclusion of a fibre. Let $\lambda_i \in \{S^{2n+k}, T(\zeta) \wedge K_n\}$ be the image of the generator μ under

$$\{S^{2n}, K_n\} = \{S^{2n+k}, S^k \wedge K_n\} \xrightarrow{a_i*} \{S^{2n+k}, T(\zeta) \wedge K_n\}.$$

THEOREM (1.7). *If (X, ζ, α) is a $2n$ -Poincaré triple with fundamental class $x \in H_{2n}(X)$, the functions $\varphi: H^n(X) \rightarrow \mathbf{Z}_4$ satisfying*

$$(1.8) \quad \varphi(u + v) = \varphi(u) + \varphi(v) + j(u \cup v)(x)$$

are in one-to-one correspondence with homomorphisms

$$h: \{S^{2n+k}, T(\zeta) \wedge K_n\} \longrightarrow \mathbf{Z}_4$$

such that $h(\lambda_i) = 2$ under the correspondence $\varphi(u) = hA_\alpha F(u, 0)$ where F and A_α are given in (1.3) and (1.6), respectively.

Proof. Note A_α in (1.6) is natural in Y and it can be applied to the commutative diagram,

$$\begin{array}{ccc} \{X_i, S^{2n}\} & \xrightarrow{A_\alpha} & \{S^{2n+k}, T(\zeta|X_i) \wedge S^{2n}\} \\ \downarrow \mu_* & & \downarrow \mu_* \\ \{X, K_i\} & \xrightarrow{A_\alpha} & \{S^{2n+k}, T(\zeta|X_i) \wedge K_n\}. \end{array}$$

If $g: X_i \rightarrow S^{2n}$, $\mu_* A_\alpha\{g\} = (g^* s_{2n})(x)\lambda_i$. Hence $A_\alpha F(0, v) = v(x)\lambda_i$ if $v \in H^n(X_i)$. (1.7) now follows from (1.3) and (1.4).

Theorem (1.7) suggests a method of constructing quadratic functions for manifolds in a cobordism theory as follows: Suppose $MG = \{MG_k\}$ is a Thom spectrum. Let $\lambda \in \{S^{2n+k}, MG_k \wedge K_n\}$ be the image of $\mu \in \{S^{2n+k}, S^k \wedge K_n\}$ under the inclusion of a fibre. Choose a homomorphism

$$h: \{S^{2n+k}, MG_k \wedge K_n\} \longrightarrow \mathbf{Z}_4$$

such that $h(\lambda) = 2$. If X is a $2n$ -manifold, ζ is its normal bundle, α is obtained from the Thom construction and $W: T(\zeta) \rightarrow MG_k$ comes from a G structure on ζ , then

$$\varphi(u) = hW_* A_\alpha(F(u, 0))$$

gives a function on $H^n(X)$ satisfying the formula in (1.3). This method works so long as $\lambda \neq 0$. We would of course choose h above for each k in a consistent way. This amounts to choosing

$$h: H_{2n}(K_n; MG) \rightarrow \mathbf{Z}_4.$$

Our next result describes when $\lambda \neq 0$ and h exists.

Suppose $Y = \{Y_k, \mu_k\}$ is a spectrum such that Y_k is $k - 1$ connected and $H^0(Y) \approx \mathbf{Z}_2$. Let $U \in H^0(Y)$ be a generator and let $p: S \rightarrow Y$ be a map of the sphere spectrum into Y such that $p^*U \in H^0(S)$ is the generator. Let $\lambda \in H_{2n}(K_n, Y)$ be the element represented by

$$S^{2n+k} \xrightarrow{\mu} S^k \wedge K_n \xrightarrow{p_k \wedge \text{id}} Y_k \wedge K_n.$$

PROPOSITION (1.9). $\lambda \neq 0$ if and only if $\chi(Sq^{n+1})U = 0$, where χ is the canonical antiautomorphism of the Steenrod algebra. Furthermore, if $\lambda \neq 0$, it is at most divisible by 2.

Definition (1.10). A Wu n -spectrum is a pair (Y, h) where Y is as above, $\chi(Sq^{n+1})U = 0$ and $h: H_{2n}(K_n, Y) \rightarrow \mathbf{Z}_4$ is a homomorphism such that $h(\lambda) = 2$. A (Y, h) orientation of a fibration ζ is a map $W: T(\zeta) \rightarrow Y_k$ such that W^*U_k is the Thom class of $T(\zeta)$.

COROLLARY (1.11). If (X, ζ, α) is a $2n$ -Poincaré triple, W is (Y, h) orientation of ζ where (Y, h) is a Wu n -spectrum and $\varphi: H^n(X) \rightarrow \mathbf{Z}_4$ is given by

$$\varphi(u) = h W_* A_\alpha F(u, 0),$$

then

$$\varphi(u + v) = \varphi(u) + \varphi(v) + j(u \cup v)(x)$$

where $x \in H_{2n}(X)$ is the fundamental class.

We next describe the geometrical properties of φ . Suppose $f: S^n \rightarrow X$ and $f^*\zeta$ is trivial. Up to homotopy type $T(f^*\zeta) = S^k \vee S^{n+k}$. Let $\hat{f}: S^{n+k} \rightarrow T(\zeta)$ be the inclusion of $T(f^*\zeta)$ in $T(\zeta)$ restricted to S^{n+k} . Consider the commutative diagram:

$$\begin{array}{ccc} \{X^+, S^n\} \xrightarrow{A_\alpha} \{S^{2n+k}, T(\zeta) \wedge S^n\} & \xrightarrow{W_*} & \{S^{2n+k}, Y_k \wedge S^n\} \\ \downarrow (s_n)_* & & \downarrow \\ \{X^+, K_n\} \xrightarrow{A_\alpha} \{S^{2n+k}, T(\zeta) \wedge K_n\} & \xrightarrow{W_*} & \{S^{2n+k}, Y_k \wedge K_n\}. \end{array}$$

Let $\tilde{f} \in \{X^+, S^n\}$ be such that $A_\alpha \tilde{f} = \hat{f}$, that is, $\tilde{f}$ is the S -dual of f . Let $\bar{f} = s_{n*} \tilde{f}$ and let $\beta_f = W_* A_\alpha(\bar{f})$.

THEOREM (1.12). $\bar{f} = F(u, v)$ where u is the Poincaré dual of the element in $H_n(X)$ represented by $f: S^n \rightarrow X$ and

$$\varphi(u) = jv(x) + h(\beta_f).$$

Hence, if $h(\beta_f) = 0$, $\varphi(u)$ is the obstruction to desuspending $\bar{f}$ to a map of X^+ to K_n .

Proof. Let $B: \{X^+, K_n\} \rightarrow H^n(X)$ be given by $S^l B(g) = g^* S^l \iota_n$ where

$g: S^1 X^+ \rightarrow S^1 K_n$. $BF(u, v) = u$ is the S -dual of the element in $H_{n+k}(T(\zeta))$ represented by $\hat{f}$. Hence u is the Poincaré dual of $f(S^n) \in H_n(X)$. Since $\bar{f} = F(u, v)$,

$$\begin{aligned} h(\beta_f) &= hW_* A_\alpha F(u, v) \\ &= hW_* A_\alpha (F(u, 0) + F(0, v)) \\ &= \varphi(u) + jv(x). \end{aligned}$$

COROLLARY (1.13). *If (X, ζ, α) is a smooth (or PL) manifold, its normal bundle and its Thom map, $f: S^n \rightarrow X$ is a smooth embedding, ν is the normal bundle of $f(S^n)$ in X , ν is stably trivial, and $u \in H^n(X)$ is the Poincaré dual of $f(S^n)$, then $\varphi(u) = \varepsilon + h(\beta_f)$ where*

$$\begin{aligned} \varepsilon &= 0 \text{ if } \nu \text{ is trivial} \\ &= 1 \text{ if } \nu \text{ is not trivial and } n \text{ is odd} \\ &= \text{Euler number of } \nu \bmod 4 \text{ if } n \text{ is even.} \end{aligned}$$

Remark. Suppose $Y = MG$ for some Thom spectrum and W comes from a map $(g', g): (\zeta, X) \rightarrow (\xi_K, BG)$ where ξ_K is the canonical bundle over BG . If $gf = 0$, $W\hat{f} = Vi \in \{S^{n+k}, MG_k\}$ where $V: S^{n+k} \rightarrow S^k$ and $i: S^k \rightarrow MG_k$. From the proof of (1.9) one sees that $\beta_f = m\lambda$ where $m = \text{Hopf invariant of } V$. Hence if $n \neq 1, 3, 7$, $h(\beta_f) = 0$.

We next consider the situation in which X is a Poincaré space boundary. Suppose $X \subset Y$, η is a spherical fibration over Y , $\eta|_X = \zeta$, $\alpha' \in \pi_{2n+k+1}(T(\eta)/T(\zeta))$ maps into $S\alpha$ under the map $T(\eta)/T(\zeta) \rightarrow ST(\zeta)$ and if $y \in H_{2n+1}(Y, X)$ is the element corresponding to α' ,

$$y \cap : H^n(Y) \approx H_{n+1}(Y, X).$$

LEMMA (1.14). *If $i: X \rightarrow Y$ is the inclusion, $\varphi i^* = 0$.*

Suppose (X_i, ζ_i, α_i) , $i = 1, 2$, are $2n$ -Poincaré triples, W_i are (Y, h) orientations of ζ_i , k_i is the fibre dimension of ζ_i ,

$$g: \zeta_1 + 0^{l_1} \longrightarrow \zeta_2 + 0^{l_2}$$

where $k_1 + l_1 = k_2 + l_2$ and 0^{l_i} is the trivial S^{l_i-1} bundle over X_i and g covers $f: X_1 \rightarrow X_2$. W_i define (Y, h) orientations $\bar{W}_i$ of $\zeta_i + 0^{l_i}$. Let φ_i be the quadratic functions on $H^n(X_i)$. The following is immediate.

PROPOSITION (1.15). *If $\bar{W}_1 = T(g)^* \bar{W}_2$ and $T(g)_* S^{l_1} \alpha_1 = S^{l_2} \alpha_2$, then $\varphi_1(f^* u) = \varphi_2(u)$, $u \in H^n(X_2)$.*

We next examine how φ depends on the choice of W and α . Let $W(n) = \{W_k(n)\}$ be the Ω -spectrum where $W_k(n)$ is the fibration over K_k with fibre K_{k+n} and k -invariant $\chi(Sq^{n+1})l_k$. Note by (1.9) there is an h making $(W(n), h)$

a Wu n -spectrum. If Y is a spectrum as in (1.10) there is a map $\tau: Y \rightarrow W(n)$ such that $(Y, h\tau_*)$ is a Wu n -spectrum. Also if (X, ζ, α) is a $2n$ -Poincaré triple, ζ is $W(n)$ orientable since $\chi(Sq^{n+1})$ of the Thom class of ζ is the Wu class v_{n+1} .

If $W_1, W_2: T(\zeta) \rightarrow W_k(n)$ are $(W(n), h)$ orientations, they differ by an element

$$d(W_1, W_2) \in H^n(X) \approx H^{n+k}(T(\zeta)) .$$

PROPOSITION (1.16). *If φ_1 and φ_2 are the quadratic functions from (X, ζ, α) and W_1 and W_2 , respectively, then*

$$\varphi_1(u) = \varphi_2(u) + j(u \cup d(W_1, W_2))(x) .$$

Note if φ_1 and φ_2 are two quadratic functions for (X, ζ, α) , $\varphi_1 - \varphi_2$ is linear and hence $\varphi_1 - \varphi_2 = j(y \cup \quad)$ for some $y \in H^n(X)$. This yields:

COROLLARY (1.17). *The $(W(n), h)$ orientations of (X, ζ, α) are in one-to-one correspondence with functions on $H^n(X)$ satisfying (1.8).*

Let G_k be the H space of unbased maps of S^{k-1} to itself of degree one. Recall if (X, ζ, α_i) , $i = 1, 2$, are Poincaré triples, there is a map $g: X \rightarrow G_k$ which defines an automorphism $\bar{g}: \zeta \rightarrow \zeta$ such that $T(\bar{g})\alpha_2 = \alpha_1$ (see proof of (1.17)). Let $u_i \in H^i(G_k)$ be the classes which transgress to $w_{i+1} \in H^{i+1}(BG_k)$, the $i + 1$ Stiefel-Whitney class. If φ_1, φ_2 , and φ_3 are the quadratic functions associated to (X, ζ, α_1, W) , (X, ζ, α_2, W) , and $(X, \zeta, \alpha_1, T(g)^*W)$, respectively, $\varphi_2 = \varphi_3$ by (1.15) and hence

$$\varphi_1 = \varphi_2 + j(x \cup \quad)$$

where $x = d(T(g)^*W, W)$.

THEOREM (1.18). *If (X, ζ, α_i) , $i = 1, 2$ are $2n$ -Poincaré triples, W is a $(W(n), h)$ orientation of ζ , φ_i are the associated quadratic functions and $g: X \rightarrow G_k$ is a map such that $T(\bar{g})\alpha_1 = \alpha_2$, then*

$$\varphi_1(u) = \varphi_2(u) + j(x \cup u)$$

where

$$x = d(W, T(g)^*W) = \sum v_{n+1-2i} \cup g^*u_{2i-1}$$

where $v_i = v_i(\zeta)$ are the Wu classes.

To obtain numerical invariants for (X, ζ, α, W) we construct an algebraic invariant as follows:

Definition (1.19). Suppose V is a finite dimensional vector space over $\mathbf{Z}_2$. A function $\varphi: V \rightarrow \mathbf{Z}_2$ is quadratic if

$$\varphi(u + v) = \varphi(u) + \varphi(v) + jt(u, v)$$

where $t: V \otimes V \rightarrow \mathbf{Z}_2$ is a bilinear pairing. φ is *nonsingular* if t is. If $\varphi_i: V_i \rightarrow \mathbf{Z}_4$ are two such functions, φ_1 is *isomorphic* to φ_2 if there is a linear isomorphism $T: V_1 \rightarrow V_2$ such that $\varphi_1 = \varphi_2 T$. $(\varphi_1 + \varphi_2): V_1 \oplus V_2 \rightarrow \mathbf{Z}_4$ is defined by $(\varphi_1 + \varphi_2)(u, v) = \varphi_1(u) + \varphi_2(v)$. $(-\varphi_1)(u) = -\varphi_1(u)$. $\varphi_1 \varphi_2: V_1 \otimes V_2 \rightarrow \mathbf{Z}_4$ is the unique quadratic function such that $\varphi_1 \varphi_2(u \otimes u) = \varphi_1(u) \varphi_2(v)$.

THEOREM (1.20). *There is a unique function σ from non-singular quadratic functions as in (1.9) to $\mathbf{Z}_8$ satisfying*

- (i) *If $\varphi_1 \approx \varphi_2$, $\sigma(\varphi_1) = \sigma(\varphi_2)$.*
- (ii) *$\sigma(\varphi_1 + \varphi_2) = \sigma(\varphi_1) + \sigma(\varphi_2)$.*
- (iii) *$\sigma(-\varphi_1) = -\sigma(\varphi_1)$.*
- (iv) *$\sigma(\gamma) = 1$ where $\gamma: \mathbf{Z}_2 \rightarrow \mathbf{Z}_4$ by $\gamma(0) = 0$, $\gamma(1) = 1$.*

Furthermore σ satisfies:

- (v) *$\sigma(\varphi_1 \varphi_2) = \sigma(\varphi_1) \sigma(\varphi_2)$.*
- (vi) *If $\varphi: V \rightarrow \mathbf{Z}_4$, $\sigma(\varphi) = (\dim V) \bmod 2$.*
- (vii) *If $\varphi = j\varphi'$,*

$$\sigma(\varphi) = l(\text{Arf } \varphi')$$

where $l: \mathbf{Z}_2 \rightarrow \mathbf{Z}_8$ is the homomorphism sending 1 to 4.

(viii) *If U is a finitely generated free abelian group, $\theta: U \otimes U \rightarrow \mathbf{Z}$ is a symmetric, unimodular bilinear form, $\psi(u) = \theta(u, u)$ and $\varphi: U/2U \rightarrow \mathbf{Z}_4$ is defined by $\varphi(u) = \psi(u) \bmod 4$, then φ is quadratic and*

$$\sigma(\varphi) = (\text{signature } \psi) \bmod 8.$$

(ix) *Suppose $t: V \otimes V \rightarrow \mathbf{Z}_2$ is the bilinear form associated to $\varphi: V \rightarrow \mathbf{Z}_4$,*

$$V_1 \xrightarrow{\nu} V \xrightarrow{\delta} V_2$$

is an exact sequence of $\mathbf{Z}_2$ vector spaces, and $t': V_1 \otimes V_2 \rightarrow \mathbf{Z}_2$ is a nonsingular bilinear form such that $t'(u, \delta v) = t(\nu u, v)$. If $\varphi \nu = 0$, $\sigma(\varphi) = 0$.

(x) *If $\varphi_1, \varphi_2: V \rightarrow \mathbf{Z}_4$ have the same bilinear form t , then $\varphi_2(u) = \varphi_1(u) + jt(u, x)$ for some x and*

$$\sigma(\varphi_1) - \sigma(\varphi_2) = m(\varphi_1(x))$$

where $m: \mathbf{Z}_4 \rightarrow \mathbf{Z}_8$ sends 1 to 2.

(xi) *$\sigma(\varphi)$ is related to φ by the formula*

$$\sum_{u \in V} i^{\varphi(u)} = \sqrt{2^{\dim V}} e^{\frac{\pi i \sigma(\varphi)}{4}}$$

where $i = \sqrt{-1}$.

Definition (1.21). If (X, ζ, α) is a $2n$ -Poincaré triple, (Y, h) is a Wu

n -spectrum and W is a Y orientation of ζ , we define the Kervaire invariant of (X, ζ, α, W) to be

$$K(X, \zeta, \alpha, W) = \sigma(\varphi) \in \mathbf{Z}_8$$

where $\varphi: H^n(X) \rightarrow \mathbf{Z}_4$ is given by

$$\varphi(u) = hW_* A_\alpha F(u, 0) .$$

(1.20) and (1.14) yield:

COROLLARY (1.22). *If MG is a Thom spectrum and (MG, h) is a Wu n -spectrum, K defines a homomorphism*

$$K: \Omega_{2n}(G) \rightarrow \mathbf{Z}_8$$

where $\Omega_{2n}(G)$ denotes the cobordism group based on G orientable manifolds.

Suppose $(X_i, \zeta_i, \alpha_i, W_i)$ are $2n$ -Poincaré triples with $(W(n), h)$ orientations and $(g, f): (\zeta_1, X_1) \rightarrow (\zeta_2, X_2)$ is a fibre map, ζ_1 and ζ_2 have some fibre dimension and g is homotopy equivalence on fibres. Then $H^n(X_1)$ splits over the cup product pairing into $f^* H^n(X_2) \oplus V$. Furthermore, f^* is a monomorphism (see [3]). (1.15) gives:

COROLLARY (1.23). *If $T(g)^* W_2 = W_1$ and $T(g)_* \alpha_1 = \alpha_2$,*

$$\sigma(\varphi_1|V) = K(X_1, \zeta_1, \alpha_1, W_1) - K(X_2, \zeta_2, \alpha_2, W_2) .$$

If the Wu classes $v_j(\zeta_2) = 0$ for $j = n - (2^j - 1)$, then the above is independent of the choice of α_1 and α_2 . If, in the above, one replaces W_1 by $T(g)^* W_2$, then $\sigma(\varphi|V)$ is independent of the choice of W_2 .

Remark. If X_2 is 1-connected, n is odd, and (X_1, ζ_1, α_1) is a smooth or PL manifold, its normal bundle, and its Thom construction, then $\sigma(\varphi|V)$ is the surgery obstruction to making f a homotopy equivalence [3].

We conclude this section with some examples of cobordism theories in which K is defined.

Example (1.24). If $G_k = \{e\}$, $MG_k = S^k$, and (S, h) is a $W(n)$ spectrum for all n with h the unique map taking $H_{2n}(K_n, S) \approx \mathbf{Z}_2 \rightarrow \mathbf{Z}_4$ such that $\lambda \rightarrow 2$. $\Omega_{2n}(e)$ is framed cobordism and

$$K: \Omega_{2n}(e) \rightarrow \mathbf{Z}_8$$

is 0 or 4 according as the Kervaire invariant [4] is 0 or 1. φ in this case has a somewhat simpler form, namely if $u \in H^n(X)$, $\varphi(u) \in \{S^{2n}, K_n\} \approx \mathbf{Z}_2 \subset \mathbf{Z}_4$ is the composition

$$S^{2n+k} \xrightarrow{\alpha} T(\zeta) \xrightarrow{\Delta} T(\zeta) \wedge X^+ \xrightarrow{T \wedge u} S^k \wedge K_n$$

where T comes from a framing of ζ .

Example (1.25). If $G_k = \text{Spin}_k$, there is a Wu n -spectrum $(M \text{Spin}, h)$ for $n \equiv 1 \pmod{4}$, since

$$\chi(Sq^{n+1}) = \chi(Sq^{n-1})Sq^2 + \chi(Sq^n)Sq^1$$

and $Sq^i U_k = W_i U_k = 0$, $i = 1, 2$. For certain choices of h , the $\bar{K}$ obtained is equivalent to the Kervaire invariant defined in [1].

Example (1.26). If $G_k = SU_k$, the situation is as in (1.25) except h is unique because $H_n(BSU) = 0$.

Example (1.27). Suppose $\{MG_k\}$ is a sequence of classifying spaces. Let $BG_k(v_{n+1})$ be the fibration over BG_k with fibre K_n and k -invariant v_{n+1} . Let $MG_k(v_{n+1})$ be the associated spectrum. Clearly there is an h in this case. These are the cobordism theories studied in [2], particularly $BO_k(v_{n+1})$. Suppose M is a $2n$ -manifold with a v_{n+1} orientation. Each choice of h for $MG(v_{n+1})$ gives a function

$$\varphi_h: H^n(M) \rightarrow \mathbf{Z}_4.$$

Let $L = \{u \mid \varphi_n(u) \text{ is independent of } h\}$. Then $\varphi|L$ is the quadratic function studied in [2]. In [2] the Kervaire invariant of M is defined if φ is zero on the radical of L and is $\sigma(\varphi|L/R)$, where R is the radical. In general this will be different from the invariant we have defined.

Example (1.28). If $G_k = \mathbf{Z}_2$ and $n = 1$, $\Omega_2(\mathbf{Z}_2)$ is the cobordism group of surfaces immersed in $\mathbf{R}^3$. (A cobordism is a 3-manifold immersed in $\mathbf{R}^3 \times I$.) $H_2(K_1, M(\mathbf{Z}_2)) \approx \mathbf{Z}_4$, so that h is unique up to a sign. One may prove:

$$K: \Omega_2(\mathbf{Z}_2) \approx \mathbf{Z}_8.$$

φ has the following geometric interpretation suggested to me by Dennis Sullivan. Let $i: S \rightarrow \mathbf{R}^3$ be an immersion of a compact, closed surface S . If $u \in H^1(S)$, choose an embedded circle $S^1 \subset S$ representing the dual of u . Let T be a tubular neighborhood of S^1 in S . $i(T)$ is a twisted strip in $\mathbf{R}^3$. Then $\varphi(u)$ is the number of half twists of $i(T)$. (Möbius band has ± 1 half twists depending on whether its twist is right- or left-handed.) The number of half twists of $i(T)$ only makes sense modulo four because one must frame the normal bundle of $i(S^1)$ in $\mathbf{R}^3$ in order to count twists. This framing is determined up to 720° since $\pi_1(SO_3) = \mathbf{Z}_2$. In this situation one easily sees that $K((S, i)) = \sigma(\varphi)$ is the surgery obstruction to making S immersion cobordant to S^2 .

Example (1.29). If $G_k = SO_k$ and n is even, there is an h making (MSO_k, h) a Wu n -spectrum for all even n which may be chosen as follows: Let $\bar{U}_k: MSO_k \rightarrow K(\mathbf{Z}, k)$ be the Thom class and let $p_2: K(\mathbf{Z}_2, n) \rightarrow K(\mathbf{Z}_4, 2n)$ represent the Pontrjagin (cohomology) square. Let h be

$$(p_{2*} \bar{U}_*): H_{2n}(K_n, MSO) \longrightarrow H_{2n}(K(\mathbf{Z}_4, 2n); \mathbf{Z}) = \mathbf{Z}_4.$$

With this choice of h , $\varphi(u)$ is $p_2(u)(\bar{x})$ where $\bar{x} \in H_{2n}(X; \mathbf{Z})$ is the fundamental class. S. Morita has shown in [5] that $\sigma(\varphi)$, in this case, is the index of X modulo 8. Hence

$$K: \Omega_{4l}(SO) \rightarrow \mathbf{Z}_8$$

is the index modulo 8.

2. Proofs

Let N be a large integer and $p: E_N \rightarrow K_N$ be the fibration with fibre K_{N+n} and k -invariant $Sq^{n+1} \iota_N$. Let $R: S^N K_n \rightarrow E_{N+n}$ be a map such that $pR = S^N \iota_N$. Let $E_k = \Omega^{N-k} E_N$ and let $R_k: S^k K_n \rightarrow E_{k+n}$ be the adjoint of R . Since $Sq^{n+1} \iota_n = 0$, there is a homotopy equivalence $\iota_n \times \iota_{2n}: E_n \rightarrow K_n \times K_{2n}$.

The following is well known:

LEMMA (2.1). *If $\mu: E_n \times E_n \rightarrow E_n$ is the loop multiplication map,*

$$\mu^*(\iota_{2n}) = \iota_{2n} \otimes 1 + 1 \otimes \iota_{2n} + \iota_n \otimes \iota_n.$$

The following is easily checked:

LEMMA (2.2). *If $m > 1$ and Y is a CW complex such that $\dim Y \leq 2n + m$,*

$$\{Y, S^m \wedge K_n\} = [Y, S^m \wedge K_n]$$

and

$$[Y, S^m \wedge K_n] \stackrel{R_{m*}}{\approx} [Y, E_{n+m}].$$

Proof of (1.2). By (2.1) and (2.2)

$$\{S^{2n}, K_n\} \approx [S^{2n+2}, E_{n+2}] \approx \mathbf{Z}_2.$$

Proof of (1.3). Let T be the isomorphism

$$\{X, K_n\} = [S^2 X, S^2 K_n] \approx [S^2 X, E_{n+2}] \approx [X, E_n] \approx [X, K_n \times K_{2n}].$$

If $u \in H^n(X)$, $v \in H^{2n}(X)$, and $g: X \rightarrow S^{2n}$ is a map such that $g^* s_{2n} = v$,

$$TF(u, v) = T(\{u\} + \{\mu g\}) = u \times v$$

(where $\mu \in \{S^{2n}, K_n\}$ is the generator). With respect to the H space structure on $K_n \times K_{2n}$ coming from E_n , (2.1) yields

$$u \times v + u' \times v' = (u + u') \times (u \cup u' + v + v').$$

Proof of (1.9). We wish to show

$$\{S^{2n}, K_n\} \rightarrow \{S^{2n+k}, S^k \wedge K_n\} \rightarrow \{S^{2n+k}, Y_k \wedge K_n\}$$

is nonzero if and only if $\chi(Sq^{n+1})U_k \neq 0$. Since

$$\lim \{S^{2n+k}, Y^a \wedge K_n\} \approx \{S^{2n+k}, Y_k \wedge K_n\}$$

where Y^a are the finite subcomplexes of Y_k , we may assume Y_k is a finite complex. Let Y^* be an m S -dual of Y_k . Let $g: Y^* \rightarrow S^{m-k}$ be the S -dual of the generator $\nu: S^k \rightarrow Y_k$. S -duality gives a commutative diagram

$$\begin{array}{ccc} \{S^{2n+k}, S^k \wedge K_n\} & \xrightarrow{p_*} & \{S^{2n+k}, Y_k \wedge K_n\} \\ \wr & & \wr \\ \{S^{2n+m}, S^m \wedge K_n\} & \xrightarrow{g^*} & \{S^{2n+k} \wedge Y^*, S^m \wedge K_n\} \\ \wr & & \wr \\ [S^{2n+m}, E_{n+m}] & \xrightarrow{g^*} & [S^{2n+k} \wedge Y^*, E_{n+m}]. \end{array}$$

The fibration $K_{m+2n} \rightarrow E_{n+m} \rightarrow K_m$ gives exact sequences:

$$\begin{array}{ccccc} \longrightarrow & H^{2n+m}(S^{2n+m}) \approx [S^{2n+m}, E_{n+m}] & \longrightarrow & H^m(S^{2n+m}) \\ & \wr & & \downarrow g^* \\ H^{m+n-1}(Y^*) & \xrightarrow{Sq^{n+1}} & H^{2n+m}(Y^*) & \longrightarrow & [S^{2n+k} \wedge Y^*, E_{n+m}] \longrightarrow H^m(Y^*). \end{array}$$

Hence p_* above is nonzero if and only if $Sq^{n+1} H^{m+n-1}(Y^*) = 0$, if and only if $\chi(Sq^{n+1})H^k(Y_k) = 0$. ($\chi(Sq^{n+1})$ corresponds to Sq^{n+1} under S -duality.)

Proof of (1.13). In this situation $T(f^*\zeta)$ is the S -dual of $T(\nu)$. S -duality gives a commutative diagram:

$$\begin{array}{ccc} \{T(\nu), K_n\} & \xrightarrow{d} & \{S^{2n+k}, T(f^*\zeta) \wedge K_n\} \\ \downarrow T^* & & \downarrow i_* \\ \{X^+, K_n\} & \xrightarrow{A_\alpha} & \{S^{2n+k}, T(\zeta) \wedge K_n\} \end{array}$$

where $T: X^+ \rightarrow T(\nu)$ is the Thom construction. If V is the Thom class of ν , $T^*V = F(u, 0)$ where u is the Poincaré dual of $f(S^n) \in H_n(X)$. $T(f^*\zeta) = S^k \vee S^{n+k}$. Hence $dV = \alpha_1 + \alpha_2$ where $\alpha_1 \in \{S^{2n+k}, S^k \wedge K_n\}$ and $\alpha_2 \in \{S^{2n+k}, S^{n+k} \wedge K_n\}$. Note α_2 is the generator because V is the Thom class.

$$\begin{aligned} \varphi(u) &= hW_* A_\alpha F(u, 0) \\ &= hW_* i_*(\alpha_1 + \alpha_2) \\ &= hW_* i_* \alpha_1 + h(\beta_f). \end{aligned}$$

Thus $\varphi(u) - h(\beta_f) = 0$ or 2 according as $\alpha_1 = 0$ or the generator of $\{S^{2n+k}, S^k \wedge K_n\}$. Note α_1 depends only on ν ; hence to compute $\varphi(u)$ we can choose any X^+ containing S^n with normal bundle ν .

Suppose ν is trivial. Referring to (1.12), $\tilde{f} \in \{X^+, S^n\}$ desuspends to $X^+ \rightarrow T(\nu) \rightarrow S^n$ and hence $\tilde{f}$ desuspends to u . Hence by (1.12), $\varphi(u) - h(\beta_f) = 0$.

Suppose ν is not trivial and n is odd. Let $f: S^n \rightarrow S^n \times S^n$ be the diagonal. Then ν is the normal bundle of $f(S^n)$. Take $Y_k = S^k$. Then $\beta_f = 0$, $u = s_n \otimes 1 + 1 \otimes s_n$ and

$$\varphi(u) = \varphi(s_n \otimes 1) + \varphi(1 \otimes s_n) + j(s_n \otimes s_n)(S^n \times S^n) = 2.$$

$\varphi(s_n \otimes 1) = 0$ because $(S^n, *) \subset S^n \times S^n$ has trivial normal bundle.

The same argument as above works for n even if one takes f to be a multiple of the diagonal.

Proof of (1.14). We use $T(\eta) \cup \widehat{T(\zeta)}$, the mapping cone of $T(\zeta) \subset T(\eta)$, instead of $T(\eta)/T(\zeta)$. Consider the following commutative diagram:

$$\begin{array}{ccccc} T(\eta) \cup \widehat{T(\zeta)} & \longrightarrow & T(\eta) \wedge (Y \cup \hat{X}) & & \\ \downarrow & & \downarrow & & \\ S^{2n+k+1} & \xrightarrow{\alpha'} & T(\eta) \wedge SX^+ & \longrightarrow & T(\eta) \wedge SY^+ \xrightarrow{c} Y^k \wedge SK_n \\ & \searrow \alpha & \uparrow & & \\ & & ST(\zeta) & \longrightarrow & T(\zeta) \wedge SX^+ \end{array}$$

where $c = W \wedge Sv$, $v: Y^+ \rightarrow K_n$ and the unlabeled maps are the obvious maps. If $\beta: S^{2n+k+1} \rightarrow Y^k \wedge SK_n$ denotes the composition of the bottom line, $\varphi(i^*v) = h(\beta)$. The top line is zero because $Y \cup \hat{X} \rightarrow SX^+ \rightarrow SY^+$ is zero. Hence $\varphi(i^*v) = 0$.

Proof of (1.16). If W_1 and W_2 are $(W(n), h)$ orientations of ζ and $v = d(W_1, W_2)$, W_2 is the composition

$$T(\zeta) \xrightarrow{\Delta} T(\zeta) \times T(\zeta) \xrightarrow{W_1 \times (vU_k)} W_k(n) \times K_{n+k} \xrightarrow{\mu} W_k(n)$$

where μ is the action of K_{n+k} on $W_k(n)$. Consider:

$$\begin{array}{ccc} (T(\zeta) \wedge X^+) \vee (T(\zeta) \wedge X^+) & & \\ \nearrow \alpha' & \parallel & \nwarrow a \\ (T(\zeta) \vee T(\zeta)) \wedge X^+ & & \\ \cap & & \\ S^{2n+k} \xrightarrow{\Delta\alpha} (T(\zeta) \times T(\zeta)) \wedge X^+ \xrightarrow{b} W_k(n) \wedge K_n \end{array}$$

where α' is a lifting of $\Delta\alpha$, $b = \mu(W_1 \times vU_k) \wedge u$, $a = (W_1 \wedge u) \vee c$, and $c = i(vU_k) \wedge u$, $i: K_{n+k} \rightarrow W_k(n)$ the inclusion. $\alpha' = \alpha_1 + \alpha_2$ where α_1 and α_2 are on the two factors of the wedge.

$$\begin{aligned} \varphi_2(u) &= h(b\Delta\alpha) \\ &= h(a\alpha_1) + h(a\alpha_2) \\ &= \varphi_1(u) + h(a\alpha_2). \end{aligned}$$

The generator of $\pi_{2n+k}(K_{n+k} \wedge K_n)$ goes into the generator of $\{S^{2n+k}, W_k(n) \wedge K_n\}$ (see proof of (1.3)). Hence

$$\begin{aligned} h(a\alpha_2) &= j(vU_k \otimes u)(S^{2n+k}) \\ &= j(v \cup u)(x). \end{aligned}$$

Proof of (1.18). Let l be a large integer, F_l the H -space of degree one, based maps of S^l to itself and G_l the unbased, degree one maps of S^{l-1} to itself. The unreduced suspension gives a map $\rho: G_l \rightarrow F_l$. Consider

$$[X, G_l] \approx [X, F_l] \subset [S^l X^+, S^l] \approx \{X^+, S^0\} \xrightarrow{A_\alpha} \{S^{2n+k}, T(\zeta)\}$$

where A_α is S -duality with respect to α . Under this map $[X, G_l]$ maps to those $\bar{\alpha}$'s such that $(X, \zeta, \bar{\alpha})$ is a Poincaré triple. Hence if (X, ζ, α_i) , $i = 1, 2$ are Poincaré triples, there is a $g: X \rightarrow G_l$ which goes to α_2 under A_{α_1} . g gives a stable automorphism of ζ by

$$\text{id} \times \bar{g}: \zeta + 0^l \rightarrow \zeta + 0^l$$

where 0^l is the trivial bundle $X \times S^{l-1}$ and $\bar{g}(x, s) = (x, g(x)(s))$. $T(\text{id} \times \bar{g})$ is the composition

$$\hat{g}: S^l T(\zeta) \xrightarrow{S^l \Delta} S^l(T(\zeta) \wedge X^+) = T(\zeta) \wedge S^l X^+ \xrightarrow{\text{id} \wedge \tilde{g}} T(\zeta) \wedge S^l = S^l T(\zeta)$$

where $\tilde{g} = \rho g$. By the definition of g , $\hat{g}_* \alpha_1 = \alpha_2$. If W is $W(n)$ orientation for $T(\zeta)$,

$$W': S^l T(\zeta) \xrightarrow{\hat{g}} S^l T(\zeta) \xrightarrow{S^l W} S^l W_k(n) \longrightarrow W_{k+l}(n)$$

is the new orientation produced by g . We wish to determine $d(W, W')$.

Choose a base point for X . Identifying $S^l X^+$ with $S^l \vee S^l X$, the adjoint of $\rho g: X^+ \rightarrow F_l$ gives a map $g': S^l X \rightarrow S^l$. Identify $T(\zeta) \wedge S^l X^+$ with

$$T(\zeta) \wedge (S^l \vee S^l X) = (T(\zeta) \wedge S^l) \vee (T(\zeta) \wedge S^l X).$$

Under this identification, $\text{id} \wedge \tilde{g}$ becomes $(\text{id} \wedge \text{id}) \vee (\text{id} \wedge g')$.

$$T(\zeta) \wedge S^l X \xrightarrow{\text{id} \wedge g'} T(\zeta) \wedge S^l = S^l T(\zeta) \xrightarrow{W} W_{k+l}(n)$$

factors through the inclusion of the fibre $i: K_{k+l+n} \rightarrow W_{k+l}(n)$. Let $W(\text{id} \wedge g') = iv$, $v: T(\zeta) \wedge S^l X \rightarrow K_{k+l+n}$. v will have the form

$$v = \sum x_i U_k \otimes S^l y_i$$

and $d(W, W') = \sum x_i y_i$. By the fibre space definition of functional operations

$$v \in \chi(Sq^{n+1})_{\text{id} \wedge g'}(U_k \otimes s_l).$$

One easily checks that this operation and $\chi(Sq^i)_{g'}(s_l)$, $i > 0$, has zero indeterminacy. Using the Cartan formula, $\chi(Sq^i)U_k = v_i U_k$ and the exact sequence definition of functional operations one easily checks that

$$v = \sum v_{n+1-i} U_k \otimes \chi(Sq^i)_{g'}(s_l).$$

We complete the proof of (1.18) by proving

LEMMA (2.3).

$$\begin{aligned}\chi(Sq^i)_{g'}(s_l) &= S^l(u_{2^{j-1}}) && \text{if } i = 2^j \\ &= 0 && \text{if } i \neq 2^j.\end{aligned}$$

Proof. Let γ be the S^{l-1} fibration over SX defined by $g: X \rightarrow G_l$. It is well known that

$$T(\gamma) = S^l \cup g'S^l X$$

where S^l corresponds to the "fibre" of $T(\gamma)$. Hence

$$S(\chi(Sq^i)_{g'}(s_l))U_l = \chi(Sq^i)U_l$$

where U_l is the Thom class of $T(\gamma)$. $\chi(Sq^i)U_l = v_i(\gamma)U_l$. Let $\hat{g}: SX \rightarrow BG_l$ be the classifying map of γ and $\bar{v}_i \in H^i(BG_l)$ the Wu classes. Then

$$S(\chi(Sq^i)g(s_l)) = \hat{g}^*\bar{v}_i.$$

$\chi(Sq^{2^i}) = Sq^{2^i} + \text{decomposables}$ and $\chi(Sq^j)$ is decomposable for $j \neq 2^i$. Hence $\bar{v}_j$ is decomposable for $j \neq 2^i$ and is $W_{2^i} + \text{decomposables}$ for $j = 2^i$. $H^*(SX)$ has zero cup products. Hence $\hat{g}^*(v_j) = 0$, $j \neq 2^i$ and w_{2^i} for $j = 2^i$. w_j suspends to $u_{j-1} \in H^*(G_l)$ and the lemma is proved.

3. Proof of Theorem (1.20)

Suppose V is a finite dimensional vector space over $\mathbf{Z}_2$ and $\varphi: V \rightarrow \mathbf{Z}_4$ is any function. Let $\lambda\varphi$ be the complex number defined by

$$\lambda(\varphi) = \sum_{u \in V} i^{\varphi(u)}.$$

$\lambda(\varphi)$ is well-defined since $i^4 = 1$.²

LEMMA (3.1). *If φ is linear,*

$$\begin{aligned}\lambda(\varphi) &= 2^{\dim V} && \text{if } \varphi = 0 \\ &= 0 && \text{if } \varphi \neq 0.\end{aligned}$$

Proof. For any $v \in V$

$$\sum_u i^{\varphi(u)} = \sum_u i^{\varphi(u+v)} = i^{\varphi(v)} \sum_u i^{\varphi(u)}.$$

Hence $\lambda(\varphi) = i^{\varphi(v)} \lambda(\varphi)$. Hence $\lambda(\varphi) = 0$ or $\varphi(v) = 0$ for all v .

The following is immediate

LEMMA (3.2).

$$\begin{aligned}\lambda(\varphi_1 + \varphi_2) &= \lambda(\varphi_1)\lambda(\varphi_2) \\ \lambda(-\varphi) &= \overline{\lambda(\varphi)}.\end{aligned}$$

LEMMA (3.3). *If φ is quadratic and non-singular, $\lambda(\varphi)^8$ is a positive real number.*

Proof. Let $t: V \otimes V \rightarrow \mathbf{Z}_2$ be the bilinear form of φ . For some $x \in V$,

² This method of constructing an algebraic invariant was suggested to me by Paul Monsky.

$$t(u, u) = t(u, x)$$

for all u .

$$0 = \varphi(2u) = 2\varphi(u) + jt(u, u).$$

Hence $2\varphi(u) = jt(u, x)$. Therefore

$$\begin{aligned}\varphi(u + v) + \varphi(v) &= \varphi(u) + 2\varphi(v) + jt(u, v) \\ &= \varphi(u) + jt(u + x, v); \\ \lambda(2\varphi) &= \sum_{u,v} i^{\varphi(u) + \varphi(v)} \\ &= \sum_{u,v} i^{\varphi(u+v) + \varphi(v)} \\ &= \sum_u i^{\varphi(u)} \sum_v i^{jt(u+x,v)}.\end{aligned}$$

$t(u + x, v)$ is linear in v and zero for all v if and only if $u = x$. Therefore

$$\lambda(2\varphi) = 2^{\dim V} i^{\varphi(x)}.$$

Hence

$$\lambda(\varphi)^8 = \lambda(2\varphi)^4 = 2^{4 \dim V}.$$

Therefore

$$\lambda(\varphi) = \sqrt[4]{2}^{\dim V} e^{\frac{\pi i}{4} n}$$

for some $n \in \mathbf{Z}_8$. Define $\sigma(\varphi) = n$. By the definition of σ and (3.2), it satisfies (i), (ii), (iii), and (xi) (1.20). If $\gamma: \mathbf{Z}_2 \rightarrow \mathbf{Z}_4$ by $\gamma(0) = 0$, $\sigma(\gamma) = 1$, and hence (iv) is satisfied. We next show that σ is unique. This follows by induction on $\dim V$ and the following lemma.

LEMMA (3.4). *If φ is non-singular quadratic,*

$$\gamma + \varphi \approx \varepsilon_1 \gamma + \varepsilon_2 \gamma + \varphi'$$

where $\varepsilon_i = \pm 1$.

Proof. Choose $u, v \in V$ such that $t(u, v) = 1$ and $u = v$ if possible. Let $V_1 = \{u, v\}$, $V_2 = \{w \in V \mid t(u, w) = t(v, w) = 0\}$. V is a direct sum of V_1 and V_2 and $\varphi|V \approx \varphi|V_1 + \varphi|V_2$. If $u = v$, $\varphi|V_1 \approx \pm \gamma$. If $u \neq v$, let U_1 , U_2 and U_3 be the subspaces of $\mathbf{Z}_2 + V_1$ spanned by $(1, v)$, $(1, u)$, and $(1, u + v)$ respectively. One easily checks that

$$\gamma + \varphi|V_1 = \sum (\gamma + \varphi)|U_i$$

and $(\gamma + \varphi)|U_i = \pm \gamma$.

The multiplicativity of σ easily follows from (3.4) and the fact that $\gamma\varphi = \varphi$ for any φ .

If $\varphi = j\varphi'$, $\lambda(\varphi) = (n - m)$ where n and m are the number of elements in V at which φ' is 0 and 1, respectively. Hence $\sigma(\varphi)$ is 0 or 4 according as $\text{Arf } \varphi'$ is 0 or 1.

To prove (1.20) (ix), there is no loss in generality if we assume

$$0 \longrightarrow V_1 \xrightarrow{\nu} V_1 \xrightarrow{\delta} V_2 \longrightarrow 0$$

is exact. Choose a map $\beta: V_2 \rightarrow V$ such that $\delta\beta = \text{id}$. If $u \in V_1$, and $\varphi\nu = 0$

$$\begin{aligned}\varphi((\nu) + \beta(v)) &= \varphi(\beta(u)) + jt(\nu(u), \beta(v)) \\ &= \varphi(\beta(u)) + jt'(u, v) .\end{aligned}$$

Hence

$$\begin{aligned}\lambda(\varphi) &= \sum i^{\varphi(\nu(u) + \beta(v))} \\ &= \sum_u i^{\varphi(\beta(u))} \sum_v i^{jt'(u, v)} \\ &= \text{positive real number}\end{aligned}$$

since $t'(u, v)$ is linear in v and zero for all v if and only if $u = 0$. Hence if $\varphi\nu = 0$, $\sigma(\varphi) = 0$.

If $\varphi_1, \varphi_2: V \rightarrow \mathbf{Z}_4$, and $\varphi_2(u) = \varphi_1(u) + jt(u, x)$ for some x as in (x),

$$\begin{aligned}\lambda(\varphi_2) &= \sum i^{\varphi_1(u) + jt(u, x)} \\ &= \sum i^{\varphi_1(u+x) - \varphi_1(x)} \\ &= i^{-\varphi_1(x)} \lambda(\varphi_1) .\end{aligned}$$

Hence $\sigma(\varphi_2) = \sigma(\varphi_1) - l\varphi_1(x)$ where $l: \mathbf{Z}_4 \rightarrow \mathbf{Z}_8$ takes 1 to 2.

Finally we prove (1.20) (vii). (viii) is obviously true if θ is a diagonal form. Recall the Grothendieck group of symmetric unimodular bilinear forms over the integers is isomorphic to $\mathbf{Z} + \mathbf{Z}$ and the isomorphism is given by the rank + the signature [6]. This means that if θ is such a form, there are forms D, θ_1 , and θ_2 , where D is diagonal, such that $\theta + (\theta_1 + (-\theta_1)) \approx D + (\theta_2 + (-\theta_2))$. The desired result now follows.

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(Received September 14, 1971)